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ASYMPTOTIC MEASURES OF SYSTEM PERFORMANCE

UNDER ALTERNATIVE OPERATING RULES, II

by

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(SEE ABSTRACT)

ABSTRACT

This is the second part of a study of stochastic processes generated by failures and repairs of components in a series system. In particular, we consider a two component series system where failure of component 1 shuts off component 2 but not vice versa. In Part I, selected measures of system performance were calculated for general failure and repair distributions as implicit functions of system availability. In order to obtain explicit results, we assume in this part that either component 1 or component 2 has exponential failure and repair distributions. For the case where component 1 has a general failure distribution, we obtain counter-intuitive results concerning the long run average of system uptimes. Intuitively, it is best to shut off an operating component when the system is down. However, this is not correct if the component failure distribution is IFR (or more generally NBUE) and we wish to maximize the long run average of system uptimes. We use the method of supplementary variables to obtain explicit results for important special cases.

ASYMPTOTIC MEASURES OF SYSTEM PERFORMANCE
UNDER ALTERNATIVE OPERATING RULES, II

by

Richard E. Barlow and Esther Sid Hudes

1. INTRODUCTION AND SUMMARY

In this paper we study stochastic processes generated by failures and repairs of components in a series system. A series system of k components operates if and only if each of the k components operates. However, depending on the shut-off rule, some components may continue to operate with the system down. For example, failure of the power supply may shut down a computer but not vice versa. Only failed components are repaired or replaced, and repair or replacement takes a random time. Repaired components are assumed to function like new components. Furthermore, components are separately maintained. Failure and repair times are statistically independent.

We are interested in the asymptotic (as time becomes infinite) values of selected measures of system performance. Some of these quantities are:

(i) The limiting system availability; that is, the limiting probability that the system is functioning;

(ii) The limiting system failure rate;

(iii) The limiting average of system uptimes and downtimes;

(iv) The limiting average number of system failures due to a specified component.

These measures have various uses. For example, (iv) might be used to evaluate the relative importance of different components in the system. In Part 1 [1], we computed these measures implicitly, for systems with two components, as functions of limiting system availability. See [1] for notation and related work. In this part we assume that either component 1 or component 2 has exponential failure and repair distributions and give explicit formulas for the above measures.

The basic results obtained are that:

- Asymptotic measures of system performance defined above depend on the Laplace transforms of probability distributions at specified values of the argument as well as means;
- Sharp bounds on asymptotic measures of system performance can be given in terms of means assuming certain distributions belong to the class of "new better (worse) than used in expectation" and "new better (worse) than used." [Marshall and Proschan (1972).]
- If component 1 is NBUE and we want to maximize the long run average of system uptimes, then we *should not* shut off 1 when 2 fails. If component 1 is NWUE, then we *should* shut it off when 2 fails.

Classes of life distributions described in Barlow and Proschan (1975) are the key to obtaining bounds on asymptotic measures of system performance. A life distribution F with mean μ is said to be

(i) Increasing (Decreasing) failure rate or IFR (DFR) if $\log \bar{F}(x)$ is concave (convex) for $x > 0$;

(ii) Increasing (Decreasing) failure rate average or IFRA (DFRA) if $-[\log \bar{F}(x)]/x$ is increasing (decreasing) in $x > 0$;

(iii) New better (worse) than used or NBU (NWU) if

$$\bar{F}(x + y) \leq (\geq) \bar{F}(x)\bar{F}(y)$$

for all $x, y \geq 0$;

(iv) New better (worse) than used in expectation or NBUE (NWUE) if

$$\mu \geq (\leq) \int_0^\infty \bar{F}(t + x)dx/\bar{F}(t)$$

for all $t \geq 0$ such that $\bar{F}(t) > 0$.

(v) Discounted life is less (greater) than the exponential or DLL (DLG) if

$$\int_0^\infty e^{-st}dF(t) \leq (\geq) \frac{1}{1 + s\mu}$$

for all $s > 0$.

The chains of implications

$$\text{IFR} \Rightarrow \text{IFRA} \Rightarrow \text{NBU} \Rightarrow \text{NBUE} \Rightarrow \text{DLL}$$
$$\text{DFR} \Rightarrow \text{DFRA} \Rightarrow \text{NWU} \Rightarrow \text{NWUE} \Rightarrow \text{DLG}$$

are easy to establish. For completeness we prove the last implication in each chain.

Lemma 1.1:

If F is NBUE (NWUE) with mean μ, then

$$\int_0^\infty e^{-st}dF(t) \leq (\geq) \frac{1}{1+s\mu}$$

for all $s > 0$.

Proof:

For F NBUE,

$$\mu\bar{F}(t) \geq \int_0^\infty \bar{F}(t+x)dx = \int_t^\infty \bar{F}(x)dx .$$

Hence,

$$\mu\int_0^\infty e^{-st}\bar{F}(t)dt \geq \int_0^\infty e^{-st}\int_t^\infty \bar{F}(x)dxdt .$$

Interchanging order of integration on the right and reducing yields $\int_0^\infty e^{-st}\bar{F}(t)dt \geq \frac{\mu}{1+\mu s}$. But $\int_0^\infty e^{-st}\bar{F}(t)dt = \left[1 - \int_0^\infty e^{-st}dF(t)\right]/s$ so that $\int_0^\infty e^{-st}dF(t) \leq \frac{1}{1+\mu s}$.

