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Optimal Firing Policy for the Defense

Part I: Confirmation Option

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FOR THE COMMANDER

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OPTIMAL FIRING POLICY FOR THE DEFENSE

PART I: CONFIRMATION OPTION

A. A. GROMETSTEIN

Group 92

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Art is long
And Time is short --
Occasion imminent,
Decision difficult,
And experiment perilous.

Hippocrates

LEXINGTON MASSACHUSETTS

ABSTRACT

This report deals with a task arising during an offense-defense battle, namely, efficient allocation by the defense of interceptor weapons from a limited stockpile when the attacking forces, composed in part of missiles and in part of decoys, must be engaged one by one on a sequential basis. The central factors are:

(a) Observation of the attacking units, and the defense's decisions whether to engage each, is sequential in time as the units come into view, one by one.

(b) The interceptors are limited in quantity and must be employed sparingly.

(c) The nature of each attacking unit (i.e., whether it be missile or decoy) is known only imperfectly at the time of its engagement, but is confirmed immediately after engagement.

Under these assumptions, a firing policy for the defense is derived which is most efficient in destroying the attacking missiles.

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OPTIMAL FIRING POLICY FOR THE DEFENSE
PART I: CONFIRMATION OPTION

I. INTRODUCTION

We study an offense-defense battle, adaptive on the part of the defense, nonadaptive on the part of the offense. The gist of the battle is that the offense prepares and launches a number of components (targets), some of which are missiles and some of which are decoys, to attack a defended region. The defense utilizes an adaptive firing policy to determine which targets to intercept and which to pass.

This problem arose during a study of anti-ballistic missile tactics, and we pose it in corresponding terms:

> A ballistic missile attack, consisting of missiles [war-heads, or re-entry vehicles (RVs)] and of mock missiles (decoys) is launched against a defended region. The defense commander has a stockpile of interceptor weapons as well as a long-range sensor which supplies discrimination information about the approaching targets. The structure and timing of the attack is such that the defense must engage one target at a time, i.e., the defense gathers information on a target from the sensor and decides either to commit one interceptor or to pass the target. Then the defense observes a second target, etc. The decision on a target must be made immediately after examining it and is final; the target cannot be reexamined nor the engagement decision delayed.

The defense suffers damage for each RV passed, either because it is believed to be a decoy or because the interceptors are exhausted. The interceptors in stock may or may not exceed the number of RVs in the attack, but in any case are fewer than the total number of targets, so that the defense cannot afford to fire indiscriminately. However, the discrimination information supplied by the sensor is imperfect and the commander must make his† decisions under some uncertainty as to the nature of the target under observation. What firing doctrine ought he to adopt to maximize the number of RVs destroyed? (Such a doctrine, of course, minimizes the number of RVs which penetrate the defense.)

Emphasis is placed on the sequential and adaptive aspects of the commander's decisions. The engagement policy is developed by an application of Dynamic Programming.

II. ASSUMPTIONS AND DISCUSSION

We assume that the interceptors are inerrant and lethal, so that commitment of one against a target ensures that the target is destroyed. Of course, such an action diminishes the interceptor stockpile and facilitates penetration of RVs which may appear later in the battle; on the other hand, if a target is passed it may prove to be an RV and cause severe damage to the defended region. The commander endeavors to commit his interceptors according to a policy which achieves a suitable compromise between these risks. The measure of performance of a policy is the expected number of RVs destroyed, given a specific attack.

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† For "his," read "his or her," throughout.

Of crucial importance is the amount and type of information available to the commander about the battle, both in advance and during the course. We assume the following:

(a) He knows the initial composition of the attack (i.e., how many RVs and how many decoys are involved), although he does not know the identity of specific targets.

(b) The defense sensor, upon observing a target, produces a score, ω, whose magnitude provides information about the nature of the target.† The commander knows the cumulative distribution function (CDF) of ω, conditional upon the target type; i.e., he knows the functions:

$$P_r(w) = Pr(\omega \leqslant w \mid RV)$$

$$P_d(w) = Pr(\omega \leqslant w \mid DY)$$

together with their derivatives, the conditional probability density functions (PDFs):

$$\pi_r(w) = dP_r(w)/dw$$

$$\pi_d(w) = dP_d(w)/dw \quad .$$

(c) The commander knows how many interceptors remain at any stage in the battle, and how many attacking targets. In addition, he recalls the scores of all targets already observed.

A fourth factor which influences the commander's actions is whether he knows the nature of the targets already engaged. For example, an interceptor committed against a target might disclose (by, say, the nature of the resulting explosion) whether that target had been an RV; conversely, a target passed by the defense might reveal its nature when it reaches the defended region. In such circumstances, the commander knows – after engagement, but otherwise in real-time during the battle – the nature of the target engaged, and therefore the composition of the remaining targets. On the other hand, it may be that the commander remains uncertain about the nature of a target even after engagement.

We refer to this as the Confirmation Factor. After deciding whether to engage a target, the commander may or may not receive confirmation of the nature of that target. This factor then has two options:

(d-1) The commander receives confirmation of the results of each decision, and hence knows the precise composition of the remaining targets throughout the battle (but not, to repeat, their individual identities).

(d-2) The commander's decisions are not confirmed, and his knowledge of the composition of the remaining targets decays with the course of the battle.

† The score is a stochastic scalar quantity. It might be simply a sensor reading (e.g., the amplitude of a returned radar pulse), or it might be the output of a discrimination algorithm whose inputs are sensor readings. The commander must estimate the nature of a new target from the *a priori* probability of occurrence of RVs and decoys, from the scores of previous targets, and from the score of the new target itself. We assume that the score has the range $0 \leqslant \omega \leqslant 1$.

This report deals with the Confirmation Option; it is planned that a later paper will address the more interesting and more difficult non-Confirmation Option.

