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AN IMPLICIT FUNCTION THEOREM FOR GROUP EQUATIONS GENERATED BY A--ETC(U)
AUG 79 P R THRIFT
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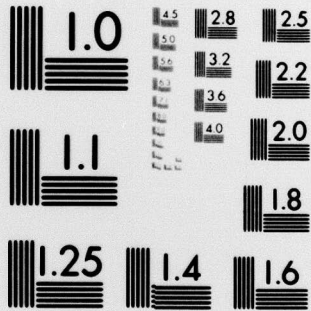
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AN IMPLICIT FUNCTION THEOREM FOR GROUP EQUATIONS
GENERATED BY A FINITE AUTOMATON.

by

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Princeton University

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1. INTRODUCTION

The study of mathematical semantics [3] involves the study of mappings from a formal language L to a configuration space C . (The reader is referred to [2] and [5] for pattern and automata theoretic notions.) In some of the cases that have been investigated, we can consider the set C as a set of operations G on some set of objects. The algebraic nature of these operations and sets of objects can be very simple or complex, depending on a particular application. In [3] these operations are called connectors when they deal with establishing or breaking relationships between these objects.

In the case where $L = L(D)$, $D =$ a deterministic finite state diagram (or wiring diagram for some finite state automaton) over some alphabet Σ , and G (the set of operations) is a group, then we can define a semantic map by first associating an element of G with each production $p: i \xrightarrow{x} j$ of D , where $x \in \Sigma$ and i, j are states of D . If we write this map as $p \rightarrow \theta(p) \in G$, p a production of D , then for each sentence $x_1 \dots x_k$ in $L(D)$, we define

$$\phi(\theta)(x_1 \dots x_k) = \theta(p_1) \dots \theta(p_k) \quad (\text{multiplication in } G),$$

where $s \xrightarrow{x_1} i_1 \xrightarrow{x_2} i_2 \dots i_{k-1} \xrightarrow{x_k} i_k$.

$$p_j: i_{j-1} \xrightarrow{x_j} i_j \quad \text{are productions of } D.$$

$S =$ start state, and $i_k \in$ final state set F of D . Thus $\phi(\theta)$ is a semantic map $\phi(\theta): L(D) \rightarrow G$. A semantic map $\gamma: L(D) \rightarrow G$ is called sequential if there is some map θ , $p \rightarrow \theta(p)$ p a production of D , such that $\gamma = \phi(\theta)$.

ABSTRACT

The study of mathematical semantics involves the study of mappings from a formal language L to a configuration space C . In this paper we consider the special case where $L = L(D)$, D is a finite automaton, C is a group G , and the semantic map is defined by associating each production of D with a group element of G , $p \rightarrow \theta(p)$. A sequence of productions would then be mapped to a product of corresponding group elements. When the semantic map is observed on sentences in $L(D)$, we discuss a graph-theoretic method of solving for θ .

theta

approaches theta(p)



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In this paper we consider the following problem: suppose the structure of the deterministic finite automaton \mathcal{D} is known and we can observe $\gamma(w)$, $w \in L(\mathcal{D})$. Under the assumption that γ is sequential, what can we say about all possible solutions of $\phi(\theta) = \gamma$? The answer is stated in the form of an implicit function theorem and a procedure is given for calculating solutions.

2. FINITE STATE DIAGRAMS AND GRAPH-THEORETIC NOTIONS

Let F be a directed pseudograph with vertex set

$$V(F) = \{v_1, \dots, v_p\} \quad \text{and arc set}$$

$$A(F) = \{a_1, \dots, a_q\}. \quad (\text{See [4] for graph-theoretic notions.})$$

To each arc a_j corresponds an ordered pair

$$(\alpha(a_j), \beta(a_j)), \quad \alpha(a_j), \beta(a_j) \in V(F). \quad \text{If } \alpha(a_j) = \beta(a_j) \text{ then we call } a_j \text{ a loop.}$$

If Σ is some finite alphabet $\{a, b, c, \dots\}$ and there is a map $A(F) \xrightarrow{\sigma} \Sigma$ satisfying the property:

$$\alpha(a_j) = \alpha(a_k), \quad j \neq k \implies \sigma(a_j) \neq \sigma(a_k), \quad \text{and if } S \text{ is a single vertex of } V(F), \quad F \text{ a (non-empty) subset of } V(F), \text{ then}$$

$\mathcal{D} = (F, \Sigma, \sigma, S, F)$ is called a finite state diagram with state set

$$V(F) \text{ and productions } \alpha(a_j) \xrightarrow{\sigma(a_j)} \beta(a_j), \quad j = 1, \dots, q. \quad \text{An example appears in Figure 1 below.}$$

In this paper we shall consider, for simplicity, only the case where F is also a single state of $V(F)$ and $S \neq F$, so that $p = |V(F)| \geq 2$.

