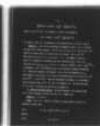


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Extreme Points of Certain Convex Subsets of
Log-convex functions.

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Abstract

A general technique is presented for identifying the extreme points of a convex set C of log-convex functions and the extreme points of certain types of its convex subsets. For K one these subsets, it is shown that $\text{ext } K = K \cap \text{ext } C$ even though K is not an extremal subset of C .

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1. Introduction. The identification of the extreme points of a convex set is an important problem in mathematics. One reason for identifying the extreme points is that they are, in many instances, the basic building blocks for the convex set. For instance, when the convex set is a compact subset of a locally convex space it is the closed convex hull of its extreme points (Krein-Milman Theorem) and if in addition the set is metrizable, any point in it has an integral representation in terms of the extreme points (Choquet's Theorem). *Recently*

Recently, Langberg, León, Lynch, and Proschan (hereafter, referred to as L³P) (1978, 1979) have identified the extreme points for certain types of convex compact sets of probability distributions which occur in reliability theory. *have been identified.* L³P (1979) identify the extreme points for the set of discrete decreasing failure rate distributions, *are identified* and, without appealing to Choquet's Theorem, give an explicit representation for these distributions in terms of the extreme points. In L³P (1978), they identify the extreme points for the decreasing failure rate distributions in the continuous case. It turns out that techniques employed in this last paper can be used to identify the extreme points for certain convex sets of log-convex functions. The purpose of this paper is to present these results.

In Section 2, some preliminary material is presented and the main result - Theorem 2.1, which identifies the extreme points - is stated. The proof of Theorem 2.1 is given in Section 4. In Section 3, certain types of convex subsets of log-convex functions are examined and their extreme points are identified.

2. Preliminaries and the Main Result. Throughout let a, b ($-\infty \leq a < b \leq \infty$) be a fixed interval. Denote the space of continuous functions on a, b by $C[a, b]$ and denote the space of finite signed measures on a, b by $C^*[a, b]$, which is the dual of $C[a, b]$. Let L denote the set of functions which are log-convex on a, b , i.e., those functions $x(\cdot)$ which are positive on a, b and for which $\log x(\cdot)$ is convex on $[a, b]$. For a fixed constant c , let

$$C = \{x \in L: x(a) = c \text{ and } x \text{ is continuous at } a \text{ and } b\}.$$

It is well known that L is closed (under pointwise convergence) and convex (see Roberts and Varberg, 1973, Section 13); thus C is a closed convex subset of $C[a, b]$ with the usual norm.

For each $x \in L$ and $t \in [a, b]$, let $X(t) = \log x(t)$. Since X is convex, its first and second order derivatives, X' and X'' , exist almost everywhere (a.e.) on $[a, b]$. Let

$$E_x = \{t \in [a, b]: X'' \text{ exists at } t\}.$$

Denote the Lebesgue measure by m and let

$$E = \{x \in C: m(\{t \in E_x: a' \leq t \leq b' \text{ and } X''(t) < \delta\}) > 0 \text{ for all } a \leq a' < b' \leq b \text{ and all } \delta > 0\}.$$

For each $x \in C$, let $I(x)$ denote an interval of positive length contained in $[a, b]$ such that for some $\delta_x > 0$, $X'' > \delta_x$ a.e. on $I(x)$. If no such interval exists, let $I(x) = \emptyset$, the empty set. Note that if $x \in C \cap E^c$, then $I(x) = \emptyset$ and for some $\delta_x > 0$ we have $X'' > \delta_x$ a.e. on $I(x)$.

Recall that a point x in a convex set K is an extreme point of K if $x = \alpha y + (1 - \alpha)z$, $0 < \alpha < 1$, and $y, z \in K$ imply that $y = z = x$. The set of extreme points of a convex set K will be denoted by $\text{ext } K$. In the following theorem, the proof of which is given in Section 4, the extreme points of C are identified.

Theorem 2.1. $\text{ext } C = E$.

