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ON THE WEAK LAW OF LARGE NUMBERS FOR $D(0,1)$. (U)
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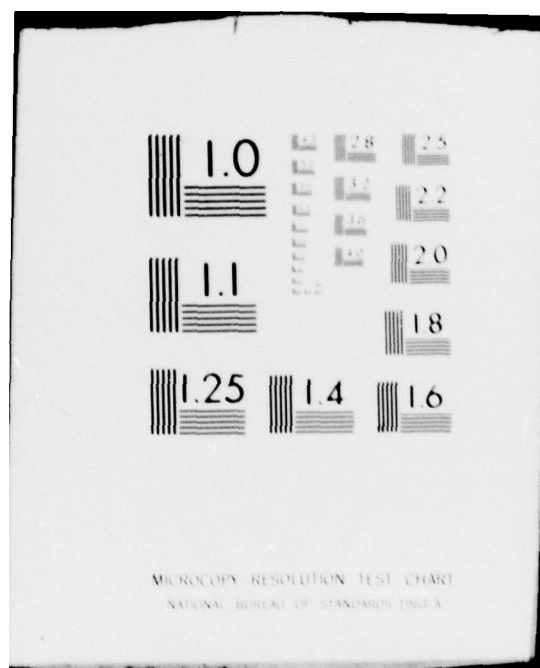
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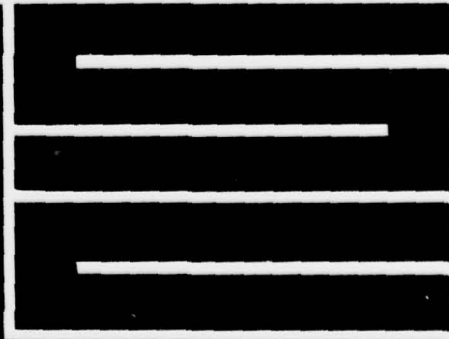
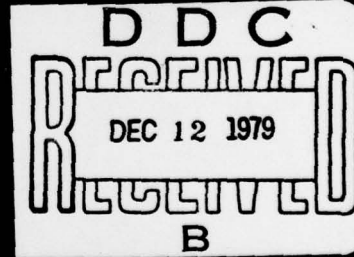
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⑥ ON THE WEAK LAW OF LARGE
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by

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University of South Carolina
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Abstract

Weak laws of large numbers are obtained for random elements in $D[0,1]$ where the convergence is in the sup-norm topology. For identically distributed random elements satisfying a compact integral condition, the weak law of large numbers holding pointwise is shown to be necessary and sufficient for the weak law of large numbers. In addition to a discussion of the compact integral condition, a weak law of large numbers is obtained for monotone increasing random elements, and convergence of weighted sums of independent, identically distributed random elements is obtained.

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1. Introduction and Preliminaries. Let $D=D[0,1]$ denote the space of real-valued functions on $[0,1]$ which are right-continuous and possess left-hand limits at each $t \in [0,1]$. Let the linear space D be equipped with the topology generated by the Skorohod metric d and let $|| \cdot ||$ denote the uniform norm, $||x|| = \sup_{0 \leq t \leq 1} |x(t)|$ for $x \in D$ (see Billingsley (1968), pages 109-153 for detailed topological and probabilistic properties of D). With the Skorohod metric topology, the linear space D is separable but not a linear metric space (addition is not continuous). However, the major obstacle in developing laws of large numbers is the absence of local convexity for D with the Skorohod topology.

Ranga Rao (1963) proved a strong law of large numbers for independent, identically distributed random elements in D while Sethuraman (1965) proved a large deviation result for independent, identically distributed random elements in D . Their methods of proof used truncation to a compact set and the following lemma.

