

AD-A070 795

MASSACHUSETTS INST OF TECH LEXINGTON LINCOLN LAB

F/G 9/3

SIGNAL RECONSTRUCTION FROM PHASE OR MAGNITUDE.(U)

SEP 79 M HAYES , J S LIM , A V OPPENHEIM

F19628-78-C-0002

UNCLASSIFIED

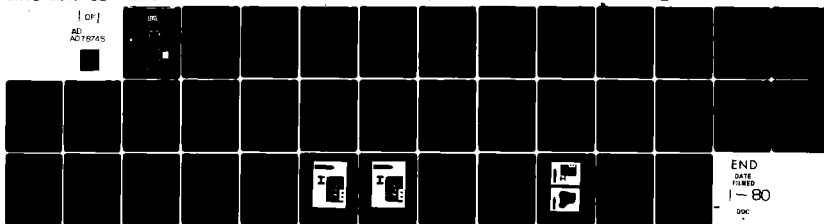
TN-1979-64

ESD-TR-79-226

NL

1 of 1

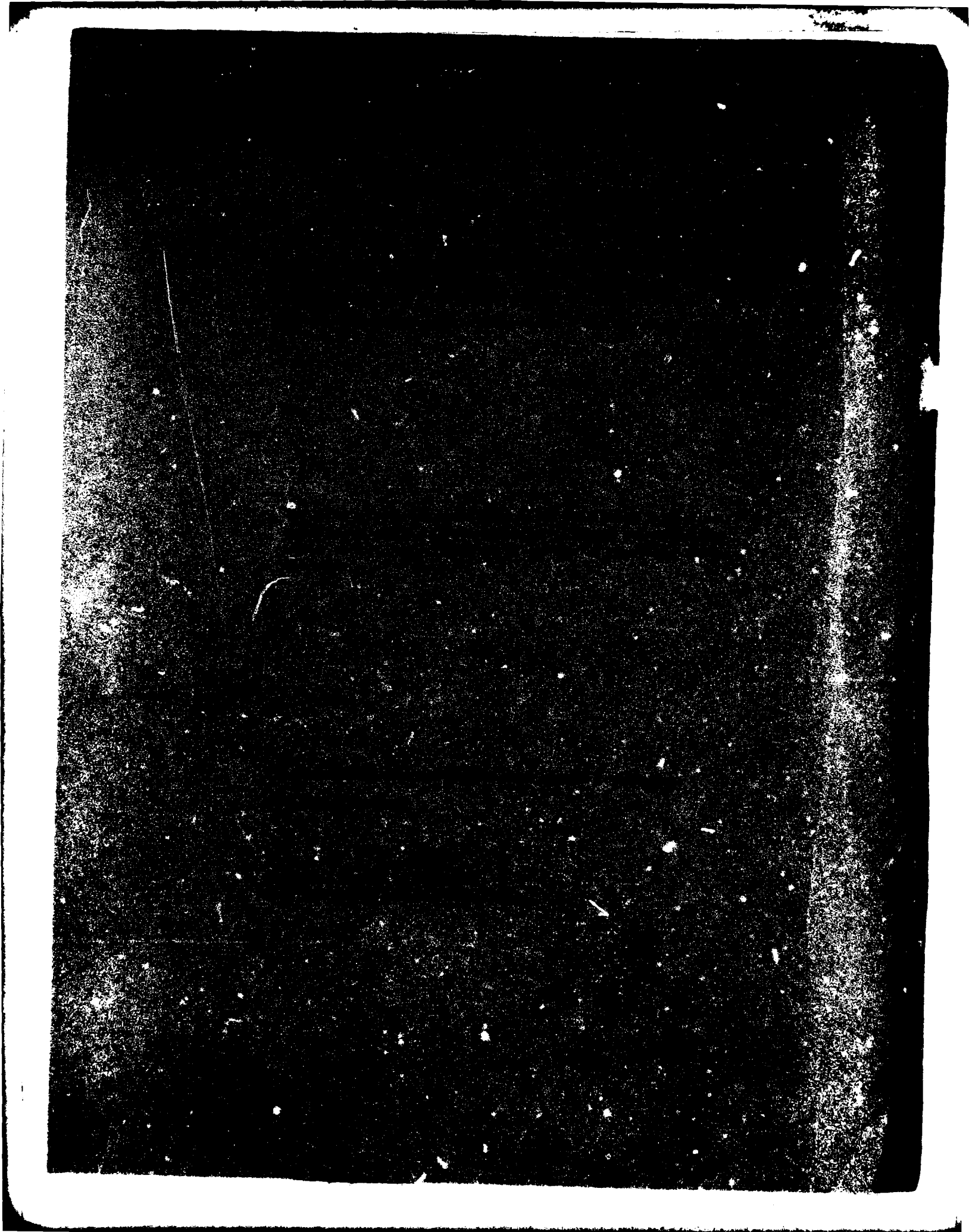
AD
A070795



END
DATE
FILMED
1-80

DOC

OPY ADAL 78715



MASSACHUSETTS INSTITUTE OF TECHNOLOGY
LINCOLN LABORATORY

12

SIGNAL RECONSTRUCTION FROM PHASE OR MAGNITUDE

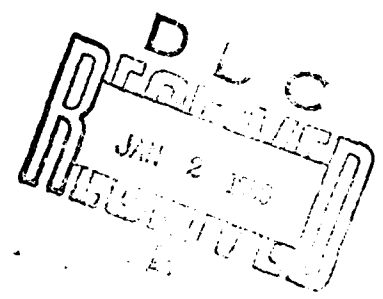
M. H. HAYES

J. S. LIM

Group 27

A. V. OPPENHEIM

Division 2



TECHNICAL NOTE 1979-64

18 SEPTEMBER 1979

Approved for public release; distribution unlimited.

LEXINGTON

MASSACHUSETTS

ABSTRACT

In this paper, we develop a set of conditions under which a sequence is uniquely specified by the phase or samples of the phase of its Fourier transform, and a similar set of conditions under which a sequence is uniquely specified by the magnitude of its Fourier transform. These conditions are distinctly different from the maximum or minimum phase conditions, and are applicable to both one-dimensional and multi-dimensional sequences. Under the specified conditions, we also develop several algorithms which may be used to reconstruct a sequence from its phase or magnitude. As a potential application area, the results of this paper are applied to the blind deconvolution problem of digital images blurred by a symmetric point spread function.

Accession For	
NTIS GRA&I	☒
DIC TAB	☐
Unannounced	☐
Justification	
By	
Date	
Approved for Release	
Dist	Avail and/or Special
A	

TABLE OF CONTENTS

	ABSTRACT	iii
I.	INTRODUCTION	1
II.	UNIQUENESS OF A SEQUENCE WITH A PHASE FUNCTION SPECIFIED AT ALL FREQUENCIES	2
III.	UNIQUENESS OF A SEQUENCE WITH A PHASE FUNCTION SPECIFIED AT DISCRETE FREQUENCIES	7
IV.	UNIQUENESS OF A SEQUENCE WITH A SPECIFIED MAGNITUDE FUNCTION	11
V.	NUMERICAL ALGORITHMS FOR RECONSTRUCTION FROM SAMPLES OF A PHASE FUNCTION	15
	A. Iterative Algorithm	16
	B. Closed Form Solution	19
VI.	EXTENSION TO MULTI-DIMENSIONAL SEQUENCES	23
	References	33

I. INTRODUCTION

For both continuous-time and discrete-time signals, the magnitude and phase of the Fourier transform are, in general, independent functions, i.e., the signal cannot be recovered from knowledge of either alone. Under certain conditions, however, relationships exist between these components. For example, when the signal is a minimum phase or maximum phase signal both the log magnitude and phase can be obtained from the other through the Hilbert transform. This relationship has been exploited in a variety of ways in many fields including network theory, communications and signal processing (1,2,3).

In this paper we develop a set of conditions under which a discrete-time sequence is completely specified to within a scale factor by the phase of its Fourier transform, without the restriction of minimum or maximum phase, and propose several algorithms for implementing the reconstruction of a signal from the phase of its Fourier transform. In Section II we consider the case in which the phase is specified at all frequencies, and in Section III the case where the phase is specified at a discrete set of frequencies. Algorithms for implementing the reconstruction are developed in Section V. In Section IV, we develop a different set of conditions, again without the restriction of minimum or maximum phase, in which a discrete-time sequence is completely specified by the magnitude of its Fourier transform. In Section VI, we extend the results of Sections II, III, IV and V to the case of multi-dimensional sequences and illustrate an application of the results of this paper to blind deconvolution.

