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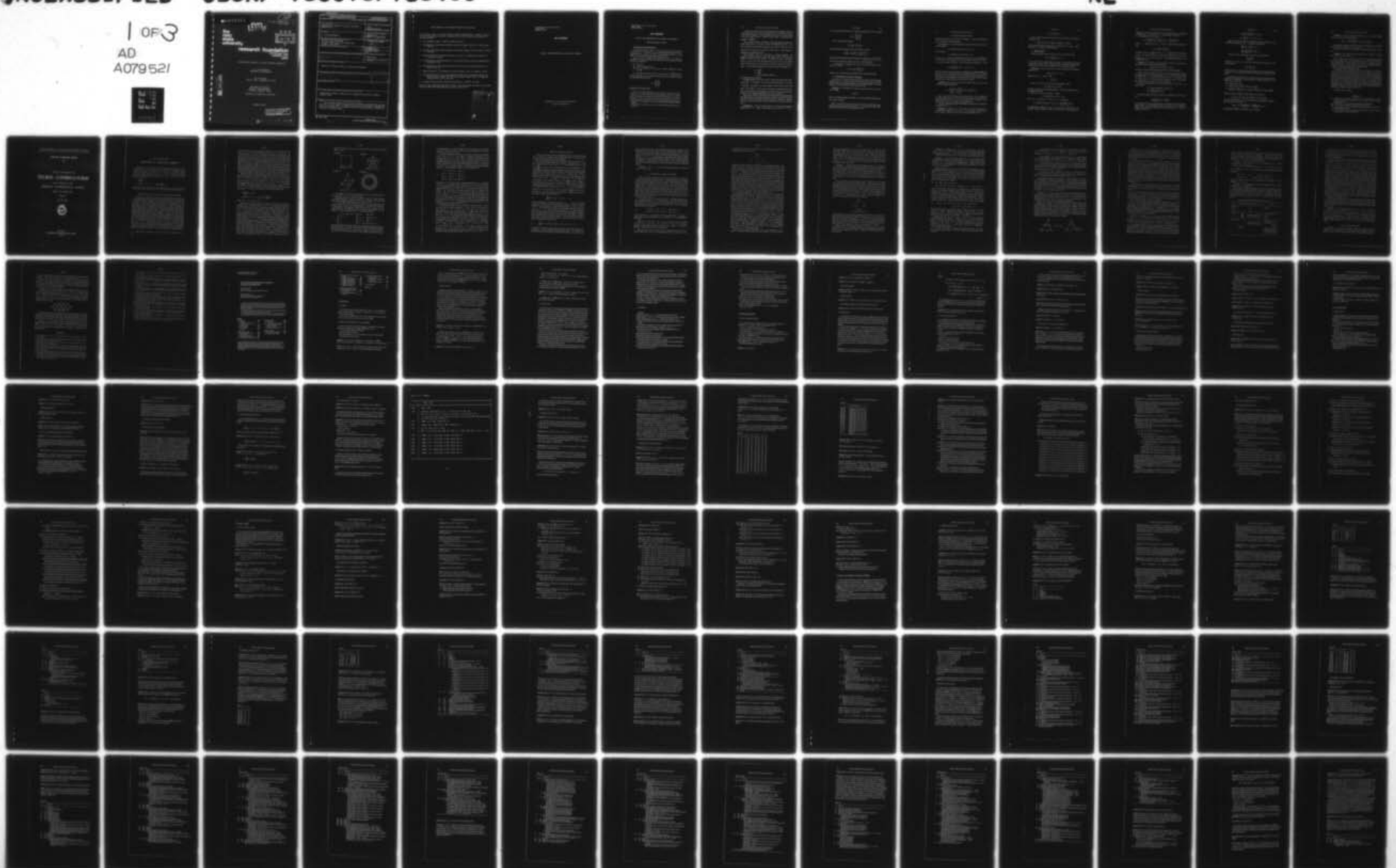

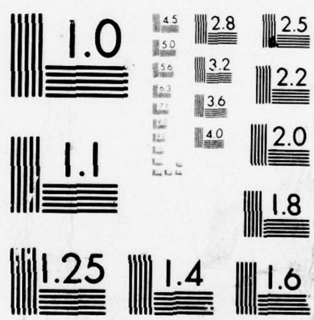

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COMBINATORIAL PROBLEMS OF APPLIED DISCRETE MATHEMATICS

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For the Period
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This research program studied several important combinatorial problems that arise in the fields of statistical design of experiments, mathematical theory of communication, information theory, switching theory, information retrieval and graph theory.

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FINAL REPORT OF ONR CONTRACT N00014-67-A-0232-0016

As the final report of the ONR Contract N00014-67A-0232-0016, I submit copies of the following papers which were supported by the ONR contract and appeared in various journals:

1. On t-Designs, Osaka J. Math, 12(1975), 737-744.

2. Uniqueness of Association Schemes, Atti Dei Convegni Lincei (17) 1976, Roma, 465-479.

3. Balanced Incomplete Block Designs and Related Designs, Discrete Math 11(1975), 255-369.

4. Characterization of Projective Graphs, Journal of Combinatorial Theory, B, 24(1978), 294-300.

5. A Combinatorial Characterization of Attenuated Spaces, Utilitas Mathematica, 15(1979), 3-29.

6. Characterization of Projective Incidence Structures, Geometria Dedicata, 5 (1976), 361-376.

7. Characterization of Linegraph of an Affine Space, JCT, A, 26(1979), 48-64.

8. A Characterization of the line-hyperplane Design of a Projective Space and Some Extremal Theorems for Matroid Designs, Number Theory and Algebra, Academic Press, 1977, 289-301.

9. Strongly Regular Graphs, Discrete Mathematics, 13(1975), 357-381.

Copies of all unpublished manuscripts which resulted during the course of the ONR contract N00014-67A-0232-0016 were already sent to ONR.

Ray-Chaudhuri, D.K. and Wilson, R.M.
Osaka J. Math.
12 (1975), 737–744

ON t-DESIGNS

DIJEN K. RAY-CHAUDHURI AND RICHARD M. WILSON

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ON t-DESIGNS

DIJEN K. RAY-CHAUDHURI* AND RICHARD M. WILSON**

(Received January 14, 1975)

Introduction and preliminaries

An *incidence structure* is a triple $S=(X, \mathcal{A}, \mathcal{I})$ where X and $\mathcal{A}$ are disjoint sets and $\mathcal{I}\subseteq X\times\mathcal{A}$. Elements $x\in X$ are called *points* and elements $A\in\mathcal{A}$ are called *blocks* of S. A point x and a block A are *incident* iff $(x, A)\in\mathcal{I}$. For any block A, (A) will denote the set of points incident with A.

Let v, k, t and λ be integers with $v\geq k\geq t\geq 0$ and $\lambda\geq 1$. An $S_\lambda(t, k, v)$ (*a t-design* on v points with block size k and index λ) is an incidence structure $D=(X, \mathcal{A}, \mathcal{I})$ such that

(i) $|X|=v$,
(ii) $|(A)|=k$ for every $A\in\mathcal{A}$,
(iii) for every t-subset T of X, there are exactly λ blocks $A\in\mathcal{A}$ with $T\subseteq(A)$.

It is well known that every $S_\lambda(t, k, v)$ has exactly $b=\lambda\binom{v}{t}\Big/\binom{k}{t}$ blocks and more generally, for any i-subset I of points $(0\leq i\leq t)$, the number of blocks A of the design with $I\subseteq(A)$ is

$$b_i=\lambda\frac{\binom{v-i}{t-i}}{\binom{k-i}{t-i}},$$

independent of the subset I [2].

Abstract: We present the generalization (conjectured by A. Ja. Petrenjuk) of Fisher's Inequality $b\geq v$ for 2-designs and Petrenjuk's Inequality $b\geq\binom{v}{2}$ for 4-designs. The t-designs satisfying the inequality with equality may be considered as generalizations of the symmetric 2-designs ($b=v$) and have the property that there are exactly $\frac{1}{2}t$ possible values for the size of the intersection of two distinct blocks, these values being computable from the parameters.

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An $S_\lambda(t, k, v)$, say $D=(X, \mathcal{A}, \mathcal{I})$, is *simple* when the mapping $A \mapsto (A)$ from $\mathcal{A}$ into $\mathcal{P}_k(X)$ (the class of all k-element subsets of X) is injective; and D is *trivial* when the mapping $A \mapsto (A)$ is (surjective and) m-to-one for some integer m, *i.e.* each k-subset "occurs as a block" exactly m times. In this latter case, evidently $\lambda = m\binom{v-t}{k-t}$.

The well known Fisher's Inequality (see [2]) asserts that the number b of blocks of an $S_\lambda(2, k, v)$ is at least v, under the assumption $v \geq k+1$. A. Ja. Petrenjuk [4] proved in 1968 that $b \geq \binom{v}{2}$ for any $S_\lambda(4, k, v)$ with $v \geq k+2$ and conjectured that $b \geq \binom{v}{s}$ in any $S_\lambda(2s, k, v)$ with $v \geq k+s$. This conjecture is established in the following section.

This condition shows the nonexistence of certain t-designs. For example, Petrenjuk's Inequality shows that $S_5(4, 22, 79)$ do not exist even though the b_i's $(0 \leq i \leq 4)$ are integral. We might note that a hypothetical $S_2\left(4, k, 2+\frac{1}{2}(k-1)(k-2)\right)$ would satisfy $b=\binom{v}{2}$ (and the b_i's are integral when $k \not\equiv 1 \pmod 4$), but no such designs exist by the corollary of Theorem 5 below. The inequality $b \geq \binom{v}{3}$ rules out the entire family of 6-designs with

$$v = 120m\,,$$
$$k = 60m\,,$$
$$\lambda = (20m-1)(15m-1)(12m-1)\,,$$

(for which the b_i's are integral).

By a *tight* t-desigh (t even, say $t=2s$) we mean an $S_\lambda(t, k, v)$ with $v \geq k+s$ and $b=\binom{v}{s}$. As examples, we have the trivial designs $S_\lambda(2s, k, k+s)$ where $\lambda=\binom{k-s}{k-2s}$. An example of a tight 4-design is the well known $S_1(4, 7, 23)$ where $b=253=\binom{23}{2}$. N. Ito [3] has recently shown, using Theorem 5 below, that the only nontrivial tight 4-desighns are the $S_1(4, 7, 23)$ and its complement, an $S_{52}(4, 16, 23)$. Tight t-designs with $t \geq 4$ seem to be very rare.

Our proof of Petrenjuk's conjecture uses only elementary linear algebra and the observation that the nunber of blocks of an $S_\lambda(t, k, v)$ which are incident with some i points and not incident some other j points is constant (i.e., depends only on i, j, and the parameters; not the particular sets of points) whenever $i+j \leq t$.

Proposition 1. *Let* $(X, \mathcal{A}, \mathcal{I})$ *be an* $S_\lambda(t, k, v)$. *Let i and j be nonnegative integers with* $i+j \leq t$. *Then for any subsets* $I, J \subseteq X$ *with* $|I|=i$, $|J|=j$,

$I \cap J = \phi$, the number of blocks $A \in \mathcal{A}$ such that $I \subseteq (A)$ and $J \cap (A) = \phi$ is exactly

$$b_i^j = \lambda \frac{\binom{v-i-j}{k-i}}{\binom{v-t}{k-t}}.$$

Proof. By inclusion-exclusion,

$$b_i^j = \sum_{r=0}^{j} (-1)^r \binom{j}{r} b_{i+r}.$$

In view of the above expression for b_i, we have $b_i^j = \lambda c$ where

$$c = \sum_{r=0}^{j} (-1)^r \binom{j}{r} \binom{v-i-r}{t-i-r} \binom{k-i-r}{t-i-r}^{-1}.$$

But in the case of the trivial design $(X, \mathcal{P}_k(X), \in)$, $\lambda = \binom{v-t}{k-t}$ and $b_i^j = \binom{v-i-j}{k-i}$, from which we deduce the simpler expression $c = \binom{v-i-j}{k-i}\binom{v-t}{k-t}^{-1}$.

As a corollary, the *complement* $(X, \mathcal{A}, (X \times \mathcal{A}) - \mathcal{I})$ of an $S_\lambda(t, k, v)$ is an $S_{\lambda^*}(t, v-k, v)$ with

$$\lambda^* = b_0^t = \lambda \binom{v-t}{k} \binom{v-t}{k-t}^{-1}$$

(unless $v < k+t$, in which case the original $S_\lambda(t, k, v)$ is evidently trivial).

2. Generalizations of Fisher's inequality

For any set Y, we denote by $V(Y)$ the free vector space over the rationals generated by Y, i.e. $V(Y)$ consists of all formal sums $\alpha = \sum_{y \in Y} a_y y$ with rational coefficients a_y and formal addition and scalar multiplication. The "unit vectors" y, $y \in Y$, by definition provide a basis for $V(Y)$.

Theorem 1. *The existence of an $S_\lambda(t, k, v)$ with t even, say $t = 2s$, and $v \geq k+s$ implies*

$$b \geq \binom{v}{s},$$

where b is the number of blocks of the design. In fact, the number of distinct subsets (A) is itself at least $\binom{v}{s}$.

Proof. Let $D = (X, \mathcal{A}, \mathcal{I})$ be an $S_\lambda(t, k, v)$ and put $V_s = V(\mathcal{P}_s(X))$, where $\mathcal{P}_s(X)$ is the class of all s-element subsets of X. For each block A of D, define a vector $\hat{A} \in V_s$ as the "sum" of all s-subsets of (A), i.e.

$$\hat{A} = \sum(S\colon S \in \mathcal{P}_s(X),\ S \subseteq (A))$$

We claim that the set of vectors $\{\hat{A}\colon A \in \mathcal{A}\}$ spans V_s. Since V_s has dimension $\binom{v}{s}$, the theorem follows immediately.

Let $S_0 \in \mathcal{P}_s(X)$. To show S_0 belongs to the span of $\{\hat{A}\colon A \in \mathcal{A}\}$, we introduce the vectors

$$E_i = \sum(S\colon S \in \mathcal{P}_s(X),\ |S \cap S_0| = s-i)$$

(*so* $E_0 = S_0$) and

$$F_i = \sum(\hat{A}\colon A \in \mathcal{A},\ |(A) \cap S_0| = s-i)$$

for $i=0, 1, \cdots, s$. Now for $S_1 \in \mathcal{P}_s(X)$ with $|S_1 \cap S_0| = s-i$, the coefficient of S_1 in the sum F_r is the number of blocks A such that $S_1 \subseteq (A)$ and $|(A) \cap S_0| = s-r$; and this number is $\binom{i}{r} b^r_{s-r+i}$ with the notation of Proposition 1. Thus

$$F_r = \sum_{i=r}^{s} \binom{i}{r} b^r_{s-r+i} E_i \qquad (r = 0, 1, \cdots, s).$$

The above system of linear equations is triangular and the diagonal coefficients b^r_s $(r=0, 1, \cdots, s)$ are all nonzero under our hypothesis $v \geq k+s$. Thus we can solve for the E_i's (in particular, for $E_0 = S_0$) as linear combinations of the F_r's. Since the F_r's are by definition in the span of $\{\hat{A}\colon A \in \mathcal{A}\}$, we have $S_0 \in$ span $\{\hat{A}\colon A \in \mathcal{A}\}$ for every $S_0 \in \mathcal{P}_s(X)$, and our claim is verified.

Corollary. *The existence of an* $S_\lambda(t, k, v)$ *with* t *odd, say* $t = 2s+1$ *and* $(v-1) \geq k+s$ *implies the inequality*

$$b = \frac{\lambda\binom{v}{2s+1}}{\binom{k}{2s+1}} \geq \frac{\lambda\binom{v-1}{2s}}{\binom{k-1}{2s}} + \binom{v-1}{s} \geq 2\binom{v-1}{s}.$$

Proof. Let $D=(X, \mathcal{A}, \mathcal{I})$ be an $S_\lambda(t, k, v)$ and $x \in X$. Let $\mathcal{A}'$ be the class of blocks incident with x and $\mathcal{A}''$ be the class of blocks not incident with x. Observe that both $D'=(X', \mathcal{A}', \mathcal{I} \cap (X' \times \mathcal{A}'))$ and $D''=(X', \mathcal{A}'', \mathcal{I} \cap (X' \times \mathcal{A}''))$, where $X' = X - \{x\}$, are $2s$-designs and apply Theorem 1.

The above inequality also rules out infinitely many parameters for which b_i's are integers, $i=0, 1, \cdots, t$.

Theorem 2. *Let* $D=(X, \mathcal{A}, \mathcal{I})$ *be an* $S_\lambda(t, k, v)$ *where* $t=2s$ *and* $v \geq k+s$. *If there exists a partition* $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \cdots \cup \mathcal{A}_r$ *such that each substructure* $(X, \mathcal{A}_i, \mathcal{I} \cap (X \times \mathcal{A}_i))$ *is an* $S_{\lambda_i}(s, k, v)$ *for some positive integers* λ_i, *then*

$$b = |\mathcal{A}| \geq \binom{v}{s} + r - 1 .$$

Proof. With the notation of Theorem 1, the vectors $\{\hat{A}: A \in \mathcal{A}\}$ span V. But observe that

$$\sum \{\hat{A}: A \in \mathcal{A}_i\} = \lambda_i \sum (S: S \in \mathcal{P}_s(X)) = \lambda_i \hat{X}, \text{ say.}$$

So if we choose one block A_i from each $\mathcal{A}_i$, then $\{\hat{A}: A \in \mathcal{A} - \{A_1, \cdots, A_r\}\} \cup \{\hat{X}\}$ spans V. The stated inequality follows.

3. Tight t-designs

Recall that a *tight* t-design ($t=2s$) is an $S_\lambda(t, k, v)$ with $v \geq k+s$ and

$$b = \lambda \binom{v}{t} \Big/ \binom{k}{t} = \binom{v}{s}.$$

In view of Theorem 1, tight designs are simple. In this section we extend the well known result that two distinct blocks of a symmetric design (tight 2-design) have exactly λ common incident points (see Theorem 4 below).

Theorem 3. *Let X be a v-set and $\mathcal{A}$ a class of k-subsets of X such that for distinct $A, B \in \mathcal{A}$,*

$$|A \cap B| \in \{\mu_1, \mu_2, \cdots, \mu_s\}$$

where $k > \mu_1 > \mu_2 > \cdots > \mu_s \geq 0$. Then

$$|\mathcal{A}| \leq \binom{v}{s}.$$

Proof. Let $V = V(\mathcal{A})$. For each $S \in \mathcal{P}_s(X)$, define a vector

$$\bar{S} = \sum (A: A \in \mathcal{A}, A \supseteq S) .$$

We claim that the vectors $\{\bar{S}: S \in \mathcal{P}_s(X)\}$ span V. Since V has dimension $|\mathcal{A}|$, the theorem will follow.

Write $\mu_0 = k$. Let $A_0 \in \mathcal{A}$ be given. Define

$$H_i = \sum (B: B \in \mathcal{A}, |B \cap A_0| = \mu_i)$$

for $i=0, 1, \cdots, s$ (note $H_0 = A_0$). For $r=0, 1, \cdots, s$, we see that

$$G_r = \sum (\bar{S}: S \in \mathcal{P}_s(X), |S \cap A_0| = r) = \sum_{i=0} \binom{\mu_i}{r} \binom{k-\mu_i}{s-r} H_i ,$$

by comparing the coefficient of each $A \in \mathcal{A}$ on both sides of the equation. We now show that the coefficient matrix of this system of $s+1$ linear equations is nonsingular, so that we can solve for the H_i's in terms of the G'_rs. In particular, we then have $H_0=A_0\in\text{span}\ \{G_0, G_1, \cdots, G_r\}\subseteq\text{span}\ \{\bar{S}: S\in\mathcal{P}_s(X)\}$.

So consider the $s+1$ row vectors

$$v_r=\left(\binom{\mu_0}{r}\binom{k-\mu_0}{s-r}, \binom{\mu_1}{r}\binom{k-\mu_1}{s-r}, \cdots, \binom{\mu_s}{r}\binom{k-\mu_s}{s-r}\right),$$

$r=0, 1, \cdots, s$. Suppose $c_0v_0+c_1v_1+\cdots+c_sv_s=0$. This means that the polynomial

$$p(x)=\sum_{r=0}^{s} c_r\binom{x}{r}\binom{k-x}{s-r}$$

of degree $\leq s$ has $s+1$ distinct roots $\mu_0, \mu_1, \cdots, \mu_s$ and hence is the zero polynomial. Now $p(0)=c_0\binom{k}{s}$, so $c_0=0$; then $p(1)=c_1\binom{k-1}{s-1}$, so $c_1=0$; and, inductively, $c_0=c_1=\cdots=c_s=0$. That is, $v_0, \cdots, v_s$ are linearly independent. This completes the proof.

Theorem 4. *Let $D=(X, \mathcal{A}, \mathcal{I})$ be an $S_\lambda(t, k, v)$ with $t=2s$ and $v\geq k+s$. Then there are at least s distinct elements in the set*

$$\{|(A)\cap(B)|: A\in\mathcal{A}, B\in\mathcal{A}, A\neq B\},$$

and there are exactly s distinct elements if and only if D is a tight t-design.

Proof. In view of Theorems 1 and 3, it remains only to show that for any tight t-design, there exist s integers $\mu_1, \mu_2, \cdots, \mu_s$ with $0\leq\mu_i<k$ so that $|(A)\cap(B)|\in\{\mu_1, \cdots, \mu_s\}$ for distinct blocks A and B. Let $D=(X, \mathcal{A}, \mathcal{I})$ be a tight $S_\lambda(t, k, v)$. With the notation of Theorem 1, the $b=\binom{v}{s}$ vectors $\{\hat{A}: A\in\mathcal{A}\}$ must, since they span V_s, be a basis for V_s.

Fix $A_0\in\mathcal{A}$ and for $B\in\mathcal{A}$, write $\mu_B=|(B)\cap(A_0)|$. For $i=0, 1, \cdots, s$, define vectors

$$M_i=\sum(S: S\in\mathcal{P}_s(X), |S\cap(A_0)|=i),$$
$$N_i=\sum\left(\binom{\mu_B}{i}\hat{B}: B\in\mathcal{A}\right).$$

Now given $S\in\mathcal{P}_s(X)$ with $|S\cap(A_0)|=i$, the coefficient of S in the sum N_r is

$$\sum\left(\binom{\mu_B}{r}: B\in\mathcal{A}, S\subseteq(B)\right),$$

i.e., the number of ordered pairs (B, R) in $\mathcal{A}\times\mathcal{P}_r(X)$ such that $S\subseteq(B)$ and $R\subseteq(A_0)\cap(B)$. For any r-subset $R\subseteq(A_0)$ with $|R\cap S|=j$, the number of blocks B such that (B, R) satisfies the above conditions is b_{s+r-j}. Thus the coefficient of S in N_r is

$$c_r^i = \sum_{j=0}^{i}\binom{i}{j}\binom{k-i}{r-j}b_{s+r-j}; \text{ and so}$$

$$N_r = \sum_{i=0}^{s} c_r^i M_i \qquad (r = 0, 1, \cdots, s).$$

The $s+1$ vectors $N_r - c_r^s M_s$ are contained in the span of $M_0, M_1, \cdots, M_{s-1}$; hence there exist rationals $a_0, a_1, \cdots, a_s$, not all zero, such that

$$\sum_{r=0}^{s} a_r(N_r - c_r^s M_s) = 0, \quad \text{or}$$

$$\sum_{r=0}^{s} a_r \sum_{B\in\mathcal{A}}\left(\binom{\mu_B}{r}\hat{B} - c_r^s \hat{A}_0\right) = 0.$$

Now $\{\hat{A}: A\in\mathcal{A}\}$ is a basis for V_s, so for $B\neq A_0$, the coefficient

$$\sum_{r=0}^{s} a_r\binom{\mu_B}{r}$$

of $\hat{B}$ must be 0. That is, for any $B\neq A_0$, the intersection number μ_B is a root of the polynomial

$$f(x) = \sum_{r=0}^{s} a_r\binom{x}{r}$$

of degree at most s. Finally, note that the coefficients c_r^i are (and hence $f(x)$ can be chosen to be) independent of the block A_0: all intersection numbers are roots of $f(x)$.

The polynomials $f(x)$ described in the proof of Theorem 4 have been found explicitly by P. Delsarte [1]. As an example, we consider the case $t=4$. The equations of Theorem 4 are

$$N_0 = b_2M_0 + b_2M_1 + b_2M_2,$$
$$N_1 = kb_3M_0 + (b_2 + (k-1)b_3)M_1 + (2b_2 + (k-2)b_3)M_2,$$
$$N_2 = \binom{2}{k}b_4M_0 + \left(\binom{k-1}{2}b_4 + (k-1)b_3\right)M_1 + \left(\binom{k-2}{2}b_4 + 2(k-2)b_3 + b_2\right)M_2.$$

Using the relation $b_2 = \binom{k}{2}$ in a tight 4-design, one verifies that

$$(b_2-b_3)N_2 - (k-1)(b_3-b_4)N_1 + (2b_3(b_3-b_4) - b_4(b_2-b_3))N_0$$

is a scalar multiple of $M_2 = \hat{A}_0$. For a block $B\neq A_0$, the coefficient of $\hat{B}$ in the above expression must be zero, i.e.,

$$\mu_B(\mu_B - 1) - \frac{2(k-1)(b_3-b_4)}{(b_2-b_3)}\mu_B + \frac{4b_3(b_3-b_4)}{(b_2-b_3)} - 2b_4 = 0.$$

Rewriting the coefficients in terms of v, k, and λ, we have

Theorem 5. *The two "intersection numbers" μ_1, μ_2 of a tight 4-design $S_\lambda(4, k, v)$ are the roots of the polynomial*

$$f(x) = x^2 - \left(\frac{2(k-1)(k-2)}{(v-3)} + 1\right)x + \lambda\left(2 + \frac{4}{k-3}\right).$$

Application of Theorem 5 yields the well known fact that any two distinct blocks of an $S_1(4, 7, 23)$ meet in 1 or 3 points.

Since $f(x)$ has integral roots, it must have integral coefficients, and we have the

Corollary. *The existence of a tight 4-desigh $S_\lambda(4, k, v)$ implies $v-3$ divides $2(k-1)(k-2)$, and $k-3$ divides 4λ.*

In [1], Delsarte observes that Theorems 4 and 5 are similar to Lloyd's Theorem on perfect codes. Indeed, Delsarte develops a theory of designs and codes (emphasizing a "formal duality") in the context of association schemes. Contained therein are results analogous to the above for orthogonal arrays of strength t, the analogue of Theorem 1 being Rao's bound.

We conclude with the following remarks.

Let $D=(X, \mathcal{A}, \mathcal{I})$ be a tight $S_\lambda(t, k, v)$ with $t=2s$ and $v \geq k+s$. Let $J(s, v)$ denote the association scheme whose points are the s-element subsets of X (see [1]). Let N be a $(0-1)$-matrix whose rows are indexed by elements of $\mathcal{P}_s(X)$ and columus are indexed by the blocks of D. At the row corresponding to S and column corresponding to a block A, the entry of N is 1 iff $S \subseteq (A)$. The matrix NN^T belongs to the Bose-Mesner algebra of the scheme $J(s, v)$. The matrix NN^T is obviously rationally congruent to the identity matrix. Using the properties of the algebra of $J(s, v)$, it is possible to compute the Hasse-Minkowski invariant of NN^T and obtain some more necessary conditions for the existence of tight $2s$-designs. (See also [5].)

THE OHIO STATE UNIVERSITY

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UNIQUENESS OF ASSOCIATION SCHEMES (*)

Riassunto. — Sia V un insieme finito. Una relazione binaria simmetrica su V è una funzione $R : V \times V \rightarrow \{0, 1\}$ dove $R(x, y) = R(y, x)$ $\forall x, y \in V$. Tale relazione può essere vista come una matrice $(v \times v)$ simmetrica di zeri e uni dove v è il numero di elementi dell'insieme V. Siano $v, m, p^i_{jk}, i, j, k = 0, 1, \cdots, m$, interi non negativi. Uno schema di associazione $\mathcal{A}$ con parametri $(v, m, p^i_{jk}, i, j, k = 0, 1, \cdots, m)$ consiste di un insieme finito V e di relazioni binarie simmetriche non nulle $R_0, R_1, \cdots, R_m$ su V tali che R_0 sia la relazione identità e

(i) $\sum_{i=0}^{m} R_i = J$ e

(ii) $\forall j, k = 0, 1, 2 \cdots m, \qquad R_j R_k = \sum_{i=0}^{m} p^i_{jk} R_i$

dove $J(x, y) = 1$, per tutti gli $x, y \in V$. Il presente lavoro descrive vari risultati riguardanti certe famiglie di schemi di associazione che sono caratterizzati da pochi dei loro parametri.

1. Graphs, Incidence structures and Association schemes

A graph G is a triple (V, E, I) where V and E are disjoint sets and I is a mapping from E to the subsets of vertices such that for all $e \in E$, $I(e)$ contains at most two elements. Elements of V and E are respectively called vertices and edges. For an edge e, the vertices of the set $I(e)$ are called the ends of e. The edge e is said to be joining its ends together. An edge e with only one end is called a loop. If two edges have the same set of ends p, then they are called parallel edges or multiple edges. A graph without loops and multiple edges is called a simple graph. The degree (or valence) of a vertex v in a simple graph is the number of edges e which have v as an end. If all vertices have the same degree, then the graph is said to be regular. Two vertices are said to be adjacent iff there exists an edge joining them. The complete graph K_v is a simple graph on v vertices in which any two distinct vertices are adjacent. A path P is an ordered tuple $(v_0, e_1, v_1, e_2, v_2, \cdots, e_n, v_n)$ such that for $i = 1, \cdots, n$, v_{i-1} and v_i are the ends of e_i. The path P is said to join the vertices v_0 and v_n. The integer n is the length of the path n. The graph is said to be connected iff there exists a path joining any two vertices of the graph. The distance between two vertices x and y is the smallest integer n

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for which there exists a path of length n joining x and y. An incidence structure π is a triple (P, B, I) where P and B are disjoint sets and $I \subseteq P \times B$. The elements of P and B are respectively called points (or treatments) and lines (or blocks). For $p \in P$ and $b \in B$ if $(p, b) \in I$, we say that the point p and the block b are mutually incident and the ordered pair (p, b) is called a flag. The flag graph of π is a simple graph whose vertices are the flags of π and two flags (p, b) and (p', b') are adjacent iff either $p = p'$, $b \neq b'$ or $p \neq p'$ and $b = b'$. Let v, k, λ be positive integers. A (v, k, λ)-balanced incomplete block design (bibd) is an incidence structure π with v points such that every block is incident with exactly k points and for any two distinct points x and y, there are exactly λ blocks incident with both x and y. If moreover the number of blocks is equal to the number of points, then the b i b d is called a symmetric b i b d.

Association schemes were implicitly considered by Bose and Nair in [5]. Association schemes were explicitly introduced by Bose and Shinamato [6]. Let V be a finite set. A binary symmetric relation on V is a mapping $R: V \times V \to \{0, 1\}$ where $R(x, y) = R(y, x)\ \forall x, y \in V$. Such a relation can be viewed as a $(v \times v)$-symmetric $0 - 1$ matrix where v is the number of elements of the set V. Let $v, m, p^i_{jk}, i, j, k = 0, 1, \cdots, m$, be non negative integers. An association scheme $\mathscr{A}$ with parameters $(v, m, p^i_{jk}, i, j, k = 0, 1, \cdots, m)$ consists of a finite set V and non null binary symmetric relations $R_0, R_1, \cdots, R_m$ on V such that R_0 is the identity relation and

$$\text{(i)} \quad \sum_{i=0}^{m} R_i = J \text{ and}$$

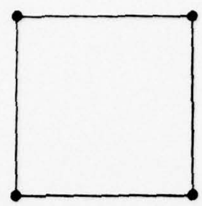
$$\text{(ii)} \quad \forall j, k = 0, 1, \cdots, m, \quad R_j R_k = \sum_{i=0}^{m} p^i_{jk} R_i$$

where $J(x, y) = 1$, for all $x, y \in V$.

Elements of V are called vertices or treatments. If $R_i(x, y) = 1$, then we say that x and y are ith associates, $j = 0, 1, \cdots, m$. The condition (i) states that for any two vertices x and y, there exists exactly one integer i such that $0 \leq i \leq m$ and x and y are ith associates. For two vertices x and y and $j, k = 0, 1, \cdots, m$, let $p_{jk}(x, y)$ denote the number of vertices z such that z and x are jth associates and z and y are kth associates. The matrices $R_1, \cdots, R_m$ define an edge coloring of K_v, the complete graph on v vertices by m colors. If the vertices x and y are ith associates, then the edge joining them is colored by the ith color, $i = 1, 2, \cdots, m$. The graph consisting of the edges of the ith color is called the ith associate graph. The parameters of an association scheme are not all independent. For instance for a 2-class scheme, it is sufficient to specify the 4 parameters v, p^0_{11}, p^1_{11} and p^2_{11}. Graphs of the first associates in a 2-class scheme are also called strongly regular graphs. A strongly regular graph with parameters $(v, p^0_{11}, p^1_{11}, p^2_{11})$ contains v vertices such that (1) every vertex is incident with p^0_{11} edges, (2) for any two adjacent vertices x and y, there are exactly p^1_{11} vertices z which are adjacent to both x and y and (3) for any two nonadjacent vertices x and y there are exactly p^2_{11} vertices z

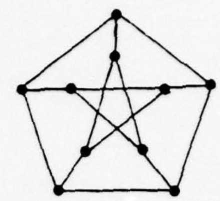
which are adjacent to both x and y. We give below some examples of strongly regular graphs.

Example 1



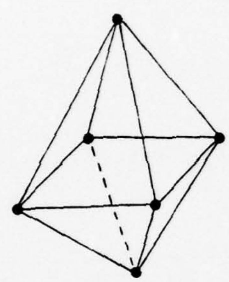
parameters (4, 2, 0, 1)

Example 2



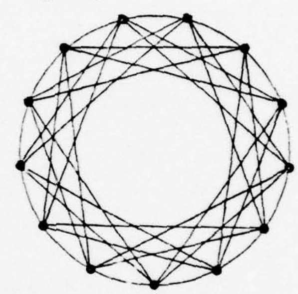
parameters (10, 3, 0, 1)

Example 3



parameters (6, 4, 2, 4)

Example 4



In example 4 vertices are the elements of GF (13). Two elements x and y are adjacent iff $x - y$ is a square in GF (13).

Association schemes are readily available in nature. We only describe a few infinite families of association schemes. An exhaustive survey of 2-class schemes (or strongly regular graphs) can be found in [7].

(1) Three class schemes of projective planes. Let π be a finite projective plane of order n. Let G (π) be the flag graph of π. We define two vertices to be ith associates iff the distance between them in G (π) is i where i is a non negative integer. This definition of the association relations satisfy the properties of a 3-class association scheme $\mathscr{A}(\pi)$ with parameters

$$\begin{array}{llll} & v = (n+1)(n^2+n+1)\,, & p^0_{11} = 2n\,, & p^0_{22} = 2n^2\,, \\ (1) & p^1_{11} = n - 1\,, & p^1_{12} = n\,, & p^1_{22} = n(n-1)\,, \\ & p^2_{11} = 1\,, & p^2_{12} = n - 1\,, & p^2_{22} = n\,, \\ & p^3_{11} = 0\,, & p^3_{12} = 2 \quad \text{and} & p^3_{22} = 4(n-1)\,. \end{array}$$

The remaining parameters of the scheme can be expressed in terms of the parameters given above. Conversely it can be shown that for any association scheme $\mathscr{A}$ with parameters given in (1), there exists a projective plane π of order n such that $\mathscr{A}$ and $\mathscr{A}(\pi)$ are isomorphic. In other words an association scheme

$\mathcal{A}$ with parameters (1) is really nothing but the projective plane π. It is interesting to note that the association scheme $\mathcal{A}$ does not distinguish between points and lines of the projective plane.

(2) Three class schemes of symmetric balanced incomplete block designs (bibd). Let $v' > k > \lambda > 0$ be integers. Consider a symmetric b i b d π with parameters (v', k, λ). Let $G(\pi)$ be a bipartite graph whose vertices are points and lines of π and two vertices are adjacent iff one of them is a treatment and the other is a block incident with the treatment. Two vertices x and y are defined to be ith associates iff the distance between x and y in $G(\pi)$ is i where i is a non negative integer. The association relations so defined produce a 3-class association scheme $\mathcal{A}(\pi)$ with parameters

$$
\begin{aligned}
&v = 2v', \quad p^0_{11} = k, \quad p^0_{22} = v' - 1, \\
&p^1_{11} = 0, \quad p^1_{12} = k - 1, \quad p^1_{22} = 0, \\
&p^2_{11} = \lambda, \quad p^2_{12} = 0, \quad p^2_{22} = v' - 2, \\
&p^3_{11} = 0, \quad p^3_{12} = k \text{ and } p^3_{22} = 0.
\end{aligned}
\tag{2}
$$

Conversely if $\mathcal{A}$ is a 3-class scheme with parameters given by (2), then there exists a (v', k, λ)-symmetric bidb π such that $\mathcal{A}$ and $\mathcal{A}(\pi)$ are isomorphic.

(3) Association schemes of the projective spaces. Let m and d be positive integers satisfying $m \leq d/2$ and $d \geq 4$. Let π denote the projective space $PG(d-1, q)$. Construct an association scheme with $(m-1)$-flats as the vertices. Two $(m-1)$-flats are ith associates iff their intersection is an $(m-1-i)$-flat, $i = 0, 1, \cdots, m$. The association relations so defined satisfy the properties of an m-class scheme. This scheme will be denoted by $P(m, q, d)$. This scheme can be described in terms of $G(\pi)$, the graph of the first associates. Two vertices are ith associates iff the distance between them in $G(\pi)$ is i, $i = 0, 1, 2, \cdots, m$. The graph of the first associates of $P(2, q, d)$ is also called the line graph of $PG(d-1, q)$.

(4) Association schemes of the restriction of projective spaces. Let m and d be positive integers satisfying $m \leq \frac{m+d}{2}$. Let π be the projective space $PG(m+d-1, q)$. Let Σ_{d-1} be a $(d-1)$-flat of π. Construct an association scheme whose vertices are the $(m-1)$-flats, which do not intersect Σ_{d-1}. Two flats are ith associates iff they intersect in an $(m-1-i)$-flat, $i = 0, 1, \cdots m.$, The association relations so defined satisfy the properties of an m-class scheme. This scheme is denoted by $R(m, q, d)$. $R(m, q, d)$ can be described in another way. Let V be a vector space of dimension d over $GF(q)$. The vertices of $R(m, q, d)$ are m-tuples $(x_1, x_2, \cdots, x_m)$ belonging to $\underbrace{V \times V \times \cdots \times V}_{m}$. Two m-tuples $(x_1, x_2, \cdots, x_m)$ and $(y_1, y_2, \cdots, y_m)$ are ith associates iff the dimension of the subspace spanned by the vectors $x_1 - y_1, \cdots, x_m - y_m$ is i, $i = 0, 1, \cdots, m$. $R(m, q, d)$ also can be described in terms of its graph of the first associates G. Two vertices are ith associates iff the distance between them in G is i, $i = 0, 1, \cdots, m$.

2. Study of association schemes

There had been three kinds of investigations about association schemes; (1) non existence of schemes with certain parameters, (2) construction of schemes and (3) uniqueness of certain schemes.

Bose and Mesner [4] introduced the algebra of the association matrices. Let $\mathscr{A}$ be an association scheme with association matrices $R_0, R_1, \cdots, R_m$ and parameters $(v, m, p^i_{jk}, i, j, k = 0, 1, \cdots, m)$. Consider the set of matrices $\sum_{i=0}^{m} c_i R_i$ where c_i's are arbitrarily chosen rational coefficients. From the defining properties it is easily seen that this set is closed under addition and multiplication. Therefore we get an algebra of matrices called the association algebra. Let $P_k = ((p^j_{ik}))$ be an $(m+1)\times(m+1)$-matrix whose entry in the ith row and jth column is the parameter p^j_{ik}. It can be seen that the parameter matrices $P_0, P_1, \cdots, P_m$ generate an algebra over the rationals which is isomorphic to the association algebra. Let $R = \sum_{i=0}^{m} c_i R_i$ and $P = \sum_{i=0}^{m} c_i P_i$. The matrices R and P have the same minimum polynomials and the same set of distinct eigen values $\theta_0, \theta_1, \cdots, \theta_u, u \leq m$. Let α_i be the multiplicity of the eigen value θ_i in the matrix $R, i = 0, 1, \cdots, u$. For any integer p, matrix R^p can be expressed as a linear combination $\sum_{i=0}^{m} c_{pi} R_i$ where c_{pi}'s depend on c_i's and the parameters of the scheme. Computing trace R^p in two different ways, we get the equation

$$\sum_{i=0}^{m} \alpha_i \theta_i^p = v c_{p0}, \quad p = 0, 1, \cdots, u. \tag{3}$$

In the equations (3), all quantities except $\alpha_0, \alpha_1, \cdots, \alpha_u$ can be computed explicitly as functions of the parameters of the scheme. Hence a necessary condition for the existence of a scheme with parameters $(v, m, p^i_{jk}, i, j, k = 0, 1, \cdots, m)$ is that the equations (3) have integral solutions for the unknowns $\alpha_0, \alpha_1, \cdots, \alpha_u$. This necessary condition is a very strong condition and eliminates many parameter sets. Since the association algebra is commutative, the algebra of the parameter matrices is also commutative. Therefore the parameter matrices commute pairwise. The commutativity of the parameter matrices also imply several relations among the parameters. For instance some necessary conditions are

$$\sum_{k=0}^{m} p^i_{jk} = p^0_{jj}, \; p^0_{ii}\, p^i_{jk} = p^0_{jj}\, p^j_{ik}. \quad \forall i, j, k = 0, 1, \cdots, m. \tag{4}$$

The algebra of association matrices had been used successfully to prove the nonexistence of Moore graphs of diameter greater than 2. The diameter of a connected graph is the largest possible distance between two vertices of the graph. The girth of a graph is the smallest possible number of edges in a polygon of the graph if such a polygon exists. A Moore graph of diameter k and valence k is a graph with valence d, diameter k and girth $(2k+1)$. In such a graph we can define two vertices to be ith associates iff the distance between them is i, $i = 0, 1, \cdots, k$. This defines a k-class association scheme. Hoffman and Singleton [16] exploited the association algebra to prove that Moore graphs of valence $d > 2$ and diameter 3 do not exist. Vijayan [23], Bannai and Ito [1] and Damerell [10] used the association algebra to prove the following theorem.

THEOREM 1. *For $d > 2$, $k \geq 3$, Moore graphs of valence d and girth $(2k+1)$ do not exist.*

3. Uniqueness of association schemes

Study of uniqueness of association schemes was started by Connor [9] in connection with the triangular scheme. Shrikhande [19] did pioneering work in proving the uniqueness of the L_2-scheme and Bruck [8] in a certain sense proved the uniqueness of the L_r-scheme. Bose [2] generalized the methods of these workers and proved an important theorem for partial geometries. Let r, k and t be positive integers. An (r, k, t)-partial geometry π is an incidence structure of points and lines such that (1) every line is incident with exactly k points, (2) every point is incident with exactly r lines, (3) two distinct points are incident with at most one common line and (4) given a point p and a nonincident line l, there are exactly t lines which are incident with p and also a point of l.

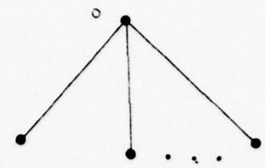
It is easy to see that the dual of an (r, k, t)-partial geometry is a (k, r, t)-partial geometry. For a partial geometry π, we define a simple graph $G(\pi)$ whose vertices are the points of π and two points are adjacent in the graph iff there is a line in π incident with both the points. $G(\pi)$ is a strongly regular graph with parameters

$$v = k/t\,((r-1)(k-1)+t)\ ,\quad p^0_{11} = r(k-1)\,,$$
$$p^1_{11} = (k-2)+(r-1)(t-1) \quad \text{and} \quad p^2_{11} = rt \tag{5}$$

$G(\pi)$ is called an (r, k, t)-geometric strongly regular graph. A strongly regular graph with parameters given in (5) is called an (r, k, t)-pseudogeometric graph.

THEOREM 2. (Bose [2]): *Let r, k and t be positive integers satisfying $k > \frac{1}{2}(r(r-1)+t(r+1)(r^2-2r+2))$ and G be an (r, k, t)-pseudogeometric strongly regular graph. Then there exists a unique (r, k, t)-partial geometry π such that G and $G(\pi)$ are isomorphic.*

The concepts of claw and clique play an important role in the proof of Bose's theorem. An s-claw of G is an ordered pair $(0, U)$ where U is a set of s-vertices such that no two vertices of U are pairwise adjacent and o is a vertex adjacent to all vertices of U.



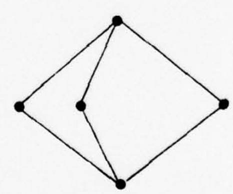
A clique is a subset of vertices such that any two are pairwise adjacent. Let (o , U) be an $(r - 1)$ - claw of G and $f(i)$ be the number of vertices x of G which are adjacent to o and also adjacent to exactly i of the $(r - 1)$-vertices in the set U. Using the parameters, one gets bounds for the power sums $\Sigma\, i^t f(i)$ for $t = 0 , 1 , 2$ which in turn gives information about the frequencies $f(0)$ and $f(1)$. This method is commonly known as the method of moments. One then proves that G contains no $(r + 1)$-claw and that every pair of adjacent vertices is contained in a maximal clique of size at least $k - (r - 1)^2 (t - 1)$. Such cliques are called grand cliques. After some simple manipulations one proves that (i) these grand cliques have size exactly equal to k, (ii) every vertex is contained in exactly r grand cliques, (iii) every pair of adjacent vertices is contained in a unique grand clique and (iv) given a clique C and a vertex o not in C, exactly t vertices of C are adjacent to o. Hence if one takes the vertices of G to be points and the grand cliques as lines, one easily gets an (r , k , t)-partial geometry π with G (π) isomorphic to G.

Recently Bose, Shrikhande and Singhi [7] made an important generalization of Theorem 2. Consider an incidence structure (P , B , I) where P and B are disjoint sets and $I \subseteq P \times B$. Elements of P and B are respectively called points and blocks. For two points p and p', let $m(p , p')$ denote the number of blocks b incident with both p and p'. For $p \in P$, $b \in B$, define $n(p , b) = \Sigma m(p , p')$ where the sum is over all points p' incident with b. Let r , k , t and c be non negative integers. An incidence structure π of points and blocks is said to be an (r , k , t , c)-partial geometric design iff (1) every block is incident with exactly k points, (2) every point is incident with exactly r blocks, (3) for an incident pair (p , b) , $n(p , b) = r + k - 1 + c$ and (4) for a nonincident pair (p , b) , $n(p , b) = t$. It is easy to see that an $(r, k , t , 0)$-partial geometric design is an (r , k , t)-partial geometry. Define a graph $G(\pi)$ whose vertices are the points of π and two points p and p' are joined by $m(p , p')$ distinct edges. The graph $G(\pi)$ is called an (r , k , t , c)-geometric graph and possesses some " regularity properties ". An arbitrary graph G with similar " regularity properties " is called an (r , k , t , c)-pseudo-geometric graph. Bose, Shrikhande and Singhi proves that if k is greater than a certain function of r , t and c, then every (r , k , t , c)-pseudogeometric graph is an (r , k , t , c)-geometric graph.

Bose's theorem played an important role in the development of the subject. A theorem of the present author and a theorem of Alan J. Hoffman also played important roles. Let G be a simple graph, i.e. a graph without loops and multiple edges. The line graph L (G) is a graph whose vertex set is the edge set of G. Two vertices e and e' of L (G) are adjacent if the corresponding edges of G have a common incident vertex. For a simple graph G with v vertices, the adjacency matrix of G is a $(v \times v)$-matrix $A = ((a_{ij}))$ where $a_{ij} = 1$ (0) iff the ith vertex and the jth vertex are adjacent (not adjacent), $i, j = 1, 2, \cdots, v$. The eigen values of the adjacency matrix are called the eigen values of the graph.

THEOREM 3. (Ray-Chaudhuri [18], Characterization of line graphs). *Let* G *be a finite simple graph such that the number of edges of* G *is greater than the number of vertices of* G. *Then the minimum eigen value of* L (G) *is* -2. *Conversely, let H be a simple graph with the minimum eigen value equal to* -2, *the minimum valence not less than* 46 *and the property that for any two adjacent vertices x and y, there are at least two distinct vertices z and z' adjacen to x and not adjacent to y. Then there exists a simple graph* G *with* L (G) *isomorphic to* H.

