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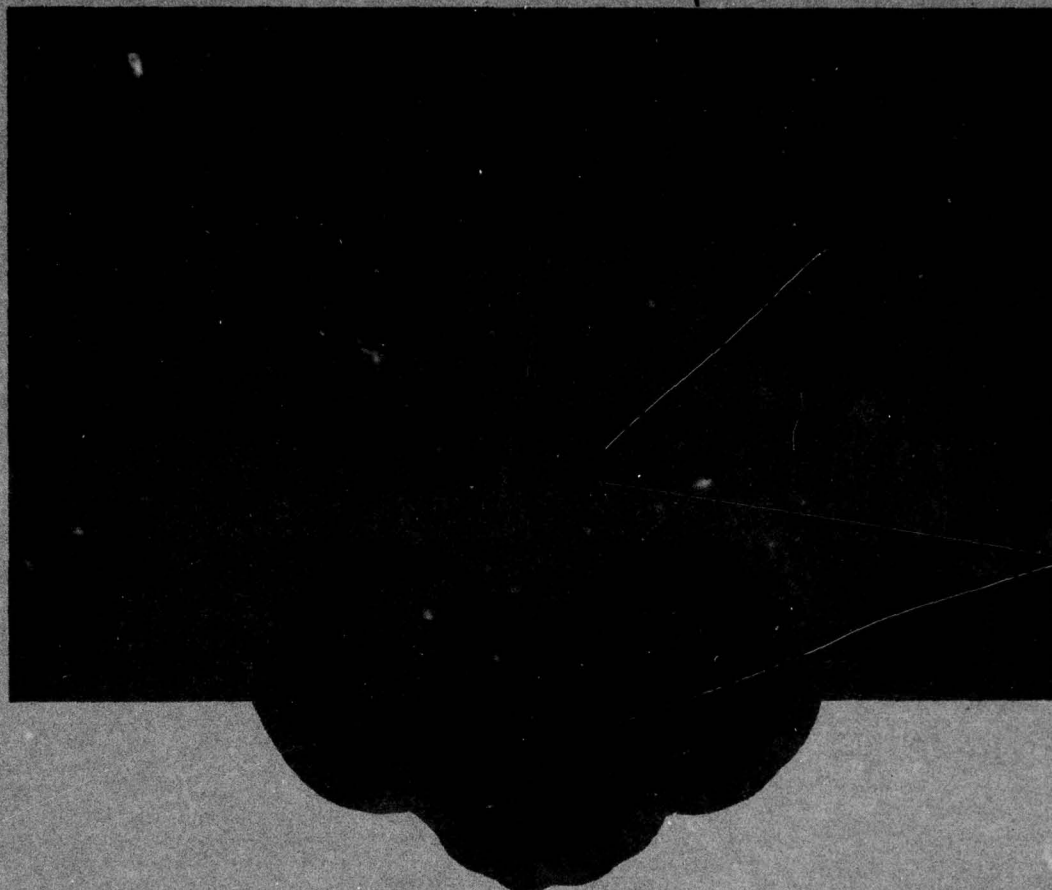
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Nonparametric Regression Based on the Concomitants of Order Statistics

by

Gordon Johnston
University of North Carolina at Chapel Hill

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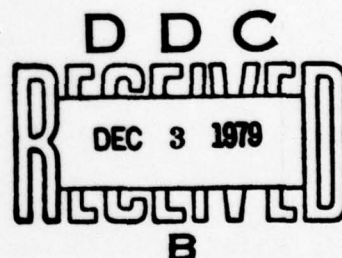
by

Gordon Johnston*

Abstract

We investigate the properties of nonparametric regression function estimates based on the concomitants of order statistics.

Key Words and Phrases: Regression, nonparametric estimation, density estimation, concomitant order statistics, Gaussian processes.



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1. Introduction.