Lemma 1 (Ranga Rao (1963)): Let X be a random element in D with $E||X|| < \infty$. For each $\epsilon > 0$ there exists a partition $0 = t_0 < t_1 < \dots < t_m = 1$ of $[0,1]$ such that

$$\max_{1 \leq i \leq m} \sup_{t_{i-1} \leq s, t < t_i} E|X(t) - X(s)| \leq \epsilon.$$

To compensate for the absence of local convexity, Taylor and Daffer (1978) & (1979) required the random elements to be convex tight and obtained laws of large numbers in the Skorohod topology. In this paper a weak law of large numbers is obtained in the uniform norm topology for identically distributed random elements satisfying the following compact integral condition.

Property MT: A random element X in $D[0,1]$ has property MT if for each $\epsilon_1 > 0$ and $\epsilon_2 > 0$ there exists a compact set K and a partition $0 = t_0 < \dots < t_m = 1$ such that

$$E \left[\max_{0 \leq i \leq m-1} \sup_{t_i \leq t < t_{i+1}} |X(t) - X(t_i)| I_{[X \in K]} \right] < \epsilon_1$$

and where $P[X \in K] > 1 - \epsilon_2$.

Since convex tightness and a finite first moment is sufficient for property MT, the convergence in the $|| \cdot ||$ - topology significantly improves the laws of large numbers for convex tight random elements. A brief discussion of property MT will be included, and a weak law for the convergence of weighted sums of random elements will also be obtained in Section 3.

2. A Weak Law of Large Numbers. Given a partition $0 = t_0 < t_1 < \dots < t_m = 1$, for notational convenience let

$$Tx = \sum_{i=0}^m x(t_i) I_{[t_i, t_{i+1})}(t) \quad (2.1)$$

where $x = x(t) \in D[0,1]$ and $[t_m, t_{m+1}) = \{1\}$. It is easy to see that T is a linear, Borel measurable (with respect to the Skorohod topology) function from D into D and that $||Tx|| \leq ||x||$ for all $x \in D$.

Theorem 1: Let $\{X_n\}$ be identically distributed random elements in D such that $E||X_1|| < \infty$ and let X_1 have property MT. Then

$$\frac{1}{n} \sum_{k=1}^n X_k(t) \rightarrow EX_1(t) \text{ in probability}$$

for each $t \in [0,1]$ if and only if

$$||\frac{1}{n} \sum_{k=1}^n X_k - EX_1|| \rightarrow 0 \text{ in probability.}$$

Proof: The "if" part is immediate. For the "only if" part let $\epsilon > 0$ and $0 < \eta \leq 1$ be given. Since $E||X_1|| < \infty$ and X_1 has property MT, there exists a compact set K such that

$$E(||X_1|| I_{[X_1 \notin K]}) < \frac{\epsilon \eta}{12} \quad (2.2)$$

and a partition $0 = t_0 < t_1 < \dots < t_m = 1$ such that

$$E[\max_{0 \leq i \leq m-1} \sup_{t_i \leq t < t_{i+1}} |X_1(t) - X_1(t_i)| I_{[X_1 \in K]}] < \frac{\epsilon \eta}{12}. \quad (2.3)$$

For each n

$$\begin{aligned} P[||\frac{1}{n} \sum_{k=1}^n X_k - EX_1|| > \epsilon] \\ \leq P[||\frac{1}{n} \sum_{k=1}^n TX_k - E(TX_1)|| > \frac{\epsilon}{4}] \\ + P[||\frac{1}{n} \sum_{k=1}^n X_k - \frac{1}{n} \sum_{k=1}^n TX_k|| > \frac{\epsilon}{2}] \\ + P[||EX_1 - E(TX_1)|| > \frac{\epsilon}{4}]. \end{aligned} \quad (2.4)$$