Proof of the first part of the theorem is easy. The proof of the converse part of the theorem uses some interesting ideas. It is easily seen that a line graph L (G) has a class of cliques $\mathscr{C}$ such that every vertex of L (G) is contained in exactly two cliques of $\mathscr{C}$ and every edge is contained in exactly one clique of $\mathscr{C}$. Conversely one can show that if a simple graph H contains such a class of cliques $\mathscr{C}$, then H will be a line graph. To build the class of cliques $\mathscr{C}$ in H, first one shows that H does not contain a 3-claw. Since the minimum eigen value of H is —2, many graphs can not occur as induced subgraphs of G. To give an example, H can not contain the graph



As an induced subgraph.

A graph H is said to be an induced subgraph of G iff V (H) $\subseteq$ V (G) and every edge of G with ends belonging to V(H) is an edge of H. Suppose the minimum eigen value of a graph F is smaller than -2. If F were an induced subgraph of H, then by the minimum principle the minimum eigen value of H will be strictly smaller than -2. One builds up a list of inadmissible subgraphs for H and uses these subgraphs to prove the nonexistence of a 3-claw.

Disjoint unions of the complete graph K_v is easily seen to be a strongly regular graph. The class of these graphs and their compliments is called the class of trivial strongly regular graphs. Hoffman proved the following important theorem.

THEOREM 4. (Hoffman [13]). *Let m be a positive integer. Then there exists a function $f(m)$ such that for every non trivial strongly regular graph* G *with minimum eigen value $-m$, the parameter p_{11}^2 is not greater than $f(m)$.*

Using the theorems stated above, one can prove the following Theorem.

THEOREM 5. *Let m be a positive integer and* G *be a non trivial strongly regular graph with minimum eigen value equal to $-m$. Then with finitely many exceptions* G *is either an (m, k, m)-pseudogeometric strongly regular graph or an $(m, k, m-1)$-pseudogeometric strongly regular graph where k is a positive integer depending on* G.

Using the relations (3), one can see that the three parameters p_{11}^0, p_{11}^1 and p_{11}^2 determine the parameter v for a $(v, p_{11}^0, p_{11}^1, p_{11}^2)$-strongly regular graph. For a fixed parameter triple $(p_{11}^0, p_{11}^1, p_{11}^2)$, there are only finitely many nontrivial strongly regular graphs. Using the association algebra, one can show that the eigen values of G are p_{11}^0, $-m$ and $p_{11}^1 - p_{11}^2 + m$. Also denoting by r the multiplicity of the eigen value $p_{11}^1 - p_{11}^2 + m$, we get the the following two equations

$$(6) \qquad m^2 + (p_{11}^1 - p_{11}^2)\, m + p_{11}^2 = p_{11}^0$$

$$p_{11}^2\, r\,(p_{11}^1 - p_{11}^2 + 2m) = (m-1)\, p_{11}^0\, (m p_{11}^1 - (m-1)\, p_{11}^2 + m(m+1)).$$

Given m, p_{11}^2 and p_{11}^1, the equation (6) determines p_{11}^0. Therefore there are finitely many nontrivial strongly regular graphs with fixed values for the minimum eigen value $-m$ and the parameters p_{11}^1 and p_{11}^2. By Hoffman's theorem, the parameter p_{11}^2 of G will be bounded by a function $f(m)$. Hence we need to consider only finitely many values for the parameter p_{11}^2. Suppose we fix m and p_{11}^2 and try to find the possible values for p_{11}^1. Let x be the unknown value of p_{11}^1. From the equations (6), we can derive the equation

$$(7) \qquad p_{11}^2\, r\,(x - p_{11}^2 + 2m) = (m-1)\,(mx - (m-1)\, p_{11}^2 + m^2) \times$$
$$(mx - (m-1)\, p_{11}^2 + m\,(m+1)).$$

We use the fact that r is a positive integer. If $p_{11}^2 \neq m^2$ or $m(m-1)$, the equation (7) leads to finitely many possible values for p_{11}^1. Hence if $p_{11}^2 \neq m^2$ or $m(m-1)$, there are finitely many nontrivial strongly regular graphs with fixed values for the parameters $-m$ and p_{11}^2.

Suppose $p_{11}^2 = m^2$. If we let $k = p_{11}^1 + 2 - (m-1)^2$, then a strongly regular graph with parameters $(m, p_{11}^2 = m^2, p_{11}^1)$ is an (m, k, m)-pseudogeometric strongly regular graph. Similarly if $p_{11}^2 = m(m-1)$, we let $k = p_{11}^1 + 2 - (m-1)(m-2)$. A non trivial strongly regular graph with parameters $(m, p_{11}^2 = m(m-1), p_{11}^1)$ is then an $(m, k, m-1)$-pseudogeometric strongly regular graph.

C. C. Sims [20] proved a beautiful characterization theorem for a large class of strongly regular graphs. We define 4 classes of strongly regular graphs as follows.

(1) Line graph of the complete graph on n vertices where n is a positive integer. This graph is a $(2, n-1, 2)$-geometric strongly regular graph with parameters $v = \binom{n}{2}$, $p^0_{11} = 2(n-2)$, $p^1_{11} = n-2$ and $p^2_{11} = 4$.

(2) Line graph of the complete bipartite graph on $n + n$ vertices where n is a positive integer. This graph is a $(2, n, 1)$-geometric strongly regular graph with parameters $v = n^2$, $p^0_{11} = 2(n-1)$, $p^1_{11} = n-2$ and $p^2_{11} = 2$.

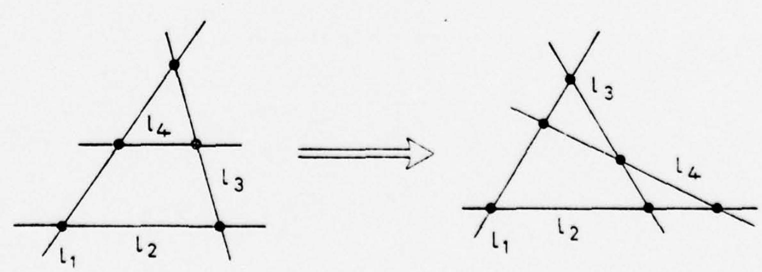
(3) Line graph of the projective space $PG(d-1, q)$ where q is a prime power and d is a positive integer not less than 4. This is the graph of the first associates of the $P(2, q, d)$-scheme. This graph is a $(q+1, k, q+1)$-geometric strongly regular graph with $k = \frac{q^{d-1}-1}{q-1}$.

(4) Graph of the first associates of the $R(2, q, d)$-scheme where q is a prime power and d is a positive integer not less than 2. This graph is a $(q+1, k, q)$-geometric strongly regular graph with $k = q^{d-1}$.

All the graphs defined above satisfy the 4-vertex condition. Let x and y be two vertices of a strongly regular graph G. Let S_{xy} be the set of vertices z which are adjacent to both x and y. Let $\alpha(x, y)$ denote the number of edges of G with both ends belonging to S_{xy}. The 4-vertex condition requires that there exists two numbers α_1 and α_2 such that if the distance between x and y is i, then $\alpha(x, y) = \alpha_i$, $i = 1, 2$. Actually the graphs of the 4-classes defined above have rank 3 automorphism groups.

THEOREM 6. (C. S. Sims [20]). *Let m be a positive integer greater than* 1. *Then there exists a finite class of graphs $\mathcal{G}$ such that if* G *is a non trivial strongly regular graph with minimum eigen value equal to $-m$ and satisfying the 4-vertex condition, then either* $G \in \mathcal{G}$ *or $m-1 = q$ is a prime power or* 1 *and* G *belongs to one of the 4-classes of graphs defined above.*

Sims proved his theorem under the assumption that G has a rank 3 automorphism group. Higman [12] and also some other workers observed that Sims' proof remains valid under the weaker 4-vertex condition. The proof of Sims' theorem depends on the previous theorems. For the case $m = 2$, the proof comes out fairly quickly as an application of Theorem 3. Using the Theorem 2 and 3, one can see that G can be assumed to be either an (m, k, m)-geometric strongly regular graph or an $(m, k, m-1)$-geometric strongly regular graph where k is some positive integer.



Suppose $G = G(\pi)$ where π is an (m, k, m)-partial geometry. Then π^* the dual of π is a (k, m, m)-partial geometry. The 4-vertex condition for G is used to show that π^* satisfies the Pasche axiom, i.e. if l_1, l_2, l_3 and l_4 are lines such that no three have a common point and 5 pairs of lines intersect, then the 6th pair also must intersect. The condition can be described by the above picture.

It is then easily seen that π^* is a projective space. If $G = G(\pi)$ where π is an $(m, k, m-1)$-partial geometry, then π^* satisfies the Pasche axiom. Sims then essentially shows that π^* is isomorphic to an incidence structure (P, L, I) where P is the set of points of $\Sigma_n - \Sigma_{n-2}$, Σ_n is a projective space of dimension n, Σ_{n-2} is an $(n-2)$-flat of Σ_n, L is the set of lines of Σ_n which do not intersect Σ_{n-2}, incidence relation is that in Σ_n and n is a suitable positive integer. This result shows that if we assume the Pasche axiom, then the projective space Σ_n for $n \geq 3$ can be reconstructed from the lines which do not intersect a distinguished $(n-2)$-flat Σ_{n-2}. This result itself is very interesting. Unfortunately the proof given by Sims is very long. For the case $m = 2$, the 4-vertex condition is not necessary and one can prove the following theorem.

THEOREM 7. *There exists a finite class of graphs $\mathcal{G}$ such that if* G *is a strongly regular graph with* -2 *as the minimum eigen value, then either* $G \in \mathcal{G}$ *or* G *is a line graph of a complete graph or* G *is a line graph of a complete bipartite graph.*

Let G be a finite connected graph. For two vertices x and y, and integers j and k, $p_{jk}(x, y)$ will denote the number of vertices z which have distance j from x and distance k from y. If for all pairs of vertices (x, y) with distance i, the numbers $p_{jk}(x, y)$'s are equal, the common value is denoted by p^i_{jk} and we say that the distance parameter p^i_{jk} exists.

Let n and m be positive integers satisfying the inequality $n \geq 2m$ and X be an n-set. We can define an association scheme with the m-subsets of X as the vertices. Two m element subsets Y and Y′ are defined to be ith associates iff their intersection contains exactly $m - i$ elements, $i = 0, 1, \cdots, m$. Dowling [11] denotes the graph of the first associates of this scheme by G^n_m. Dowling proves that the graph G^n_m (or the corresponding association scheme) can be reconstructed from a few of its properties.

THEOREM 8. (Dowling [11]). *Let n and m be positive integers such that* $n > 2m(m-1)+1$. *Let* G *be a connected graph on* $\binom{n}{m}$ *vertices with distance parameters* $p^0_{11} = m(n-m)$, $p^1_{11} = n-2$ *and* $p_{11}(x, y) \leq 4$ *for all pairs of vertices* (x, y) *with distance more than* 1. *Then* G *is isomorphic to* G^n_m.

Alan Sprague and the author in a certain sense generalized the results of Sims. Our results are about reconstruction of the $P(m, q, d)$ and $R(m, q, d)$-schemes for $m \geq 3$. We can prove that if d is large compared to m and q, then such reconstruction is possible. We do not need to assume all the parameters of the scheme. The existence and correct values for the parameters in the P_1-matrix are sufficient for the purpose of reconstruction of these schemes. The investigation is not complete yet. We think we will be able to reduce the number of parameters considerably. Also we need only assume the existence of the graph of the first associates with certain properties. For the case $m = 3$, we have the following theorems.

THEOREM 9. (Alan Sprague and D. K. Ray-Chaudhuri [21]. *Uniqueness of the* $P(3, q, d)$*-schemes*).

Let q and d be integers satisfying $q \geq 2$, $d \geq 9$ and $(q, d) \neq (2, 9)$. Let $k = \frac{q^d - q^2}{q^3 - q^2}$. Let G be a finite simple connected graph with distance parameters

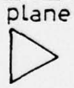
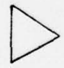
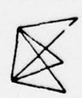
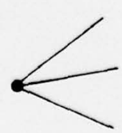
$$p^0_{11} = (q^2 + q + 1)(k - 1) \qquad p^1_{11} = (k - 2) + q^2(q + 1),$$
$$p^2_{11} = (q + 1)^2 \qquad \text{and} \qquad p^2_{31} = q^2(k - q^2 - q - 1).$$

Then q is a prime power and G is isomorphic to the graph of the first associates of $P(3, q, d)$.

THEOREM 10. (Alan Sprague and D. K. Ray-Chaudhuri [22]. *Uniqueness of the* $R(3, q, d)$*-schemes*).

Let q and d be integers satisfying $q \geq 2$, $d \geq 6$, $(q, d) \neq (2, 6)$. Let G be a finite simple connected graph which satisfies the 4-vertex condition and has the distance parameters $p^0_{11} = (q^2 + q + 1)(q^d - 1)$, $p^1_{11} = q^d + q^3 - q - 2$. $p^2_{11} = q^2 + q$, $p^2_{31} = q^{d+2} - q^4$ and $p^3_{21} = q^2(q^2 + q + 1)$. Then G is isomorphic to the graph of the first associates of the $R(3, q, d)$-scheme and q is a prime power.

CORRESPONDENCE BETWEEN OBJECTS IN π AND $G(\pi)$

π	Class of associated flats of π	$G(\pi)$
plane 		vertex
line 	 the pencil of planes containing the line	A clique of size $\frac{q^{d-2} - 1}{q - 1}$
• point	 the pencil of lines containing the point	A class of cliques containing $\frac{q^{d-1} - 1}{q - 1}$ cliques such that any two cliques of the class have exactly one common vertex.

Theorem 8 and Theorem 9 are deep theorems and their proofs are unfortunately very long. I try to give below an idea about the structure of the proof. Let π and G (π) respectively denote the projective space PG ($d-1$, q) and the graph of the first associates of the P (3, q, d)-scheme. The we have the following correspondences.

Given a graph G with the properties stated in Theorem 9, we first prove the existence of large cliques. We prove the existence of a family of cliques of size $\frac{q^{d-2}-1}{q-1}$ such that every edge of the graph is contained in exactly one of these "grand cliques". Next one proves the existence of complexes of cliques. A complex of cliques is a class of $\frac{q^{d-1}-1}{q-1}$ cliques such that any two cliques of the class have exactly one common vertex. Complexes and cliques will respectively correspond to points and lines of the projective space. One proves that the incidence structure of complexes and cliques satisfies the axioms of the projective space. One of the interesting by-product of Theorem 9 is the fact that the projective space can be reconstructed from the incidence structure of the lines and the planes. One needs to assume very few properties of this incidence structure for this reconstruction problem. The proof of Theorem 10 is much more difficult than that of Theorem 9. In fact this is not surprising. It can be easily seen that the R (3, q, d)-scheme in a certain sense is a sub-scheme of the P (3, q, $d+3$)-scheme. Theorem 10 also produces a reconstruction theorem for projective spaces. Let Σ_{d+2} be a finite projective space of dimension ($d+2$) and Σ_{d-1} be a ($d-1$) flat. Let π be the incidence structure of the lines and planes of Σ_{d+2} which do not intersect Σ_{d-1}. It can be shown that a few properties of the incidence structure π are sufficient for the reconstruction of the projective space.

In the proofs of Theorems 9 and 10, to prove the existence of large cliques we apply the Bose-Laskar theorem. The theorem of Bose and Laskar shows that if in a graph G the parameters p^0_{11} and p^1_{11} exist and the remaining numbers $p_{11}(x, y)$ are not too large, then one can prove the existence of a nice family of large cliques.

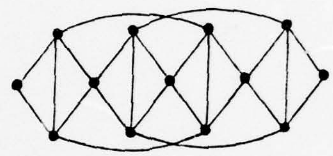
THEOREM 11. (Bose and Laskar [3]). *Let* $r \geq 1$, $k \geq 2$, $e \geq 0$ *and* $b \geq 0$ *be integers such that* $k > \max(1+b+e(2r-1),\ 1+\frac{1}{2}(r+1)(rb-2e))$. *Let* G *be a graph such that the distance parameters* p^0_{11} *and* p^1_{11} *are given by* $p^0_{11} = r(k-1)$, $p^1_{11} = k-2+e$ *and* $p_{11}(x, y) \leq 1+b$ *for pairs of vertices* (x, y) *with distance greater than* 1. *Then every vertex is contained in exactly r grand cliques and every edge is contained in exactly one gran clique where a maximal clique containing at least* $k-(r-1)\,e$ *vertices is called a grand clique.*

4. NEAR ASSOCIATION SCHEMES

Frequently we come across situations where the symmetric relations defined on a set satisfy many but not all of the properties required of an associations scheme. Such structures could be called "near association schemes".

A few uniqueness theorems have been proved about such "near association schemes". I give only two illustrations of such theorems. Let $v > k > \lambda > 0$ be integers and π be a symmetric (v, k, λ)-bibd. Let G (π) denote the flag graph of π. The distance relations in the graph G (π) satisfy many but not all the properties required of an association scheme. A theorem of Hoffman and the present author shows the graph G (π) can be reconstructed from its distinct eigen values and connectedness.

THEOREM 12. (A. J. Hoffman and D. K. Ray-Chaudhuri [15]). *Let* $v > k > \lambda > 0$ *be integers and* $(v, k, \lambda) \neq (4, 3, 2)$. *Let* G *be a regular simple connected graph with distinct eigen values* $2k - 2$, -2, $k - 2 + \sqrt{k - \lambda}$ *and* $k - 2 - \sqrt{k - \lambda}$. *Then there exists a symmetric bibd* π *such that* G *is isomorphic to* G (π). *Further, if* $(v, k, \lambda) = (4, 3, 2)$ *and* G *is not isomorphic to the graph drawn below then the statement of the theorem holds. In the diagram e and e' represent the same vertex.*



The distance relations in the flag graph of a finite affine plane also satisfy many but not all the properties required of an association scheme. A theorem of Hoffman and the present author proves that the flag graph of an affine plane can be reconstructed from its distinct eigen values and connectedness.

THEOREM 13. (A. J. Hoffman and D. K. Ray-Chaudhuri [14]). *Let n be a positive integer and* G *be a regular connected simple graph with distinct eigen values* $2n - 1$, -2, $1/2\,(2n - 3 + \sqrt{(4n + 1)})$, $1/2\,(2n - 3 - \sqrt{(4n + 1)})$ *and* $n - 2$. *Then there exists a finite affine plane* π *such that* G *is isomorphic to the flag graph of* π.

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BALANCED INCOMPLETE BLOCK DESIGNS AND RELATED DESIGNS*

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Preface. No special mathematical preparation is necessary in order to read this paper. The reader would be well advised to have some knowledge of primitive roots and residua classes modulo primes, as well as of finite Galois fields. All other preparatory information necessary to understand the problems of balanced incomplete block designs is given in Sections 2, 3.1 and 3.2.

The reader is advised to read carefully Section 1.5 (notation) before proceeding further. Some of the notations used in this paper are not standard.

The author will be grateful for remarks regarding mistakes or misprints in this paper as well as for suggestions for simpler or more elegant proofs of the theorems proved here.

Contents

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1. Introduction

1.1. Designs

Let X be a finite set of *points* and let $\mathcal{B} = \{B_i : i \in I\}$ be a family of – not necessarily distinct – subsets B_i – called *blocks* – of X. The pair $(X, \mathcal{B})$ is called a *design*.

The *order* of a design $(X, \mathcal{B})$ is $|X|$ (the cardinality of X) and the set $\{|B_i| : B_i \in \mathcal{B}\}$ is the set of *block-sizes* of the design.

1.2. Balanced incomplete block designs (BIBD)

Let $v \geqslant k \geqslant 2$ and λ be positive integers. A design $(X, \mathcal{B})$ is called a *balanced incomplete block design* (*BIBD*) $B[k, \lambda; v]$ if

(i) $|X| = v$ (the design is of order v);

(ii) the blocks are of size k;

(iii) every pairset $\{x, y\} \subset X$ is contained in exactly λ blocks of $\mathcal{B}$.

A well-known theorem states:

Theorem 1.1. *A necessary condition for the existence of a BIBD $B[k, \lambda; v]$ is that* $\lambda(v-1) \equiv 0 \pmod{k-1}$ *and* $\lambda v(v-1) \equiv 0 \pmod{k(k-1)}$.

Proof. $r = \lambda(v-1)/(k-1)$ is the replication number of every point of the design and $b = \lambda v(v-1)/(k(k-1))$ is the total number of blocks.

For $k = 2$ the construction of BIBD $B[2, \lambda; v]$ is trivial; we obtain such design by taking all the pairsets $\{x, y\} \subset X$, λ times each. For $k = 3$ and $k = 4$ it has been proved [16] that the condition of Theorem 1.1 is also sufficient for the existence of a BIBD $B[k, \lambda; v]$. For $k > 4$ the condition of Theorem 1.1 is in general not sufficient.

1.3. Known results

Balanced incomplete block designs $B[3, 1; v]$ with $k = 3$ and $\lambda = 1$ were introduced by Kirkman and Steiner [30] more than a hundred years ago and are also known as *Steiner triple systems.* Reiss proved [25] that $v \equiv 1$ or 3 (mod 6) is a necessary and sufficient condition for the existence of a Steiner triple system. In more modern period Bose proved [2] that the condition of Theorem 1.1 is sufficient if $k = 3$ and $\lambda = 2$. Further it has been proved [16,18] that the condition of Theorem 1.1 is sufficient for the existence of BIBD with $k = 3$, 4 and 5 and every λ, with the exception of the design $B[5, 2; 15]$.

On the other hand it has been shown by Tarry [32] that BIBD's $B[6, 1; 36]$ and $B[7, 1; 43]$ do not exist. Later, more general results have been obtained on non-existence of BIBD's with parameters v, k and λ satisfying the condition of Theorem 1.1. We give here the relevant lemmas without proof and the interested reader may find the proofs in the book of Hall [14]. We start with the well-known inequality of Fisher [14, p. 103].

Lemma 1.1. *A necessary condition for the existence of a BIBD $B[k, \lambda; v]$ is that $b = \lambda v(v-1)/(k(k-1)) \geq v$.*

It follows from Lemma 1.1 that e.g. BIBD's $B[6, 1; 16]$, $B[6, 1; 21]$, $B[10, 3; 25]$, $B[12, 2; 34]$, $B[15, 2; 36]$ and $B[15, 4; 36]$ do not exist.

As mentioned, $b = \lambda v(v-1)/(k(k-1))$ is the number of blocks in $B[k, \lambda; v]$. A BIBD is *symmetric* if $b = v$. The following lemma on the existence of symmetric BIBD's is due to Bruck, Ryser and Chowla [5,7] (see also [14, p. 107]).

Lemma 1.2. *If a symmetric BIBD $B[k, \lambda; v]$ exists, then*

(a) *if v is even, then $(k - \lambda)$ is a square;*

(b) *if v is odd, then $z^2 = (k - \lambda)x^2 + (-1)^{(v-1)/2}\lambda y^2$ has a solution in integers x, y, z not all zero.*

It follows that e.g. BIBD's $B[7, 1; 43]$, $B[7, 2; 22]$, $B[8, 2; 29]$, $B[10, 2; 46]$, $B[12, 4; 34]$ and $B[15, 5; 43]$ do not exist.

Further Hall and Connor [9,15] (see also [14, p. 257]) proved the following lemma on symmetric BIBD's:

Lemma 1.3. *Let v, k, λ satisfy $\lambda(v - 1) = k(k - 1)$ and let $\lambda = 1$ or 2, then $B[k - \lambda, \lambda, v - k]$ exists if and only if $B[k, \lambda, v]$ exists.*

It follows that e.g. BIBD's $B[5, 2; 15]$, $B[6, 1; 36]$, $B[6, 2; 21]$ and $B[8, 2; 36]$ do not exist.

1.4. On this paper

The main purpose of this paper is to determine necessary and sufficient conditions for the existence of BIBD's. For BIBD's with block-sizes 3, 4 and 5 such conditions are already known [16,18] and simpler proofs than the old ones are given herewith. For BIBD's with block-size 6 a necessary and sufficient condition is determined for $\lambda > 1$. At the present state of knowledge it seems hopeless to find a reasonable necessary and sufficient condition for the existence of BIBD's with $k = 6$ and $\lambda = 1$. However, a list of known BIBD's with $k = 6$, $\lambda = 1$ and $v < 2000$ is given in Section 5.4. Further, a necessary and sufficient condition is given for the existence of BIBD's with block-size 7 and $\lambda = 6$, 7 and 42. In Section 5.6 a table is given of all BIBD's of order $v \leqslant 43$.

In order to prove that the known necessary condition of Theorem 1.1 is (with few exemptions) also sufficient for the mentioned values of k and λ, some auxiliary designs had to be introduced, namely, the pairwise balanced designs, group divisible designs and transversal designs. In Section 2 the most important properties of the pairwise balanced designs and of the group divisible designs will be found. In Section 3 a more detailed discussion on transversal designs is given.

This paper is selfsustained and all the existence theorems are proved in it completely. However, the nonexistence lemmas of Section 1.3 are proved by methods different from those used in this paper. Consequently, they are not proved here and the reader interested in them may find the proofs in the book of Hall [14].

For designs with block-size 3 some further properties are given in Sections 6 and 7. In Section 6 a necessary and sufficient condition is determined for the existence of a group divisible design with $k = 3$ and in Section 7 the problems of covering pairs by triples and of packing of triples in pairs are discussed.

As mentioned before some parts of this paper have already been published – in different form – before. For the first time are published the following parts: Section 3.4, several lemmas in Section 4.2, Sections 5.4 and 5.5, most of Section 5.6, and Sections 6 and 7.

Finally, it should be mentioned a recent proof by Wilson [35] that the necessary condition of Theorem 1.1 is also sufficient for every k and every λ if v is sufficiently large.

1.5. Notation

Usually:

lower case letters ($a, k, m, ...$) will denote points or integers;

capital letters ($A, K, M, ...$) will denote sets of points or sets of integers;

script capital letters ($\mathcal{B}, \mathcal{G}, \mathcal{A}, ...$) will denote families of sets.

q denotes exclusively a prime-power (an integer which is a power of a prime).

$|S|$ denotes the cardinality of S (the number of points in the set S).

I denotes the set of non-negative integers $I = \{0, 1, 2, ...\}$.

$I(n)$ denotes the set of non-negative integers smaller than n, e.g., $I(5) = \{0, 1, 2, 3, 4\}$.

$Z(n)$ denotes the cycle of residua mod n.

$\mathrm{GF}(q)$ denotes Galois field of order q.

In general theorems for prime-powers q, the Galois field $\mathrm{GF}(q)$ will be used covering also the case that q is a prime.

$Z(p, x)$ (only when p is a prime) denotes $Z(p)$ with the additional information that x is the primitive root used.

$\mathrm{GF}(q, f(x) = 0)$ denotes $\mathrm{GF}(q)$ with the additional information that x is the primitive mark used.

In all cases that $X = Z(p, x)$ or $X = \mathrm{GF}(q, f(x) = 0)$, the points are denoted by *exponents* of x and so the symbol a denotes the point x^a. For the residuum (or mark) 0 the symbol $\emptyset$ is used.

If $X = I(n)$ or in other cases that the integer itself is used and not the exponent, the integer appears with a prime sign, e.g., a'.

When $X = Y \times Z$, then the points are denoted by a symbol (a, b) where a is an exponent of an element in Y and b an exponent of an element in Z.

In case of group divisible designs and transversal designs, $X = Y \times Z$, Y denotes the set of points in a group and Z the set of the groups. In such case a semicolon is used in the symbol $(a; b)$.

The brackets $\langle\ \rangle$ are used exclusively to denote blocks.

The words mod (q_1, q_2) after a block denote that all the elements of the block should be taken cyclically by adding to them all the residua of $Z(q)$ or all the marks of $\mathrm{GF}(q)$.

If S is a set of integers, then $S + 1 = \{s + 1 : s \in S\}$ similarly $mS = \{ms : s \in S\}$.

2. Combinatorial designs

2.1. Pairwise balanced designs

Let v and λ be positive integers and K a set of prositive integers. A design $(X, \mathcal{B})$ is a *pairwise balanced design* $B[K, \lambda; v]$ if

(i) $|X| = v$ (the design is of order v);

(ii) $\{|B_i| : B_i \in \mathcal{B}\} \subset K$ (the block-sizes are from K);

(iii) every pairset $\{x, y\} \subset X$ is contained in exactly λ blocks of $\mathcal{B}$.

A pairwise balanced design $B[K, \lambda; v]$, where $K = \{k\}$ consists of exactly one integer, is a BIBD $B[k, \lambda; v]$.

The set of integers v, for which pairwise balanced designs $B[K, \lambda; v]$ exist, will be denoted by $B(K, \lambda)$. Similarly, the set of integers v, for which BIBD's $B[k, \lambda; v]$ exist, will be denoted by $B(k, \lambda)$.

The following lemmas are evident:

Lemma 2.1. $K \subset B(K, \lambda)$.

Lemma 2.2. *If* $K' \subset K$, *then* $B(K', \lambda) \subset B(K, \lambda)$.

Lemma 2.3. *If* λ' *divides* λ, *then* $B(K, \lambda') \subset B(K, \lambda)$.

And more generally:

Lemma 2.4. $B(K, \lambda) \cap B(K, \lambda') \subset B(K, n\lambda + n'\lambda')$, *where* n *and* n' *are any non-negative integers.*

Further we have:

Lemma 2.5. *If* $v \in B(K', \lambda')$ *and* $K' \subset B(K, \lambda)$, *then* $v \in B(K, \lambda\lambda')$ *holds.*

The following special case of Lemma 2.5 will be useful:

Lemma 2.6. *If* $v \in B(K, 1)$ *and* $K \subset B(k, \lambda)$, *then* $v \in B(k, \lambda)$ *holds.*

2.2. Finite planes

As an illustrative and important example of BIBD's may serve the finite planes. Those interested in the geometric aspect of finite planes may find full description in the book of Hall [14, pp. 167–188]. We deal here only with the combinatorial aspect of finite planes connecting them with the BIBD's.

A finite projective plane PG$(2, q)$ (projective geometry of *dimension* 2 and *order* q) is a BIBD $[q + 1, 1; q^2 + q + 1]$. The blocks of such designs are usually called *lines.* By definition of BIBD's (see also the proof of Theorem 1.1), a projective plane PG$(2,q)$ has $q^2 + q + 1$ lines and every point is included in $q + 1$ lines. Considering any given line, we see that it intersects in each of its points q other lines and all together all the other $q^2 + q$ lines. It follows that in PG$(2, q)$ every two lines intersect. It is known – since Galois – that PG$(2, q)$ exists whenever q is a prime-power. So far no projective plane has been constructed of non-prime-power order. We prove:

Theorem 2.1. *If* q *is a prime-power, then* $q^2 + q + 1 \in B(q + 1, 1)$; *in other words there exists a finite projective plane* PG$(2, q)$.

Proof.

$$X = \mathrm{GF}(q, f(x)=0) \times \mathrm{GF}(q, f(x)=0) \cup \{(\infty_i): i = 0, 1, ..., q\}.$$

$$\mathcal{B} = \langle(\infty_0), (\infty_1), ..., (\infty_q)\rangle$$

$$\langle(\infty_q), (\emptyset,\emptyset), (0,\emptyset), (1,\emptyset), ..., (q-2,\emptyset)\rangle \bmod (-,q),$$

$$\langle(\infty_{q-1}), (\emptyset,\emptyset), (\emptyset,0), (\emptyset,1), ..., (\emptyset,q-2)\rangle \bmod (q,-),$$

$$\langle(\infty_\alpha), (\emptyset,\emptyset), (\alpha,0), (\alpha+1,1), ..., (\alpha+q-2, q-2)\rangle \bmod (q,-),$$

$$\alpha = 0, 1, ..., q-2.$$

By deleting the points (∞_i), $i = 0, 1, ..., q$, from the described design, we obtain a BIBD $B[q, 1; q^2]$ which is also called a *finite affine plane* $\mathrm{AG}(2,q)$ (affine geometry of dimension 2 and order q). It follows immediately:

Theorem 2.2. *If q is a prime-power, then $q^2 \in B(q, 1)$; in other words there exists a finite affine plane* $\mathrm{AG}(2,q)$.

2.3. Group divisible designs

Let a design $(X, \mathcal{B})$ be given. A *parallel class* of blocks is a subfamily $\mathcal{G} \subset \mathcal{B}$ of disjoint blocks, the union of which equals X.

We shall consider at length designs of the form $(X, \mathcal{G}, \mathcal{P})$, where X is a finite set of *points*, $\mathcal{G}$ is a parallel class of subsets of X called *groups* and $\mathcal{P}$ is a family of subsets of X called *proper blocks*, or for short – *blocks*.

Let k, m, λ and v be positive integers. A design $(X, \mathcal{G}, \mathcal{P})$ is a *group divisible design* $\mathrm{GD}[k, \lambda, m; v]$ if

(i) $|X| = v$;

(ii) $|G_i| = m$ for every $G_i \in \mathcal{G}$;

(iii) $|B_j| = k$ for every $B_j \in \mathcal{P}$;

(iv) $|G_i \cap B_j| \leqslant 1$ for every $G_i \in \mathcal{G}$ and every $B_j \in \mathcal{P}$;

(v) every pairset $\{x, y\} \subset X$, such that x and y belong to distinct groups, is contained in exactly λ blocks of $\mathcal{P}$.

It follows immediately that a BIBD $B[k, \lambda; v]$ is a group divisible design $\mathrm{GD}[k, \lambda, 1; v]$.

The set of integers v for which group divisible designs $GD[k, \lambda, m; v]$ exist, will be denoted by $GD(k, \lambda, m)$. Clearly $GD(k, \lambda, m) \subset mI$.

Some of the properties of the group divisible designs are given in the following lemmas:

Lemma 2.7. *If* λ' *divides* λ, *then* $GD(k, \lambda', m) \subset GD(k, \lambda, m)$.

Lemma 2.8. $GD(k, \lambda, m) \subset B(\{k, m\}, \lambda)$.

Proof. Take the groups of the group divisible design as additional blocks, λ times each.

Lemma 2.9. $GD(k, \lambda, m) + 1 \subset B(\{k, m+1\}, \lambda)$.

Proof. Take as additional blocks the groups of the group divisible design, each λ times with a fixed additional point adjoint.

Putting in Lemmas 2.8 and 2.9, $m = k$ and $m = k - 1$, respectively, we obtain the special results which are most useful.

Lemma 2.10. $GD(k, \lambda, k) \subset B(k, \lambda)$.

Lemma 2.11. $GD(k, \lambda, k-1) + 1 \subset B(k, \lambda)$.

In the case $\lambda = 1$ we have the stronger result:

Lemma 2.12. $GD(k, 1, k-1) + 1 = B(k, 1)$.

Proof. $GD(k, 1, k-1) + 1 \subset B(k, 1)$ follows from Lemma 2.11. To prove $B(k, 1) \subset GD(k, 1, k-1) + 1$, choose any fixed point of the BIBD and denote it by (∞). Consider all the blocks containing the point (∞), delete this point and take the truncated blocks as groups of a group divisible design. The remaining blocks of the BIBD serve as blocks in the group divisible design.

Let us assume that in Lemmas 2.8 and 2.9, respectively, $m \in B(k, \lambda)$ and $m + 1 \in B(k, \lambda)$. In these cases instead of taking the blocks having

m and $m + 1$ points, respectively, we take the blocks of $B[k, \lambda; m]$ or $B[k, \lambda; m + 1]$ and get:

Lemma 2.13. *If* $m \in B(k, \lambda)$, *then* $\mathrm{GD}(k, \lambda, m) \subset B(k, \lambda)$.

Lemma 2.14. *If* $m + 1 \in B(k, \lambda)$, *then* $\mathrm{GD}(k, \lambda, m) + 1 \subset B(k, \lambda)$.

Further we prove:

Lemma 2.15. *If* $m + k \in B(k, 1)$, *then* $\mathrm{GD}(k, \lambda, m) + k \subset B(k, \lambda)$.

Proof. Add to the group divisible design k fixed points. On these points together with each of the groups form a BIBD $B[k, 1; m + k]$ in such way that these k points form a block.

Lemma 2.16. *If* $n \in B(K, \lambda)$ *and* $mK \subset \mathrm{GD}(k, \lambda', m)$, *then* $mn \in \mathrm{GD}(k, \lambda\lambda', m)$.

Proof. Consider the groups of the group divisible design as points of the BIBD $B[K, \lambda; n]$ and form a group divisible design on every block of $B[K, \lambda; n]$.

Lemma 2.17. *If* $n \in B(K, \lambda)$ *and* $(k - 1)K + 1 \subset B(k, 1)$, *then* $(k - 1)n + 1 \in B(k, \lambda)$.

Proof. By Lemma 2.12, $(k-1)K \subset \mathrm{GD}(k, 1, k-1)$ and by Lemma 2.16, $(k-1)n \in \mathrm{GD}(k, \lambda, k-1)$. Further use Lemma 2.11.

2.4. Pairwise group divisible designs

The pairwise group divisible designs are a generalization of the group divisible designs by allowing both the groups and the blocks to attain several sizes. Like in Section 2.3 let a design $(X, \mathcal{G}, \mathcal{P})$ be given. Further let v and λ be positive integers and K and M sets of positive integers. A design $(X, \mathcal{G}, \mathcal{P})$ is a *pairwise group divisible design* $\mathrm{GD}[K, \lambda, M; v]$ if

(i) $|X| = v$;
(ii) $\{|G_i|: G_i \in \mathcal{G}\} \subset M$;
(iii) $\{|B_j|: B_j \in \mathcal{P}\} \subset K$;

(iv) $|G_i \cap B_j| \leq 1$ for every $G_i \in \mathcal{G}$ and every $B_j \in \mathcal{P}$;

(v) every pairset $\{x, y\} \subset X$ such that x and y belong to distinct groups is contained in exactly λ blocks of $\mathcal{P}$.

A pairwise group divisible design $GD[K, \lambda, M; v]$, where $K = \{k\}$ and $M = \{m\}$ consists of exactly one integer, is a group divisible design $GD[k, \lambda, m; v]$.

The set of integers v for which pairwise group divisible designs $GD[K, \lambda, M; v]$ exist, will be denoted by $GD(K, \lambda, M)$.

Evidently we have:

Lemma 2.18. $M \subset GD(K, \lambda, M)$.

Lemma 2.19. *If* $M' \subset M$ *and* $K' \subset K$, *then* $GD(K', \lambda, M') \subset GD(K, \lambda, M)$.

Lemma 2.20. *If* λ' *divides* λ, *then* $GD(K, \lambda', M) \subset GD(K, \lambda, M)$.

Lemma 2.21. *If* $v \in GD(K, \lambda', M')$ *and* $M' \subset GD(K, \lambda, M)$, *then* $v \in GD(K, \lambda\lambda', M)$.

As a special case, when $\lambda = \lambda' = 1$, we obtain from Lemma 2.21:

Lemma 2.22. *If* $v \in GD(K, 1, M')$ *and* $M' \subset GD(K, 1, M)$, *then* $v \in GD(K, 1, M)$.

As immediate generalizations of Lemmas 2.8 and 2.9 we get:

Lemma 2.23. $GD(K, \lambda, M) \subset B(K \cup M, \lambda)$.

Lemma 2.24. $GD(K, \lambda, M) + 1 \subset B(K \cup (M+1), \lambda)$.

Further we prove:

Lemma 2.25. *If* $n \in GD(S, 1, R)$, $mR \subset B(k, \lambda)$ *and* $mS \subset GD(k, \lambda, m)$, *then* $mn \in B(k, \lambda)$.

Proof. Consider the set $I(m) \times I(n)$. On $I(n)$ form a pairwise group divisible design $GD[S, 1, R; n]$. For every group G of this design, $|G| \in R$ and therefore a BIBD $B[k, \lambda; m|G|]$ may be constructed on $I(m) \times G$. For every block B of the pairwise group divisible design, form a group divisible design $\mathrm{GD}[k, \lambda, m; m|B|]$ on $I(m) \times B$.

In a similar way can be proved:

Lemma 2.26. *If* $n \in \mathrm{GD}(S, 1, R)$, $mR + 1 \subset B(k, \lambda)$ *and* $mS \subset \mathrm{GD}(k, \lambda, m)$, *then* $mn + 1 \in B(k, \lambda)$.

Remark. Lemmas 2.25 and 2.26 may be easily generalized as follows:

(a) If $n \in \mathrm{GD}(S, \lambda', R)$, $mR \subset B(K, \lambda\lambda')$ and $mS \subset \mathrm{GD}(K, \lambda, m)$, then $mn \in B(K, \lambda\lambda')$.

(b) If $n \in \mathrm{GD}(S, \lambda', R)$, $mR + 1 \subset B(K, \lambda\lambda')$ and $mS \subset \mathrm{GD}(K, \lambda, m)$, then $mn + 1 \in B(K, \lambda\lambda')$.

3. Transversal designs

3.1. Introduction

Transversal designs are a special case of group divisible designs, satisfying the additional condition, that every block of $\mathcal{B}$ intersects every group of $\mathcal{G}$.

Formally, like in Section 2.3, let a design $(X, \mathcal{G}, \mathcal{B})$ be given. Further, let s, r and λ be positive integers. A design $(X, \mathcal{G}, \mathcal{B})$ is a *transversal design* $\mathrm{T}[s, \lambda; r]$ if

(i) $|G_i| = r$ for every $G_i \in \mathcal{G}$;

(ii) $|\mathcal{G}| = s$;

(iii) $|G_i \cap B_j| = 1$ for every $G_i \in \mathcal{G}$ and every $B_j \in \mathcal{B}$;

(iv) every pairset $\{x, y\} \subset X$, such that x and y belong to distinct groups, is contained in exactly λ blocks of $\mathcal{B}$.

It follows immediately that in a transversal design $\mathrm{T}[s, \lambda; r]$, $|X| = sr$, $|B_j| = s$ for every $B_j \in \mathcal{B}$, and $|\mathcal{B}| = r^2\lambda$. Further, a transversal design $\mathrm{T}[s, \lambda; r]$ is a group divisible design $\mathrm{GD}[s, \lambda, r; rs]$.

The set of integers r for which transversal designs $\mathrm{T}[s, \lambda; r]$ exist, will be denoted by $\mathrm{T}(s, \lambda)$.

Evidently we have:

Lemma 3.1. *If* $s' \leqslant s$, *then* $\mathrm{T}(s, \lambda) \subset \mathrm{T}(s', \lambda)$.

Lemma 3.2. *If* λ' *divides* λ, *then* $\mathrm{T}(s, \lambda') \subset \mathrm{T}(s, \lambda)$.

And more generally:

Lemma 3.3. $\mathrm{T}(s, \lambda) \cap \mathrm{T}(s, \lambda') \subset \mathrm{T}(s, n\lambda + n'\lambda')$, *where* n *and* n' *are any non-negative integers.*

Further we have:

Lemma 3.4. *If* $r \in \mathrm{T}(s, \lambda)$ *and* $r' \in \mathrm{T}(s, \lambda')$, *then* $rr' \in \mathrm{T}(s, \lambda\lambda')$.

Proof. For every block B of $\mathrm{T}[s, \lambda; r]$, consider as elements of B the groups of $\mathrm{T}[s, \lambda'; r']$ and form on them the blocks of $\mathrm{T}[s, \lambda'; r']$.

For $\lambda = 1$ the following further results are known:

Lemma 3.5. *For every prime-power* q, $q \in \mathrm{T}(q+1, 1)$.

Proof. Consider the projective plane $\mathrm{PG}(2, q)$ constructed in Theorem 2.1. Delete the point (∞_q) and consider as groups the truncated lines which contained this point. The other lines of $\mathrm{PG}(2, q)$ are the blocks of $\mathrm{T}[q+1, 1; q]$.

From Lemmas 3.5, 3.1 and 3.4 follows the theorem of MacNeish [20].

Theorem 3.1. *Let* $r = \Pi p_i^{\alpha_i}$ *be the factorization of* r *into powers of distinct primes, then* $r \in \mathrm{T}(s, 1)$, *where* $s = 1 + \min p_i^{\alpha_i}$.

In some applications, transversal designs are used in which the blocks can be partitioned into parallel classes. Accordingly we define:

A *resolvable transversal design* $\mathrm{RT}[s, \lambda; r]$ is a transversal design $\mathrm{T}[s, \lambda; r]$ in which the family $\mathcal{B}$ of blocks can be partitioned into λr parallel classes. As usually, we shall denote by $\mathrm{RT}(s, \lambda)$ the set of integers r for which resolvable transversal designs $\mathrm{RT}[s, \lambda; r]$ exist.

For $\lambda = 1$, we have:

Lemma 3.6. RT$(s, 1)$ = T$(s+1, 1)$.

Proof. If RT$[s, 1; r]$ exists, adjoin a fixed additional point to all the blocks of any given parallel class. Proceeding in this way for every parallel class, r additional points are obtained. Form a group from these r additional points and get T$[s+1, 1; r]$. If, on the other hand, T$[s+1, 1; r]$ is given, delete any one of the groups. The blocks which contained a fixed deleted point form now a parallel class and the design is RT$[s, 1; r]$.

In general, we may prove in the same way:

Lemma 3.7. RT$(s, \lambda) \subset$ T$(s+1, \lambda)$.

We shall now prove the following:

Theorem 3.2. *If* $r \in$ GD(K, λ, M), $M \subset$ T(s, λ) *and* $K \subset$ RT$(s, 1)$, *then* $r \in$ T(s, λ).

Proof. Denote the set of points of a $r \times s$ matrix by $X = I(r) \times I(s)$, where $I(r)$ is the set of points of GD$[K, \lambda, M; r]$. We shall prove that a transversal design T$[s, \lambda; r]$ can be constructed on X with $I(r) \times \{i\}$, $i \in I(s)$ as groups. Let $G \subset I(r)$ be a group of GD$[K, \lambda, M; r]$. Clearly $|G| = m \in M$ and by hypothesis we may form T$[s, \lambda; m]$ on $G \times I(s)$. We perform this operation for every group of GD$[K, \lambda, M; r]$. Further let $B \subset I(r)$ be a block of GD$[K, \lambda, M; r]$. Clearly $|B| = k \in K$ and by hypothesis we may form RT$[s, 1; k]$ on $B \times I(s)$. Considering that we deal with a resolvable transversal design we may assume that one of the parallel classes is composed of the blocks $\{j\} \times I(s)$, $j \in B$; these blocks we omit. Again we perform this operation for every block of GD$[K, \lambda, M; r]$. So obtained blocks are all the blocks of T$[s, \lambda; r]$ which is hereby constructed.

Putting in Theorem 3.2, $\lambda = 1$ and $M = \{1\}$, we obtain:

Theorem 3.3. *If* $r \in B(K, 1)$ *and* $K \subset$ RT$(s, 1)$, *then* $r \in$ T$(s, 1)$.

The transversal designs are strongly connected with other combinatorial designs. We have already seen (Lemma 3.5) their connection with the finite projective planes and (Lemmas 3.5 and 3.6) with the finite affine planes. Further, the existence of transversal designs $T[s, 1; r]$ is equivalent to the existence of $s - 2$ mutually orthogonal Latin squares of order r and the transversal designs $T[4\lambda - 1, \lambda; 2]$ are equivalent to the Hadamard matrices of order 4λ. The interested reader may find these subjects dealt with in the book of Hall [14, pp. 189–222].

3.2. *Truncated transversal designs*

Let a transversal design $T[s + t, \lambda; r]$ exist. Delete some points from at most t of its groups. The obtained design is a pairwise group divisible design

$$\mathrm{GD}\left[\{s, s+1, \ldots, s+t\}, \lambda, \{r, r_1, r_2, \ldots, r_t\}; rs + \sum_{i=1}^{t} r_i\right],$$

where r_i are the sizes of the truncated groups. It follows (for $\lambda = 1$):

Lemma 3.8. *If* $r \in T(s + t, 1)$ *and* $0 \leqslant r_1 \leqslant r_2 \leqslant \ldots \leqslant r_t \leqslant r$, *then*

$$rs + \sum_{i=1}^{t} r_i \in \mathrm{GD}(\{s, s+1, \ldots, s+t\}, 1, \{r, r_1, r_2, \ldots, r_t\}).$$

From Lemma 2.23 and Lemma 2.24, respectively, considering Lemma 2.5, we have:

Lemma 3.9. *If* $r \in T(s + t, 1)$, $0 \leqslant r_1 \leqslant r_2 \leqslant \ldots \leqslant r_t \leqslant r$, *and* $\{s, s+1, \ldots, s+t, r, r_1, r_2, \ldots, r_t\} \subset B(K, \lambda)$, *then*

$$rs + \sum_{i=1}^{t} r_i \in B(K, \lambda).$$

Lemma 3.10. *If* $r \in T(s + t, 1)$, $0 \leqslant r_1 \leqslant r_2 \leqslant \ldots \leqslant r_t \leqslant r$, *and* $\{s, s+1, \ldots, s+t, r+1, r_1+1, r_2+1, \ldots, r_t+1\} \subset B(K, \lambda)$, *then*

$$rs + \sum_{i=1}^{t} r_i + 1 \in B(K, \lambda).$$

In the special case $t = 0$ we get:

Lemma 3.11. *If* $r \in T(s, 1)$ *and if* $\{s, r\} \subset B(K, \lambda)$, *then* $rs \in B(K, \lambda)$.