In this report an optimal engagement policy is formulated and evaluated for the Confirmation Option – optimal in the sense that under that policy the expected number of RVs destroyed is maximized.

III. CONFIRMATION OPTION

If, at any time during the battle, s targets remain, we say that the battle is in Stage s, and we refer to the target addressed at that stage as Target s, or the s^{th} target.

We define the state of the battle at any time by the triplet of non-negative integers, $\langle snr \rangle$, where s is the stage and where

$$n \equiv \text{No. interceptors left to the defense}$$

$$r \equiv \text{No. RVs remaining in the attack.}$$

(There are, then, $d = s - r$ decoys remaining.) Of course, we have $s \geqslant r \geqslant 0$, and we lose no generality by assuming that $s \geqslant n$.

Under the Confirmation Option, the defense knows the state throughout the battle.†

We measure the value of an engagement by the expected number of RVs which, under the defense's firing policy, are destroyed, and denote this quantity by $v_{sn}(r)$, when applied to a battle in State $\langle snr \rangle$. An optimal policy (which may not be unique) is one with an associated optimal value, $v^*_{sn}(r)$, such that:

$$v^*_{sn}(r) \geqslant v_{sn}(r)$$

where $v_{sn}(r)$ is the value of any alternative feasible policy. We will determine $v^*_{sn}(r)$ recursively through an approach associated with the discipline of Dynamic Programming.

We first note that the optimal value can be found trivially if either r or n equal 0 or s:

$v^*_{ss}(r) = r$	The defense fires at each target and intercepts every RV.
$v^*_{s0}(r) = 0$	Each RV will penetrate without interception.
$v^*_{sn}(s) = n$	The defense will intercept n RVs but cannot prevent the rest from penetrating.
$v^*_{sn}(0) = 0$	There being no RVs, the value is nil.

The battle begins in, say, state $\langle SNR \rangle$. As the defense reacts to the approaching targets, one by one, the stage progresses uniformly from S to $(S - 1)$ to $(S - 2)$, etc., whereas the state varies stochastically. If $\langle snr \rangle$ and $\langle \tilde{s}\tilde{n}\tilde{r} \rangle$ are successive states, corresponding to successive stages, then

$$\tilde{s} = (s - 1)$$

$\tilde{n} = n$ or $(n - 1)$ according as the defense passes the s^{th} target or fires at it

$\tilde{r} = r$ or $(r - 1)$ according as Target s is a decoy or RV .

† It is sometimes convenient to distinguish between the initial state of a battle and a state which might be reached during its course; when the distinction is to be made we denote the former by upper-case symbols, e.g., $\langle SNR \rangle$, and the latter by lower-case symbols, e.g., $\langle snr \rangle$.

The battle effectively or actually ends when one or several of the following conditions obtains:

$s = 0$ No targets remain.

$n = 0$ No interceptors are left; hence the value of the remaining engagements is nil.

$r = 0$ No RVs are left; hence the remaining engagements do not influence the value.

We see that a battle cannot have more than $(S + 1)$ stages, counting the initial and final ones, and may have as few as $[1 + \min(R, N)]$, depending on the particular course of the attack. At any stage except the last, transitions are possible from the current state to one of four other states; the sequence from r to $\tilde{r}$ is governed by the mix of targets since we assume that they are examined in random order,† while the sequence from n to $\tilde{n}$ is governed by the scores of the targets and by the firing doctrine. Thus, in the temporal evolution of the state vector, the defense actions affect only the component n; the components s and r change in a purely Markovian manner, independent of what the defense does.

It is informative to examine the possible states for small values of the stage variable.

<u>Stage 0</u>:

There is one state, $\langle 000 \rangle$, for which

$$v^*_{00}(0) = 0$$

and the battle is at an end.

<u>Stage 1</u>:

Four states are possible‡ and the value of each is found trivially.

$$v^*_{10}(0) = 0 \qquad v^*_{10}(1) = 0$$

$$v^*_{11}(0) = 0 \qquad v^*_{11}(1) = 1 \quad .$$

The battle is at an end except for State $\langle 111 \rangle$, in which the defense will fire at Target 1, regardless of its score.

<u>Stage 2</u>:

Of the nine states, the values of eight are found trivially.

$$v^*_{20}(0) = 0 \qquad v^*_{21}(0) = 0 \qquad v^*_{22}(0) = 0$$

$$v^*_{20}(1) = 0 \qquad v^*_{21}(1) = \kappa \qquad v^*_{22}(1) = 1$$

$$v^*_{20}(2) = 0 \qquad v^*_{21}(2) = 1 \qquad v^*_{22}(2) = 2 \quad .$$

The battle is at an end except for the states $\langle 211 \rangle$, $\langle 212 \rangle$, $\langle 221 \rangle$, and $\langle 222 \rangle$. For the last three of these the firing threshold can be taken to be 0, which is equivalent to "fire at the next target, whatever its score."

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† The case in which the offense deliberately adjusts the order of targets so that, e.g., the RVs preferentially appear late in the attack can be treated by a modification of the approach described below.

‡ $(s + 1)^2$ states are possible in Stage s.

We now examine the nontrivial state, ⟨211⟩, to the value of which, lying between 0 and 1, we assign the symbol κ. If we assume that high scores are preferentially associated with RVs – that decoys, in other words, tend to give low scores – a rational firing policy for this engagement takes the form:

> Fire at Target 2 if it shows a score, $\omega \geqslant t$;
> else, fire at Target 1.

How do we find the optimal level t^* of the firing threshold t? We recognize that the battle, if in State ⟨211⟩ at Stage 2, will by Stage 1 be in one of four states:

> ⟨100⟩ if Target 2 is the RV and the defense fires
>
> ⟨110⟩ if Target 2 is the RV and the defense passes
>
> ⟨111⟩ if Target 2 is the decoy and the defense passes
>
> ⟨101⟩ if Target 2 is the decoy and the defense fires.