The third term of (2.4) is zero since

$$\begin{aligned} ||EX_1 - E(TX_1)|| \\ \leq E(||X_1 - TX_1|| I_{[X_1 \in K]}) + E(||X_1 - TX_1|| I_{[X_1 \notin K]}) \\ < E[\max_i \sup_{t_i \leq t < t_{i+1}} |X_1(t) - X_1(t_i)| I_{[X_1 \in K]}] + \frac{2\epsilon \eta}{12} \\ < \frac{\epsilon}{12} + \frac{2\epsilon}{12} = \frac{\epsilon}{4} \end{aligned} \quad (2.5)$$

by (2.2) and (2.3). Using Markov's inequality and the identical distributions for the second term of (2.4),

$$\begin{aligned}
 & P\left[\left|\frac{1}{n} \sum_{k=1}^n X_k - \frac{1}{n} \sum_{k=1}^n TX_k\right| > \frac{\epsilon}{2}\right] \\
 & \leq \frac{2}{\epsilon n} \sum_{k=1}^n E|X_k - TX_k| \\
 & \leq \frac{2}{\epsilon} (E[\max_i \sup_{t_1 \leq t < t_{i+1}} |X_1(t) - X_1(t_1)| I_{[X_1 \in K]}] \\
 & \quad + 2E(|X_1| I_{[X_1 \notin K]})) \\
 & < \frac{2}{\epsilon} \left(\frac{\epsilon n}{12} + \frac{2\epsilon n}{12}\right) = \frac{n}{2}.
 \end{aligned} \tag{2.6}$$

The weak law of large numbers holds for $\{X_k(t_i): k \geq 1\}$ with $i = 0, 1, \dots, m$ by hypothesis. Thus, for the first term of (2.4),

$$\begin{aligned}
 & P\left[\left|\frac{1}{n} \sum_{k=1}^n TX_k - E(TX_1)\right| > \frac{\epsilon}{4}\right] \\
 & \leq P\left[\max_{0 \leq i \leq m} \left|\frac{1}{n} \sum_{k=1}^n X_k(t_i) - E(X_1(t_i))\right| > \frac{\epsilon}{4}\right] \\
 & < \frac{n}{2}
 \end{aligned} \tag{2.7}$$

for all $n \geq N(\epsilon, n)$. Combining (2.4), (2.5), (2.6), and (2.7) yields

$$P\left[\left|\frac{1}{n} \sum_{k=1}^n X_k - EX_1\right| > \epsilon\right] < \frac{n}{2} + \frac{n}{2} = n$$

for all $n \geq N(\epsilon, n)$.

///

A random element X in D is said to be convex tight if for each $\epsilon > 0$ there exists a compact, convex set K such that $P[X \in K] > 1 - \epsilon$. Characterizations of convex tightness were given by Daffer and Taylor (1979).

The following lemma shows that convex tightness will imply the property MT.

Lemma 2: If a random element X in D is convex tight and $E\|X\| < \infty$, then X has property MT.

Proof: Given $\epsilon_1 > 0$ and $\epsilon_2 > 0$, choose K convex and compact such that $P[X \in K] > 1 - \epsilon_2$. By Theorem 6 of Daffer and Taylor (1979),

$$\{t \in [0,1]: \sup_{x \in K} |x(t) - x(t-0)| > \frac{\epsilon_1}{2}\}$$

is finite ($= \{s_1, \dots, s_r\}$). Let

$$\{t_0, t_1, \dots, t_m\} = \{s_1, \dots, s_r\} \cup \{0, \frac{1}{k}, \dots, \frac{k}{k}\}.$$

Then, $k \leq m \leq k+r$, and

$$\limsup_{m \rightarrow \infty} \{ \max_1 \sup_{t_1 \leq t < t_{i+1}} |x(t) - x(t_1)| \} \leq \frac{\epsilon_1}{2}.$$

By the bounded convergence theorem

$$\begin{aligned} \limsup_{m \rightarrow \infty} E[\max_1 \sup_{t_1 \leq t < t_{i+1}} |X(t) - X(t_1)| I_{[X \in K]}] &\leq \frac{\epsilon_1}{2} \\ &\leq E[\limsup_{m \rightarrow \infty} \max_1 \sup_{t_1 \leq t < t_{i+1}} |X(t) - X(t_1)| I_{[X \in K]}] \leq \frac{\epsilon_1}{2}. \end{aligned}$$

Thus, there exists m such that

$$E[\max_{0 \leq i \leq m-1} \sup_{t_1 \leq t < t_{i+1}} |X(t) - X(t_1)| I_{[X \in K]}] < \epsilon_1. \quad ///$$

From Lemma 2 it follows that the weak law of large numbers with convergence in the $|| ||$ -topology holds for many situations. The most standard condition may be pointwise uncorrelated identically distributed random elements which satisfy a convexity condition and hence yields an appropriate partition of $[0,1]$.

