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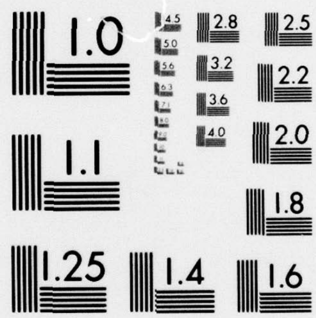

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APPLICATION OF DIGITAL TECHNOLOGY

TO

GUIDANCE AND CONTROL THEORY

by

S.M. Seltzer, Principal Investigator

Technical Report for Task I:

" A Mathematical Means of Spreading Clustered System Roots"

This research was performed for the
United States Army Missile Command
Redstone Arsenal, Alabama 35809
under
Contract No. DAAK40-79-C-0213

Control Dynamics Company
701 Corlett Drive, Suite 2
Huntsville, Alabama 35801
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ABSTRACT

The purpose of the study reported upon herein is to investigate the potential existence of a clustering of system poles and zeros in the z-domain. This potential is shown to exist. A means of separating the clustering is proposed by mapping from the complex z-domain onto the complex w-domain. The reason for so-doing is to ameliorate computational difficulties that can arise as the result of closely clustered roots.

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SECTION I. INTRODUCTION.

A. Study Objective.

The purpose of this study is to investigate the potential existence of a clustering of system poles and zeros in the z-domain. If the potential exists, a mathematical means of obviating the problem is to be determined.

The study effort is composed of a single task described in the Scope of Work contained in Technical Requirement No. T-0208 entitled "Application of Digital Technology to Guidance and Control Theory" dated 21 May 1979.[1] The requirements imposed by this task are contained in the referenced document and re-stated (for the convenience of the reader) below.

Task I requirements:

"2.1 Investigate to determine if a sensitivity problem due to clustering of the system poles and zeros exists for the parameter space technique of digital system analysis/design.

2.2 If a sensitivity problem (i.e., clustering of the system poles and zeros) is found in the investigation of Para. 2.1, a technique shall be developed, to mathematically spread apart the system poles and zeros via intermediate complex transform techniques to eliminate the sensitivity."

B. Study Schedule.

The requirements of Task I were completed during the period 1 August 1979 (contract initiation) through 30 September 1979. They are reported upon in this report.

C. Contents of Report.

This technical report covers the efforts performed in completing the requirements of Task I of the study. As stated in the referenced contract requirements: "The required documentation shall be a technical report."

SECTION II. PROBLEM INVESTIGATION.

As indicated in Section 2.1 of the referenced Technical Requirement, an investigation was made to determine if a sensitivity problem due to clustering of the system poles and zeros exists for the parameter space technique of digital system analysis/design. As will be shown in the sequel, any technique that utilizes design in the complex z-domain is subject to this problem. If this problem is shown to exist, then the parameter plane technique (among others) is subject to it.[2] Hence it is prudent to describe what the nature of the problem is (Section A, below) and see how likely it is to occur in practice (Section B, below).

A. Nature of Problem.

The nature of the problem can be seen in the complex z-domain. Consider that the dynamics of a system are represented by three roots, such as system poles: two that are complex conjugates and one that is real. Further suppose that they are all located arbitrarily close to the point $z = +1$. This situation is shown in Fig. 1, where the three roots are designated z_1 , z_2 , and $z_3 = \bar{z}_2$ (the overbar is used to indicate the complex conjugate of a quantity):

$$z_1 = d, \qquad (2\text{-}1)$$

and

$$z_2 = \rho\, e^{i\phi} = x + i\, y, \qquad (2\text{-}2)$$

where d, x, and y are real quantities and

$$\rho = (x^2 + y^2)^{\frac{1}{2}}, \qquad (2\text{-}3)$$

$$\phi = \tan^{-1} (y/x). \qquad (2\text{-}4)$$

The complex variable, z, is defined in the usual way, i.e.

$$z = r\, e^{i\theta}, \qquad (2\text{-}5)$$

where r is the modulus and the argument θ is defined as

$$\theta = \cos^{-1} (\omega_n T \sqrt{1-\zeta^2}), \qquad (2\text{-}6)$$

The symbol ω_n represents the natural frequency associated with a complex pair of roots, T represents the sampling period, and ζ represents the damping ratio associated with the pair of complex roots. When θ equals 0 or π, then z is a real (rather than complex) variable.