A finite state diagram can be used to generate sentences in Σ^* (that is, sequences $x_1 \dots x_k$ from Σ). To generate a sentence in Σ^* we take a walk $a_{i_1} \dots a_{i_k}$ from S to F in F (that is, a sequence of arcs such that $S = \alpha(a_{i_1}), \beta(a_{i_j}) = \alpha(a_{i_{j+1}})$, $j = 1, \dots, k-1, \beta(a_{i_k}) = F$). We then produce the sentence

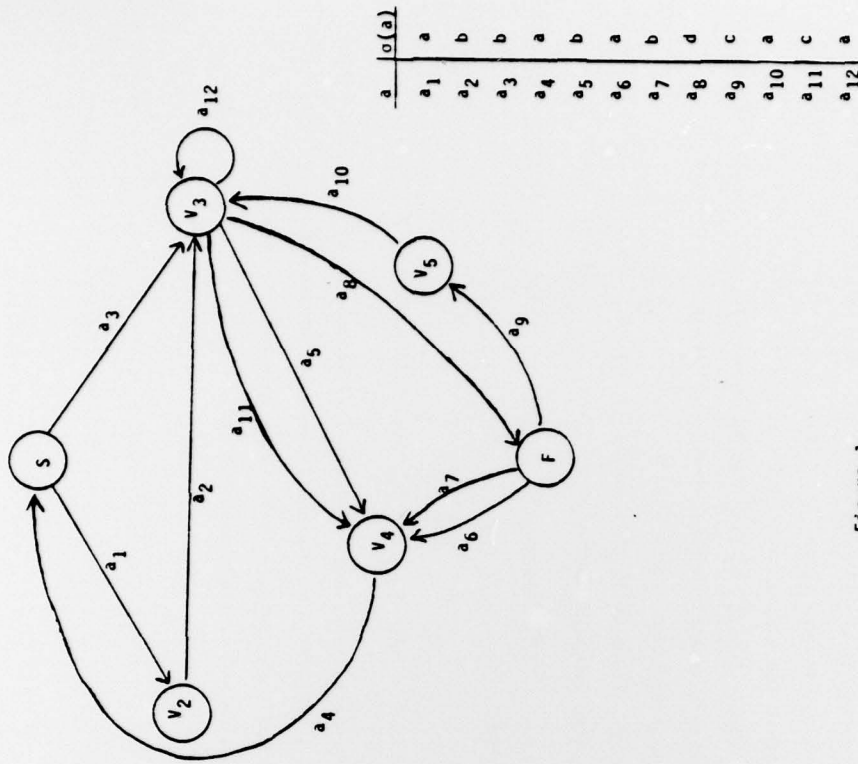


Figure 1.

$\sigma(a_{i_1} \dots a_{i_k}) = \sigma(a_{i_1}) \dots \sigma(a_{i_k})$ by concatenation. The language generated by \mathcal{D} is defined to be

$$L(\mathcal{D}) = \{w \in \Sigma^* \mid w = \sigma(a_{i_1} \dots a_{i_k}) \text{ for some walk } a_{i_1} \dots a_{i_k} \text{ from } S \text{ to } F\}.$$

$L(\mathcal{D})$ is called a finite state language.

In addition, given \mathcal{D} , and any $w \in L(\mathcal{D})$ we can obtain the (unique) walk $a_{i_1} \dots a_{i_k}$ that generated w by using the uniqueness property of σ .

It may happen that there are no walks from S to F , in which case $L(\mathcal{D}) = \emptyset$. However, we shall assume that F has the following connectedness property: for every $v_j \in \{v_1, \dots, v_p\}$, there is a walk $a_{i_1} \dots a_{i_k}$ from S to F that "passes through" v_j . It is apparent if a given state diagram \mathcal{D} has a vertex v_j that does not have this property, then it can be eliminated from \mathcal{D} (along with any incident arcs) and $L(\mathcal{D})$ will remain the same.