Lemma 3.12. *If* $r \in T(s, 1)$ *and if* $\{s, r + 1\} \subset B(K, \lambda)$, *then* $rs + 1 \in B(K, \lambda)$.

Tchebychef, Sylvester and others (see, e.g., [10, p. 435]) proved that for every positive s and every sufficiently large integer $d \geqslant d_0 = d_0(s)$, there exists a prime between d and $d(s + 1)/s$. Using this result we have:

Lemma 3.13. *Let s and t be positive integers and let D be a finite set of integers, then there exists an integer* v_0 *such that for every* $v \geqslant v_0$ *we can find integers* $r, r_1, \ldots, r_t$ *satisfying the following conditions:*

(i) $0 \leqslant r_1 \leqslant r_2 \leqslant \ldots \leqslant r_t \leqslant r$,
(ii) $v = rs + \Sigma_{i=1}^{t} r_i$,
(iii) $r \in T(s + t, 1)$,
(iv) $\{r, r_1, r_2, \ldots, r_t\} \cap D = \emptyset$,
(v) *it may also be assumed that for given a,* $r \equiv a \pmod{s}$.

Lemma 3.13 will be used in Section 5 with $t \in \{1, 2\}$. In each case the value of v_0 will be stated and it has to be checked that actually for every $v \geqslant v_0$ the conditions of the lemma are satisfied. If $s + 1 \leqslant 8$, upper bounds for v_0 are given by Theorems 3.7, 3.8, 3.9 and 3.10. For values of v below these upper bounds usually Theorem 3.1 may be applied.

3.3. Transversal designs with $\lambda = 1$ (*Wilson's theorem*)

It has been proved by Chowla, Erdös and Straus [6] that for every positive integer s, there exists an integer $r_0(s)$ such, that for every $r > r_0(s)$, $r \in T(s, 1)$ holds. They proved that $r_0(s) \leqslant (3s)^{91}$. This result has been improved by Rogers [26] who proved $r_0(s) \leqslant s^{42+\epsilon}$, with $\epsilon \to 0$ as $s \to \infty$. Lately Wilson proved [34]:

Theorem 3.4. *For every sufficiently large s, if* $r > s^{17}$, *then* $r \in T(s, 1)$ *holds.*

This theorem as well as the forthcoming Theorems 3.5 and 3.6 shall not be used in sequel and are cited here without proof.

Table 5.14 - addenda.

v	B [6, 1; v]
106	See p. 368 .
306	We prove $305 \in GD(6, 1, 5)$. $X = Z(5, 2) \times Z(61, 2)$. $\mathcal{P} = \langle (\phi; 6\alpha + 3\beta), (\phi; 6\alpha + 3\beta + 30), (\beta; 6\alpha + 3\beta + 4),$ $(\beta; 6\alpha + 3\beta + 34), (\beta + 2; 6\alpha + 3\beta + 26), (\beta + 2; 6\alpha + 3\beta + 56)\rangle \bmod (5; 61)$ $\alpha = 0, 1, 2, 3, 4, \quad \beta = 0, 1$.
531	Lemma 2.17 and $106 \in B(6, 1)$.
636	Lemma 3.11, $106 \in B(6, 1)$ and $106 \in T(6, 1)$.
841	$X = GF(841, x^2 = x + 26)$. $\mathcal{B} = \langle 5\alpha, 5\alpha + 8, 5\alpha + 280, 5\alpha + 288, 5\alpha + 568\rangle \bmod 841, \alpha = 0, 1, \ldots, 27$.
876	Lemma 2.15 , $151 \in B(6, 1)$ and $145 \in T(6, 1)$.
1236	Lemma 2.15 , $211 \in B(6, 1)$ and $205 \in T(6, 1)$.
1416	Lemma 2.15 , $241 \in B(6, 1)$ and $235 \in T(6, 1)$.
1596	Lemma 2.15 , $271 \in B(6, 1)$ and $265 \in T(6, 1)$.
1836	Lemma 3.11 , $306 \in B(6, 1)$ and $306 \in T(6, 1)$.
1956	Lemma 2.15 , $331 \in B(6, 1)$ and $325 \in T(6, 1)$.

* * *

For specific values of s, stronger results have been obtained and so Bose, Parker and Shrikhande proved [23,4,3] (for a proof see also [14, pp. 194–203] or [34]):

Theorem 3.5. *For every* $r > 6$, $r \in T(4, 1)$ *holds.*

Further it is known [17]:

Theorem 3.6. *For every* $r > 34\,115\,553$, $r \in T(31, 1)$ *holds.*

We shall now proceed to prove that $r_0(8) \leqslant 90$. The proof is due to Wilson [34] and will be given here in full. From this results the bounds of $r_0(s)$ for $s \in \{5, 6, 7\}$ are easily obtained. We start with the almost obvious lemma:

Lemma 3.14. *If* $r \in T(s, 1)$ *and* $r > 1$, *then* $r \geqslant s - 1$.

Proof. Suppose $s \geqslant r + 2$. Every point of the design is included in r blocks. Any given block B has s points and therefore it intersects $s(r-1) \geqslant (r+2)(r-1) \geqslant r^2$ blocks, but there are only $r^2 - 1$ blocks in addition to B, which leads to a contradiction.

Lemma 3.15. *If* $r \in T(s, 1)$ *and* $r \geqslant s$, *then in the design* $T[s, 1; r]$ *there are at least two disjoint blocks.*

Proof. Every point of the design is included in exactly r blocks. Accordingly any given block B intersects $s(r-1) \leqslant r(r-1)$ other blocks. But the transversal design has r^2 blocks and accordingly there exists a block which does not intersect B.

We shall now describe a truncated transversal design as follows. Let $r \in T(s+t, 1)$ and let t groups be truncated in this design. We denote the set of points of this design by $X = (I(r) \times I(s)) \cup T'$, where $(I(r) \times I(s))$ is the untruncated part and

$$T' = (I(r_1) \times \{1'\}) \cup (I(r_2) \times \{2'\}) \cup \ldots \cup \{I(r_t) \times \{t'\})$$

is the truncated part. We prove:

Lemma 3.16. *Let a transversal design* $T[s+t, 1; r]$ *exist, and let t groups in this design be truncated, these truncated groups having* r_i, $i = 1, 2, ..., t$, *points, provided* $\{r_i: i = 1, 2, ..., t\} \subset T(s, 1)$. *Let* T' *be the set of points of the truncated part* ($|T'| = \Sigma_{i=1}^{t} r_i$) *and let m be a positive integer such that* $\{m, m+1, m+2\} \subset T(s, 1)$. *If for every block B of* $T[s+t, 1; r]$, $|B \cap T'| \leq 2$, *then* $\rho = mr + |T'| \in T(s, 1)$.

Proof. Denote the set of points of a $(mr + |T'|) \times s$ matrix by $X = (I(m) \times I(r) \cup T') \times I(s)$, where $Y = I(r) \times I(s)$ is the untruncated part of the (existing) design $T[s+t, 1; r]$. We shall prove that a transversal design $T[s, 1; \rho]$ can be constructed on X with $(I(m) \times I(r) \cup T') \times \{i\}$, $i \in I(s)$, as groups. For every block B of $T[s+t, 1; r]$, let $B' = B \cap T'$ and $B_0 = B \cap Y$; clearly $|B_0| = s$ and $0 \leq |B'| \leq 2$. Considering that $\{m, m+1, m+2\} \subset T(s, 1)$, we may form a transversal design $T[s, 1; m+|B'|]$ on $(I(m) \times B_0) \cup (B' \times I(s)) \subset X$. By Lemma 3.14, $m \geq s - 1$ and therefore $m + 2 > s$. Accordingly, if $|B'| = 2$, there are in our transversal design at least 2 disjoint blocks; we may choose them as $\{b\} \times I(s)$, $b \in B'$. These blocks we delete. (If $|B'| < 2$, this procedure is trivial.) Further for every $i = 1, 2, ..., t$ we form $T[s, 1; r_i]$ on $(I(r_i) \times \{i'\}) \times I(s) \subset (T' \times I(s))$. All blocks so obtained form a transversal design $T[s, 1; \rho]$.

From Lemma 3.16 follows easily:

Lemma 3.17. *If* $r \in T(s+2, 1)$, $0 \leq r_1 \leq r_2 \leq r$ *and* $\{m, m+1, m+2, r_1, r_2\} \subset T(s, 1)$, *then* $mr + r_1 + r_2 \in T(s, 1)$.

Proof. Put in Lemma 3.16, $t = 2$.

Lemma 3.18. *If* $r > \frac{1}{2}(t-1)(t-2)$, $r \in T(s+t, 1)$ *and* $\{m, m+1, m+2\} \subset T(s, 1)$, *then* $mr + t \in T(s, 1)$.

Proof. Put in Lemma 3.16, $r_1 = r_2 = ... = r_t = 1$. In order to apply Lemma 3.16 it has to be proved that under the hypothesis of our lemma the points of the truncated groups can be so chosen that no three of them are in a block. This is shown by induction. Suppose we have already chosen $t' < t$ such points. There are exactly $\frac{1}{2}t'(t'-1) \leq \frac{1}{2}(t-1)(t-2)$ blocks containing two of these points each. However, every not truncated group has at least $\frac{1}{2}(t-1)(t-2)+1$ points and therefore an additional point can be chosen so that no three points of the truncated groups are in a block.

Lemma 3.19. *For any integer n, at least one of the integers* $n, n+1, n+2, ..., n+9$ *is not divisible by any of the numbers* 2, 3, 5 *and* 7.

Proof. Let n' be the odd integer of $\{n, n+1\}$. Of the integers $n', n'+2, n'+4, n'+6, n'+8$, at most two are divisible by 3, at most one by 5, and at most one by 7; hence at least one is divisible by neither 3, 5, 7, nor 2.

For integers w, $7 \leq w \leq 76$, define u_w and v_w as in Table 3.1. Note that for each w, $7 \leq w \leq 76$, we have $\{u_w, v_w\} \subset I(64)$, and that by Theorem 3.1, $\{u_w, v_w\} \subset T(8,1)$ (if $v_w \in \{0,1\}$, $v_w \in T(8,1)$ is trivial).

Table 3.1.

w	u_w	v_w	w	u_w	v_w	w	u_w	v_w
7	7	0	31	23	8	54	27	27
8	8	0	32	16	16	55	32	23
9	9	0	33	17	16	56	29	27
10	9	1	34	17	17	57	32	25
11	11	0	35	19	16	58	29	29
12	11	1	36	19	17	59	32	27
13	13	0	37	29	8	60	31	29
14	7	7	38	19	19	61	32	29
15	8	7	39	23	16	62	31	31
16	8	8	40	23	17	63	32	31
17	9	8	41	25	16	64	32	32
18	9	9	42	23	19	65	49	16
19	11	8	43	27	16	66	37	29
20	11	9	44	25	19	67	56	11
21	13	8	45	29	16	68	37	31
22	11	11	46	23	23	69	37	32
23	16	7	47	31	16	70	41	29
24	13	11	48	25	23	71	63	8
25	16	9	49	32	17	72	41	31
26	13	13	50	25	25	73	41	32
27	16	11	51	32	19	74	37	37
28	17	11	52	27	25	75	43	32
29	16	13	53	37	16	76	47	29
30	17	13						

Table 3.2

r	Proof of $r \in T(8, 1)$
455–516	Lemma 3.20, $n = 64$, $7 \leq w \leq 68$
420–454	Lemma 3.20, $n = 59$, $7 \leq w \leq 41$
378–419	Lemma 3.20, $n = 53$, $7 \leq w \leq 48$
350–377	Lemma 3.20, $n = 49$, $7 \leq w \leq 34$
308–349	Lemma 3.20, $n = 43$, $7 \leq w \leq 48$
266–307	Lemma 3.20, $n = 37$, $7 \leq w \leq 48$
224–265	Lemma 3.20, $n = 31$, $7 \leq w \leq 48$
196–223	Lemma 3.20, $n = 27$, $7 \leq w \leq 34$
168–195	Lemma 3.20, $n = 23$, $7 \leq w \leq 34$
164–167	Lemma 3.18, $m = 7$, $r = 23$, $3 \leq t \leq 6$
140–163	Lemma 3.20, $n = 19$, $7 \leq w \leq 30$
119–139	Lemma 3.20, $n = 16$, $7 \leq w \leq 27$
114–118	Lemma 3.18, $m = 7$, $r = 16$, $2 \leq t \leq 6$
98–113	Lemma 3.20, $n = 13$, $7 \leq w \leq 22$
91– 97	Lemma 3.20, $n = 11$, $14 \leq w \leq 20$
84– 89	Lemma 3.20, $n = 11$, $7 \leq w \leq 12$
83	Theorem 3.1
77– 81	Lemma 3.18, $m = 7$, $r = 11$, $0 \leq t \leq 4$

Lemma 3.20. *If* $n \in T(10, 1)$, $7 \leq w \leq 76$ *and* $u_w, v_w \leq n$, *then* $7n + w \in T(8, 1)$.

Proof. Apply Lemma 3.17 with $m = 7$, $r_1 = u_w$ and $r_2 = v_w$.

Theorem 3.7. *For every* $r > 90$, $r \in T(8, 1)$ *holds.*

Remark. We prove moreover that if $r > 76$ and $r \notin \{82, 90\}$, then $r \in T(8, 1)$ holds.

Proof. By Lemma 3.19, an integer n may be found which is relatively prime to 210 and satisfies $r - 76 \leq 7n \leq r - 7$. If $r \geq 517$, then putting $w = r - 7n$ we have $7 \leq w \leq 76$, and $n \geq \frac{1}{7}(r - 76) \geq 63$. By Theorem 3.1, we have $n \in T(12, 1)$ and applying Lemma 3.20, we get $r = 7n + w \in T(8, 1)$. If $r < 517$, the proof of $r \in T(8, 1)$ is given in Table 3.2.

Theorem 3.8. *For every* $r > 62$, $r \in T(7, 1)$ *holds.*

Remark. We prove moreover that $\{47, 49, 50, 53, 56, 57, 58, 59, 61\} \subset \mathrm{T}(7, 1)$.

Proof (See [17]). By the remark to Theorem 3.7, it remains to prove this theorem for the set of integers $\{47, 49, 50, 53, 56, 57, 58, 59, 61, 63, 64, ..., 76, 82, 90\}$. By Theorem 3.1, $\{47, 49, 53, 56, 59, 61, 63, 64, 67, 71, 72, 73\} \subset \mathrm{T}(8, 1)$. For other values of r namely $r \in \{50, 57, 58, 65, 66, 68, 69, 70, 74, 75, 76, 82, 90\}$, we shall prove that $r \in B(K, 1)$ with $K \subset \mathrm{RT}(7, 1)$ and use Theorem 3.3.

$50 \in B(\{7, 8\}, 1)$. Lemma 3.12 and $7 \in \mathrm{T}(7, 1)$.

$57 \in B(8, 1)$. Lemma 3.12 and $7 \in \mathrm{T}(8, 1)$.

$58 \in B(\{7, 8, 9\}, 1)$. In $\mathrm{T}[9, 1; 8]$ delete 7 points from each of 2 groups; further apply Lemma 3.9.

$65 \in B(\{8, 9\}, 1)$. Lemma 3.12 and $8 \in \mathrm{T}(8, 1)$.

$66 \in B(\{7, 8, 9\}, 1)$. In $\mathrm{T}[10, 1; 9]$ delete 8 points from each of 3 groups in such a way that the remaining 3 points be not in one block (this is possible, as explained in the proof of Lemma 3.18); apply Lemma 3.9.

$68 \in B(\{7, 8, 9\}, 1)$. In $\mathrm{T}[9, 1; 8]$ delete 1 point from each of 4 groups in such a way that no 3 deleted points be in one block; apply Lemma 3.9.

$69 \in B(\{7, 8, 9\}, 1)$. In $\mathrm{T}[9, 1; 8]$ delete 1 point from each of 3 groups, not all 3 deleted points from the same block; apply Lemma 3.9.

$70 \in B(\{7, 8, 9\}, 1)$. In $\mathrm{T}[8, 1; 9]$ delete 2 points from one of the groups; apply Lemma 3.9.

$74 \in B(\{7, 8, 9\}, 1)$. Let G_0, G_1, G_2, G_3 be any 4 groups in $\mathrm{T}[9, 1; 9]$ and let $a_0 \in G_0$ and B_0, B_1 be any 2 blocks containing a_0. Delete the 4 points $(B_0 \cup B_1) \cap (G_2 \cup G_3)$ and one point on G_1 which is not in a block from which already two points have been deleted; further delete from G_0 two points, none of them in a block from which already two points have been deleted; apply Lemma 3.9.

$75 \in B(\{7, 8, 9\}, 1)$. Like above, but in the last step delete from G_0 only one point.

$76 \in B(\{7, 8, 9\}, 1)$. Like above, but delete no points from G_0.

$82 \in B(\{7, 8, 9, 11\}, 1)$. In $\mathrm{T}[12, 1, 11]$ delete 10 points from each of 5 groups in such a way that 4 of the remaining 5 points be in one block and the fifth point be not in this block; apply Lemma 3.9.

$90 \in B(\{7,8,9,11\},1)$. In T[9, 1; 11] delete 3 points from one group and 2 points from another one; then delete 4 points from a third group, none of them in a block from which already 2 points have been deleted; apply Lemma 3.9.

A different proof of Theorem 3.8 – independent of Theorem 3.7 – may be found in [17].

Before proceeding we prove the following lemma [11] (see also [14, p. 203]):

Lemma 3.21. $12 \in \mathrm{RT}(6,1)$.

Proof. By $C(2, a^2 = 1)$, $C(6, b^6 = 1)$ we denote cyclic groups of orders 2 and 6 with generators a and b, respectively. As usually we shall use the exponents of the generators to denote the points of the design.

$$X = (C(2, a^2 = 1) \times C(6, b^6 = 1)) \times I(6).$$

$$\begin{aligned}
\mathcal{P} = {} & \langle(0,0;0'),(0,0;1'),(0,0;2'),(0,0;3'),(0,0;4'),(0,0;5')\rangle \bmod (2,6;-),\\
& \langle(0,0;0'),(0,1;1'),(1,0;2'),(0,3;3'),(1,2;4'),(0,4;5')\rangle \bmod (2,6;-),\\
& \langle(0,0;0'),(0,2;1'),(1,2;2'),(1,0;3'),(0,1;4'),(1,5;5')\rangle \bmod (2,6;-),\\
& \langle(0,0;0'),(0,3;1'),(0,2;2'),(0,1;3'),(1,5;4'),(1,4;5')\rangle \bmod (2,6;-),\\
& \langle(0,0;0'),(0,4;1'),(1,1;2'),(1,3;3'),(0,5;4'),(0,2;5')\rangle \bmod (2,6;-),\\
& \langle(0,0;0'),(0,5;1'),(0,1;2'),(1,5;3'),(1,3;4'),(1,1;5')\rangle \bmod (2,6;-),\\
& \langle(0,0;0'),(1,0;1'),(1,3;2'),(0,2;3'),(0,3;4'),(1,2;5')\rangle \bmod (2,6;-),\\
& \langle(0,0;0'),(1,1;1'),(1,5;2'),(1,2;3'),(1,4;4'),(1,0;5')\rangle \bmod (2,6;-),\\
& \langle(0,0;0'),(1,2;1'),(0,4;2'),(0,5;3'),(0,2;4'),(1,3;5')\rangle \bmod (2,6;-),\\
& \langle(0,0;0'),(1,3;1'),(1,4;2'),(0,4;3'),(1,1;4'),(0,1;5')\rangle \bmod (2,6;-),\\
& \langle(0,0;0'),(1,4;1'),(0,5;2'),(1,1;3'),(1,0;4'),(0,3;5')\rangle \bmod (2,6;-),\\
& \langle(0,0;0'),(1,5;1'),(0,3;2'),(1,4;3'),(0,4;4'),(0,5;5')\rangle \bmod (2,6;-).
\end{aligned}$$

Theorem 3.9. *For every* $r > 52$, $r \in \mathrm{T}(6,1)$ *holds.*

Remark. We prove moreover that $\{41, 43, 45, 47, 49, 50\} \subset T(6, 1)$.

Proof. By the remark to Theorem 3.8, it remains to prove this theorem for the set of integers $\{41, 43, 45, 54, 55, 60, 62\}$. By Theorem 3.1, $\{41, 43, 45, 55\} \subset T(6, 1)$. Further,

$54 \in T(6, 1)$ by Lemma 3.17 with $r = 7$, $s = 6$, $m = 7$, $r_1 = 0$, $r_2 = 5$.

$60 \in T(6, 1)$ by Lemma 3.4 and Lemma 3.21.

$62 \in T(6, 1)$ by Lemma 3.17 with $r = 8$, $s = 6$, $m = 7$, $r_1 = 1$, $r_2 = 5$.

Theorem 3.10. *For every* $r > 42$, $r \in T(5, 1)$ *holds.*

Proof. By the remark to Theorem 3.9, it remains to prove that $\{44, 46, 48, 51, 52\} \subset T(5,1)$. By Theorem 3.1, $\{44, 52\} \subset T(5, 1)$. For other values of r we have:

$46 \in T(5, 1)$ (see [35]). We prove $46 \in B(\{4, 5, 7, 8\}, 1)$ with exactly one block size of 4. Take

$$X = Z(7, 3) \times Z(3, 2) \times Z(2) \cup \{(\infty_i)\colon i = 0, 1, 2, 3\},$$

$$\begin{aligned}\mathcal{B} = {} & \langle(\infty_0), (\infty_1), (\infty_2), (\infty_3)\rangle, \\ & \langle(\infty_\alpha), (2\alpha, \emptyset, \emptyset), (2\alpha + 2, 0, \emptyset), (2\alpha + 4, 1, \emptyset), (4\alpha, \emptyset, 0), \\ & \quad (4\alpha + 2, 0, 0), (4\alpha + 4, 1, 0)\rangle \bmod (7, -, -), \quad \alpha = 0, 1, 2, \\ & \langle(\infty_3), (\emptyset, \emptyset, \emptyset), (0, \emptyset, \emptyset), (1, \emptyset, \emptyset), (2, \emptyset, \emptyset), (3, \emptyset, \emptyset), (4, \emptyset, \emptyset), \\ & \quad (5, \emptyset, \emptyset)\rangle \bmod (-, 3, 2), \\ & \langle(\emptyset, \emptyset, \emptyset), (\emptyset, 0, \emptyset), (4, \emptyset, 0), (2, 0, 0), (0, 1, 0)\rangle \bmod (7, -, 2), \\ & \langle(\emptyset, 0, \emptyset), (\emptyset, 1, \emptyset), (2, \emptyset, 0), (0, 0, 0), (4, 1, 0)\rangle \bmod (7, -, 2), \\ & \langle(\emptyset, \emptyset, \emptyset), (\emptyset, 1, \emptyset), (0, \emptyset, 0), (4, 0, 0), (2, 1, 0)\rangle \bmod (7, -, 2).\end{aligned}$$

Further apply Theorem 3.3. It should be noted that $4 \notin RT(5, 1)$ but only $4 \in T(5, 1)$. However, it may be easily checked that the existence of only one block of size 4 (or – by the way – of any number of such disjoint blocks) does not prevent the use of Theorem 3.3.

$48 \in T(5, 1)$ by Lemma 3.17 with $r = 11$, $s = 5$, $m = 4$, $r_1 = 0$, $r_2 = 4$. Because $r_1 = 0$, the condition $m + 2 \in T(s, 1)$ is not necessary and may be omitted.

$51 \in T(5, 1)$ by Lemma 3.17 with $r = 11$, $s = 5$, $m = 4$, $r_1 = 0$, $r_2 = 7$. Because $r_1 = 0$, the condition $m + 2 \in T(s, 1)$ may be omitted.

3.4. Transversal designs with $\lambda > 1$

We prove the theorem [19]:

Theorem 3.11. *For every $r \geq 1$ and every $\lambda > 1$, $r \in T(7, \lambda)$ holds.*

Proof. For $r = 1$ the theorem is trivial. Further, it follows from Theorem 3.1 and Lemma 3.4 that it is sufficient to prove the theorem for r being factors of 60; also from Lemmas 3.1, 3.3 and 3.7 follows that it suffices to prove the theorem for $\lambda = 2$ and $\lambda = 3$ and for each of these values of λ, to prove $r \in T(s, \lambda)$ or $r \in RT(s-1, \lambda)$ for any $s \geq 7$.

$2 \in T(7, 2)$. $X = I(2) \times Z(7, 3)$.

$\mathcal{P} = \langle(0';\emptyset), (0';0), (0';1), (0';2), (0';3), (0';4), (0';5)\rangle,$
$\langle(0';0), (0';2), (0';4), (1';\emptyset), (1';1), (1';3), (1';5)\rangle \bmod (-;7)$.

$2 \in T(11, 3)$. $X = I(2) \times Z(11, 2)$.

$\mathcal{P} = \langle(0';\emptyset), (0';0), (0';1), (0';2), (0';3), (0';4), (0';5), (0';6), (0';7), (0';8), (0';9)\rangle,$
$\langle(0';0), (0';2), (0';4), (0';6), (0';8), (1';\emptyset), (1';1), (1';3), (1';5), (1';7), (1';9)\rangle \bmod (-;11)$.

$3 \in RT(6, 2)$. $X = Z(3, 2) \times (Z(3, 2) \times I(2))$.

$\mathcal{P} = \langle(\emptyset;\emptyset, 0'), (\emptyset;0, 0'), (0;1, 0'), (1;\emptyset, 1'), (1;0, 1'), (0;1, 1')\rangle \bmod (3;3, -),$
$\langle(\emptyset;\emptyset, 0'), (\emptyset;0, 0'), (\emptyset;1, 0'), (\emptyset;\emptyset, 1'), (\emptyset;0, 1'), (\emptyset;1, 1')\rangle \bmod (3;-, -),$
$\langle(\emptyset;1, 0'), (0;\emptyset, 0'), (1;0, 0'), (\emptyset;0, 1'), (0;1, 1'), (1;\emptyset, 1')\rangle \bmod (3;-, -),$
$\langle(\emptyset;0, 0'), (1;1, 0'), (0;\emptyset, 0'), (\emptyset;1, 1'), (1;\emptyset, 1'), (0;0, 1')\rangle \bmod (3;-, -)$.

$3 \in T(13, 3)$. $X = (Z(2) \cup \{\infty\}) \times Z(13, 2)$.

$\mathcal{P} = \langle(\infty;\emptyset), (\infty;0), (\infty;1), (\infty;2), (\infty;3), (\infty;4), (\infty;5), (\infty;6), (\infty;7), (\infty;8), (\infty;9), (\infty;10), (\infty;11)\rangle,$
$\langle(\infty;\emptyset), (\infty;1), (\infty;5), (\infty;9), (\emptyset;0), (\emptyset;4), (\emptyset;8), (0;2), (0;3), (0;6), (0;7), (0;10), (0, 11)\rangle \bmod (2;13)$.

$4 \in \mathrm{RT}(8,2)$. $X = \mathrm{GF}(4, x^2 = x+1) \times (Z(7,3) \cup \{\infty\})$.

$\mathcal{P} = \langle(\emptyset;\infty), (\emptyset;\emptyset), (\emptyset;0), (\emptyset;1), (\emptyset;2), (\emptyset;3), (\emptyset;4), (\emptyset;5)\rangle \bmod (4;-)$,

$\langle(\emptyset;\infty), (\emptyset;\emptyset), (0;0), (0;1), (1;2), (1;3), (2;4), (2;5)\rangle \bmod (4;7)$.

$4 \in \mathrm{RT}(8,3)$. $X = \mathrm{GF}(4, x^2 = x+1) \times \mathrm{GF}(8, x^3 = x+1)$.

$\mathcal{P} = \langle(\emptyset;\emptyset), (\emptyset;0), (\alpha;1), (\alpha+1;2), (\emptyset;3), (\alpha+1;4), (\alpha;5), (\alpha;6)\rangle$

$\bmod (4;-)$, $\alpha = 0, 1, 2$,

$\langle(\emptyset;\emptyset), (\alpha;0), (\emptyset;1), (\alpha+1;2), (\alpha+1;3), (\emptyset;4), (\alpha;5), (\alpha+1;6)\rangle$

$\bmod (4;-)$, $\alpha = 0, 1, 2$,

$\langle(\emptyset;\emptyset), (\alpha;0), (\alpha+1;1), (\emptyset;2), (\alpha+1;3), (\alpha;4), (\emptyset;5), (\alpha;6)\rangle$

$\bmod (4;-)$, $\alpha = 0, 1, 2$,

$\langle(\emptyset;\emptyset), (\alpha+1;0), (\alpha+1;1), (\alpha+1;2), (\alpha;3), (\alpha;4), (\alpha;5), (\emptyset;6)\rangle$

$\bmod (4;-)$, $\alpha = 0, 1, 2$.

$5 \in \mathrm{RT}(10,2)$. $X = Z(5,2) \times (Z(5,2) \times I(2))$.

$\mathcal{P} = \langle(\emptyset;\emptyset,0'), (\emptyset;\emptyset,1'), (0;1,0'), (0;3,0'), (1;0,1'), (1;2,1'), (2;0,0')$,

$(2;2,0'), (3;1,1'), (3;3,1')\rangle \bmod (5;5,-)$,

$\langle(\emptyset;\emptyset,0'), (\emptyset;\emptyset,1'), (1;0,0'), (1;1,1'), (1;2,0'), (1;3,1'), (3;0,1')$,

$(3;1,0'), (3;2,1'), (3;3,0')\rangle \bmod (5;5,-)$.

$5 \in \mathrm{RT}(7,3)$: $X = Z(5,2) \times Z(7,3)$.

$\mathcal{P} = \langle(\emptyset;\emptyset), (\emptyset;0), (\emptyset;1), (\emptyset;2), (\emptyset;3), (\emptyset;4), (\emptyset;5)\rangle \bmod (5;-)$,

$\langle(\emptyset;\emptyset), (\alpha;0), (\alpha;3), (\alpha+1;2), (\alpha+1;5), (\alpha+3;1), (\alpha+3;4)\rangle$

$\bmod (5;7)$, $\alpha = 0, 1$.

$6 \in \mathrm{T}(7,2)$. $X = (Z(5,2) \cup \{\infty\}) \times Z(7,3)$.

$\mathcal{P} = \langle(\infty;\emptyset), (\infty;0), (\infty;1), (\infty;2), (\infty;3), (\infty;4), (\infty;5)\rangle$, 2 times,

$\langle(\infty;\emptyset), (\emptyset;0), (\emptyset;3), (\alpha;2), (\alpha;5), (\alpha+2;1), (\alpha+2;4)\rangle \bmod (5;7)$,

$\alpha = 0, 1$.

$6 \in \mathrm{T}(7,3)$. $X = (Z(5,2) \cup \{\infty\}) \times Z(7,3)$.

$\mathcal{P} = \langle(\infty;\emptyset), (\infty;0), (\infty;1), (\infty;2), (\infty;3), (\infty;4), (\infty;5)\rangle$, 3 times,

$\langle(\infty;\emptyset),(\emptyset;1),(\emptyset;2),(\emptyset;5),(2\alpha;4),(2\alpha+1;0),(2\alpha+3;3)\rangle \bmod (5;7)$,
$\alpha=0,1$,
$\langle(\infty;\emptyset),(\emptyset;0),(\emptyset;3),(0;2),(0;5),(2;1),(2;4)\rangle \bmod (5;7)$.

$10\in T(7,2)$. $X=(Z(9)\cup\{\infty\})\times Z(7,3)$.
$\mathcal{P}=\langle(\infty;\emptyset),(\infty;0),(\infty;1),(\infty;2),(\infty;3),(\infty;4),(\infty;5)\rangle$, 2 times,
$\langle(0';\emptyset),(0';0),(0';1),(0';2),(0';3),(0';4),(0';5)\rangle \bmod (9;-)$,
$\langle(\infty;\emptyset),(0';3\alpha),(1';3\alpha+2),(2';3\alpha+4),(4';3\alpha+5),(6';3\alpha+1),$
$(7';3\alpha+3)\rangle \bmod (9;7)$, $\alpha=0,1$,
$\langle(0';\emptyset),(1';2),(1';5),(2';0),(2';3),(7';1),(7';4)\rangle \bmod (9;7)$.

$10\in T(8,3)$. $X=(Z(3,2)\times Z(3,2)\cup\{\infty\})\times(Z(2)\times Z(2)\times Z(2))$.
$\mathcal{P}=\langle(\infty;\emptyset,\emptyset,\emptyset),(\infty;\emptyset,\emptyset,0),(\infty;\emptyset,0,\emptyset),(\infty;\emptyset,0,0),(\infty;0,\emptyset,\emptyset),$
$(\infty;0,\emptyset,0),(\infty;0,0,\emptyset),(\infty;0,0,0)\rangle$, 3 times ,
$\langle(\emptyset,\emptyset;\emptyset,\emptyset,\emptyset),(\emptyset,\emptyset;\emptyset,\emptyset,0),(\emptyset,\emptyset;\emptyset,0,\emptyset),(\emptyset,\emptyset;\emptyset,0,0),(\emptyset,\emptyset;0,\emptyset,\emptyset),$
$(\emptyset,\emptyset;0,\emptyset,0),(\emptyset,\emptyset;0,0,\emptyset),(\emptyset,\emptyset;0,0,0)\rangle \bmod (3,3;-,-,-)$,
$\langle(\infty;\emptyset,\emptyset,\emptyset),(\emptyset,\emptyset;\emptyset,\emptyset,0),(\emptyset,\emptyset;\emptyset,0,\emptyset),(\emptyset,0;\emptyset,0,0),(\emptyset,1;0,0,\emptyset),$
$(0,1;0,\emptyset,0),(0,1;0,\emptyset,\emptyset),(1,\emptyset;0,0,0)\rangle \bmod (3,3;2,2,2)$,
$\langle(\infty;\emptyset,\emptyset,\emptyset),(\emptyset,\emptyset;\emptyset,0,\emptyset),(\emptyset,\emptyset;0,\emptyset,\emptyset),(\emptyset,0;0,0,\emptyset),(\emptyset,1;0,\emptyset,0),$
$(0,0;\emptyset,0,0),(0,0;\emptyset,\emptyset,0),(1,0;0,0,0)\rangle \bmod (3,3;2,2,2)$,
$\langle(\infty;\emptyset,\emptyset,\emptyset),(\emptyset,\emptyset;0,\emptyset,\emptyset),(\emptyset,\emptyset;\emptyset,\emptyset,0),(\emptyset,0;0,\emptyset,0),(\emptyset,1;\emptyset,0,0),$
$(0,\emptyset;0,0,\emptyset),(0,\emptyset;\emptyset,0,\emptyset),(1,1;0,0,0)\rangle \bmod (3,3;2,2,2)$,
$\langle(\emptyset,\emptyset;\emptyset,0,0),(\emptyset,0;0,\emptyset,0),(\emptyset,1;0,0,\emptyset),(0,\emptyset;\emptyset,0,\emptyset),(0,0;0,\emptyset,\emptyset),$
$(0,1;\emptyset,\emptyset,0),(1,\emptyset;\emptyset,\emptyset,\emptyset),(1,\emptyset;0,0,0)\rangle \bmod (3,3;2,2,2)$.

$12\in RT(6,1)$. Lemma 3.21.
$15\in RT(7,2)$. $X=(Z(3,2)\times Z(5,2))\times Z(7,3)$.
$\mathcal{P}=\langle(\emptyset,\emptyset;\emptyset),(\emptyset,\emptyset;0),(\emptyset,\emptyset;1),(\emptyset,\emptyset;2),(\emptyset,\emptyset;3),(\emptyset,\emptyset;4),(\emptyset,\emptyset;5)\rangle$
$\bmod (3,5;-)$, 2 times,
$\langle(\emptyset,\beta+3;\emptyset),(\alpha,\emptyset;0),(\alpha+1,\emptyset;3),(\alpha,\beta;2),(\alpha+1,\beta;5),(\alpha,\beta+2;4),$
$(\alpha+1,\beta+2;1)\rangle \bmod (3,5;7)$, $\alpha=0,1$, $\beta=0,1$.

$15 \in \mathrm{RT}(7,3)$. $X = (Z(3,2) \times Z(5,2)) \times Z(7,3)$.

$$\mathcal{P} = \langle(\emptyset,\emptyset;\emptyset),(\emptyset,\emptyset;0),(\emptyset,\emptyset;1),(\emptyset,\emptyset;2),(\emptyset,\emptyset;3),(\emptyset,\emptyset;4),(\emptyset,\emptyset;5)\rangle \bmod (3,5;-),\quad 3 \text{ times},$$
$$\langle(\emptyset,\emptyset;\emptyset),(0,\emptyset;\alpha),(1,\emptyset;\alpha+1),(0,0;\alpha+2),(1,1;\alpha+3),(0,2;\alpha+4),(1,3;\alpha+5)\rangle \bmod (3,5;7),\quad \alpha=0,1,\ldots,5.$$

$20 \in \mathrm{T}(7,2)$. $X = (Z(19,2) \cup \{\infty\}) \times Z(7,3)$.

$$\mathcal{P} = \langle(\infty;\emptyset),(\infty;0),(\infty;1),(\infty;2),(\infty;3),(\infty;4),(\infty;5)\rangle,\quad 2 \text{ times},$$
$$\langle(\infty;\emptyset),(10\alpha+2;0),(10\alpha+2;3),(10\alpha+8;2),(10\alpha+8;5),(10\alpha+14;1),(10\alpha+14;4)\rangle \bmod (19;7),\quad \alpha=0,1,$$
$$\langle(\emptyset;\emptyset),(9\alpha;4\beta),(9\alpha+1;4\beta+3),(9\alpha+6;4\beta+2),(9\alpha+7;4\beta+5),(9\alpha+12;4\beta+4),(9\alpha+13;4\beta+1)\rangle \bmod (19;7),\quad \alpha=0,1,\ \beta=0,1.$$

$20 \in \mathrm{T}(7,3)$. $X = (Z(19,2) \cup \{\infty\}) \times Z(7,3)$.

$$\mathcal{P} = \langle(\infty;\emptyset),(\infty;0),(\infty;1),(\infty;2),(\infty;3),(\infty;4),(\infty;5)\rangle,\quad 3 \text{ times},$$
$$\langle(\infty;\emptyset),(\mu;0),(\mu;3),(\mu+6;2),(\mu+6;5),(\mu+12;1),(\mu+12;4)\rangle \bmod (19;7),\quad \mu=6,14,16,$$
$$\langle(\emptyset;\emptyset),(2\alpha+9\beta;0),(2\alpha+9\beta+1;3),(2\alpha+9\beta+6;2),(2\alpha+9\beta+7;5),(2\alpha+9\beta+12;4),(2\alpha+9\beta+13;1)\rangle \bmod (19;7),\quad \alpha=0,1,2,\ \beta=0,1.$$

For $r = 30$ we make use of the resolvable transversal design $\mathrm{RT}[8,\lambda;4]$, $\lambda > 1$. Deleting 2 points from one of the groups we obtain $30 \in \mathrm{GD}(\{7,8\},\lambda,\{2,4\})$. Further we apply Theorem 3.2 and observe that $\{7,8\} \subset \mathrm{RT}(7,1)$ and $\{2,4\} \subset \mathrm{T}(7,\lambda)$. Consequently, $30 \in \mathrm{T}(7,\lambda)$ for $\lambda > 1$.

For $r = 60$ we make use of $\mathrm{T}[8,1;8]$ and delete 4 points from one of the groups. We obtain $60 \in \mathrm{GD}(\{7,8\},1,\{4,8\})$. Apply again Theorem 3.2 and observe that $\{7,8\} \subset \mathrm{RT}(7,1)$ and $\{4,8\} \subset \mathrm{T}(7,\lambda)$, $\lambda > 1$. Consequently, $60 \in \mathrm{T}(7,\lambda)$ for $\lambda > 1$.

With the help of Theorem 3.11 the following – most useful – lemmas are easily proved in the same way as Lemma 3.11 and Lemma 3.12.

Lemma 3.22. *If* $\lambda > 1$, $k \leqslant 7$ *and* $r \in B(k,\lambda)$, *then* $kr \in B(k,\lambda)$.

Lemma 3.23. *If* $\lambda > 1$, $k \leqslant 7$ *and* $r+1 \in B(k,\lambda)$, *then* $kr+1 \in B(k,\lambda)$.

4. Auxiliary designs

4.1. General auxiliary designs

If $v = q$ is a power of a prime, then some BIBD's may be easily constructed with the help of Galois fields $GF(q)$. Let x be a primitive mark of $GF(q)$ satisfying the equation $f(x) = 0$. To shorten the notation we shall denote a Galois field $GF(q)$ with $f(x) = 0$ by $GF(q, f(x) = 0)$ (see Notation, Section 1.5). Further we shall denote by $f = (a, b)$ the greatest common factor of a and b.

The Lemmas 4.1–4.4 are generalizations of observations made originally by Bose [2].

Lemma 4.1. *If q is a prime-power and $f = (q-1, k)$, then $q \in B(k, k(k-1)/f)$.*

Proof. Let $d = (q-1)/f$. $X = GF(q, f(x) = 0)$.
$\mathcal{B} = \langle \alpha, \alpha+1, ..., \alpha-1+k/f, \alpha+d, \alpha+d+1, ..., \alpha+d-1+k/f, ...,$
$\alpha+(f-1)d, \alpha+(f-1)d+1, ..., \alpha+(f-1)d-1+k/f \rangle \bmod q,$
$\alpha = 0, 1, ..., d-1.$

Lemma 4.2. *If q is a prime-power and $f = (q-1, k-1)$, then $q \in B(k, k(k-1)/f)$.*

Proof. Let $d = (q-1)/f$. $X = GF(q, f(x) = 0)$.
$\mathcal{B} = \langle \emptyset, \alpha, \alpha+1, ..., \alpha-1+(k-1)/f, \alpha+d, \alpha+d+1, ..., \alpha+d-1+(k-1)/f,$
$..., \alpha+(f-1)d, \alpha+(f-1)d+1, ..., \alpha+(f-1)d-1+(k-1)f \rangle$
$\bmod q, \quad \alpha = 0, 1, ..., d-1.$

Lemma 4.3. *If q is a power of an odd prime, k is odd and $f = (q-1, k)$, then $q \in B(k, k(k-1)/2f)$.*

Proof. Let $d = (q-1)/f$. $X = GF(q, f(x) = 0)$.
$\mathcal{B} = \langle \alpha, \alpha+1, ..., \alpha-1+k/f, \alpha+d, \alpha+d+1, ..., \alpha+d-1+k/f, ...,$
$\alpha+(f-1)d, \alpha+(f-1)d+1, ..., \alpha+(f-1)d-1+k/f \rangle \bmod q,$
$\alpha = 0, 1, ..., \frac{1}{2}d-1.$

Lemma 4.4. *If q is a power of an odd prime, k is even and $f = (q-1, k-1)$, then $q \in B(k, k(k-1)/2f)$.*

Proof. Let $d = (q-1)/f$. $X = \mathrm{GF}(q, f(x) = 0)$.
$\mathcal{B} = \langle \emptyset, \alpha, \alpha+1, \ldots, \alpha - 1 + (k-1)/f, \alpha + d + 1, \ldots, \alpha + d - 1 + (k-1)/f, \ldots,$
$\alpha + (f-1)d, \alpha + (f-1)d + 1, \ldots, \alpha + (f-1)d - 1 + (k-1)/f \rangle \bmod q,$
$\alpha = 0, 1, \ldots, \frac{1}{2}d - 1.$

Further, by taking the complement of each block we obtain the following almost selfevident lemma:

Lemma 4.5. *If* $2 \leqslant k \leqslant v - 2$ *and if* $v \in B(k, \lambda)$, *then* $B[v-k, \lambda'; v]$ *exists, where* $\lambda' = \lambda(v-k)(v-k-1)/(k(k-1))$.

Mullin and Stanton [22] proved:

Lemma 4.6. *If both* $B[k, \lambda; v]$ *and* $B[k+1, r-\lambda; v]$ *exist, where* $r = \lambda(v-1)/(k-1)$, *then also* $B[k+1, r; v+1]$ *exists.*

Proof. $X = I(v) \cup \{\infty\}$. $\mathcal{B}$ = blocks of $B[k, \lambda; v]$ on $I(v)$ with the point ∞ adjoint to each block and blocks of $B[k+1, r-\lambda; v]$ on $I(v)$.

From Theorem 2.1 and Lemma 2.12 follows:

Lemma 4.7. *If* $k-1$ *is a prime-power, then* $(k-1)k \in \mathrm{GD}(k, 1, k-1)$.

From Theorem 2.2 and Lemma 2.12 follows:

Lemma 4.8. *If* k *is a prime-power, then* $(k-1)(k+1) \in \mathrm{GD}(k, 1, k-1)$.

4.2. Special auxiliary designs

Lemma 4.9. $\{6, 8\} \subset \mathrm{GD}(3, 1, 2)$.

Proof. Follows from Lemmas 4.7 and 4.8.

Lemma 4.10. $\{12, 15\} \subset \mathrm{GD}(4, 1, 3)$.

Proof. Follows from Lemmas 4.7 and 4.8.

Lemma 4.11. $\{20, 24\} \subset \mathrm{GD}(5,1,4)$.

Proof. Follows from Lemmas 4.7 and 4.8.

Lemma 4.12. *If q is a power of an odd prime, then $5q \in \mathrm{GD}(5,2,5)$ and $5q \in B(5,2)$.*

Proof. We prove $5q \in \mathrm{GD}(5,2,5)$. Let $d = \frac{1}{2}(q-1)$.
$X = Z(5,2) \times \mathrm{GF}(q, f(x) = 0)$.
$\mathcal{P} = \langle(\emptyset;\emptyset), (0;\alpha), (0;\alpha+d), (2;\alpha+1), (2;\alpha+d+1)\rangle \bmod (5;q)$,
$\alpha = 0, 1, \ldots, d-1$.
$5q \in B(5,2)$ follows from Lemma 2.10.

Lemma 4.13. *If $q \equiv 1 \pmod 4$ is a prime-power, then $5q \in \mathrm{GD}(5,1,5)$ and $5q \in B(5,1)$.*

Proof. We prove $5q \in \mathrm{GD}(5,1,5)$. Let $d = \frac{1}{4}(q-1)$.
$X = Z(5,2) \times \mathrm{GF}(q, f(x) = 0)$.
$\mathcal{P} = \langle(\emptyset;\emptyset), (0;\alpha), (0;\alpha+2d), (2;\alpha+d), (2;\alpha+3d)\rangle \bmod (5;q)$,
$\alpha = 0, 1, \ldots, d-1$.
$5q \in B(5,1)$ follows from Lemma 2.10.

Lemma 4.14. $\{40, 44\} \subset \mathrm{GD}(5,1,4)$.

Proof. We prove $41 \in B(5,1)$. $X = Z(41,6)$.
$\mathcal{B} = \langle 2\alpha, 2\alpha+8, 2\alpha+16, 2\alpha+24, 2\alpha+32\rangle \bmod 41, \quad \alpha = 0, 1$.
$45 \in B(5,1)$ holds by Lemma 4.13. Further apply Lemma 2.12.

Lemma 4.15. $\{10, 12\} \subset \mathrm{GD}(5,2,2)$.

Proof. $10 \in \mathrm{GD}(5,2,2)$ follows from Theorem 3.11 and Lemma 3.1.
$12 \in \mathrm{GD}(5,2,2)$. $X = Z(2) \times (Z(5,2) \cup \{\infty\})$.
$\mathcal{P} = \langle(\emptyset;\infty), (\emptyset;0), (\emptyset;2), (0;1), (0;3)\rangle \bmod (2;5)$,
$\langle(\emptyset;\emptyset), (\emptyset;0), (\emptyset;1), (\emptyset;2), (\emptyset;3)\rangle \bmod (2;-)$.