If it is assumed that z_1, z_2, and z_3 comprise the roots of a particular system, then the characteristic equation (C.E.) may be written as

$$\begin{aligned} \text{C.E.}\ (z) &= (z-z_1)(z-z_2)(z-z_3) \\ &= (z-d)(z-\rho e^{i\phi})(z-\rho e^{-i\phi}) \\ &= z^3 - (d + 2\rho\cos\phi)\ z^2 + (\rho^2 + 2d\rho\cos\phi)\ z \\ &\quad - d\rho^2 \\ &= z^3 - (d + 2x)\ z^2 + (\rho^2 + 2dx)\ z - d\rho^2 = 0 \qquad (2\text{-}7) \end{aligned}$$

One observes that as the locations of the three roots approach each other, i.e. as ρ approaches d, one obtains (in the limit) from Eq. (2-7) the expression

$$\lim_{\rho \to d} \text{C.E.}\ (z) = \lim_{\substack{x \to d \\ y \to 0}} \text{C.E.}\ (z) = (z-d)^3 = 0, \qquad (2\text{-}8)$$

as expected. Computer and scaling problems can occur if d approaches the absolute value of unity, as is explained below.

As can be seen with the aid of Fig. 1, when d approaches either +1 or -1, the three (in this example) clustered roots approach one of the two singularities (i.e. $z=+1$, $z=-1$) in the z-domain. If it is desired to solve the system characteristic equation for its roots (the so-called eigenvalue problem) in order to assess system stability, computational techniques often run into difficulty if the roots (eigenvalues) are close to each other. Sometimes this problem can be circumvented by increasing the scale of the system, i.e. — in the case of Fig. 1 — by increasing the scale of the figure. However, because the origin of the vectors to the roots is at $z=0$, when the roots are clustered closely and on the unit circle (such as at $z = \pm 1$), it is difficult to separate them sufficiently for the computer to solve the associated eigenvalue problem. It can be induced that if the problem at hand has even more (than three) clustered roots, the computational problem becomes more severe. Thus a means of avoiding this "pathological" (computer jargon) problem is to be desired if it appears that such problems might arise in actual practice.