II. UNIQUENESS OF A SEQUENCE WITH A PHASE FUNCTION SPECIFIED AT ALL FREQUENCIES

The sequences that we consider are real with rational z-transforms. Since we are interested in conditions under which the sequence can be uniquely specified by the phase of its Fourier transform, the Fourier transform is assumed to converge i.e., the region of convergence of the z-transform includes the unit circle.

For such sequences, we first show that a finite length sequence is uniquely specified by the phase of its Fourier transform if its z-transform has no zeroes in reciprocal pairs or on the unit circle.¹ More specifically, denoting the phase of $x[n]$ and $y[n]$ by $\theta_x(\omega)$ and $\theta_y(\omega)$ respectively, we demonstrate the following:

Theorem 1: Let $x[n]$ and $y[n]$ be two finite length sequences whose z-transforms have no zeroes in reciprocal pairs or on the unit circle. If $\theta_x(\omega) = \theta_y(\omega)$ for all ω , then $x[n] = Ay[n]$ for some positive constant A . If $\tan \theta_x(\omega) = \tan \theta_y(\omega)$ for all ω , then $x[n] = Ay[n]$ for some real constant A .

To demonstrate the validity of Theorem 1, we note first of all, that if a finite-length sequence $g[n]$ with z-transform $G(z)$ has a phase which is zero or π for all ω , then $g[n]$ is an even sequence, i.e., $g[n] = g[-n]$, and consequently, if $G(z)$ has a zero at $z=z_0$, then $G(z)$ must also have

¹ Since we are considering only sequences which are real, zeroes occur in complex conjugate pairs. In the following discussions, this symmetry is implicitly assumed, particularly in reference to reciprocal zeroes.

a zero at $z=1/z_0$. Now assume that $x[n]$ and $y[n]$ both satisfy the conditions of Theorem 1 and define $g[n]$ as

$$g[n]=x[n]*y[-n] \quad (1)$$

so that $G(z)=X(z)Y(1/z) \quad (2)$

If $\theta_x(\omega) = \theta_y(\omega)$ or if $\tan \theta_x(\omega) = \tan \theta_y(\omega)$, then the phase of $g[n]$ is zero or π . Therefore, $g[n]$ is an even sequence. Since the zeroes of $G(z)$ are collectively the zeroes of $X(z)$ and $Y(1/z)$, if $X(z_0)=0$ then either $X(1/z_0)=0$ or $Y(z_0)=0$. However, because the conditions of Theorem 1 exclude reciprocal zeroes or zeroes on the unit circle, $X(z_0)$ and $X(1/z_0)$ cannot both be zero. Thus, if $X(z_0)=0$ then $Y(z_0)=0$ and vice versa, i.e., the zeroes of $X(z)$ and $Y(z)$ are identical. Consequently, since $g[n]$ is an all zero sequence,²

$$X(z)=AY(z) \quad (3)a$$

or $x[n]=Ay[n] \quad (3)b$

Combining equations 1 and 3b, we have

$$g[n]=Ax[n]*x[-n] \quad (4)$$

Since the phase of $x[n]*x[-n]$ is always zero, if $\theta_x(\omega) = \theta_y(\omega)$ then the phase of $g[n]$ is zero so A must be a positive constant. If $\tan \theta_x(\omega) = \tan \theta_y(\omega)$ then the phase of $g[n]$ is zero or π so A must be real.

An interpretation of Theorem 1 is suggested by the observation that for a rational z -transform, in general, a zero at $z=z_0$ and a pole at $z=1/z_0^*$ contribute the same phase but different magnitude to the Fourier

²When we refer to a sequence as an all-zero (all-pole) sequence this should be interpreted to mean that the z -transform has only zeroes (poles) except possibly at $z=0$ or $z=\infty$.

transform. Thus with phase information alone, there is an inherent ambiguity in the z -transform in the sense that a zero (pole) at $z=z_0$ associated with the original sequence can potentially only be identified from the given phase as either a zero (pole) at $z=z_0$ or a pole (zero) at $z=1/z_0^*$ and this ambiguity cannot be further resolved without additional information or conditions. The finite length condition in Theorem 1 resolves this ambiguity by restricting the z -transform to have only zeroes except possibly at $z=0$ or at $z=\infty$. The additional condition that the z -transform have no zeroes in reciprocal pairs eliminates the possibility of zero phase components in the z -transform which, of course, could never be recovered from phase information alone. The conditions in Theorem 1 also eliminate the possibility of zeroes on the unit circle. While the Theorem can be modified to allow for the possibility of zeroes on the unit circle, the result becomes somewhat more complicated and we have chosen not to include this additional generality.

Although Theorem 1 requires that $x[n]$ be an all-zero sequence, a dual to Theorem 1 can be formulated for an all-pole sequence. Specifically, let $\tilde{x}[n]$ denote the convolutional inverse of a sequence $x[n]$, i.e.

$$x[n] * \tilde{x}[n] = \delta[n] \quad (5)$$

Then:

Theorem 2: Let $x[n]$ and $y[n]$ be two sequences whose z -transforms have no poles in reciprocal pairs, and which have finite duration convolutional inverses. If $\angle O_x(\omega) = \angle O_y(\omega)$ for all ω , then $x[n] = Ay[n]$ for some positive constant A . If $\tan \angle O_x(\omega) = \tan \angle O_y(\omega)$ for all ω , then $x[n] = Ay[n]$ for some real constant A .

Theorem 2 follows directly from Theorem 1. Since the phase of the Fourier transform of $\tilde{x}[n]$ is specified by the phase of the Fourier transform of $x[n]$, $\tilde{x}[n]$ is uniquely specified to within a positive scale factor by the phase of the Fourier transform of $x[n]$, by virtue of Theorem 1. Then, $x[n]$ is uniquely determined from the reciprocal of the z-transform of $\tilde{x}[n]$.

In Section IV we will consider a number of numerical algorithms which can be implemented on a digital computer for reconstructing a sequence from its phase under the condition of Theorem 1 or Theorem 2. At this point, however, we discuss a conceptual algorithm which may potentially have a practical implementation but which, more importantly, serves to lend insight into Theorem 1 and 2. We outline the algorithm under the conditions of Theorem 1 since it is easily modified for the conditions of Theorem 2.

Let $\theta_x(\omega)$ denote the specified phase function from which the sequence is to be reconstructed and $\hat{\theta}_x(\omega)$ the associated unwrapped phase (3). From the conditions of Theorem 1, $X(z)$ is restricted to be of the form

$$X(z) = Cz^{n_0} \prod_{k=1}^{N_1} (1 - a_k z^{-1}) \prod_{k=1}^{N_2} (1 - b_k z) \quad (6)$$

with C real, n_0 an integer, $|a_k| < 1$, $|b_k| < 1$ for all k and $a_k \neq b_\ell^*$ for any k and ℓ .

Step 1:

The algebraic sign of C is obtained from $\theta_x(\omega)$ using the fact that $\theta_x(0)$ is zero if and only if C is positive (3). The value of n_0 in eq. (6) is obtained from the unwrapped phase as

$$n_0 = \frac{1}{\pi} [\hat{\theta}_x(\pi) - \hat{\theta}_x(0)] \quad (7)$$

Step 2:

From the unwrapped phase function and the value of n_0 obtained in Step 1, a new phase function is specified as

$$\phi_x(\omega) \triangleq \hat{\theta}_x(\omega) - n_0 \omega - \hat{\theta}_x(0) \quad (8)$$

Using the Hilbert transform, a minimum phase sequence $x_{\min}[n]$ can be specified which has the phase $\phi_x(\omega)$. The z-transform $X_{\min}(z)$ of $x_{\min}[n]$ is given by (3,4)

$$X_{\min}(z) = \frac{\prod_{k=1}^N (1 - a_k z^{-1})}{\prod_{k=1}^N (1 - b_k^* z^{-1})} \quad (9)$$

where the coefficients a_k and b_k are identical to those in eq. (6).

Since pole-zero cancellations cannot occur in eq. (9) by virtue of the condition in Theorem 1 which implies that $a_k \neq b_\ell^*$ for any k or ℓ , the coefficients a_k in eq. (6) can be obtained from the zeroes of $X_{\min}(z)$ and the coefficients b_k^* , and thus b_k in eq. (6), can be obtained from the poles of $X_{\min}(z)$.

The condition in Theorem 1 that there are no zeroes in reciprocal pairs ensures that there are no pole-zero cancellations in eq. (9). If the original sequence has reciprocal zeroes, then the algorithm above may still be applied to recover all but those zeroes in $X(z)$ which are in reciprocal pairs.

III. UNIQUENESS OF A SEQUENCE WITH A PHASE FUNCTION SPECIFIED AT DISCRETE FREQUENCIES

In Theorems 1 and 2 we assumed that the phase function was specified at all frequencies. A similar set of Theorems can be stated if the phase is specified at a sufficient number of discrete frequencies. As in Section 11 we assume the sequences are real with rational z-transforms with a region of convergence that includes the unit circle.