The optimal values associated with the resulting states are, as we have determined, respectively 0, 0, 1, and 0, since only in the third state, ⟨111⟩, is the RV intercepted as Target 1. However, reaching the first state, ⟨100⟩, accrues a unit of value because the RV is destroyed in Stage 2, so that that state has a <u>transition value</u> associated with reaching it. The value of State ⟨211⟩ is a weighted average of the values of the resulting states, taking into account the value of transitions to those states. The weighting factors are the probabilities of transition, as we set forth below.

The probability that Target 2 is the RV is $r/s = 1/2$ and, of course, the probability that it is the decoy is the complement, 1/2. The CDF of the score from an RV is $P_r(w)$; hence, the probability that an RV score will exceed a threshold t is $1 - P_r(t)$; for a decoy, the probability is $1 - P_d(t)$. The defense, as we have noted, fires only if the score exceeds the threshold.

Combining the values of the four subsequent states (including the value of transition to State ⟨100⟩) with the transition probabilities we have, for the value of State ⟨211⟩ under a threshold t:

$$\begin{aligned} V_{21}(1) &= \frac{1}{2}\left\{[1 - P_r(t)] \cdot 1 + P_r(t) \cdot 0\right\} \\ &\quad + \frac{1}{2}\left\{P_d(t) \cdot 1 + [1 - P_d(t)] \cdot 0\right\} \\ &= \frac{1}{2}\left\{1 + [P_d(t) - P_r(t)]\right\} \quad . \end{aligned} \tag{1}$$

The optimal threshold is chosen to maximize this value; we find that t^* is the root of the equation:†

$$\pi_r(t) - \pi_d(t) = 0 \quad .$$

Equivalently, t^* is the root of the equation established by setting the <u>Likelihood Function</u> $\lambda(t) \equiv \pi_r(t)/\pi_d(t)$, equal to 1. For this value of the threshold, we define $\delta \equiv P_d(t^*) - P_r(t^*)$, and we have:

$$V^*_{21}(1) = \frac{1}{2}(1 + \delta) \quad . \tag{1'}$$

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† See the Appendix for a discussion of cases when the root is not unique.

We have thus determined the optimal values for Stages 2, 1, and 0, as well as the optimal thresholds, in a recursive manner. For higher-order stages, we continue this approach.

Stage s:

If at Stage s the battle is in State ⟨snr⟩, and if the defense adopts a threshold t, then at the next stage the battle will be in one of four states.

Subsequent State	Probability of Transition
⟨s'nr'⟩	$(r/s)\ P_r(t)$
⟨s'n'r'⟩	$(r/s)\ [1 - P_r(t)]$
⟨s'nr⟩	$(d/s)\ P_d(t)$
⟨s'n'r⟩	$(d/s)\ [1 - P_d(t)]$

where

$$s' \equiv s - 1 \quad ; \quad n' \equiv n - 1 \quad ; \quad r' \equiv r - 1 \quad .$$

To find the value of State ⟨snr⟩, we take a weighted average of the values of the resulting states, accounting as well for any transition values. (Only one state provides a value of transition: in the above list, the change to State ⟨s'n'r'⟩ comes about when Target s is an RV and the defense fires, thus adding 1 to the value of the battle.) Employing such considerations, we find [writing, for convenience, $P_r(t)$ as P_r, and similarly for $P_d(t)$]:

$$v_{sn}(r) = (r/s)\left\{P_r v_{s'n}(r') + (1 - P_r)\,[1 + v_{s'n'}(r')]\right\}$$
$$+ (1 - r/s)\left\{P_d v_{s'n}(r) + (1 - P_d)\ v_{s'n'}(r)\right\} \quad . \qquad (2)$$

If the firing policy is such that the succeeding states result in optimal values, Eq.(2) can be rewritten as:

$$v_{sn}(r) = (r/s)\left\{P_r v^*_{s'n}(r') + (1 - P_r)\,[1 + v^*_{s'n'}(r')]\right\}$$
$$+ (1 - r/s)\left\{P_d v^*_{s'n}(r) + (1 - P_d)\ v^*_{s'n'}(r)\right\} \quad . \qquad (3)$$

It is convenient to introduce the difference operator Δ:

$$\Delta v_{\cdot n}(\cdot) \equiv v_{\cdot n+1}(\cdot) - v_{\cdot n}(\cdot)$$

(i.e., Δ operates only on the interceptor subscript n). Then Eq.(3) becomes:

$$v_{sn}(r) = (r/s)\,[1 + v^*_{s'n'}(r')] + (1 - r/s)\ v^*_{s'n'}(r)$$
$$- (r/s)\,[1 - \Delta v^*_{s'n'}(r')]\ P_r + (1 - r/s)\ \Delta v^*_{s'n'}(r)\ P_d \quad . \qquad (3')$$

Clearly, the firing threshold should be chosen to maximize the last line of Eq.(3'); i.e., it is the solution of:

$$\lambda(t) = (s/r - 1)\ \Delta v^*_{s'n'}(r)/[1 - \Delta v^*_{s'n'}(r')] \quad . \qquad (4)$$

Designating the root of Eq.(4) by t^*, we have:

$$v^*_{sn}(r) = (r/s)\,[1 + v^*_{s'n'}(r')] + (1 - r/s)\, v^*_{s'n'}(r)$$

$$- (r/s)\,[1 - \Delta v^*_{s'n'}(r')]\, P_r(t^*) + (1 - r/s)\, \Delta v^*_{s'n'}(r)\, P_d(t^*) \quad . \tag{5}$$

Equation (5) is the basic recursive relation for the optimal value. It expresses the value of State ⟨snr⟩ in terms of known quantities, assuming that the values of later stages (s−1, s−2, etc.) have been found. In passing, we note that the optimal policy – the set of best thresholds – does not depend upon whether a stage under consideration is the first of the battle or an intermediate one reached during the course of the attack. This is in conformity with Bellman's Principle of Optimality which asserts, in effect, that sub-policies of an optimal policy are themselves optimal policies. That is, if an optimal policy takes a system from an initial State A to a final State Z through an intermediate State Q, then the path from A is optimal for reaching Q, and the path from Q is optimal for reaching Z.