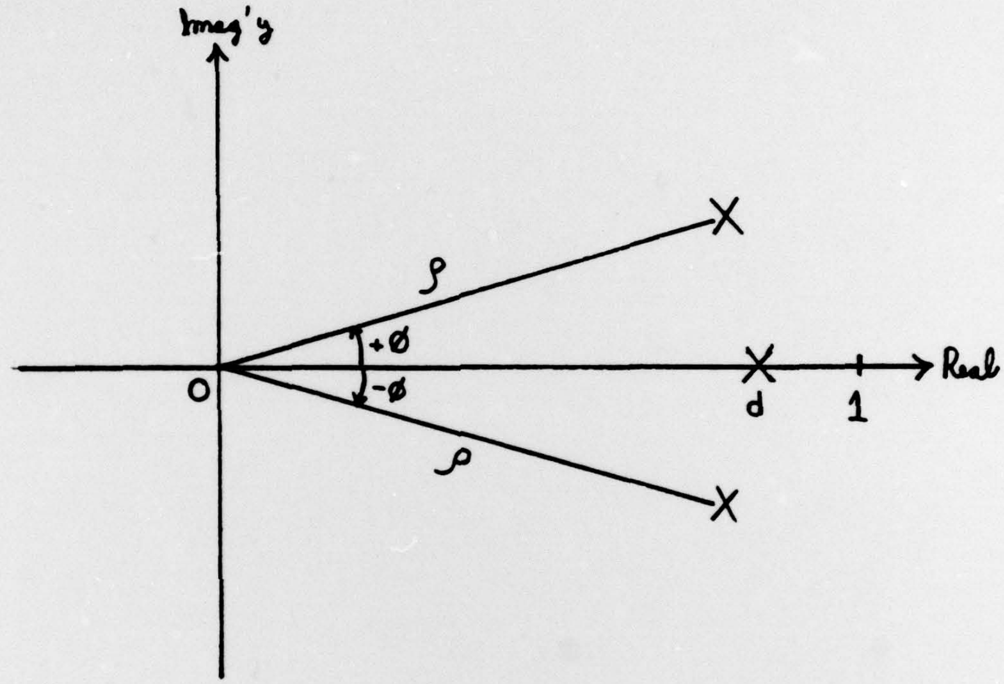

FIGURE 1. CLUSTERED ROOTS IN Z-DOMAIN

B. Likelihood of Occurance.

Before attempting to attack the problem of clustered roots, it is desirable to assess the possibility of their actually occuring in practice. Digital design experience shows that the probability of system roots occuring at the $z = -1$ singularity is unlikely, unless deliberately placed there by a digital filter. Since the system designer has the option of removing such roots by altering the characteristics of his digital filter, this problem can be solved readily. However, the problem is not so readily solved if the roots are clustered at the $z = +1$ singularity.

Suppose that a system plant has a second order characteristic, such as one would find typically with a bending mode or in the denominator of a transfer function (T.F.) describing the characteristics of some sensors or actuators. A typical transfer function might be

$$\mathrm{T.F.}(s) = \frac{N(s)}{(s+a)^2 + \omega^2}\,, \qquad (2\text{-}9)$$

where the numerator N(s) may have any form (in this discussion). An example for a bending mode would have N(s) represent the modal gain, "a" would be the modal damping ratio, and "ω" would be the square root of the difference of the squares of the modal natural frequency (ω_n) and "a", i.e.

$$\omega = (\omega_n^2 - a^2)^{\frac{1}{2}}\,. \qquad (2\text{-}10)$$

The z-transform of the transfer function of Eq. (2-9) is seen to be

$$\mathrm{T.F.}(z) = \frac{N(s)}{z^2 - 2ze^{-aT}\cos \omega T + e^{-2aT}}\,, \qquad (2\text{-}11)$$

where T represents the sampling period of the digital system. The nature of numerator N(z) depends on the form of N(s). The important thing to observe is that if a or T or both, (i.e. if their product) approach zero, then the denominator of Eq. (2-11) can approach the form,

$$D(z) = (z-1)^2\,. \qquad (2\text{-}12)$$

In this unfortunate case, the roots would be clustered near (or on, in the limit) the singularity at z = +1, giving rise to the problems brought forth in Section A above. Thus it unfortunately appears possible for the clustering problem to actually arise in a physical system. The probability of such an occurance depends on the actual dynamic nature of the plant's sensors, actuators, and dynamic bending properties. Judicious engineering design can of course obviate such problems brought forward because of actuators or sensors, if desired. However, changing the bending characteristics of the missile plant can be difficult. All is not lost, however, because it is the product of a and T in Eq. (2-11) that gives rise to the problem. Hence, again the designer can select the numerical value of T that will preclude such a clustering problem. If other design constraints conspire against the digital system designer, he might desire another analytical approach that might solve the clustering problem. Should that desire materialize, another approach is proposed and discussed in the following sections of this report.

SECTION III. PROPOSED SOLUTION.

The problem investigated in Section II is attacked in this section by applying a transformation from the complex z-domain to the complex w-domain. As desired in Section 2.2 of the referenced Technical Requirement, it is intended that this technique "mathematically spread the system poles and zero via intermediate complex transform techniques to eliminate the sensitivity."