Then:

Theorem 3: Let $x[n]$ and $y[n]$ be two finite length sequences which are zero outside the interval ³ $0 \leq n \leq N-1$ with z-transforms which have no zeroes in reciprocal pairs or on the unit circle. If $\theta_x(\omega) = \theta_y(\omega)$ at $(N-1)$ distinct frequencies greater than zero and less than π , then $x[n] = Ay[n]$ for some positive constant A. If $\tan \theta_x(\omega) = \tan \theta_y(\omega)$ at $(N-1)$ distinct frequencies greater than zero and less than π , then $x[n] = Ay[n]$ for some real constant A.

The validity of Theorem 3 follows in an almost identical manner to that of Theorem 1. Specifically, consider a finite length sequence $g[n]$ for which $g[n]=0$ outside the interval $(-N+1) \leq n \leq (N-1)$. Let $G(\omega) = |G(\omega)| e^{j\theta_g(\omega)}$ denote the Fourier transform of $g[n]$ with $\theta_g(\omega)$ zero or π at $N-1$ discrete frequencies $\omega_1, \omega_2, \dots, \omega_{N-1}$ between $\omega=0$ and $\omega=\pi$, i.e.,

³More generally, $x[n]$ need only be zero outside any finite interval of length N . This added generality, however, is not considered in order to simplify the following discussions.

$$\theta_g(\omega_k) = 0 \text{ or } \pi \text{ for } k=1, \dots, N-1 \quad (10)a$$

$$\text{with } \omega_k \neq \omega_\ell \text{ for } k \neq \ell \quad (10)b$$

$$\text{and } 0 < \omega_k < \pi \text{ for } k=1, \dots, N-1 \quad (10)c$$

Then, $G(\omega_k)$ is real and

$$\text{Im}\{G(\omega_k)\} = \sum_{n=-N+1}^{N-1} g[n] \sin n\omega_k = 0 \quad (11)$$

Or, equivalently

$$\sum_{n=1}^{N-1} \{g[n] - g[-n]\} \sin n\omega_k = 0 \quad k=1, 2, \dots, N-1 \quad (12)$$

$$0 < \omega_k < \pi$$

Equation (12) implies that (5)

$$[g[n] - g[-n]] = 0 \quad n=1, \dots, N-1 \quad (13)$$

i.e., $g[n]$ is an even sequence. Now, consider two sequences $x[n]$ and $y[n]$ satisfying the conditions of Theorem 3 and having the same phase at $(N-1)$ distinct frequencies between zero and π . As with Theorem 1, we form the sequence

$$g[n] = x[n] * y[-n] \quad (14)$$

Since $g[n] = 0$ outside the interval $(-N+1) \leq n \leq (N-1)$ and $\theta_g(\omega_k)$ satisfies eqs. (10), then $g[n]$ is an even sequence. For reasons identical to those used in justifying Theorem 1, it then follows that

$$x[n] = Ay[n] \quad (15)$$

where A is a positive constant if the phase samples of $x[n]$ and $y[n]$ are equal and a real constant if tangents of the phase samples are equal.

Although Theorem 3 requires that $x[n]$ be an all-zero sequence, a dual to Theorem 3 for an all-pole sequence is easily formulated in terms of the convolutional inverse. Specifically:

Theorem 4: Let $x[n]$ and $y[n]$ be two sequences whose z-transforms have no poles in reciprocal pairs, and which have convolutional inverses that are zero outside the interval $0 \leq n \leq N-1$. If $\angle x(\omega) = \angle y(\omega)$ at $(N-1)$ distinct frequencies greater than zero and less than π then $x[n] = Ay[n]$ for some positive constant A . If $\tan \angle x(\omega) = \tan \angle y(\omega)$ at $(N-1)$ distinct frequencies greater than zero and less than π , then $x[n] = Ay[n]$ for some real constant A .

Theorem 4 follows from Theorem 3 in the same manner that Theorem 2 follows from Theorem 1.

It should be noted from Theorem 1, that if the phase is specified at all frequencies, then the interval outside of which the sequence is zero need not be known whereas from Theorem 3, if the phase is specified at $N-1$ frequencies then this interval must also be specified. In both cases, except for a positive scale factor, the phase in the case of Theorem 1 or phase samples in the case of Theorem 2 uniquely specify a sequence within the class of sequences which have no zeroes in reciprocal pairs or on the unit circle. However, if the additional constraint that $x[0] \neq 0$ is included in the conditions of Theorem 3, then as we now show, the phase samples uniquely specify a sequence within a broader class of sequences. Specifically:

Theorem 5: Let $x[n]$ be a sequence that is zero outside the interval $0 \leq n \leq N-1$ with $x[0] \neq 0$ and which has a z-transform with no zeroes in reciprocal pairs or on the unit circle. Let $y[n]$ be any sequence

which is zero outside the interval $0 \leq n \leq N-1$. If $O_y(\omega) = O_x(\omega)$ at $(N-1)$ distinct frequencies greater than zero and less than π , then $y[n] = Ax[n]$ for some positive constant A . If $\tan O_y(\omega) = \tan O_x(\omega)$ at $(N-1)$ distinct frequencies greater than zero and less than π , then $x[n] = Ay[n]$ for some real constant A .

To demonstrate the validity of Theorem 5, we first form the sequence $g[n] = x[n] * y[-n]$. As discussed in Section III, since $g[n] = 0$ outside the interval $(-N+1) \leq n \leq (N-1)$ and satisfies equation (10), $g[n]$ is an even sequence. Now let $N_1 - 1$ represent the location of the last non-zero point in $x[n]$, i.e., $x[n] = 0$ for $n \geq N_1$ and $x[N_1 - 1] \neq 0$. Then

$$G(z) = X(z)Y(z^{-1}) = \sum_{n=0}^{N_1-1} x[n]z^{-n} \sum_{n=0}^{N-1} y[n]z^{-n} \quad (16)$$

Since $g[n]$ is even and $x[0] \neq 0$, $y[n] = 0$ for $n \geq N_1$ so that the number of zeroes of $y[n]$ is less than or equal to the number of zeroes of $x[n]$. Now, for reasons identical to those used in justifying Theorem 1, if $g[n]$ is an even sequence and if $x[n]$ has no zeroes in reciprocal pairs, then for each zero of $x[n]$, $y[n]$ must also have the same zero. Even though $y[n]$ is not restricted to the class of sequences with no zeroes in reciprocal pairs or on the unit circle, from our previous result, $y[n]$ cannot have more zeroes than $x[n]$, and therefore, $y[n] = Ax[n]$. For reasons identical to those used in justifying Theorem 1, A is a positive constant if the phase samples of $x[n]$ and $y[n]$ are equal whereas A is a real constant if the tangent of the phase samples are equal.

Although Theorem 5 requires that the sequence be an all-zero sequence, a dual to Theorem 3 for all all-pole sequence is easily formulated in terms of the convolutional inverse. Specifically:

Theorem 6: Let $x[n]$ be a sequence whose z -transform has no poles in reciprocal pairs or on the unit circle, and whose convolutional inverse is zero outside the interval $0 \leq n \leq N-1$ and non-zero at $n=0$. Let $y[n]$ be any sequence whose convolutional inverse is zero outside the interval $0 \leq n \leq N-1$. If $\theta_y(\omega) = \theta_x(\omega)$ at $(N-1)$ distinct frequencies greater than zero and less than π , then $y[n] = Ax[n]$ for some positive constant A . If $\tan \theta_y(\omega) = \tan \theta_x(\omega)$ at $(N-1)$ distinct frequencies greater than zero and less than π , then $x[n] = Ay[n]$ for some real constant A .

Theorem 6 follows from Theorem 5 in the same manner that Theorem 2 follows from Theorem 1.

IV. UNIQUENESS OF A SEQUENCE WITH A SPECIFIED MAGNITUDE FUNCTION

In Section II, several sets of conditions are presented which establish a uniqueness between a sequence and its phase function. Unlike the case for minimum or maximum phase sequences, there is no dual statement of uniqueness between a sequence and its magnitude function under the same set of conditions. However, under a different set of conditions a sequence is uniquely specified to within a sign and a time shift by the magnitude of the Fourier transform. The conditions are embodied in the following Theorems:

Theorem 7: Let $x[n]$ and $y[n]$ be two sequences whose z-transforms contain no reciprocal pole-zero pairs and which have all poles, not at $z=\infty$, inside the unit circle and all zeroes, not at $z=0$, outside the unit circle. If the magnitudes of the Fourier transforms of $x[n]$ and $y[n]$ are equal then $x[n] = \pm y[n+m]$ for some integer m .