Equation (5) can be used to determine the optimal threshold for State ⟨snr⟩ once the values of the immediately succeeding states, ⟨s'..⟩, are known. This is an example of the backward solution – moving from terminal states retrogressively toward initial states – which is common in Dynamic Programming (or Recursive Optimization, as the technique is sometimes and more appropriately known).

IV. EXAMPLES

Since the calculations involved in determining the thresholds and values are extended and laborious, a computer program, WECOM (Weapon Commitment) was written to produce these quantities for as many as S = 40 targets.† To illustrate the work we choose a simple case.

Example 1:

We take the PDFs of the scores from RV and decoy to be:

$$\pi_r(w) = 2w \quad ; \qquad 0 \leqslant w \leqslant 1$$

$$\pi_d(w) = 2(1 - w) \quad ; \qquad 0 \leqslant w \leqslant 1 \tag{6}$$

with corresponding CDFs:

$$P_r(w) = w^2$$

$$P_d(w) = 1 - (1 - w)^2 \quad . \tag{7}$$

The Likelihood Function is then:

$$\lambda(w) = w/(1 - w) \tag{8}$$

and the condition of monotonicity of the Appendix is met (i.e., $d\lambda/dw \geqslant 0$), so that the Likelihood Equation will have at most one root for any condition of interest.

† WECOM is written in FORTRAN and runs on an IBM 370/65. Its operational code is about 500 statements long; storage for various arrays is additional. Its running time is less than one second for any initial state permitted. The limitation to 40 targets mainly arises from underflow problems when calculating thresholds for states with numerous targets.

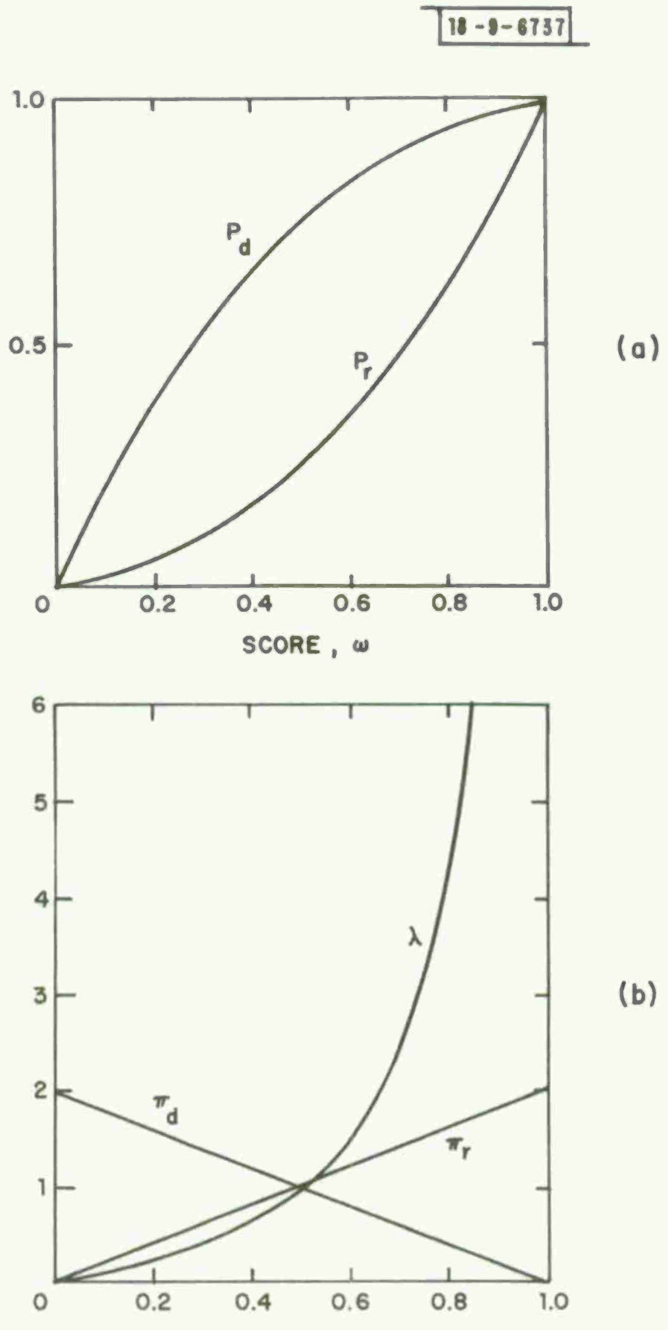

Fig. 1. (a) CDFs, Example 1 and (b) PDFs and Likelihood Function, Example 1.

Figure 1(a) shows the CDFs for the two targets, while Fig.1(b) shows the PDFs and the Likelihood Function.

WECOM was run with these choices of CDFs and PDFs for an initial state, ⟨16, 10, 10⟩. Figure 2(a) shows a portion of the Threshold Matrix printout.† Note that the threshold for State ⟨211⟩ is 0.500; this is found by solving the Likelihood Equation

$$\lambda(t) = t/(1 - t) = 1$$

to obtain the root $t^* = 0.500$.

Figure 2(b) shows the corresponding portion of the Value Matrix printout; the arrangement on the page parallels that of Fig.2(a). The value of State ⟨211⟩ is $v^*_{21}(1) = 0.750$; this is found by inserting t* (above) into Eq.(1) to obtain [since $\delta = P_d(0.5) - P_r(0.5) = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$]:

$$\kappa = \frac{1}{2}(1 + \frac{1}{2}) = 0.750 \quad .$$

This value is to be contrasted with the value nr/s, which is the expected number of RVs destroyed if the defense ignores the target scores and fires at the targets indifferently. (We refer to the value of such a firing policy as a blind value.) In the present case, the blind value is $1 \cdot \frac{1}{2} = \frac{1}{2}$. Hence the optimal policy has a value which is $0.750/\frac{1}{2} = 1.5$ superior to the blind value.