S. S. Yang (1977) proposed as an estimation of the regression function $m(u) = E[Y|X = u]$ of a bivariate random vector (X, Y) the statistic M_n defined by

$$M_n(u) = (n\epsilon_n)^{-1} \sum_{i=1}^n K\left(\frac{i/n - F_n(u)}{\epsilon_n}\right) Y_{[i:n]}$$

Here $\{\epsilon_n^{-1}K(x/\epsilon_n)\}$ is a δ -function sequence of kernel type (Watson and Leadbetter (1964)) (X_i, Y_i) , $i=1, \dots, n$ are i.i.d. observations on (X, Y) , F_n is the empirical distribution function (EDF) of the X -observations, and $Y_{[i:n]}$ is the Y -observation corresponding to the i -th order statistic of the X -observations, i.e., the i -th concomitant of the X -values n (see, e.g., Yong (1977)).

Our purpose here is to find conditions under which

$$(1.1) \quad (n\epsilon_n \log n)^{\frac{1}{2}} \left[\sup_{a \leq u \leq b} \left| \frac{(n\epsilon_n)^{\frac{1}{2}} [M_n(u) - m(u)]}{[s(u) \int k^2(t) dt]^{\frac{1}{2}}} \right| - d_n \right]$$

$\xrightarrow{L} E$ as $n \rightarrow \infty$, where E is a random variable with density $e^{-2e^{-x}}$, $x > 0$,

a, b , are constants, $\{\epsilon_n\}$ and $\{d_n\}$ are appropriate real sequences and

$s(u) = E[Y^2|X = a]$. Bickel and Rosenblatt (1973) proved a similar result

for kernel estimates of a density function. A large sample confidence interval for $m(u)$, based on $M_n(u)$ is given, using (1.1).

We also give conditions under which

$$(1.2) \quad (n\epsilon_n)^{\frac{1}{2}} [M_n(u) - m(u)] \xrightarrow{L} N(0, s(u) \int k^2(t) dt) \text{ as } n \rightarrow \infty$$

for appropriate points u and sequence $\{\epsilon_n\}$.

Our method of proof is to show that

$$(1.3) \quad (n\epsilon_n \log n)^{\frac{1}{2}} \sup_{a \leq u \leq b} |M_n(u) - M_n^{**}(u)| \xrightarrow{P} 0,$$

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where M_n^{**} is defined by

$$(1.4) \quad M_n^{**}(u) = (n\epsilon_n)^{-1} \sum_{i=1}^n Y_i K((F(X_i) - F(u))/\epsilon_n).$$

M_n^{**} is a special case of the regression function estimation proposed by Watson (1964). Johnston (1979) gives conditions under which (1.1) and (1.2) hold for M_n^{**} in place of M_n , and (1.1) and (1.2) will thus hold by virtue of (1.3).

2. Asymptotic Equivalence of M_n and M_n^{**} .

In this section we verify (1.3). The proof is given in the Appendix since it is rather technical and lengthy. Define

$$M_n^*(u) = (n\epsilon_n)^{-1} \sum_{i=1}^n Y_i K((F_n(X_i) - F(u))/\epsilon_n)$$

Then Lemma 2.1 gives conditions under which

$$(2.1) \quad (n\epsilon_n \log n)^{\frac{1}{2}} \sup_{a \leq u \leq b} |M_n^*(u) - M_n(u)| \xrightarrow{P} 0$$

$$(2.2) \quad (n\epsilon_n \log n)^{\frac{1}{2}} \sup_{a \leq u \leq b} |M_n^{**}(u) - M_n^*(u)| \xrightarrow{P} 0,$$

which together imply (1.3).

Lemma 2.1 Suppose $\{\epsilon_n^{-1} K(x/\epsilon_n)\}$ is a δ -function sequence such that $(\log n)^{-1} (n\epsilon_n^{\frac{1}{2}}) \rightarrow \infty$, K has bounded support and 3 bounded continuous derivatives on the support. Suppose $\int |K''(t)| dt < \infty$ and K and K' are of bounded variation.