A dual to this Theorem is:

Theorem 8: Let $x[n]$ and $y[n]$ be two sequences whose z-transforms contain no reciprocal pole-zero pairs and which have all poles, not at $z=0$, outside the unit circle and all zeroes, not at $z=\infty$, inside the unit circle. If the magnitudes of the Fourier transforms of $x[n]$ and $y[n]$ are equal then $x[n] = \pm y[n+m]$ for some integer m .

Since the justification of Theorem 7 is almost identical to that of Theorem 8, we will focus only on the first. The validity of Theorem 7 is suggested by noting that a zero (pole) at $z=z_0$ and a zero (pole) at $z=1/z_0^*$ contribute the same magnitude to the Fourier transform. Therefore, with magnitude information alone, there is an inherent ambiguity in the specification of the sequence in that a zero (pole) of the original sequence can potentially only be identified from the magnitude as either a zero (pole) at $z=z_0$ or at $z=1/z_0^*$. In Theorem 7 this ambiguity is resolved by restricting the poles to lie inside the unit circle and the zeroes to lie outside while in Theorem 8, the condition is the reverse. The additional condition that there are no conjugate reciprocal pole-zero

pairs eliminates the possibility of all-pass terms which contribute only to the phase and not the magnitude.

To more formally demonstrate Theorem 7, we define the set of sequences C_m to consist of all sequences $x[n]$ which satisfy the conditions of Theorem 7 so that their z -transform are restricted to be of the form

$$X(z) = Cz^{n_0} \frac{\prod_{k=1}^{N_1} (1 - a_k z)}{\prod_{k=1}^{P_1} (1 - b_k z^{-1})} \quad (17)$$

with $a_k \neq b_\ell^*$ for any k and ℓ , and where C is a real constant, n_0 is an integer, and $|a_k| < 1$, $|b_k| < 1$ for all k . Now consider two sequences $x[n]$ and $y[n]$, both in the set C_m , with z -transforms $X(z)$ and $Y(z)$ respectively. We wish to show that if $X(z)$ and $Y(z)$ both have the same magnitude on the unit circle then $x[n] = y[n+m]$ for some integer m .

Consider $G(z)$ defined as the ratio $X(z)/Y(z)$. Since $X(z)$ and $Y(z)$ both have the same magnitude on the unit circle, $G(z)$ must be entirely all-pass with unity magnitude, i.e., for a zero at $z=z_0$ there must be a pole at $z=1/z_0^*$ and vice versa. Therefore, $G(z)$ consists only of poles and/or zeroes at $z=0$ or at $z=\infty$ and conjugate reciprocal pole-zero pairs.

Because of the conditions in Theorem 7, this in turn requires that for any zero (or pole) of $X(z)$ at $z=z_0$ there must be a zero (or pole) of $Y(z)$ at $z=1/z_0^*$ which, for $z_0 \neq 0$ or ∞ , violates the conditions in Theorem 7 since one will always be inside and the other outside the unit circle. Thus, $G(z)$ must be of the form

$$G(z) = \pm z^m \quad (18)$$

or, equivalently,

$$x[n] = \pm y[n+m] \quad (19)$$

for some integer m .

A conceptual algorithm similar to that considered in Section II can be developed for reconstruction of a sequence to within an algebraic sign and a delay from the magnitude of its Fourier transform under the conditions of Theorem 7 or Theorem 8. We outline the procedure below under the conditions of Theorem 7. It is easily modified for the conditions of Theorem 8.

Let $|X(\omega)|$ denote the specified magnitude function. Using the Hilbert transform, a minimum phase sequence $x_{\min}[n]$ can be specified which has the same magnitude function. The z -transform $X_{\min}(z)$ of $x_{\min}[n]$ is given by

$$X_{\min}(z) = |C| \frac{\prod_{k=1}^{N_1} (1 - a_k^* z^{-1})}{\prod_{k=1}^P (1 - b_k z^{-1})} \quad (20)$$

i.e., it has the same poles as $X(z)$ and the zeroes are reflected inside the unit circle. Since the conditions of Theorem 7 exclude the possibility of pole-zero cancellation, the coefficients a_k^* , and thus a_k in eq. (17), can be obtained from the zeroes of $X_{\min}(z)$ and the coefficients b_k in eq. (17) can be obtained from the poles of $X_{\min}(z)$.

V. NUMERICAL ALGORITHMS FOR RECONSTRUCTION FROM SAMPLES OF A
PHASE FUNCTION

In Section II, we presented two sets of conditions, embodied in Theorems 1 and 2, under which a sequence is uniquely specified to within a positive scale factor by the phase of its Fourier transform. In this section, we describe two numerical algorithms which can be used to reconstruct a sequence satisfying the requirements of Theorem 1 from samples of its phase function when the location of the first non-zero point of $x[n]$ is known. Although these algorithms will only be discussed in terms of reconstructing sequences satisfying the conditions of Theorem 1, the reconstruction of sequences meeting the requirements of Theorem 2 may be accomplished by simply reconstructing the finite length sequence $\tilde{x}[n]$ defined in eq. (5) using the negative of the specified phase samples and then computing the convolutional inverse sequence.

The first algorithm presented below is an iterative technique in which the estimate of $x[n]$ is improved in each iteration. This algorithm has been developed by slightly modifying an iterative technique proposed by Quatieri (6) for reconstructing a minimum phase or maximum phase sequence from its phase function. The second algorithm is a closed form solution which is obtained by solving a set of linear equations. Under the conditions specified in Theorem 1, this algorithm provides the desired sequence $x[n]$ to within a scale factor when the location of the first non-zero point of $x[n]$ is known.

In the discussions which follow, $x[n]$ is used to denote a sequence which satisfies the conditions of Theorem 1 and is zero outside the interval $0 \leq n \leq N-1$ with $x[0] \neq 0$. In the more general case (see Footnote 3), a linear phase term may be added to the given phase to accomplish this.

A. Iterative Algorithm

The M-point discrete Fourier transform (DFT) of $x[n]$ will be denoted as

$$X(k) = |X(k)| e^{j\theta_x(k)} \quad (21)$$

where it is assumed that $M \geq 2N$. Then, an iterative technique to reconstruct the sequence $x[n]$ from the M samples of its phase, $\theta_x(k)$ $k=0,1,\dots,M-1$, is as follows:

Step 1:

We begin with $|X_0(k)|$, an initial guess of the unknown DFT magnitude and form the first estimate, $X_1(k)$, of $X(k)$ using the specified phase function, i.e.,

$$X_1(k) = |X_0(k)| e^{j\theta_x(k)} \quad (22)$$

Computing the inverse DFT of $X_1(k)$ provides the first estimate, $x_1[n]$, of $x[n]$. Since an M-point DFT is used, $x_1[n]$ is an M-point sequence which is, in general, non-zero for $N \leq n \leq M-1$.

Step 2:

From $x_1[n]$, another sequence, $y_1[n]$, is defined by

$$y_1[n] = \begin{cases} x_1[n] & 0 \leq n \leq N-1 \\ 0 & N \leq n \leq M-1 \end{cases} \quad (23)$$

Step 3:

The magnitude $|Y_1(k)|$ of the M-point DFT of $y_1[n]$ is then considered as a new estimate of $|X(k)|$ and a new estimate of $X(k)$ is formed by

$$x_2(k) = |Y_1(k)| e^{j\theta_x(k)} \quad (24)$$

From this, a new estimate $x_2[n]$ is obtained from the inverse DFT of $x_2(k)$. Repetitive application of steps two and three defines the iteration.

