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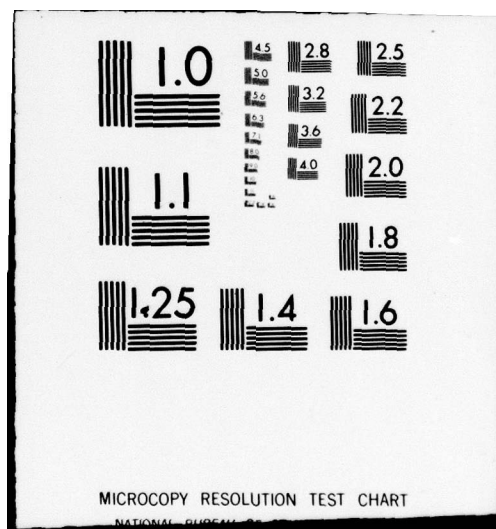
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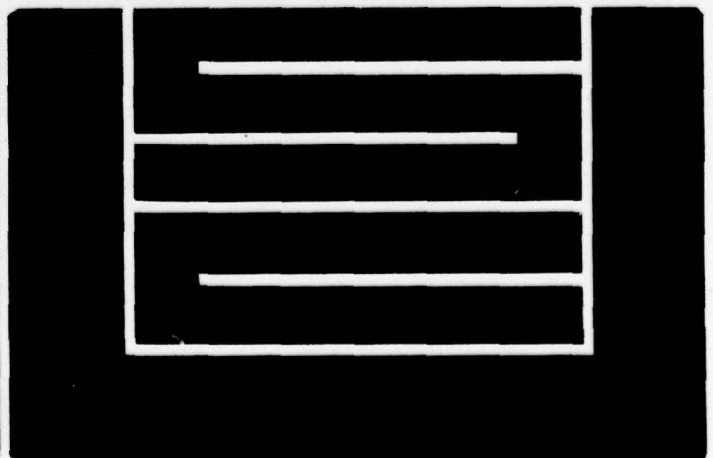
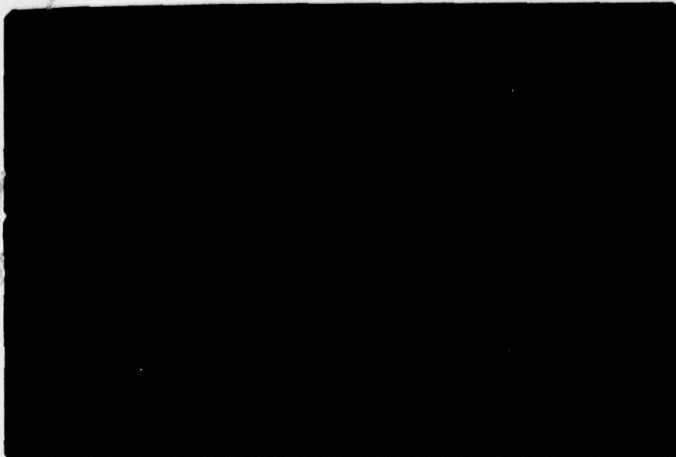
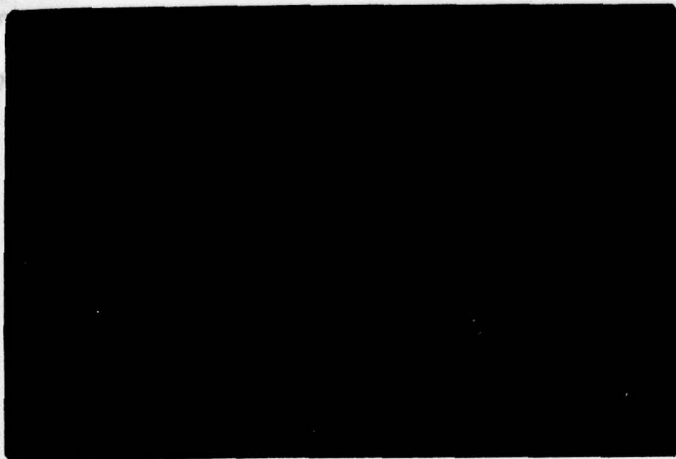
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**Convergence of Weighted Sums of Random
Elements in Type p Spaces**

by

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**University of South Carolina
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Abstract

Let $\{X_{nk} : n \geq 1, k \geq 1\}$ be an array of row-wise independent random elements in a Banach space of type p , $1 < p \leq 2$. The convergence of $\sum_{k=1}^{\infty} a_{nk} X_{nk}$ in probability and almost surely is obtained under varying moment and distribution conditions on $\{X_{nk}\}$. In particular, laws of large numbers are obtained for triangular arrays of random elements. Finally, the direct applications of these results in obtaining consistency of the kernel density estimates are indicated.