3. Related Results. An example of a random element which does not have property MT is somewhat (by Lemma 2) related to nonseparable support with respect to the $|| ||$ -topology. Let $\Omega = [0,1]$, A be the Borel subsets, and $X(t) = I_{[\omega,1]}(t)$. The uniform probability measure on $[0,1]$ is assumed, that is, $P[X \in \{I_{[s,1]} : 0 \leq s_1 \leq s \leq s_2 \leq 1\}] = s_2 - s_1$. Let K be any compact set such that $P[X \notin K] < \epsilon$. For any partition $\{t_0, \dots, t_m\}$ of $[0,1]$

$$\sup_{t_1 \leq t < t_{i+1}} |X(t) - X(t_i)| = 1 \quad \text{for } \omega \in (t_i, t_{i+1}).$$

Thus,

$$\max_{0 \leq i \leq m-1} \sup_{t_i \leq t < t_{i+1}} |X(t) - X(t_i)| I_{[X \in K]} = 1$$

if and only if $\omega \in [X \in K] \cap \left(\bigcup_{i=0}^{m-1} (t_i, t_{i+1}) \right)$. Hence,

$$\begin{aligned} E \left[\max_{0 \leq i \leq m-1} \sup_{t_i \leq t < t_{i+1}} |X(t) - X(t_i)| I_{[X \in K]} \right] \\ = P \left[[X \in K] \cap \left(\bigcup_{i=0}^{m-1} (t_i, t_{i+1}) \right) \right] \\ = P[X \in K] > 1 - \epsilon. \end{aligned} \tag{3.1}$$

By (3.1) X does not have property MT.

The property MT is certainly not necessary for the weak law of large numbers in $D[0,1]$. Note that X is monotone increasing with probability one. The following weak law of large numbers for monotone increasing random elements can be proved in the same manner as Theorem 2 of Daffer and Taylor (1979).

Theorem 2: Let $D^{\dagger}$ denote the cone of nondecreasing elements of D . Let $\{X_n\}$ be a sequence of random elements in D such that $X_n \in D^{\dagger}$ with probability one and $EX_n = EX_1$ for each n . Then,

$$\left| \frac{1}{n} \sum_{k=1}^n X_k(t) - EX_1(t) \right| \rightarrow 0 \text{ in probability}$$

for each $t \in [0,1]$ if and only if

$$\left| \left| \frac{1}{n} \sum_{k=1}^n X_k - EX_1 \right| \right| \rightarrow 0 \text{ in probability.}$$

The major use of property MT in obtaining the weak law of large numbers for D is in developing Inequality (2.6). It is instructive (for applications) to consider

$$\begin{aligned} P \left[\left| \frac{1}{n} \sum_{k=1}^n X_k - \frac{1}{n} \sum_{k=1}^n TX_k \right| > \frac{\epsilon}{2} \right] \\ = P \left[\max_{0 \leq i \leq m-1} \sup_{t_i \leq t < t_{i+1}} \left| \frac{1}{n} \sum_{k=1}^n (X_k(t) - X_k(t_i)) \right| > \frac{\epsilon}{2} \right] \end{aligned}$$

when the $\{X_k(t)\}$ are stochastic processes with independent increments. For these applications, Levy's inequalities may yield Inequality (2.6).

The generality of the pointwise condition is balanced by a compact, smooth type condition (property (MT)). Additional probability structure on the random elements can eliminate the need for property MT as is illustrated in Theorem 3 for the convergence of weighted sums. Only the essential steps of the proof for Theorem 3 will be listed for comparison with the proof of Theorem 1.