In this iterative procedure, the total squared error between $x[n]$ and its estimate is non-increasing with each iteration. To see this, let $x_r[n]$ denote the estimate after the r th iteration and define the error, E_r , as

$$E_r = \sum_{n=0}^{M-1} [x[n] - x_r[n]]^2 \quad (25)$$

From Parseval's theorem,

$$\begin{aligned} E_r &= \frac{1}{M} \sum_{k=0}^{M-1} |X(k) - |X_r(k)| e^{j\theta_x(k)}|^2 \\ &= \frac{1}{M} \sum_{k=0}^{M-1} |X(k) - |Y_{r-1}(k)| e^{j\theta_x(k)}|^2 \end{aligned} \quad (26)$$

Since $\theta_x(k)$ is the phase of $X(k)$, then

$$|X(k) - |Y_{r-1}(k)| e^{j\theta_x(k)}|^2 \leq |X(k) - Y_{r-1}(k)|^2 \quad (27)$$

with equality if and only if $\theta_x(k) = \theta_{r-1}(k)$ where $\theta_{r-1}(k)$ is the phase

of $Y_{r-1}(k)$. Combining eq. (26) and (27) gives

$$E_r \leq \frac{1}{M} \sum_{k=0}^{M-1} |X(k) - Y_{r-1}(k)|^2$$

or, with Parseval's theorem:

$$E_r \leq \sum_{n=0}^{M-1} [x[n] - y_{r-1}[n]]^2 \quad (28)$$

Since

$$y_{r-1}[n] = x_{r-1}[n] \text{ for } 0 \leq n \leq N-1$$

$$y_{r-1}[n] = x[n] = 0 \text{ for } N \leq n \leq M-1$$

then,

$$\sum_{n=0}^{M-1} [x[n] - y_{r-1}[n]]^2 \leq \sum_{n=0}^{M-1} [x[n] - x_{r-1}[n]]^2 = E_{r-1} \quad (29)$$

with equality if and only if $y_{r-1}[n] = x_{r-1}[n]$. Therefore,

$$E_r \leq E_{r-1} \quad (30)$$

Although eq. (30) is not sufficient to guarantee the existence of a converging solution, if $\hat{x}[n]$ is a converging solution then $\hat{x}[n] = Ax[n]$ with $A > 0$. To show this, we note that the equality in equation (30) holds if and only if $0_{r-1}(k) = 0_x(k)$ from equation (27) and if and only if $y_{r-1}[n] = x_{r-1}[n]$ from equation (29). Therefore, if the algorithm converges, then

the convergent solution $\hat{x}[n]$ is zero outside the interval $0 \leq n \leq N-1$ and has the same phase samples as $x[n]$ at $\omega = \frac{2\pi}{M} k$ for $k=0,1,\dots,M-1$. Since $M \geq 2N$, there exist at least $N-1$ distinct values of ω in the interval $0 < \omega < \pi$ at which $\Theta_{\hat{x}}(\omega)$ equals $\Theta_x(\omega)$. Then from Theorem 5, $\hat{x}[n] = Ax[n]$ for some positive constant A .

It is not yet known if this iterative procedure always leads to a converging solution. However, in all the examples that we have considered so far, we have empirically observed that the algorithm converges to the correct solution when $M \geq 2N$ and that the number of iterations required to achieve a small total squared error is, in general, quite large. We have also observed that increasing M may increase the rate of convergence of the algorithm, but such an increase obviously results in an increase in the number of computations required for each iteration.

Two examples of the iterative procedure applied to a mixed phase sequence, $x[n]$, of length 8 are shown in Table 1. In the first example an FFT of length 16 was used. In the second example, the FFT length was extended to 128 points. In both cases, the initial guess of the unknown magnitude was chosen to be a constant, and the scaling factor A was chosen so that the resulting sequences have the same value at the origin as $x[n]$. The results after 10, 100, 500, and 1000 iterations are presented along with the values of the total squared error.

B. Closed Form Solution⁴

A closed form solution⁴ for reconstructing a sequence $x[n]$ from samples of its phase, $\Theta_x(\omega)$, follows from the definition of $\Theta_x(\omega)$. With $\Theta_x(\omega)$ defined so that $-\pi \leq \Theta_x(\omega) < \pi$, we have

⁴A closed form solution similar to the one presented in this section can be obtained by expressing the real and imaginary parts of $X(\omega)$ in terms of the given phase and then relating the real and imaginary parts through the discrete Hilbert transform relations for causal sequences.

TABLE 1:
Iterative Reconstruction of a Sequence from its Phase

FFT LENGTH	NUMBER OF ITERATIONS	x[0]	x[1]	x[2]	x[3]	x[4]	x[5]	x[6]	x[7]	TOTAL SQUARED ERROR
16	10	4.000	1.892	-9.935	3.331	5.113	7.115	13.886	-4.961	11.961
	100	4.000	2.022	-10.335	3.615	4.845	6.679	14.123	-5.379	7.050
	500	4.000	2.040	-10.804	4.456	4.272	5.577	14.642	-5.854	$8.925 \cdot 10^{-1}$
	1000	4.000	2.012	-10.947	4.849	4.075	5.159	14.901	-5.961	$6.792 \cdot 10^{-2}$
128	10	4.000	1.647	-10.245	4.875	5.015	6.563	15.286	-4.640	6.117
	100	4.000	1.855	-10.671	4.902	4.459	5.723	15.104	-5.412	1.229
	500	4.000	1.996	-10.991	4.997	4.013	5.019	15.003	-5.984	$9.109 \cdot 10^{-2}$
	1000	4.000	2.000	-11.000	5.000	4.000	5.000	15.000	-6.000	$4.118 \cdot 10^{-5}$
ORIGINAL SEQUENCE		4.0	2.0	-11.0	5.0	4.0	5.0	15.0	-6.0	

$$\tan \Theta_x(\omega) = \frac{-\sum_{n=0}^{N-1} x[n] \sin n\omega}{\sum_{n=0}^{N-1} x[n] \cos n\omega} \quad (31)$$

For the case in which $\Theta_x(\omega) = \pm\pi/2$ so that $\tan \Theta_x(\omega) = \pm\infty$, eq. (31) is equivalent to:

$$\sum_{n=0}^{N-1} x[n] \cos n\omega = 0 \quad (32)$$

Sampling $\tan \Theta_x(\omega)$ at $N-1$ distinct frequencies $\omega_1, \omega_2, \dots, \omega_{N-1}$ with $0 < \omega_k < \pi$ for $k=1, 2, \dots, N-1$, eq. (31) and (32) can be written as:

$$\begin{aligned} \sum_{n=1}^{N-1} x[n] \left[\cos n\omega_k \tan \Theta_x(\omega_k) + \sin n\omega_k \right] \\ = -x[0] \tan \Theta_x(\omega_k) \quad \text{if } \Theta_x(\omega_k) \neq \pm\frac{\pi}{2} \end{aligned} \quad (33)a$$

$$\sum_{n=1}^{N-1} x[n] \cos n\omega_k = -x[0] \quad \text{if } \Theta_x(\omega_k) = \pm\frac{\pi}{2} \quad (33)b$$

for $k=1, 2, \dots, N-1$. Equation (33) represents $(N-1)$ linear equations in the $(N-1)$ unknowns of $x[n]$ and can be expressed in matrix form as;

$$S \underline{x} = x[0] \underline{b} \quad (34)$$

where $\underline{x}$ represents the vector of elements of $x[n]$ excluding the first element of the sequence, $x[0]$. Any solution $s[n]$, to eq. (34) has the property that $s[n]$ is zero outside the interval $0 \leq n \leq N-1$ and has the same

tangent of the phase as $x[n]$ for $N-1$ distinct frequencies between zero and π . Thus, from Theorem 5, we conclude that $\hat{x}[n] = Ax[n]$ for some real constant A . Therefore, there is only one independent solution to eq. (34). This implies that S^{-1} , the inverse of the matrix S , exists and that $\hat{x}[n]$ is given by

$$\underline{\hat{x}} = AS^{-1}\underline{b} \quad (35)$$

for some real constant A . To specify $\hat{x}[n]$ to within a positive scale factor, we first assume that $A > 0$ and determine the phase of $\underline{\hat{x}}$ in eq. (35). If the resulting phase does not differ from $\theta_x(\omega)$, then A is positive. Otherwise, A is negative.

In reconstructing $\hat{x}[n]$ from $\theta_x(\omega)$ using eq. (35), it should be noted that we have some control over the matrix S . Since the elements of the matrix S are functions of the samples of $\theta_x(\omega)$, S can be changed by choosing a different set of frequency samples. This control over S may be useful in avoiding those frequency samples for which $\tan \theta_x(\omega)$ is very large or in avoiding potential numerical instabilities in computing the inverse matrix S^{-1} .

Compared with the iterative algorithm, the closed form solution presented above has the advantage that the desired sequence is guaranteed to be the solution to eq. (34) and, in addition, no iterations are required in order to reach a solution. On the other hand, equation (34) requires computing the inverse of an $(N-1) \times (N-1)$ matrix which may lead to numerical problems and severe round-off errors, particularly as N becomes large.

The algorithm discussed above has been applied to a variety of different examples. Consistent with our theoretical results, in all

examples we have considered, the desired solutions have been obtained. Specifically, for the sequence shown in Table 1, the closed form solution was used to reconstruct the sequence from its phase. The phase samples used to define the matrix S were first chosen to be equally spaced between zero and π , and then were randomly selected within the interval zero to π . Within the limits of finite precision arithmetic, in both cases the sequence was reconstructed exactly when the scaling factor was chosen so that the solutions obtained had the correct value at the origin.