Lemma 4.16. *If q is a power of an odd prime, then $4q \in \mathrm{GD}(5,5,4)$ and $4q+1 \in B(5,5)$.*

Proof. We prove $4q \in \mathrm{GD}(5, 5, 4)$. Let $d = \frac{1}{2}(q-1)$.
$X = \mathrm{GF}(4, x^2 = x+1) \times \mathrm{GF}(q, f(y) = 0)$.
$\mathcal{P} = \langle(\emptyset;\emptyset), (0;\alpha), (0;\alpha+d), (1;\alpha+1), (1;\alpha+d+1)\rangle \bmod (4;q)$,
$\alpha = 0, 1, \ldots, d-1$,
$\langle(\emptyset;\emptyset), (\emptyset;\alpha), (\emptyset;\alpha+d), (0;\alpha+1), (1;\alpha+d+1)\rangle \bmod (4;q)$,
$\alpha = 0, 1, \ldots, d-1$.

$4q + 1 \in B(5, 5)$ follows from Lemma 2.11.

Lemma 4.17. $\{8, 14, 32\} \subset B(\{5, 6\}, 5)$.

Proof. $8 \in B(\{5, 6\}, 5)$. $X = Z(2) \times Z(4)$.
$\mathcal{B} = \langle(0', 0'), (0', 1'), (0', 2'), (1', 0'), (1', 1')\rangle \bmod (-, 4)$,
$\langle(0', 0'), (0', 3'), (1', 0'), (1', 1'), (1', 2')\rangle \bmod (-, 4)$,
$\langle(0', 0'), (0', 1'), (0', 2'), (0', 3'), (1', \alpha'), (1', (\alpha+2)')\rangle \bmod (2, -)$,
$\alpha = 0, 1$.

$14 \in B(\{5, 6\}, 5)$. $X = Z(13, 2) \cup \{\infty\}$.
$\mathcal{B} = \langle\infty, \emptyset, 2, 5, 8, 11\rangle \bmod 13$,
$\langle\emptyset, \alpha, \alpha+3, \alpha+6, \alpha+9\rangle \bmod 13$, $\alpha = 0, 1$.

$32 \in B(\{5, 6\}, 5)$. $X = Z(31, 3) \cup \{\infty\}$.
$\mathcal{B} = \langle\infty, 5, 11, 17, 23, 29\rangle \bmod 31$,
$\langle 0, 3, 10, 13, 20, 23\rangle \bmod 31$,
$\langle\alpha, \alpha+6, \alpha+12, \alpha+18, \alpha+24\rangle \bmod 31$, $\alpha = 0, 1, 2, 3, 4$.

Lemma 4.18. $14 \in \mathrm{GD}(5, 10, 2)$.

Proof. $X = Z(2) \times Z(7, 3)$.
$\mathcal{P} = \langle(\emptyset;\emptyset), (\emptyset;\alpha), (\emptyset;\alpha+3), (0;\alpha+1), (0;\alpha+4)\rangle \bmod (2;7)$, $\alpha = 0, 1, 2$,
$\langle(\emptyset;\alpha), (\emptyset;\alpha+1), (\emptyset;\alpha+3), (\emptyset;\alpha+4), (0;\emptyset)\rangle \bmod (2;7)$, $\alpha = 0, 1, 2$.

Lemma 4.19. *If* $q \equiv 1 \pmod 6$ *is a prime-power, then* $5q \in \mathrm{GD}(6, 2, 5)$ *and* $5q + 1 \in B(6, 2)$.

Proof. We prove $5q \in \mathrm{GD}(6, 2, 5)$. Let $d = \frac{1}{6}(q-1)$.
$X = Z(5, 2) \times \mathrm{GF}(q, f(x) = 0)$.
$\mathcal{P} = (\emptyset, \alpha), (\emptyset, \alpha+3d), (\beta, \alpha+d), (\beta, \alpha+4d), (\beta+2, \alpha+2d), (\beta+2, \alpha+5d)\rangle$
$\bmod (5, q)$, $\alpha = 0, 1, \ldots, d-1$, $\beta = 0, 1$.

$5q + 1 \in B(6,2)$ follows from Lemma 2.11.

Lemma 4.20. $30 \in GD(6,1,5)$.

Proof. Follows from Lemma 4.7.

Lemma 4.21. $\{35,45,50,60,65\} \subset GD(6,2,5)$.

Proof. $35 \in GD(6,2,5)$ follows from Lemma 4.19.
$45 \in GD(6,2,5)$. $X = I(5) \times GD(9, x^2 = 2x+1)$.
$\mathcal{P}_1 = \langle(0';\alpha),(0';\alpha+4),(1';\alpha+1),(2';\alpha+2),(3';\alpha+6),(4';\alpha+5)\rangle$ mod $(-;9)$, $\alpha = 0,1,\ldots,7$.
Change the notation of X to $X = I(5) \times (Z(3,2) \times Z(3,2))$.
$\mathcal{P}_2 = \langle(1';\emptyset,\emptyset),(1';\emptyset,0),(2';0,\emptyset),(2';1,0),(3';0,1),(3';1,1)\rangle$ mod $(-;3,3)$,
$\langle(1';\emptyset,\emptyset),(1';0,1),(2';\emptyset,0),(2';0,0),(4';1,\emptyset),(4';1,1)\rangle$ mod $(-;3,3)$,
$\langle(1';\emptyset,\emptyset),(1';0,0),(3';\emptyset,0),(3';0,\emptyset),(4';\emptyset,1),(4';0,1)\rangle$ mod $(-;3,3)$,
$\langle(2';\emptyset,\emptyset),(2';0,1),(3';1,\emptyset),(3';1,1),(4';\emptyset,1),(4';0,\emptyset)\rangle$ mod $(-;3,3)$,
$\langle(1';\emptyset,\emptyset),(1';\emptyset,0),(1';0,\emptyset),(1';0,0),(1';1,\emptyset),(1';1,0)\rangle$ mod $(-;-;3)$,
$\langle(2';\emptyset,\emptyset),(2';\emptyset,0),(2';\emptyset,1),(2';0,\emptyset),(2';0,0),(2';0,1)\rangle$ mod $(-;3;-)$,
$\langle(3';\emptyset,\emptyset),(3';\emptyset,0),(3';0,0),(3';0,1),(3';1,1),(3';1,\emptyset)\rangle$ mod $(-;-;3)$,
$\langle(4';\emptyset,\emptyset),(4';\emptyset,0),(4';0,1),(4';0,\emptyset),(4';1,0),(4';1,1)\rangle$ mod $(-;-,3)$,

$50 \in GD(6,2,5)$. $X = Z(5,2) \times Z(10)$.
$\mathcal{P} = \langle(\emptyset;0'),(\emptyset;1'),(\emptyset;2'),(0;3'),(2;5'),(3;6')\rangle$ mod $(5;10)$,
$\langle(\emptyset;0'),(\emptyset;2'),(\emptyset;6'),(0;1'),(0;8'),(3;3')\rangle$ mod $(5;10)$,
$\langle(\emptyset;0'),(\emptyset;5'),(1;1'),(1;4'),(2;2'),(3;8')\rangle$ mod $(5;10)$.

$60 \in GD(6,2,5)$. $X = Z(5,2) \times (Z(11,2) \cup \{\infty\})$.
$\mathcal{P} = \langle(\emptyset;\infty),(\emptyset;9),(2\alpha;4),(2\alpha+1;\emptyset),(2\alpha+1;0),(2\alpha+2;6)\rangle$ mod $(5;11)$, $\alpha = 0,1$,
$\langle(\emptyset;\emptyset),(\emptyset;1),(\emptyset;4),(2\alpha;7),(2\alpha;8),(2\alpha+3;2)\rangle$ mod $(5;11)$, $\alpha = 0,1$.

$65 \in GD(6,2,5)$ follows from Lemma 4.19.

Lemma 4.22. $\{35,40\} \subset GD(6,3,5)$.

Proof. $35 \in GD(6,3,5)$. $X = Z(5,2) \times Z(7,3)$.
$\mathcal{P} = \langle(\emptyset;0),(\emptyset;2),(\emptyset;4),(2\alpha;\emptyset),(2\alpha+1;1),(2\alpha+3;5)\rangle$ mod $(5;7)$. $\alpha = 0,1$,
$\langle(0;0),(0;3),(1;2),(1;5),(3;1),(3;4)\rangle$ mod $(5;7)$.

$40 \in GD(6, 3, 5)$. $X = Z(5, 2) \times GF(8, x^3 = x + 1)$.

$\mathcal{P} = \langle(\emptyset;\emptyset), (\emptyset;\alpha), (\emptyset;\alpha+3), (0;\alpha+6), (1;\alpha+4), (3;\alpha+1)\rangle \bmod (5;-)$,
$\alpha = 0, 1, \ldots, 6$,
$\langle(\emptyset;\emptyset), (\emptyset;\alpha+5), (0;\alpha+6), (2;\alpha), (2;\alpha+1), (2;\alpha+3)\rangle \bmod (5;-)$,
$\alpha = 0, 1, \ldots, 6$,
$\langle(\emptyset;\emptyset), (0;\alpha+3), (1;\alpha+2), (1;\alpha+6), (3;\alpha), (3;\alpha+1)\rangle \bmod (5;-)$,
$\alpha = 0, 1, \ldots, 6$,
$\langle(\emptyset;\alpha+1), (\emptyset;\alpha+3), (1;\alpha+2), (1;\alpha+4), (2;\alpha), (2;\alpha+6)\rangle \bmod (5;-)$,
$\alpha = 0, 1, \ldots, 6$.

Lemma 4.23. $\{18, 21, 24\} \subset GD(6, 5, 3)$.

Proof. $18 \in GD(6, 5, 3)$ follows from Theorem 3.11 and Lemma 3.1.

$21 \in GD(6, 5, 3)$. $X = Z(3, 2) \times Z(7, 3)$.

$\mathcal{P} = \langle(\alpha;0), (\alpha;2), (\alpha;4), (\alpha+1;1), (\alpha+1;3), (\alpha+1;5)\rangle \bmod (3;7)$, $\alpha = 0, 1$,
$\langle(\emptyset;0), (\emptyset;3), (0;1), (0;4), (1;2), (1;5)\rangle \bmod (3;7)$.

$24 \in GD(6, 5, 3)$. $X = Z(3, 2) \times (Z(7, 3) \cup \{\infty\})$.

$\mathcal{P} = \langle(\alpha;\infty), (\emptyset;1), (\emptyset;3), (\emptyset;5), (0;2\alpha+2), (1;2\alpha+4)\rangle \bmod (3;7)$, $\alpha = 0, 1$,
$\langle(\emptyset;\infty), (\emptyset;1), (\emptyset;3), (\emptyset;5), (0;0), (1;2)\rangle \bmod (3;7)$,
$\langle(0;1), (0;3), (0;5), (1;0), (1;2), (1;4)\rangle \bmod (3;7)$.

Lemma 4.24. $48 \in GD(7, 1, 6)$.

Proof. Follows from Lemma 4.8.

Lemma 4.25. $\{42, 54\} \subset GD(7, 7, 6)$.

Proof. $42 \in GD(7, 7, 6)$ follows from Theorem 3.11.

$54 \in GD(7, 7, 6)$ follows from Lemma 2.16, $9 \in B(8,7)$ by Lemma 4.1 and $48 \in GD(7, 1, 6)$ by Lemma 4.24.

Lemma 4.26. *If $q \geq 7$ is a power of an odd prime, then* $7q \in GD(7, 3, 7)$.

Remark. More generally, if $k \equiv 3 \pmod 4$ is a prime-power and $q \geq k$ is a power of an odd prime, then $kq \in GD(k, \frac{1}{2}(k-1), k)$.

Proof. Let $d = \frac{1}{2}(q-1)$.
$X = Z(7,3) \times \mathrm{GF}(q, f(x)=0)$.
$\mathcal{P} = \langle(\emptyset;\emptyset), (0;\alpha), (0;\alpha+d), (2;\alpha+1), (2;\alpha+d+1), (4;\alpha+2), (4;\alpha+d+2)\rangle$
mod $(7,q)$, $\alpha = 0, 1, ..., d-1$.

Lemma 4.27. $49 \in \mathrm{GD}(7,1,7)$.

Proof. Follows from Theorem 2.2.

Lemma 4.28. $\{56, 63\} \subset \mathrm{GD}(7,6,7)$.

Proof. $56 \in \mathrm{GD}(7,6,7)$ follows from Lemma 2.16, $8 \in B(7,6)$ by Lemma 4.1 and $49 \in \mathrm{GD}(7,1,7)$ by Lemma 4.27.

$63 \in \mathrm{GD}(7,3,7)$ follows from Lemma 4.26.

Lemma 4.29. $\{14, 16\} \subset \mathrm{GD}(7,3,2)$.

Proof. $14 \in \mathrm{GD}(7,3,2)$ follows from Theorem 3.11.
$16 \in \mathrm{GD}(7,3,2)$. $X = Z(2) \times (Z(7,3) \cup \{\infty\})$.
$\mathcal{P} = \langle(\emptyset;\infty), (\emptyset;0), (\emptyset;2), (\emptyset;4), (0;1), (0;3), (0;5)\rangle$ mod $(2;7)$,
$\langle(0;\emptyset), (0;0), (0;1), (0;2), (0;3), (0;4), (0;5)\rangle$ mod $(2;-)$.

5. Necessary and sufficient conditions for BIBD's

In this section we shall prove that the necessary condition of Theorem 1.1 is also sufficient for the existence of BIBD's $B[k,\lambda;v]$ with $k = 3,4,5$ and every λ with the exception of the nonexisting [9,15] BIBD, $B[5,2;15]$. Further we shall prove that this condition is sufficient for the existence of BIBD's with $k = 6$ and $\lambda > 1$ with the exception of the nonexisting [9,15] design $B[6,2,21]$, and also for the existence of BIBD's with $k = 7$ and $\lambda = 6, 7$ and 42.

In the proof of the relevant lemmas and theorems extensive use will be made of the Lemmas 2.1, 2.2, 2.3, 2.7, 2.10, 2.11, 2.18, 2.19, 2.20, 2.22, 3.1, 3.2, 3.5, 3.11, 3.12 and of the Theorems 2.1, 2.2, 3.1 *often without mentioning them.*

5.1. BIBD's with block-size 3

Lemma 5.1. *If* $u \equiv 0$ *or* 1 (mod 3), *then* $u \in \mathrm{GD}(\{3,4\},1,M_3)$, *where* $M_3 = \{1,3,4,6,7,19\}$.

Proof. According to Lemma 3.13 with $t = 1$, $s = 3$, $r \equiv 0$ or 1 (mod 3), it may be checked that if $u \geqslant 21$, then there exists (use Theorems 3.1 and 3.10) a transversal design $T[3+1, 1; r]$ such that $r \equiv 0$ or 1 (mod 3) and one of its groups can be truncated to obtain $3r + r_1 = u$. Evidently also $r_1 \equiv 0$ or 1 (mod 3). Further apply Lemma 2.22.

For $u < 21$ use the truncated transversal designs $T[3+1, 1; r]$ as follows: for $9 \leqslant u \leqslant 12$, $r = 3$, for $13 \leqslant u \leqslant 16$, $r = 4$ and for $u = 18$, $r = 6$.

Lemma 5.2. *For every positive integer* u, $u \in \mathrm{GD}(\{3,4\}, 1, M_3')$ *holds, where* $M_3' = \{1, 2, ..., 8\}$.

Proof. According to Lemma 3.13 with $t = 1$, $s = 3$, it may be checked that if $u \geqslant 9$, then there exists (use Theorem 3.1) a transversal design $T[3+1, 1; r]$ such that one of its groups can be truncated to obtain $3r + r_1 = u$.

Lemma 5.3. *For every integer* $v \geqslant 3$, $v \in B(K_3, 1)$ *holds, where* $K_3 = \{3,4,5,6,8\}$.

Proof. According to Lemma 3.13 with $t = 1$, $s = 3$, $D = \{2\}$, it may be checked that if $v \notin K_3' = \{5,6,7,8,11,14,17,23\}$, then there exists (use Theorems 3.1 and 3.10) a transversal design $T[3+1, 1; r]$ such that one of its groups can be truncated to obtain $3r + r_1 = v$ and $r \neq 2 \neq r_1$. By Lemmas 2.22 and 3.9, it follows that $v \in B(K_3' \cup \{3,4\}, 1)$. It remains to prove that if $v \in \{7, 11, 14, 17, 23\}$, then $v \in B(K_3, 1)$. $7 \in B(3, 1)$ follows from Lemma 4.3.

$11 \in B(\{3,5\}, 1)$. $X = I(2) \times Z(5,2) \cup \{(\infty)\}$.

$\mathcal{B} = \langle (\infty), (0', \emptyset), (1', \emptyset) \rangle \bmod (-, 5)$,

$\langle (0', \emptyset), (1', \alpha), (1', \alpha+2) \rangle \bmod (-, 5)$, $\alpha = 0, 1$,

$\langle (0', \emptyset), (0', 0), (0', 1), (0', 2), (0', 3) \rangle$.

$14 \in B(\{3,4,5\},1)$. $X = GF(4, x^2 = x+1) \times Z(3,2) \cup \{(\infty_i): i = 0,1\}$.

$\mathcal{B} = \langle(\infty_0),(\infty_1),(\emptyset,\emptyset),(\emptyset,0),(\emptyset,1)\rangle$,

$\langle(\infty_0),(\alpha,\emptyset),(\alpha,0),(\alpha,1)\rangle$, $\alpha = 0,1,2$,

$\langle(\infty_1),(\alpha,\emptyset),(\alpha+1,0),(\alpha+2,1)\rangle$, $\alpha = 0,1,2$,

$\langle(\emptyset,\emptyset),(0,\emptyset),(1,\emptyset),(2,\emptyset)\rangle$ mod $(-,3)$,

$\langle(\emptyset,\emptyset),(\alpha,0),(\alpha+2,1)\rangle$ mod $(-,3)$, $\alpha = 0,1,2$.

$17 \in B(\{4,5\},1)$. Lemma 3.12 and $4 \in T(4,1)$.

$23 \in B(\{3,5\},1)$. $X = I(2) \times Z(11,2) \cup \{(\infty)\}$.

$\mathcal{B}$ = Blocks of $B[\{3,5\},1;11]$ on $\{0'\} \times Z(11)$, as above,

$\langle(\infty),(0',\emptyset),(1',\emptyset)\rangle$ mod $(-,11)$,

$\langle(0',\emptyset),(1',\alpha),(1',\alpha+5)\rangle$ mod $(-,11)$, $\alpha = 0,1,2,3,4$.

Lemma 5.4. *If* $v \equiv 1$ *or* 3 (mod 6), *then* $v \in B(3,1)$ *holds.*

Proof. Let $v = 2u+1$, where $u \equiv 0$ or 1 (mod 3). By Lemma 5.1, $u \in GD(\{3,4\},1,M_3)$. By Lemmas 2.26 and 4.9, it suffices to show that $v = 2\mu+1 \in B(3,1)$ for every $\mu \in M_3$. This is done in Table 5.1.

Lemma 5.5. *If* $v \equiv 1$ (mod 2), *then* $v \in B(3,3)$ *holds.*

Proof. Let $v = 2u+1$, where u is a positive integer. By Lemma 5.2, $u \in GD(\{3,4\},1,M'_3)$. By Lemmas 2.26 and 4.9, it suffices to show that $v = 2\mu+1 \in B(3,3)$ for every $\mu \in M'_3$. For $\mu \equiv 0$ or 1 (mod 3) this is already proved in Lemma 5.4, and for other values of μ, namely $\mu \in \{2,5,8\}$, we have $v \in \{5,11,17\} \subset B(3,3)$ by Lemma 4.3.

Lemma 5.6. *If* $v \equiv 0$ *or* 1 (mod 3), *then* $v \in B(3,2)$ *holds.*

Table 5.1.

μ	v	$B[3,1;v]$
1	3	Trivial
3	7	Lemma 4.3.
4	9	Theorem 2.2.
6	13	Lemma 4.3.
7	15	Lemma 2.17 and $7 \in B(3,1)$ as above.
19	39	Lemma 2.17 and $19 \in B(3,1)$ by Lemma 4.3.

Proof. By Lemma 5.1, $v \in \mathrm{GD}(\{3,4\}, 1, M_3)$ and by Lemma 2.23, $v \in B(M_3, 1)$. By Lemma 2.6, it suffices to show that $\mu \in B(3,2)$ for every $\mu \in M_3$. For $\mu \equiv 1$ or $3 \pmod 6$ this follows from Lemma 5.4, and for other values of $\mu \in M_3$, namely for $\mu \in \{4, 6\}$, we have:

$4 \in B(3,2)$ by Lemma 4.1.

$6 \in B(3,2)$. $X = Z(5,2) \cup \{\infty\}$.
$\mathcal{B} = \langle \infty, 0, 2\rangle \bmod 5, \langle \emptyset, 0, 2\rangle \bmod 5$.

Lemma 5.7. *For every integer* $v \geq 3$, $v \in B(3,6)$ *holds.*

Proof. By Lemma 5.3, $v \in B(K_3, 1)$ and it suffices to show that $v \in B(3,6)$ for every $v \in K_3$. For $v \equiv 1 \pmod 2$ this follows from Lemma 5.5 and for $v \equiv 0$ or $1 \pmod 3$ this follows from Lemma 5.6. It remains to show that $8 \in B(3,6)$ and this follows from Lemma 4.1.

Theorem 5.1. *Let* λ *and* $v \geq 3$ *be positive integers. A necessary and sufficient condition for the existence of a BIBD* $B[3, \lambda; v]$ *is that*

$$\lambda(v-1) \equiv 0 \pmod 2 \quad \textit{and} \quad \lambda v(v-1) \equiv 0 \pmod 6.$$

Proof. The necessity follows from Theorem 1.1. To prove sufficiency we note that λ determines the values of v for which the condition of the theorem is satisfied. By Lemma 2.3, it suffices to consider only those values of λ which are factors of 6 and we obtain

for $\lambda = 1$, $v \equiv 1$ or $3 \pmod 6$,
for $\lambda = 2$, $v \equiv 0$ or $1 \pmod 3$,
for $\lambda = 3$, $v \equiv 1 \pmod 2$,
for $\lambda = 6$, every v.

In all these cases the existence of the relevant BIBD's is proved in Lemmas 5.4, 5.6, 5.5 and 5.7, respectively.

5.2. BIBD's with block-size 4

Lemma 5.8. *If* $u \equiv 0$ *or* $1 \pmod 4$, *then* $u \in \mathrm{GD}(\{4,5\}, 1, M_4)$, *where* $M_4 = \{1, 4, 5, 8, 9, 12, 13, 28, 29\}$.

Proof. According to Lemma 3.13 with $t = 1$, $s = 4$, $r \equiv 0$ or $1 \pmod 4$, it may be checked that if $u \geqslant 52$, then there exists (use Theorems 3.1 and 3.10) a transversal design $T[4+1, 1; r]$ such that by truncating one of its groups, $4r + r_1 = u$ is obtained. Evidently $r_1 \equiv 0$ or $1 \pmod 4$. Further apply Lemma 2.22.

For $u \leqslant 49$ use the truncated transversal designs $T[4+1, 1; r]$ as follows: for $16 \leqslant u \leqslant 20$, $r = 4$, for $21 \leqslant u \leqslant 25$, $r = 5$, for $32 \leqslant u \leqslant 40$, $r = 8$, for $41 \leqslant u \leqslant 45$, $r = 9$, and for $48 \leqslant u \leqslant 49$, $v = 12$ (Lemma 3.21).

Lemma 5.9. *For every positive integer u,* $u \in GD(\{4, 5\}, 1, M_4')$ *holds, where* $M_4' = \{1, 2, ..., 15, 26, 27\}$.

Proof. According to Lemma 3.13 with $t = 1$, $s = 4$, it may be checked that if $u \geqslant 28$, then there exists (use Theorems 3.1 and 3.10) a transversal design $T[4+1, 1; r]$ such that one of its groups may be truncated to obtain $4r + r_1 = u$. If $u < 27$, use for $16 \leqslant u \leqslant 20$ the truncated transversal design $T[4+1, 1; r]$ with $r = 4$ and for $21 \leqslant u \leqslant 25$, $r = 5$.

Lemma 5.10. *For every integer* $v \geqslant 4$, $v \in B(K_4, 1)$ *holds, where* $K_4 = \{4, 5, ..., 12, 14, 15, 18, 19, 23, 27\}$.

Proof. According to Lemma 3.13 with $t = 1$, $s = 4$, $D = \{2, 3\}$, it may be checked that if $v \geqslant 68$, then there exists (use Theorems 3.1 and 3.10) a transversal design $T[4+1, 1; r]$ such that one of its groups can be truncated to obtain $4r + r_1 = v$, further make use of Lemma 3.9. For $v = 13$ and $v = 31$, $13 \in B(4, 1)$ and $31 \in B(6, 1)$ follows from Theorem 2.1. For $v = 22$ we have:

$22 \in B(\{4, 7\}, 1)$. $X = Z(3, 2) \times Z(7, 3) \cup \{(\infty)\}$.

$\mathcal{B} = \langle(\infty), (\emptyset, \emptyset), (0, \emptyset), (1, \emptyset)\rangle \bmod (-, 7)$,

$\langle(\emptyset, \emptyset), (0, 0), (0, 2), (0, 4)\rangle \bmod (-, 7)$,

$\langle(\emptyset, 2\alpha), (0, \emptyset), (1, 2\alpha+4), (1, 2\alpha+5)\rangle \bmod (-, 7)$, $\alpha = 0, 1, 2$,

$\langle(\emptyset, \emptyset), (\emptyset, 0), (\emptyset, 1), (\emptyset, 2), (\emptyset, 3), (\emptyset, 4), (\emptyset, 5)\rangle$.

For other values of v we apply Lemma 3.9 with $t = 1$ and choose integers r and s as specified in Table 5.2.

Lemma 5.11. *If* $v \equiv 1$ *or* $4 \pmod{12}$, *then* $v \in B(4, 1)$ *holds.*

Table 5.2.

v	r	s	v	r	s
16–17	4	4	36–40	8	4
20–21	5	4	41–45	9	4
24–25	5	4	46–48	8	5
26	5	5	49–55	11	4
28–29	7	4	56–65	13	4
30	5	5	66–67	11	6
32–35	7	4			

Table 5.3.

μ	v	$B[4,1;v]$
1	4	Trivial
4	13	Theorem 2.1.
5	16	Theorem 2.2.
8	25	$X = \mathrm{GF}(25, x^2 = 2x + 2)$.
		$\mathcal{B} = \langle \emptyset, 2\alpha, 2\alpha + 8, 2\alpha + 16 \rangle \bmod 25, \quad \alpha = 0, 1.$
9	28	We prove $27 \in \mathrm{GD}(4, 1, 3)$, $X = Z(3, 2) \times \mathrm{GF}(9, x^2 = 2x + 1)$.
		$\mathcal{P} = \langle (0;\alpha), (0;\alpha + 4), (1;\alpha + 2), (1;\alpha + 6) \rangle \bmod (3;9), \quad \alpha = 0, 1.$
12	37	$X = Z(37, 2)$.
		$\mathcal{B} = \langle \emptyset, 12\alpha, 12\alpha + 11, 12\alpha + 14 \rangle \bmod 37, \quad \alpha = 0, 1, 2.$
13	40	Lemmas 2.17, 4.10 and $13 \in B(4,1)$ as above.
28	85	Lemmas 2.17, 4.10 and $28 \in B(4,1)$ as above.
29	88	By Theorem 2.1, $21 \in B(5,1)$ and by Theorem 3.3, $21 \in T(4,1)$.
		Apply Lemma 2.15 and $25 \in B(4,1)$ as above.

Proof. Let $v = 3u + 1$, where $u \equiv 0$ or $1 \pmod 4$. By Lemma 5.8, $u \in \mathrm{GD}(\{4,5\}, 1, M_4)$. By Lemmas 2.26 and 4.10, it suffices to show that $v = 3\mu + 1 \in B(4, 1)$ for every $\mu \in M_4$. That is done in Table 5.3.

Lemma 5.12. *If* $v \equiv 1 \pmod 3$, *then* $v \in B(4, 2)$ *holds.*

Proof. Let $v = 3u + 1$, where u is a positive integer. By Lemma 5.9, $u \in \mathrm{GD}(\{4,5\}, 1, M_4')$. By Lemmas 2.26 and 4.10, it suffices to show that $v = 3\mu + 1 \in B(4, 2)$ for every $\mu \in M_4'$. For $\mu \equiv 0$ or $1 \pmod 4$ this follows from Lemma 5.11; for other values of $\mu \in M_4'$ the solution is given in Table 5.4.

Table 5.4.

μ	v	$B[4,2;v]$
2	7	Lemma 4.4.
3	10	$X = I(2) \times Z(5,2)$. $\mathcal{B} = \langle(0',\emptyset),(1',\emptyset),(\alpha',\alpha),(\alpha',\alpha+2)\rangle \bmod (-,5), \quad \alpha = 0,1,$ $\langle(0',0),(0',2),(1',1),(1',3)\rangle \bmod (-,5)$.
6	19	Lemma 4.4.
7	22	Lemma 2.17 and $7 \in B(4,2)$ as above.
10	31	Lemma 4.4.
11	34	$11 \in B(5,2)$. $X = Z(11,2)$. $\mathcal{B} = \langle 0,2,4,6,8\rangle \bmod 11$. Further use Lemma 2.17 and $16 \in B(4,1)$ by Lemma 5.11.
14	43	Lemma 4.4.
15	46	We prove $45 \in \mathrm{GD}(4,2,3)$ and apply Lemma 2.11. $X = Z(3,2) \times (Z(3,2) \times Z(5,2))$. $\mathcal{P} = \langle(\emptyset;\alpha,\emptyset),(\emptyset;\alpha+1,\alpha+2\beta),(0;\alpha+1,\emptyset),(1;\alpha,\alpha+2\beta+3)\rangle \bmod (3;3,5),$ $\alpha = 0,1, \quad \beta = 0,1,$ $\langle(\emptyset;\emptyset,\emptyset),(\emptyset;0,\emptyset),(0;1,\alpha),(0;1,\alpha+2)\rangle \bmod (3;3,5), \quad \alpha = 0,1,$ $\langle(\emptyset;\emptyset,0),(\emptyset;\emptyset,2),(\emptyset;0,1),(\emptyset;0,3)\rangle \bmod (3;3,5)$.
26	79	Lemma 4.4.
27	82	We prove $81 \in \mathrm{GD}(4,2,3)$ and apply Lemma 2.11. $X = Z(3,2) \times \mathrm{GF}(27, x^3 = x+2)$. $\mathcal{P} = \langle(\emptyset;\alpha),(\emptyset;\alpha+13),(0;\alpha+1),(0;\alpha+14)\rangle \bmod (3;27), \quad \alpha = 0,1,\ldots,12$.

Table 5.5.

v	$B[4,3;v]$
5	Lemma 4.1.
8	Lemma 4.6, $7 \in B(3,1)$ by Lemma 5.4 and $7 \in B(4,2)$ by Lemma 5.12.
9	Lemma 4.1.
12	$X = Z(11,2) \cup \{\infty\}$. $\mathcal{B} = \langle \infty,0,1,9\rangle \bmod 11,$ $\langle \alpha,\alpha+1,\alpha+5,\alpha+6\rangle \bmod 11, \quad \alpha = 0,1$.
29	Lemma 4.1.

Lemma 5.13. *If* $v \equiv 0$ *or* 1 (mod 4), *then* $v \in B(4,3)$ *holds.*

Proof. By Lemma 5.8, $v \in \mathrm{GD}(\{4,5\},1,M_4)$ and by Lemma 2.23, $v \in B(M_4,1)$. By Lemma 2.6, it suffices to show that $v \in B(4,3)$ for every $v \in M_4$. For $v \equiv 1$ or 4 (mod 12) this follows from Lemma 5.11; for other values of $v \in M_4$ this is proved in Table 5.5.

Table 5.6.

v	$B[4, 6; v]$
6	Lemma 4.5 and $6 \in B(2, 1)$ trivially.
11	Lemma 4.1.
14	Lemma 4.6, $13 \in B(3,1)$ by Lemma 5.4 and $13 \in B(4,1)$ by Lemma 5.11.
15	$X = Z(3, 2) \times Z(5, 2)$. $\mathcal{B} = \langle (0,\alpha), (0, \alpha+2), (1, \alpha+1), (1, \alpha+3) \rangle \bmod (3, 5), \quad \alpha = 0, 1, 2, 3, 4,$ $\langle (\emptyset, \emptyset), (\emptyset, \beta), (0, \emptyset), (1, \emptyset) \rangle \bmod (3, 5), \quad \beta = 0, 1.$
18	$X = Z(17, 3) \cup \{\infty\}$. $\mathcal{B} = \langle \infty, 4\alpha+2, 4\alpha+3, 4\alpha+4 \rangle \bmod 17, \quad \alpha = 0, 1,$ $\langle \beta, \beta+4, \beta+8, \beta+12 \rangle \bmod 17, \quad \beta = 0, 1, ..., 6.$
23	Lemma 4.1.
27	Lemma 4.1.

Lemma 5.14. *For every integer* $v \geqslant 4$, $v \in B(4, 6)$ *holds.*

Proof. By Lemma 5.10, it suffices to show that $v \in B(4, 6)$ for every $v \in K_4$. For $v \equiv 1 \pmod 3$ this is proved in Lemma 5.12 and for $v \equiv 0$ or $1 \pmod 4$ this is proved in Lemma 5.13. For other values of $v \in K_4$ the proof is given in Table 5.6.

Theorem 5.2. *Let* λ *and* $v \geqslant 4$ *be positive integers. A necessary and sufficient condition for the existence of a BIBD* $B[4, \lambda; v]$ *is that*

$$\lambda(v-1) \equiv 0 \pmod 3 \quad \textit{and} \quad \lambda v(v-1) \equiv 0 \pmod{12}.$$

Proof. The necessity follows from Theorem 1.1. To prove sufficiency we note that λ determines the values of v for which the condition of the theorem is satisfied. By Lemma 2.3, it suffices to consider only those values of λ which are factors of 6 and we obtain

for $\lambda = 1$, $v \equiv 1$ or $4 \pmod{12}$,
for $\lambda = 2$, $v \equiv 1 \pmod 3$,
for $\lambda = 3$, $v \equiv 0$ or $1 \pmod 4$,
for $\lambda = 6$, every v.

In all these cases the existence of the relevant BIBD's is proved in Lemmas 5.11, 5.12, 5.13 and 5.14, respectively.

5.3. BIBD's with block-size 5

Lemma 5.15. *If* $u \equiv 0$ *or* 1 (mod 5), *then* $u \in \mathrm{GD}(\{5, 6, 10, 11\}, 1, M_5)$, *where* $M_5 = \{1, 5, 6, 10, 11, 15, 16, 20, 21, 31, 35, 36, 40, 41, 45, 46, 50, 51, 70, 71, 75, 76, 100, 101, 105, 106, 151\}$.

Proof. According to Lemma 3.13 with $t = 1$, $s = 5$, $r \equiv 0$ or 1 (mod 5), it may be checked that if $u \geqslant 275$, then there exists (use Theorems 3.1 and 3.9) a transversal design $\mathrm{T}[5 + 1, 1; r]$ such that by truncating one of its groups $5r + r_1 = u$ is obtained. Evidently, $r_1 \equiv 0$ or 1 (mod 5). Further apply Lemma 2.22. For $u < 275$ use the truncated transversal design $\mathrm{T}[s + 1, 1; r]$ with values of r and s as in Table 5.7.

Lemma 5.16. *If* $u \equiv 0$ *or* 2 (mod 5) *and* $u \neq 7$, *then* $u \in \mathrm{GD}(\{5, 6\}, 1, M'_5)$ *holds, where* $M'_5 = \{2, 5, 10, 12, 15, 17, 20, 22, 32, 35, 37, 40, 42, 45, 47, 50, 52, 55, 57, 67, 75, 77, 80, 82, 92, 105, 107, 110, 112, 115, 117, 120, 122, 132, 167\}$.

Proof. According to Lemma 3.13 with $t = 1$, $s = 5$, $r \equiv 0$ or 2 (mod 5), it may be checked that if $u \geqslant 335$, then there exists (use Theorem 3.9) a transversal design $\mathrm{T}[5+1, 1; r]$ such that by truncating one of its groups, $5r + r_1 = u$ is obtained. Clearly $r_1 \equiv 0$ or 2 (mod 5). For $u < 335$ use the truncated transversal design $\mathrm{T}[5 + 1, 1; r]$ with values of r as in Table 5.8. It should be remembered that $12 \in \mathrm{T}(7, 1)$ by Lemma 3.21, and that $57 \in B(8, 1)$ by Theorem 2.1 and therefore $57 \in \mathrm{T}(8, 1)$ by Theorem 3.3.

Table 5.7.

u	r	s
25– 30	5	5
55– 66	11	5
80– 96	16	5
110–121	11	10
125–150	25	5
155–186	31	5
190–210	35	5
211–246	41	5
250–271	25	10

Table 5.8.

u	r	u	r
25– 30	5	165	32
60– 65	12	170–192	32
70– 72	12	195–222	37
85– 90	17	225–240	40
95–102	17	242–270	45
125–130	25	272–282	47
135–150	25	285–330	55
152–162	27	332	57

Lemma 5.17. *For every integer* $u \geq 2$, $u \in \mathrm{GD}(\{5,6,7\}, 1, M_5'')$ *holds, where* $M_5'' = \{2, 3, \ldots, 24, 26, 31, 32, 33, 34, 36\}$.

Proof. According to Lemma 3.13 with $t = 2$, $s = 5$ and $D = \{1\}$, it may be checked that if $u \geq 37$, there exists (use Theorems 3.1 and 3.8) a transversal design $T[5+2, 1; r]$ such that by truncating two of the groups, $5r + r_1 + r_2 = u$ is obtained. For $u < 37$ use the truncated transversal design $T[5+1, 1; r]$ with $r = 5$ for $u \in \{25, 27, 28, 29, 30\}$, and $r = 7$ for $u = 35$.

Lemma 5.18. *For every integer* $v \geq 5$, $v \in B(K_5, 1)$ *holds, where* $K_5 = \{5, 6, \ldots, 20, 22, 23, 24, 27, 28, 29, 32, 33, 34, 39\}$.

Proof. According to Lemma 3.13 with $t = 4$, $s = 5$ and $D = \{2, 3, 4\}$, it may be checked that if $v \geq 40$, there exists (use Theorems 3.1 and 3.7) a transversal design $T[5+4, 1; r]$ such that by truncating 4 of the groups $5r + r_1 + r_2 + r_3 + r_4 = v$ is obtained. Further apply Lemma 3.9. For $v \in \{21, 31\}$ we have by Theorem 2.1, $21 \in B(5, 1)$ and $31 \in B(6, 1)$, and for other values of v we apply Lemma 3.9 as follows:

for $v \in \{25, 26, 30\}$, $t = 1$, $s = 5$, $r = 5$;
for $v \in \{35, 36\}$, $t = 1$, $s = 5$, $r = 7$;
for $v = 37$, $t = 2$, $s = 5$, $r = 7$; and
for $v = 38$, $t = 3$, $s = 5$, $r = 7$.

Lemma 5.19. *If* $v \equiv 1$ *or* 5 (mod 20), *then* $v \in B(5, 1)$ *holds.*

Table 5.9.

μ	v	$B[5,1;v]$
1	5	Trivial
5	21	Theorem 2.1.
10	41	Lemma 4.14.
15	61	$X = Z(61,2)$. $\mathfrak{B} = \langle 2\alpha, 2\alpha+12, 2\alpha+24, 2\alpha+36, 2\alpha+48\rangle \bmod 61, \quad \alpha = 0,1,2.$
20	81	$X = GF(81, x^4 = 2x^3 + 2x^2 + x + 1)$. $\mathfrak{B} = \langle \gamma, \gamma+16, \gamma+32, \gamma+48, \gamma+64\rangle \bmod 81, \quad \gamma = 0,1,4,5.$
35	141	$X = GF(4, x^2 = x+1) \times Z(5,2) \times Z(7,3) \cup (\infty)$. $\mathfrak{B} = B[5,1;21]$ as above on $GF(4) \times Z(5) \times \{i\} \cup \{(\infty)\}, \quad i \in Z(7)$, $\langle(0,\emptyset,\emptyset), (\emptyset,1,2\alpha), (0,1,2\alpha+3), (\emptyset,3,2\alpha+1), (\emptyset,3,2\alpha+4)\rangle \bmod (-,5,7)$, $\alpha = 0,1,2$, $\langle(\emptyset,\emptyset,\emptyset), (1,1,2\alpha), (2,1,2\alpha+3), (\emptyset,3,2\alpha+1), (1,3,2\alpha+4)\rangle \bmod (-,5,7)$, $\alpha = 0,1,2$, $\langle(1,\emptyset,\emptyset), (0,1,2\alpha), (1,1,2\alpha+3), (0,3,2\alpha+1), (0,3,2\alpha+4)\rangle \bmod (-,5,7)$, $\alpha = 0,1,2$, $\langle(0,\emptyset,\emptyset), (2,1,2\alpha), (\emptyset,1,2\alpha+3), (0,3,2\alpha+1), (2,3,2\alpha+4)\rangle \bmod (-,5,7)$, $\alpha = 0,1,2$, $\langle(2,\emptyset,\emptyset), (0,1,2\alpha), (\emptyset,1,2\alpha+3), (1,3,2\alpha+1), (1,3,2\alpha+4)\rangle \bmod (-,5,7)$, $\alpha = 0,1,2$, $\langle(\emptyset,\emptyset,\emptyset), (2,1,2\alpha), (1,1,2\alpha+3), (1,3,2\alpha+1), (\emptyset,3,2\alpha+4)\rangle \bmod (-,5,7)$, $\alpha = 0,1,2$, $\langle(1,\emptyset,\emptyset), (1,1,2\alpha), (0,1,2\alpha+3), (2,3,2\alpha+1), (2,3,2\alpha+4)\rangle \bmod (-,5,7)$, $\alpha = 0,1,2$, $\langle(2,\emptyset,\emptyset), (\emptyset,1,2\alpha), (2,1,2\alpha+3), (2,3,2\alpha+1), (0,3,2\alpha+4)\rangle \bmod (-,5,7)$, $\alpha = 0,1,2$.
40	161	$X = Z(7,3) \times Z(23,5)$. $\mathfrak{B} = \langle(\emptyset,\emptyset), (2\alpha,11\beta), (2\alpha+1,11\beta+4), (2\alpha+4,11\beta+8), (2\alpha+3,11\beta+12)\rangle$ $\bmod (7,23), \quad \alpha = 0,1,2, \quad \beta = 0,1$, $\langle(\emptyset,3), (\emptyset,14), (0,\emptyset), (2,\emptyset), (4,\emptyset)\rangle \bmod (7,23)$, $\langle(\emptyset,\emptyset), (\emptyset,2), (\emptyset,6), (\emptyset,10), (\emptyset,14)\rangle \bmod (7,23)$.
41	165	Lemma 2.17 and $41 \in B(5,1)$ as above.
45	181	Lemma 2.17 and $45 \in B(5,1)$ by Lemma 4.13.
50	201	Lemma 2.14 ($m = 40$) and $41 \in B(5,1)$ as above.
70	281	$X = Z(281,3)$. $\mathfrak{B} = \langle 2\alpha, 2\alpha+56, 2\alpha+112, 2\alpha+168, 2\alpha+224\rangle \bmod 281, \quad \alpha = 0,1,\ldots,13.$
71	285	Lemma 2.15 ($m = 56$) and $61 \in B(5,1)$ as above.
75	301	As above $61 \in B(5,1)$; by Theorem 3.10, $60 \in T(5,1)$; use Lemma 2.14.
100	401	Lemma 2.14 ($m = 80$) and $81 \in B(5,1)$ as above.
105	421	Lemma 2.14 ($m = 84$), $84 \in T(5,1)$ by Theorem 3.10 and $85 \in B(5,1)$ by Lemma 4.13.
106	425	Lemma 2.13 ($m = 85$) and $85 \in B(5,1)$ by Lemma 4.13.

Table 5.10.

μ	v	$B[5,2;v]$
37	75	We prove $75 \in GD(5,2,5)$. $X = Z(5,2) \times (Z(3,2) \times Z(5,2))$. $\mathcal{P} = \langle(\emptyset;\emptyset,\emptyset),(0;\alpha,\emptyset),(0;\alpha+1,\alpha+1),(2;\alpha,\alpha+2),(2;\alpha+1,\alpha+3)\rangle \bmod (5;3,5)$, $\alpha = 0,1,2,3$, $\langle(\emptyset;\emptyset,\emptyset),(1;0,\emptyset),(1;1,\emptyset),(3;\beta,\beta+1),(3;\beta,\beta+3)\rangle \bmod (5;3,5)$, $\beta = 0,1$, $\langle(\emptyset;\emptyset,\emptyset),(1;0,1),(1;0,3),(3;0,0),(3;0,2)\rangle \bmod (5;3,5)$.
45	91	Lemma 2.26 ($n = 45$); $45 \in GD(5,1,5)$ by Lemma 4.13, $11 \in B(5,2)$ by Lemma 4.3 and $10 \in GD(5,2,2)$ by Lemma 4.15.
55	111	$X = Z(3,2) \times Z(37,2)$. $\mathcal{B} = \langle(\emptyset,\emptyset),(0,3\alpha),(0,3\alpha+18),(1,3\alpha+3),(1,3\alpha+21)\rangle \bmod (3,37)$, $\alpha = 0,1,\ldots,5$, $\langle(\emptyset,6\beta+1),(\emptyset,6\beta+8),(\emptyset,6\beta+19),(\emptyset,6\beta+26),(0,\emptyset)\rangle \bmod (3,37)$, $\beta = 0,1,2$, $\langle(\emptyset,\gamma+4),(\emptyset,\gamma+16),(\emptyset,\gamma+28),(0,\emptyset),(1,\emptyset)\rangle \bmod (3,37)$, $\gamma = 0,1$.
115	231	Lemma 2.16 ($n = 21$), $11 \in T(5,1)$ and $21 \in B(5,1)$ by Lemma 5.19; further apply Lemma 2.13 and $11 \in B(5,2)$ by Lemma 4.3.

Proof. Let $v = 4u + 1$, where $u \equiv 0$ or $1 \pmod 5$. By Lemma 5.15, $u \in GD(\{5,6,10,11\}, 1, M_5)$. By Lemmas 2.26, 4.11 and 4.14, it suffices to show that $v = 4\mu + 1 \in B(5,1)$ for every $\mu \in M_5$. For $\mu \in \{6, 11, 16, 21, 31, 36, 46, 51, 76, 101, 151\}$ we have $v \in \{25, 45, 65, 85, 125, 145, 185, 205, 305, 405, 605\}$ and by Lemma 4.13, $v \in B(5,1)$. For other values of v the solution is given in Table 5.9.

Lemma 5.20. *If* $v \equiv 1$ *or* $5 \pmod{10}$ *and* $v \neq 15$, *then* $v \in B(5,2)$ *holds.*

Proof. Let $v = 2u + 1$, where $u \equiv 0$ or $2 \pmod 5$ and $u \neq 7$. By Lemma 5.16, $u \in GD(\{5,6\}, 1, M'_5)$. By Lemmas 2.26 and 4.15, it suffices to show that $v = 2\mu + 1 \in B(5,2)$ for every $\mu \in M'_5$. For $\mu \equiv 0$ or $2 \pmod{10}$ this follows from Lemma 5.19; for $\mu \in \{17, 47, 57, 67, 77, 107, 117, 167\}$ we have $v \in \{35, 95, 115, 135, 155, 215, 235, 335\}$ and by Lemma 4.12, $v \in B(5,2)$; for $\mu \in \{5, 15, 35, 75, 105\}$ we have $v \in \{11, 31, 71, 151, 211\}$ and by Lemma 4.3, $v \in B(5,2)$. For other values of v the solution is given in Table 5.10.

Lemma 5.21. *If* $v \equiv 1 \pmod 4$, *then* $v \in B(5,5)$ *holds.*

Proof. Let $v = 4u + 1$, where u is a positive integer. For $u = 1$ the lemma is trivial. For $u \geq 2$, by Lemma 5.17, $u \in GD(\{5,6,7\}, 1, M''_5)$. By

Table 5.11.