Theorem 3: Let $\{X_n\}$ be independent, identically distributed random elements in D such that $E||X_1|| < \infty$. Let $\{a_{nk}\}$ be a double array of constants such

that (i) $\sum_{k=1}^{\infty} |a_{nk}| \leq 1$ for each n and (ii) $\lim_{n \rightarrow \infty} \sum_{k=1}^n a_{nk} = 1$. A necessary and sufficient condition that

$$||\sum_{k=1}^n a_{nk} X_k - EX_1|| \rightarrow 0 \text{ in probability}$$

is that $\max_{1 \leq k \leq n} |a_{nk}| \rightarrow 0$.

Outline of Proof: Since real numbers can be identified with constant functions in D , $\max_{1 \leq k \leq n} |a_{nk}| \rightarrow 0$ is necessary. For sufficiency pick K compact such that (2.2) holds. Choose δ so that $\alpha \leq t < \beta < \alpha + \delta$ implies that $|x(t) - x(\alpha)| \leq |x(\beta - 0) - x(\alpha)| + \frac{\epsilon \eta}{8}$ uniformly for $x \in K$. In place of (2.3), Lemma 1 provides a partition such that

$$\max_{0 \leq i \leq m-1} \sup_{t_i \leq t, s < t_{i+1}} E|X(s) - X(t)| \leq \frac{\epsilon \eta}{12},$$

and it can be assumed that $t_{i+1} - t_i < \delta$ for all $0 \leq i \leq m-1$. For (2.5)

$$\begin{aligned} & ||EX_1 - ETX_1|| \\ &= \max_{0 \leq i \leq m-1} \sup_{t_i \leq t < t_{i+1}} E|X(t) - X(t_i)| \leq \frac{\epsilon \eta}{12} < \frac{\eta}{6}. \quad (3.2) \end{aligned}$$

Using the independence of $\{X_k(t_{i+1} - 0) - X_k(t_i): k = 1, 2, 3, \dots\}$ for each i , (2.6) becomes

$$\begin{aligned} & P[||\sum_{k=1}^n a_{nk} X_k - \sum_{k=1}^n a_{nk} TX_k|| > \frac{\epsilon}{2}] \\ & \leq P[\max_{0 \leq i \leq m} \sup_{t_i \leq t < t_{i+1}} |\sum_{k=1}^n a_{nk} (X_k(t) - X_k(t_i)) I_{[X_k \in K]}| > \frac{\epsilon}{4}] \\ & + P[\sum_{k=1}^n |a_{nk}| ||X_k - TX_k|| I_{[X_k \notin K]} > \frac{\epsilon}{4}] \end{aligned}$$

$$\begin{aligned}
 & \leq P \left[\max_{0 \leq i \leq m-1} \sum_{k=1}^n |a_{nk}| |X_k(t_{i+1} - 0) - X_k(t_i)| I_{[X_k \in K]} > \frac{\epsilon}{8} \right] \\
 & \quad + \frac{4}{\epsilon} \sum_{k=1}^n |a_{nk}| 2E(|X_k|) I_{[X_k \notin K]} \\
 & \leq \sum_{i=0}^{m-1} P \left[\sum_{k=1}^n |a_{nk}| (|X_k(t_{i+1} - 0) - X_k(t_i)| I_{[X_k \in K]} \right. \\
 & \quad \left. - E(|X_k(t_{i+1} - 0) - X_k(t_i)| I_{[X_k \in K]}) > \frac{\epsilon}{8} - \frac{\epsilon}{12} \right] \\
 & \quad + \frac{8}{\epsilon} \cdot \frac{\epsilon \eta}{12}. \tag{3.3}
 \end{aligned}$$

Since the random variables $\{|X_k(t_{i+1} - 0) - X_k(t_i)| I_{[X_k \in K]} : k = 1, 2, \dots\}$ are independent, identically distributed random elements, the first term can be made less than $\frac{\eta}{6}$. ///

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