Remark. Since $\text{ext } C = E$, $\text{ext } C$ is dense (under uniform convergence on $[a, b]$) in C . This can be seen as follows. Let $x \in C$. Then x is convex on $[a, b]$. For each positive integer n , partition $[a, b]$ into a countable number of intervals each of length less than $1/n$ and denote the set of endpoints of these intervals by $\{t_i^n\}$. Let $X_n(t) = X_n(t_i^n)$ if $t = t_i^n$ for some i and linear between endpoints of an interval in the partition. Then X_n is convex and $x_n = e^{X_n} \in E$ since $X_n'' = 0$ a.e. on $[a, b]$. Since $x_n \rightarrow x$ uniformly on $[a, b]$, $E = \text{ext } C$ is dense in C .

The proof of Theorem 2.1 and the examples in the next section require the technical lemma given below.

We need the following notation. For the interval $[a', b']$ ($a < a' < b' < b$), let $C^{(2)}[a', b']$ denote the set of functions on $[a', b']$ which have continuous second derivatives. The norm on $C^{(2)}[a', b']$ is given by $\|x\|_{[a', b']} = \max \{|x(t)|, |x'(t)|, |x''(t)| : t \in [a', b']\}$. The subspace of $C^{(2)}[a', b']$ of functions which vanish at a' and b' and whose first and second order derivatives also vanish at a' and b' will be denoted by $C_0^{(2)}[a', b']$. The set of all functions in $C[a, b]$ that vanish off $[a', b']$ and have restrictions to $[a', b']$ which are in $C_0^{(2)}[a', b']$ will be denoted by $V[a', b']$. For $t < \ln 2$, let $c(t) = \ln(2 - e^t)$.

Lemma 2.2. Let X be a convex function on $[a, b]$ which has its second derivative a.e. uniformly bounded away from zero on some subinterval $[a', b']$ ($a' < b'$) of $[a, b]$. Let $\{\Delta_n\}$ be a sequence of functions defined on $[a, b]$ which are in $V[a', b']$ and such that $\|\Delta_n\|_{[a', b']} \rightarrow 0$ as $n \rightarrow \infty$. Then the functions

$$(2.1) \quad X_{1n} \stackrel{\text{def}}{=} X + \Delta_n \text{ and } X_{2n} \stackrel{\text{def}}{=} X + c(\Delta_n)$$

are well-defined and convex for all sufficiently large n . In particular, for all sufficiently large n , the functions

$$(2.2) \quad x_{1n} \equiv e^{X_{1n}} \text{ and } x_{2n} \equiv e^{X_{2n}}$$

are log-convex on $[a, b]$ and

$$(2.3) \quad x \equiv e^X = (x_{1n} + x_{2n})/n.$$

Proof. Let X_{1n} and X_{2n} be as given in (2.1). Then X_{1n} is always well-defined while $X_{2n}(t)$ makes sense only when $\Delta_n < \ln 2$. Since $\|\Delta_n\| \rightarrow 0$ as $n \rightarrow \infty$ and Δ_n vanishes off $[a', b']$, it follows that X_{1n} and X_{2n} are well-defined for all sufficiently large n . Throughout the remainder of this proof, it will be assumed that n has already been chosen so that X_{2n} is well-defined.

Since X'' is a.e. bounded away from zero on $[a', b']$, there is a $\delta > 0$ such that $X'' > \delta$ a.e. on $[a', b']$. Since X' and X'' exist a.e. on $[a, b]$, X'_{in} and X''_{in} ($i = 1, 2$) exists a.e. and satisfy the following identities:

$$\begin{aligned} X'_{1n} &= X' + \Delta'_n, \\ X'_{2n} &= X' + c'(\Delta_n)\Delta'_n, \end{aligned}$$