μ	v	$B[5,10;v]$
7	15	$X = Z(3,2) \times Z(5,2)$. $\mathfrak{B} = \langle(\emptyset,\emptyset),(0,\alpha),(0,\alpha+2),(1,\alpha),(1,\alpha+2)\rangle$ mod $(3,5)$, $\alpha = 0,1,2,3$, $\langle(\emptyset,\emptyset),(0,0),(0,2),(1,1),(1,3)\rangle$ mod $(3,5)$, $\langle(\emptyset,\emptyset),(\emptyset,0),(\emptyset,2),(0,0),(0,2)\rangle$ mod $(3,5)$, $\langle(\emptyset,0),(\emptyset,1),(\emptyset,2),(\emptyset,3),(0,\emptyset)\rangle$ mod $(3,5)$.
19	39	$X = Z(3,2) \times Z(13,2)$. $\mathfrak{C} = \langle(\emptyset,\alpha),(\emptyset,\alpha+4),(\emptyset,\alpha+8),(0,\alpha+2),(1,\alpha+6)\rangle$ mod $(3,13)$, $\alpha = 0,1,\ldots,11$, $\langle(\emptyset,\beta),(\emptyset,\beta+4),(\emptyset,\beta+8),(0,\emptyset),(1,\emptyset)\rangle$ mod $(3,13)$, $\beta = 0,1,2,3$, $\langle(\emptyset,\emptyset),(\emptyset,2\gamma),(\emptyset,2\gamma+6),(0,2\gamma),(0,2\gamma+6)\rangle$ mod $(3,13)$, $\gamma = 0,1,2$.
31	63	By $9 \in T(7,1)$ and Lemma 3.11, $63 \in B(\{7,9\},1)$; further – by Lemma 5.21 – $9 \in B(5,5)$ and by Lemma 4.3, $7 \in B(5,10)$.

Lemmas 2.26, 4.11 and 4.16 ($q = 7$), it suffices to show that $v = 4\mu + 1 \in B(5,5)$ for every $\mu \in M_5''$. For $\mu \equiv 0$ or 1 (mod 5) this follows from Lemma 5.19; for $\mu \in \{2,3,4,7,9,12,13,18,22,24,34\}$ we have $v \in \{9,13,17,29,37,49,53,73,89,97,137\}$ and by Lemma 4.2, $v \in B(5,5)$; for $\mu \in \{17,19,23\}$, $v \in \{69,77,93\}$ and by Lemma 4,16, $v \in B(5,5)$; for $\mu \in \{8,14,32\}$ apply Lemma 4.17 to Lemma 2.17 considering $\{21,25\} \subset B(5,1)$ by Lemma 5.19 and for $\mu = 33$ we use the obtained result $33 (= 4.8 + 1) \in B(5,5)$ and use again Lemma 2.17.

Lemma 5.22. *If* $v \equiv 1$ (mod 2) *and* $v \geq 5$, *then* $v \in B(5,10)$ *holds.*

Proof. Let $v = 2u + 1$, where $u \geq 2$. By Lemma 5.17, $u \in \mathrm{GD}(\{5,6,7\},1,M_5'')$. By Lemmas 2.26, 4.15 and 4.18, it suffices to show that $v = 2\mu + 1 \in B(5,10)$ for every $\mu \in M_5''$. For $\mu \equiv 0$ (mod 2) this follows from Lemma 5.21 and for $\mu \equiv 0$ or 2 (mod 5) and $u \neq 7$ this follows from Lemma 5.20. For $\mu \in \{3,9,11,13,21,23,33\}$ we have $v \in \{7,19,23,27,43,47,67\}$ and by Lemma 4.3, $v \in B(5,10)$. It remains to prove our lemma for $\mu \in \{7,19,31\}$ which is performed in Table 5.11.

Lemma 5.23. *If* $v \equiv 0$ *or* 1 (mod 5), *then* $v \in B(5,4)$ *holds.*

Proof. By Lemma 5.15, $v \in \mathrm{GD}(\{5,6,10,11\},1,M_5)$ and by Lemma 2.23, $v \in B(M_5,1)$. By Lemma 2.6, it suffices to show that $v \in B(5,4)$

Table 5.12.

v	$B[5,4;v]$
6	Lemma 4.6, $5 \in B(4,3)$ by Lemma 5.13 and $5 \in B(5,1)$ trivially.
10	$X = GF(9, x^2 = 2x+1) \cup \{\infty\}$.
	$\mathcal{B} = \langle \infty, 0, 2, 4, 6 \rangle \bmod 9$,
	$\langle \emptyset, 0, 2, 4, 6 \rangle \bmod 9$.
15	$X = (Z(2) \cup \{\infty\}) \times Z(5,2)$.
	$\mathcal{B} = \langle (\infty, \emptyset), (\infty, 0), (\infty, 1), (\infty, 2), (\infty, 3) \rangle$, 2 times,
	$\langle (\infty, \alpha), (\infty, \alpha+2), (\emptyset, \emptyset), (0, \alpha+1), (0, \alpha+3) \rangle \bmod (2,5)$, $\alpha = 0, 1$,
	$\langle (\infty, \emptyset), (\emptyset, \emptyset), (\emptyset, \alpha), (\emptyset, \alpha+2), (0, \emptyset) \rangle \bmod (2,5)$, $\alpha = 0,1$.
16	Lemma 4.1.
20	$X = Z(19,2) \cup \{\infty\}$.
	$\mathcal{B} = \langle \infty, \emptyset, 0, 6, 12 \rangle \bmod 19$,
	$\langle 6\alpha, 6\alpha+2, 6\alpha+4, 6\alpha+6, 6\alpha+8 \rangle \bmod 19$, $\alpha = 0, 1, 2$.
36	By Lemma 4.12, $35 \in GD(5,2,5)$; use Lemma 2.14 and $6 \in B(5,4)$ as above.
40	We prove $40 \in GD(5,4,5)$. $X = Z(5,2) \times GF(8, x^3 = x+1)$.
	$\mathcal{P} = \langle (\emptyset; \emptyset), (0; \alpha), (0; \alpha+1), (2; \alpha+2), (2; \alpha+3) \rangle \bmod (5;8)$, $\alpha = 0, 1, \ldots, 6$.
46	Lemma 3.12 ($r = 9$) and $10 \in B(5,4)$ as above.
50	Lemma 3.22 ($r = 10$) and $10 \in B(5,4)$ as above.
70	We prove $70 \in GD(5,4,5)$. $X = Z(5,2) \times (Z(13,2) \cup \{\infty\})$.
	$\mathcal{P} = \langle (\emptyset; \infty), (\emptyset; \emptyset), (\emptyset; 0), (\emptyset; 4), (\emptyset; 8) \rangle \bmod (5,13)$,
	$\langle (\emptyset; \infty), (0; \emptyset), (1; 3\alpha), (2; 3\alpha+8), (3; 3\alpha+4) \rangle \bmod (5;13)$, $\alpha = 0, 1, 2, 3$,
	$\langle (\emptyset; \emptyset), (0; \beta), (0; \beta+6), (2; \beta+3), (2; \beta+9) \rangle \bmod (5;13)$, 3 times, $\beta = 0, 1, 2$.
76	Lemma 3.23 ($r = 15$) and $16 \in B(5,4)$ as above.
100	Lemma 3.22 ($r = 20$) and $20 \in B(5,4)$ as above.
106	By Lemma 2.16 and $21 \in B(5,1)$, $105 \in GD(5,1,5)$. Use Lemma 2.14 and $6 \in B(5,4)$ as above.

for every $v \in M_5$. For $v \equiv 1$ or $5 \pmod{10}$ and $v \neq 15$ this follows from Lemma 5.20. For other values of v the proof is given in Table 5.12.

Lemma 5.24. *For every integer $v \geq 5$, $v \in B(5, 20)$ holds.*

Proof. By Lemma 5.18, $v \in B(K_5, 1)$ and it suffices to show that $v \in B(5, 20)$ for every $v \in K_5$. For $v \equiv 1 \pmod 2$ this follows from Lemma 5.22 and for $v \equiv 0$ or $1 \pmod 5$ this follows from Lemma 5.23. For other values of $v \in K_5$ the proof is given in Table 5.13.

Lemma 5.25. *If $\lambda \equiv 0 \pmod 2$ and $\lambda > 2$, then $15 \in B(5, \lambda)$ holds.*

Proof. For $\lambda = 4$, $15 \in B(5,4)$ follows from Lemma 5.23. For $\lambda = 6$ we prove

Table 5.13.

v	$B[5,20;v]$
8	Lemma 4.1.
12	Lemma 4.6, $11 \in B(4,6)$ by Lemma 5.14 and $11 \in B(5,2)$ by Lemma 5.20.
14	Lemma 4.6, $13 \in B(4,1)$ by Lemma 5.11 and $13 \in B(5,5)$ by Lemma 5.21.
18	$X = Z(17,3) \cup \{\infty\}$.
	$\mathcal{B} = \langle \infty, 2\alpha, 2\alpha+3, 2\alpha+4, 2\alpha+7 \rangle \bmod 17, \quad \alpha = 0,1,2,3,4,$
	$\langle \emptyset, \beta, \beta+4, \beta+8, \beta+12 \rangle \bmod 17, \quad \beta = 0,1,\ldots,11,$
	$\langle \emptyset, 4, 7, 8, 11 \rangle \bmod 17.$
22	Lemma 4.6, $21 \in B(4,3)$ by Lemma 5.13 and $21 \in B(5,1)$ by Lemma 5.19.
24	$X = Z(23,5) \cup \{\infty\}$.
	$\mathcal{B} = \langle \infty, \emptyset, 0, 20, 21 \rangle \bmod 23, \quad$ 2 times,
	$\langle \infty, \emptyset, 0, 19, 21 \rangle \bmod 23,$
	$\langle \infty, \emptyset, 0, 1, 15 \rangle \bmod 23,$
	$\langle \infty, \emptyset, 1, 2, 8 \rangle \bmod 23,$
	$\langle \emptyset, \alpha, \alpha+1, \alpha+2, \alpha+3 \rangle \bmod 23, \quad \alpha = 0,1,\ldots,18.$
28	$X = \mathrm{GF}(4, x^2 = x+1) \times Z(7,3)$.
	$\mathcal{B} = \langle (\emptyset,0), (\emptyset,2), (\emptyset,4), (\alpha,\emptyset), (\alpha+1,\emptyset) \rangle \bmod (4,7), \quad$ 4 times, $\quad \alpha = 0,1,2,$
	$\langle (\emptyset,\emptyset), (\alpha,\beta), (\alpha,\beta+3), (\alpha+1,\beta+1), (\alpha+1,\beta+4) \rangle \bmod (4,7), \quad \alpha = 0,1,2, \quad \beta = 0,1,2,$
	$\langle (\emptyset,\beta), (\emptyset,\beta+3), (0,\emptyset), (1,\emptyset), (2,\emptyset) \rangle \bmod (4,7), \quad$ 2 times, $\quad \beta = 0,1,2.$
32	Lemma 4.1.
34	$X = Z(3,2) \times Z(11,2) \cup \{(\infty)\}$.
	$\mathcal{B} = \langle (\infty), (\emptyset,\alpha), (\emptyset,\alpha+5), (0,\emptyset), (1,\emptyset) \rangle \bmod (3,11), \quad \alpha = 0,1,2,3,4,$
	$\langle (\emptyset,\emptyset), (0,\alpha), (0,\alpha+5), (1,\alpha+1), (1,\alpha+6) \rangle \bmod (3,11), \quad$ 4 times, $\quad \alpha = 0,1,2,3,4,$
	$\langle (\emptyset,\emptyset), (\emptyset,\alpha), (\emptyset,\alpha+5), (0,\emptyset), (1,\emptyset) \rangle \bmod (3,11), \quad \alpha = 0,1,2,3,4,$
	$\langle (\emptyset,0), (\emptyset,2), (\emptyset,4), (\emptyset,6), (\emptyset,8) \rangle \bmod (3,11), \quad$ 4 times.

$15 \in B(5,6)$. $X = Z(3,2) \times Z(5,2)$.

$\mathcal{B} = \langle (\emptyset,\emptyset), (\emptyset,\alpha), (\emptyset,\alpha+2), (0,\emptyset), (1,\emptyset) \rangle \bmod (3,5), \quad \alpha = 0,1,$

$\langle (\emptyset,\emptyset), (0,\alpha), (0,\alpha+2), (1,\alpha+1), (1,\alpha+3) \rangle \bmod (3,5), \quad \alpha = 0,1,$

$\langle (\emptyset,\emptyset), (\emptyset,0), (\emptyset,1), (\emptyset,2), (\emptyset,3) \rangle \bmod (3,-).$

For other values of $\lambda \equiv 0 \pmod 2$ apply Lemma 2.4.

Theorem 5.3. *Let λ and $v \geqslant 5$ be positive integers. A necessary and sufficient condition for the existence of a BIBD $B[5,\lambda;v]$ is that the design is not $B[5,2;15]$ and that*

$$\lambda(v-1) \equiv 0 \pmod 4 \quad \textit{and} \quad \lambda v(v-1) \equiv 0 \pmod{20}.$$

Proof. The necessity follows from Theorem 1.1. To prove sufficiency we note that λ determines the values of v for which the condition of the theorem is satisfied. By Lemma 2.3, it suffices to consider only those values of λ which are factors of 20 and we obtain

for $\lambda = 1$, $v \equiv 1$ or 5 (mod 20),
for $\lambda = 2$, $v \equiv 1$ or 5 (mod 10),
for $\lambda = 4$, $v \equiv 0$ or 1 (mod 5),
for $\lambda = 5$, $v \equiv 1$ (mod 4),
for $\lambda = 10$, $v \equiv 1$ (mod 2),
for $\lambda = 20$, every v.

In all these cases – with exception of $\lambda = 2$, $v = 15$ – the existence of the relevant BIBD's is proved in Lemmas 5.19, 5.20, 5.23, 5.21, 5.22 and 5.24, respectively. Considering Lemma 5.25, the theorem is proved completely.

It has been proved by Hall and Connor [9, 15] that the exceptional BIBD $B[5, 2; 15]$ does not exist, see Lemma 1.3.

5.4. BIBD's with block-size 6

In this section we shall determine a necessary and sufficient condition for the existence of BIBD's with $k = 6$ and $\lambda > 1$. To find such condition for BIBD's with $k = 6$ and $\lambda = 1$ is at the present state of knowledge almost hopeless. We mention, for instance, that the necessary condition of Theorem 1.1 in the case of $\lambda = 1$ and k a primepower states that $v \equiv 1$ or k (mod $k(k-1)$). However, if k is not a primepower, additional values of v satisfy this condition. In the case $\lambda = 1$ and $k = 6$ a necessary condition for the existence of BIBD $B[6, 1; v]$ – according to Theorem 1.1 – is that $v \equiv 1, 6, 16$ or 21 (mod 30). So far no BIBD $B[6, 1; v]$ with $v \equiv 16$ or 21 (mod 30) have been known and only recently Wilson [35] announced the existence of such BIBD's with very large values of v. Moreover, even when $v \equiv 1$ or 6 (mod 30) there are still many values of v for which it is unknown whether BIBD $B[6, 1; v]$ exist.

The known negative results are that $16 \in B(6, 1)$ and $21 \in B(6, 1)$ by Lemma 1.1, and further that $36 \in B(6, 1)$ by Lemma 1.3.

The known to exist BIBD's with $k = 6$, $\lambda = 1$ and $v < 2000$ are listed in Table 5.14. The BIBD's $B[6, 1; v]$ with $v \in \{91, 121\}$ have been constructed by Mills [21] and those with prime $v > 200$ by Wilson [33].

Table 5.14.

v	$B[6,1;v]$
31	Theorem 2.1.
91	$X = Z(91)$. $\mathcal{B} = \langle 0', 1', 3', 7', 25', 38' \rangle$ mod (91), $\langle 0', 5', 20', 32', 46', 75' \rangle$ mod 91, $\langle 0', 8', 17', 47', 57', 80' \rangle$ mod 91.
121	$X = Z(11,2) \times Z(11,2)$. $\mathcal{B} = \langle (\emptyset,\emptyset), (\emptyset,2), (\emptyset,8), (0,0), (0,7), (2,9) \rangle$ mod (11, 11), $\langle (\emptyset,\emptyset), (\emptyset,1), (1,4), (2,7), (3,\emptyset), (9,2) \rangle$ mod (11, 11), $\langle (\emptyset,\emptyset), (0,4), (1,\emptyset), (2,0), (7,1), (9,\emptyset) \rangle$ mod (11, 11), $\langle (\emptyset,\emptyset), (0,\emptyset), (2,3), (6,4), (8,6), (9,0) \rangle$ mod (11, 11).
126	We prove $125 \in \mathrm{GD}(6,1,5)$. $X = Z(5,2) \times \mathrm{GF}(25, x^2 = 2x+2)$. $\mathcal{P} = \langle (\emptyset; 6\alpha+3\beta), (\emptyset; 6\alpha+3\beta+12), (\beta; 6\alpha+3\beta+4), (\beta; 6\alpha+3\beta+16), (\beta+2; 6\alpha+3\beta+8),$ $(\beta+2; 6\alpha+3\beta+20) \rangle$ mod (5; 25), $\alpha = 0,1$, $\beta = 0,1$.
151	$X = Z(151,7)$. $\mathcal{B} = \langle 30\alpha, 30\alpha+1, 30\alpha+50, 30\alpha+51, 30\alpha+100, 30\alpha+101 \rangle$ mod 151, $\alpha = 0,1,2,3,4$.
156	Lemma 2.17 and $31 \in B(6,1)$ as above.
181	$X = Z(181,2)$. $\mathcal{B} = \langle \emptyset, 15\alpha, 15\alpha+36, 15\alpha+72, 15\alpha+108, 15\alpha+144 \rangle$ mod 181, $\alpha = 0,1,\ldots,5$.
186	Lemma 3.11, $31 \in \mathrm{T}(6,1)$ and $31 \in B(6,1)$ as above.
211	$X = Z(211,2)$. $\mathcal{B} = \langle 5\alpha, 5\alpha+16, 5\alpha+70, 5\alpha+86, 5\alpha+140, 5\alpha+156 \rangle$ mod 211, $\alpha = 0,1,\ldots,6$.
241	$X = Z(241,7)$. $\mathcal{B} = \langle 5\alpha, 5\alpha+18, 5\alpha+80, 5\alpha+98, 5\alpha+160, 5\alpha+178 \rangle$ mod 241, $\alpha = 0,1,\ldots,7$.
271	$X = Z(271,6)$. $\mathcal{B} = \langle 5\alpha, 5\alpha+54, 5\alpha+90, 5\alpha+144, 5\alpha+180, 5\alpha+234 \rangle$ mod 271, $\alpha = 0,1,\ldots,8$.
331	$X = Z(331,3)$. $\mathcal{B} = \langle 5\alpha, 5\alpha+34, 5\alpha+110, 5\alpha+144, 5\alpha+220, 5\alpha+254 \rangle$ mod 331, $\alpha = 0,1,\ldots,10$.
361	$X = \mathrm{GF}(361, x^2 = 9x+12)$. $\mathcal{B} = \langle 5\alpha, 5\alpha+1, 5\alpha+120, 5\alpha+121, 5\alpha+240, 5\alpha+241 \rangle$ mod 361, $\alpha = 0,1,\ldots,11$.
421	$X = Z(421,2)$. $\mathcal{B} = \langle 5\alpha, 5\alpha+62, 5\alpha+140, 5\alpha+202, 5\alpha+280, 5\alpha+342 \rangle$ mod 421, $\alpha = 0,1,\ldots,13$.
456	Lemma 2.17 and $91 \in B(6,1)$ as above.
516	Lemma 2.17, $103 \in B(\{6,18\},1)$ by Lemma 3.12 ($r = 17$, $s = 6$), and $\{31, 91\} \subset B(6,1)$ as above.
541	Lemma 3.12, $90 \in \mathrm{T}(6,1)$ by Theorem 3.9, and $91 \in B(6,1)$.
546	Lemma 3.11, $91 \in \mathrm{T}(6,1)$ by Theorem 3.1, and $91 \in B(6,1)$.
571	$X = Z(571,3)$. $\mathcal{B} = \langle 5\alpha, 5\alpha+182, 5\alpha+190, 5\alpha+372, 5\alpha+380, 5\alpha+562 \rangle$ mod 571, $\alpha = 0,1,\ldots,18$.
601	$X = Z(601,7)$. $\mathcal{B} = \langle 5\alpha, 5\alpha+134, 5\alpha+200, 5\alpha+334, 5\alpha+400, 5\alpha+534 \rangle$ mod 601, $\alpha = 0,1,\ldots,19$.
606	Lemma 2.17 and $121 \in B(6,1)$ as above.
631	Lemma 2.17 and $126 \in B(6,1)$ as above.
661	$X = Z(661,2)$. $\mathcal{B} = \langle 5\alpha, 5\alpha+131, 5\alpha+220, 5\alpha+351, 5\alpha+440, 5\alpha+571 \rangle$ mod 661, $\alpha = 0,1,\ldots,21$.
691	$X = Z(691,3)$. $\mathcal{B} = \langle 5\alpha, 5\alpha+103, 5\alpha+230, 5\alpha+333, 5\alpha+460, 5\alpha+563 \rangle$ mod 691, $\alpha = 0,1,\ldots,22$.
696	Lemma 2.17, $139 \in B(\{6,24\},1)$ by Lemma 3.12 ($r = 23$, $s = 6$), and $\{31, 121\} \subset B(6,1)$ as above.

Table 5.14 (cont.).

v	$B[6,1;v]$
721	Lemma 3.12, $120 \in T(6,1)$ by Theorem 3.9, and $121 \in B(6,1)$ as above.
726	Lemma 3.11, $121 \in T(6,1)$ by Theorem 3.1, and $121 \in B(6,1)$.
751	Lemma 3.12, $125 \in T(6,1)$ by Theorem 3.1, and $126 \in B(6,1)$ as above.
756	Lemma 3.11, $126 \in T(6,1)$ by Theorem 3.9, and $126 \in B(6,1)$.
781	Lemma 2.17 and $156 \in B(6,1)$ as above.
811	$X = Z(811,3)$.
	$\mathcal{B} = \langle 5\alpha, 5\alpha+178, 5\alpha+270, 5\alpha+448, 5\alpha+540, 5\alpha+718\rangle \bmod 811, \quad \alpha = 0, 1, \ldots, 26.$
901	Lemma 3.12, $150 \in T(6,1)$ by Theorem 3.9, and $151 \in B(6,1)$ as above.
906	Lemma 3.11, $151 \in T(6,1)$ by Theorem 3.1, and $151 \in B(6,1)$.
931	Lemma 3.12, $155 \in T(6,1)$ by Theorem 3.1, and $156 \in B(6,1)$ as above.
936	Lemma 3.11, $156 \in T(6,1)$ by Theorem 3.9, and $156 \in B(6,1)$.
961	$961 \in B(31,1)$ by Theorem 2.2 and $31 \in B(6,1)$ as above.
991	$X = Z(991,6)$.
	$\mathcal{B} = \langle 5\alpha, 5\alpha+112, 5\alpha+330, 5\alpha+442, 5\alpha+660, 5\alpha+772\rangle \bmod 991, \quad \alpha = 0, 1, \ldots, 32.$
1021	$X = Z(1021,10)$.
	$\mathcal{B} = \langle 5\alpha, 5\alpha+182, 5\alpha+340, 5\alpha+522, 5\alpha+680, 5\alpha+862\rangle \bmod 1021, \quad \alpha = 0, 1, \ldots, 33.$
1051	$X = Z(1051,7)$
	$\mathcal{B} = \langle 5\alpha, 5\alpha+66, 5\alpha+350, 5\alpha+416, 5\alpha+700, 5\alpha+766\rangle \bmod 1051, \quad \alpha = 0, 1, \ldots, 34.$
1056	Lemma 2.17 and $211 \in B(6,1)$ as above.
1081	Lemma 3.12, $180 \in T(6,1)$ by Theorem 3.9, and $181 \in B(6,1)$ as above.
1086	Lemma 3.11, $181 \in T(6,1)$ by Theorem 3.1, and $181 \in B(6,1)$.
1111	Lemma 3.12, $185 \in T(6,1)$ by Theorem 3.1, and $186 \in B(6,1)$ as above.
1116	Lemma 3.11, $186 \in T(6,1)$ by Theorem 3.9, and $186 \in B(6,1)$.
1171	$X = Z(1171,2)$.
	$\mathcal{B} = \langle 5\alpha, 5\alpha+29, 5\alpha+390, 5\alpha+419, 5\alpha+780, 5\alpha+809\rangle \bmod 1171, \quad \alpha = 0, 1, \ldots, 38.$
1201	$X = Z(1201,11)$.
	$\mathcal{B} = \langle 5\alpha, 5\alpha+38, 5\alpha+400, 5\alpha+438, 5\alpha+800, 5\alpha+838\rangle \bmod 1201, \quad \alpha = 0, 1, \ldots, 39.$
1206	Lemma 2.17 and $241 \in B(6,1)$ as above.
1231	$X = Z(1231,3)$.
	$\mathcal{B} = \langle 5\alpha, 5\alpha+61, 5\alpha+410, 5\alpha+471, 5\alpha+820, 5\alpha+881\rangle \bmod 1231, \quad \alpha = 0, 1, \ldots, 40.$
1261	Lemma 3.12, $210 \in T(6,1)$ and $211 \in B(6,1)$ as above.
1266	Lemma 3.11, $211 \in T(6,1)$ and $211 \in B(6,1)$.
1291	$X = Z(1291,2)$.
	$\mathcal{B} = \langle 5\alpha, 5\alpha+1, 5\alpha+430, 5\alpha+431, 5\alpha+860, 5\alpha+861\rangle \bmod 1291, \quad \alpha = 0, 1, \ldots, 42.$
1321	$X = Z(1321,13)$.
	$\mathcal{B} = \langle 5\alpha, 5\alpha+8, 5\alpha+440, 5\alpha+448, 5\alpha+880, 5\alpha+888\rangle \bmod 1321, \quad \alpha = 0, 1, \ldots, 43.$
1356	Lemma 2.17 and $271 \in B(6,1)$ as above.
1381	$X = Z(1381,2)$.
	$\mathcal{B} = \langle 5\alpha, 5\alpha+38, 5\alpha+460, 5\alpha+498, 5\alpha+920, 5\alpha+958\rangle \bmod 1381, \quad \alpha = 0, 1, \ldots, 45.$
1441	Lemma 3.12, $240 \in T(6,1)$ and $241 \in B(6,1)$.
1446	Lemma 3.11, $241 \in T(6,1)$ and $241 \in B(6,1)$.
1471	$X = Z(1471,6)$.
	$\mathcal{B} = \langle 5\alpha, 5\alpha+77, 5\alpha+490, 5\alpha+567, 5\alpha+980, 5\alpha+1057\rangle \bmod 1471, \quad \alpha = 0, 1, \ldots, 48.$
1531	$X = Z(1531,2)$.
	$\mathcal{B} = \langle 5\alpha, 5\alpha+184, 5\alpha+510, 5\alpha+694, 5\alpha+1020, 5\alpha+1204\rangle \bmod 1531, \quad \alpha = 0, 1, \ldots, 50.$
1536	Lemma 2.17, $307 \in B(18,1)$ by Theorem 2.1, and $91 \in B(6,1)$ as above.
1621	Lemma 3.12, $270 \in T(6,1)$ and $271 \in B(6,1)$.

Table 5.14 (cont.).

v	$B[6,1;v]$
1626	Lemma 3.11, 271 ∈ T(6,1) and 271 ∈ $B(6,1)$.
1656	Lemma 2.17 and 331 ∈ $B(6,1)$.
1681	$X = \mathrm{GF}(1681, x^2 = x+7)$.
	$\mathfrak{B} = \langle 5\alpha, 5\alpha+12, 5\alpha+560, 5\alpha+572, 5\alpha+1120, 5\alpha+1132\rangle$, $\alpha = 0, 1, ..., 55$.
1741	$X = Z(1741, 2)$.
	$\mathfrak{B} = \langle 5\alpha, 5\alpha+9, 5\alpha+580, 5\alpha+589, 5\alpha+1160, 5\alpha+1169\rangle \bmod 1741$, $\alpha = 0, 1, ..., 57$.
1801	$X = Z(1801, 11)$.
	$\mathfrak{B} = \langle 5\alpha, 5\alpha+577, 5\alpha+600, 5\alpha+1177, 5\alpha+1200, 5\alpha+1777\rangle \bmod 1801$, $\alpha = 0, 1, ..., 59$.
1806	Lemma 2.17 and 361 ∈ $B(6,1)$.
1831	$X = Z(1831, 3)$.
	$\mathfrak{B} = \langle 5\alpha, 5\alpha+481, 5\alpha+610, 5\alpha+1091, 5\alpha+1220, 5\alpha+1701\rangle \bmod 1831$, $\alpha = 0, 1, ..., 60$.
1861	$X = Z(1861, 2)$.
	$\mathfrak{B} = \langle 5\alpha, 5\alpha+422, 5\alpha+620, 5\alpha+1042, 5\alpha+1240, 5\alpha+1662\rangle \bmod 1861$, $\alpha = 0, 1, ..., 61$.
1951	$X = Z(1951, 3)$.
	$\mathfrak{B} = \langle 5\alpha, 5\alpha+3, 5\alpha+650, 5\alpha+653, 5\alpha+1300, 5\alpha+1303\rangle \bmod 1951$, $\alpha = 0, 1, ..., 64$.
1981	Lemma 3.12, 330 ∈ T(6,1) by Theorem 3.9, and 331 ∈ $B(6,1)$.
1986	Lemma 3.11, 331 ∈ T(6,1) and 331 ∈ $B(6,1)$.

Lemma 5.26. *If* $u \equiv 0$ *or* 1 (mod 3) *and* $u \neq 4$, *then* $u \in \mathrm{GD}(\{6,7,9,10\}, 1, M_6)$, *where* $M_6 = \{1, 3, 6, 7, 9, 10, 12, 13, 15, 16, 18, 19, 21, 22, 24, 25, 27, 28, 30, 31, 33, 34, 36, 37, 39, 40, 46, 51, 52, 58, 64, 66, 67, 69, 70, 76, 93, 94, 100, 135, 136, 138, 139, 141, 142, 148, 219, 220\}$.

Proof. According to Lemma 3.13 with $t = 1$, $s = 6$ and $r \equiv 0$ or 1 (mod 3) and by Theorem 3.8, whenever $u \geq 384$, there exists a transversal design $\mathrm{T}[6+1, 1; r]$ such that by truncating one of its groups, $6r + r_1 = u$ is obtained. For $u < 384$ and $u \equiv 0$ or 1 (mod 3) use the truncated transversal design $\mathrm{T}[s+1, 1; r]$ with $t = 1$ and values of r and s as in Table 5.15. It should be remembered that 12 ∈ T(7,1) by Lemma 3.21, and that by Theorem 2.1, 57 ∈ $B(8, 1)$, and therefore by Theorem 3.3, 57 ∈ T(8,1).

Lemma 5.27. *For every positive integer* u, $u \in \mathrm{GD}(\{6, 7, 8\}, 1, I(42))$ *holds.*

Proof. Apply Lemma 3.13 with $s = 6$ and $t = 2$, Theorems 3.1 and 3.7.

Table 5.15.

u	r	s	u	r	s
42–45	7	6	144–147	16	9
48–49	7	6	150–160	16	9
54–57	9	6	162–175	25	6
60–63	9	6	177–190	19	9
72–75	12	6	192–217	31	6
78–79	13	6	222–224	37	6
81–82	9	9	225–226	25	9
84–91	13	6	228–259	37	6
96–99	16	6	261–270	27	9
102–112	16	6	271–301	43	6
114–115	19	6	303–343	49	6
117–118	13	9	345–370	37	9
120–133	19	6	371–382	57	6

In a similar way it can be checked:

Lemma 5.28. *For every integer* $u \geq 2$, $u \in \mathrm{GD}(\{6,7,8\},1,M'_6)$ *holds, where* $M'_6 = \{2,3,...,43\}$.

Further we prove:

Lemma 5.29. *For every integer* $v \geq 6$, $v \in B(K_6,1)$ *holds, where* $K_6 = \{6,7,...,41,45,46,47\}$.

Proof. For $v \geq 66$ it is easily checked by Lemma 3.13 with $s = 6$ and $t = 2$, using Theorems 3.1 and 3.7, that $v \in \mathrm{GD}(\{6,7,8\},1,\{1,7,8,...,65\})$ holds. For $v \leq 65$, $v \notin K_6$ we have:

for $62 \leq v \leq 65$, Lemma 3.13 with $r = 8$, $s = 7$, $t = 2$;

for $60 \leq v \leq 61$, Lemma 3.13 with $r = 9$, $s = 6$, $t = 1$;

for $58 \leq v \leq 59$, delete in T[8,1;8], 2 points from each of 2 groups and then delete 1 or 2 points from a third group in such way that non of them should be in a block from which already 2 points have been deleted;

for $54 \leq v \leq 57$, Lemma 3.13 with $r = 8$, $s = 6$, $t = 2$;

for $51 \leq v \leq 53$, delete in T[8,1;7] one point from each of 3, 4 or 5 groups respectively, no three of these points being in the same block (this is possible as shown in the proof of Lemma 3.18);

for $42 \leq v \leq 44$ and $48 \leq v \leq 50$, Lemma 3.13 with $r = 7$, $s = 6$, $t = 2$.

Lemma 5.30. *If* $v \equiv 0$ *or* 1 (mod 3) *and* $6 \leqslant v \leqslant 130$, *then* $v \in B(K'_6, 1)$, *where* $K'_6 = (M_6 \cup \{45, 57, 75, 99\}) \cap \{6, 7, \ldots, 130\}$.

Proof. We make use of Lemma 5.26 and remark that 45, 57, 75 and 99 are the only integers $u \leqslant 130$ in Table 5.15 which correspond to $r_1 = 3$. In all other cases either $r_1 = 1$ or $r_1 \geqslant 6$.

Lemma 5.31. *If* $v \equiv 1$ *or* 6 (mod 15) *and* $v \neq 21$, *then* $v \in B(6, 2)$ *holds.*

Proof. Let $v = 5u + 1$, where $u \equiv 0$ or 1 (mod 3) and $u \neq 4$. By Lemma 5.26, $u \in \mathrm{GD}(\{6, 7, 9, 10\}, 1, M_6)$. By Lemmas 2.26, 4.20 and 4.21, it suffices to show that $v = 5\mu + 1 \in B(6, 2)$ for every $\mu \in M_6$. This is performed in Table 5.16.

Table 5.16.

μ	v	$B[6, 2; v]$
1	6	Trivial.
3	16	$X = \mathrm{GF}(16, x^4 = x + 1)$. $\mathcal{B} = \langle \emptyset, 0, 3, 6, 9, 12 \rangle \bmod 16$.
6	31	Table 5.14.
7	36	Lemmas 4.21 and 2.11.
9	46	Lemmas 4.21 and 2.11.
10	51	Lemmas 4.21 and 2.11.
12	61	Lemmas 4.21 and 2.11.
13	66	Lemmas 4.21 and 2.11.
15	76	$X = I(2) \times Z(2) \times Z(19, 2)$. $\mathcal{B} = \langle (0', \emptyset, \emptyset), (0', \emptyset, 6\alpha+1), (0', 0, 6\alpha+2), (0', 0, 6\alpha+6), (1', \emptyset, 6\alpha+8), (1', 0, 6\alpha+13)\rangle$ mod $(-, 2, 19)$, $\alpha = 0, 1, 2$, $\langle (0', \emptyset, 6\alpha+13), (0', 0, 6\alpha+8), (1', \emptyset, \emptyset), (1', \emptyset, 6\alpha+1), (1', 0, 6\alpha+2), (1', 0, 6\alpha+6)\rangle$ mod $(-, 2, 19)$, $\alpha = 0, 1, 2$, $\langle (0', \emptyset, 0), (0', \emptyset, 6), (0', \emptyset, 12), (1', 0, 4), (1', 0, 10), (1', 0, 16)\rangle$ mod $(-, 2, 19)$, $\langle (0', \emptyset, 4), (0', \emptyset, 10), (0', \emptyset, 16), (1', \emptyset, 0), (1', \emptyset, 6), (1', \emptyset, 12)\rangle$ mod $(-, 2, 19)$, $\langle (0', \emptyset, \emptyset), (0', \emptyset, 0), (0', \emptyset, 6), (0', \emptyset, 12), (0', 0, \emptyset), (1', 0, \emptyset)\rangle$ mod $(-, 2, 19)$, $\langle (0', \emptyset, \emptyset), (1', \emptyset, \emptyset), (1', \emptyset, 0), (1', \emptyset, 6), (1', \emptyset, 12), (1', 0, \emptyset)\rangle$ mod $(-, 2, 19)$.
16	81	Lemmas 2.17, 4.20 and $16 \in B(6, 2)$ as above.
18	91	Table 5.14.
19	96	Lemma 4.19.
21	106	$X = Z(3, 2) \times Z(5, 2) \times Z(7, 3) \cup \{(\infty)\}$. $\mathcal{B}$ = Blocks of $B[6, 2; 16]$ on $Z(3) \times Z(5) \times \{i\} \cup \{(\infty)\}$, $i \in Z(7)$, $\langle (\emptyset, \emptyset, 2\alpha+1), (\emptyset, \emptyset, 2\alpha+2), (\emptyset, 1, 2\alpha+5), (0, \emptyset, 2\alpha+4), (1, 0, 2\alpha+3), (1, 3, 2\alpha)\rangle$ mod $(3, 5, 7)$, $\alpha = 0, 1, 2$, $\langle (\emptyset, \emptyset, 2\alpha+1), (\emptyset, \emptyset, 2\alpha+4), (\emptyset, 0, 2\alpha+3), (\emptyset, 2, 2\alpha), (0, 1, 2\alpha+2), (1, \emptyset, 2\alpha+5)\rangle$ mod $(3, 5, 7)$, $\alpha = 0, 1, 2$.

Table 5.16 (cont.).

μ	v	$B[6,2;v]$
22	111	$X = Z(5,2) \times (Z(19,2) \cup \{\infty_i: i = 0,1,2\}) \cup \{(\infty,\infty)\}$. $\mathfrak{B}$ = Blocks of $B[6,2;16]$ on $Z(5) \times \{\infty_i: i = 0,1,2\} \cup \{(\infty,\infty)\}$, $\langle(\infty,\infty),(\emptyset,\emptyset),(0,\emptyset),(1,\emptyset),(2,\emptyset),(3,\emptyset)\rangle$ mod $(-,19)$, 2 times, $\langle(\emptyset,\infty_\alpha),(\emptyset,\emptyset),(\beta,6\alpha+2-\beta),(\beta,6\alpha+3),(\beta+2,6\alpha+1+\beta),(\beta+2,6\alpha+13)\rangle$ mod $(5,19)$, $\alpha = 0,1,2$, $\beta = 0,1$, $\langle(\emptyset,0),(\emptyset,9),(\beta,3),(\beta,12),(\beta+2,6),(\beta+2,15)\rangle$ mod $(5,19)$, $\beta = 0,1$.
24	121	Table 5.14.
25	126	Table 5.14.
27	136	$X = Z(3,2) \times Z(5,2) \times \mathrm{GF}(9, x^2 = 2x+1) \cup \{(\infty)\}$. $\mathfrak{B}$ = Blocks of $B[6,2;16]$ on $Z(3) \times Z(5) \times \{i\} \cup \{(\infty)\}$, $i \in \mathrm{GF}(9)$, $\langle(\emptyset,\emptyset,2\alpha),(\emptyset,\emptyset,2\alpha+1),(\emptyset,1,2\alpha+4),(0,\emptyset,2\alpha+2),(1,0,2\alpha+3),(1,3,2\alpha+6)\rangle$ mod $(3,5,9)$, $\alpha = 0,1,2,3$, $\langle(\emptyset,\emptyset,2\alpha),(\emptyset,\emptyset,2\alpha+2),(\emptyset,0,2\alpha+1),(\emptyset,2,2\alpha+6),(0,1,2\alpha+5),(1,\emptyset,2\alpha+3)\rangle$ mod $(3,5,9)$, $\alpha = 0,1,2,3$.
28	141	$X = Z(5,2) \times (\mathrm{GF}(25, x^2 = 2x+2) \cup \{\infty_i: i = 0,1,2\}) \cup \{(\infty,\infty)\}$. $\mathfrak{B}$ = Blocks of $B[6,2;16]$ on $Z(5) \times \{\infty_i: i = 0,1,2\}) \cup \{(\infty,\infty)\}$, $\langle(\infty,\infty),(\emptyset,\emptyset),(0,\emptyset),(1,\emptyset),(2,\emptyset),(3,\emptyset)\rangle$ mod $(-,25)$, 2 times, $\langle(\emptyset,\infty_\alpha),(\emptyset,\emptyset),(\gamma,4\alpha),(\gamma,4\alpha+12),(\gamma+2,4\alpha+2),(\gamma+2,4\alpha+14)\rangle$ mod $(5,25)$, $\alpha = 0,1,2$, $\gamma = 0,1$, $\langle(\emptyset,2\beta+1),(\emptyset,2\beta+13),(\gamma,2\beta+5),(\gamma,2\beta+17),(\gamma+2,2\beta+9),(\gamma+2,2\beta+21)\rangle$ mod $(5,25)$, $\beta = 0,1$, $\gamma = 0,1$.
30	151	Table 5.14.
31	156	Table 5.14.
33	166	$X = Z(3,2) \times Z(5,2) \times Z(11,2) \cup \{(\infty)\}$. $\mathfrak{B}$ = Blocks of $B[6,2;16]$ on $Z(3) \times Z(5) \times \{i\} \cup \{(\infty)\}$, $i \in Z(11)$, $\langle(\emptyset,\emptyset,2\alpha),(\emptyset,\emptyset,2\alpha+1),(\emptyset,1,2\alpha+5),(0,\emptyset,2\alpha+4),(1,0,2\alpha+2),(1,3,2\alpha+8)\rangle$ mod $(3,5,11)$, $\alpha = 0,1,2,3,4$, $\langle(\emptyset,\emptyset,2\alpha+1),(\emptyset,\emptyset,2\alpha+2),(\emptyset,0,2\alpha+9),(\emptyset,2,2\alpha+3),(0,1,2\alpha+6),(1,\emptyset,2\alpha+4)\rangle$ mod $(3,5,11)$, $\alpha = 0,1,2,3,4$.
34	171	$X = Z(5,2) \times (Z(31,3) \cup \{\infty_i: i = 0,1,2\}) \cup \{(\infty,\infty)\}$. $\mathfrak{B}$ = Blocks of $B[6,2;16]$ on $Z(5) \times \{\infty_i: i = 0,1,2\} \cup \{(\infty,\infty)\}$, $\langle(\infty,\infty),(\emptyset,\emptyset),(0,\emptyset),(1,\emptyset),(2,\emptyset),(3,\emptyset)\rangle$ mod $(-,31)$, 2 times, $\langle(\emptyset,\infty_\alpha),(\emptyset,\emptyset),(\beta,5\alpha),(\beta,5\alpha+15),(\beta+2,5\alpha+1),(\beta+2,5\alpha+16)\rangle$ mod $(5,31)$, $\alpha = 0,1,2$, $\beta = 0,1$, $\langle(\emptyset,\alpha+2),(\emptyset,\alpha+17),(\beta,\alpha+7),(\beta,\alpha+22),(\beta+2,\alpha+12),(\beta+2,\alpha+27)\rangle$ mod $(5,31)$, $\alpha = 0,1,2$, $\beta = 0,1$.
36	181	Table 5.14.
37	186	Table 5.14.
39	196	$X = Z(3,2) \times Z(5,2) \times Z(13,2) \cup \{(\infty)\}$. $\mathfrak{B}$ = Blocks of $B[6,2;16]$ on $Z(3) \times Z(5) \times \{i\} \cup \{(\infty)\}$, $i \in Z(13)$, $\langle(\emptyset,\emptyset,2\alpha),(\emptyset,\emptyset,2\alpha+1),(\emptyset,1,2\alpha+10),(0,\emptyset,2\alpha+9),(1,0,2\alpha+4),(1,3,2\alpha+6)\rangle$ mod $(3,5,13)$, $\alpha = 0,1,\ldots,5$, $\langle(\emptyset,\emptyset,2\alpha+1),(\emptyset,\emptyset,2\alpha+2),(\emptyset,0,2\alpha+5),(\emptyset,2,2\alpha+7),(0,1,2\alpha+11),(1,\emptyset,2\alpha+10)\rangle$ mod $(3,5,13)$, $\alpha = 0,1,\ldots,5$.
40	201	$X = Z(5,2) \times (Z(37,2) \cup \{\infty_i: i = 0,1,2\}) \cup \{(\infty,\infty)\}$. $\mathfrak{B}$ = Blocks of $B[6,2,16]$ on $Z(5) \times \{\infty_i: i = 0,1,2\} \cup \{(\infty,\infty)\}$, $\langle(\infty,\infty),(\emptyset,\emptyset),(0,\emptyset),(1,\emptyset),(2,\emptyset),(3,\emptyset)\rangle$ mod $(-,37)$, 2 times,

Table 5.16 (cont.).