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1. Introduction and Preliminaries. The study of laws of large numbers in Banach spaces has led to the geometric considerations of Banach spaces such as Beck convexity (Beck (1963)), G_α (Woyczynski (1973)), and type-p (Hoffmann-Jørgensen and Pisier (1976))-as well as many other authors. The study of probability density estimation led to estimates in the form of averages or weighted sums of random variables whose values are in function spaces (Parzen (1962) and Rosenblatt (1971)). As a application of the law of the iterated logarithm in linear measurable spaces, Kuelbs (1978) considered the rates of convergence in these density estimates. The estimates are not always averages of sequences of random elements in a Banach space but are more often weighted sums of arrays of random elements where the weights are not necessarily Toeplitz matrices.

In this paper the convergence of weighted sums of arrays of random elements in Banach spaces of type p is obtained both in probability and almost surely. As corollaries these results have forms of Pruitt's (1963) and Rohatgi's (1971) results for Banach spaces and also have extensions for the results of Padgett and Taylor (1976). However, exact statements of the theorems are related to the possible applications for density estimation. These applications are indicated in Sections 3 and 4.

Let E denote a real separable Banach space with norm $|| \cdot ||$. Let $(\Omega, \mathcal{A}, P)$ denote a probability space. A random element X in E is a function from Ω into E which is $\mathcal{A}$ -measurable with respect to the Borel subsets of E . The expected value of X is defined to be the Pettis integral (when it exists) and is denoted by EX . The moments of a random element X are $E(||X||^p)$ where E is the expected value of the (real-valued) random

variable $||X||^p$. The concepts of independence, identical distributions, and convergence have direct extensions to E.

Definition 1: A separable Banach space E is said to be of type p, $1 \leq p \leq 2$, if and only if there exists $C_1 \in \mathbb{R}^+$ such that

$$E(||\sum_{k=1}^n X_k||^p) \leq C_1 \sum_{k=1}^n E(||X_k||^p)$$

for every finite collection of independent random elements $X_1, \dots, X_n$ in E with mean 0 and finite pth moments.

Hoffmann-Jørgensen and Pisier (1976) had several equivalent statements for type p spaces including the strong law of large numbers. For the results of this paper, the following analogue of Marcinkiewicz-Zygmund inequality by Woycznski (preprint) will be used.

Proposition 1: Let $1 \leq p \leq 2$ and $q \geq 1$. Then E is of type p if and only if there exists $C_2 \in \mathbb{R}^+$ such that

$$E(||\sum_{k=1}^n X_k||^q) \leq C_2 E[(\sum_{k=1}^n ||X_k||^p)^{q/p}]$$

for all independent random elements $X_1, \dots, X_n$ with 0 means and finite qth moments.

A collection of random elements $\{X_\alpha\}$ in E is said to be stochastically bounded by a random variable X, $||X_\alpha|| \leq |X|$, if for each α and for each t

$$P[||X_\alpha|| > t] \leq P[|X| > t]. \quad (1.1)$$

If the random elements $\{X_\alpha\}$ are identically distributed, then they are

stochastically bounded by the random variable $\|X_{\alpha_0}\|$ for any α_0 .

Also, by Lemma 5.22 of Taylor (1978) uniformly bounded r th moments (for some $r > 0$) is sufficient for stochastic boundedness by a random variable X and for the existence of s ($s < r$) moments for the random variable X .

2. Almost Sure Convergence of Weighted Sums. In this section the almost sure convergence for Toeplitz weighted sums of arrays of random elements is obtained. Recall that $\{a_{nk} : n \geq 1, k \geq 1\}$ is a Toeplitz array if

$$(i) \quad \lim_{n \rightarrow \infty} a_{nk} = 0 \text{ for each } k \quad (2.1)$$

$$(ii) \quad \sum_{k=1}^{\infty} |a_{nk}| \leq \Gamma \text{ for each } n \quad (2.2)$$

where it can be assumed that $\Gamma=1$). When the random elements $\{X_{nk}\}$ in E are stochastically bounded by the random variable X and $E|X| < \infty$, then

$$\sum_{k=1}^{\infty} E|a_{nk} X_{nk}| \leq \sum_{k=1}^{\infty} |a_{nk}| E|X| \leq E|X|$$

and $\sum_{k=1}^{\infty} a_{nk} X_{nk}$ is convergent almost surely for each n . Theorem 1 will be stated for complete convergence (which implies almost sure convergence) since the proof will consist of showing $\sum_{n=1}^{\infty} P[|\sum_{k=1}^{\infty} a_{nk} X_{nk}| \geq \epsilon] < \infty$ for each $\epsilon > 0$.