As in Section II (for the sake of simplicity), one may again assume the dynamics of a system are represented by the three roots of Eqs. (2-1) - (2-2). The usual definition used in mapping from the z-domain to the w-domain is:

$$z = \frac{(1 + w)}{(1 - w)}. \qquad (3\text{-}1)$$

The complex variable w is composed of real and imaginary parts which may be denoted in the usual manner,

$$w = \sigma_w + i\,\omega_w. \qquad (3\text{-}2)$$

Applying Eq. (3-1) to root z_1 (2-1), one may obtain its equivalent location in the w-domain:

$$z = \frac{(1 + w)}{(1 - w)} = z_1 = d\ . \qquad (3\text{-}3)$$

If one solves Eq. (3-1) for w, one obtains the root location

$$w_1 = -\frac{(1 - d)}{(1 + d)}. \qquad (3\text{-}4)$$

It may be observed in passing that the real root location in the z-domain has been transformed into a real root location in the w-domain. In a similar manner the equivalent w-domain expression for the complex conjugate root of Eq. (2-2) may be obtained:

$$z = \frac{(1 + w)}{(1 - w)} = z_2 = \rho\, e^{i\phi} = x + iy\ . \qquad (3\text{-}5)$$

When rearranged and solved for w, one may obtain

$$w_2 = -\frac{(1 - \rho e^{i\phi})}{(1 + \rho e^{i\phi})}$$

$$= \frac{-(1 - x^2 - y^2) + i\,2y}{(1 + x)^2 + y^2} . \qquad (3\text{-}6)$$

The expression for w_3 is the complex conjugate of Eq. (3-6). The three roots may now be mapped onto the complex w-domain as shown in Fig. 2.

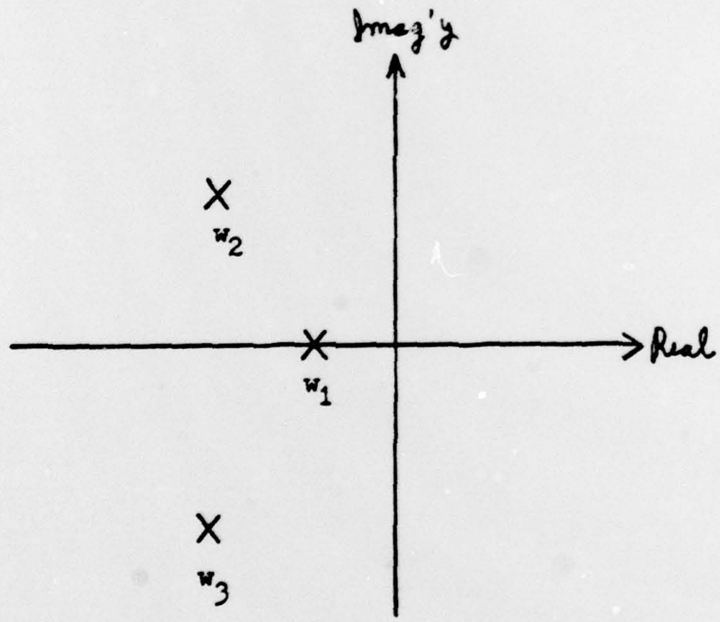

FIGURE 2. CLUSTERED ROOTS IN W-DOMAIN

If it is assumed that w_1 , w_2 , and w_3 comprise the roots of the same system analyzed in Section II, then the characteristic equation may be written as

$$C.E.(w) = (w-w_1)(w-w_2)(w-w_3) = 0. \qquad (3\text{-}7)$$

Alternatively, the characteristic equation may be determined by subsituting the value for z of Eq. (3-1) into C.E.(z) of Eq. (2-7) to obtain

$$C.E.(w) = (1+d+2x+\rho^2+2dx+d\rho^2)w^3 + (3+d+2x-\rho^2-2dx-3d\rho^2)w^2 + (3-d-2x-\rho^2-2dx+3d\rho^2)w + (1-d-2x+\rho^2+2dx-d\rho^2) = 0. \qquad (3\text{-}8)$$