Suppose we list the vertices $V(F)$ as $\{s, v_2, \dots, v_{p-1}, f\}$. V_S is defined to be the following subset of V : $S \in V_S$ and also any $v_j \in \{v_2, \dots, v_{p-1}\}$ is in V_S if there is some path from S to v_j (that is, a walk from S to v_j with no vertex repeated) that does not pass through F . V_F is defined to be $V(F) - V_S$, so that $V(F) = V_S \cup V_F$. Note that neither v_S nor v_F are \emptyset since $S \in V_S$ and $f \in V_F$. Note also that elements v_k in $V_F - \{f\}$ have the property that any path from S to v_k passes through F .

For any directed pseudograph F , a subgraph T is called a directed tree if:

- (a) There is exactly one vertex of T called the root which no arc of T enters.
- (b) Every vertex of T except the root has exactly one entering arc.
- (c) There is a unique path from the root to each vertex in T .

If $V(T)$ is the vertex set of T , then we say that T spans $V(T)$.

We now wish to construct two directed trees in F : T_S rooted at S and spanning V_S , and T_F rooted at F and spanning V_F .

We construct T_S as follows. Start with S and define $V_S^{(0)} = S$. Let $V_S^{(1)}$ be the set of all vertices in $V_S - V_S^{(0)}$ that can be reached from $V_S^{(0)}$ in a single arc. Let $A_S^{(1)}$ be a set of arcs starting from $V_S^{(0)}$ and having the property that every element of $V_S^{(1)}$ is reached by exactly one arc of $A_S^{(1)}$. In general, let $V_S^{(k)}$ be the set of vertices in $V_S - V_S^{(k-1)} - \dots - V_S^{(k-1)}$ that can be reached from a vertex of $V_S^{(k-1)}$ in a single arc, and $A_S^{(k)}$ a set of arcs starting from $V_S^{(k-1)}$ and having the property that every element of $V_S^{(k)}$ is reached by exactly one arc of $A_S^{(k)}$. Continue until $V_S^{(k)}$ is empty. Let

$$T_S = (V_S^{(0)} \cup V_S^{(1)} \cup \dots \cup A_S^{(1)} \cup A_S^{(2)} \cup \dots).$$
 Clearly, T_S is a tree.

Proposition. T_S spans V_S .

<Proof>. Let $v_j \in V_S$. If $v_j = S$ then we are done. Other-

wise, suppose $S \xrightarrow{b_1} v_{j_1} \xrightarrow{b_2} \dots \xrightarrow{b_k} v_{j_k} = v_j$ where

v_{j_1}, \dots, v_{j_k} are also in V_S . Then
 $v_{j_1} \in V_S^{(0)} \cup V_S^{(1)}$
 \vdots
 $v_{j_k} \in V_S^{(0)} \cup \dots \cup V_S^{(k)}$

can be proven inductively. ||

We construct T_F in exactly the same manner (substituting F for S).

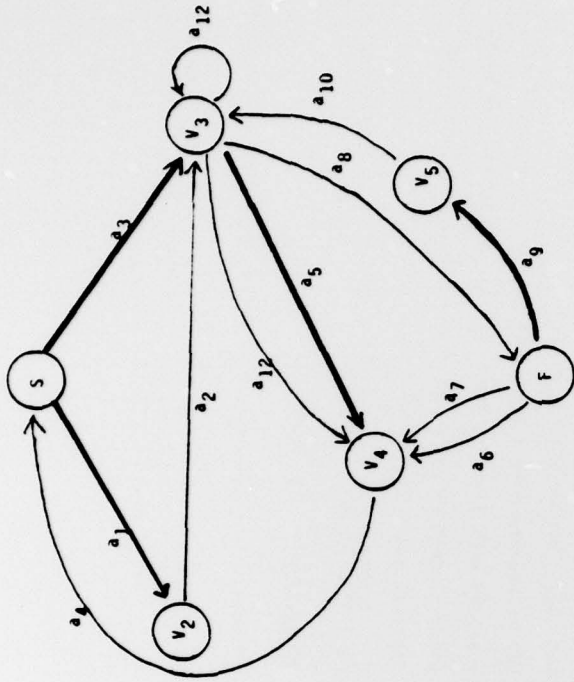
Proposition. T_F spans V_F .

<Proof>. Let $v_k \in V_F$. Then there is a path

$$F \xrightarrow{c_1} v_{k_1} \xrightarrow{c_2} \dots \xrightarrow{c_j} v_{k_j} = v_k$$

where $v_{k_1}, \dots, v_{k_{j-1}}$ are also in V_F , as is apparent from the definition of V_F . The proof then follows exactly as before. ||

In our previous example, we have



$$A(T_S) = (a_1, a_3, a_5)$$

$$A(T_F) = (a_9)$$

Figure 2.