$$X''_{1n} = X'' + \Delta''_n,$$

$$X''_{2n} = X'' + c'(\Delta_n)\Delta''_n + c''(\Delta_n)(\Delta'_n)^2.$$

Since $X''(t) > \delta$ a.e. on $[a', b']$ and $||\Delta_n|| \rightarrow 0$ as $n \rightarrow \infty$, it follows from the continuity of c' and c'' that $X''_{in}(t) > \delta/2$ ($i = 1, 2$) for all sufficiently large n , say $n \geq n_0$. Let $n \geq n_0$. Thus X_{in} is strictly convex on $[a', b']$ for $n \geq n_0$. Also X and X_{in} and their derivatives agree off (a', b') since Δ_n vanishes off (a', b') and $c'(0) = 0$. It follows that X_{in} ($i = 1, 2$) are convex on $[a, b]$, and consequently, the functions x_{1n} and x_{2n} defined in (2.2) are log-convex.

Finally,

$$\begin{aligned} \frac{x_{1n} + x_{2n}}{2} &= (\exp\{X + \Delta_n\} + \exp\{X + c(\Delta_n)\})/2 \\ &= \exp\{X\}(\exp \Delta_n + \exp c(\Delta_n))/2 \\ &= x(\exp \Delta_n + 2 - \exp \Delta_n)/2 \\ &= x, \end{aligned}$$

which proves identity (2.3). ||

3. Extreme Points of Convex Subsets of C. Let K be a convex subset of C . Then clearly

$$K \cap \text{ext } C \subset \text{ext } K.$$

It is of interest to know when there is equality above rather than just containment; i.e., when is (3.1) below satisfied?

$$(3.1) \quad K \cap \text{ext } C = \text{ext } K.$$

It is an elementary observation that (3.1) holds when K is an extremal subset of C . Recall that a subset of K of convex set C is an extremal subset of C if $x, y \in C$, $0 < \alpha < 1$, and $\alpha x + (1 - \alpha)y \in K$ imply that $x, y \in K$.

In this section, we shall see that condition (3.1) holds for some subsets K of C which are not extremal subsets.

Lemma 3.1. Let K be a convex subset of C and $x_0 \in K \cap E^C$. Then $x_0 \notin \text{ext } K$ provided that there exists (i) a set of functions, K_1 , defined on $[a, b]$ for which $K = K_1 \cap E^C$, and (ii) a proper subinterval $[a', b']$ of $I(x_0)$ such that the zero function is not an isolated point of the set $\{\Delta \in V[a', b'] : x_0 e^{\Delta}$ and $x_0 e^{c(\Delta)}$ are both in $K_1\}$.

Proof. By the hypothesis of the theorem there is a sequence $\{\Delta_n\} \subset V[a', b']$ ($\Delta_n \neq 0$) such that $\|\Delta_n\|_{[a', b']} \rightarrow 0$ and $x_0 e^{\Delta_n}$ and $x_0 e^{c(\Delta_n)}$ are both in K_1 . By Lemma 2.2 for all sufficiently large n , say $n \geq n_0$, $x_0 e^{\Delta_n}$ and $x_0 e^{c(\Delta_n)}$ are both log-convex and therefore also in C since Δ_n vanishes at a . Thus $x_0 e^{\Delta_{n_0}}$ and $x_0 e^{c(\Delta_{n_0})}$ are in K . Now by

identity (2.3), we have $x_0 = (x_0 e^{\Delta_{n_0}} + x_0 e^{c(\Delta_n)})/2$. This shows that $x_0 \notin \text{ext } K$ since Δ_{n_0} is not identically zero. ||

Theorem 3.2. If the hypothesis of Lemma 3.1 is satisfied for every $x_0 \in K \cap E^C$, then (3.1) holds; i.e. $K \cap \text{ext } C = \text{ext } K$.

Proof. By Theorem 3.1, $\text{exp } K \subset K \cap \text{ext } C$, while the reverse inclusion is obvious. ||

In Examples 1 and 2 below we use Theorem 3.2 to identify the extreme points of two interesting convex subsets of C . Two other applications are given later in the paper. For another interesting application of importance in statistics, see L³P (1978).