One observes in this case that, as the locations of the three roots coalesce, i.e. as ρ approaches d, one obtains (in the limit) from Eq. (3-8) the expression

$$\lim_{\rho\to d} C.E.(w) = \lim_{\substack{\rho\to d \\ x\to d \\ y\to 0}} C.E.(w) = (1+d)^3w^3 + 3(1+d)^2(1-d)w^2 + 3(1+d)(1-d)^2w + (1-d)^3 = 0. \qquad (3\text{-}9)$$

If d approaches unity (as before), then

$$\lim_{d\to 1} C.E.(w) = (2w)^3 = 0, \qquad (3\text{-}10)$$

and the three roots approach (as a cluster) the origin of the w-plane. Now, however, if the scale of the system, i.e. -- in the case of Fig. 2--is increased, success is met in spreading apart the roots because the vector to each originates at the <u>nearby</u> origin. Recall, in the z-domain, increasing the scale did not substantially spread the roots apart because the vectors to the roots went (approximately) from the origin to the z=+1 point.

SECTION IV. COMPARISON.

The scaling problem and proposed solution alluded to above in the w-domain may not be immediately apparent. Therefore several numerical comparisons are made in this section. Ten runs were made for selected values of roots in the z-domain. The corresponding roots in the w-domain were calculated for comparison. The results were obtained using the program of the Appendix on the Hewlett-Packard 9100B Calculator. They are summarized in the following Table.

TABLE. ROOT LOCATIONS IN Z- & W- DOMAINS

Run	z_1=d	z_2=x+iy	W_1	W_2
1	0.9	0.8+i0.2	-0.05	-0.10+i0.12
2	0.96	0.95+i0.02	-0.02	-0.01+i0.03
3	0.98	0.97+i0.02	-0.010	-0.015+i0.010
4	0.99	0.98+i0.01	-0.005	-0.010+i0.005
5	0.98	0.99+i0.01	-0.010	-0.005+i0.005
6	0.996	0.95+0.001	-0.002	$-0.00251+i5X10^{-4}$
7	0.995	0.996+i0.001	-0.00251	$-0.00200+i5X10^{-4}$
8	0.999	0.999+i0.001	$-5X10^{-4}$	$-5X10^{-4}+i5X10^{-4}$
9	0.9999	0.9999+i0.0001	$-5X10^{-5}$	$-5X10^{-5}+i5X10^{-5}$
10	0.99999	0.99999+0.00001	$-5X10^{-6}$	$-5X10^{-6}+i5X10^{-6}$

It is seen that the distance between the z roots and the singular point at z=+1 is about twice the distance between the w roots and the origin of the w-plane, even though (in each case) the distances are quite small. However, the effect of increasing the scaling in the z-domain is not nearly so great as the effect of increasing the scaling in the w-domain. This is seen by example on Fig. 3 where the z- and w- domain roots are plotted on an increases scale.