μ	v	$B[6,2;v]$
		$\langle(\emptyset,\infty_\alpha),(\emptyset,\emptyset),(\gamma,6\alpha),(\gamma,6\alpha+18),(\gamma+2,6\alpha+7),(\gamma+2,6\alpha+25)\rangle$ mod $(5,37)$, $\alpha=0,1,2,\quad \gamma=0,1$, $\langle(\emptyset,\beta+2),(\emptyset,\beta+20),(\gamma,\beta+8),(\gamma,\beta+26),(\gamma+2,\beta+14),(\gamma+2,\beta+32)\rangle$ mod $(5,37)$, $\beta=0,1,2,3,\quad \gamma=0,1$.
46	231	Lemmas 2.17, 4.20 and $46\in B(6,2)$ as above.
51	256	Lemmas 2.17, 4.20 and $51\in B(6,2)$ as above.
52	261	$X=Z(5,2)\times(\mathrm{GF}(49,x^2=x+4)\cup\{\infty_i: i=0,1,2\})\cup\{(\infty,\infty)\}$. $\mathfrak{B}$ = Blocks of $B[6,2;16]$ on $Z(5)\times\{\infty_i: i=0,1,2\}\cup\{(\infty,\infty)\}$, $\langle(\infty,\infty),(\emptyset,\emptyset),(0,\emptyset),(1,\emptyset),(2,\emptyset),(3,\emptyset)\rangle$ mod $(-,49)$, 2 times, $\langle(\emptyset,\infty_\alpha),(\emptyset,\emptyset),(\gamma,8\alpha+\gamma),(\gamma,8\alpha+\gamma+24),(\gamma+2,8\alpha+\gamma+8),(\gamma+2,8\alpha+\gamma+32)\rangle$ mod $(5,49)$, $\alpha=0,1,2,\quad \gamma=0,1$, $\langle(\emptyset,\beta+2),(\emptyset,\beta+26),(\gamma,\beta+10),(\gamma,\beta+34),(\gamma+2,\beta+18),(\gamma+2,\beta+42)\rangle$ mod $(5,49)$, $\beta=0,1,\ldots,5,\quad \gamma=0,1$.
58	291	$X=Z(5,2)\times(\mathrm{GF}(49,x^2=x+4)\cup\{\infty_i: i=0,1,\ldots,8\})\cup\{(\infty,\infty)\}$. $\mathfrak{B}$ = Blocks of $B[6,2;46]$ on $Z(5)\times\{\infty_i: i=0,1,\ldots,8\}\cup\{(\infty,\infty)\}$, $\langle(\infty,\infty),(\emptyset,\emptyset),(0,\emptyset),(1,\emptyset),(2,\emptyset),(3,\emptyset)\rangle$ mod $(-,49)$, 2 times, $\langle(\emptyset,\infty_{3\alpha+\beta}),(\emptyset,\emptyset),(\gamma,8\alpha+\beta+4\gamma),(\gamma,8\alpha+\beta+4\gamma+24),(\gamma+2,8\alpha+\beta+4\gamma+8)$, $(\gamma+2,8\alpha+\beta+4\gamma+32)\rangle$ mod $(5,49)$, $\alpha=0,1,2,\quad \beta=0,1,2,\quad \gamma=0,1$, $\langle(\emptyset,4\delta+3),(\emptyset,4\delta+27),(\gamma,4\delta+11),(\gamma,4\delta+35),(\gamma+2,4\delta+19),(\gamma+2,4\delta+43)\rangle$ mod $(5,49)$, $\gamma=0,1,\quad \delta=0,1$.
64	321	$X=Z(5,2)\times(Z(61,2)\cup\{\infty_i: i=0,1,2\})\cup\{(\infty,\infty)\}$. $\mathfrak{B}$ = Blocks of $B[6,2;16]$ on $Z(5)\times\{\infty_i: i=0,1,2\}\cup\{(\infty,\infty)\}$, $\langle(\infty,\infty),(\emptyset,\emptyset),(0,\emptyset),(1,\emptyset),(2,\emptyset),(3,\emptyset)\rangle$ mod $(-,61)$, 2 times, $\langle(\emptyset,\infty_\alpha),(\emptyset,\emptyset),(\gamma,10\alpha+4),(\gamma,10\alpha+34),(\gamma+2,10\alpha+19),(\gamma+2,10\alpha+49)\rangle$ mod $(5,61)$, $\alpha=0,1,2,\quad \gamma=0,1$, $\langle(\emptyset,5\beta+\delta),(\emptyset,5\beta+\delta+30),(\gamma,5\beta+\delta+10),(\gamma,5\beta+\delta+40),(\gamma+2,5\beta+\delta+20)$, $(\gamma+2,5\beta+\delta+50)\rangle$ mod $(5,61)$, $\beta=0,1,\quad \gamma=0,1,\quad \delta=0,1,2,3$.
66	331	Table 5.14.
67	336	Lemma 4.19.
69	346	$X=Z(3,2)\times Z(5,2)\times Z(23,5)\cup\{(\infty)\}$. $\mathfrak{B}$ = Blocks of $B[6,2;16]$ on $Z(3)\times Z(5)\times\{i\}\cup\{(\infty)\}$, $i\in Z(23)$, $\langle(\emptyset,\emptyset,2\alpha),(\emptyset,\emptyset,2\alpha+1),(\emptyset,1,2\alpha+4),(0,\emptyset,2\alpha+5),(1,0,2\alpha+2),(1,3,2\alpha+9)\rangle$ mod $(3,5,23)$, $\alpha=0,1,\ldots,10$, $\langle(\emptyset,\emptyset,2\alpha),(\emptyset,\emptyset,2\alpha+1),(\emptyset,0,2\alpha+2),(\emptyset,2,2\alpha+6),(0,1,2\alpha+4),(1,\emptyset,2\alpha+5)\rangle$ mod $(3,5,23)$, $\alpha=0,1,\ldots,10$.
70	351	$X=Z(5,2)\times(Z(61,2)\cup\{\infty_i: i=0,1,\ldots,8\})\cup\{(\infty,\infty)\}$. $\mathfrak{B}$ = Blocks of $B[6,2;46]$ on $Z(5)\times\{\infty_i: i=0,1,\ldots,8\}\cup\{(\infty,\infty)\}$, $\langle(\infty,\infty),(\emptyset,\emptyset),(0,\emptyset),(1,\emptyset),(2,\emptyset),(3,\emptyset)\rangle$ mod $(-,61)$, 2 times, $\langle(\emptyset,\infty_{3\alpha+\beta}),(\emptyset,\emptyset),(\gamma,20\alpha+\beta+2),(\gamma,20\alpha+\beta+32),(\gamma+2,20\alpha+\beta+17)$, $(\gamma+2,20\alpha+\beta+47)\rangle$ mod $(5,61)$, $\alpha=0,1,2,\quad \beta=0,1,2,\quad \gamma=0,1$, $\langle(\emptyset,5\delta+\eta),(\emptyset,5\delta+\eta+30),(\gamma,5\delta+\eta+10),(\gamma,5\delta+\eta+40),(\gamma+2,5\delta+\eta+20)$, $(\gamma+2,5\delta+\eta+50)\rangle$ mod $(5,61)$, $\gamma=0,1,\quad \delta=0,1,\quad \eta=0,1$.
76	381	Lemmas 2.17, 4.20 and $76\in B(6,2)$ as above.
93	466	$465\in \mathrm{GD}(6,2,15)$ by $31\in B(6,1)$ (Table 5.14) and $15\in \mathrm{T}(6,2)$ (Theorem 3.11). Apply Lemma 2.14 and $16\in B(6,2)$ as above.

Table 5.16 (cont.).

μ	v	$B[6,2;v]$
94	471	$X = Z(5,2) \times (Z(79,3) \cup \{\infty_i : i = 0,1,\ldots,14\}) \cup \{(\infty,\infty)\}$. $\mathcal{B}$ = Blocks of $B[6,2;76]$ on $Z(5) \times \{\infty_i : i = 0,1,\ldots,14\} \cup \{(\infty,\infty)\}$, $\langle(\infty,\infty),(\emptyset,\emptyset),(0,\emptyset),(1,\emptyset),(2,\emptyset),(3,\emptyset)\rangle$ mod $(-,79)$, 2 times, $\langle(\emptyset,\infty_{\alpha+3\beta+6\gamma}),(\emptyset,\emptyset),(\delta,26\alpha+\beta+7\gamma),(\delta,26\alpha+\beta+7\gamma+39)$, $(\delta+2,26\alpha+3\beta+7\gamma+15),(\delta+2,26\alpha+3\beta+7\gamma+54)\rangle$ mod $(5,79)$, $\alpha = 0,1,2$, $\beta = 0,1$, $\gamma = 0,1$, $\delta = 0,1$, $\langle(\emptyset,\infty_{\alpha+12}),(\emptyset,\emptyset),(\delta,26\alpha+11),(\delta,26\alpha+50),(\delta+2,26\alpha+29),(\delta+2,26\alpha+68)\rangle$ mod $(5,79)$, $\alpha = 0,1,2$, $\delta = 0,1$, $\langle(\emptyset,\mu),(\emptyset,\mu+39),(\delta,\mu+13),(\delta,\mu+52),(\delta+2,\mu+26),(\delta+2,\mu+65)\rangle$ mod $(5,79)$, $\delta = 0,1$, $\mu = 4,6,10$.
100	501	Considering $11 \subset T(9,1)$, by Lemma 3.12, $100 \in B(\{9,12\},1)$; by Lemma 4.21, Lemma 2.16 may be applied. Further make use of Lemma 2.11.
135	676	$X = Z(5,2) \times (Z(113,3) \cup \{\infty_i : i = 0,1,\ldots,21\}) \cup \{(\infty,\infty)\}$. $\mathcal{B}$ = Blocks of $B[6,2;111]$ on $Z(5) \times \{\infty_i : i = 0,1,\ldots,21\} \cup \{(\infty,\infty)\}$, $\langle(\infty,\infty),(\emptyset,\emptyset),(0,\emptyset),(1,\emptyset),(2,\emptyset),(3,\emptyset)\rangle$ mod $(-,113)$, 2 times, $\langle(\emptyset,\infty_{\alpha+2\beta}),(\emptyset,\emptyset),(\gamma,28\alpha+2\beta+5),(\gamma,28\alpha+2\beta+61),(\gamma+2,28\alpha+2\beta+10)$, $(\gamma+2,28\alpha+2\beta+66)\rangle$ mod $(5,113)$, $\alpha = 0,1$, $\beta = 0,1$, $\gamma = 0,1$, $\langle(\emptyset,\infty_{\alpha+2\beta+4}),(\emptyset,\emptyset),(\gamma,28\alpha+3\beta+23),(\gamma,28\alpha+3\beta+79),(\gamma+2,28\alpha+\beta+28)$, $(\gamma+2,28\alpha+\beta+84)\rangle$ mod $(5,113)$, $\alpha = 0,1$, $\beta = 0,1$, $\gamma = 0,1$, $\langle(\emptyset,\infty_{\alpha+8}),(\emptyset,\emptyset),(\gamma,2\alpha+9),(\gamma,2\alpha+65),(\gamma+2,2\alpha+37),(\gamma+2,2\alpha+93)\rangle$ mod $(5,113)$, $\alpha = 0,1$, $\gamma = 0,1$, $\langle(\emptyset,\infty_{\alpha+10}),(\emptyset,\emptyset),(\gamma,\alpha+24),(\gamma,\alpha+80),(\gamma+2,\alpha+52),(\gamma+2,\alpha+108)\rangle$ mod $(5,113)$, $\alpha = 0,1$, $\gamma = 0,1$, $\langle(\emptyset,\infty_{\delta+12}),(\emptyset,\emptyset),(\gamma,2\delta+2),(\gamma,2\delta+58),(\gamma+2,2\delta+30),(\gamma+2,2\delta+86)\rangle$ mod $(5,113)$, $\gamma = 0,1$, $\delta = 0,1,2$, $(\emptyset,\infty_{\delta+15}),(\emptyset,\emptyset),(\gamma,\delta+14),(\gamma,\delta+70),(\gamma+2,\delta+42),(\gamma+2,\delta+98)\rangle$ mod $(5,113)$, $\gamma = 0,1$, $\delta = 0,1,2$, $\langle(\emptyset,\infty_{\eta+18}),(\emptyset,\emptyset),(\gamma,\eta+19),(\gamma,\eta+75),(\gamma+2,\eta+47),(\gamma+2,\eta+103)\rangle$ mod $(5,113)$, $\gamma = 0,1$, $\eta = 0,1,2,3$, $\langle(\emptyset,\mu),(\emptyset,\mu+56),(\gamma,\mu+5),(\gamma,\mu+61),(\gamma+2,\mu+14),(\gamma+2,\mu+70)\rangle$ mod $(5,113)$, $\gamma = 0,1$, $\mu = 3,13,31,41$.
136	681	Lemmas 2.17, 4.20 and $136 \in B(6,2)$ as above.
138	691	Table 5.14.
139	696	Lemma 4.19.
141	706	Lemmas 2.17, 4.20 and $141 \in B(6,2)$ as above.
142	711	$X = Z(5,2) \times (\mathrm{GF}(121, x^2 = 4x+9) \cup \{\infty_i : i = 0,1,\ldots,20\}) \cup \{(\infty,\infty)\}$. $\mathcal{B}$ = Blocks of $B[6,2;106]$ on $Z(5) \times \{\infty_i : i = 0,1,\ldots,20\} \cup \{(\infty,\infty)\}$, $\langle(\infty,\infty),(\emptyset,\emptyset),(0,\emptyset),(1,\emptyset),(2,\emptyset),(3,\emptyset)\rangle$ mod $(-,121)$, 2 times, $\langle(\emptyset,\infty_{\alpha+6\beta}),(\emptyset,\emptyset),(\gamma,10\alpha+8\beta),(\gamma,10\alpha+8\beta+60),(\gamma+2,10\alpha+8\beta+15)$, $(\gamma+2,10\alpha+8\beta+75)\rangle$ mod $(5,121)$, $\alpha = 0,1,\ldots,5$, $\beta = 0,1$, $\gamma = 0,1$, $\langle(\emptyset,\infty_{\alpha+12}),(\emptyset,\emptyset),(\gamma,10\alpha+9),(\gamma,10\alpha+69),(\gamma+2,10\alpha+17),(\gamma+2,10\alpha+77)\rangle$ mod $(5,121)$, $\alpha = 0,1,\ldots,5$, $\gamma = 0,1$, $\langle(\emptyset,\infty_{\delta+18}),(\emptyset,\emptyset),(\gamma,20\delta+6),(\gamma,20\delta+66),(\gamma+2,20\delta+36),(\gamma+2,20\delta+96)\rangle$ mod $(5,121)$, $\gamma = 0,1$, $\delta = 0,1,2$, $\langle(\emptyset,\mu),(\emptyset,\mu+60),(\gamma,\mu+20),(\gamma,\mu+80),(\gamma+2,\mu+40),(\gamma+2,\mu+100)\rangle$ mod $(5,121)$, $\gamma = 0,1$, $\mu = 1,2,4,11,12,14$.

Table 5.16 (cont.).

μ	v	$B[6,2;v]$
148	741	$X = Z(5,2) \times (Z(127,3) \cup \{\infty_i : i = 0,1,\ldots,20\}) \cup \{(\infty,\infty)\}$.
		$\mathfrak{B}$ = Blocks of $B[6,2;106]$ on $Z(5) \times \{\infty_i : i = 0,1,\ldots,20\} \cup \{(\infty,\infty)\}$,
		$\langle(\infty,\infty),(\emptyset,\emptyset),(0,\emptyset),(1,\emptyset),(2,\emptyset),(3,\emptyset)\rangle$ mod $(-,127)$, 2 times,
		$\langle(\emptyset,\infty_\alpha),(\emptyset,\emptyset),(\gamma,3\alpha),(\gamma,3\alpha+63),(\gamma+2,3\alpha+14),(\gamma+2,3\alpha+77)\rangle$
		mod $(5,127)$, $\alpha = 0,1,\ldots,20$, $\gamma = 0,1$,
		$\langle(\emptyset,9\beta+1),(\emptyset,9\beta+64),(\gamma,9\beta+22),(\gamma,9\beta+85),(\gamma+2,9\beta+43),(\gamma+2,9\beta+106)\rangle$
		mod $(5,127)$, $\beta = 0,1,\ldots,6$, $\gamma = 0,1$.
219	1096	$X = Z(3,2) \times Z(5,2) \times Z(73,5) \cup \{(\infty)\}$.
		$\mathfrak{B}$ = Blocks of $B[6,2;16]$ on $Z(3) \times Z(5) \times \{i\} \cup \{(\infty)\}$, $i \in Z(73)$,
		$\langle(\emptyset,\emptyset,2\alpha),(\emptyset,\emptyset,2\alpha+1),(\emptyset,1,2\alpha+9),(0,\emptyset,2\alpha+3),(1,0,2\alpha+4),(1,3,2\alpha+15)\rangle$
		mod $(3,5,73)$, $\alpha = 0,1,\ldots,35$,
		$\langle(\emptyset,\emptyset,2\alpha+1),(\emptyset,\emptyset,2\alpha+2),(\emptyset,0,2\alpha+3),(\emptyset,2,2\alpha+16),(0,1,2\alpha+6)$,
		$(1,\emptyset,2\alpha+8)\rangle$ mod $(3,5,73)$, $\alpha = 0,1,\ldots,35$.
220	1101	$X = Z(5,2) \times (Z(181,2) \cup \{\infty_i : i = 0,1,\ldots,38\}) \cup \{(\infty,\infty)\}$.
		$\mathfrak{B}$ = Blocks of $B[6,2;196]$ on $Z(5) \times \{\infty_i : i = 0,1,\ldots,38\} \cup \{(\infty,\infty)\}$,
		$\langle(\infty,\infty),(\emptyset,\emptyset),(0,\emptyset),(1,\emptyset),(2,\emptyset),(3,\emptyset)\rangle$ mod $(-,181)$, 2 times,
		$\langle(\emptyset,\infty_{\alpha+6\beta}),(\emptyset,\emptyset),(\gamma,15\alpha+2\beta+5),(\gamma,15\alpha+2\beta+95),(\gamma+2,15\alpha+2\beta+16)$,
		$(\gamma+2,15\alpha+2\beta+106)\rangle$ mod $(5,181)$, $\alpha = 0,1,\ldots,5$, $\beta = 0,1$, $\gamma = 0,1$,
		$\langle(\emptyset,\infty_{\delta+3\beta+12}),(\emptyset,\emptyset),(\gamma,30\delta+2\beta+4),(\gamma,30\delta+2\beta+94),(\gamma+2,30\delta+2\gamma+49)$,
		$(\gamma+2,30\delta+2\beta+139)\rangle$ mod $(5,181)$, $\beta = 0,1$, $\gamma = 0,1$, $\delta = 0,1,2$,
		$\langle(\emptyset,\infty_{\delta+2\eta+18}),(\emptyset,\emptyset),(\gamma,30\delta+\eta+8),(\gamma,30\delta+\eta+98),(\gamma+2,30\delta+\eta+53)$,
		$(\gamma+2,30\delta+\eta+143)\rangle$ mod $(5,181)$, $\gamma = 0,1$, $\delta = 0,1,2$, $\eta = 0,1,\ldots,6$,
		$\langle(\emptyset,\mu),(\emptyset,\mu+90),(\gamma,\mu+30),(\gamma,\mu+120),(\gamma+2,\mu+60),(\gamma+2,\mu+150)\rangle$
		mod $(5,181)$, $\gamma = 0,1$, $\mu = 0,2,15,17$.

Lemma 5.32. *If* $v \equiv 1 \pmod 5$, *then* $v \in B(6,3)$ *holds.*

Proof. Let $v = 5\mu + 1$, where u is a positive integer. By Lemma 5.27, $u \in \mathrm{GD}(\{6,7,8\},1,I(42))$. By Lemmas 2.26, 4.20 and 4.22, it suffices to show that $v = 5\mu + 1 \in B(6,3)$ for every $\mu \in I(42)$. For $\mu = 1$ the lemma is trivial. If $v = 5\mu + 1$ is a power of an odd prime, $v \in B(6,3)$ follows from Lemma 4.4. The proof of $v \in B(6,3)$ for other values of μ is given in Table 5.17.

Table 5.17.

μ	v	$B[6,3;v]$
3	16	[1] and [14, p. 256]. $X = \mathrm{GF}(16, x^4 = x+1)$. $\mathcal{B} = \langle \emptyset,0,5,6,12,13\rangle, \langle 1,3,5,6,9,11\rangle, \langle 0,1,6,7,11,14\rangle,$ $\langle 1,3,6,7,10,12\rangle, \langle \emptyset,4,5,7,10,11\rangle, \langle 0,3,5,8,11,13\rangle,$ $\langle 1,2,5,8,10,14\rangle, \langle 1,2,4,11,12,13\rangle, \langle 2,3,5,7,10,13\rangle,$ $\langle 0,2,7,8,9,11\rangle, \langle 1,4,5,8,9,12\rangle, \langle \emptyset,0,1,2,3,4\rangle,$ $\langle \emptyset,2,5,6,9,14\rangle, \langle 0,2,6,8,10,12\rangle, \langle 3,4,6,8,13,14\rangle,$ $\langle \emptyset,4,6,8,10,11\rangle, \langle \emptyset,0,1,9,10,13\rangle, \langle 0,4,5,7,12,14\rangle,$ $\langle \emptyset,2,3,11,12,14\rangle, \langle 0,3,4,9,10,14\rangle, \langle \emptyset,1,7,8,13,14\rangle,$ $\langle 2,4,6,7,9,13\rangle, \langle \emptyset,3,7,8,9,12\rangle, \langle 9,10,11,12,13,14\rangle.$
4	21	[24]. $X = Z(3,2) \times Z(7,3)$. $\mathcal{B} = \langle (0,0),(0,2),(0,4),(1,0),(1,2),(1,4)\rangle \bmod (3,7),$ $\langle (\emptyset,0),(\emptyset,3),(0,1),(0,4),(1,2),(1,5)\rangle \bmod (3,7).$
5	26	[24]. $X = Z(5,2) \times Z(5,2) \cup \{(\infty)\}$. $\mathcal{B} = \langle (\infty),(\emptyset,\emptyset),(0,\emptyset),(1,\emptyset),(2,\emptyset),(3,\emptyset)\rangle \bmod (-,5),$ $\langle (\infty),(\emptyset,\emptyset),(0,1),(1,2),(2,3),(3,0)\rangle \bmod (-,5),$ $\langle (\infty),(\emptyset,\emptyset),(0,3),(1,0),(2,1),(3,2)\rangle \bmod (-,5),$ $\langle (\emptyset,0),(\emptyset,2),(1,1),(1,3),(3,1),(3,3)\rangle \bmod (5,5),$ $\langle (\emptyset,1),(\emptyset,3),(0,0),(0,2),(2,0),(2,2)\rangle \bmod (5,5).$
7	36	Lemmas 4.22 and 2.11.
9	46	We prove $45 \in \mathrm{GD}(6,3,5)$. $X = Z(5,2) \times \mathrm{GF}(9, x^2 = 2x+1)$. $\mathcal{P} = \langle (\emptyset;0),(\emptyset;4),(0;3),(0;7),(2;2),(2;6)\rangle \bmod (5;9),$ $\langle (\emptyset;1),(\emptyset;5),(0;2),(0;6),(2;0),(2;4)\rangle \bmod (5;9),$ $\langle (\emptyset;0),(\emptyset;4),(1;1),(1;5),(3;3),(3;7)\rangle \bmod (5;9),$ $\langle (\emptyset;1),(\emptyset;5),(1;2),(1;6),(3;3),(3;7)\rangle \bmod (5;9).$
10	51	$X = Z(3,2) \times Z(17,3)$. $\mathcal{B} = \langle (\emptyset,\alpha),(\emptyset,\alpha+3),(\emptyset,\alpha+6),(\emptyset,\alpha+9),(0,\emptyset),(1,\emptyset)\rangle \bmod (3,17),\quad \alpha = 0,1,2,$ $\langle (\emptyset,1),(\emptyset,9),(0,2),(0,10),(1,2\beta+3),(1,2\beta+11)\rangle \bmod (3,17),\quad \beta = 0,1.$
11	56	Lemmas 2.17, 4.20 and $11 \in B(6,3)$ by Lemma 4.4.
13	66	Lemma 3.11, $11 \in \mathrm{T}(6,1)$ and $11 \in B(6,3)$ by Lemma 4.4.
15	76	$X = Z(4) \times Z(19,2)$. $\mathcal{B} = \langle (0',6\alpha+1),(0',6\alpha+10),(1',6\alpha+5),(1',6\alpha+14),(2',6\alpha+4),(2',6\alpha+13)\rangle$ $\bmod (4,19),\quad \alpha = 0,1,2,$ $\langle (0',6\alpha),(0',6\alpha+1),(1',6\alpha+7),(1',6\alpha+8),(2',6\alpha+11),(2',6\alpha+12)\rangle$ $\bmod (4,19),\quad \alpha = 0,1,2,$ $\langle (0',\emptyset),(0',1),(0',4),(1',\emptyset),(1',1),(1',4)\rangle \bmod (4,19),$ $\langle (\beta',\emptyset),(\beta',0),(\beta',3),((\beta+2)',\emptyset),((\beta+2)',0),((\beta+2)',3)\rangle \bmod (-,19),$ $\beta = 0,1.$
17	86	We prove $85 \in \mathrm{GD}(6,3,5)$. $X = Z(5,2) \times Z(17,3)$. $\mathcal{P} = \langle (\emptyset;2\alpha+\beta+1),(\emptyset;2\alpha+\beta+9),(\beta;2\alpha+\beta+2),(\beta;2\alpha+\beta+10),(\beta+2;2\alpha+\beta),$ $(\beta+2;2\alpha+\beta+8)\rangle \bmod (5;17),\quad \alpha = 0,1,2,3,\quad \beta = 0,1.$
18	91	Table 5.14.
19	96	Lemma 3.11, $16 \in \mathrm{T}(6,1)$ and $16 \in B(6,3)$ as above.
21	106	Lemmas 2.17, 4.20 and $21 \in B(6,3)$ as above.

Table 5.17 (cont.).

μ	v	$B[6,3;v]$
22	111	$X = Z(3,2) \times Z(37,2)$. $\mathcal{B} = \langle(\emptyset,3\alpha+2),(\emptyset,3\alpha+20),(0,3\alpha+9),(0,3\alpha+27),(1,3\alpha+16),(1,3\alpha+34)\rangle$ mod $(3,37)$, $\alpha = 0,1,\ldots,5$, $\langle(\emptyset,\emptyset),(\emptyset,6\beta),(\emptyset,6\beta+9),(\emptyset,6\beta+18),(\emptyset,6\beta+27),(0,\emptyset)\rangle$ mod $(3,37)$, $\beta = 0,1,2$, $\langle(\emptyset,3\gamma),(\emptyset,3\gamma+18),(0,3\gamma+6),(0,3\gamma+24),(1,3\gamma+12),(1,3\gamma+30)\rangle$ mod $(3,37)$, $\gamma = 0,1$.
23	116	$X = Z(5,2) \times (Z(19,2) \cup \{\infty_i: i = 0,1,2,3\}) \cup \{(\infty,\infty)\}$. $\mathcal{B}$ = Blocks of $B[6,3;21]$ on $Z(5) \times \{\infty_i: i = 0,1,2,3\} \cup \{(\infty,\infty)\}$, $\langle(\infty,\infty),(\emptyset,\emptyset),(0,\emptyset),(1,\emptyset),(2,\emptyset),(3,\emptyset)\rangle$ mod $(-,19)$, 3 times, $\langle(\emptyset,\infty_\alpha),(\emptyset,\emptyset),(\beta,3\alpha),(\beta,3\alpha+9),(\beta+2,3\alpha+4\beta+4),(\beta+2,3\alpha+4\beta+13)\rangle$ mod $(5,19)$, $\alpha = 0,1,2$, $\beta = 0,1$, $\langle(\emptyset,\infty_\alpha),(\emptyset,\emptyset),(0,3\alpha),(1,3\alpha+2),(2,3\alpha+11),(3,3\alpha+1)\rangle$ mod $(5,19)$, $\alpha = 0,1,2$, $\langle(\emptyset,\infty_3),(\emptyset,\emptyset),(0,3\alpha+9),(1,3\alpha+11),(2,3\alpha+2),(3,3\alpha+10)\rangle$ mod $(5,19)$, $\alpha = 0,1,2$, $\langle(\emptyset,0),(\emptyset,1),(\emptyset,6),(\emptyset,7),(\emptyset,12),(\emptyset,13)\rangle$ mod $(5,19)$.
25	126	Table 5.14.
27	136	$X = Z(5,2) \times (\mathrm{GF}(25, x^2 = 2x+2) \cup \{\infty_i: i = 0,1\}) \cup \{(\infty,\infty)\}$. $\mathcal{B}$ = Blocks of $B[6,3;11]$ on $Z(5) \times \{\infty_i: i = 0,1\} \cup \{(\infty,\infty)\}$, $\langle(\infty,\infty),(\emptyset,\emptyset),(0,\emptyset),(1,\emptyset),(2,\emptyset),(3,\emptyset)\rangle$ mod $(-,25)$, 3 times, $\langle(\emptyset,\infty_\alpha),(\emptyset,\emptyset),(0,12\alpha+4\beta),(1,12\alpha+4\beta+7),(2,12\alpha+4\beta+13),(3,12\alpha+4\beta+5)\rangle$ mod $(5,25)$, $\alpha = 0,1$, $\beta = 0,1,2$, $\langle(\emptyset,4\gamma+1),(\emptyset,4\gamma+13),(0,4\gamma+2),(1,4\gamma+16),(2,4\gamma+5),(3,4\gamma+3)\rangle$ mod $(5,25)$, $\gamma = 0,1,\ldots,5$, $\langle(\emptyset,0),(\emptyset,4),(\emptyset,8),(\emptyset,12),(\emptyset,16),(\emptyset,20)\rangle$ mod $(5,25)$, $\langle(\emptyset,0),(\emptyset,1),(\emptyset,8),(\emptyset,9),(\emptyset,16),(\emptyset,17)\rangle$ mod $(5,25)$.
28	141	$X = Z(20) \times Z(7,3) \cup \{(\infty)\}$. $\mathcal{B}$ = Blocks of $B[6,3;21]$ on $Z(20) \times \{i\} \cup \{(\infty)\}$, $i \in Z(7)$, $\langle(0',2\alpha),(0',2\alpha+3),(1',2\alpha+4),(4',\emptyset),(6',2\alpha+2),(8',2\alpha+5)\rangle$ mod $(20,7)$, $\alpha = 0,1,2$, $\langle(0',2\alpha),(0',2\alpha+3),(3',2\alpha+1),(4',2\alpha+5),(9',\emptyset),(15',2\alpha+2)\rangle$ mod $(20,7)$, $\alpha = 0,1,2$, $\langle(0',2\alpha),(0',2\alpha+3),(2',2\alpha+2),(5',2\alpha+5),(6',2\alpha+4),(13',2\alpha+1)\rangle$ mod $(20,7)$, $\alpha = 0,1,2$, $\langle(0',\emptyset),(1',2\alpha),(3',2\alpha+4),(10',2\alpha+5),(11',2\alpha+3),(13',2\alpha+1)\rangle$ mod $(20,7)$, $\alpha = 0,1,2$.
29	146	We prove $145 \in \mathrm{GD}(6,3,5)$. $X = Z(5,2) \times Z(29,2)$. $\mathcal{P} = \langle(\emptyset;2\alpha+\beta+1),(\emptyset;2\alpha+\beta+15),(\beta;2\alpha+\beta+2),(\beta;2\alpha+\beta+16),(\beta+2;2\alpha+\beta),$ $(\beta+2;2\alpha+\beta+14)\rangle$ mod $(5,29)$, $\alpha = 0,1,\ldots,6$, $\beta = 0,1$.
31	156	Table 5.14.
32	161	We prove $160 \in \mathrm{GD}(6,3,5)$. $X = Z(5,2) \times (Z(31,3) \cup \{\infty\})$. $\mathcal{P} = \langle(\emptyset;\infty),(\emptyset;2\alpha),(0;2\alpha+24),(1;2\alpha+18),(2;2\alpha+6),(3;2\alpha+12)\rangle$ mod $(5;31)$, $\alpha = 0,1,2$, $\langle(\emptyset;\emptyset),(\emptyset;6\beta+2\gamma+2),(0;6\beta+\gamma),(1;6\beta+4\gamma+10),(2;6\beta+14\gamma+18),$ $(3;6\beta-2\gamma+28)\rangle$ mod $(5;31)$, $\beta = 0,1,2,3,4$, $\gamma = 0,1$,

Table 5.17 (cont.).

μ	v	$B[6,3;v]$
		$\langle(\emptyset;\gamma+4),(\emptyset;\gamma+10),(\emptyset;\gamma+16),(\emptyset;\gamma+22),(\emptyset;\gamma+28),(\gamma;\emptyset)\rangle \bmod (5;31)$, $\gamma=0,1$, $\langle(\emptyset;\emptyset),(\emptyset;0),(\emptyset;6),(\emptyset,12),(\emptyset,18),(\emptyset,24)\rangle \bmod (5;31)$.
33	166	$X=Z(5,2)\times(Z(31,3)\cup\{\infty_i: i=0,1\})\cup\{(\infty,\infty)\}$. $\mathscr{B}$ = Blocks of $B[6,3;11]$ on $Z(5)\times\{\infty_i: i=0,1\}\cup\{(\infty,\infty)\}$, Blocks of $B[6,1;31]$ on $\{j\}\times Z(31)$, $j\in Z(5)$, Blocks of GD$[6,1,5;155]$ (exist by Table 5.14 and Lemma 2.12) on $Z(5)\times Z(31)$, 2 times, $\langle(\infty,\infty),(\emptyset,\emptyset),(0,\emptyset),(1,\emptyset),(2,\emptyset),(3,\emptyset)\rangle \bmod (-,31)$, 3 times, $\langle(\emptyset,\infty_\alpha),(\emptyset,3\alpha+\beta),(0,3\alpha+\beta+6),(1,3\alpha+\beta+12),(2,3\alpha+\beta+24)$, $(3,3\alpha+\beta+18)\rangle \bmod (5,31)$, $\alpha=0,1$, $\beta=0,1,2$.
34	171	$X=Z(9,2)\times Z(19,2)$, in the cycle $Z(9,2)$ the powers of 2 will be denoted by exponents, but the multiples of 3 by $0',3',6'$ respectively. $\mathscr{B}=\langle(0',3\alpha+7),(0',3\alpha+16),(\alpha+3\beta,3\alpha+4),(\alpha+3\beta+1,3\alpha+5),(\alpha+3\beta+3,3\alpha+9)$, $(\alpha+3\beta+4,3\alpha+17)\rangle \bmod (9,19)$, $\alpha=0,1,2$, $\beta=0,1$, $\langle(0',3\alpha+2),(0',3\alpha+11),(\alpha,3\alpha+16),(\alpha+3,3\alpha+16),(3',3\alpha+3),(6',3\alpha+3)\rangle$ $\bmod (9,19)$, $\alpha=0,1,2$, $\langle(0',3\alpha+2),(0',3\alpha+11),(\alpha,3\alpha+1),(\alpha+1,3\alpha+8),(\alpha+3,3\alpha+1),(\alpha+4,3\alpha+8)\rangle$ $\bmod (9,19)$, $\alpha=0,1,2$, $\langle(0',0),(0',9),(\alpha,3),(\alpha,12),(\alpha+1,6),(\alpha+1,15)\rangle \bmod (9,19)$, $\alpha=0,1,2$, $\langle(0,\beta+1),(0,\beta+10),(2,\beta+4),(2,\beta+13),(4,\beta+7),(4,\beta+16)\rangle \bmod (9,19)$, $\beta=0,1$.
35	176	Lemma 3.11, $16\in T(11,1)$ and $\{11,16\}\subset B(6,3)$.
37	186	Table 5.14.
39	196	$X=Z(5,2)\times(Z(37,2)\cup\{\infty_i: i=0,1\})\cup\{(\infty,\infty)\}$. $\mathscr{B}$ = Blocks of $B[6,3;11]$ on $Z(5)\times\{\infty_i: i=0,1\}\cup\{(\infty,\infty)\}$, $\langle(\infty,\infty),(\emptyset,\emptyset),(0,\emptyset),(1,\emptyset),(2,\emptyset),(3,\emptyset)\rangle \bmod (-,37)$, 3 times, $\langle(\emptyset,\infty_\alpha),(\emptyset,\emptyset),(0,18\alpha+6\beta),(1,18\alpha+6\beta+13),(2,18\alpha+6\beta+22),(3,18\alpha+6\beta+6)\rangle$ $\bmod (5,37)$, $\alpha=0,1$, $\beta=0,1,2$, $\langle(\emptyset,6\beta+3\gamma+10),(\emptyset,6\beta+3\gamma+28),(\gamma,6\beta+3\gamma+5),(\gamma,6\beta+3\gamma+23)$, $(\gamma+2,6\beta+3\gamma),(\gamma+2,6\beta+3\gamma+18)\rangle \bmod (5,37)$, $\beta=0,1,2$, $\gamma=0,1$, $\langle(2\alpha-\gamma+1,3\gamma),(2\alpha-\gamma+1,3\gamma+12),(2\alpha-\gamma+1,3\gamma+24),(2\alpha-\gamma+3,3\gamma+2)$, $(2\alpha-\gamma+3,3\gamma+14),(2\alpha-\gamma+3,3\gamma+26)\rangle \bmod (5,37)$, $\alpha=0,1$, $\gamma=0,1$, $\langle(\emptyset,3\alpha+7),(\emptyset,3\alpha+25),(\gamma,3\alpha+13),(\gamma,3\alpha+31),(\gamma+2,3\alpha+1),(\gamma+2,3\alpha+19)\rangle$ $\bmod (5,37)$, $\alpha=0,1$, $\gamma=0,1$.
40	201	$X=Z(3,2)\times Z(67,2)$. $\mathscr{B}=\langle(\emptyset,\alpha),(\emptyset,\alpha+33),(0,\alpha+11),(0,\alpha+44),(1,\alpha+22),(1,\alpha+55)\rangle \bmod (3,67)$, $\alpha=0,1,\ldots,10$, $\langle(\emptyset,11\beta),(\emptyset,11\beta+33),(0,11\beta+1),(0,11\beta+34),(1,11\beta+16),(1,11\beta+49)\rangle$ $\bmod (3,67)$, $\beta=0,1,2$, $\langle(\emptyset,5\gamma+2),(\emptyset,5\gamma+35),(0,5\gamma+13),(0,5\gamma+46),(1,5\gamma+24),(1,5\gamma+57)\rangle$ $\bmod (3,67)$, $\gamma=0,1$, $\langle(\emptyset,2\gamma),(\emptyset,3\gamma+1),(\emptyset,2\gamma+22),(\emptyset,3\gamma+23),(\emptyset,2\gamma+44),(\emptyset,3\gamma+45)\rangle$ $\bmod (3,67)$, $\gamma=0,1$, $\langle(\emptyset,3),(\emptyset,19),(\emptyset,25),(\emptyset,41),(\emptyset,47),(\emptyset,63)\rangle \bmod (3,67)$, $\langle(0,6),(0,28),(0,50),(1,6),(1,28),(1,50)\rangle \bmod (3,67)$.
41	206	Lemmas 2.17, 4.20 and $41\in B(6,3)$ by Lemma 4.4.

Lemma 5.33. *If* $v \equiv 0$ *or* 1 (mod 3) *and* $v \geqslant 6$, *then* $v \in B(6, 5)$ *holds.*

Proof. Let $v = 3u + \epsilon$, where $u \geqslant 2$ and $\epsilon = 0$ or 1. By Lemma 5.28, $u \in \mathrm{GD}(\{6, 7, 8\}, 1, M'_6)$. By Lemmas 2.25, 2.26 and 4.23, it suffices to show that $v = 3\mu + \epsilon \in B(6, 5)$ for every $\mu \in M'_6$ and $\epsilon \in \{0, 1\}$. By definition of M'_6, this means that we have to prove $v \in B(6, 5)$ for $v \equiv 0$ or 1 (mod 3) and $6 \leqslant v \leqslant 130$. By Lemmas 5.30 and 2.5, it is sufficient to prove our lemma for $v \in K'_6$. If $v \equiv 1$ or 6 (mod 15) and $v \neq 21$ it follows by Lemmas 5.31 and 5.32, that both $v \in B(6, 2)$ and $v \in B(6, 3)$ hold and consequently by Lemma 2.4, $v \in B(6, 5)$. If $v \equiv 1$ (mod 6) is a prime-power, $v \in B(6, 5)$ follows from Lemma 4.1, and for $v = 6$ the lemma is trivial. Accordingly it remains to prove the lemma for $v \in \{9, 10, 12, 15, 18, 21, 22, 24, 27, 28, 30, 33, 34, 39, 40, 45, 52, 57, 58, 64, 69, 70, 75, 93, 94, 99, 100\}$. This is performed in Table 5.18.

Table 5.18.

v	$B[6, 5; v]$
9	Lemma 4.5 and $9 \in B(3, 1)$ by Lemma 5.4.
10	Lemma 4.5 and $10 \in B(4, 2)$ by Lemma 5.12.
12	Lemma.4.6, $11 \in B(5, 2)$ by Lemma 5.20 and $11 \in B(6, 3)$ by Lemma 5.32.
15	[24]. $X = Z(3, 2) \times Z(5, 2)$. $\mathcal{B} = \langle(0, \emptyset), (0, 1), (0, 3), (1, \emptyset), (1, 0), (1, 2)\rangle$ mod (3, 5), $\langle(\emptyset, 0), (\emptyset, 2), (0, 0), (0, 2), (1, 1), (1, 3)\rangle$ mod (3, 5), $\langle(\emptyset, 0), (\emptyset, 1), (0, 0), (0, 1), (1, 0), (1, 1)\rangle$ mod (–, 5).
18	$X = Z(17, 3) \cup \{\infty\}$. $\mathcal{B} = \langle\infty, \emptyset, 0, 4, 8, 12\rangle$ mod 17, $\langle\emptyset, 4\alpha, 4\alpha+3, 4\alpha+7, 4\alpha+10, 4\alpha+15\rangle$ mod 17, $\alpha = 0, 1$.
21	$X = Z(3, 2) \times Z(7, 3)$. $\mathcal{B} = \langle(\emptyset, 0), (\emptyset, 3), (0, 1), (0, 4), (1, 2), (1, 5)\rangle$ mod (3, 7), $\langle(\emptyset, \emptyset), (0, \emptyset), (0, 1), (0, 4), (1, 0), (1, 3)\rangle$ mod (3, 7), $\langle(0, 1), (0, 2), (0, 4), (0, 5), (1, 2), (1, 5)\rangle$ mod (3, 7), $\langle(\emptyset, 0), (\emptyset, 3), (0, 0), (0, 3), (1, 0), (1, 3)\rangle$ mod (–, 7).
22	$X = Z(2) \times Z(11, 2)$. $\mathcal{B} = \langle(\emptyset, \emptyset), (\emptyset, \alpha), (\emptyset, \alpha+5), (0, \emptyset), (0, \alpha+1), (0, \alpha+6)\rangle$ mod (–, 11), $\alpha = 0, 1, 2, 3, 4$, $\langle(\emptyset, \emptyset), (0, 0), (0, 2), (0, 4), (0, 6), (0, 8)\rangle$ mod (2, 11).
24	$X = Z(23, 5) \cup \{\infty\}$. $\mathcal{B} = \langle\infty, \emptyset, 0, 10, 19, 20\rangle$ mod 23, $\langle\emptyset, 0, 2, 6, 9, 18\rangle$ mod 23, $\langle\emptyset, 0, 3, 4, 6, 14\rangle$ mod 23, $\langle\emptyset, 0, 8, 9, 16, 19\rangle$ mod 23.
27	$X = Z(2) \times Z(13, 2) \cup \{(\infty)\}$. $\mathcal{B} = \langle(\infty), (\emptyset, \emptyset), (0, \emptyset), (0, 0), (0, 4), (0, 8)\rangle$ mod (2, 13), $\langle(\emptyset, 4\alpha+1), (\emptyset, 4\alpha+5), (0, \emptyset), (0, 0), (0, 4), (0, 8)\rangle$ mod (2, 13), $\alpha = 0, 1, 2$, $\langle(\emptyset, 0), (\emptyset, 4), (\emptyset, 8), (0, 0), (0, 4), (0, 8)\rangle$ mod (–, 13).

Table 5.18 (cont.).

v	$B[6,5;v]$
28	$X = Z(4) \times Z(7,3)$.
	$\mathfrak{B} = \langle (0',2\alpha),(0',2\alpha+3),(1',2\alpha+2),(1',2\alpha+5),(2',\emptyset),(3',\emptyset)\rangle$ mod $(4,7)$, $\alpha = 0,1,2$,
	$\langle (0',\emptyset),(0',0),(0',2),(0',4),(1',\emptyset),(3',\emptyset)\rangle$ mod $(4,7)$,
	$\langle (\beta',0),(\beta',2),(\beta',4),((\beta+2)',0),((\beta+2)',2),((\beta+2)',4)\rangle$ mod $(-,7)$, $\beta = 0,1$.
30	$X = Z(29,2) \cup \{\infty\}$.
	$\mathfrak{B} = \langle \infty,\emptyset,0,7,14,21\rangle$ mod 29,
	$\langle \emptyset,\mu,\mu+1,\mu+3,\mu+15,\mu+17\rangle$ mod 29, $\mu = 0,3,7,10$.
33	$X = Z(3,2) \times Z(11,2)$.
	$\mathfrak{B} = \langle (\emptyset,\emptyset),(\emptyset,1),(\emptyset,6),(0,\alpha+3),(0,\alpha+8),(1,\emptyset)\rangle$ mod $(3,11)$, $\alpha = 0,1$,
	$\langle (\emptyset,1),(\emptyset,8),(\alpha,2),(\alpha,7),(\alpha+1,0),(\alpha+1,9)\rangle$ mod $(3,11)$, $\alpha = 0,1$,
	$\langle (0,\emptyset),(0,0),(0,2),(0,5),(0,7),(1,\emptyset)\rangle$ mod $(3,11)$,
	$\langle (\emptyset,2),(\emptyset,3),(0,2),(0,3),(1,2),(1,3)\rangle$ mod $(-,11)$.
34	$X = Z(3,2) \times Z(11,2) \cup \{(\infty)\}$.
	$\mathfrak{B} = \langle (\infty),(\emptyset,0),(\emptyset,2),(\emptyset,4),(0,\emptyset),(1,\emptyset)\rangle$ mod $(3,11)$,
	$\langle (\emptyset,0),(\emptyset,1),(\emptyset,2),(\emptyset,3),(\alpha,\emptyset),(\alpha+1,6)\rangle$ mod $(3,11)$, $\alpha = 0,1$,
	$\langle (\emptyset,6),(\emptyset,7),(\emptyset,9),(\alpha,3),(\alpha,8),(\alpha+1,\emptyset)\rangle$ mod $(3,11)$, $\alpha = 0,1$,
	$\langle (\emptyset,1),(\emptyset,5\alpha+2),(0,1),(0,5\alpha+2),(1,1),(1,5\alpha+2)\rangle$ mod $(-,11)$, $\alpha = 0,1$.
39	$X = Z(3,2) \times Z(11,2) \cup \{(\infty_i: i = 0,1,\dots,5\}$.
	$\mathfrak{B} = \langle (\infty_0),(\infty_1),(\infty_2),(\infty_3),(\infty_4),(\infty_5)\rangle$, 5 times,
	$\langle (\infty_\alpha),(\emptyset,\emptyset),(\emptyset,0),(\emptyset,4),(\alpha,6),(\alpha+1,7)\rangle$ mod $(3,11)$, $\alpha = 0,1$,
	$\langle (\infty_{\alpha+2}),(\emptyset,\emptyset),(\emptyset,6),(\emptyset,8),(\alpha,3),(\alpha+1,4)\rangle$ mod $(3,11)$, $\alpha = 0,1$,
	$\langle (\infty_{\alpha+4}),(\emptyset,5),(\emptyset,2\alpha+1),(\emptyset,2\alpha+7),(0,\emptyset),(1,\emptyset)\rangle$ mod $(3,11)$, $\alpha = 0,1$,
	$\langle (0,\emptyset),(0,0),(0,5),(1,\emptyset),(1,2),(1,7)\rangle$ mod $(3,11)$,
	$\langle (\emptyset,\emptyset),(\emptyset,0),(0,\emptyset),(0,0),(1,\emptyset),(1,0)\rangle$ mod $(-,11)$.
40	$X = Z(3,2) \times Z(11,2) \cup \{(\infty_i): i = 0,1,\dots,6\}$.
	$\mathfrak{B} =$ Blocks of $B[6,5;7]$ on $\{(\infty_i): i = 0,1,\dots,6\}$,
	$\langle (\infty_\alpha),(\emptyset,\emptyset),(\emptyset,0),(\emptyset,8),(\alpha,1),(\alpha,3)\rangle$ mod $(3,11)$, $\alpha = 0,1$,
	$\langle (\infty_{\alpha+2}),(\emptyset,\emptyset),(\emptyset,0),(\emptyset,2),(\alpha,5),(\alpha+1,7)\rangle$ mod $(3,11)$, $\alpha = 0,1$,
	$\langle (\infty_{\alpha+4}),(\emptyset,\emptyset),(\emptyset,1),(\emptyset,9),(\alpha,3),(\alpha+1,6)\rangle$ mod $(3,11)$, $\alpha = 0,1$,
	$\langle (\infty_6),(\emptyset,0),(\emptyset,1),(\emptyset,2),(0,\emptyset),(1,\emptyset)\rangle$ mod $(3,11)$,
	$\langle (\emptyset,0),(\emptyset,5\alpha+4),(0,0),(0,5\alpha+4),(1,0),(1,5\alpha+4)\rangle$ mod $(-,11)$, $\alpha = 0,1$.
45	$X = Z(2) \times Z(19,2) \cup \{(\infty_i): i = 0,1,\dots,6\}$.
	$\mathfrak{B} =$ Blocks of $B[6,5;7]$ on $\{(\infty_i): i = 0,1,\dots,6\}$,
	$\langle (\infty_\alpha),(\emptyset,6\alpha+5),(0,\emptyset),(0,0),(0,6),(0,12)\rangle$ mod $(2,19)$, $\alpha = 0,1,2$,
	$\langle (\infty_{\alpha+3}),(\emptyset,3\alpha),(\emptyset,3\alpha+9),(0,2),(0,8),(0,14)\rangle$ mod $(2,19)$, $\alpha = 0,1$,
	$\langle (\infty_6),(\emptyset,\emptyset),(0,\emptyset),(0,0),(0,6),(0,12)\rangle$ mod $(2,19)$,
	$\langle (\emptyset,1),(\emptyset,7),(\emptyset,13),(0,2),(0,8),(0,14)\rangle$ mod $(2,19)$,
	$\langle (\emptyset,0),(\emptyset,6),(\emptyset,12),(0,0),(0,6),(0,12)\rangle$ mod $(-,19)$.
52	$X = Z(43,3) \cup \{\infty_i: i = 0,1,\dots,8\}$.
	$\mathfrak{B} =$ Blocks of $B[6,5;9]$ on $\{\infty_i: i = 0,1,\dots,8\}$,
	$\langle \infty_\alpha,\emptyset,7\alpha+1,7\alpha+20,7\alpha+22,7\alpha+41\rangle$ mod 43, $\alpha = 0,1,2$,
	$\langle \infty_{\alpha+3},\emptyset,7\alpha+2,7\alpha+3,7\alpha+23,7\alpha+24\rangle$ mod 43, $\alpha = 0,1,2$,
	$\langle \infty_{\alpha+6},\emptyset,7\alpha+4,7\alpha+19,7\alpha+25,7\alpha+40\rangle$ mod 43, $\alpha = 0,1,2$,
	$\langle 0,7,14,21,28,35\rangle$ mod 43.
57	Lemma 3.12, $8 \in T(7,1)$ and $\{7,9\} \subset B(6,5)$.