Other entries in the matrices are computed analogously.

Note that the entries must be interpreted as averages over a series of trials. We define a partie as a battle beginning in a specified state and fought to a conclusion according to a particular firing policy. Another partie beginning in the same state and following the same policy might have a different outcome, depending on the random sequence in which the targets are observed and the stochastic nature of the scores. The value, then, is the mean number of RVs destroyed when reckoned over a large and random selection of parties. For example, in Fig.2(b), the entry for State ⟨633⟩ indicates that, if the defense has 3 interceptors to defend against an attack of 3 RVs and 3 decoys, 2.163 RVs will be destroyed on the average. In any one partie, of course, an integral number of RVs will be destroyed, ranging from 0 to 3;‡ but, as remarked above, the mean number destroyed is 2.163 if the defense firing policy is optimal.

Examination of Fig.2(b) shows that the entries change in an expected manner with s, n, and r:

(a) Reading across any row of constant s and n, the entries increase with r until, when r = s, the entry is equal to n ("each interceptor kills an RV, since all targets are RVs")§

(b) Reading down the entries of a given subcolumn of constant s and r, we note that increasing the number of interceptors increases the value of the partie.¶

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† Column and row headings and caption have been supplied by typewriter. The trivial rows, n = 0 and n = s, are omitted, as is the trivial first column, r = 0.

‡ The number of RVs killed in a partie can be as few as max(0, R + N − S) or as many as min(R, N). Of course, the numbers of penetrating and killed RVs add to R.

§ The column corresponding to r = 8 for s = 8 has been deleted.

¶ The lines corresponding to n = s have been omitted; the entries would be equal to r.

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s	n	r = 1	2	3	4	5	6	7
2	1	0.500	0.0					
3	1	0.600	0.667	0.0				
	2	0.333	0.400	0.0				
4	1	0.655	0.708	0.750	0.0			
	2	0.434	0.500	0.566	0.0			
	3	0.250	0.292	0.345	0.0			
5	1	0.692	0.736	0.771	0.800	0.0		
	2	0.497	0.558	0.612	0.659	0.0		
	3	0.341	0.388	0.442	0.503	0.0		
	4	0.200	0.229	0.264	0.308	0.0		
6	1	0.719	0.756	0.787	0.812	0.833	0.0	
	2	0.542	0.597	0.644	0.685	0.719	0.0	
	3	0.402	0.449	0.500	0.551	0.598	0.0	
	4	0.281	0.315	0.356	0.403	0.458	0.0	
	5	0.167	0.188	0.213	0.244	0.281	0.0	
7	1	0.739	0.772	0.800	0.822	0.841	0.857	0.0
	2	0.576	0.626	0.669	0.704	0.735	0.761	0.0
	3	0.448	0.493	0.540	0.585	0.626	0.661	0.0
	4	0.339	0.374	0.415	0.460	0.507	0.552	0.0
	5	0.239	0.265	0.296	0.331	0.374	0.424	0.0
	6	0.143	0.159	0.178	0.200	0.228	0.261	0.0
8	1	0.756	0.786	0.810	0.831	0.848	0.863	0.875
	2	0.604	0.650	0.688	0.720	0.748	0.771	0.792
	3	0.485	0.527	0.570	0.611	0.647	0.679	0.707
	4	0.383	0.419	0.458	0.500	0.542	0.581	0.617
	5	0.293	0.321	0.353	0.389	0.430	0.473	0.515
	6	0.208	0.229	0.252	0.280	0.312	0.350	0.396
	7	0.125	0.137	0.152	0.169	0.190	0.214	0.244

Fig. 2(a). Matrix of thresholds. $\langle SNR \rangle = \langle 16, 10, 10 \rangle$.

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s	n	r = 1	2	3	4	5	6	7
2	1	0.750	1.000					
3	1	0.633	0.889	1.000				
	2	0.889	1.633	2.000				
4	1	0.561	0.815	0.938	1.000			
	2	0.808	1.447	1.808	2.000			
	3	0.938	1.815	2.561	3.000			
5	1	0.511	0.760	0.889	0.960	1.000		
	2	0.747	1.325	1.677	1.882	2.000		
	3	0.882	1.677	2.325	2.747	3.000		
	4	0.960	1.889	2.760	3.511	4.000		
6	1	0.472	0.716	0.850	0.927	0.972	1.000	
	2	0.699	1.235	1.578	1.790	1.920	2.000	
	3	0.834	1.571	2.163	2.571	2.834	3.000	
	4	0.920	1.790	2.578	3.235	3.699	4.000	
	5	0.972	1.927	2.850	3.716	4.472	5.000	
7	1	0.442	0.681	0.817	0.898	0.948	0.980	1.000
	2	0.660	1.164	1.500	1.715	1.853	1.942	2.000
	3	0.794	1.486	2.040	2.436	2.705	2.883	3.000
	4	0.883	1.705	2.436	3.040	3.486	3.794	4.000
	5	0.942	1.853	2.715	3.500	4.164	4.660	5.000
	6	0.980	1.948	2.898	3.817	4.681	5.442	6.000
8	1	0.417	0.651	0.789	0.873	0.926	0.961	0.984
	2	0.627	1.107	1.434	1.652	1.796	1.892	1.956
	3	0.759	1.415	1.942	2.328	2.598	2.785	2.913
	4	0.850	1.632	2.321	2.889	3.321	3.632	3.850
	5	0.913	1.785	2.598	3.328	3.942	4.415	4.759
	6	0.956	1.891	2.796	3.652	4.434	5.107	5.627
	7	0.984	1.961	2.926	3.873	4.789	5.651	6.417

Fig.2(b). Matrix of values. ⟨SNR⟩ = ⟨16, 10, 10⟩.