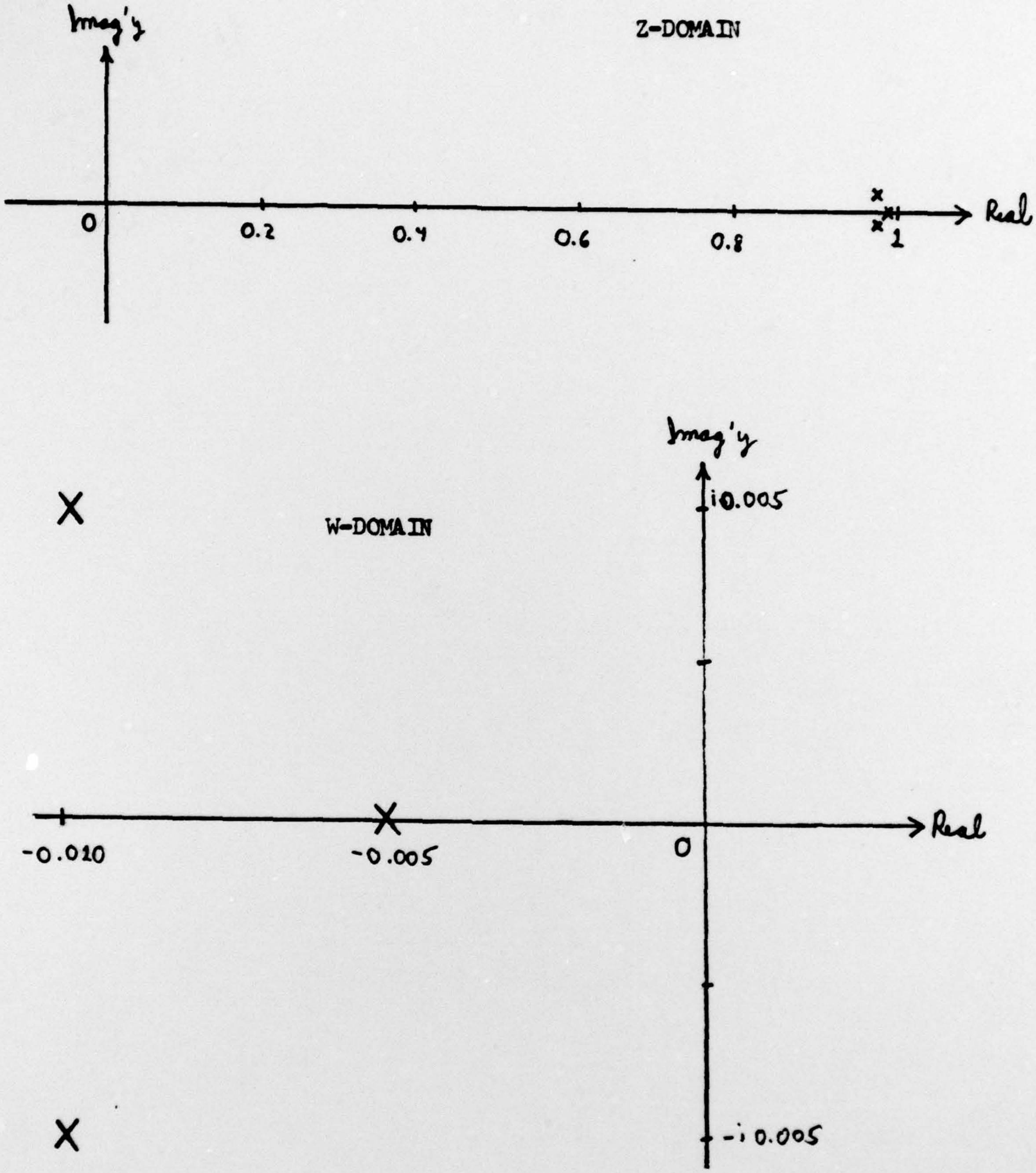

FIGURE 3. CLUSTERED ROOTS OF RUN #4 (SHOWS EFFECT OF SCALING)

It is readily observed that increases in scaling in the z-domain do not have a significant effect on roots clustered about the z=+1 singularity. This is because the vectors to these roots eminate from origin. The same is not true in the w-domain. Here it is seen that large increases in scale have profound effect on spreading out the roots clustered about the origin. This is because (in this case) the vectors to these roots eminate from the origin to the (in this case) nearby root locations.

SECTION V. CONCLUSION

An investigation has been made to investigate the potential existence of a clustering of system poles and zeros in the z-domain. It was shown by example in Section IIA how such a problem might arise. In Section IIB the likelihood of such an occurance was assessed. It was shown that such a situation could occur, but that there are design techniques to prevent it. If these fail, an alternate approach is described in Section II whereby the system poles and zeros are transformed from the complex z-domain to the complex w-domain. It then is possible to increase the scaling (in the w-domain) with significanteffect in separating the clustered poles and zeros, thereby obviating computational difficulties that might have arisen because of the clustering in the z-domain.