Example 1. Fix $t_0 \in [a, b]$ and let K denote the set of all functions in C whose minimum occurs at t_0 . Then K is convex and $K = K_1 \cap C$, where K_1 is the set of functions on $[a, b]$ whose minimum occurs at t_0 . Let $x \in K \cap E^C$. Then there is an interval $[a', b'] \subset I(x)$ ($-\infty < a' < b' < \infty$) such that $\inf \{x(t) : t \in [a', b']\} > x(t_0)$. Notice that $t_0 \notin [a', b']$. Choose any sequence $\{\Delta_n\} \subset V[a', b']$ ($\Delta_n \neq 0$) such that $\|\Delta_n\|_{[a', b']} \rightarrow 0$. Since $\|\Delta_n\|_{[a', b']} \rightarrow 0$ as $n \rightarrow \infty$ and $\{\Delta_n\} \subset V[a', b']$, then $\sup \{|c(\Delta_n(t))| : t \in [a', b']\} \rightarrow 0$ as $n \rightarrow \infty$ and Δ_n and $c(\Delta_n)$ vanish off $[a', b']$. Since $\Delta_n(t_0) = 0$ implies that $x(t_0)e^{\Delta_n(t_0)} = x(t_0)e^{c(\Delta_n(t_0))} = x(t_0)$, it follows that for all sufficiently large n ,

$$(3.2) \quad \inf\{x(t)e^{\Delta_n(t)} : t \in [a, b]\} = x(t_0) \\ = \inf\{x(t)e^{c(\Delta_n(t))} : t \in [a, b]\}.$$

Equivalently, $x e^{\Delta_n}$ and $x e^{c(\Delta_n)}$ are in K_1 for all sufficiently large n .

Thus $\Delta \equiv 0$ is not an isolated point of $\{\Delta \in V[a', b'] : x e^{\Delta}, x e^{c(\Delta)} \in K_1\}$.

Thus the hypothesis of Lemma 3.1 is satisfied for every $x \in K \cap E^C$, and thus (3.1) holds by Theorem 3.2.

Example 2. Let K denote the set of all functions in C which are decreasing on $[a, b]$. Then K is convex and $K = K_1 \cap C$, where K_1 is the set of decreasing functions on $[a, b]$. Let $x \in K \cap E^C$. Since $x \in E^C$ we can choose $[a', b'] \subset I(x)$ such that $-\infty < a' < b' < \infty$. Let $\{\Delta_n\}$ be any sequence in $V[a', b']$, where $\Delta_n \neq 0$ and $\|\Delta_n\|_{[a', b']} \rightarrow 0$ as $n \rightarrow \infty$. By Lemma 2.1, there exists an integer n_0 such that xe^{Δ_n} and $xe^{c(\Delta_n)}$ are log-convex for $n \geq n_0$. Since x is decreasing and since xe^{Δ_n} and $xe^{c(\Delta_n)}$ are log-convex and agree with x off $[a', b']$, it follows that xe^{Δ_n} and $xe^{c(\Delta_n)}$ are decreasing and so are in K_1 . Thus this sequence $\{\Delta_n: n \geq n_0\} \subset \{\Delta \in V[a', b']: xe^{\Delta}, xe^{c(\Delta)} \in K_1\}$, and so $\Delta \equiv 0$ is not an isolated point of this set. By Theorem 3.2, (3.1) holds.

Remark. In Example 2, let a be finite. If $b = \infty$, it is easy to see that K is an extremal subset of C , in which case, (3.1) holds by the remark following (3.1). However if $b < \infty$, the above argument does not hold since in this case K is not an extremal subset of C .

Throughout the remainder of this section, we consider convex subsets K of C of the form

$$K = \{x \in C: L(x) = k\},$$

where L is a linear functional and k is a fixed constant. For L a continuous linear functional on $C[a, b]$ (i.e., $L \in C^*[a, b]$), we prove in Theorem 3.5 that (3.1) holds. The following two lemmas are needed for the proof of Theorem 3.5.