All inequalities are reversed if F is NWUE.■

We assume that component i has failure distribution F_i with mean $\mu_i = 1/\lambda_i$ and repair distribution G_i with mean $\nu_i = 1/\theta_i$, $i = 1,2$. Let $U_1, U_2, \ldots, U_n$ be successive system uptimes under our policy that 1 shuts off 2 but not vice versa; i.e.,

$$\underset{1}{o} \rightarrow \underset{2}{o} .$$

Let

$$\lim_{n\to\infty} \frac{U_1 + \cdots + U_n}{n} = \mu_{\rightarrow}$$

when this limit exists almost surely. In Part 1 [1], we showed that when F_1 is exponential

$$\mu_{\rightarrow} = \frac{1}{\lambda_1 + \lambda_2} .$$

This is also true for general distributions and the shut-off policy

$$\underset{1}{o} \quad \underset{2}{o}$$

i.e., neither component shuts off the other, as well as the shut-off policy

$$\underset{1}{o} \leftrightarrow \underset{2}{o}$$

i.e., failure of either component shuts off the other.

A major result of this paper is that if F_1 is NBUE (NWUE) and F_2, G_2 are exponential, then

$$\mu_{\rightarrow} \geq (\leq) \frac{1}{\lambda_1 + \lambda_2} .$$

Actually, the inequalities hold for F_1 DLL (DLG). Hence, in terms of limiting average of system uptimes, it is better not to shut off 1 when 2 fails and F_1 is NBUE.

In Part I [1], we showed that in general

$$\mu_{\rightarrow} = \left(\lambda_1 \frac{\theta_1 \pi_{1'}}{\lambda_1 \pi_0} + \lambda_2\right)^{-1} .$$

Theorem 2.1 provides explicit expressions for $\pi_{1'}$ and π_0 when F_2 and G_2 are exponential. Figure 1.1 is a graph of $\mu_{\rightarrow}$ as a function of α where F_1 has the gamma density

$$f_1(x) = \frac{\alpha^{\alpha} x^{\alpha-1} e^{-\alpha x}}{\Gamma(\alpha)} .$$

Note that F_1 has mean 1. The graph is for $\lambda_1 = \lambda_2 = \theta_1 = \theta_2 = 1$ and G_1, F_2, G_2 exponential. F_1 in this case is IFR (and hence, NBUE) for $\alpha > 1$. For $0 < \alpha < 1$, F_1 is DFR and hence, NWUE. Figure 1.2 is a graph of limiting availability $A_{\rightarrow}$ as a function of α umder the same assumptions.

Of course, other considerations such as the cost of repairing component 1 may make our solution less attractive. In general, the limiting availability seems always to be greatest for the policy $\underset{1}{o} \leftarrow \underset{2}{o}$.