Let (X, Y) be such that $E|Y| < \infty$, $g(u) = E[Y|X = F^{-1}(u)]$ has 2 bounded derivatives on $[0, 1]$ and $h(u) = E[(Y)|X = F^{-1}(u)]$ is bounded on $[0, 1]$.

Assume there exists a real sequence $\{a_n\}$ such that $a_n \rightarrow \infty$,

$$a_n^2 \log n / (n\epsilon_n^3) \rightarrow 0 \quad \text{and}$$

$$n^{\frac{1}{2}} \int |y| dF^Y(y) \rightarrow 0 \quad \text{as } n \rightarrow \infty, \\ |y| > a_n$$

Then, for $0 < F(a) < F(b) < 1$, (2.1) and (2.2) hold. $\square$

3. Applications.

We will assume throughout this section that the assumptions of Theorem 2.1 are in force. We first note that M_n^{**} may be written as

$$M_n^{**}(u) = (m_n)^{-1} \sum_{i=1}^n Y_i K((Z_i - F(u))/\epsilon_n)$$

where

$$Z_i = F(X_i) \sim U(0,1).$$

According to Theorem 2.5.2 of Johnston (1979), under certain conditions,

$$(n\epsilon_n)^{\frac{1}{2}} [M_n^{**}(u) - E(Y|Z = F(u))] \xrightarrow{L} N(0, E(Y^2|Z = F(u)) \int K^2(t)dt).$$

If we assume F to be strictly increasing, then

$$E(Y|Z = F(u)) = m(u)$$

and

$$E(Y^2|Z = F(u)) = s(u).$$

Thus we have, by virtue of (1.3)

$$(n\epsilon_n)^{\frac{1}{2}} [M_n(u) - m(u)] \xrightarrow{L} N(0, s(u) \int K^2(t)dt),$$

which completes the proof of normality of M_n . We note that this asymptotic variance differs from that of Yong (1977), Theorem 6.

If the conditions of Corollary 3.2.9 of Johnston (1979) hold, then

$$(3.1) \quad (2\delta \log n)^{\frac{1}{2}} \left[\sup_{a \leq u \leq b} \left| \frac{(n\epsilon_n)^{\frac{1}{2}} [M_n^{**}(u) - m(u)]}{[s(u) \int K^2(t)dt]^{\frac{1}{2}}} \right| - d_n \right] \xrightarrow{L} E,$$

where E is a random variable with density $e^{-2e^{-x}}$, $x > 0$. Here $\epsilon_n = n^{-\delta}$, $\frac{1}{5} < \delta < \frac{1}{2}$ and d_n is the sequence of entering constants specified in Bickel and Rosenblatt (1973). By virtue of (1.3), (3.1) holds with M_n replacing M_n^{**} , as we wished to prove. Inverting (3.1) in the usual way yields an approximate $(1-\alpha) \times 100\%$ confidence band for $m(u)$ over the interval (a,b) , based on $M_n(u)$:

$$M_n(u) \pm (n\epsilon_n)^{-\frac{1}{2}} \left[s(u) \int K^2(t) dt \right]^{\frac{1}{2}} \left[d_n + \frac{c(\alpha)}{(2\delta \log n)^{\frac{1}{2}}} \right]$$

where

$$c(\alpha) = \log 2 - \log |\log (1-\alpha)|.$$

APPENDIX

Proof of Lemma 2.1.

We begin with the following preliminary lemma, which is very similar to Lemma 1 of Bhattacharyya (1967).

A1. Lemma Assume that $g(u) = E[Y|X = F^{-1}(u)]$ has r continuous derivatives on $[0,1]$, $r > 0$, and that K has bounded support and r bounded derivatives on the support. Then for a, b such that $0 < F(a) < F(b) < 1$,

$$\left| \epsilon_n^{-(r+1)} \iint y K^{(r)}((F(x) - F(z))/\epsilon_n) dF(x, y) \right| = O(1)$$

uniformly in $z \in [a, b]$ as $n \rightarrow \infty$.