Table 5.18 (cont.).

v	$B[6,5;v]$
58	$X = \mathrm{GF}(49, x^2 = x+4) \cup \{\infty_i : i = 0,1,\ldots,8\}$. $\mathcal{B}$ = Blocks of $B[6,5;9]$ on $\{\infty_i : i = 0,1,\ldots,8\}$, $\langle \infty_{\alpha+3\beta}, \emptyset, 8\alpha+2\beta, 8\alpha-6\beta+17, 8\alpha+2\beta+24, 8\alpha-6\beta+41 \rangle \bmod 49$, $\alpha = 0,1,2$, $\beta = 0,1,2$, $\langle \gamma+6, \gamma+14, \gamma+22, \gamma+30, \gamma+38, \gamma+46 \rangle \bmod 49$, $\gamma = 0,1$.
64	Lemma 3.12, $9 \in \mathrm{T}(7,1)$ and $\{7,10\} \subset B(6,5)$.
69	$X = Z(3,2) \times Z(19,2) \cup \{(\infty_i) : i = 0,1,\ldots,11\}$. $\mathcal{B}$ = Blocks of $B[6,5;12]$ on $\{(\infty_i) : i = 0,1,\ldots,11\}$, $\langle (\infty_\alpha), (\emptyset,\emptyset), (\emptyset,2), (\emptyset,12), (\alpha,5), (\alpha+1,9) \rangle \bmod (3,19)$, $\alpha = 0,1$, $\langle (\infty_{\alpha+2}), (\emptyset,\emptyset), (\emptyset,1), (\emptyset,8), (\alpha,5), (\alpha+1,12) \rangle \bmod (3,19)$, $\alpha = 0,1$, $\langle (\infty_{\alpha+4}), (\emptyset,\emptyset), (\emptyset,0), (\emptyset,16), (\alpha,6), (\alpha+1,2) \rangle \bmod (3,19)$, $\alpha = 0,1$, $\langle (\infty_{\alpha+6}), (\emptyset,\emptyset), (\emptyset,8), (\emptyset,13), (\alpha,7), (\alpha+1,17) \rangle \bmod (3,19)$, $\alpha = 0,1$, $\langle (\infty_8), (\emptyset,3), (\emptyset,7), (\emptyset,14), (0,\emptyset), (1,\emptyset) \rangle \bmod (3,19)$, $\langle (\infty_9), (\emptyset,5), (\emptyset,9), (\emptyset,11), (0,\emptyset), (1,\emptyset) \rangle \bmod (3,19)$, $\langle (\infty_{10}), (\emptyset,\emptyset), (\emptyset,5), (\emptyset,14), (0,8), (0,17) \rangle \bmod (3,19)$, $\langle (\infty_{11}), (\emptyset,\emptyset), (\emptyset,7), (\emptyset,16), (0,1), (0,10) \rangle \bmod (3,19)$, $\langle (0,\emptyset), (0,0), (0,9), (1,\emptyset), (1,3), (1,12) \rangle \bmod (3,19)$, $\langle (\emptyset,3), (\emptyset,12), (0,3), (0,12), (1,3), (1,12) \rangle \bmod (-,19)$.
70	By Theorem 3.1, $\mathrm{T}[7,1;11]$ exists; delete in this design any block and $\mathrm{GD}[\{6,7\},1,10;70]$ is obtained; apply Lemma 2.23 and $\{7,10\} \subset B(6,5)$.
75	By Lemma 3.10 with $r = 11, s = 6, t = 1$, $75 \in B(\{6,7,9,12\},1)$; further $\{7,9,12\} \subset B(6,5)$.
93	$X = Z(7,3) \times Z(11,2) \cup \{(\infty_i) : i = 0,1,\ldots,15\}$. $\mathcal{B}$ = Blocks of $B[6,5;16]$ on $\{(\infty_i) : i = 0,1,\ldots,15\}$, $\langle (\infty_{3\alpha+\beta}), (\emptyset,\emptyset), (2\beta,2\alpha), (2\beta+2,2\alpha+8), (2\beta+3,2\alpha+1), (2\beta+5,2\alpha+9) \rangle \bmod (7,11)$, $\alpha = 0,1,2,3,4$, $\beta = 0,1,2$, $\langle (\infty_{15}), (\emptyset,0), (\emptyset,2), (\emptyset,4), (\emptyset,6), (\emptyset,8) \rangle \bmod (7,11)$, $\langle (\emptyset,\emptyset), (\emptyset,0), (\emptyset,2), (\emptyset,4), (\emptyset,6), (\emptyset,8) \rangle \bmod (7,11)$, $\langle (0,\emptyset), (1,\emptyset), (2,\emptyset), (3,\emptyset), (4,\emptyset), (5,\emptyset) \rangle \bmod (7,11)$.
94	$X = Z(5,2) \times Z(17,3) \cup \{(\infty_i) : i = 0,1,\ldots,8\}$. $\mathcal{B}$ = Blocks of $B[6,5;9]$ on $\{(\infty_i) : i = 0,1,\ldots,8\}$, $\langle (\infty_\alpha), (\emptyset,\emptyset), (0,\alpha), (0,\alpha+8), (2,\alpha+4), (2,\alpha+12) \rangle \bmod (5,17)$, $\alpha = 0,1,\ldots,7$, $\langle (\infty_8), (\emptyset,\emptyset), (0,\emptyset), (1,\emptyset), (2,\emptyset), (3,\emptyset) \rangle \bmod (5,17)$, $\langle (\emptyset,\alpha+2), (\emptyset,\alpha+10), (\alpha,\alpha+5), (\alpha,\alpha+13), (\alpha+2,\alpha), (\alpha+2,\alpha+8) \rangle \bmod (5,17)$, $\alpha = 0,1,\ldots,7$.
99	$X = Z(3,2) \times Z(3,2) \times Z(11,2)$. $\mathcal{B}$ = Blocks of $B[6,5;9]$ on $Z(3) \times Z(3) \times \{i\}$, $i \in Z(11)$, $\langle (\emptyset,\emptyset,2\alpha), (\emptyset,\emptyset,2\alpha+1), (\emptyset,\emptyset,2\alpha+2), (0,\emptyset,2\alpha+3), (0,0,2\alpha+5), (0,1,2\alpha+7) \rangle$ $\bmod (3,3,11)$, $\alpha = 0,1,2,3,4$, $\langle (\emptyset,\emptyset,2\alpha), (\emptyset,\emptyset,2\alpha+5), (0,\emptyset,2\alpha+1), (0,\emptyset,2\alpha+6), (1,0,2\alpha+4), (1,1,2\alpha+9) \rangle$ $\bmod (3,3,11)$, $\alpha = 0,1,2,3,4$, $\langle (0,\emptyset,2\alpha), (0,0,2\alpha+1), (0,1,2\alpha+2), (1,\emptyset,2\alpha+4), (1,0,2\alpha+5), (1,1,2\alpha+3) \rangle$ $\bmod (3,3,11)$, $\alpha = 0,1,2,3,4$.
100	Lemma 3.12, $11 \in \mathrm{T}(9,1)$ and $\{9,12\} \subset B(6,5)$.

Table 5.19.

v	$B[6, 15; v]$
8	Lemma 4.5 and $8 \in B(2, 1)$ trivially.
14	Lemma 4.6, $13 \in B(5, 5)$ by Lemma 5.21 and $13 \in B(6, 5)$ by Lemma 5.33.
20	$X = Z(19, 2) \cup \{\infty\}$. $\mathcal{B}$ = Blocks of $B[6, 5; 19]$ on $Z(19)$, 2 times, $\langle \infty, \emptyset, 6\alpha, 6\alpha+1, 6\alpha+10, 6\alpha+14 \rangle$ mod 19, $\alpha = 0, 1, 2$, $\langle 0, 3, 6, 9, 12, 15 \rangle$ mod 19.
32	Lemma 4.6, $31 \in B(5, 2)$ by Lemma 5.20 and $31 \in B(6, 1)$ as in Table 5.14.
35	$X = Z(5, 2) \times Z(7, 3)$. $\mathcal{B} = \langle (\emptyset, 0), (\emptyset, 2), (\emptyset, 4), (\alpha, \emptyset), (\alpha+1, \emptyset), (\alpha+2, \emptyset) \rangle$ mod (5, 7), $\alpha = 0, 1, 2, 3$, $\langle (\emptyset, 0), (\emptyset, 2), (\emptyset, 4), (\alpha, 1), (\alpha, 3), (\alpha, 5) \rangle$ mod (5, 7), $\alpha = 0, 1, 2, 3$, $\langle (\emptyset, \emptyset), (\emptyset, 2\beta), (0, 2\beta+1), (1, 2\beta+4), (2, 2\beta+1), (3, 2\beta+4) \rangle$ mod (5, 7), 3 times, $\beta = 0, 1, 2$.
38	$X = Z(37, 2) \cup \{\infty\}$. $\mathcal{B}$ = Blocks of $B[6, 5; 37]$ on $Z(37)$, 2 times, $\langle \infty, \emptyset, 6\alpha, 6\alpha+2, 6\alpha+4, 6\alpha+6 \rangle$ mod 37, $\alpha = 0, 1, 2$, $\langle 2\alpha+1, 2\alpha+7, 2\alpha+13, 2\alpha+19, 2\alpha+25, 2\alpha+31 \rangle$ mod 37, $\alpha = 0, 1, 2$, $\langle 4, 10, 16, 22, 28, 34 \rangle$ mod 37.

Lemma 5.34. *For every integer $v \geqslant 6$, $v \in B(6, 15)$ holds.*

Proof. By Lemma 5.29, $v \in B(K_6, 1)$ and it suffices to show that $v \in B(6, 15)$ for every $v \in K_6$. For $v \equiv 0$ or 1 (mod 3) this follows from Lemma 5.33 and for $v \equiv 1$ (mod 5) this follows from Lemma 5.32. If v is a power of an odd prime, $v \in B(6, 15)$ follows from Lemma 4.4. For other values of v the proof is given in Table 5.19.

Lemma 5.35. *If $\lambda \geqslant 3$, then $21 \in B(6, \lambda)$ holds.*

Proof. For $\lambda = 3$, $21 \in B(6, 3)$ follows from Lemma 5.32; for $\lambda = 5$, $21 \in B(6, 5)$ follows from Lemma 5.33. For $\lambda = 4$ we prove $21 \in B(6,4)$.
$X = GF(4, x^2 = x+1) \times GF(4, x^2 = x+1) \cup \{(\infty_i): i = 0, 1, 2, 3, 4\}$.
$\mathcal{B}$ = Blocks of $B[6, 2; 16]$ on $GF(4) \times GF(4)$,

$\langle (\infty_\alpha), (\infty_{\alpha+\beta+1}), (\emptyset, \emptyset), (0, \alpha), (1, \alpha+1), (2, \alpha+2) \rangle$ mod $(-, 4)$,
$\alpha = 0, 1, 2$, $\beta = 0, 1$,
$\langle (\infty_3), (\infty_{4\beta}), (\emptyset, \emptyset), (0, \emptyset), (1, \emptyset), (2, \emptyset) \rangle$ mod $(-, 4)$, $\beta = 0, 1$,
$\langle (\infty_4), (\infty_\beta), (\emptyset, \emptyset), (\emptyset, 0), (\emptyset, 1), (\emptyset, 2) \rangle$ mod $(4, -)$, $\beta = 0, 1$.

For other values of $\lambda \geqslant 3$ apply Lemma 2.4.

Theorem 5.4. *Let $\lambda > 1$ and $v \geqslant 6$ be integers. A necessary and sufficient condition for the existence of a BIBD $B[6, \lambda; v]$ with $\lambda > 1$ is that the design is not $B[6, 2; 21]$ and that*

$$\lambda(v-1) \equiv 0 \pmod 5 \quad \textit{and} \quad \lambda v(v-1) \equiv 0 \pmod{30}.$$

Proof. The necessity follows from Theorem 1.1. To prove sufficiency note that λ determines the values of v for which the condition of the theorem is satisfied. By Lemma 2.3, it suffices to consider only those values of λ which are factors of 30 and in our case – because $k = 6$ is even – only factors of 15 and $\lambda = 2$ and we obtain

for $\lambda = 2$, $v \equiv 1$ or $6 \pmod{15}$,
for $\lambda = 3$, $v \equiv 1 \pmod 5$,
for $\lambda = 5$, $v \equiv 0$ or $1 \pmod 3$,
for $\lambda = 15$, every v.

In all these cases – with exception of $\lambda = 2$, $v = 21$ – the existence of the relevant BIBD's is proved in Lemmas 5.31, 5.32, 5.33 and 5.34, respectively. Considering Lemma 5.35 the theorem is proved completely.

It has been proved by Connor and Hall [9,15] that the exceptional BIBD $B[6, 2; 21]$ does not exist, see Lemma 1.3.

5.5. BIBD's with block-size 7

Only partial results are given here, namely necessary and sufficient conditions are obtained for BIBD's with $k = 7$ and $\lambda = 6, 7$ and 42. However, whenever available, constructions of BIBD's with smaller values of λ are performed.

Lemma 5.36. *For every positive integer u, $u \in \mathrm{GD}(\{7, 8, 9\}, 1, I(49))$ holds.*

Proof. Apply Lemma 3.13 with $t = 2$, $s = 7$, but for $49 \leqslant u < 56$ with $t = 1$, $s = 7$, $r = 7$. Check with the help of Theorem 3.1. For $u \geqslant 630$, $u \in \mathrm{GD}(\{7, 8\}, 1, I(90))$ follows by Lemma 3.9 and Theorem 3.7 and needs no checking.

Lemma 5.37. *For every integer* $v \geqslant 7$, $v \in B(K_7, 1)$ *holds, where* $K_7 = \{7, 8, ..., 48, 51, 52, 53, 54, 55, 59, 60, 61, 62\}$.

Proof. Apply Lemma 3.13 with $D = \{2, 3, 4, 5, 6\}$ and other parameters as follows: for $49 \leqslant v \leqslant 50$, $t = 1$, $s = 7$, $r = 7$; for $56 \leqslant v \leqslant 58$, $t = 2$, $s = 7$, $r = 8$; for $63 \leqslant v \leqslant 66$, $t = 3$, $s = 7$, $r = 9$; for $70 \leqslant v \leqslant 74$, $t = 3$, $s = 7$, $r = 9$; for $77 \leqslant v \leqslant 78$, $t = 1$, $s = 7$, $r = 11$; for $79 \leqslant v \leqslant 82$, $t = 2$, $s = 8$, $r = 9$; for $84 \leqslant v \leqslant 90$, $t = 3$, $s = 7$. $r = 9$; for $91 \leqslant v \leqslant 121$, $t = 4$, $s = 7$, $r = 11$; for $v \geqslant 122$ the checking may be done easily with parameters $t = 3$ and $s = 7$. We have shown that for the mentioned values of v, $v \in \mathrm{GD}(\{7, 8, 9, 10, 11\}, 1, K_7)$, and by Lemma 3.9, $v \in B(K_7, 1)$. It remains to prove the lemma for $v \in \{67, 68, 69, 75, 76, 83\}$.

If $v \in \{67, 68, 69\}$, consider a transversal design T[9, 1; 8] and delete 1 point from each of 5, 4 or 3 groups respectively, in such a way that no 3 of the deleted points be in a block. This is possible according to the proof of Lemma 3.18 and we obtain $v \in B(\{7, 8, 9\}, 1)$. For $v \in \{75, 76\}$ delete 2 points from each of 3 groups in T[9, 1; 9] and regarding $v = 83$ consider in T[9, 1; 11] any two intersecting blocks, and delete all their points with the exception of their intersection; clearly $83 \in B(\{7, 8, 9, 11\}, 1)$.

Lemma 5.38. *If* $v \equiv 1 \pmod 6$, *then* $v \in B(7, 7)$ *holds.*

Proof. Let $v = 6u + 1$, where u is a positive integer. By Lemma 5.36, $u \in \mathrm{GD}(\{7, 8, 9\}, 1, I(49))$. By Lemmas 2.26, 4.24 and 4.25, it suffices to show that $v = 6\mu + 1 \in B(7, 7)$ for every $\mu \in I(49)$. For $\mu = 1$ the lemma is trivial, if v is a power of an odd prime, then $v \in B(7, 7)$ by Lemma 4.2. For other values of v see Table 5.20.

Table 5.20.

μ	v	$B[7, 7; v]$
9	55	Lemmas 4.25 and 2.11.
14	85	Lemma 3.23 ($r = 12$) and $13 \in B(7, 7)$ by Lemma 4.2.
15	91	We prove $91 \in \mathrm{GD}(7, 1, 7)$. [24]. $X = Z(7, 3) \times Z(13, 2)$. $\mathscr{B} = \langle(\emptyset, \emptyset), (0, 3\alpha), (0, 3\alpha+6), (2, 3\alpha+4), (2, 3\alpha+10), (4, 3\alpha+2), (4, 3\alpha+8)\rangle$ mod (7, 13), $\alpha = 0, 1$.

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1.0 4.5 5.0 5.6 6.3 2.8 3.2 3.6 4.0 2.5 2.2 2.0

1.1 1.8

1.25 1.4 1.6

Table 5.20 (cont.).

μ	v	$B[7,7;v]$
19	115	We prove $114 \in GD(7,7,6)$. $X = (Z(5,2) \cup \{\infty\}) \times Z(19,2)$. $\mathcal{P}$ = Blocks of $B[7,7;19]$ on $\{\infty\} \times Z(19)$. $\langle(\infty;\emptyset),(\emptyset;3\alpha),(\emptyset;3\alpha+2),(\emptyset;3\alpha+4),(0;3\alpha+10),(1;3\alpha+5),(2;3\alpha+12)\rangle$ mod $(5;19)$, $\alpha = 0,1,\ldots,5$, $\langle(\infty;\emptyset),(\emptyset;\beta),(\emptyset;\beta+9),(\gamma;\beta+3),(\gamma;\beta+12),(\gamma+2;\beta+6),(\gamma+2;\beta+15)\rangle$ mod $(5;19)$, 2 times, $\beta = 0,1,2$, $\gamma = 0,1$, $\langle(\infty;\emptyset),(\emptyset;3\beta+1),(\emptyset;3\beta+10),(1;3\beta+5),(1;3\beta+14),(3;3\beta),(3;3\beta+9)\rangle$ mod $(5;19)$, $\beta = 0,1,2$.
22	133	Lemma 3.11, $19 \in T(7,1)$ and $19 \in B(7,7)$.
24	145	$X = I(3) \times I(6) \times I(8) \cup \{\infty\}$. $\mathcal{B}$ = Blocks of $B[7,7;19]$ on $I(3) \times I(6) \times \{i\} \cup \{\infty\}$, $i \in I(8)$, Blocks of $T[7,7;3]$ (exists by Theorem 3.11) on $I(3) \times B$ for every block B of $GD[7,1,6;48]$ (exists by Lemma 4.24) on $I(6) \times I(8)$.
29	175	Lemma 3.11, $25 \in T(7,1)$ and $25 \in B(7,7)$.
31	187	We prove $186 \in GD(7,7,6)$. $X = (Z(5,2) \cup \{\infty\}) \times Z(31,3)$. $\mathcal{P}$ = Blocks of $B[7,7;31]$ on $\{\infty\} \times Z(31)$. $\langle(\infty;\emptyset),(\emptyset;\alpha),(\emptyset;\alpha+15),(\beta;\alpha+5),(\beta;\alpha+20),(\beta+2;\alpha+10),(\beta+2;\alpha+25)\rangle$ mod $(5;31)$, 2 times, $\alpha = 0,1,2,3,4$, $\beta = 0,1$, $\langle(\infty;\emptyset),(\emptyset;3\alpha+10),(\emptyset;3\alpha+25),(1;3\alpha+5),(1;3\alpha+20),(3;3\alpha),(3;3\alpha+15)\rangle$ mod $(5;31)$, $\alpha = 0,1,2,3,4$, $\langle(\infty;\emptyset),(\emptyset;3\gamma),(\emptyset;3\gamma+10),(\emptyset;3\gamma+20),(0;3\gamma+25),(1;3\gamma+5),(2;3\gamma+15)\rangle$ mod $(5;31)$, $\gamma = 0,1,\ldots,9$.
34	205	$X = Z(6) \times Z(31,3) \cup \{(\infty_i): i = 0,1,\ldots,18\}$. $\mathcal{B}$ = Blocks of $B[7,7;19]$ on $\{(\infty_i): i = 0,1,\ldots,18\}$, $\langle(\infty_\alpha),(0',\alpha),(0',\alpha+15),(1',\alpha+5),(1',\alpha+20),(3',\alpha+10),(3',\alpha+25)\rangle$ mod $(6,31)$, $\alpha = 0,1,\ldots,14$, $\langle(\infty_\alpha),(0',\alpha),(1',\alpha+5),(2',\alpha+10),(3',\alpha+15),(4',\alpha+20),(5',\alpha+25)\rangle$ mod $(-,31)$, $\alpha = 0,1,\ldots,14$, $\langle(\infty_{\beta+15}),(0',7\beta+\gamma+15),(1',7\beta+\gamma+20),(2',7\beta+\gamma+25),(3',7\beta+\gamma)$, $(4',7\beta+\gamma+5),(5',7\beta+\gamma+10)\rangle$ mod $(-,31)$, $\beta = 0,1$, $\gamma = 0,1,\ldots,6$, $\langle(\infty_{17}),(0',29),(1',4),(2',9),(3',14),(4',19),(5',24)\rangle$ mod $(-,31)$, $\langle(\infty_{17}),(0',0),(0',3),(0',10),(0',13),(0',20),(0',23)\rangle$ mod $(6,31)$, $\langle(\infty_{18}),(0',\emptyset),(1',\emptyset),(2',\emptyset),(3',\emptyset),(4',\emptyset),(5',\emptyset)\rangle$ mod $(-,31)$, 7 times, $\langle(0',2\alpha),(0',2\alpha+10),(0',2\alpha+20),(1',2\alpha+21),(2',2\alpha+2),(4',2\alpha+11)$, $(5',2\alpha+1)\rangle$ mod $(6,31)$, $\alpha = 0,1,\ldots,14$.
36	217	Lemma 3.11, $31 \in T(7,1)$ and $31 \in B(7,7)$.
39	235	$X = Z(7,3) \times Z(31,3) \cup \{(\infty_i): i = 0,1,\ldots,17\}$. $\mathcal{B}$ = Blocks of $B[7,1;49]$ on $\{j\} \times Z(31) \cup \{(\infty_i): i = 0,1,\ldots,17\}, j \in Z(7)$ $\langle(\infty_\alpha),(\emptyset,0),(\emptyset,3),(\emptyset,10),(\emptyset,13),(\emptyset,20),(\emptyset,23)\rangle$ mod $(7,31)$, $\alpha = 0,1,\ldots,5$, Form a resolvable transversal design $RT[7,1;31]$ on $Z(31) \times Z(7)$, adjoin each of the points $(\infty_{\alpha+6})$, $\alpha = 0,1,\ldots,11$, to all blocks of a parallel class of blocks; so are obtained blocks of size 8; by Lemma 4.1, $8 \in B(7,6)$; further take once the blocks of these parallel classes without the adjoint point $(\infty_{\alpha+6})$. Blocks of other parallel classes take 7 times each.
41	247	By Lemma 4.1, $41 \in B(8,7)$; apply Lemmas 2.16, 4.24 and 2.11.
42	253	Lemma 3.23 ($r = 36$) and $37 \in B(7,7)$ by Lemma 4.2.
43	259	Lemma 3.22 ($r = 37$).

Table 5.20 (cont.).

μ	v	$B[7,7;v]$
44	265	$X = Z(6) \times Z(37,2) \cup \{(\infty_i): i = 0,1,\ldots,42\}$. $\mathcal{B}$ = Blocks of $B[7,7;43]$ on $\{(\infty_i): i = 0,1,\ldots,42\}$, $\langle(\infty_{\alpha+7}),(0',\alpha),(1',\alpha+6),(2',\alpha+12),(3',\alpha+18),(4',\alpha+24),(5',\alpha+30)\rangle$ mod $(-,37)$, 7 times, $\alpha = 0,1,\ldots,35$, $\langle(\infty_\beta),(0',\beta),(0',\beta+2),(0',\beta+12),(0',\beta+14),(0',\beta+24),(0',\beta+26)\rangle$ mod $(6,37)$, $\beta = 0,1,\ldots,6$, $\langle(\infty_\beta),(0',\emptyset),(1',\emptyset),(2',\emptyset),(3',\emptyset),(4',\emptyset),(5',\emptyset)\rangle$ mod $(-,37)$, $\beta = 0,1,\ldots,6$, $\langle(0',\emptyset),(0',4),(0',7),(0',16),(0',19),(0',28),(0',31)\rangle$ mod $(6,37)$.

Lemma 5.39. *If* $v \equiv 0$ *or* 1 (mod 7), *then* $v \in B(7,6)$ *holds.*

Proof. Let $v = 7u + \epsilon$, where u is a positive integer and $\epsilon = 0$ or 1. By Lemma 5.36, $u \in \mathrm{GD}(\{7,8,9\},1,I(49))$. By Lemmas 2.25, 2.26, 4.27 and 4.28, it suffices to show that $v = 7\mu + \epsilon \in B(7,6)$ for every $\mu \in I(49)$. For $v = 7$ the lemma is trivial, $\{8,64\} \subset B(7,6)$ by Lemma 4.1, $\{29,43,71,113,127,169,197,211,239,281,337\} \subset B(7,3)$ by Lemma 4.3, $\{63,77,119,133,161,175,189,203,217,259,287,301,329\} \subset B(7,3)$ by Lemmas 4.26 and 2.10, $\{50,78,92,120,134,162,176,190,204,218,260,288,302,330\} \subset B(7,6)$ by Lemmas 4.26 and 2.14, considering $8 \in B(7,6)$. For other values of v see Table 5.21.

Table 5.21.

v	$B[7,6;v]$
14	$X = Z(13,2) \cup \{\infty\}$. $\mathcal{B} = \langle\infty,0,2,4,6,8,10\rangle$ mod 13, $\langle\emptyset,0,2,4,6,8,10\rangle$ mod 13.
15	$15 \in B(7,3)$. [12]. $X = Z(3,2) \times Z(5,2)$. $\mathcal{B} = \langle(\emptyset,\emptyset),(0,\emptyset),(0,0),(0,2),(1,\emptyset),(1,1),(1,3)\rangle$ mod $(3,5)$.
21	$21 \in B(7,3)$. [12]. $X = Z(3,2) \times Z(7,3)$. $\mathcal{B} = \langle(\emptyset,0),(\emptyset,2),(\emptyset,4),(0,\alpha),(0,\alpha+3),(1,\alpha+1),(1,\alpha+4)\rangle$ mod $(-,7)$, $\alpha = 0,1,2$, $\langle(\emptyset,\emptyset),(0,1),(0,3),(0,5),(1,1),(1,3),(1,5)\rangle$ mod $(-,7)$, $\langle(\beta,\emptyset),(\beta,0),(\beta,1),(\beta,2),(\beta,3),(\beta,4),(\beta,5)\rangle$, $\beta = 0,1$.
22	$X = Z(3,2) \times Z(7,3) \cup \{(\infty)\}$. $\mathcal{B} = \langle(\infty),(\emptyset,0),(\emptyset,3),(0,1),(0,4),(1,2),(1,5)\rangle$ mod $(3,7)$, $\langle(\emptyset,\emptyset),(0,0),(0,2),(0,4),(1,0),(1,2),(1,4)\rangle$ mod $(3,7)$, 2 times, $\langle(\emptyset,\emptyset),(\emptyset,0),(\emptyset,1),(\emptyset,2),(\emptyset,3),(\emptyset,4),(\emptyset,5)\rangle$ mod $(3,-)$.

Table 5.21 (cont.).

v	$B[7,6;v]$
28	$28 \in B(7,2)$. [12]. Form $B[9,2;37]$ with $X = Z(37,2)$, and $\mathcal{B} = \langle 0,4,8,12,16,20,24,28,32\rangle \bmod 37$, and delete any one block and all its points.
35	$X = (Z(4) \cup \{\infty\}) \times Z(7,3)$. $\mathcal{B} = \langle(\infty,\emptyset),(\infty,0),(\infty,1),(\infty,2),(\infty,3),(\infty,4),(\infty,5)\rangle$, 2 times, $\langle(\infty,2\alpha),(\infty,2\alpha+3),(0',\emptyset),(0',2\alpha),(0',2\alpha+3),(1',\emptyset),(2',\emptyset)\rangle \bmod (4,7)$, $\alpha = 0,1$, $\langle(\infty,1),(\infty,4),(0',\emptyset),(0',1),(0',4),(1',\emptyset),(3',\emptyset)\rangle \bmod (4,7)$, $\langle(\infty,\emptyset),(0',2\alpha+2),(0',2\alpha+5),(1',2\alpha),(1',2\alpha+3),(2',2\alpha+1),(2',2\alpha+4)\rangle \bmod (4,7)$, $\alpha = 0,1$, $\langle(0',0),(0',3),(1',1),(1',4),(2',2),(2',5),(3',\emptyset)\rangle \bmod (4,7)$.
36	$X = Z(5,2) \times Z(7,3) \cup \{(\infty)\}$. $\mathcal{B} = \langle(\infty),(\emptyset,2),(\emptyset,5),(0,1),(0,4),(1,0),(1,3)\rangle \bmod (5,7)$, $\langle(\emptyset,2\alpha),(\emptyset,2\alpha+3),(0,2\alpha+2),(0,2\alpha+5),(1,\emptyset),(3,2\alpha+1),(3,2\alpha+4)\rangle \bmod (5,7)$, $\alpha = 0,1$, $\langle(\emptyset,0),(\emptyset,2),(\emptyset,4),(0,\emptyset),(1,\emptyset),(2,\emptyset),(3,\emptyset)\rangle \bmod (5,7)$, 2 times, $\langle(\emptyset,\emptyset),(\emptyset,0),(\emptyset,1),(\emptyset,2),(\emptyset,3),(\emptyset,4),(\emptyset,5)\rangle \bmod (5,-)$.
42	$X = (Z(5,2) \cup \{\infty\}) \times Z(7,3)$. $\mathcal{B} = \langle(\infty,\emptyset),(\infty,0),(\infty,1),(\infty,2),(\infty,3),(\infty,4),(\infty,5)\rangle$, $\langle(\infty,\alpha),(\infty,\alpha+3),(\emptyset,\emptyset),(0,\alpha+2),(0,\alpha+5),(2,\alpha),(2,\alpha+3)\rangle \bmod (5,7)$, $\alpha = 0,1,2$, $\langle(\infty,\emptyset),(\emptyset,2),(\emptyset,5),(\beta,1),(\beta,4),(\beta+2,0),(\beta+2,3)\rangle \bmod (5,7)$, $\beta = 0,1$, $\langle(\emptyset,0),(\emptyset,2),(\emptyset,4),(0,\emptyset),(1,\emptyset),(2,\emptyset),(3,\emptyset)\rangle \bmod (5,7)$, 2 times.
49	$49 \in B(7,1)$ by Theorem 2.2.
56	Lemma 3.11, $8 \in T(7,1)$ and $8 \in B(7,6)$.
57	$57 \in B(7,3)$. $X = Z(3,2) \times Z(19,2)$. $\mathcal{B} = \langle(\emptyset,\emptyset),(0,2\alpha),(0,2\alpha+6),(0,2\alpha+12),(1,2\alpha+2),(1,2\alpha+8),(1,2\alpha+14)\rangle \bmod (3,19)$, $\alpha = 0,1$, $\langle(\emptyset,\emptyset),(0,0),(0,6),(0,12),(1,0),(1,6),(1,12)\rangle \bmod (3,19)$, $\langle(0,\emptyset),(0,4),(0,10),(0,16),(1,1),(1,7),(1,13)\rangle \bmod (3,19)$.
70	$X = Z(2) \times Z(31,3) \cup \{(\infty_i): i = 0,1,\ldots,7\}$. $\mathcal{B}$ = Blocks of $B[7,6;8]$ on $\{(\infty_i): i = 0,1,\ldots,7\}$, $\langle(\infty_\alpha),(\emptyset,\emptyset),(0,\emptyset),(0,5\alpha),(0,5\alpha+4),(0,5\alpha+15),(0,5\alpha+19)\rangle \bmod (2,31)$, $\alpha = 0,1,2$, $\langle(\infty_{\alpha+3}),(\emptyset,5\alpha+13),(\emptyset,5\alpha+28),(0,5\alpha),(0,5\alpha+3),(0,5\alpha+15),(0,5\alpha+18)\rangle$ $\bmod (2,31)$, $\alpha = 0,1,2$, $\langle(\infty_{\beta+6}),(\emptyset,0),(\emptyset,10),(\emptyset,20),(0,9),(0,19),(0,29)\rangle \bmod (2,31)$, $\beta = 0,1$, $\langle(\emptyset,\emptyset),(\emptyset,5\alpha+2),(\emptyset,5\alpha+17),(0,5\alpha+4),(0,5\alpha+7),(0,5\alpha+19),(0,5\alpha+22)\rangle$ $\bmod (2,31)$, $\alpha = 0,1,2$.
84	We prove $84 \in GD(7,6,7)$. $X = Z(7,3) \times Z(11,2) \cup \{(\infty_i): i = 0,1,\ldots,6\}$. $\mathcal{P} = \langle(\infty_\alpha),(0;\alpha+2),(0;\alpha+7),(2;\alpha+1),(2;\alpha+6),(4;\alpha+3),(4;\alpha+8)\rangle \bmod (7;11)$, $\alpha = 0,1,2,3$, $\langle(\infty_{\beta+4}),(0;\emptyset),(1;5\beta+2),(2;5\beta),(3;5\beta+8),(4;5\beta+4),(5;5\beta+6)\rangle \bmod (7;11)$, $\beta = 0,1$, $\langle(\infty_6),(0;0),(0;5),(2;2),(2;7),(4;1),(4;6)\rangle \bmod (7;11)$, $\langle(\emptyset;\emptyset),(0;\alpha+1),(0;\alpha+6),(2;\alpha+3),(2;\alpha+8),(4;\alpha+2),(4;\alpha+7)\rangle \bmod (7;11)$, $\alpha = 0,1,2,3$, $\langle(\emptyset;\emptyset),(0;1),(0;6),(2;0),(2;5),(4;2),(4;7)\rangle \bmod (7;11)$.
85	Lemma 2.14, $8 \in B(7,6)$ and $84 \in GD(7,6,7)$ as above.
91	$91 \in B(7,1)$, see Table 5.20.

Table 5.21 (cont.).

v	$B[7,6;v]$
98	Lemma 2.16, $14 \in B(7,6)$ and Lemma 4.27.
99	$99 \in B(7,3)$. Lemma 3.23 ($r = 14$) and $15 \in B(7,3)$ as above.
105	$105 \in \mathrm{GD}(7,3,7)$ by Lemma 2.16, $15 \in B(7,3)$ and Lemma 4.27.
106	Lemma 2.14, $8 \in B(7,6)$ and $105 \in \mathrm{GD}(7,3,7)$ as above.
112	We prove $112 \in \mathrm{GD}(7,2,7)$. $X = Z(7,3) \times \mathrm{GF}(16, x^4 = x+1)$. $\mathcal{P} = \langle(\emptyset;\emptyset),(0;3\alpha),(0;3\alpha+3),(2;3\alpha+5),(2;3\alpha+8),(4;3\alpha+10),(4;3\alpha+13)\rangle \bmod (7;16)$, $\alpha = 0,1,2,3,4$.
126	We prove $126 \in \mathrm{GD}(7,6,7)$. $X = Z(7,3) \times Z(17,3) \cup \{(\infty_i): i = 0,1,\dots,6\}$. $\mathcal{P} = \langle(\infty_\alpha),(0;\alpha),(0;\alpha+8),(2;\alpha+3),(2;\alpha+11),(4;\alpha+4),(4;\alpha+12)\rangle \bmod (7;17)$, $\alpha = 0,1,2,3$, $\langle(\infty_{\beta+4}),(0;8\beta+1),(1;8\beta+3),(2;8\beta+2),(3;8\beta+6),(4;8\beta+4),(5;8\beta+5)\rangle$ $\bmod (7;17)$, $\beta = 0,1$, $\langle(\infty_6),(0;7),(0;15),(2;5),(2;13),(4;3),(4;11)\rangle \bmod (7;17)$, $\langle(\emptyset;\emptyset),(0;\gamma),(0;\gamma+8),(2;\gamma+6),(2;\gamma+14),(4;\gamma+4),(4;\gamma+12)\rangle \bmod (7;17)$, $\gamma = 0,1,\dots,6$, $\langle(\emptyset;\emptyset),(0;\alpha+4),(0;\alpha+12),(2;\alpha+7),(2;\alpha+15),(4;\alpha),(4;\alpha+8)\rangle \bmod (7;17)$, $\alpha = 0,1,2,3$.
140	We prove $140 \in \mathrm{GD}(7,6,7)$. $X = Z(7,3) \times Z(19,2) \cup \{(\infty_i): i = 0,1,\dots,6\}$. $\mathcal{P} = \langle(\infty_\alpha),(0;\alpha+6),(0;\alpha+15),(2;\alpha+1),(2;\alpha+10),(4;\alpha+3),(4;\alpha+12)\rangle \bmod (7;19)$, $\alpha = 0,1,2$, $\langle(\infty_{\beta+3}),(0;\beta),(0;\beta+9),(2;\beta+3),(2;\beta+12),(4;\beta+8),(4;\beta+17)\rangle \bmod (7;19)$, $\beta = 0,1$, $\langle(\infty_{\beta+5}),(0;9\beta+6),(1;9\beta+8),(2;9\beta+7),(3;9\beta+11),(4;9\beta+9),(5;9\beta+10)\rangle$ $\bmod (7;19)$, $\beta = 0,1$, $\langle(\emptyset;\emptyset),(0;\gamma+2),(0;\gamma+11),(2;\gamma+5),(2;\gamma+14),(4;\gamma+1),(4;\gamma+10)\rangle \bmod (7;19)$, $\gamma = 0,1,\dots,6$, $\langle(\emptyset;\emptyset),(0;\delta),(0;\delta+9),(2;\delta+4),(2;\delta+13),(4;\delta+6),(4;\delta+15)\rangle \bmod (7;19)$, $\delta = 0,1,\dots,5$.
141	$141 \in B(7,3)$. Lemma 3.23 and $21 \in B(7,3)$ as above.
147	$147 \in \mathrm{GD}(7,3,7)$ by Lemma 2.16, $21 \in B(7,3)$ and Lemma 4.27.
148	Lemma 2.14, $8 \in B(7,6)$ and $147 \in \mathrm{GD}(7,3,7)$ as above.
154	$154 \in \mathrm{GD}(7,6,7)$ by Lemma 2.16, $22 \in B(7,6)$ and Lemma 4.27.
155	Lemma 2.14, $8 \in B(7,6)$ and $154 \in \mathrm{GD}(7,6,7)$ as above.
168	$X = I(8) \times I(21)$. $\mathcal{B}$ = Blocks of $B[7,6;8]$ on $I(8) \times \{i\}$, $i \in I(21)$, Form $B[7,3;21]$ on $I(21)$ and for every block B of this design take 2 times the blocks of $\mathrm{T}[7,1;8]$ on $I(8) \times B$.
182	Lemma 2.25 (with $m = 2$), $91 \in \mathrm{GD}(7,1,7)$ as in Table 5.20, $14 \in B(7,6)$ as above and $14 \in \mathrm{GD}(7,3,2)$ by Lemma 4.29.
183	$183 \in B(7,3)$ by Lemma 2.26 (with $m = 2$), $91 \in \mathrm{GD}(7,1,7)$ as in Table 5.20, $15 \in B(7,3)$ as above and $14 \in \mathrm{GD}(7,3,2)$ by Lemma 4.29.
196	$196 \in B(7,2)$ by Lemma 2.16, $28 \in B(7,2)$ as above, and Lemma 4.27.
210 224 225 231 232	Lemma 3.9, $s = 7$, $t = 1$, $r = 29$, $r_1 = v - 203$; r_1 has the values 7, 21, 22, 28, 29; further $\{7,8,29,21,22,28\} \subset B(7,6)$.

Table 5.21 (cont.).

v	$B[7,6;v]$
238	$X = Z(7,3) \times Z(31,3) \cup \{(\infty_i): i = 0,1,\dots,20\}$. $\mathcal{B}$ = Blocks of $B[7,6;21]$ on $\{(\infty_i): i = 0,1,\dots,20\}$, $\langle(\infty_\alpha), (\emptyset,0), (\emptyset,3), (\emptyset,10), (\emptyset,13), (\emptyset,20), (\emptyset,23)\rangle$ mod (7.31), $\alpha = 0,1,\dots,5$, Form RT[7, 1; 31] on $Z(31) \times Z(7)$; for $\beta = 0,1,\dots,14$ adjoin the point $(\infty_{\beta+6})$ to all blocks of a distinct parallel class of blocks and form $B[7,6;8]$ on each so enlarged block; the blocks of other parallel classes of blocks take 6 times each.
245	$245 \in GD(7,6,7)$ by Lemma 2.16, $35 \in B(7,6)$ and Lemma 4.27.
246	Lemma 2.14, $8 \in B(7,6)$ and $245 \in GD(7,6,7)$ as above.
252	$252 \in GD(7,6,7)$ by Lemma 2.16, $36 \in B(7,6)$ and Lemma 4.27.
253	Lemma 2.14, $8 \in B(7,6)$ and $252 \in GD(7,6,7)$ as above.
266	We prove $266 \in GD(7,6,7)$. $X = Z(7,3) \times Z(37,2) \cup \{(\infty_i): i = 0,1,\dots,6\}$. $\mathcal{P} = \langle(\infty_\alpha), (0;2\alpha+12), (0;2\alpha+30), (2;2\alpha+15), (2;2\alpha+33), (4;2\alpha+16), (4;2\alpha+34)\rangle$ mod (7;37), $\alpha = 0,1,2$, $\langle(\infty_{\beta+3}), (0;2\beta), (0;2\beta+18), (2;2\beta+3), (2;2\beta+21), (4;2\beta+6), (4;2\beta+24)\rangle$ mod (7;37), $\beta = 0,1$, $\langle(\infty_{\beta+5}), (0;18\beta+4), (1;18\beta+8), (2;18\beta 6), (3;18\beta+14), (4;18\beta+10), (5;18\beta+12)\rangle$ mod (7,37), $\beta = 0,1$, $\langle(\emptyset;\emptyset), (0;2\gamma+1), (0;2\gamma+19), (2;2\gamma+4), (2;2\gamma+22), (4;2\gamma+2\beta+5), (4;2\gamma+2\beta+23)\rangle$ mod (7,37), $\beta = 0,1$, $\gamma = 0,1,\dots,8$. $\langle(\emptyset;\emptyset), (0;2\delta+4), (0;2\delta+22), (2;2\delta+7), (2;2\delta+25), (4;2\delta+10), (4;2\delta+28)\rangle$ mod (7;37), $\delta = 0,1,\dots,6$, $\langle(\emptyset;\emptyset), (0;2\eta), (0;2\eta+18), (2;2\eta+3), (2;2\eta+21), (4;2\eta+4), (4;2\eta+22)\rangle$ mod (7;37), $\eta = 0,1,\dots,5$.
267	Lemma 2.14, $8 \in B(7,6)$ and $266 \in GD(7,6,7)$ as above.
273	$X = Z(7,3) \times Z(37,2) \cup \{(\infty_i): i = 0,1,\dots,13\}$. $\mathcal{B}$ = Blocks of $B[7,6;14]$ on $\{(\infty_i): i = 0,1,\dots,13\}$, $\langle(\emptyset,\emptyset), (0,\emptyset), (1,\emptyset), (2,\emptyset), (3,\emptyset), (4,\emptyset), (5,\emptyset)\rangle$ mod $(-,37)$, 6 times, The family $\mathcal{A}$ of blocks as below. $\mathcal{A} = \langle(\infty_\alpha), (0,2\alpha), (0,2\alpha+18), (2,2\alpha+3), (2,2\alpha+21), (4,2\alpha+4), (4,2\alpha+22)\rangle$ mod (7,37), $\alpha = 0,1,\dots,5$, $\langle(\infty_{2\beta+\gamma+6}), (0,6\beta+2\gamma+6), (0,6\beta+2\gamma+24), (2,6\beta+2\gamma+9), (2,6\beta+2\gamma+27), (4,6\beta+2\gamma+12), (4,6\beta+2\gamma+30)\rangle$ mod (7,37), $\beta = 0,1$, $\gamma = 0,1$, $\langle(\infty_{2\beta+\gamma+10}), (0,18\beta+6\gamma+10), (1,18\beta+6\gamma+14), (2,18\beta+6\gamma+12), (3,18\beta+6\gamma+20), (4,18\beta+6\gamma+16), (5,18\beta+6\gamma+18)\rangle$ mod (7,37), $\beta = 0,1$, $\gamma = 0,1$, $\langle(\emptyset,\emptyset), (0,2\delta+1), (0,2\delta+19), (2,2\delta+4), (2,2\delta+22), (4,2\delta+2\beta+5), (4,2\delta+2\beta+23)\rangle$ mod (7,37), $\beta = 0,1$, $\delta = 0,1,\dots,8$, $\langle(\emptyset,\emptyset), (0,2\eta+12), (0,2\eta+30), (2,2\eta+15), (2,2\eta+33), (4,2\eta+16), (4,2\eta+34)\rangle$ mod (7,37), $\eta = 0,1,2$, $\langle(\emptyset,\emptyset), (0,2\mu), (0,2\mu+18), (2,2\mu+3), (2,2\mu+21), (4,2\mu+6), (4,2\mu+24)\rangle$ mod (7,37), $\mu = 0,1,2,5,8$.
274	$X = Z(7,3) \times Z(37,2) \cup \{(\infty_i): i = 0,1,\dots,14\}$. $\mathcal{B}$ = Blocks of $B[7,6;15]$ on $\{(\infty_i): i = 0,1,\dots,14\}$, Blocks of $B[7,6;8]$ on $Z(7) \times \{j\} \cup \{(\infty_{14})\}$, $j \in Z(37)$, The family $\mathcal{A}$ of blocks as above.
280	$X = I(8) \times I(35)$. $\mathcal{B}$ = Blocks of $B[7,6;8]$ on $I(8) \times \{i\}$, $i \in I(35)$, Form $B[7,6;35]$ on $I(35)$ and for every block B of this design take the blocks of T[7, 1; 8] on $I(8) \times B$.

Table 5.21 (cont.).

v	$B[7,6;v]$
294	$294 \in GD(7,6,7)$ by Lemma 2.16, $42 \in B(7,6)$ and Lemma 4.27.
295	Lemma 2.14, $8 \in B(7,6)$ and $294 \in GD(7,6,7)$ as above.
308 309 315 316 322 323 336	Lemma 3.9, $s = 7$, $t = 1$, $r = 43$, $r_1 = v - 301$; r_1 has the values $7, 8, 14, 15, 21, 22, 35$; further $\{7, 8, 43, 14, 15, 21, 22, 35\} \subset B(7,6)$.

Lemma 5.40. *For every integer* $v \geq 7$, $v \in B(7,42)$ *holds.*

Proof. By Lemma 5.37, it is sufficient to show that $v \in B(7,42)$ for every $v \in K_7$. If $v \equiv 1 \pmod 6$, $v \in B(7,42)$ follows from Lemma 5.38; if $v \equiv 0$ or $1 \pmod 7$, $v \in B(7,42)$ follows from Lemma 5.39, and if v is a prime-power this follows from Lemma 4.1. For other values of $v \in K_7$, $v \in B(7,42)$ is proved in Table 5.22.