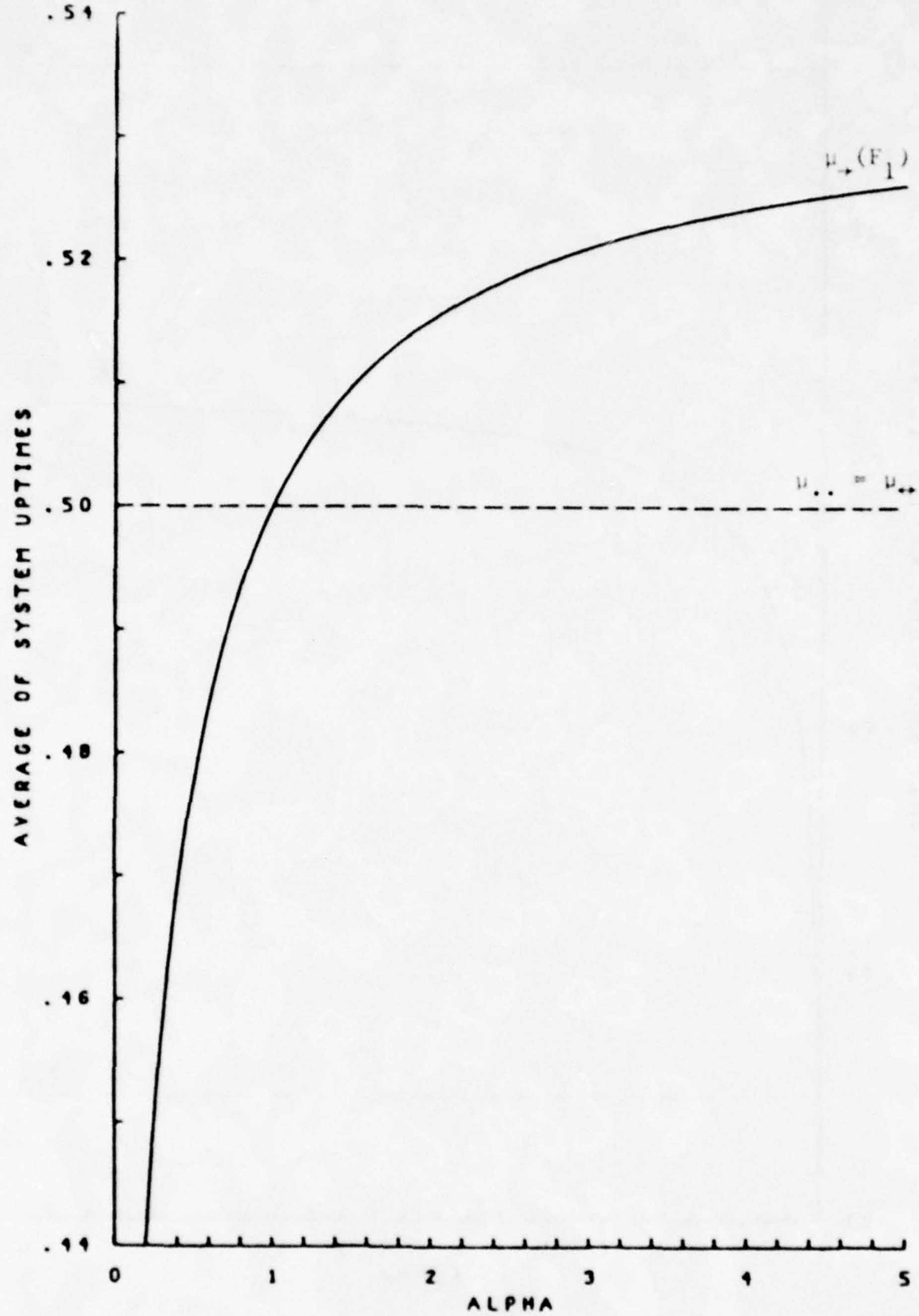

FIGURE 1.1
PLOT OF AVERAGE OF SYSTEM UPTIMES
VERSUS GAMMA SHAPE PARAMETER ALPHA, FOR F_1

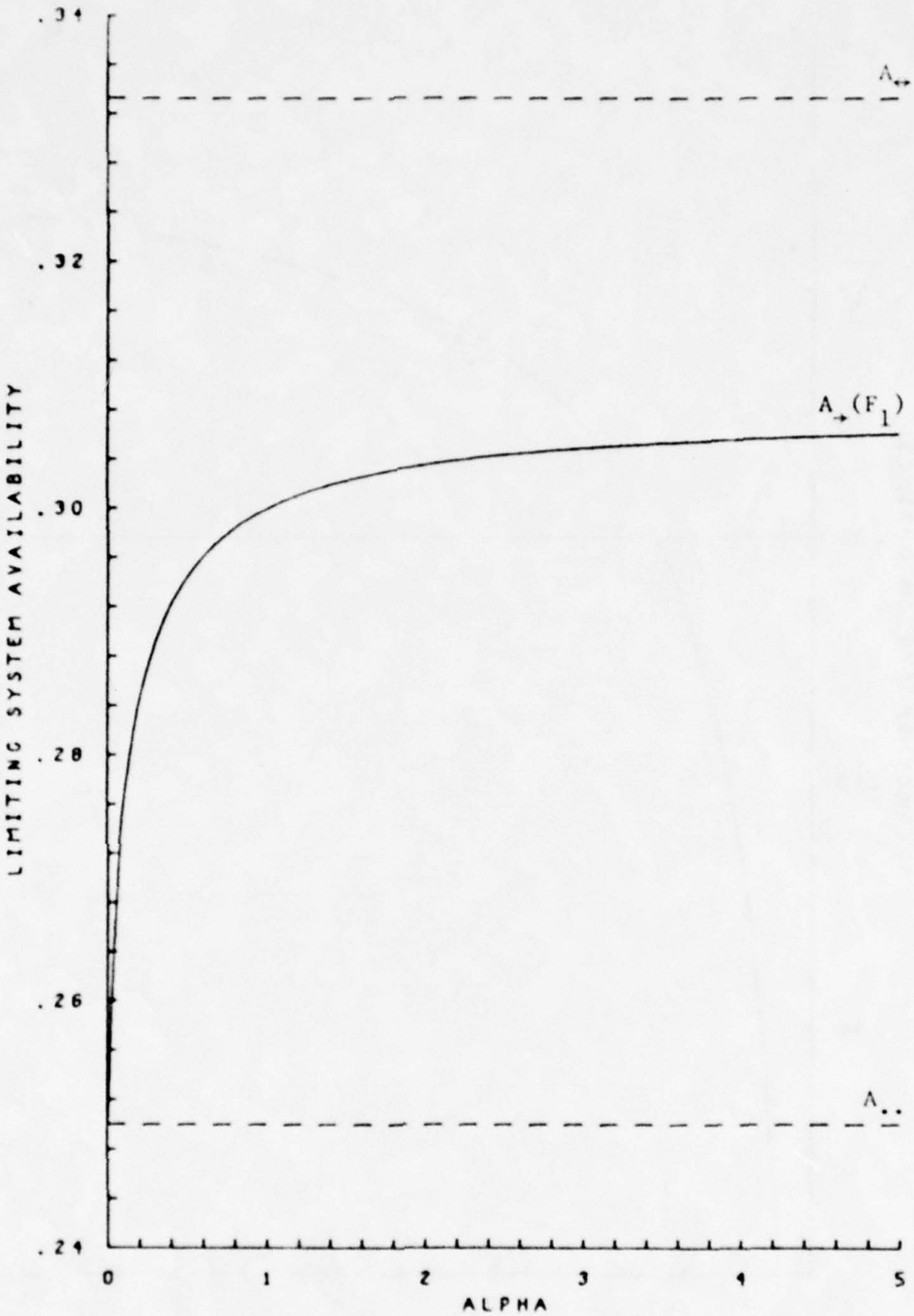

FIGURE 1.2
PLOT OF LIMITING SYSTEM AVAILABILITY
VERSUS GAMMA SHAPE PARAMETER ALPHA, FOR F_1

2. F_2 , G_2 EXPONENTIAL

In this section, we assume that both failure and repair distributions of component 2 are exponential, and F_1 , G_1 are absolutely continuous with failure rates $\lambda_1(t)$, $\theta_1(t)$, respectively. We obtain explicit expressions for the stationary state probabilities in general, and for the limiting availability in particular.

We obtain bounds on the availability $A(F_1,G_1)$ for our model. We also compare the other stationary probabilities as well as μ and ν with the corresponding quantities for the all-exponential case.

Finally, we compare the quantities A , μ , ν with the corresponding quantities in Models A and B of Part 1. We use the method of supplementary variables described in Cox (1955) to obtain stationary probabilities.

Following the notation in Part I, π_i is the limiting probability that component i $(i = 1,2)$ alone is under repair, π_0 is the probability no component is under repair and π_3 is the probability both components are under repair. See Part I for additional notation.

We need the notation $\bar{F}_i^*(s) = \int_0^\infty e^{-st}\bar{F}_i(t)dt = \dfrac{1 - f_i^*(s)}{s}$ where

$f_i^*(s) = \int_0^\infty e^{-st}dF_i(t)$.

Theorem 2.1:

If F_2 , G_2 are exponential and $\bar{F}_1(t) = e^{-\int_0^t \lambda_1(u)du}$,

$\bar{G}_1(t) = e^{-\int_0^t \theta_1(u)du}$, then the stationary state probabilities are given by