VI. EXTENSION TO MULTI-DIMENSIONAL SEQUENCES

In this section, we extend the results of Sections II, III, IV and V to the case of multi-dimensional sequences. This extension is achieved by mapping a multi-dimensional sequence into a 1-D (one dimensional) sequence and then applying the results for 1-D sequences. Since the extension of the 2-D case to sequences of higher dimension is straightforward, our discussions in the section will concentrate on the 2-D case. Again, we consider only sequences which are real, have rational z -transforms, and have Fourier transforms that converge.

Let $x[n_1, n_2]$ represent a 2-D sequence which has a rational z -transform with $X(\omega_1, \omega_2)$ given by

$$X(\omega_1, \omega_2) = \frac{A(\omega_1, \omega_2)}{B(\omega_1, \omega_2)} \quad (36)$$

where $A(\omega_1, \omega_2)$ is a 2-D polynomial of degree M_1 in $\exp[j\omega_1]$ and M_2 in $\exp[j\omega_2]$ and where $B(\omega_1, \omega_2)$ is a 2-D polynomial of degree N_1 in $\exp[j\omega_1]$ and N_2 in $\exp[j\omega_2]$. Suppose we form a 1-D sequence $\hat{x}_1[n]$ or $\hat{x}_2[n]$ by

$$\hat{x}_1[n] = \sum_{m=-\infty}^{\infty} x[m, n-Nm] \quad (37)a$$

$$\text{or } \hat{x}_2[n] = \sum_{m=-\infty}^{\infty} x[n-Mm, m] \quad (37)b$$

where $M \geq \max(N_1, M_1)$ and $N \geq \max(M_2, N_2)$. Then it can be shown (7,8) that the transformation in equation (37) is invertible and that

$$\hat{X}_1(\omega) = X(\omega_1, \omega_2) \Big|_{\omega_1 = \omega N, \omega_2 = \omega} \quad (38)a$$

$$\text{and } \hat{X}_2(\omega) = X(\omega_1, \omega_2) \Big|_{\omega_1 = \omega, \omega_2 = \omega M} \quad (38)b$$

From equation (38), it is clear that the phase of $\hat{x}_1[n]$ or $\hat{x}_2[n]$ is specified by the phase of $x[n_1, n_2]$ and the magnitude of $\hat{x}_1[n]$ or $\hat{x}_2[n]$ is specified by the magnitude of $x[n_1, n_2]$. Therefore, all the theorems and numerical algorithms developed in Sections II, III, IV and V for 1-D sequences may be extended to 2-D sequences by first transforming them into 1-D sequences using equation (37) and then applying the 1-D results to the resulting 1-D sequences. Thus, for example, Theorem 1 may be extended to 2-D sequences as follows: Let $x[n_1, n_2]$ and $y[n_1, n_2]$ be two 2-D sequences such that the two 1-D sequences $\hat{x}[n]$ and $\hat{y}[n]$ obtained from transforming $x[n_1, n_2]$ and $y[n_1, n_2]$ using equation (37) are finite in length and have no zeroes in reciprocal pairs or on the unit circle. If $\theta_x(\omega_1, \omega_2) = \theta_y(\omega_1, \omega_2)$ for all ω_1 and ω_2 , then $x[n_1, n_2] = Ay[n_1, n_2]$ for some positive constant A. If $\tan \theta_x(\omega_1, \omega_2) = \tan \theta_y(\omega_1, \omega_2)$ for all ω_1 and ω_2 , then $x[n_1, n_2] = Ay[n_1, n_2]$ for some real constant A.

If $x[n_1, n_2]$ is a 2-D sequence with finite support, then the transformation given by equation (37) can be reduced to a simpler form. Specifically, let $x[n_1, n_2]$ be zero outside the region $0 \leq n_1 \leq N_1 - 1$ and $0 \leq n_2 \leq N_2 - 1$. Then equation (37) can be rewritten as

$$\hat{x}_1[n_1N+n_2] = x[n_1, n_2] \quad \text{with } N \geq N_1 \quad (39)a$$

$$\text{or} \quad \hat{x}_2[n_1+n_2M] = x[n_1, n_2] \quad \text{with } M \geq N_2 \quad (39)b$$

Clearly, the transformation in equation (39) is invertible and it can be easily shown (8) that $\hat{X}_1(\omega)$ and $\hat{X}_2(\omega)$ are given by equation (38).

As an illustration of the results of this section, a 2-D sequence representing the intensity of an image, $x[n_1, n_2]$, was created which is zero outside the region $0 \leq n_1 \leq 11$ and $0 \leq n_2 \leq 11$. From the phase of $\hat{X}_1(\omega)$ defined by equation (38)a with $N=12$, the closed form solution was used to reconstruct $x[n_1, n_2]$. With the scale factor chosen so that the reconstructed image had the same value at the origin as $x[n_1, n_2]$, the result, shown in Figure 1, is indistinguishable from the original. For illustration, the image shown has been enlarged by means of a zero-order hold.

Finally, we present another example which illustrates a potential application of the results of this paper to blind deconvolution. In image processing, a model which frequently arises to describe image blur in the absence of noise is:

$$y[n_1, n_2] = x[n_1, n_2] * h[n_1, n_2] \quad (40)$$

where $x[n_1, n_2]$ is the original unblurred image and $h[n_1, n_2]$ is a (real) symmetric 2-D sequence, i.e., $h[n_1, n_2] = h[-n_1, -n_2]$. This model may be used, under the appropriate band-limited assumptions, to account for uniform motion blur, severely defocussed lenses with circular aperture stops, and long-term exposure of atmospheric turbulence [9]. In the context of blind deconvolution, the desired objective is to recover $x[n_1, n_2]$ in eq. (40) from the blurred image $y[n_1, n_2]$ with no detailed

knowledge of $h[n_1, n_2]$. However, if $h[n_1, n_2]$ is known to be symmetric, one possible approach to the deconvolution of eq. (40) is suggested by the results of this section and section V. Specifically, let $x[n_1, n_2]$ denote an original unblurred image and $h[n_1, n_2]$ a symmetric point spread function. With $y[n_1, n_2]$ representing the blurred image as expressed by equation (40), define the coordinate system for $y[n_1, n_2]$ in such a way that $y[n_1, n_2]$ is zero outside the region $0 \leq n_1 \leq N_1 - 1$ and $0 \leq n_2 \leq N_2 - 1$. Since $h[n_1, n_2]$ is symmetric, its phase is zero or π for all ω_1 and ω_2 . Therefore,

$$\tan \Theta_y(\omega_1, \omega_2) = \tan \Theta_x(\omega_1, \omega_2) \quad (41)$$

for all ω_1 and ω_2 . Suppose there exists a sequence $\hat{x}_1[n]$ or $\hat{x}_2[n]$ defined by equation (39) which satisfies the conditions of Theorem 3 and that the location of the first non-zero point of this sequence is known. To simplify the following discussion, we denote this sequence simply by $\hat{x}[n]$ and define the sequences $\hat{y}[n]$ and $\hat{h}[n]$ by performing the same transformation on $y[n_1, n_2]$ and $h[n_1, n_2]$ that is used to obtain $\hat{x}[n]$. From the results of (8),

$$\hat{y}[n] = \hat{x}[n] * \hat{h}[n] \quad (42)$$

and therefore, it follows from equations (38) and (41) that

$$\tan \Theta_{\hat{y}}(\omega) = \tan \Theta_{\hat{x}}(\omega) \quad (43)$$

for all ω . Since the location of the first non-zero point of $\hat{x}[n]$ is assumed to be known, by adding the appropriate linear phase to $\Theta_{\hat{x}}(\omega)$,