Lemma 3.3. Let L be a linear functional on $C[a, b]$, $[a', b'] \subset [a, b]$, $\Delta \in V[a', b']$, and $x \in C$. Let $L(xe^{\Delta}) = L(x)$ and $\sup \{\Delta(t): t \in [a', b']\} < \ln 2$. Then $L(xe^{c(\Delta)}) = L(x)$.

Proof. First note that since $\Delta \in V[a', b']$ and $\sup\{\Delta(t) : t \in [a', b']\} < \ln 2$, both xe^Δ and $xe^{c(\Delta)}$ are in $C[a, b]$, and consequently both $L(xe^\Delta)$ and $L(xe^{c(\Delta)})$ are well-defined. Since $L(xe^\Delta) = L(x)$ and by the definition of $c(\cdot)$, $e^{c(\Delta)} = 2 - e^\Delta$, we conclude that $L(xe^{c(\Delta)}) = 2L(x) - L(xe^\Delta) = L(x)$, as was to be shown. ||

Lemma 3.4. Let $L \in C^*[a', b']$ ($a \leq a' < b' \leq b$) and $x_0 \in C[a', b']$. Then (a) there exists a sequence $\{\Delta_n\} \subset C_0^{(2)}[a', b']$ of nonzero functions such that $\|\Delta_n\|_{[a', b']} \rightarrow 0$ as $n \rightarrow \infty$, and (b) $L(x_0) = (x_0 e^{\Delta_n})$.

Proof. We know that $L(x) = \int_{[a, b]} x(t) \mu(dt) = \int_{[a', b']} x^+(t) \mu^+(dt) - \int_{[a', b']} x^-(t) \mu^-(dt) = \int_{[a', b']} x^+(t) \mu^+(dt) + \int_{[a', b']} x^-(t) \mu^-(dt)$ where $x^+ \equiv x \vee 0$, $x^- \equiv -(x \wedge 0)$, and where μ^+ and μ^- are the positive and negative parts given in the Jordan-Hahn decomposition of some finite signed measure μ defined by L . Thus it is sufficient to prove the theorem for x a positive function and μ a positive measure.

For simplicity assume that both a' and b' are finite. A modification of the argument given below will hold when a' and b' are not necessarily finite. Let $m = (a' + b')/2$.

Case 1. $\mu[a', m] = 0$ or $\mu(m, b'] = 0$. Without loss of generality, assume that $\mu[a', m] = 0$. Define $\Delta_n(t) = (m - t)^3(t - a')^3$ for $a' \leq t \leq m$ and 0 otherwise. Then $\{\Delta_n\}$ satisfies the conditions stated in the conclusion of the theorem.

Case 2. $\mu[a', m] > 0$ and $\mu(m, b'] > 0$. For nonnegative numbers M and N , let

$$\begin{aligned} \Delta(M, N) &= M(a' - t)^3 \text{ for } a' \leq t \leq m, \\ &= N(b' - t)^3 \text{ for } m < t \leq b', \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Consider the function $H(M, N) = L(x_0 e^{\Delta(M, N)})$, $M, N \geq 0$. Then (i) H is a continuous function such that $H(0, 0) = L(x_0)$, and (ii) for each $M > 0$, $H(M, N)$ increases as a function of N from a value less than $L(x_0)$ to the value ∞ . Property (i) follows from the Bounded Convergence theorem, while Property (ii) follows from the definition of $\Delta(M, N)$ and the fact that both $\mu[a', m)$ and $\mu(m, b']$ are greater than zero. From these two properties, it is easy to see that for each positive integer n there exists a positive integer N_n such that $L(1/n, N_n) = L(x_0)$ and N_n converges to zero as $n \rightarrow \infty$. If we let $\Delta_n \equiv \Delta(1/n, N_n)$, the conclusion of the theorem follows. ||

Theorem 3.5. Let $L \in C^*[a, b]$ and k be a fixed constant. Let $K = \{x \in C: L(x) = k\}$. Then K is convex and $\text{ext } K = K \cap \text{ext } C$.