Theorem 1: Let $\{X_{nk}\}$ be an array of random elements in a Banach space of type p , $1 < p \leq 2$, which are row-wise independent and such that $EX_{nk} = 0$ for all k and n . If $\{X_{nk}\}$ are stochastically bounded by a random variable X and if $\{a_{nk}\}$ is a Toeplitz array such that $\max_k |a_{nk}| = O(n^{-\gamma})$, $\gamma > 0$, then $E|X|^{1+1/\gamma} < \infty$ implies that

$$\left| \sum_{k=1}^{\infty} a_{nk} X_{nk} \right| \rightarrow 0 \quad \text{completely.}$$

The proof of Theorem 1 is accomplished in three lemmas. The first two lemmas follow directly from Pruitt (1966) and Rohatgi (1971) since $\|a_{nk} X_{nk}\| = |a_{nk}| \|X_{nk}\|$ and $\{|a_{nk}|\}$ & $\{\|X_{nk}\|\}$ are weights and random variables satisfying the corresponding hypotheses.

Lemma 2: If $E|X|^{1+1/\gamma} < \infty$ and $\max_k |a_{nk}| \leq Bn^{-\gamma}$, then for every $\epsilon > 0$

$$\sum_{n=1}^{\infty} P[\|a_{nk} X_{nk}\| \geq \epsilon \text{ for some } k] < \infty.$$

Lemma 3: If $E|X|^{1+1/\gamma} < \infty$ and $\max_k |a_{nk}| \leq Bn^{-\gamma}$, then for $\alpha < \gamma/2(\gamma+1)$,

$$\sum_{n=1}^{\infty} P[\|a_{nk} X_{nk}\| \geq n^{-\alpha} \text{ for at least two values of } k] < \infty.$$

Lemma 4: If $EX_{nk} = 0$ for all n and k , $E|X|^{1+1/\gamma} < \infty$, and $\max_k |a_{nk}| \leq Bn^{-\gamma}$, then for every $\epsilon > 0$,

$$\sum_{n=1}^{\infty} P\left[\left|\sum_{k=1}^{\infty} a_{nk} X_{nk} I[\|a_{nk} X_{nk}\| < n^{-\alpha}]\right| \geq \epsilon\right] < \infty$$

where $0 < \alpha < \gamma$.

Proof: When $a_{nk} = 0$, then $|a_{nk}|^{-1}$ is understood to be ∞ . Since $EX_{nk} = 0$,

$$\begin{aligned} & \left| E(X_{nk} I[\|a_{nk} X_{nk}\| < n^{-\alpha}]) \right| \\ &= \left| -E(X_{nk} I[\|a_{nk} X_{nk}\| \geq n^{-\alpha}]) \right| \\ &\leq E(\|X_{nk}\| I[\|X_{nk}\| \geq n^{-\alpha} |a_{nk}|^{-1}]) \end{aligned}$$

$$\leq \int_{|x| \geq n^{-\alpha} B^{-1} n^{\gamma}} |x| dP[|X| \leq x] dx. \quad (2.3)$$

Thus, $\gamma > \alpha$ yields $||E(X_{nk} I[||a_{nk} X_{nk}|| < n^{-\alpha}])|| \rightarrow 0$ uniformly in k as $n \rightarrow \infty$.

Hence,

$$\begin{aligned} & ||\sum_{k=1}^{\infty} a_{nk} E(X_{nk} I[||a_{nk} X_{nk}|| < n^{-\alpha}])|| \\ & \leq \sum_{k=1}^{\infty} |a_{nk}| E(||X_{nk}|| I[||a_{nk} X_{nk}|| < n^{-\alpha}]) \rightarrow 0 \end{aligned} \quad (2.4)$$

as $n \rightarrow \infty$. Define,

$$Z_{nk} = X_{nk} I[||a_{nk} X_{nk}|| < n^{-\alpha}] - E(X_{nk} I[||a_{nk} X_{nk}|| < n^{-\alpha}]). \quad (2.5)$$