Proof. Note that

$$\begin{aligned} & \epsilon_n^{-(r+1)} \iint y K^{(r)}((F(x) - F(z))/\epsilon_n) dF(x, y) \\ &= \epsilon_n^{-(r+1)} E Y K^{(r)}((F(X) - F(z))/\epsilon_n) \\ &= \epsilon_n^{-(r+1)} \int m(x) K^{(r)}((F(x) - F(z))/\epsilon_n) dF(x) \\ &= \epsilon_n^{-(r+1)} \int_0^1 g(u) K^{(r)}((u - F(z))/\epsilon_n) du. \end{aligned}$$

Now write

$$\begin{aligned} & \epsilon_n^{-(r+1)} g(u) K^{(r)}((u - F(z))/\epsilon_n) \\ &= \epsilon_n^{-1} g^{(r)}(u) K((u - F(z))/\epsilon_n) \\ &= \frac{d}{du} \sum_{s=0}^{r-1} \epsilon_n^{-(s+1)} g^{(r-s-1)}(u) K^{(s)}((u - F(z))/\epsilon_n). \end{aligned}$$

Hence

$$\begin{aligned} & \sup_z \left| \epsilon_n^{-(r+1)} \int_0^1 g(u) K^{(r)}((u - F(z))/\epsilon_n) du \right| \\ &\leq \sup_z \left| \epsilon_n^{-1} \int_0^1 g^{(r)}(u) K((u - F(z))/\epsilon_n) du \right| \end{aligned}$$

$$+ \sup_z \left| \left[\sum_{s=0}^{r-1} \epsilon_n^{-(s+1)} g^{(r-s-1)}(u) k^{(s)}((u-F(z))/\epsilon_n) \right]_{u=0}^1 \right|$$

The second term above is zero for large n since the argument of $k^{(s)}$ is eventually outside the support of k . Write

$$\begin{aligned} & \sup_z \left| \epsilon_n^{-1} \int_0^1 g^{(r)}(u) K((u-F(z))/\epsilon_n) du \right| \\ &= \sup_z \left| \int_{-F(z)/\epsilon_n}^{(1-F(z))/\epsilon_n} K(v) g^{(r)}(\epsilon_n v + F(z)) dv \right| \end{aligned}$$

$$\leq \sup_t |g^{(r)}(t)| \int |K(v)| dv < \infty.$$

□

We now proceed with the proof of Lemma 2.1. It is convenient to rewrite

$$M_n(u) = \epsilon_n^{-1} \iint y K((F_n(x) - F_n(u))/\epsilon_n) dF_n(x, y),$$

and similarly for M_n^* and M_n^{**} . Thus, letting $Z_n(x, y) = F_n(x, y) - F(x, y)$, we may write

$$\begin{aligned} & M_n^*(u) - M_n(u) \\ &= \epsilon_n^{-1} \iint y \left[K\left(\frac{F_n(x) - F(u)}{\epsilon_n}\right) - K\left(\frac{F_n(x) - F_n(u)}{\epsilon_n}\right) \right] dZ_n(x, y) \\ &+ \epsilon_n^{-1} \iint y \left[K\left(\frac{F_n(x) - F(u)}{\epsilon_n}\right) - K\left(\frac{F_n(x) - F_n(u)}{\epsilon_n}\right) \right] dF(x, y) \end{aligned}$$

$= J_1 + J_2$, say. We first show $(n \epsilon_n \log n)^{\frac{1}{2}} |J_2| \xrightarrow{P} 0$. Since, by assumption, K has 3 continuous derivatives, we may write (by expanding $K((F_n(x) - F_n(u))/\epsilon_n)$ about $(F_n(x) - F(u))/\epsilon_n$)

$$\begin{aligned} J_2 &= \epsilon_n^{-2} [F_n(u) - F(u)] \iint y K' \left(\frac{F_n(x) - F(u)}{\epsilon_n} \right) dF(x, y) \\ &+ \epsilon_n^{-3} [F_n(u) - F(u)]^2 \iint y K'' \left(\frac{F_n(x) - F(u)}{\epsilon_n} \right) dF(x, y) \end{aligned}$$

$$\begin{aligned}
& + \epsilon_n^{-4} [F_n(u) - F(u)]^3 \iint y K''' \left(\frac{F_n(x) + w_n(u)}{\epsilon_n} \right) dF(x, y) \\
& = J_2^{(1)} + J_2^{(2)} + J_2^{(3)}, \text{ say, where } w_n(u) \text{ is between } F_n(u) \text{ and } F(u).
\end{aligned}$$