$$\pi_0 = \left(\frac{\theta_1}{\lambda_1 + \theta_1}\right)\left(\frac{\theta_2 m}{(\lambda_2 + \theta_2)d}\right) \tag{2.1}$$

$$\pi_1 = \rho_1(1 + \rho_2)\left(\frac{d - \lambda_2\theta_1\bar{F}_1^*(\lambda_2 + \theta_2)\bar{G}_1^*(\theta_2)}{m}\right)\pi_0 \tag{2.2}$$

$$\pi_2 = \rho_2\left(\frac{d - \lambda_1\theta_2\bar{F}_1^*(\lambda_2 + \theta_2)\bar{G}_1^*(\theta_2)}{m}\right)\pi_0 \tag{2.3}$$

$$\pi_3 = (1 + \rho_2)\left(\frac{\lambda_1\lambda_2\bar{F}_1^*(\lambda_2 + \theta_2)\bar{G}_1^*(\theta_2)}{m}\right)\pi_0 \tag{2.4}$$

where

$$\begin{aligned} d &= 1 - f_1^*(\lambda_2 + \theta_2)g_1^*(\theta_2) \\ &= (\lambda_2 + \theta_2)\bar{F}_1^*(\lambda_2 + \theta_2) + \theta_2\bar{G}_1^*(\theta_2) - \theta_2(\lambda_2 + \theta_2)\bar{F}_1^*(\lambda_2 + \theta_2)\bar{G}_1^*(\theta_2) \end{aligned} \tag{2.5}$$

$$m = d + \lambda_1\lambda_2\bar{F}_1^*(\lambda_2 + \theta_2)\bar{G}_1^*(\theta_2) \ . \tag{2.6}$$

Also,

$$\pi_{1'} = \rho_1\left(\frac{d + \lambda_2\bar{G}_1^*(\theta_2)[1 - (\lambda_2 + \theta_2)\bar{F}_1^*(\lambda_2 + \theta_2)]}{m}\right)\pi_0 \tag{2.7}$$

$$\pi_{1''} = \rho_1(1 + \rho_2)\left(\frac{\lambda_2\bar{F}_1^*(\lambda_2 + \theta_2)[1 - \theta_1\bar{G}_1^*(\theta_2)]}{m}\right)\pi_0 \ . \tag{2.8}$$

Proof:

By adding the supplementary variables t_1 = age of component 1 at time t, and s_1 = repair time of component 1 at t the process becomes Markovian. Let $Z(t) = 0$ if both components are operating, $Z(t) = i$ if component i $(i = 1,2)$ is under repair and $Z(t) = 3$ if both components are under repair. Let $p_n(t,t_1)$ be, at time t, the joint probability of $Z(t) = n$ and t_1 for $n = 0,2$, and similarly $p_n(t,s_1)$ for $n = 1,3$. The Kolmogorov differential equations are