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s	n	r = 1	2	3	4	5	6	7
2	1	0.500	0.0					
3	1	0.626	0.667	0.0				
	2	0.333	0.374	0.0				
4	1	0.693	0.723	0.750	0.0			
	2	0.459	0.500	0.541	0.0			
	3	0.250	0.277	0.307	0.0			
5	1	0.736	0.760	0.781	0.800	0.0		
	2	0.536	0.573	0.606	0.637	0.0		
	3	0.363	0.394	0.427	0.464	0.0		
	4	0.200	0.219	0.240	0.264	0.0		
6	1	0.766	0.786	0.804	0.820	0.833	0.0	
	2	0.591	0.622	0.651	0.676	0.700	0.0	
	3	0.439	0.469	0.500	0.531	0.561	0.0	
	4	0.300	0.324	0.349	0.378	0.409	0.0	
	5	0.167	0.180	0.196	0.214	0.234	0.0	
7	1	0.789	0.806	0.821	0.835	0.846	0.857	0.0
	2	0.631	0.659	0.683	0.705	0.726	0.744	0.0
	3	0.495	0.523	0.551	0.578	0.604	0.628	0.0
	4	0.372	0.396	0.422	0.449	0.477	0.505	0.0
	5	0.256	0.274	0.295	0.317	0.341	0.369	0.0
	6	0.143	0.154	0.165	0.179	0.194	0.211	0.0
8	1	0.807	0.822	0.835	0.847	0.857	0.867	0.875
	2	0.663	0.687	0.709	0.728	0.746	0.762	0.776
	3	0.539	0.565	0.590	0.614	0.636	0.657	0.677
	4	0.427	0.451	0.475	0.500	0.525	0.549	0.573
	5	0.323	0.343	0.364	0.386	0.410	0.435	0.461
	6	0.224	0.238	0.254	0.272	0.291	0.313	0.337
	7	0.125	0.133	0.143	0.153	0.165	0.178	0.193

Fig.3(a). Matrix of thresholds. $\langle SNR \rangle = \langle 16, 10, 10 \rangle$; $\mu = 1.5$.

(c) Skipping down the table for various s but for the same n and r shows the entries decreasing as s increases, since the change is of an increasing number of decoys, which dilutes the defense's action against a fixed number of RVs.

(d) The superiority of the optimal value to the blind value is greatest when there are few RVs and few interceptors (i.e., when the ratios n/s and r/s are small). Conversely, when there are few decoys or there are many interceptors, optimal value is not much greater than the blind value.

Figure 2(a) shows trends which, again, meet expectations, and which can be summarized by saying that the threshold increases as the fraction of RVs (r/s) increases or as the number of interceptors decreases.

To exhibit the manner in which the values and thresholds vary as the degree of discriminability of RVs and decoys changes, we introduce:

Example 2:

We extend the CDFs of Example 1 [see Eq.(7)] by taking:

$$P_r(w) = w^{\mu}$$

$$P_d(w) = 1 - (1 - w)^{\mu} \tag{7'}$$

whence:

$$\pi_r(w) = \mu w^{\mu-1} \quad , \quad \pi_d(w) = \mu(1 - w)^{\mu-1} \tag{6'}$$

and

$$\lambda(w) = [w/(1 - w)]^{\mu-1} \tag{8'}$$

[cf. Eqs.(6) and (8)].

This example serves as a generalization of Example 1 by introducing the parameter, $\mu \geqslant 1$. Discrimination grows progressively easier for larger values of this parameter and, of course, Example 1 is equivalent to taking $\mu = 2$.

Figures 3(a) and (b) list the values and thresholds for the case, $\mu = 1.5$, and Figs.4(a) and (b) for the case, $\mu = 5.0$. It is evident that discrimination is more difficult in the former case than for Example 1, while it is more assured in the second case.

WECOM can be used to list the values and thresholds of any inserted CDFs and PDFs for any initial state (limited, as has been mentioned, to 40 targets or fewer). Of course, Examples 1 and 2 are somewhat simple in that π_r and π_d are mirror images of each other, through the line, $w = \frac{1}{2}$. However, introduction of asymmetric CDFs introduces no essentially new factor, and simply biases the threshold, compared with the symmetric case.

The algebraic expressions underlying the entries in the matrices of values and thresholds become exceedingly complicated even for small s, n, and r; we have, e.g., for $\langle 211 \rangle$:

$$\text{Value} = 1 - 2^{-\mu}$$

$$\text{Threshold} = \tfrac{1}{2}$$

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s	n	r = 1	2	3	4	5	6	7
2	1	0.646	1.000					
3	1	0.501	0.808	1.000				
	2	0.808	1.501	2.000				
4	1	0.417	0.691	0.875	1.000			
	2	0.686	1.256	1.686	2.000			
	3	0.875	1.691	2.417	3.000			
5	1	0.362	0.612	0.786	0.911	1.000		
	2	0.603	1.099	1.486	1.780	2.000		
	3	0.780	1.486	2.099	2.603	3.000		
	4	0.911	1.786	2.612	3.362	4.000		
6	1	0.323	0.553	0.719	0.841	0.932	1.000	
	2	0.541	0.987	1.342	1.619	1.834	2.000	
	3	0.706	1.337	1.882	2.337	2.706	3.000	
	4	0.834	1.619	2.342	2.987	3.541	4.000	
	5	0.932	1.841	2.719	3.553	4.323	5.000	
7	1	0.293	0.507	0.666	0.785	0.876	0.946	1.000
	2	0.493	0.902	1.232	1.495	1.704	1.869	2.000
	3	0.647	1.223	1.720	2.140	2.487	2.771	3.000
	4	0.771	1.487	2.140	2.720	3.223	3.647	4.000
	5	0.869	1.704	2.495	3.232	3.902	4.493	5.000
	6	0.946	1.876	2.785	3.666	4.507	5.293	6.000
8	1	0.269	0.470	0.622	0.739	0.829	0.900	0.956
	2	0.454	0.834	1.144	1.395	1.598	1.762	1.894
	3	0.600	1.131	1.593	1.986	2.315	2.589	2.815
	4	0.718	1.380	1.981	2.515	2.981	3.380	3.718
	5	0.815	1.589	2.315	2.986	3.593	4.131	4.600
	6	0.894	1.762	2.598	3.395	4.144	4.834	5.454
	7	0.956	1.900	2.829	3.739	4.622	5.470	6.269

Fig.3(b). Matrix of values. $\langle SNR \rangle = \langle 16, 10, 10 \rangle$; $\mu = 1.5$.