3. SEMANTIC MAPS AND AN IMPLICIT FUNCTION THEOREM

We now consider a group G and a map

$$\theta: A(F) \rightarrow G.$$

Since there are q arcs in $A(F)$ we shall consider θ as an element of G^q with $\theta_j = \theta(a_j)$ $j = 1, \dots, q$ called the arc values.

Let Γ be the set of all semantic maps $\gamma: L(D) \rightarrow G$ and let

$$\phi: G^q \rightarrow \Gamma$$

$$\phi(\theta)\omega = \theta(a_{i_1}) \theta(a_{i_2}) \dots \theta(a_{i_p}) \text{ if } a_{i_1}, \dots, a_{i_k}$$

is the walk from S to F determined by the sentence ω . Let us suppose the arcs are listed so that a_1, \dots, a_{p-2} are the arcs in $T_S \cup T_F$ with a_{p-1}, \dots, a_q remaining. Let $r = p-2$ and $m = q-r$. We now state the following implicit function theorem.

Theorem. Assume that $\gamma_0 \in \Gamma$ is sequential. Then there exist functions $\phi_1, \dots, \phi_m: G^r \rightarrow G$ (depending, of course, on γ_0) such that

$$(\theta \in G^q: \phi(\theta) = \gamma_0) =$$

$$\{(\theta \in G^q: (\theta_1, \dots, \theta_r) \in G^r, \theta_{r+1} = \phi_1(\theta_1, \dots, \theta_r) \text{ for } l=1, \dots, m)\}.$$

Remark. In the case where $r = 0$ (that is, the case where T_S and T_F consist only of single vertices S and F) then there is a unique solution of $\phi(\theta) = \gamma_0$.

<Proof>. By assumption there is some $\hat{\theta} \in G^q$ satisfying $\phi(\hat{\theta}) = \gamma_0$. For each $(\theta_1, \dots, \theta_r) \in G^r$ we shall first show that there exists a $(\theta_{r+1}, \dots, \theta_q) \in G^m$ such that $\phi(\theta) = \gamma_0$, and then show that $(\theta_{r+1}, \dots, \theta_q)$ is uniquely determined. The uniqueness argument gives a procedure for constructing the maps ϕ_1, \dots, ϕ_m given γ_0 .

Let v_j be an element of $V_S^{(1)}$. Then v_j appears in F as

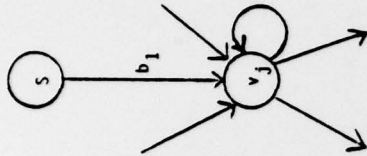


Figure 3.

with incoming arcs b_1, \dots, b_r , loops l_1, \dots, l_t , and outgoing arcs c_1, \dots, c_s . Assume that b_1 is the arc that appears in T_S . Note that $\{c_1, \dots, c_s\}$ is not empty if $A(T_S)$ is not empty. Update the arc values by the following system of assignments:

$$\hat{\theta}(b_i) \leftarrow \hat{\theta}(b_i) \hat{\theta}(b_1)^{-1} \theta(b_1) \quad i = 1, \dots, r$$

$$\hat{\theta}(c_j) \leftarrow \theta(b_1)^{-1} \hat{\theta}(b_1) \hat{\theta}(c_j) \quad j = 1, \dots, s$$

$$\hat{\theta}(l_k) \leftarrow \theta(b_1)^{-1} \hat{\theta}(b_1) \hat{\theta}(l_k) \hat{\theta}(b_1)^{-1} \theta(b_1) \quad k = 1, \dots, t.$$

Note that the arc value of b_1 is changed to $\theta(b_1)$. This operation is performed for each vertex v in $V_S^{(1)}$ and then repeated for $V_S^{(2)}, V_S^{(3)}, \dots$ until the vertex set of V_S is exhausted, and noting that once an arc value of an arc of T_S is set to the appropriate

is w_1, \dots, w_m , then the finite number of equations

$$\phi(\theta)w_j = \gamma_0(w_j) \quad j = 1, \dots, m$$

determine $\theta_{r+1} = \phi_1(\theta_1, \dots, \theta_r)$ for $1 = 1, \dots, m$ in the statement of the theorem. Also, any additional sentences will not reduce the solution set. In this sense, we can say that $\phi(\theta) = \gamma_0$ involves m "independent" group equations.

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