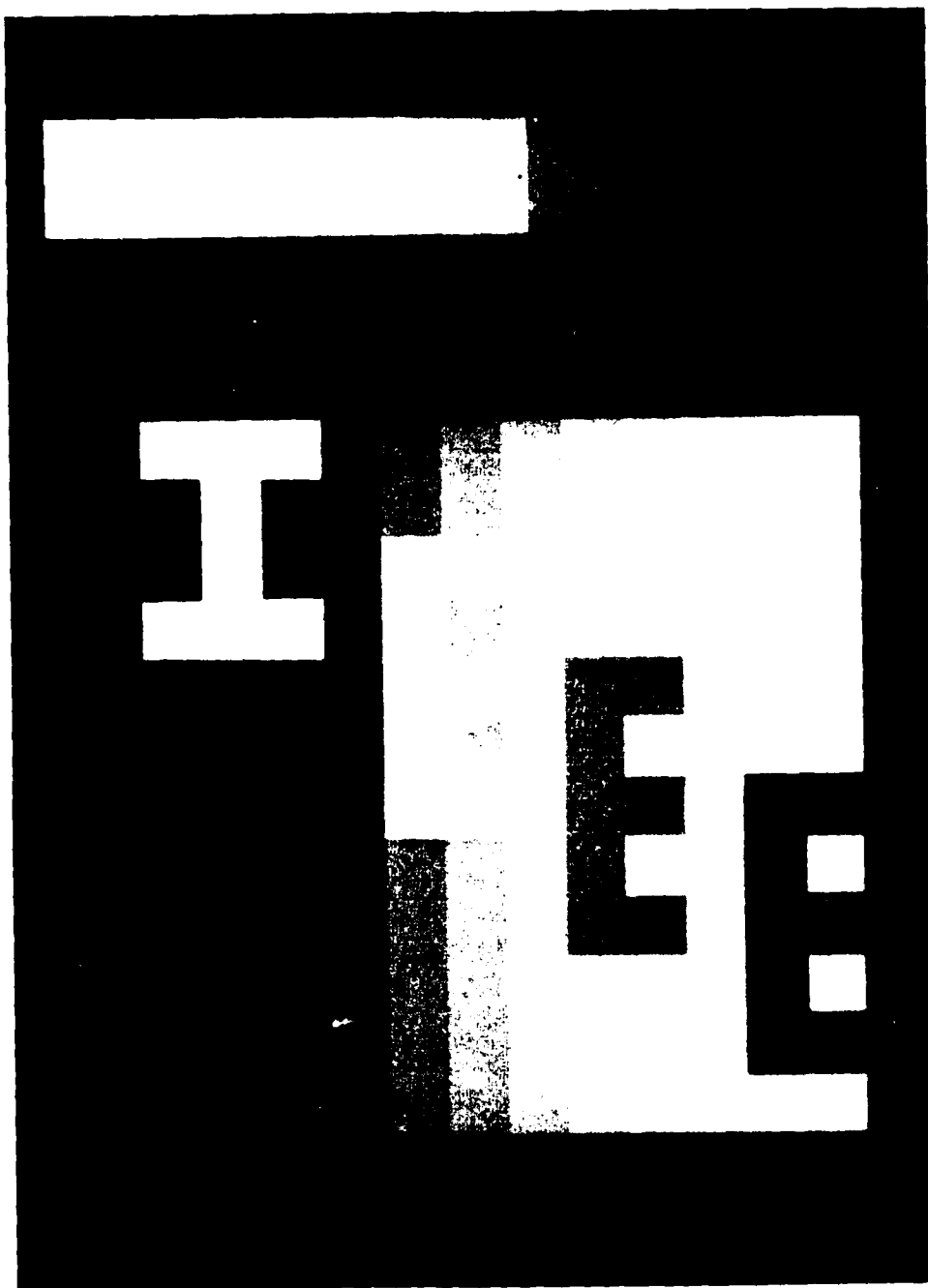


Fig.1. Image reconstruction from phase information.

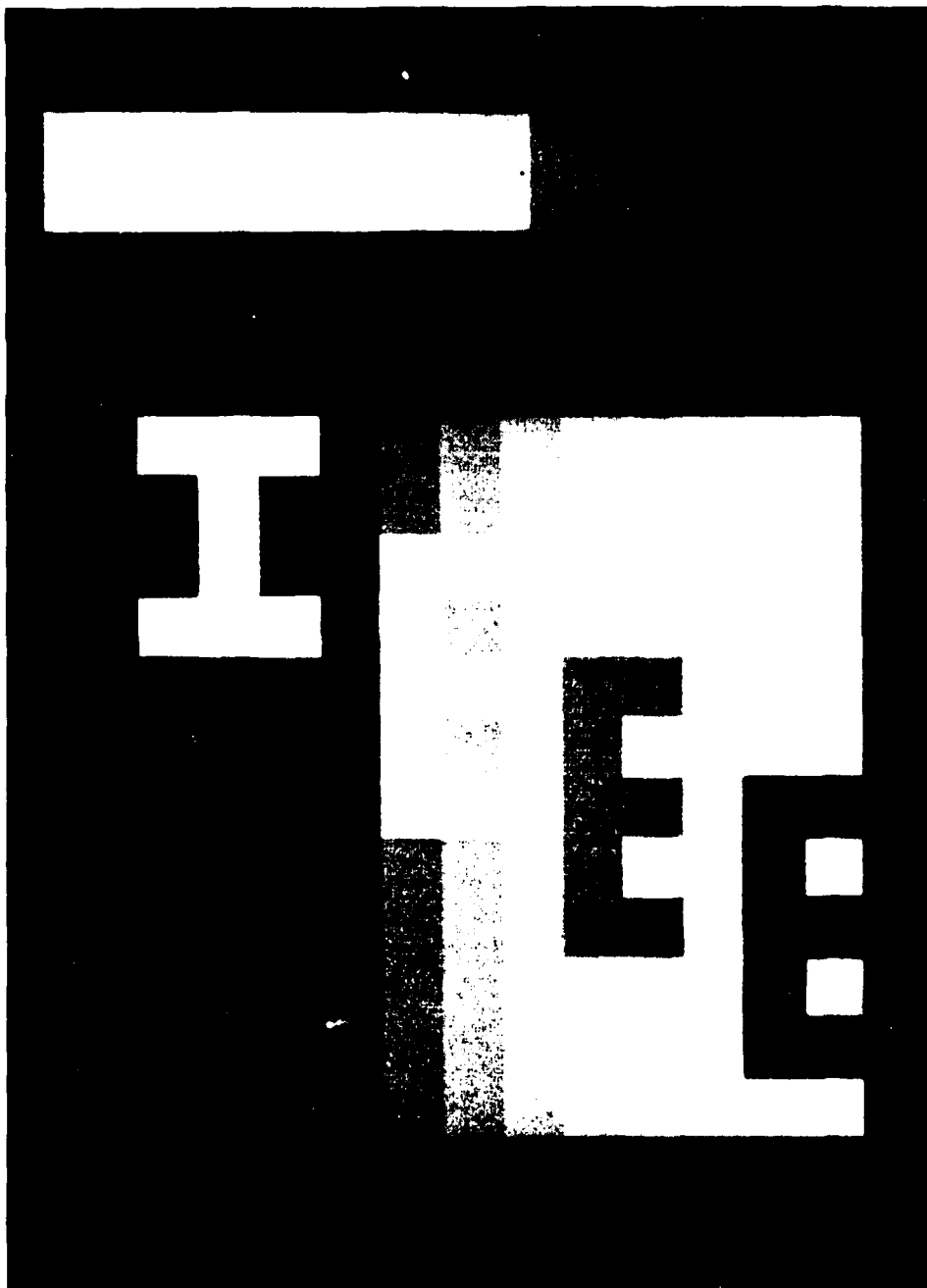


Fig.1. Image reconstruction from phase information.

the corresponding sequence, $\tilde{x}[n]$, satisfies the conditions of Theorem 5, i.e., $\tilde{x}[n]$ satisfies the conditions of Theorem 3 with $\tilde{x}[0] \neq 0$. Consequently, the closed form solution given by equation (35) specifies $\tilde{x}[n]$ to within a scale factor from samples of its phase or, equivalently, from samples of the phase of $y[n_1, n_2]$. Therefore, since equation (39) represents an invertible transformation, $x[n_1, n_2]$ can be recovered from $\tilde{x}[n]$ and knowledge of the location of the first non-zero point of $\hat{x}[n]$.

Since it is unreasonable to assume that the location of the first non-zero point of $\hat{x}[n]$ is always known, we outline one possible procedure for finding this point from $\hat{y}[n]$ which is given by equation (42). Specifically, with $\hat{y}[n]$ equal to zero outside the interval $0 \leq n \leq N-1$, suppose we assume that $\hat{x}[0] \neq 0$ and proceed to solve equation (34) for the vector $\hat{x}$ using $(N-1)$ samples of $\Theta_{\hat{y}}(\omega)$. Then, it is easy to show that the $(N-1) \times (N-1)$ matrix S in equation (34) is singular if $x[0] \neq 0$. More importantly, however, it can be shown that n_0 , the location of the first non-zero point of $\hat{x}[n]$, is related to the rank of S by:

$$n_0 = (N-1) - \text{rank}[S] \quad (44)$$

Therefore, suppose we are given the 2-D sequence $y[n_1, n_2]$ in equation (40) with $y[n_1, n_2]$ taken to be zero outside the region $0 \leq n_1 \leq N_1$ and $0 \leq n_2 \leq N_2-1$. Then, if $\hat{x}_1[n]$ or $\hat{x}_2[n]$ satisfy the conditions of Theorem 3, equation (44) may be used to determine the location of the first non-zero point of $\hat{x}_1[n]$ or $\hat{x}_2[n]$. With this information, equation (34) may be solved to recover $\hat{x}_1[n]$ or $\hat{x}_2[n]$ to within a scale factor from $\Theta_{\hat{y}}(\omega_1, \omega_2)$. Then, performing the inverse of the transformation given by equation (39), $x[n_1, n_2]$ can be reconstructed to within a scale factor.