-8-8742

s	n	r = 1	2	3	4	5	6	7
2	1	0.500	0.0					
3	1	0.541	0.667	0.0				
	2	0.333	0.459	0.0				
4	1	0.565	0.679	0.750	0.0			
	2	0.378	0.500	0.622	0.0			
	3	0.250	0.321	0.435	0.0			
5	1	0.582	0.688	0.755	0.800	0.0		
	2	0.406	0.524	0.635	0.708	0.0		
	3	0.292	0.365	0.476	0.594	0.0		
	4	0.200	0.245	0.312	0.418	0.0		
6	1	0.595	0.695	0.760	0.803	0.834	0.0	
	2	0.426	0.541	0.645	0.714	0.762	0.0	
	3	0.320	0.393	0.500	0.607	0.680	0.0	
	4	0.238	0.286	0.355	0.459	0.574	0.0	
	5	0.167	0.197	0.240	0.305	0.405	0.0	
7	1	0.605	0.701	0.763	0.805	0.835	0.858	0.0
	2	0.443	0.554	0.652	0.718	0.765	0.799	0.0
	3	0.341	0.413	0.517	0.617	0.686	0.735	0.0
	4	0.265	0.314	0.383	0.483	0.587	0.659	0.0
	5	0.201	0.235	0.282	0.348	0.446	0.557	0.0
	6	0.143	0.165	0.195	0.237	0.299	0.395	0.0
8	1	0.614	0.706	0.766	0.807	0.837	0.859	0.877
	2	0.456	0.565	0.658	0.722	0.767	0.800	0.826
	3	0.358	0.429	0.530	0.625	0.691	0.738	0.774
	4	0.285	0.335	0.403	0.500	0.597	0.665	0.715
	5	0.226	0.262	0.309	0.375	0.470	0.571	0.642
	6	0.175	0.200	0.233	0.278	0.342	0.435	0.544
	7	0.125	0.142	0.164	0.193	0.234	0.294	0.386

Fig. 4(a). Matrix of thresholds. $\langle SNR \rangle = \langle 16, 10, 10 \rangle$; $\mu = 5.0$.

-8-8743

s	n	r = 1	2	3	4	5	6	7
2	1	0.969	1.000					
3	1	0.951	0.996	1.000				
	2	0.996	1.951	2.000				
4	1	0.937	0.993	0.999	1.000			
	2	0.992	1.921	1.992	2.000			
	3	0.999	1.993	2.937	3.000			
5	1	0.927	0.990	0.998	1.000	1.000		
	2	0.988	1.899	1.985	1.998	2.000		
	3	0.998	1.985	2.899	2.988	3.000		
	4	1.000	1.998	2.990	3.927	4.000		
6	1	0.918	0.988	0.997	0.999	1.000	1.000	
	2	0.985	1.881	1.980	1.996	1.999	2.000	
	3	0.996	1.978	2.870	2.978	2.996	3.000	
	4	0.999	1.996	2.980	3.881	3.985	4.000	
	5	1.000	1.999	2.997	3.988	4.918	5.000	
7	1	0.911	0.986	0.997	0.999	1.000	1.000	1.000
	2	0.981	1.867	1.975	1.994	1.998	2.000	2.000
	3	0.995	1.972	2.847	2.970	2.993	2.999	3.000
	4	0.999	1.993	2.970	3.847	3.972	3.995	4.000
	5	1.000	1.998	2.994	3.975	4.867	4.981	5.000
	6	1.000	2.000	2.999	3.997	4.986	5.911	6.000
8	1	0.904	0.984	0.996	0.999	1.000	1.000	1.000
	2	0.978	1.854	1.971	1.992	1.998	1.999	2.000
	3	0.994	1.966	2.828	2.963	2.990	2.997	2.999
	4	0.998	1.991	2.961	3.820	3.961	3.991	3.998
	5	0.999	1.997	2.990	3.963	4.828	4.966	4.994
	6	1.000	1.999	2.998	3.992	4.971	5.854	5.978
	7	1.000	2.000	3.000	3.999	4.996	5.984	6.904

Fig. 4(b). Matrix of values. $\langle SNR \rangle = \langle 16, 10, 10 \rangle$; $\mu = 5.0$.

while for $\langle 311 \rangle$:

$$\text{Value} = (1/3)\left\{1 - t^{*\mu} + 2(1 - 2^{-\mu})\left[1 - (1 - t^*)^{\mu}\right]\right\}$$

where

$$\text{Threshold, } t^* = \left[1 + 2(2^{\mu} - 1)^{-1/(\mu-1)}\right]^{-1} \quad .$$

Expressions for earlier stages (larger s) are difficult to write explicitly.

V. CONCLUDING COMMENTS

We have derived and presented an algorithm for an optimal firing policy by the defense in the face of uncertain target identification and confirmation after engagement.

Numerical examples have been given of partie values and firing thresholds for illustrative probability densities of the targets' scores. Calculation of the thresholds for the defense commander is feasible either in real-time during an engagement, or by means of pre-calculated tables.