Note that $EZ_{nk} = 0$, $E||Z_{nk}||^{1+1/\gamma} \leq K_1 E||X_{nk}||^{1+1/\gamma} \leq K_1 E|X|^{1+1/\gamma} = d$ for some $d > 0$, and $||a_{nk} Z_{nk}|| \leq 2n^{-\alpha}$. From (2.4)

$$\begin{aligned} & [||\sum_{k=1}^{\infty} a_{nk} X_{nk} I[||a_{nk} X_{nk}|| < n^{-\alpha}])|| > \epsilon] \\ & \subset [||\sum_{k=1}^{\infty} a_{nk} Z_{nk}|| \geq \frac{\epsilon}{2}] \end{aligned} \quad (2.6)$$

for sufficiently large n . For each n , pick $s(n)$ so that

$$\sum_{k=s(n)+1}^{\infty} |a_{nk}| \leq n^{-2}.$$

Let v be chosen so that

$$s = \frac{2v}{p} \text{ is an integer and } v > [(1 - \frac{1}{p})(2\gamma)]^{-1}. \quad (2.7)$$

Since $||a_{nk} Z_{nk}|| \leq 2n^{-\alpha}$, $E(||\sum_{k=1}^{s(n)} a_{nk} Z_{nk}||^{2v})$ is finite. Using Proposition 1

$$E(||\sum_{k=1}^{s(n)} a_{nk} Z_{nk}||^{2v}) \leq C_2 E(||\sum_{k=1}^{s(n)} a_{nk} Z_{nk}||^p)^{\frac{2v}{p}}$$

$$= C_2 \sum_{k_1, \dots, k_s} E \left[\prod_{j=1}^s \|a_{nk_j} z_{nk_j}\|^p \right]. \quad (2.8)$$

In the general term of (2.8), let

$$q_1 \text{ of the } k\text{'s} = C_1, \dots, q_m \text{ of the } k\text{'s} = c_m$$

$$r_1 \text{ of the } k\text{'s} = b_1, \dots, r_t \text{ of the } k\text{'s} = b_t$$

where $p \leq pq_1 \leq 1 + \frac{1}{\gamma}$, $pr_j > 1 + \frac{1}{\gamma}$, and

$$\sum_{i=1}^m q_i + \sum_{j=1}^t r_j = s = \frac{2v}{p}.$$

Thus, in (2.8)

$$\begin{aligned} E \left[\prod_{j=1}^s \|a_{nk_j} z_{nk_j}\|^p \right] &= E \left[\prod_{i=1}^m \|a_{nc_i} z_{nc_i}\|^{pq_i} \prod_{j=1}^t \|a_{nb_j} z_{nb_j}\|^{pr_j} \right] \\ &= \left[\prod_{i=1}^m |a_{nc_i}|^{pq_i} E(\|z_{nc_i}\|^{pq_i}) \right] \left[\prod_{j=1}^t E(\|a_{nb_j} z_{nb_j}\|^{pr_j}) \right] \\ &\leq (1+K_1 E|X|^{1+1/\gamma})^{m+t} \left(\prod_{i=1}^m |a_{nc_i}|^{pq_i-1} \right) \\ &\quad \times \left(\prod_{j=1}^t |a_{nb_j}|^{1+1/\gamma} (2n^{-\alpha})^{pr_j-1-\frac{1}{\gamma}} \right) \\ &\leq (1+d)^{t+m} \left(\prod_{i=1}^m |a_{nc_i}| \right) \left(\prod_{j=1}^t |a_{nb_j}| \right) (Bn^{-\gamma})^{\sum_{i=1}^m (pq_i-1) + \frac{t}{\gamma}} \\ &\quad \times (2n^{-\alpha})^{\sum_{j=1}^t (pr_j-1-\frac{1}{\gamma})}. \end{aligned} \quad (2.9)$$

For the last expression in (2.9), the power of n^{-1} is

$$\gamma \sum_{i=1}^m (pq_i-1) + t + \sum_{j=1}^t (pr_j-1-\frac{1}{\gamma}). \quad (2.10)$$

If $t \geq 1$, then

$$\gamma \sum_{i=1}^m (pq_i-1) + t + \alpha \sum_{j=1}^t (pr_j-1-\frac{1}{\gamma})$$

$$\geq 1 + \alpha \sum_{j=1}^t (pr_j - 1 - 1/\gamma) \geq 1 + \alpha\delta \quad (2.11)$$

for some fixed $\delta > 0$ since r_j is an integer greater than $\frac{1}{p} + \frac{1}{p\gamma}$. If $t = 0$, then

$$\begin{aligned} & \gamma \sum_{i=1}^m (pq_i - 1) + t + \alpha \sum_{j=1}^t (pr_j - 1 - \frac{1}{\gamma}) \\ &= \gamma \sum_{i=1}^m (pq_i - 1) \\ &= \gamma (2v - \frac{2v}{p}) \\ &= \gamma (2v) (1 - \frac{1}{p}) > 1 \end{aligned}$$

by (2.7). By (2.7) and (2.11) $r = \min\{\gamma(2v)(1 - \frac{1}{p}), 1 + \alpha\delta\} > 1$. From (2.8) and (2.9)

$$E(|\sum_{k=1}^{s(n)+1} a_{nk} Z_{nk}|^{2v}) \leq K_2 n^{-r}$$

where the constant K_2 depends only on d, γ, p , and B .