An example which illustrates the results of this section to blind deconvolution is shown in Figure 2. The image in Figure 2a has a support of 16 pels x 16 pels which, for illustration, has been enlarged using a zero-order hold and represents the original image, $x[n_1, n_2]$, which has been blurred by a symmetrical 2-D sequence $h[n_1, n_2]$ as in eq. (40). After determining the support of $h[n_1, n_2]$ from the blurred image using a variation of the procedure outlined above for determining the location of the first non-zero point of $\hat{x}_1[n]$ or $\hat{x}_2[n]$, the image shown in Figure 2b was obtained from the closed form solution. Within the finite precision of digital computation, the reconstruction is exact.

Although the iterative technique may also be used in blind deconvolution, it has an important limitation as compared to the closed form solution. Specifically, the iterative algorithm requires that the phase of the blurred image be identical to the phase of the original image as opposed to the tangents of the phase being equal as required in the case of the closed form solution. This, in turn, restricts the blurring function $h[n_1, n_2]$, to be zero phase.

Although we have presented one possible approach to blind deconvolution when the degrading function is symmetric, it does not constitute a general solution to the problem of blind deconvolution since, in practice, eq. (40) is only an approximation. More generally, a blurred image is given by

$$y[n_1, n_2] = \left\{ x[n_1, n_2] * h[n_1, n_2] \right\} w[n_1, n_2] + v[n_1, n_2] \quad (46)$$

where $v[n_1, n_2]$ is additive noise and $w[n_1, n_2]$ is a window function. The sensitivity of our proposed deconvolution technique to the approximation in eq. (40) as well as possible extensions to the more general case in eq. (46) are currently under investigation.

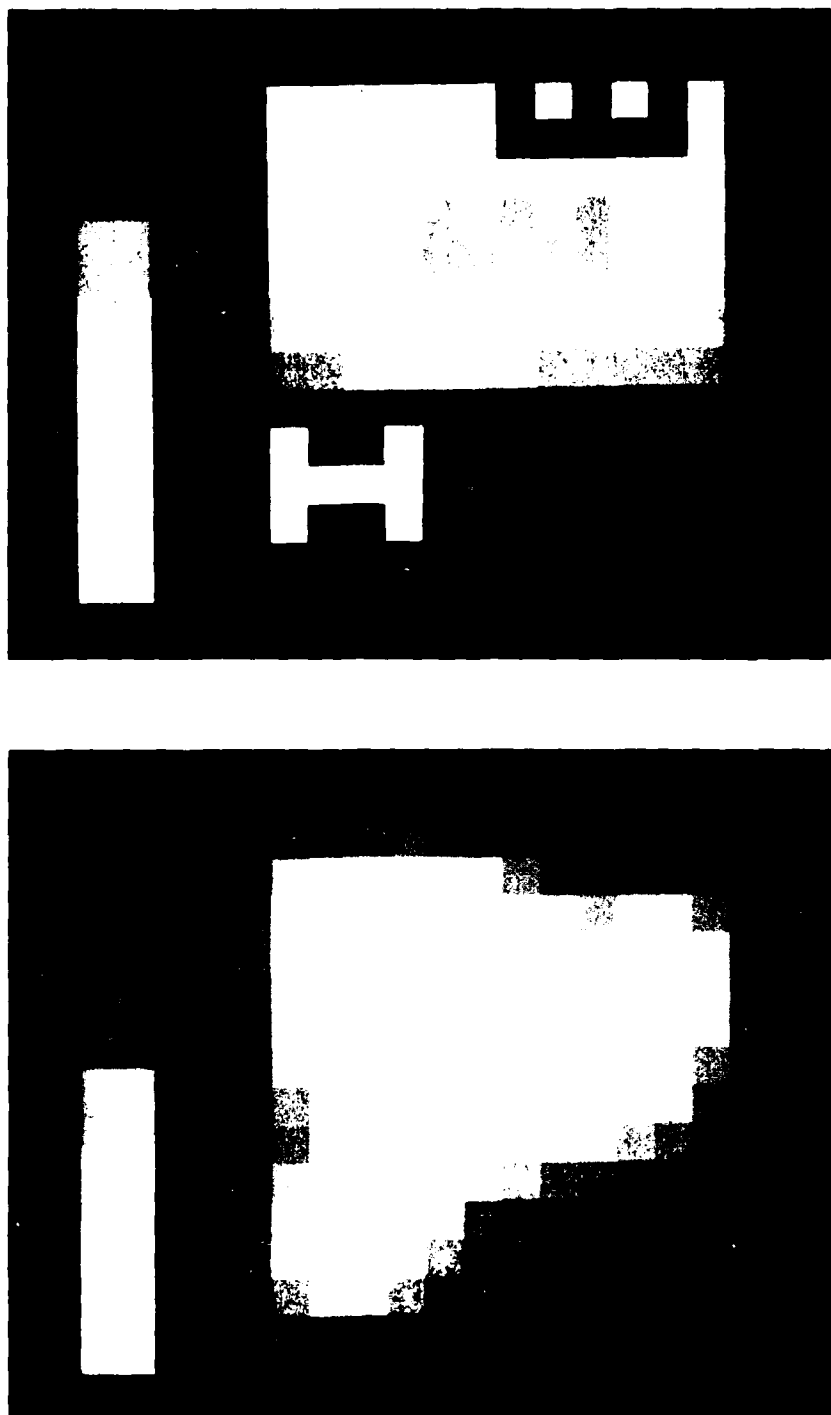


Fig.2(a-b). Blind deconvolution of images: (a) Image blurred by symmetric point spread function. (b) Result of blind deconvolution of image (a).

References

1. W. D. Gregg, Analog and Digital Communication, (Wiley, New York, 1977).
2. E. Guillemin, Theory of Linear Physical Systems, (Wiley, New York, 1963).
3. A. V. Oppenheim and R. W. Schaffer, Digital Signal Processing, (Prentice-Hall, Englewood Cliffs, N.J., 1975).
4. R. W. Schaffer, "Echo Removal by Discrete Generalized Linear Filtering," PhD Dissertation, M.I.T., Dept. of Elect. Eng., 1968.
5. S. Karlin and W. J. Studden, Tchebycheff Systems; With Applications in Analysis and Statistics, (Interscience/Wiley, New York, 1966).
6. T. Quatieri, "Mixed-Phase Estimation with Applications to Speech Analysis/Synthesis," PhD Dissertation, M.I.T. Dept. of Elect. Eng., 1979.
7. R. M. Mersereau, "The Digital Reconstruction of Multi-Dimensional Signals From Their Projections," ScD Dissertation, M.I.T., Dept. of Elect. Eng., 1973.
8. R. M. Mersereau and D. E. Dudgeon, "The Representation of Two-Dimensional Sequences as One-Dimensional Sequences," IEEE Trans. Acoust., Speech, and Signal Processing, ASSP-22, (1974).
9. H. C. Andrews and B. P. Hunt, Digital Image Restoration, (Prentice-Hall Englewood Cliffs, N.J., 1977).

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER E5D-1R-79-228	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Signal Reconstruction from Phase or Magnitude		5. TYPE OF REPORT & PERIOD COVERED Technical Note
		6. PERFORMING ORG. REPORT NUMBER Technical Note 1979-64
7. AUTHOR Monte Hayes Alan V. Oppenheim Jae S. Lim		8. CONTRACT OR GRANT NUMBER(s) F19628-78-C-0002
9. PERFORMING ORGANIZATION NAME AND ADDRESS Lincoln Laboratory, M.I.T. P.O. Box 73 Lexington, MA 02173		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Program Element No. 657051 Project No. 6491
11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Systems Command, USAF Andrews AFB Washington, DC 20331		12. REPORT DATE 11/18 September 1979
		13. NUMBER OF PAGES 40
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) Electronic Systems Division Hanscom AFB Bedford, MA 01731		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited. 1417N - 6491		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES None		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) signal restriction Fourier transform blind deconvolution signal reconstruction Fourier magnitude		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) In this paper, we develop a set of conditions under which a sequence is uniquely specified by the phase or samples of the phase of its Fourier transform, and a similar set of conditions under which a sequence is uniquely specified by the magnitude of its Fourier transform. These conditions are distinctly different from the maximum or minimum phase conditions, and are applicable to both one-dimensional and multi-dimensional sequences. Under the specified conditions, we also develop several algorithms which may be used to reconstruct a sequence from its phase or magnitude. As a potential application area, the results of this paper are applied to the blind deconvolution problem of digital images blurred by a symmetric point spread function.		

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)