If the commander observes the optimal thresholds then, under the given assumptions as to the nature of the engagement, he will use his interceptors most effectively and will destroy the largest possible number of threatening targets.

APPENDIX
NON-MONOTONIC LIKELIHOOD RATIO

The assumption that high scores are preferentially associated with RVs may be taken to mean that $\lambda(x)$ increases monotonically with x. There is no loss of generality in assuming that scores lie in the interval 0 to 1 and restricting our attention to this interval. Then the Likelihood Equation, $\lambda(x) = L$, say, has at most one root in the interval. If a root exists there, it is the optimal threshold; if no root exists in the interval, then either $\lambda(x) > L$ or $\lambda(x) < L$ throughout. The former situation corresponds to a case in which interceptors are so plentiful that the defense ought to fire at the next target regardless of its score, and we take $t^* = 0$; the latter situation corresponds to a case in which interceptors are so scarce that the next target should be passed, regardless of score, and we take $t^* = 1$.

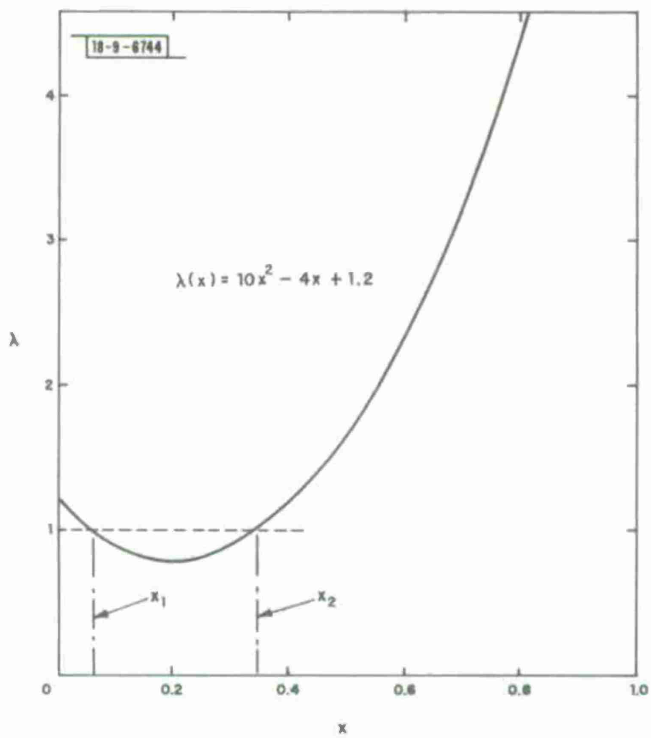

Fig. A-1. Non-Monotonic Likelihood Ratio.

If $\lambda(x)$ is not monotonic increasing in x, a complication arises in that there may be multiple roots of the Likelihood Equation.† Each instance must be treated in detail depending on the precise form of the Likelihood Function, and we illustrate the treatment for a particular form. Suppose we have:

$$\lambda(x) = 10x^2 - 4x + 1.2 \quad ; \quad 0 \leqslant x \leqslant 1$$

(see Fig. A-1). Then, in general, two thresholds must be found, with the defense firing if the score is either lower than x_1 or higher than x_2. An equation analogous to Eq. (1) in the text can be written:

† The case in which $\lambda(x)$ decreases monotonically presents no difficulty.

$$v_{21}(1) = \frac{1}{2}\ \{[P_r(x_2) - P_r(x_1)] \cdot 1 + [1 - P_r(x_2) + P_r(x_1)] \cdot 0$$

$$+ [P_d(x_2) - P_d(x_1)] \cdot 0 + [1 - P_d(x_2) + P_d(x_1)] \cdot 1\}$$

$$= \frac{1}{2}\ \{1 + [P_r(x_2) - P_d(x_2)] - [P_r(x_1) - P_d(x_1)]\}$$

and the thresholds must be chosen to satisfy

$$\pi_r(x_2) - \pi_d(x_2) = 0 \quad ; \quad \pi_r(x_1) - \pi_d(x_1) = 0$$

or, equivalently,

$$\lambda(x_2) = \lambda(x_1) = 1 \quad .$$

(For states other than $\langle 211 \rangle$, of course, the Likelihood Equations would take the form $\lambda(x_2) = \lambda(x_1) = L \neq 1$.)

In our example, where the Likelihood Function has a quadratic form, we find (e.g., see Fig. A-1):

(a) If $L = 1$ there are two roots, $(2 \pm 2^{1/2})/10$, so that

$$x_1 = 0.059 \quad ; \quad x_2 = 0.341$$

and the defense fires if the target score is 0.059 or less or if it is 0.341 or greater.

(b) If $L = 2$, we would have

$$x_1 = 0.000 \quad ; \quad x_2 = 0.547$$

and the defense fires only if the score is at least as large as 0.547.

(c) If the value of L were 0.6, there would be no real roots, and the defense would fire regardless of score.

(d) Finally, if L = , say, 8.0, there are again no real roots, and the defense passes regardless of score.

More complicated forms of the Likelihood Equation can be treated similarly. Examples 1 and 2 in this report deal with a Likelihood Function which increases monotonically and for which there is therefore at most one root.

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20. ABSTRACT *(Continue on reverse side if necessary and identify by block number)*

This report deals with a task arising during an offense-defense battle; namely, efficient allocation by the defense of interceptor weapons from a limited stockpile when the attacking forces, composed in part of missiles and in part of decoys, must be engaged one by one on a sequential basis. The central factors are: (a) Observation of the attacking units, and the defense's decisions whether to engage each, is sequential in time as the units come into view, one by one. (b) The interceptors are limited in quantity and must be employed sparingly. (c) The nature of each attacking unit (i.e., whether it be missile or decoy) is known only imperfectly at the time of its engagement, but is confirmed immediately after engagement.

Under these assumptions, a firing policy for the defense is derived which is most efficient in destroying the attacking missiles.

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