Proof. Let $K_1 = \{x \in C[a, b]: L(x) = k\}$. Then $K = K_1 \cap C$ and K is convex since K_1 and C are convex. Let $x_0 \in K \cap E^C$. Then there is an interval $[a', b'] \subset I(x_0)$ with $-\infty < a' < b' < \infty$. By Lemma 3.4, there is a sequence of nonzero functions $\{\Delta_n\} \subset V[a', b']$ such that

$\|\Delta_n\|_{[a', b']} \rightarrow 0$ as $n \rightarrow \infty$ and such that $L(x_0 e^{\Delta_n}) = k$ for each n . By Lemma 3.3, $L(x_0 e^{C(\Delta_n)}) = k$ for each n . Thus $\Delta \equiv 0$ is not an isolated point of $\{\Delta \in V[a', b']: x_0 e^{\Delta} \text{ and } x_0 e^{C(\Delta)} \text{ are both in } K_1\}$. Thus the hypothesis of Lemma 3.1 is satisfied for every $x_0 \in K \cap E^C$ and so $\text{ext } K = K \cap \text{ext } C$ by Theorem 3.2. ||

Example 3. Let $[a, b]$ be a bounded interval, n a fixed integer greater than 1, and k a fixed constant. Let

$$K \equiv \{x \in C: \int_a^b t^n x(dt) = k\}.$$

An integration by parts yields

$$\int_a^b t^n x(dt) = x(b)b^n - x(a)a^n - \int_a^b nx(t)t^{n-1}dt.$$

Thus $K = \{x \in C: L(x) = k\}$, where $L \in C^*([a, b])$ is given by

$$L(x) \equiv x(b)b^n - x(a)a^n - \int_a^b nx(t)t^{n-1}dt$$

for each $x \in C[a, b]$. By Theorem 3.5 we conclude that $K \cap \text{ext } C = \text{ext } K$.

Example 4. Let K be as specified in Examples 1 and 2, let $L \in C^*[a, b]$, and let k be a fixed constant. Let $\tilde{K} = \{x \in K: L(x) = k\} = K \cap \{x: L(x) = k\}$. Then $\tilde{K}$ is convex since K is convex and L is linear. Let $x \in \tilde{K} \cap E^C$.

Then there is an $[a', b'] \subset I(x)$ with $a' < b'$. By Lemmas 3.3 and 3.4, there is a sequence of nonidentically zero functions $\{\Delta_n\} \subset V[a', b']$ such that (i) $\|\Delta_n\|_{[a', b']} \rightarrow 0$ as $n \rightarrow \infty$, and (ii) $L(xe^{\Delta_n}) = L(x) = L(xe^{c(\Delta_n)})$. In Examples 1 and 2 the sequence $\{\Delta_n\}$ is an arbitrary sequence satisfying

(i). Using the arguments in Examples 1 and 2 and the properties of Δ_n given above, we see that $\Delta \equiv 0$ is not an isolated point of $\{\Delta \in V[a', b]: xe^{\Delta} \text{ and } xe^{c(\Delta)} \text{ both in } \tilde{K}\}$. Thus the hypothesis of Lemma 3.1 is satisfied and we conclude that $\text{ext } \tilde{K} = \tilde{K} \cap \text{ext } C$ by Theorem 3.2. We note that here $K \cap \{x: L(x) = k\}$ plays the role of K_1 in Lemma 3.1.

4. Proof of the Main Result. We prove Theorem 2.1 by showing that (i) $E \subset \text{ext } C$ (Lemma 4.1) and (ii) $\text{ext } C \subset E$ (Lemma 4.2).

Lemma 4.1. $E \subset \text{ext } C$.