Next,

$$\begin{aligned} & \sum_{n=1}^{\infty} P[|\sum_{k=1}^{\infty} a_{nk} Z_{nk}| \geq \frac{\epsilon}{2}] \\ & \leq \sum_{n=1}^{\infty} P[|\sum_{k=1}^{s(n)+1} a_{nk} Z_{nk}| \geq \frac{\epsilon}{4}] \\ & \quad + \sum_{n=1}^{\infty} P[|\sum_{k=s(n)+1}^{\infty} a_{nk} Z_{nk}| \geq \frac{\epsilon}{4}] \\ & \leq (\frac{4}{\epsilon})^{2v} \sum_{n=1}^{\infty} E[|\sum_{k=1}^{s(n)+1} a_{nk} Z_{nk}|^{2v}] \\ & \quad + \frac{4}{\epsilon} \sum_{n=1}^{\infty} \sum_{k=s(n)+1}^{\infty} E|a_{nk} Z_{nk}| \\ & \leq (\frac{4}{\epsilon})^{2v} \sum_{n=1}^{\infty} K_2 n^{-r} + \frac{4}{\epsilon} \sum_{n=1}^{\infty} \sum_{k=s(n)+1}^{\infty} |a_{nk}| (1+d) \end{aligned}$$

$$= \left(\frac{4}{\epsilon}\right)^{2\nu} K_2 \sum_{n=1}^{\infty} n^{-r} + \frac{4}{\epsilon}(1+d) \sum_{n=1}^{\infty} n^{-2} < \infty. \quad (2.12)$$

Finally, (2.6) and (2.12) yield

$$\sum_{n=1}^{\infty} P[||\sum_{k=1}^{\infty} a_{nk} X_{nk}|| \geq \epsilon] < \infty. \quad ///$$

The proof of Theorem 1 follows directly from Lemmas 2, 3, and 4 since

$$\begin{aligned} & [||\sum_{k=1}^{\infty} a_{nk} X_{nk}|| \geq \epsilon] \\ & \subset ([||\sum_{k=1}^{\infty} a_{nk} X_{nk}|| \geq \frac{\epsilon}{2} \text{ and } ||a_{nk} X_{nk}|| < n^{-\alpha}] \cup [||a_{nk} X_{nk}|| \geq \frac{\epsilon}{2} \text{ for some } k] \\ & \cup [||a_{nk} X_{nk}|| \geq n^{-\alpha} \text{ for at least two } k\text{'s}]) \end{aligned}$$

A version of Rohatgi's (1968) result for type p , $1 < p \leq 2$, spaces is obtained directly from Theorem 1 as a corollary. It is interesting to observe that the moment condition is related to the weights rather than the type p condition. Typically, $\gamma \leq 1$, and hence $1 + 1/\gamma \geq 2 \geq p$.

Corollary 1: Let $\{X_n\}$ be a sequence of independent random elements in a Banach space of type p , $1 < p \leq 2$, such that $EX_n = 0$ for each n , and let $\{X_n\}$ be stochastically bounded by a random variable X . If $\{a_{nk}\}$ is a Toeplitz array such that $\max_k |a_{nk}| = O(n^{-\gamma})$, $\gamma > 0$, then $E|X|^{1+1/\gamma} < \infty$ implies that

$$||\sum_{k=1}^{\infty} a_{nk} X_k|| \rightarrow 0 \text{ completely.}$$

The geometric property of E significantly relaxes the condition imposed in Padgett and Taylor (1976), Theorem 2, to extend Rohatgi's results to Banach spaces. The type p condition is not needed in the corresponding results for independent, identically distributed random elements since the uniform truncation is accomplished by the identical distributions.

3. Convergence in Probability and Density Estimates. Convergence in probability of the weighted sums will be obtained in this section under relaxed conditions on the summability of the weights. The general application of these results to density estimates will be indicated.