Now, expanding $K' \left(\frac{F_n(x) - F(u)}{\epsilon_n} \right)$ about $(F(x) - F(u))/\epsilon_n$ yields

$$\begin{aligned}
(A1) \quad & (n\epsilon_n \log n)^{\frac{1}{2}} \sup_u |J_2^{(1)}| \\
& \leq (n\epsilon_n \log n)^{\frac{1}{2}} \epsilon_n^{-2} \sup_u |F_n(u) - F(u)| \\
& \times \left\{ \left| \iint y K' \left(\frac{F(x) - F(u)}{\epsilon_n} \right) dF(x, y) \right| \right. \\
& + \left| \iint \left[\frac{F_n(x) - F(x)}{\epsilon_n} \right] y K'' \left(\frac{F(x) - F(u)}{\epsilon_n} \right) dF(x, y) \right| \\
& + \left. \left| \iint \left[\frac{F_n(x) - F(x)}{\epsilon_n} \right] y K''' \left(\frac{v_n(x, u)}{\epsilon_n} \right) dF(x, y) \right| \right\}
\end{aligned}$$

where $v_n(x, u)$ is between $F_n(x) - F(u)$ and $F(x) - F(u)$.

Using the fact that $\sup_u |F_n(u) - F(u)| = O_p(n^{-\frac{1}{2}})$ and applying Lemma A1 implies that the first term on the RHS of inequality A1 goes to zero. For the second term, note that

$$\begin{aligned}
& \epsilon_n^{-1} \iint \left| y K'' \left(\frac{F(x) - F(u)}{\epsilon_n} \right) \right| dF(x, y) \\
& = \epsilon_n^{-1} \int_0^1 h(t) \left| K'' \left(\frac{t - F(u)}{\epsilon_n} \right) \right| dt \\
& = \int_{-F(u)/\epsilon_n}^{(1-F(u))/\epsilon_n} |K''(v)| h(\epsilon_n v + F(u)) dv,
\end{aligned}$$

which is a bounded sequence since h is bounded and K'' has bounded supports.

Thus the second term on the RHS of (A1) is equal to

$(n\epsilon_n \log n)^{\frac{1}{2}} \epsilon_n^{-2} O_p(n^{-1}) O(1)$, which converges to zero in probability if $(n\epsilon_n \log n)^{\frac{1}{2}}/n\epsilon_n^2 \rightarrow 0$, i.e., if $(n\epsilon_n^3)(\log n)^{-1} \rightarrow \infty$, which is true by assumption.

For the third term on the RHS of (A1) note

$$\int \left| y K''' \left(\frac{v_n(x,u)}{\epsilon_n} \right) \right| dF(x,y)$$

$$\leq \sup_v |K'''(v)| E|Y| < \infty.$$

Thus the third term is a $(n\epsilon_n \log n)^{\frac{1}{2}} \epsilon_n^{-4} O_p(n^{-3/2})$ sequence, and converges to zero in probability since $(\log n)^{-1} n\epsilon_n^{7/2} \rightarrow \infty$. Similar arguments apply to $J_2^{(2)}$ and $J_2^{(3)}$, and we have shown $(n\epsilon_n \log n)^{\frac{1}{2}} \sup |J_2| \xrightarrow{P} 0$.