$$(2.9)\quad \begin{cases} \dfrac{\partial p_0}{\partial t} + \dfrac{\partial p_0}{\partial t_1} = -(\lambda_1(t_1) + \lambda_2)p_0(t,t_1) + \theta_2 p_2(t,t_1) \\[2ex] \dfrac{\partial p_1}{\partial t} + \dfrac{\partial p_1}{\partial s_1} = -\theta_1(s_1)p_1(t,s_1) + \theta_2 p_3(t,s_1) \\[2ex] \dfrac{\partial p_2}{\partial t} + \dfrac{\partial p_2}{\partial t_1} = \lambda_2 p_0(t,t_1) - (\lambda_1(t_1) + \theta_2)p_2(t,t_1) \\[2ex] \dfrac{\partial p_3}{\partial t} + \dfrac{\partial p_3}{\partial s_1} = -(\theta_1(s_1) + \theta_2)p_3(t,s_1) \end{cases}$$

with the boundary conditions

$$(2.10)\quad \begin{cases} p_0(t,0) = \displaystyle\int_0^\infty \theta_1(u)p_1(t,u)du \\[2ex] p_1(t,0) = \displaystyle\int_0^\infty \lambda_1(u)p_0(t,u)du \\[2ex] p_2(t,0) = \displaystyle\int_0^\infty \theta_1(u)p_3(t,u)du \\[2ex] p_3(t,0) = \displaystyle\int_0^\infty \lambda_1(u)p_2(t,u)du \end{cases}$$

(and the convention $p_n(t,u) = 0$ for $u > t$) and $\int_0^\infty p_0(t,t_1)dt_1 + \int_0^\infty p_1(t,s_1)ds_1 + \int_0^\infty p_2(t,t_1)dt_1 + \int_0^\infty p_3(t,s_1)ds_1 = 1$.

Since we are only interested in the stationary probabilities, assuming they exist, we note that

$$\lim_{t\to\infty} \frac{\partial p_n(t,u)}{\partial t} = 0 \ .$$

Denote

$$p_n(u) \stackrel{\text{def}}{=} \lim_{t\to\infty} p_n(t,u)$$

and

$$\pi_n \stackrel{\text{def}}{=} \int_0^\infty p_n(u)du \ ,$$

the stationary probability of being in state n , $n = 0,1,2,3$.

Letting $t \to \infty$, we get the differential equations for the stationary joint probabilities

$$(2.11) \quad \begin{cases} p_0'(u) = -(\lambda_1(u) + \lambda_2)p_0(u) + \theta_2 p_2(u) \\ p_1'(u) = -\theta_1(u)p_1(u) + \theta_2 p_3(u) \\ p_2'(u) = \lambda_2 p_0(u) - (\lambda_1(u) + \theta_2)p_2(u) \\ p_3'(u) = -(\theta_1(u) + \theta_2)p_3(u) \end{cases}$$

with the initial conditions

$$(2.12)\quad \begin{cases} p_0(0) = \int_0^\infty \theta_1(u)p_1(u)du \\ p_1(0) = \int_0^\infty \lambda_1(u)p_0(u)du \\ p_2(0) = \int_0^\infty \theta_1(u)p_3(u)du \\ p_3(0) = \int_0^\infty \lambda_1(u)p_2(u)du \end{cases}$$

and the normalizing equation

$$\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1 .$$

Solving the last equation in (2.11) yields

$$(2.13)\quad \begin{aligned} p_3(t) &= B_3\bar{G}_1(t)e^{-\theta_2 t} \\ p_3(0) &= B_3 = \int_0^\infty \lambda_1(u)p_2(u)du \end{aligned}$$

from the last equation in (2.12).

Assuming for the moment $\theta_1(u) \equiv \theta_1$, i.e., constant repair rate, the characteristic polynomial of the second and fourth equations in (2.11)

$$(2.14)\quad \begin{cases} p_1' = -\theta_1 p_1 + \theta_2 p_3 \\ p_3' = -(\theta_1 + \theta_2)p_3 \end{cases}$$

is

$$(x + \theta_1)(x + \theta_1 + \theta_2)$$

with roots $-\theta_1$, $-(\theta_1 + \theta_2)$, so that

(2.15)
$$\begin{aligned} p_1(t) &= A_1 e^{-\theta_1 t} + B_1 e^{-(\theta_1+\theta_2)t} \\ &= \bar{G}_1(t)\left[A_1 + B_1 e^{-\theta_2 t}\right] . \end{aligned}$$

This is easily seen to be the solution in the case $\theta_1(u) \neq \theta_1$ (with the help of (2.13)), and substituting back yields conditions on A_1 , B_1 , B_3 :

(2.16)
$$B_3 = -B_1 .$$

The first and third equations in (2.11), for $\lambda_1(u) \equiv \lambda_1$, are

(2.17)
$$\begin{cases} p_0' = -(\lambda_1 + \lambda_2)p_0 + \theta_2 p_2 \\ p_2' = \lambda_2 p_0 - (\lambda_1 + \theta_2)p_2 \end{cases}$$

with the characteristic polynomial

$$(x + \lambda_1 + \lambda_2)(x + \lambda_1 + \theta_2) - \lambda_2\theta_2 = (x + \lambda_1)(x + \lambda_1 + \lambda_2 + \theta_2) .$$

Hence, the general solution of (2.17) is

(2.18)
$$\begin{aligned} p_n(t) &= A_n e^{-\lambda_1 t} + B_n e^{-(\lambda_1+\lambda_2+\theta_2)t} \\ &= \bar{F}_1(t)\left[A_n + B_n e^{-(\lambda_2+\theta_2)t}\right], \quad n = 0,2 . \end{aligned}$$