Theorem 2: Let $\{X_{nk}\}$ be an array of row-wise independent random elements in a Banach space E of type p , $1 < p \leq 2$. Let $EX_{nk} = 0$ for each n and k and let $\{X_{nk}\}$ be stochastically bounded by a random variable X . Let $\{a_{nk}\}$ be an array of constants such that $\max_{1 \leq k \leq n} |a_{nk}| \rightarrow 0$ as $n \rightarrow \infty$ and

$$\sum_{k=1}^{\infty} |a_{nk}|^r \leq \Gamma \text{ for all } n$$

where $1 \leq r < p$. If $E(|X|^r) < \infty$, then

$$||\sum_{k=1}^n a_{nk} X_{nk}|| \rightarrow 0 \text{ in probability}$$

The proof consists of incorporating the techniques of Theorem 1 into the proof of Theorem 5.3.2 of Taylor (1978), and will be omitted.

In the general density estimation problem $X_1, \dots, X_n$ are independent random variables with the same density function f . The kernel estimate for f is given by

$$\frac{1}{nh} \sum_{k=1}^n K\left(\frac{t - X_k}{h}\right) = f_n(t) \quad (3.1)$$

where K is an arbitrarily chosen density and $h_n \rightarrow 0$ as $n \rightarrow \infty$. There are numerous choices for K and h_n . Here, we will assume that $K(t)$ is a bounded (integrable) kernel with compact support. Let $w(t)$ be a bounded weight function such that

$$\int |f(t)|^p w(t) dt < \infty \quad (3.2)$$

for some $\frac{1}{p} < \liminf_n [1 + \frac{\log h_n}{\log n}]$. Typically, $h_n \sim n^{-\delta}$. Hence, $p > \frac{1}{1-\delta}$ where $0 < \delta < \frac{1}{2}$. Denote

$$E = \{g: g: \mathbb{R} \rightarrow \mathbb{R} \text{ and } \int_{-\infty}^{\infty} |g(t)|^p w(t) dt < \infty\}. \quad (3.3)$$

Then, E is a separable Banach space of type $\min\{2, p\}$ with norm

$$\|g\| = \left(\int |g(t)|^p w(t) dt \right)^{1/p}. \quad (3.4)$$

Since K is bounded with compact support, $\{K(\frac{t-X_1}{h_n}), \dots, K(\frac{t-X_n}{h_n})\}$ are i.i.d. random elements in E for each n . For $q \geq 1$

$$\begin{aligned} E\left(\left\|K\left(\frac{t-X_1}{h_n}\right)\right\|^q\right) &= E\left[\left(\int_{-\infty}^{\infty} \left|K\left(\frac{t-X_1}{h_n}\right)\right|^p w(t) dt\right)^{q/p}\right] \\ &= E\left[\left(\int_a^b |K(s)|^p w(X_1 + sh_n) h_n ds\right)^{q/p}\right] \\ &\leq (\text{bdd } K)^q [(\text{bdd } w)(b-a)h_n]^{q/p} \\ &< \infty. \end{aligned} \quad (3.5)$$

From (3.5) the expected value $E[K(\frac{t-X_1}{h_n})] \in E$, and the random elements $\{K(\frac{t-X_k}{h_n}) - E[K(\frac{t-X_1}{h_n})]: k = 1, 2, \dots, n \text{ and } n \geq 1\}$ are stochastically bounded by a random variable X . Let $a_{nk} = \frac{1}{nh_n}$ for $1 \leq k \leq n$ and $a_{nk} = 0$ for $k > n$. Choose r so that $1 \leq r < p$ and

$$\sum_{k=1}^{\infty} |a_{nk}|^r = n \left[\frac{1}{nh_n} \right]^r = n^{1-r} h_n^{-r} \leq \Gamma \quad (3.6)$$

for all n . Under the conditions of (3.1) to (3.6), Theorem 2 yields

$$\left| \left| \frac{1}{nh_n} \sum_{k=1}^n \left(K\left(\frac{t-X_k}{h_n}\right) - E\left(K\left(\frac{t-X_1}{h_n}\right)\right) \right) \right| \right| \rightarrow 0 \quad (3.7)$$

in probability. The troublesome aspect is showing that

$$\left| \left| \frac{1}{h_n} E\left(K\left(\frac{t-X_1}{h_n}\right)\right) - f(t) \right| \right| \rightarrow 0. \quad (3.8)$$

Lemma 5: Let f be piecewise continuous, bounded on each compact subset, and $\lim_{|s| \rightarrow \infty} f(s) = 0$. Then

$$\left| \left| \frac{1}{h_n} E\left(K\left(\frac{t-X_1}{h_n}\right)\right) - f(t) \right| \right| \rightarrow 0.$$