We now turn to J_1 . Let $\{a_n\}$ be a sequence as specified in the hypotheses and write

$$\begin{aligned} J_1 &= \epsilon_n^{-1} \int_{|y| > a_n} \int y G_n(x,u) Z_n(dx, dy) \\ &+ \epsilon_n^{-1} \int_{|y| \leq a_n} \int y G_n(x,u) Z_n(dx, dy) \\ &= J_1^{(1)} + J_1^{(2)}, \text{ say, where, for convenience, we write} \end{aligned}$$

$$G_n(x,u) = K \left(\frac{F_n(x) - F(u)}{\epsilon_n} \right) - K \left(\frac{F_n(x) - F_n(u)}{\epsilon_n} \right)$$

Using integration by parts, write

$$J_1^{(2)} = \epsilon_n^{-1} \int_{|y| \leq a_n} \int Z_n(x,y) dy G_n(dx,u)$$

$$\begin{aligned}
& + \lim_{t \rightarrow \infty} \epsilon_n^{-1} \int_{-a_n}^{a_n} G_n(t, u) y Z_n(t, dy) \\
& - \lim_{t \rightarrow -\infty} \epsilon_n^{-1} \int_{-a_n}^{a_n} G_n(t, u) y Z_n(t, dy) \\
& + \epsilon_n^{-1} a_n \int Z_n(x, a_n) G_n(dx, u) \\
& + \epsilon_n^{-1} a_n \int Z_n(x, -a_n) G_n(dx, u) \\
& = I_1 + I_2 + I_3 + I_4 + I_5, \text{ say.}
\end{aligned}$$

Since $Z_n(-\infty, y) = 0$ for each n and y , it is easily ascertained that $I_3 = 0$ for each n (e.g. Natanson (1964), p 233). Similarly,

$$I_2 = I_2(u) = \lim_{t \rightarrow \infty} G_n(t, u) \epsilon_n^{-1} \int_{-a_n}^{a_n} y dQ_n(y)$$

where

$$Q_n(y) = \lim_{t \rightarrow \infty} Z_n(t, y) = F_n^Y(y) - F^Y(y).$$

Now

$$\int_{-a_n}^{a_n} y dZ_n(y) = n^{-1} \sum_{i=1}^n \left\{ Y_i I_{[-a_n, a_n]}(Y_i) - E Y I_{[-a_n, a_n]}(Y) \right\} = O_p(n^{-1/2})$$

as $n \rightarrow \infty$ by standard central limit theorem

arguments. Further, using the mean value theorem,

$$\begin{aligned}
\lim_{t \rightarrow \infty} G_n(t, u) &= K \left(\frac{1 - F(u)}{\epsilon_n} \right) - K \left(\frac{1 - F_n(u)}{\epsilon_n} \right) \\
&= \frac{F_n(u) - F(u)}{\epsilon_n} K' \left(\frac{1 + q_n(u)}{\epsilon_n} \right) = \epsilon_n^{-1} O_p(n^{-1/2})
\end{aligned}$$

uniformly in u , where $q_n(u)$ is between $F_n(u)$ and $F(u)$.

Thus we have

$$(n\epsilon_n \log n)^{\frac{1}{2}} \sup_n |I_2(u)| = (n\epsilon_n \log n)^{\frac{1}{2}} \epsilon_n^{-2} O_p(n^{-1}) \rightarrow 0$$

$$\text{since } n\epsilon_n^3 / \log n \rightarrow \infty.$$

For I_4 , note that

$$\begin{aligned} & \left| \int Z_n(x, a_n) G_n(dx, u) \right| \\ & \leq \sup_x |Z_n(x, a_n)| V[G_n(\cdot, u)], \end{aligned}$$