Substituting back in (2.11), these are seen to be also the solution for $\lambda_1(u) \neq \lambda_1$, with the conditions on the coefficients

(2.19)
$$B_0 + B_2 = 0$$
$$-\lambda_2 A_0 + \theta_2 A_2 = 0 \ .$$

Substituting (2.13), (2.15), (2.18) in the initial conditions, Equation (2.12) yields the following relations among the constants

(2.20)
$$\begin{cases} A_0 + B_0 = A_1 + B_1 g_1^*(\theta_2) \\ A_1 + B_1 = A_0 + B_0 f_1^*(\lambda_2 + \theta_2) \\ A_2 + B_2 = B_3 g_1^*(\theta_2) \\ B_3 = A_2 + B_2 f_1^*(\lambda_2 + \theta_2) \ . \end{cases}$$

Integrating (2.13), (2.15), (2.18) gives four equations for the π_n's, and together with the normalizing condition, we have

(2.21)
$$\begin{cases} \pi_0 = \dfrac{A_0}{\lambda_1} + B_0 \bar{F}_1^*(\lambda_2 + \theta_2) \\ \pi_1 = \dfrac{A_1}{\theta_1} + B_1 \bar{G}_1^*(\theta_2) \\ \pi_2 = \dfrac{A_2}{\lambda_1} + B_2 \bar{F}_1^*(\lambda_2 + \theta_2) \\ \pi_3 = B_3 \bar{G}_1^*(\theta_2) \\ \pi_0 + \pi_1 + \pi_2 + \pi_3 = 1 \ . \end{cases}$$

Equations (2.19), (2.20), (2.21) form a system of eleven linearly independent equations in the eleven unknowns

$$\pi_0, \pi_1, \pi_2, \pi_3, A_0, B_0, A_1, B_1, A_2, B_2, B_3 .$$

The solution of (2.19), (2.20), (2.21) for the unknown coefficients yields

(2.22)
$$\begin{cases} A_0 = \dfrac{1 - f_1^*(\lambda_2 + \theta_2) g_1^*(\theta_2)}{\lambda_2 \bar{G}_1^*(\theta_2)} B_0 = \dfrac{d}{\lambda_2 \bar{G}_1^*(\theta_2)} B_0 \\ A_1 = (1 + \rho_2) \dfrac{d}{\lambda_2 \bar{G}_1^*(\theta_2)} B_0 \\ A_2 = \dfrac{d}{\theta_2 \bar{G}_1^*(\theta_2)} B_0 \\ B_1 = -(1 + \rho_2) \dfrac{\bar{F}_1^*(\lambda_2 + \theta_2)}{\bar{G}_1^*(\theta_2)} B_0 \\ B_2 = -B_0 \\ B_3 = (1 + \rho_2) \dfrac{\bar{F}_1^*(\lambda_2 + \theta_2)}{\bar{G}_1^*(\theta_2)} B_0 \end{cases}$$

and substituting (2.22) in (2.21) yields the expressions for π_n's in the theorem, as well as

(2.23)
$$B_0 = \frac{\lambda_1 \theta_1}{\lambda_1 + \theta_1} \frac{\lambda_2 \theta_2}{\lambda_2 + \theta_2} \frac{\bar{G}_1^*(\theta_2)}{d} .$$

To get $\pi_{1'}, \pi_{1''}$, split the equations dealing with $p_1(u)$ in (2.11), (2.12) into

$$(2.24)\quad \begin{cases} p_1'(u) = -\theta_1(u)p_{1'}(u) \\ p_{1''}(u) = -\theta_1(u)p_{1''}(u) + \theta_2 p_3(u) \\ p_{1'}(0) = \int_0^\infty \lambda_1(u)p_0(u)du \\ p_{1''}(0) = 0 \end{cases}$$

so that

$$p_{1'}(t) = A_{1'}\bar{G}_1(t)$$

$$p_{1'}(0) = A_{1'} = \int \lambda_1(u)p_0(u)du$$

$$= A_0 + B_0 f_1^*(\lambda_2 + \theta_2)$$

and

$$\pi_{1'} = \frac{A_{1'}}{\theta_1} = \frac{A_0}{\theta_1} + \frac{B_0}{\theta_1} f_1^*(\lambda_2 + \theta_2) ,$$

which, using (2.22), (2.23), can be shown to be as stated in the theorem.

Also, similarly to (2.15), we get

$$p_{1''}(t) = \bar{G}_1(t)\left[A_{1''} + B_{1''}e^{-\theta_2 t}\right]$$

$$0 = A_{1''} + B_{1''} .$$

Substituting back in the second equation of (2.24) and using (2.13) yields

$$B_{1''} = -B_3$$

so that

$$\pi_{1''} = \frac{A_{1''}}{\theta_1} + B_{1''}\bar{G}_1^*(\theta_2)$$

$$= \frac{B_3}{\theta_1}\,[1 - \theta_1\bar{G}_1^*(\theta_2)]$$

which, using (2.22), (2.23) can be shown to be as stated in the theorem.■

Remark:

From the preceding theorem, we can find explicit expressions for the $p_n(u)$, $n = 0,1,2,3$ (and $n = 1',1''$) . That is, the stationary probabilities of being in state n , and the age of component 1 being u , for $n = 0,2$, and the stationary probabilities of being in state n , and the repair time of component 1 is u , for $n = 1',1'',1,3$. This can be achieved easily by substituting the values of the constants from (2.22), (2.23) in the expressions for $p_n(u)$ (2.13), (2.15), (2.18).