Proof: First,

$$\begin{aligned} & \left| \left| \frac{1}{h_n} E\left(K\left(\frac{t-X_1}{h_n}\right)\right) - f(t) \right| \right|^p \\ &= \int_{-\infty}^{\infty} \left| \frac{1}{h_n} \int_{-\infty}^{\infty} K\left(\frac{t-s}{h_n}\right) f(s) ds - f(t) \right|^p w(t) dt \\ &= \int_{-\infty}^{\infty} \left| \int_a^b K(y) [f(t-yh_n) - f(t)] dy \right|^p w(t) dt. \end{aligned} \quad (3.9)$$

The integral with respect to t in (3.9) may need to be broken into several (improper) integrals, so that (w.l.o.g.), it can be assumed that f is continuous at each t . Thus, for each y and each t , $f(t-yh_n) \rightarrow f(t)$ as $n \rightarrow \infty$. Choose N_1 so that $|yh_n| \leq 1$ for all $n \geq N_1$. Hence,

$$|f(t-yh_n) - f(t)| \leq |f(t)| + \sup_{-1 \leq s \leq 1} |f(t-s)| \leq \Gamma(t)$$

for each t . Hence,

$$\int_a^b K(y)[f(t-yh_n) - f(t)]dy \rightarrow 0 \quad (3.10)$$

pointwise in t . Choose α and β so that

$$\begin{aligned} \int_{-\infty}^{\alpha+1} f(t)dt &\leq \epsilon/bdd w, \\ \int_{\beta-1}^{\infty} f(t)dt &\leq \epsilon/bdd w, \text{ and} \\ f(t) &\leq \frac{1}{2} \text{ for } t \notin [\alpha+1, \beta-1]. \end{aligned} \quad (3.11)$$

Then for $n \geq N_1$

$$\begin{aligned} &\int_{-\infty}^{\infty} \left| \int_a^b K(y)[f(t-yh_n) - f(t)]dy \right|^p w(t)dt \\ &\leq \int_{-\infty}^{\alpha} \left| \int_a^b K(y)[f(t-yh_n) - f(t)]dy \right| w(t)dt \\ &\quad + \int_{\alpha}^{\beta} \left| \int_a^b K(y)[f(t-yh_n) - f(t)]dy \right|^p w(t)dt \\ &\quad + \int_{\beta}^{\infty} \left| \int_a^b K(y)[f(t-yh_n) - f(t)] \right| w(t)dt. \end{aligned} \quad (3.12)$$

The first term of (3.12) becomes

$$\begin{aligned} &\int_{-\infty}^{\alpha} \left| \int_a^b K(y)[f(t-yh_n) - f(t)]dy \right| w(t)dt \\ &\leq bdd w \left(\int_{-\infty}^{\alpha} \int_a^b K(y)f(t-yh_n)dydt + \int_{-\infty}^{\alpha} \int_a^b K(y)f(t)dydt \right) \\ &< \epsilon + \epsilon \end{aligned} \quad (3.13)$$

by (3.11) and an interchange of integrals. Similarly, the third term of (3.12) is less than 2ϵ . The second term of (3.12) converges to zero by the dominated convergence theorem. Hence, there exists N such that

$$|| \frac{1}{h_n} E(K(\frac{t-X_1}{h_n})) - f(t) ||^P < 5\epsilon$$

for all $n \geq N$.

///

Theorem 2 and Lemma 5 provides the consistency

$$|| f_n(t) - f(t) || = \int_{-\infty}^{\infty} |f_n(t) - f(t)|^p w(t) dt \rightarrow 0$$

in probability where the p th integrated difference with respect to an arbitrary weight function $w(t)$ is related to the order of h_n . Before proceeding to the complete convergence of the estimates, it is important to note that $w(t) \equiv 1$ is often assumed and that various combinations of conditions on K and f will yield Lemma 5.

4. A Strong Law of Large Numbers Application. In this section a strong law of large numbers is obtained for arrays of random elements in type p spaces. This result substantially improves the density estimation application in the previous section. The proof of Theorem 3 will make use of the following lemma (Corollary 2.1 of Woycznski (pre-print)).

Lemma 6: Let E be of type p , $1 < p \leq 2$, and let $q \geq 1$. If $\{X_n\}$ are independent, identically distributed random elements in E with $E(||X_1||^q) < \infty$ and $EX_1 = 0$, then

$$E(||\sum_{k=1}^n X_k||^{pq}) \leq C_2 n^q E(||X_1||^{pq}) \quad (4.1)$$

for some constant C_2 independent of n .

Only the major points of the proof of Theorem 3 will be indicated. Again, the proof will use higher moment conditions on the random elements

than the type of the space.