REFERENCES

[1] U.S. Army Technical Requirement No. T-0208, "Application of Digital Technology to Guidance and Control Theory," 21 May 1979.

[2] Report CDC-79-2, "Application of Descrete Guidance and Control Theory," Control Dynamics Co., Huntsville, AL, August 1979.

APPENDIX.

Title: MAP FROM Z-DOMAIN TO W-DOMAIN

Date: SEP '79 page 1/2

Name: SELTZER

EQS. (3-4), (3-6)

$z_1 = d$

$z_{2,3} = x \pm iy$

	Step	Key	Code	Display x	Display y	Display z	Storage f	Storage e	Storage d	Storage c	Storage b	Storage a
0	0	a	13	x			INITIAL CONDITIONS:			d	y	x
	1	↑	27	x	x							
	2	×	36	x	x^2							
	3	b	14	y	x^2							
	4	↑	27		y	x^2						
	5	×	36		y^2							
	6	y→	40									
	7	d	17						y^2			
	8	↓	25	y^2	x^2							
	9	+	33	y^2	x^2+y^2							
	a	1	01	1	x^2+y^2							
	b	−	34		$-(1-x^2-y^2)$							
	c	y→	40									
	d	e	12					$-(1-x^2-y^2)$				
1	0	1	01	1								
	1	↑	27	1	1	$-(1-x^2-y^2)$						
	2	a	13	x	1							
	3	+	31	x	$1+x$							
	4	↓	25	$1+x$	$-(1-x^2-y^2)$							
	5	↑	27		$1+x$							
	6	×	36		$(1+x)^2$							
	7	d	17	y^2	$(1+x)^2$							
	8	+	31		$(1+x)^2+y^2$							
	9	y→	40									
	a	f	15				$(1+x)^2+y^2$					
	b	e	12	$-(1-x^2-y^2)$								
	c	CONT	47									
	d	↑	27		$-(1-x^2-y^2)$	$(1+x)^2+y^2$						

Title MAP FROM Z-DOMAIN TO W-DOMAIN

Date SEP '79 page 2/2

Name SELTZER

Step		Key	Code	Display x	Display y	Display z	Storage f	Storage e	Storage d	Storage c	Storage b	Storage a
2	0	f	15	$(1+x)^2+y^2$	$-(1-x^2-y^2)$	$(1+x)^2+y^2$	$(1+x)^2+y^2$	$-(1-x^2-y^2)$	y^2	d	y	x
	1	÷	35		$Re\{w_2\}$							
	2	y→	40									
	3	e	12					$Re\{w_2\}$				
	4	2	02	2								
	5	↑	27	2	2	$Re\{w_2\}$						
	6	b	14	y	2							
	7	×	36	y	2y							
	8	f	15	$(1+x)^2+y^2$	2y							
	9	÷	35		$Im\{w_2\}$							
	a	y→	40									
	b	f	15				$Im\{w_2\}$					
	c		01	1								
	d	↑	27	1	1	$Im\{w_2\}$						
3	0	c	16	d	1							
	1	−	34		1−d							
	2	↑	27		d	1−d						
	3	1	01	1	d							
	4	+	35	1	1+d							
	5	↓	25	1+d	1−d							
	6	CH SGN	32	−(1+d)	1−d							
	7	÷	35	−(1+d)	w_1							
	8	f	15	$Im\{w_2\}$	w_1							
	9	↑	27	$Im\{w_2\}$	$Im\{w_2\}$	w_1						
	a	e	12	$Re\{w_2\}$	$Im\{w_2\}$	w_1						
	b	PR	45	$Re\{w_2\}$	$Im\{w_2\}$	w_1						
	c	END	46									
	d											

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