In Part I, we showed in Theorem 2.2.10 that for arbitrary distributions

(2.25) $$\mu_{\rightarrow} = \frac{1}{\lambda_1 \dfrac{\theta_1\pi_{1'}}{\lambda_1\pi_0} + \lambda_2} .$$

Also, it is known that

$$\mu_{\cdot\cdot} = \mu_{\leftrightarrow} = \frac{1}{\lambda_1 + \lambda_2}$$

for arbitrary distributions. Using Theorem 2.1, we can prove

Theorem 2.2:

If F_1 is NBUE (NWUE), then, for arbitrary G_1,

(2.26) $$\mu_{\rightarrow} \geq (\leq) \frac{1}{\lambda_1 + \lambda_2} .$$

Proof:

From (2.25), we see that we need only show that $\frac{\theta_1 \pi_{1'}}{\lambda_1 \pi_0} \leq (\geq) 1$.

From Theorem 2.1,

$$\frac{\theta_1 \pi_{1'}}{\lambda_1 \pi_0} = \frac{d + \lambda_2 \bar{G}_1^*(\theta_2)[1 - (\lambda_2 + \theta_2)\bar{F}_1^*(\lambda_2 + \theta_2)]}{d + \lambda_2 \bar{G}_1^*(\theta_2)\lambda_1 \bar{F}_1^*(\lambda_2 + \theta_2)} \leq 1$$

if and only if $\bar{F}_1^*(\lambda_2 + \theta_2) \geq \frac{1}{\lambda_1 + \lambda_2 + \theta_2}$. The last inequality is true for F_1 NBUE by Lemma 1.1.■

Remark:

From Theorem 2.1, it is easy to verify that limiting system availability $\pi_0 = A(F_1, G_1)$ can be expressed as

$$A(F_1, G_1) =$$

(2.27) $$(1+\rho_1)^{-1} \left\{ 1 + \rho_2 \left[\frac{\bar{F}_1^*(\lambda_2+\theta_2) + \frac{\theta_2}{\lambda_2+\theta_2} \bar{G}_1^*(\theta_2) - \left(\theta_2 + \lambda_1 \frac{\theta_2}{\lambda_2+\theta_2}\right) \bar{F}_1^*(\lambda_2+\theta_2)\bar{G}_1^*(\theta_2)}{\bar{F}_1^*(\lambda_2+\theta_2) + \frac{\theta_2}{\lambda_2+\theta_2} \bar{G}_1^*(\theta_2) - \left(\theta_2 - \lambda_1 \frac{\lambda_2}{\theta_2+\lambda_2}\right) \bar{F}_1^*(\lambda_2+\theta_2)\bar{G}_1^*(\theta_2)} \right] \right\}^{-1} .$$

Lemma 2.3:

$A(F_1,G_1)$ is increasing in both $\bar{F}_1^*(\lambda_2+\theta_2)$ and $\bar{G}_1^*(\theta_2)$ for fixed λ_i, θ_i, $i = 1,2$ (i.e., the means of all distributions are held fixed).

Proof:

From (2.1), we see that we need only verify that $\frac{m}{d}$ has the required property. But

$$\frac{m}{d} = 1 + \lambda_1\lambda_2\bar{F}_1^*\bar{G}_1^*/[(\lambda_2+\theta_2)\bar{F}_1^* + \theta_2\bar{G}_1^* - \theta_2(\lambda_2+\theta_2)\bar{F}_1^*\bar{G}_1^*]$$

which obviously is increasing in both $\bar{F}_1^*$ and $\bar{G}_1^*$. ■

Theorem 2.4:

(i) If F_1 is NBUE (NWUE) with mean $\mu_1 = \lambda_1^{-1}$, then

$$A(F_1,G_1) \geq (\leq) \left[(1+\rho_1)[1+\rho_2(1+\lambda_1\bar{G}_1^*(\theta_2))]^{-1}\right]^{-1} \tag{2.28}$$

for any G_1 with mean ν_1. The inequalities are sharp.

(ii) If F_1 is IFRA with mean μ_1, then

$$A(F_1,G_1) \leq A\left(D_{\mu_1},G_1\right) \tag{2.29}$$

for any G_1 with mean ν_1 and D_{μ_1} is the degenerate distribution with mean μ_1. The inequality is sharp.

Proof:

(2.28) is sharp since we have equality when F_1 is exponential. The inequality follows from Lemma 2.3.