Where $V[\cdot]$ denotes total variation over R . Now

$$\sup_x |Z_n(x, a_n)| = O_p(n^{-\frac{1}{2}})$$

and it is easily verified, using the mean value theorem, that

$$V[G_n(\cdot, u)] = \epsilon_n^{-1} O_p(n^{-\frac{1}{2}})$$

uniformly in u . Thus

$$\begin{aligned} & (n\epsilon_n \log n)^{\frac{1}{2}} \sup_u |I_4(u)| \\ & = a_n (n\epsilon_n \log n)^{\frac{1}{2}} \epsilon_n^{-2} O_p(n^{-1}) \xrightarrow{P} 0 \end{aligned}$$

since $a_n^2 \log n / n\epsilon_n^3 \rightarrow 0$ by assumption. A similar argument applies to show

$$(n\epsilon_n \log n)^{\frac{1}{2}} \sup_u |I_5(u)| \xrightarrow{P} 0.$$

For I_1 , note that

$$\begin{aligned} & \left| \int_{|y| < a_n} \int Z_n(x, y) dy G_n(dx, u) \right| \\ & \leq \sup_{x, y} |Z_n(x, y)| V[y G_n(x, u)], \end{aligned}$$

where V denotes here the total variation in (x, y) over $R \times [-a_n, a_n]$. As before,

$$\sup_{x, y} |Z_n(x, y)| = O_p(n^{-\frac{k}{2}})$$

and

$$V[yG_n(x, u)] = a_n \epsilon_n^{-1} O_p(n^{-\frac{k}{2}}) \text{ uniformly in } u.$$

Thus

$$\begin{aligned} & (n \epsilon_n \log n)^{\frac{k}{2}} \sup_u |I_1(u)| \\ &= a_n \epsilon_n^{-2} (n \epsilon_n \log n)^{\frac{k}{2}} O_p(n^{-1}) \xrightarrow{P} 0 \end{aligned}$$

since $a_n^2 \log n / n \epsilon_n^3 \rightarrow 0$ by assumption.

As the final step in the proof, we must verify that $(n \epsilon_n \log n)^{\frac{k}{2}} \sup_u |J_1^{(1)}| \xrightarrow{P} 0$. Note that

$$\begin{aligned} (A2) \quad \epsilon_n |J_1^{(1)}| &\leq \left| \int_{|y| > a_n} \int y G_n(x, u) dF_n(x, y) \right| \\ &+ \left| \int_{|y| > a_n} \int y G_n(x, u) dF(x, y) \right|. \end{aligned}$$

For the first term, note

$$\begin{aligned} & \left| \int_{|y| > a_n} \int y G_n(x, u) dF_n(x, y) \right| \\ &\leq \sup_{x, u} |G_n(x, u)| \int_{|y| > a_n} |y| dF_n^Y(y). \end{aligned}$$

As before,

$$\sup_{x, u} |G_n(x, u)| = \epsilon_n^{-1} O_p(n^{-\frac{k}{2}}),$$

and

$$\int_{|y| > a_n} |y| dF_n^Y(y) = n^{-1} \sum_{i=1}^n |Y_i| 1_{(a_n, \infty)}(|Y_i|) .$$

Now, by the Markov inequality, for any $\epsilon > 0$

$$\begin{aligned} & P\left\{ \left| \sqrt{n} \int_{|y| > a_n} |y| dF_n^Y(y) \right| > \epsilon \right\} \\ & \leq \epsilon^{-1} E \left| \sqrt{n} \int_{|y| > a_n} |y| dF_n^Y(y) \right| \\ & = \sqrt{n} \int_{|y| > a_n} |y| dF^Y(y) \rightarrow 0 \end{aligned}$$

by assumption, and thus

$$\int_{|y| > a_n} |y| dF_n^Y(y) = o_p(n^{-\frac{1}{2}}).$$

A similar argument applies to the second integral on the RHS of (A2) and we thus have

$$\begin{aligned} & (n\epsilon_n \log n)^{\frac{1}{2}} \sup_u |J_1^{(1)}(u)| \\ & (n\epsilon_n \log n)^{\frac{1}{2}} \epsilon_n^{-2} o_p(n^{-1}) \xrightarrow{P} 0 \end{aligned}$$

since $n\epsilon_n^3/\log n \rightarrow \infty$ by assumption. □

The proof of (2.2) follows a similar pattern, and we omit the details.

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