Theorem 3: Let $\{X_{nk}\}$ be an array of row-wise independent, identically distributed random elements in a Banach space of type p , $1 < p \leq 2$, and let $EX_{nk} = 0$ for each n and each k . If for some $q \geq 1$

$$\sum_{n=1}^{\infty} E(|X_{n1}|^{pq})/n^{q(p-1)} < \infty, \quad (4.2)$$

then

$$|\frac{1}{n} \sum_{k=1}^n X_{nk}| \rightarrow 0 \text{ completely.}$$

Proof: For each $\epsilon > 0$ and each n

$$\begin{aligned} P[|\frac{1}{n} \sum_{k=1}^n X_{nk}| > \epsilon] &\leq n^{-pq} \epsilon^{-pq} E(|\sum_{k=1}^n X_{nk}|^{pq}) \\ &\leq n^{-pq} \epsilon^{-pq} C_2 n^q E(|X_{n1}|^{pq}) \end{aligned} \quad (4.3)$$

by Lemma 6. Thus,

$$\begin{aligned} \sum_{n=1}^{\infty} P[|\frac{1}{n} \sum_{k=1}^n X_{nk}| > \epsilon] \\ \leq \epsilon^{-pq} C_2 \sum_{n=1}^{\infty} E(|X_{n1}|^{pq})/n^{q(p-1)} < \infty \end{aligned}$$

by (4.2). ///

In the density estimation problem, let

$$X_{nk} = \frac{1}{h_n} K\left(\frac{t-X_k}{h_n}\right) - \frac{1}{h_n} E\left(K\left(\frac{t-X_1}{h_n}\right)\right).$$

Lemma 5 provides for the convergence.

$$\left| \left| \frac{1}{h_n} E(K(\frac{t-X_1}{h_n})) - f(t) \right| \right| \rightarrow 0.$$

In verifying the moment condition (4.2), Inequality (3.5) will be used.

For any $q \geq 1$

$$\begin{aligned} E(|X_{n1}|^{pq}) &\leq E\left(\left|\frac{1}{h_n} K(\frac{t-X_1}{h_n})\right| + \left|\frac{1}{h_n} E(K(\frac{t-X_1}{h_n}))\right|\right)^{pq} \\ &\leq h_n^{-pq} [E(|K(\frac{t-X_1}{h_n})|^{pq})^{1/p} + |E(K(\frac{t-X_1}{h_n}))|]^{pq} \\ &\leq h_n^{-pq} [(C_4 h_n^q)^{1/p} + (C_5 h_n^{1/p})]^{pq} \\ &\leq h_n^{-pq} C_6 h_n^q = C_6 h_n^{q(1-p)} \end{aligned} \quad (4.4)$$

where C_4 , C_5 , and C_6 are constants depending on the bounds of K and w and the support of K , $[a, b]$, and the values of p and q . Hence,

$$\begin{aligned} E(|X_{n1}|^{pq}) / n^{q(p-1)} &\leq C_6 h_n^{q(1-p)} / n^{q(p-1)} \\ &= C_6 (nh_n)^{-q(p-1)}. \end{aligned} \quad (4.5)$$

Thus, if h_n is of the order $n^{-\delta}$ for any $0 < \delta < 1$ there exists a $q \geq 1$ such that the terms in (4.5) sum. By Theorem 3 and Lemma 5

$$\int |f_n(t) - f(t)|^p w(t) dt \rightarrow 0 \text{ completely}$$

for any p , $1 < p \leq 2$, unrelated to the h_n 's when

$$\sum_{n=1}^{\infty} (nh_n)^{-q} < \infty$$

for some $\alpha > 0$. Not only is complete convergence obtained, but the h_n 's are much more general. This improvement is mainly due to the probability cancellation in type p spaces, and the p th integrated norm (in (4.4)).

Inequality (4.3) suggests an immediate weak law of large numbers for arrays of random elements in type p spaces.

Theorem 4: Let $\{X_{nk}\}$ be an array of row-wise independent, identically distributed random elements in a Banach space of type p , $1 < p \leq 2$, and let $EX_{nk} = 0$ for each n and each k . If for some $q \geq 1$

$$n^{-q(p-1)} E(\|X_{n1}\|^{pq}) \rightarrow 0, \quad (4.6)$$

then

$$\left\| \frac{1}{n} \sum_{k=1}^n X_{nk} \right\| \rightarrow 0 \text{ in probability.}$$

The condition of row-wise indentially distributed random elements in Theorems 3 and 4 is not necessary since Proposition 1 could be used instead of Lemma 6. However, expressions (4.2) and (4.6) would appear much more complicated.

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obtained for triangular arrays of random elements. Finally, the direct applications of these results in obtaining consistency of the kerney density estimates are indicated.