(2.29) is sharp since D_{μ_1} is IFRA in a limiting sense. Also, F_1 IFRA implies

$$\int_0^\infty \bar{F}_1(x)e^{-sx}dx \le \frac{1 - e^{-s\mu_1}}{s} .$$

<u>Theorem 2.5</u>:

(i) If G_1 is NBUE (NWUE) with mean $\nu_1 = \theta_1^{-1}$, then

$$A(F_1,G_1) \ge (\le) \ (1+\rho_1)^{-1}\left\{1+\rho_2\left[\frac{\frac{\theta_2}{\lambda_2+\theta_2}+\left(\theta_1-\lambda_1\frac{\theta_2}{\lambda_2+\theta_2}\right)\bar{F}_1^*(\lambda_2+\theta_2)}{\frac{\theta_2}{\lambda_2+\theta_2}+\left(\theta_1+\lambda_1\frac{\lambda_2}{\lambda_2+\theta_2}\right)\bar{F}_1^*(\lambda_2+\theta_2)}\right]\right\}^{-1}$$

for any F_1 with mean μ_1 . The inequalities are sharp.

(ii) If G_1 is IFRA with mean ν_1 , then

$$A(F_1,G_1) \le A\left(F_1,D_{\nu_1}\right)$$

for any F_1 with mean μ_1 and D_{ν_1} is the degenerate distribution with mean ν_1 . The inequalities are sharp.

<u>Theorem 2.6</u>:

If both F_1 and G_1 are NBUE, then

(2.31) $$\pi_0 \ge \pi_0(\exp)$$

(2.32) $$\pi_1 \ge \pi_1(\exp)$$

$$\pi_2 \le \pi_2(\exp) \tag{2.33}$$

$$\pi_3 \le \pi_3(\exp) \tag{2.34}$$

where $\pi_i(\exp)$ means that probabilities are computed in the all exponential case.

The inequalities are reversed if both F_2 and G_2 are NWUE.

The proof is straightforward and is omitted.

3. F_1 , G_1 EXPONENTIAL

In this section we assume that both failure and repair distributions of component 1 are exponential, and F_2 , G_2 are absolutely continuous with failure rates $\lambda_2(t)$, $\theta_2(t)$, respectively.

We obtain explicit expressions for the stationary state probabilities in general, and for the limiting availability in particular. All the results from Part I, Section 2.4 apply here, of course, since F_1 is exponential.

Theorem 3.1:

If F_1 , G_1 are exponential, and $\bar{F}_2(t) = e^{-\int_0^t \lambda_2(u)du}$, $\bar{G}_2(t) = e^{-\int_0^t \theta_2(u)du}$, the stationary state probabilities are given by

$$\pi_0 = \left(\frac{\theta_1}{\lambda_1 + \theta_1}\right)\left\{\frac{1}{1 + \rho_2\left[1 - \lambda_1 \dfrac{1 - \theta_2\bar{G}_2^*(\lambda_1 + \theta_1)}{\lambda_1 + \theta_1}\right]}\right\} \tag{3.1}$$

$$\pi_1 = \rho_1[1 + \lambda_2\bar{G}_2^*(\lambda_1 + \theta_1)]\pi_0 \tag{3.2}$$

$$\pi_2 = \left(\frac{\theta_1}{\lambda_1 + \theta_1}\right)\rho_2[1 + \rho_1\theta_2\bar{G}_2^*(\lambda_1 + \theta_1)]\pi_0 \tag{3.3}$$

$$\pi_3 = \left(\frac{\lambda_1}{\lambda_1 + \theta_1}\right)\rho_2[1 - \theta_2\bar{G}_2^*(\lambda_1 + \theta_1)]\pi_0 \ . \tag{3.4}$$

Also,

(3.5) $$\pi_{1'} = \rho_1\pi_0$$

(3.6) $$\pi_{1''} = \rho_1\lambda_2\bar{G}_2^*(\lambda_1 + \theta_1)\pi_0$$

where $\bar{G}_2^*(\lambda_1 + \theta_1) = \int_0^\infty e^{-(\lambda_1+\theta_1)t}\bar{G}_2(t)dt$.

The proof will be omitted since it is similar to that of Theorem 2.1.

Theorem 3.2:

(i) If G_2 is NBUE with mean ν_2 , then

(3.7) $$\frac{1}{(1+\rho_1)(1+\rho_2)} \le A(F_2,G_2) \le A(\exp) .$$

If G_2 is IFRA with mean ν_2 , then

$$A(F_2,G_2) \ge A\left(G_2 = D_{\nu_2}\right) .$$

(ii) If G_2 is NWUE with mean ν_2 then

(3.8) $$\frac{1}{1+\rho_1+\rho_2} \ge A(F_2,G_2) \ge A(\exp) .$$

Proof:

The left-hand bounds in (3.7) and (3.8) follow from Part I, Corollaries 2.2.13 and 2.4.2 respectively. They are always valid but not necessarily sharp.

All other inequalities are sharp and the proofs are similar to Theorem 2.4.■

Theorem 3.3:

If G_2 is NBUE with mean ν_2, then

$$\pi_0 \leq \pi_0(\exp) \tag{3.9}$$

$$\pi_1 \geq \pi_1(\exp) \tag{3.10}$$

$$\pi_2 \geq \pi_2(\exp) \tag{3.11}$$

$$\pi_3 \leq \pi_3(\exp) \tag{3.12}$$

$$\pi_{1'} \leq \pi_{1'}(\exp) \tag{3.13}$$

$$\pi_{1''} \geq \pi_{1''}(\exp) \, . \tag{3.14}$$

All *inequalities are reversed if* G_2 is NWUE with mean ν_2.

The proof is straightforward and is omitted.

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