Proof. Let $x \in E$. Since C is convex, to prove that $x \in \text{ext } C$, it suffices to prove that

$$(4.1) \quad x = (x_1 + x_2)/2, \text{ with } x_1, x_2 \in C,$$

implies

$$(4.2) \quad x_1 = x_2 = x.$$

Thus assume (4.1). We shall show (4.2).

Let $t_0 \in M \equiv \{t: X_1'' \text{ and } X_2'' \text{ exist at } t\}$. Since $m(M^c) = 0$ and $x \in E$, it follows from the definition of E that, for each positive integer n ,

$$m([t_0, t_0 + 1/n] \cap M \cap \{t: r' < 1/n\}) > 0.$$

Hence we may choose a sequence $\{t_n\} \subset M$ such that $t_n \rightarrow t_0$ and $X''(t_n) \rightarrow 0$ as $n \rightarrow \infty$. By a subsequence argument we may also deduce that the sequences $\{X_1''(t_n)\}$ and $\{X_2''(t_n)\}$ both have limits (possibly infinite). Now using the definitions of c , X_1 , and X_2 , and using (4.1), we can show that $(X_2 - X)(t_n) = c((X_1 - X)(t_n))$. Differentiating twice and taking limits, we get

$$(4.3) \quad \begin{aligned} \lim X_2''(t_n) &= c'((X_1 - X)(t_0))(\lim X_1''(t_n)) \\ &\quad + c''((X_1 - X)(t_0))(X_1'(t_0) - X'(t_0))^2, \end{aligned}$$

since $X''(t_n) \rightarrow 0$ and $t_n \rightarrow t_0$ as $n \rightarrow \infty$ and X , X_1 , X' , and X_1' are continuous at t_0 .

Now for $t < \ln 2$, $c'(t) = -e^5/(2 - e^t) < 0$ and $c''(t) = -2e^t/(2 - e^t)^2 < 0$.

Hence using the fact that $\lim X_1''(t_n) \geq 0$, we see that the expression on the right of (4.3) is nonpositive. Since the expression on the left of (4.3) is nonnegative, it follows that the expression on the right of (4.3) is equal to 0. We conclude that $X'(t) = X_1'(t)$ for all $t \in M$. Reversing the roles of X_1 and X_2 above, we get $X'(t) = X_2'(t)$ for $t \in M$. Thus

$$(4.4) \quad X' = X_1' = X_2' \text{ a.e on } [a, b].$$

Now since x , x_1 , and $x_2 \in C$, $c = x(a) = x_1(a) = x_2(a)$, and so, $X(a) = X_1(a) = X_2(a)$. It follows from (4.4) that $X = X_1 = X_2$, or equivalently, $x = x_1 = x_2$. ||

Lemma 4.2. $\text{ext } C \subset E$.

Proof. It suffices to show that if $x \in C$ and $x \notin E$, then $x \notin \text{ext } C$.

To prove that $x \notin \text{ext } C$ it suffices to show that there exists x_1 and $x_2 \in C$ such that $(x_1 + x_2)/2 = x$.

Let $x \in C$ and $x \notin E$. Then there exists an interval $[a', b'] \subset (a, b)$ such that a' and b' are continuity points of X' and such that X'' is a.e. uniformly bounded away from 0 on $[a', b']$. For each n , let $\Delta_n(t) = (t - a')^3(b' - t)^3/n$ for $a' \leq t \leq b'$ and 0 otherwise. Then this sequence of functions satisfies the hypotheses of Lemma 2.2 and so the functions x_{in} ($i = 1, 2$) defined in (2.2) are in C for all sufficiently large n , say $n \geq n_0$. Since $x = (x_{1n_0} + x_{2n_0})/2$ by identity (2.3), $x \notin \text{ext } C$. ||

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20. ABSTRACT A general technique is presented for identifying the extreme points of a convex set C of log-convex functions and the extreme points of certain types of its convex subsets. For K one these subsets, it is shown that $\text{ext } K = K \cap \text{ext } C$ even though K is not an extremal subset of C .		