Table 5.22.

v	$B[7,42;v]$
10	$10 \in B(7,14)$. Lemma 4.5 and $10 \in B(3,2)$ by Lemma 5.6.
12	Lemma 4.6, $11 \in B(6,3)$ by Lemma 5.32 and $11 \in B(7,21)$ by Lemma 4.3.
18	$X = Z(17,3) \cup \{\infty\}$. $\mathcal{B} = \langle \infty, \alpha, \alpha+4, \alpha+5, \alpha+8, \alpha+12, \alpha+13 \rangle \bmod 17$, $\alpha = 0,1,2,3$, $\langle \infty, 2, 3, 6, 10, 11, 14 \rangle \bmod 17$, 2 times, $\langle \infty, 0, 2, 3, 8, 10, 11 \rangle \bmod 17$, $\langle \emptyset, \beta+1, \beta+3, \beta+4, \beta+9, \beta+11, \beta+12 \rangle \bmod 17$, $\beta = 0, 1, ..., 6$, $\langle \emptyset, \alpha, \alpha+1, \alpha+4, \alpha+8, \alpha+9, \alpha+12 \rangle \bmod 17$, $\alpha = 0,1,2,3$.
20	$X = Z(19,2) \cup \{\infty\}$. $\mathcal{B} = \langle \infty, \alpha, \alpha+3, \alpha+6, \alpha+9, \alpha+12, \alpha+15 \rangle \bmod 19$, 2 times, $\alpha = 0,1,2$, $\langle \infty, 2, 3, 8, 9, 14, 15 \rangle \bmod 19$, $\langle \emptyset, \alpha, \alpha+3, \alpha+6, \alpha+9, \alpha+12, \alpha+15 \rangle \bmod 19$, 4 times, $\alpha = 0,1,2$, $\langle \emptyset, 0, 1, 6, 7, 12, 13 \rangle \bmod 19$.
24	$X = Z(23,5) \cup \{\infty\}$. $\mathcal{B} = \langle \infty, \mu, \mu+1, \mu+3, \mu+11, \mu+12, \mu+14 \rangle \bmod 23$, $\mu = 2,5,7,8,10,18$, $\langle \infty, 1, 3, 6, 12, 14, 17 \rangle \bmod 23$, $\langle \emptyset, \alpha+2, \alpha+4, \alpha+7, \alpha+13, \alpha+15, \alpha+18 \rangle \bmod 23$, $\alpha = 0, 1, ..., 9$, $\langle \emptyset, 3\beta, 3\beta+1, 3\beta+3, 3\beta+11, 3\beta+12, 3\beta+14 \rangle \bmod 23$, $\beta = 0,1,2,3$, $\langle \emptyset, 3\gamma+1, 3\gamma+2, 3\gamma+4, 3\gamma+12, 3\gamma+13, 3\gamma+15 \rangle \bmod 23$, $\gamma = 0,1,2$.

Table 5.22 (cont.).

v	$B[7,42;v]$
26	$X = GF(25, x^2 = 2x+2) \cup \{\infty\}$. $\mathcal{B} = \langle \infty, \mu, \mu+4, \mu+8, \mu+12, \mu+16, \mu+20 \rangle \bmod 25$, once for $\mu = 0$, 4 times for $\mu = 2$, 2 times for $\mu = 3$, $\langle \emptyset, \alpha, \alpha+4, \alpha+8, \alpha+12, \alpha+16, \alpha+20 \rangle \bmod 25$, 4 times, $\alpha = 0,1,2,3$, $\langle \emptyset, \nu, \nu+4, \nu+8, \nu+12, \nu+16, \nu+20 \rangle \bmod 25$, once for $\nu = 0$, 2 times for $\nu = 1$.
30	$X = Z(29,2) \cup \{\infty\}$. $\mathcal{B} = \langle \infty, \mu, \mu+4, \mu+7, \mu+14, \mu+18, \mu+21 \rangle \bmod 29$, $\mu = 2,8,10,12$, $\langle \infty, 2, 6, 10, 14, 18, 22 \rangle \bmod 29$, 2 times, $\langle \infty, 0, 8, 13, 14, 22, 27 \rangle \bmod 29$, $\langle \emptyset, \alpha, \alpha+1, \alpha+9, \alpha+14, \alpha+15, \alpha+23 \rangle \bmod 29$, $\alpha = 0,1,\ldots,12$, $\langle \emptyset, \nu, \nu+4, \nu+7, \nu+14, \nu+18, \nu+21 \rangle \bmod 29$, $\nu = 0,1,3,4,5,6,7,9,11,13$.
33	$33 \in B(7,21)$. $X = Z(3,2) \times Z(11,2)$. $\mathcal{B} = \langle (\emptyset,0), (\emptyset,2), (\emptyset,4), (\emptyset,6), (\emptyset,8), (0,\emptyset), (1,\emptyset) \rangle \bmod (3,11)$, 5 times, $\langle (\emptyset,2\alpha), (\emptyset,2\alpha+2), (\emptyset,2\alpha+4), (0,2\alpha+1), (0,2\alpha+3), (1,2\alpha+1), (1,2\alpha+3) \rangle \bmod (3,11)$, $\alpha = 0,1,2,3,4$, $\langle (\emptyset,3\beta), (\emptyset,2\beta+2), (\emptyset,2\beta+4), (0,\emptyset), (1,\emptyset), (\gamma,1), (\gamma+1,9) \rangle \bmod (3,11)$, $\beta = 0,1$, $\gamma = 0,1$, $\langle (\emptyset,0), (\emptyset,7), (\emptyset,8), (0,\emptyset), (1,\emptyset), (\gamma,3), (\gamma+1,5) \rangle \bmod (3,11)$, $\gamma = 0,1$.
34	$34 \in B(7,14)$. $X = Z(3,2) \times Z(11,2) \cup \{(\infty)\}$. $\mathcal{B} = \langle (\infty), (\emptyset,\emptyset), (\emptyset,\mu), (0,\emptyset), (0,\mu), (1,\emptyset), (1,\mu) \rangle \bmod (-,11)$, once for $\mu = 0$, 2 times for $\mu = 2,3,4$, $\langle (\emptyset,\emptyset), (0,\alpha), (0,\alpha+2), (0,\alpha+5), (0,\alpha+7), (1,\alpha+1), (1,\alpha+6) \rangle \bmod (3,11)$, $\alpha = 0,1,2,3,4$, $\langle (\emptyset,\emptyset), (0,\nu), (0,\nu+1), (0,\nu+5), (0,\nu+6), (1,\nu+3), (1,\nu+8) \rangle \bmod (3,11)$, 2 times for $\nu = 0$, once for $\nu = 1$. $\langle (\emptyset,\emptyset), (0,0), (0,3), (0,5), (0,8), (1,4), (1,9) \rangle \bmod (3,11)$.
38	$X = Z(37,2) \cup \{\infty\}$. $\mathcal{B} = \langle \infty, \mu, \mu+6, \mu+12, \mu+18, \mu+24, \mu+30 \rangle \bmod 37$, 2 times for $\mu = 2$, 4 times for $\mu = 4$, once for $\mu = 5$, $\langle \emptyset, \alpha, \alpha+6, \alpha+12, \alpha+18, \alpha+24, \alpha+30 \rangle \bmod 37$, 4 times, $\alpha = 0,1,\ldots,5$, $\langle \emptyset, \nu, \nu+6, \nu+12, \nu+18, \nu+24, \nu+30 \rangle \bmod 37$, 2 times for $\nu = 0,1,3$, once for $\nu = 5$.
39	$39 \in B(7,21)$. $X = Z(3,2) \times Z(13,2)$. $\mathcal{B} = \langle (\emptyset,\emptyset), (0,\alpha), (0,\alpha+4), (0,\alpha+8), (1,\alpha+1), (1,\alpha+5), (1,\alpha+9) \rangle \bmod (3,13)$, $\alpha = 0,1,\ldots,11$, $\langle (\emptyset,\emptyset), (0,3\beta), (0,3\beta+4), (0,3\beta+8), (1,3\beta), (1,3\beta+4), (1,3\beta+8) \rangle \bmod (3,13)$, $\beta = 0,1,2,3$, $\langle (\emptyset,\emptyset), (\emptyset,\gamma), (\emptyset,\gamma+3), (\emptyset,\gamma+6), (\emptyset,\gamma+9), (0,\emptyset), (1,\emptyset) \rangle \bmod (3,13)$, $\gamma = 0,1,2$.
40	$40 \in B(7,14)$. $X = Z(3,2) \times Z(13,2) \cup \{(\infty)\}$. $\mathcal{B} = \langle (\infty), (\emptyset,\emptyset), (\emptyset,\mu), (0,\emptyset), (0,\mu), (1,\emptyset), (1,\mu) \rangle \bmod (-,13)$, once for $\mu = 2,3,4$, 2 times for $\mu = 5,6$, $\langle (\emptyset,\emptyset), (0,\alpha), (0,\alpha+3), (0,\alpha+6), (0,\alpha+9), (1,\alpha+1), (1,\alpha+7) \rangle \bmod (3,13)$, $\alpha = 0,1,\ldots,5$, $\langle (\emptyset,\emptyset), (0,\beta+1), (0,\beta+2), (0,\beta+7), (0,\beta+8), (1,2\beta+3), (1,2\beta+9) \rangle \bmod (3,13)$, $\beta = 0,1,2$, $\langle (\emptyset,\emptyset), (0,\gamma), (0,\gamma+1), (0,\gamma+6), (0,\gamma+7), (1,\gamma+2), (1,\gamma+8) \rangle \bmod (3,13)$, $\gamma = 0,1$.

Table 5.22 (cont.).

v	$B[7,42;v]$
43*	$43 \in B(7,2)$. $X = (Z(5,2) \cup \{\infty\}) \times Z(7,3) \cup \{(\infty,\infty)\}$. $\mathcal{B} = \langle(\infty,\infty),(\infty,\emptyset),(\emptyset,\emptyset),(0,\emptyset),(2,\emptyset),(3,\emptyset)\rangle$ mod $(-,7)$, 2 times, $\langle(\infty,\emptyset),(\infty,0),(\infty,1),(\infty,2),(\infty,3),(\infty,4),(\infty,5)\rangle$, 2 times, $\langle(\infty,\emptyset),(\emptyset,0),(\emptyset,3),(\alpha,2),(\alpha,5),(\alpha+2,1),(\alpha+2,4)\rangle$ mod $(5,7)$, $\alpha = 0,1$.
44	Lemma 4.6, $43 \in B(6,5)$ by Lemma 5.33, $43 \in B(7,2)$ as above and $43 \in B(7,3)$ by Lemma 4.3.
45	$45 \in B(7,21)$. $X = Z(5,2) \times \mathrm{GF}(9, x^2 = 2x+1)$. $\mathcal{B} = \langle(\emptyset,\emptyset),(\beta,\emptyset),(\beta,\alpha),(\beta,\alpha+4),(\beta+2,\emptyset),(\beta+2,\alpha+2),(\beta+2,\alpha+6)\rangle$ mod $(5,9)$, $\alpha = 0,1,2,3$, $\beta = 0,1$, $\langle(\emptyset,\emptyset),(\beta,\beta+2\gamma),(\beta,\beta+2\gamma+4),(\beta+1,\beta+2\gamma+2),(\beta+1,\beta+2\gamma+6),(\beta+2,\beta+2\gamma+1),$ $(\beta+2,\beta+2\gamma+5)\rangle$ mod $(5,9)$, 2 times, $\beta = 0,1$, $\gamma = 0,1$, $\langle(\emptyset,\emptyset),(\beta,\beta),(\beta,\beta+4),(\beta+1,\emptyset),(\beta+2,\beta+2),(\beta+2,\beta+6),(\beta+3,\emptyset)\rangle$ mod $(5,9)$, 3 times, $\beta = 0,1$.
46	$X = Z(3,2) \times Z(13,2) \cup \{(\infty_i): i = 0,1,\ldots,6\}$. $\mathcal{B} = \langle(\infty_0),(\infty_1),(\infty_2),(\infty_3),(\infty_4),(\infty_5),(\infty_6)\rangle$, 42 times, $\langle(\infty_\alpha),(\emptyset,\emptyset),(0,6\alpha+\beta),(0,6\alpha+\beta+4),(0,6\alpha+\beta+8),(1,6\alpha+\beta),(1,6\alpha+\beta+6)\rangle$ mod $(3,13)$, $\alpha = 0,1,\ldots,5$, $\beta = 0,1,\ldots,5$, $\langle(\infty_\alpha),(0,\alpha+1),(0,\alpha+5),(0,\alpha+9),(1,\alpha),(1,\alpha+4),(1,\alpha+8)\rangle$ mod $(3,13)$, $\alpha = 0,1,\ldots,5$, $\langle(\infty_6),(0,\beta),(0,\beta+4),(0,\beta+8),(1,\beta+1),(1,\beta+5),(1,\beta+9)\rangle$ mod $(3,13)$, $\beta = 0,1,\ldots,5$, $\langle(\infty_6),(0,0),(0,4),(0,8),(1,0),(1,4),(1,8)\rangle$ mod $(3,13)$, $\langle(\emptyset,2\gamma),(\emptyset,2\gamma+1),(\emptyset,2\gamma+2),(\emptyset,2\gamma+4),(\emptyset,2\gamma+5),(0,\emptyset),(1,\emptyset)\rangle$ mod $(3,13)$, $\gamma = 0,1,2$.
48	$X = Z(47,5) \cup \{\infty\}$. $\mathcal{B} = \langle\infty,\mu,\mu+2,\mu+4,\mu+23,\mu+25,\mu+27\rangle$ mod 47, $\mu = 0,2,6$, $\langle\infty,\nu,\nu+2,\nu+3,\nu+23,\nu+25,\nu+26\rangle$ mod 47, $\nu = 8,11$, $\langle\infty,5,18,19,28,41,42\rangle$ mod 47, 2 times, $\langle\emptyset,\mu,\mu+2,\mu+4,\mu+23,\mu+25,\mu+27\rangle$ mod 47, $\mu = 1,3,4,5,7,8,\ldots,22$, $\langle\emptyset,\nu,\nu+2,\nu+3,\nu+23,\nu+25,\nu+26\rangle$ mod 47, $\nu = 0,1,\ldots,7,9,10,12,13,\ldots,22$.
51	$X = Z(43,3) \cup \{\infty_i: i = 0,1,\ldots,7\}$. $\mathcal{B}$ = Blocks of $B[7,42;8]$ (exists by Lemma 4.1) on $\{\infty_i: i = 0,1,\ldots,7\}$, Blocks of $B[7,2;43]$ (as above) on $Z(43)$, $\langle\infty_\alpha,\beta,\beta+7,\beta+14,\beta+21,\beta+28,\beta+35\rangle$ mod 43, $\alpha = 0,1,\ldots,7$, $\beta = 0,1,\ldots,6$.
52	$52 \in B(7,14)$. $X = \mathrm{GF}(4, x^2 = x+1) \times Z(13,2)$. $\mathcal{B} = \langle(\emptyset,\emptyset),(\emptyset,\alpha),(\emptyset,\alpha+4),(\emptyset,\alpha+8),(0,\alpha+1),(1,\alpha+5),(2,\alpha+9)\rangle$ mod $(4,13)$, $\alpha = 0,1,\ldots,5$, $\langle(\emptyset,\emptyset),(0,\alpha),(0,\alpha+6),(1,\alpha+2),(1,\alpha+8),(2,\alpha+4),(2,\alpha+10)\rangle$ mod $(4,13)$, $\alpha = 0,1,\ldots,5$, $\langle(\emptyset,\beta),(\emptyset,\beta+3),(\emptyset,\beta+6),(\emptyset,\beta+6),(\emptyset,\beta+9),(0,\emptyset),(1,\emptyset),(2,\emptyset)\rangle$ mod $(4,13)$, $\beta = 0,1,2$, $\langle(\emptyset,\emptyset),(\emptyset,\gamma),(\emptyset,\gamma+4),(\emptyset,\gamma+8),(0,\emptyset),(1,\emptyset),(2,\emptyset)\rangle$ mod $(4,13)$, $\gamma = 0,1$.
54	$X = Z(53,2) \cup \{\infty\}$. $\mathcal{B} = \langle\infty,\mu,\mu+5,\mu+7,\mu+26,\mu+31,\mu+33\rangle$ mod 53, $\mu = 10,15,17,25$, $\langle\infty,3,5,9,29,31,35\rangle$ mod 53, 2 times, $\langle\infty,20,24,25,46,50,51\rangle$ mod 53,

* For further reference.

Table 5.22 (cont.)

v	$B[7,42;v]$
	$\langle \emptyset, \mu, \mu+5, \mu+7, \mu+26, \mu+31, \mu+33\rangle$ mod 53, $\mu = 0, 1, \ldots, 9, 11, 12, 13, 14, 16, 18, 19, \ldots, 24$,
	$\langle \emptyset, \nu, \nu+4, \nu+5, \nu+26, \nu+30, \nu+31\rangle$ mod 53, $\nu = 0, 1, \ldots, 19, 21, 22, 23, 24, 25$.
60	$X = Z(3,2) \times Z(17,3) \cup \{(\infty_i): i = 0, 1, \ldots, 8\}$.
	$\mathcal{B}$ = Blocks of $B[7,42;9]$ on $\{(\infty_i): i = 0, 1, \ldots, 8\}$,
	$\langle (\infty_\alpha), (\emptyset,\emptyset), (\emptyset,\alpha), (\emptyset,\alpha+8), (0,\alpha+4), (0,\alpha+12), (\beta,\emptyset)\rangle$ mod (3, 17), $\alpha = 0, 1, \ldots, 7$, $\beta = 0, 1$,
	$\langle (\infty_\alpha), (\emptyset,\emptyset), (0,\alpha+8\beta), (0,\alpha+8\beta+1), (0,\alpha+8\beta+2), (1,\alpha+8\beta+3), (1,\alpha+8\beta+4)\rangle$ mod (3, 17), $\alpha = 0, 1, \ldots, 7$, $\beta = 0, 1$,
	$\langle (\infty_\alpha), (0,\alpha+8\beta), (0,\alpha+8\beta+1), (0,\alpha+8\beta+2), (1,\alpha+8\beta+8), (1,\alpha+8\beta+9), (1,\alpha+8\beta+10)\rangle$ mod (3, 17), $\alpha = 0, 1, \ldots, 7$, $\beta = 0, 1$,
	$\langle (\infty_\alpha), (\emptyset,\alpha), (\emptyset,\alpha+8), (0,\alpha+1), (0,\alpha+9), (1,\alpha+2), (1,\alpha+10)\rangle$ mod (3, 17), $\alpha = 0, 1, \ldots, 7$,
	$\langle (\infty_8), (0,\emptyset), (0,\beta+2), (0,\beta+10), (1,\emptyset), (1,\beta+6), (1,\beta+14)\rangle$ mod (3, 17), 2 times, $\beta = 0, 1$,
	$\langle (\infty_8), (0,\emptyset), (0,4\beta), (0,4\beta+8), (1,\emptyset), (1,4\beta), (1,4\beta+8)\rangle$ mod (3, 17), $\beta = 0, 1$,
	$\langle (\infty_8), (0,\emptyset), (0,1), (0,9), (1,\emptyset), (1,5), (1,13)\rangle$ mod (3, 17),
	$\langle (\emptyset,\emptyset), (0,\emptyset), (0,4\beta+\gamma), (0,4\beta+\gamma+8), (1,\emptyset), (1,4\beta-\gamma+7), (1,4\beta-\gamma+15)\rangle$ mod (3, 17), $\beta = 0, 1$, $\gamma = 0, 1$,
	$\langle (\emptyset,\emptyset), (0,\emptyset), (0,1), (0,9), (1,\emptyset), (1,5), (1,13)\rangle$ mod (3, 17).
62	$X = Z(61,2) \cup \{\infty\}$.
	$\mathcal{B} = \langle \infty, \mu, \mu+10, \mu+20, \mu+30, \mu+40, \mu+50\rangle$ mod 61, 2 times for $\mu = 0$, 4 times for $\mu = 2$, once for $\mu = 3$,
	$\langle \emptyset, \mu, \mu+10, \mu+20, \mu+30, \mu+40, \mu+50\rangle$ mod 61, once for $\mu = 3$, 2 times for $\mu = 1, 4, 5, 6, 7, 8, 9$,
	$\langle \emptyset, \alpha, \alpha+10, \alpha+20, \alpha+30, \alpha+40, \alpha+50\rangle$ mod 61, 4 times, $\alpha = 0, 1, \ldots, 9$.

Lemma 5.41. *If* $v \equiv 1$ *or* 7 (mod 42), *then* $v \in B(7,\lambda)$ *for every* $\lambda \geqslant 30$.

Proof. If $v \equiv 1$ or 7 (mod 42), then by Lemmas 5.39 and 5.38, $v \in B(7,6) \cap B(7,7)$. If $\lambda \geqslant 30$, then clearly $\lambda = 6n + 7n'$, where n and n' are nonnegative integers. By Lemma 2.4, $v \in B(7,\lambda)$.

Theorem 5.5. *Let* v *be an integer* $v \geqslant 7$. *A necessary and sufficient condition for the existence of a BIBD* $B[7,\lambda;v]$ *with* $\lambda \equiv 0, 6, 7, 12, 18, 24, 30, 35, 36$ (mod 42) *or with* $\lambda > 30$ *which is not divisible by* 2 *or by* 3 *is that*

$$\lambda(v-1) \equiv 0 \pmod 6 \quad \textit{and} \quad \lambda v(v-1) \equiv 0 \pmod{42}.$$

Proof. The necessity follows from Theorem 1.1. The sufficiency follows from Lemmas 5.39, 5.38, 5.40 and 5.41, with application of Lemma 2.3.

5.6. BIBD's of small order

In August 1973, Collens prepared [8] a computerised list of BIBD's arranged by their order. It appears that already for small values of v there are parameters k and λ for which the existence of the respective BIBD's has been unknown.

In Table 5.23 a complete list is given of parameters $16 \leq v \leq 43$, $8 \leq k \leq \frac{1}{2}v$ and the smallest λ for which the condition of Theorem 1.1 is satisfied. Whenever for such value of λ no BIBD exists, or if it is unknown whether such design exists, the multiples of these values of λ are listed. Also BIBD's are listed with $k = 7$ and $\lambda \notin \{6, 7, 42\}$. BIBD's with $k < 7$ and with $k = 7$ and $\lambda \in \{6, 7, 42\}$ have been dealt with in Sections 5.1–5.5. The discussion of BIBD's with $k > \frac{1}{2}v$ may be omitted by Lemma 4.5.

Some of the designs constructed in Table 5.23 are well-known and appear in existing tables of BIBD's. Others are new. There is a number of parameter sets v, k, λ for which the existence of BIBD's is still unknown.

Table 5.23.

v	k	λ	$B[k, \lambda; v]$
15	7	3	Table 5.21.
16	7	14	Lemma 4.6, $15 \in B(6,5)$ and $15 \in B(7,3)$.
16	8	7	Lemma 4.6, $15 \in B(7,3)$ and $15 \in B(8,4)$ by Lemma 4.5.
17	7	21	Lemma 4.2.
17	8	7	Lemma 4.1.
18	8	28	$X = Z(17,3) \cup \{\infty\}$. $\mathcal{B} = \langle \infty, 2\alpha, 2\alpha+2, 2\alpha+4, 2\alpha+6, 2\alpha+8, 2\alpha+10, 2\alpha+12 \rangle \bmod 17$, $\alpha = 0, 1, 2, 3$, $\langle 1, 3, 5, 7, 9, 11, 13, 15 \rangle \bmod 17$, 4 times, $\langle 0, 2, 4, 6, 8, 10, 12, 14 \rangle \bmod 17$.
18	9	8	[22]. $X = Z(17,3) \cup \{\infty\}$. $\mathcal{B} = \langle \infty, 0, 2, 4, 6, 8, 10, 12, 14 \rangle \bmod 17$, $\langle \emptyset, 0, 2, 4, 6, 8, 10, 12, 14 \rangle \bmod 17$.
19	8	28	Lemma 4.1.
19	9	4	Lemma 4.3.
20	8	14	$X = Z(19,2) \cup \{\infty\}$. $\mathcal{B} = \langle \infty, \emptyset, 2\alpha, 2\alpha+3, 2\alpha+6, 2\alpha+9, 2\alpha+12, 2\alpha+15 \rangle \bmod 19$, $\alpha = 0, 1$, $\langle 3\beta, 3\beta+1, 3\beta+4, 3\beta+5, 3\beta+9, 3\beta+10, 3\beta+13, 3\beta+14 \rangle \bmod 19$, $\beta = 0, 1, 2$.
20	9	72	Lemma 4.6, $19 \in B(8,28)$ and $19 \in B(9,4)$.
20	10	9	Lemma 4.6, $19 \in B(9,4)$ and $19 \in B(10,5)$ by Lemma 4.5.

Table 5.23 (cont.).

v	k	λ	$B[k, \lambda; v]$
21	7	3	Table 5.21.
21	8	14	$X = Z(3,2) \times Z(7,3)$. $\mathfrak{B} = \langle(\emptyset,\alpha), (\emptyset,\alpha+1), (\emptyset,\alpha+3), (\emptyset,\alpha+4), (0,\alpha), (0,\alpha+3), (1,\alpha+2), (1,\alpha+5)\rangle$ mod $(3,7)$, $\alpha = 0,1,2$, $\langle(\emptyset,\emptyset), (0,\beta), (0,\beta+2), (0,\beta+4), (1,\emptyset), (1,\beta), (1,\beta+2), (1,\beta+4)\rangle$ mod $(3,7)$, $\beta = 0,1$.
21	9	6	[12]. $X = (Z(2) \cup \{\infty\}) \times Z(7,3)$. $\mathfrak{B} = \langle(\infty,1), (\infty,2), (\infty,3), (\infty,4), (\emptyset,\emptyset), (0,\emptyset), (0,1), (0,3), (0,5)\rangle$ mod $(2,7)$, $\langle(\infty,0), (\infty,2), (\infty,4), (\emptyset,\emptyset), (\emptyset,2), (\emptyset,3), (0,0), (0,1), (0,4)\rangle$ mod $(2,7)$, $\langle(\infty,2), (\emptyset,\emptyset), (\emptyset,0), (\emptyset,2), (\emptyset,4), (0,\emptyset), (0,0), (0,2), (0,4)\rangle$ mod $(-,7)$.
21	10	9	[31]. $X = Z(3,2) \times Z(7,3)$. $\mathfrak{B} = \langle(\emptyset,\emptyset), (\emptyset,1), (0,\emptyset), (0,0), (0,2), (0,4), (1,\emptyset), (1,0), (1,2), (1,4)\rangle$ mod $(3,7)$, $\langle(\emptyset,1), (\emptyset,4), (0,\emptyset), (0,2), (0,5), (1,\emptyset), (1,0), (1,2), (1,3), (1,5)\rangle$ mod $(3,7)$.
22	7	2	Non-existing by Lemma 1.2.
22	7	4	[24]. $X = Z(22)$. $\mathfrak{B} = \langle 0', 1', 2', 6', 12', 15', 20'\rangle$ mod 22, $\langle 0', 2', 3', 4', 10', 15', 19'\rangle$ mod 22.
22	7		$\lambda \equiv 0 \pmod 2$, $\lambda > 2$. Lemma 2.4, $22 \in B(7,4)$ and $22 \in B(7,6)$ by Lemma 5.39.
22	8	4	Unknown.
22	8	8	$X = Z(3,2) \times Z(7,3) \cup \{(\infty)\}$. $\mathfrak{B} = \langle(\infty), (\emptyset,\emptyset), (0,0), (0,1), (0,3), (0,4), (1,1), (1,4)\rangle$ mod $(3,7)$, $\langle(\infty), (\emptyset,\emptyset), (\emptyset,0), (\emptyset,1), (\emptyset,2), (\emptyset,3), (\emptyset,4), (\emptyset,5)\rangle$ mod $(3,-)$, $\langle(\emptyset,0), (\emptyset,3), (0,\emptyset), (0,0), (0,3), (1,\emptyset), (1,1), (1,4)\rangle$ mod $(3,7)$, $\langle(\emptyset,1), (\emptyset,3), (0,0), (0,2), (0,4), (1,0), (1,2), (1,4)\rangle$ mod $(3,7)$.
22	8	12	$X = Z(2) \times Z(11,2)$. $\mathfrak{B} = \langle(\emptyset,4\alpha+1), (\emptyset,4\alpha+7), (\emptyset,4\alpha+9), (0,0), (0,2), (0,4), (0,6), (0,8)\rangle$ mod $(2,11)$, $\alpha = 0,1$, $\langle(\emptyset,2), (\emptyset,7), (0,0), (0,1), (0,3), (0,5), (0,6), (0,8)\rangle$ mod $(2,11)$, $\langle(\emptyset,\beta), (\emptyset,\beta+1), (\emptyset,\beta+2), (\emptyset,\beta+9), (0,\beta), (0,\beta+1), (0,\beta+2), (0,\beta+9)\rangle$ mod $(-,11)$, $\beta = 0,1,2$.
22	8		$\lambda \equiv 0 \pmod 4$, $\lambda > 4$. Lemma 2.4, $22 \in B(8,8)$ and $22 \in B(8,12)$.
22	9	24	$X = Z(3,2) \times Z(7,3) \cup \{(\infty)\}$. $\mathfrak{B} = \langle(\infty), (\emptyset,2\alpha+1), (\emptyset,2\alpha+3), (0,0), (0,2), (0,4), (1,0), (1,2), (1,4)\rangle$ mod $(3,7)$, $\alpha = 0,1,2$, $\langle(\emptyset,\emptyset), (\emptyset,0), (\emptyset,3), (0,\emptyset), (0,1), (0,4), (1,\emptyset), (1,2), (1,5)\rangle$ mod $(3,7)$, 3 times, $\langle(\emptyset,\emptyset), (\emptyset,0), (\emptyset,1), (\emptyset,2), (\emptyset,3), (\emptyset,4), (\emptyset,5), (0,\emptyset), (1,\emptyset)\rangle$ mod $(3,7)$, $\langle(\emptyset,0), (\emptyset,2), (\emptyset,4), (0,0), (0,2), (0,4), (1,0), (1,2), (1,4)\rangle$ mod $(-,7)$.
22	10	15	Lemma 4.6, $21 \in B(9,6)$ and $21 \in B(10,9)$.
22	11	10	[31]. Form $B[21,10;43]$ with $X = Z(43,3)$ and $\mathfrak{B} = \langle 0,2,4,6,8,10,12,14,16,18,20,22,24,26,28,30,32,34,36,38,40\rangle$ mod 43, and delete any one block and all its points.

Table 5.23 (cont.).

v	k	λ	$B[k, \lambda; v]$
23	7	21	Lemma 4.2.
23	8	28	Lemma 4.1.
23	9	36	Lemma 4.2.
23	10	45	Lemma 4.1.
23	11	5	Lemma 4.3.
24	8	7	$X = Z(3,2) \times (Z(7,3) \cup \{\infty\})$. $\mathcal{B} = \langle (\alpha', \infty), ((\alpha+1)', \infty), (0', \alpha), (0', \alpha+3), (1', \alpha), (1', \alpha+3), (2', \alpha), (2', \alpha+3) \rangle$ mod $(-, 7)$, $\alpha = 0, 1, 2$, $\langle (\emptyset, \infty), (\beta, \emptyset), (\beta, 0), (\beta, 1), (\beta, 2), (\beta, 3), (\beta, 4), (\beta, 5) \rangle$ mod $(3, -)$, $\beta = 0, 1$, $\langle (\emptyset, \infty), (\emptyset, \emptyset), (\emptyset, 0), (\emptyset, 3), (0, 2), (0, 5), (1, 1), (1, 4) \rangle$ mod $(3, 7)$, $\langle (\emptyset, 0), (\emptyset, 3), (0, \emptyset), (0, 2), (0, 5), (1, \emptyset), (1, 1), (1, 4) \rangle$ mod $(3, 7)$.
24	9	24	$X = Z(3,2) \times Z(7,3) \cup \{(\infty_i) : i = 0, 1, 2\}$. $\mathcal{B} = \langle (\infty_\alpha), (\infty_{\alpha+1}), (\emptyset, \emptyset), (\emptyset, 0), (\emptyset, 1), (\emptyset, 2), (\emptyset, 3), (\emptyset, 4), (\emptyset, 5) \rangle$ mod $(3, -)$, $\alpha = 0, 1, 2$, $\langle (\infty_\alpha), (\infty_{\alpha+1}), (\emptyset, \emptyset), (0, \emptyset), (0, \alpha), (0, \alpha+3), (1, \emptyset), (1, \alpha+1), (1, \alpha+4) \rangle$ mod $(3, 7)$, $\alpha = 0, 1, 2$, $\langle (\infty_\alpha), (\emptyset, \alpha+1), (\emptyset, \alpha+2), (\emptyset, \alpha+4), (\emptyset, \alpha+5), (0, \alpha), (0, \alpha+3), (1, \alpha),$ $(1, \alpha+3) \rangle$ mod $(3, 7)$, $\alpha = 0, 1, 2$, $\langle (\emptyset, \emptyset), (\emptyset, 0), (\emptyset, 3), (0, \emptyset), (0, 1), (0, 4), (1, \emptyset), (1, 2), (1, 5) \rangle$ mod $(3, 7)$, 2 times, $\langle (\emptyset, 0), (\emptyset, 2), (\emptyset, 4), (0, 0), (0, 2), (0, 4), (1, 0), (1, 2), (1, 4) \rangle$ mod $(-, 7)$.
24	10	45	$X = Z(3,2) \times Z(7,3) \cup \{(\infty_i) : i = 0, 1, 2\}$. $\mathcal{B} = \langle (\infty_0), (\infty_1), (\infty_2), (\emptyset, \emptyset), (\emptyset, 0), (\emptyset, 1), (\emptyset, 2), (\emptyset, 3), (\emptyset, 4), (\emptyset, 5) \rangle$ mod $(3, -)$, $\langle (\infty_\alpha), (\infty_{\alpha+1}), (\emptyset, 2\alpha+1), (\emptyset, 2\alpha+3), (0, 0), (0, 2), (0, 4), (1, 0), (1, 2),$ $(1, 4) \rangle$ mod $(3, 7)$, 2 times, $\alpha = 0, 1, 2$, $\langle (\infty_\alpha), (\emptyset, \emptyset), (\emptyset, 0), (\emptyset, 3), (0, \emptyset), (0, 1), (0, 4), (1, \emptyset), (1, 2), (1, 5) \rangle$ mod $(3, 7)$, $\alpha = 0, 1, 2$, $\langle (\infty_\alpha), (\emptyset, 0), (\emptyset, 2), (\emptyset, 4), (0, 0), (0, 2), (0, 4), (1, 0), (1, 2), (1, 4) \rangle$ mod $(-, 7)$, $\alpha = 0, 1, 2$, $\langle (\emptyset, \emptyset), (0, \emptyset), (0, \alpha), (0, \alpha+3), (1, 0), (1, 1), (1, 2), (1, 3), (1, 4), (1, 5) \rangle$ mod $(3, 7)$, $\alpha = 0, 1, 2$.
24	11	110	Lemma 4.6, $23 \in B(10, 45)$ and $23 \in B(11, 5)$.
24	12	11	Lemma 4.6, $23 \in B(11, 5)$ and $23 \in B(12, 6)$ by Lemma 4.5.
25	8	7	Lemma 4.1.
25	9	3	[12]. $X = \mathrm{GF}(25, x^2 = 2x + 2)$. $\mathcal{B} = \langle \emptyset, 0, 1, 12, 13, 16, 17, 20, 21 \rangle$, $\langle \emptyset, 0, 1, 14, 15, 18, 19, 22, 23 \rangle$, $\langle 2, 3, 4, 12, 14, 16, 18, 20, 22 \rangle$, $\langle 2, 3, 4, 13, 15, 17, 19, 21, 23 \rangle$, $\langle 5, 6, 7, 12, 15, 16, 19, 20, 23 \rangle$, $\langle 5, 6, 7, 13, 14, 17, 18, 21, 22 \rangle$, $\langle \emptyset, 2, 5, 8, 9, 12, 13, 18, 19 \rangle$, $\langle \emptyset, 2, 5, 10, 11, 14, 15, 16, 17 \rangle$, $\langle 0, 3, 6, 8, 10, 12, 14, 17, 19 \rangle$, $\langle 0, 3, 6, 9, 11, 13, 15, 16, 18 \rangle$, $\langle 1, 4, 7, 8, 11, 12, 15, 17, 18 \rangle$, $\langle 1, 4, 7, 9, 10, 13, 14, 16, 19 \rangle$, $\langle \emptyset, 3, 7, 8, 9, 16, 17, 22, 23 \rangle$, $\langle \emptyset, 3, 7, 10, 11, 18, 19, 20, 21 \rangle$, $\langle 0, 4, 5, 8, 10, 16, 18, 21, 23 \rangle$, $\langle 0, 4, 5, 9, 11, 17, 19, 20, 22 \rangle$, $\langle 1, 2, 6, 8, 11, 16, 19, 21, 22 \rangle$, $\langle 1, 2, 6, 9, 10, 17, 18, 20, 23 \rangle$, $\langle \emptyset, 4, 6, 8, 9, 14, 15, 20, 21 \rangle$, $\langle \emptyset, 4, 6, 10, 11, 12, 13, 22, 23 \rangle$, $\langle 0, 2, 7, 8, 10, 13, 15, 20, 22 \rangle$, $\langle 0, 2, 7, 9, 11, 12, 14, 21, 23 \rangle$, $\langle 1, 3, 5, 8, 11, 13, 14, 20, 23 \rangle$, $\langle 1, 3, 5, 9, 10, 12, 15, 21, 22 \rangle$, $\langle \emptyset, 0, 1, 2, 3, 4, 5, 6, 7 \rangle$.

Table 5.23 (cont.).

v	k	λ	$B[k, \lambda; v]$
25	10	3	Non existing by Lemma 1.1.
25	10	6	$X = Z(5,2) \times Z(5,2)$.
			𝔅 = ⟨(∅, ∅), (∅, 1), (∅, 3), (2α, ∅), (2α, 0), (2α, 2), (2α+1, ∅), (2α+2, ∅),
			(2α+3, 1), (2α+3, 3)⟩ mod (−, 5), α = 0, 1,
			⟨(∅, ∅), (∅, 0), (∅, 2), (2α, 0), (2α, 2), (2α+1, ∅), (2α+2, ∅), (2α+3, ∅),
			(2α+3, 1), (2α+3, 3)⟩ mod (−, 5), α = 0, 1,
			⟨(∅, ∅), (2α, 0), (2α, 1), (2α, 2), (2α, 3), (2α+1, 0), (2α+1, 2), (2α+2, 0),
			(2α+2, 2), (2α+3, ∅)⟩ mod (−, 5), α = 0, 1,
			⟨(∅, ∅), (2α, 1), (2α, 3), (2α+1, 0), (2α+1, 1), (2α+1, 2), (2α+1, 3),
			(2α+2, ∅), (2α+3, 1), (2α+3, 3)⟩ mod (−, 5), α = 0, 1.
25	10	9	$X = Z(5,2) \times Z(5,2)$.
			𝔅 = ⟨(α, ∅), (α, α), (α, α+2), (α+1, α+1), (α+1, α+3), (α+2, ∅), (α+2, α),
			(α+2, α+2), (α+3, α+1), (α+3, α+3)⟩ mod (5, 5), α = 0, 1,
			⟨(∅, α), (∅, α+2), (0, α), (0, α+2), (1, α), (1, α+2), (2, α), (2, α+2), (3, α),
			(3, α+2)⟩ mod (−, 5), α = 0, 1.
25	10	$\lambda \equiv 0 \pmod 3$, $\lambda > 3$.	Lemma 2.4, $25 \in B(10,6)$ and $25 \in B(10,9)$.
25	11	55	Lemma 4.2.
25	12	11	Lemma 4.1.
26	8	28	Lemma 4.6, $25 \in B(7,7)$ by Lemma 5.38 and $25 \in B(8,7)$.
26	9	72	Lemma 4.6, $25 \in B(8,7)$ and $25 \in B(9,3)$.
26	10	9	Lemma 4.6, $25 \in B(9,3)$ and $25 \in B(10,6)$.
26	11	22	$X = Z(5,2) \times Z(5,2) \cup \{(\infty)\}$.
			𝔅 = ⟨(∞), (∅, 0), (∅, 2), (0, 0), (0, 2), (1, 1), (1, 3), (2, 0), (2, 2), (3, 1), (3, 3)⟩
			mod (5, 5),
			⟨(∞), (0, ∅), (0, 1), (0, 3), (1, 0), (1, 2), (2, ∅), (2, 1), (2, 3), (3, 1), (3, 3)⟩
			mod (5, 5),
			⟨(∞), (0, ∅), (0, 0), (0, 1), (0, 2), (0, 3), (2, ∅), (2, 0), (2, 1), (2, 2), (2, 3)⟩
			mod (5, −),
			⟨(∅, ∅), (∅, 1), (∅, 3), (0, 0), (0, 2), (1, 1), (1, 3), (2, 1), (2, 3), (3, 1), (3, 3)⟩
			mod (5, 5),
			⟨(∅, ∅), (0, 1), (0, 3), (1, ∅), (1, 1), (1, 3), (2, 1), (2, 3), (3, ∅), (3, 0), (3, 2)⟩
			mod (5, 5),
			⟨(∅, ∅), (∅, 0), (∅, 1), (∅, 2), (∅, 3), (0, 0), (0, 2), (1, ∅), (2, 1), (2, 3), (3, ∅)⟩
			mod (5, 5).
26	12	66	$X = \mathrm{GF}(25, x^2 = 2x+2) \cup \{\infty\}$.
			𝔅 = ⟨∞, ∅, 2α, 2α+2, 2α+4, 2α+6, 2α+8, 2α+12, 2α+14, 2α+16,
			2α+18, 2α+20⟩ mod 25, α = 0, 1, ..., 5,
			⟨0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22⟩ mod 25, 7 times.
26	13	12	$X = \mathrm{GF}(25, x^2 = 2x+2) \cup \{\infty\}$.
			𝔅 = ⟨∞, 0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22⟩ mod 25,
			⟨∅, 0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22⟩ mod 25.
27	7	21	Lemma 4.2.
27	8	28	Lemma 4.1.
27	9	4	$X = Z(3,2) \times \mathrm{GF}(9, x^2 = 2x+1)$.
			𝔅 = ⟨(∅, ∅), (∅, α), (∅, α+4), (0, ∅), (0, α+1), (0, α+5), (1, ∅), (1, α+2), (1, α+6)⟩
			mod (−, 9), α = 0, 1, 2, 3,
			⟨(∅, ∅), (∅, 0), (∅, 1), (∅, 2), (∅, 3), (∅, 4), (∅, 5), (∅, 6), (∅, 7)⟩ mod (3, −).

Table 5.23 (cont.).

v	k	λ	$B[k, \lambda; v]$
27	10	45	Lemma 4.1.
27	11	55	Lemma 4.2.
27	12	22	$X = Z(3,2) \times GF(9, x^2 = 2x+1)$. $\mathfrak{B} = \langle(\emptyset, 2\alpha), (\emptyset, 2\alpha+1), (\emptyset, 2\alpha+4), (\emptyset, 2\alpha+5), (0, \emptyset), (0, 2\alpha+1), (0, 2\alpha+5),$ $(1, \emptyset), (1, 2\alpha), (1, 2\alpha+2), (1, 2\alpha+4), (1, 2\alpha+6)\rangle$ mod $(3,9)$, $\alpha = 0, 1$, $\langle(\emptyset, 2\alpha+1), (\emptyset, 2\alpha+5), (0, 2\alpha), (0, 2\alpha+3), (0, 2\alpha+4), (0, 2\alpha+7), (1, 2\alpha+1),$ $(1, 2\alpha+2), (1, 2\alpha+3), (1, 2\alpha+5), (1, 2\alpha+6), (1, 2\alpha+7)\rangle$ mod $(3,9)$, $\alpha = 0, 1$, $\langle(\emptyset, 0), (\emptyset, 2), (\emptyset, 4), (\emptyset, 6), (0,0), (0,2), (0,4), (0,6), (1,0), (1,2), (1,4),$ $(1,6)\rangle$ mod $(-, 9)$.
27	13	6	Lemma 4.3.
28	7	2	Table 5.21.
28	8	14	$X = GF(4, x^2 = x+1) \times Z(7,3)$. $\mathfrak{B} = \langle(\emptyset, 2\alpha+2), (\emptyset, 2\alpha+3), (\alpha, \emptyset), (\alpha, 2\alpha+4), (\alpha, 2\alpha+5), (\alpha+1, \emptyset), (\alpha+1, 2\alpha),$ $(\alpha+1, 2\alpha+1)\rangle$ mod $(4,7)$, $\alpha = 0, 1, 2$, $\langle(\emptyset, 2\alpha), (\emptyset, 2\alpha+1), (\emptyset, 2\alpha+2), (\alpha, \emptyset), (\alpha, 2\alpha+3), (\alpha, 2\alpha+4), (\alpha+1, \emptyset),$ $(\alpha+2, \emptyset)\rangle$ mod $(4,7)$, $\alpha = 0, 1, 2$, $\langle(\emptyset, \alpha), (\emptyset, \alpha+3), (0, \alpha), (0, \alpha+3), (1, \alpha), (1, \alpha+3), (2, \alpha), (2, \alpha+3)\rangle$ mod $(-, 7)$, $\alpha = 0, 1, 2$.
28	9	8	$X = Z(3,2) \times GF(9, x^2 = 2x+1) \cup \{(\infty)\}$. $\mathfrak{B} = \langle(\infty), (\emptyset, 0), (\emptyset, 2), (\emptyset, 4), (\emptyset, 6), (0,1), (0,5), (1,3), (1,7)\rangle$ mod $(3,9)$, $\langle(\emptyset, 0), (\emptyset, 2), (\emptyset, 4), (\emptyset, 6), (0,0), (0,4), (1,2), (1,6), (\alpha, \emptyset)\rangle$ mod $(3,9)$, $\alpha = 0, 1$, $\langle(\emptyset, \emptyset), (\emptyset, 0), (\emptyset, 1), (\emptyset, 2), (\emptyset, 3), (\emptyset, 4), (\emptyset, 5), (\emptyset, 6), (\emptyset, 7)\rangle$ mod $(3, -)$.
28	10	5	Unknown.
28	10	10	$X = GF(4, x^2 = x+1) \times Z(7,3)$. $\mathfrak{B} = \langle(\emptyset, \emptyset), (2\alpha, 2\alpha), (2\alpha, 2\alpha+2), (2\alpha+1, 2\alpha), (2\alpha+1, 2\alpha+1), (2\alpha+1, 2\alpha+3),$ $(2\alpha+2, \emptyset), (2\alpha+2, 2\alpha+1), (2\alpha+2, 2\alpha+2), (2\alpha+2, 2\alpha+3)\rangle$ mod $(4,7)$, $\alpha = 0, 1, 2$.
28	10	15	$X = Z(3,2) \times GF(9, x^2 = 2x+1) \cup \{(\infty)\}$. $\mathfrak{B} = \langle(\infty), (\emptyset, \emptyset), (0,0), (0,2), (0,4), (0,6), (1,1), (1,3), (1,5), (1,7)\rangle$ mod $(3,9)$, $\langle(\infty), (\emptyset, \emptyset), (\emptyset, 2\alpha), (\emptyset, 2\alpha+4), (0, \emptyset), (0, 2\alpha), (0, 2\alpha+4), (1, \emptyset), (1, 2\alpha),$ $(1, 2\alpha+4)\rangle$ mod $(-, 9)$, $\alpha = 0, 1$, $\langle(\emptyset, \emptyset), (\emptyset, 1), (\emptyset, 3), (\emptyset, 5), (\emptyset, 7), (\alpha, \emptyset), (\alpha, 0), (\alpha, 3), (\alpha+1, 5), (\alpha+1, 6)\rangle$ mod $(3,9)$, $\alpha = 0, 1$, $\langle(\emptyset, \emptyset), (\emptyset, 0), (\emptyset, 2), (\emptyset, 4), (\emptyset, 6), (0, \emptyset), (0,1), (0,5), (1,1), (1,5)\rangle$ mod $(3,9)$.
28	10	$\lambda \equiv 0 \pmod 5$, $\lambda > 5$.	Lemma 2.4, $28 \in B(10,10)$ and $28 \in B(10,15)$.
28	11	110	Lemma 2.6, $28 \in B(12,11)$ as below and $12 \in B(11,10)$ trivially.
28	12	11	$X = Z(3,2) \times GF(9, x^2 = 2x+1) \cup \{(\infty)\}$. $\mathfrak{B} = \langle(\infty), (\emptyset, \emptyset), (0, \emptyset), (0,0), (0,2), (0,4), (0,6), (1, \emptyset), (1,1), (1,3). (1,5), (1,7)\rangle$ mod $(3,9)$, $\langle(\emptyset, 0), (\emptyset, 2), (\emptyset, 4), (\emptyset, 6), (0,1), (0,3), (0,5), (0,7), (1,1), (1,3), (1,5),$ $(1,7)\rangle$ mod $(3,9)$, $\langle(\emptyset, 0), (\emptyset, 2), (\emptyset, 4), (\emptyset, 6), (0,0), (0,2), (0,4), (0,6), (1,0), (1,2), (1,4),$ $(1,6)\rangle$ mod $(-, 9)$.
28	13	52	Lemma 4.6, $27 \in B(12,22)$ and $27 \in B(13,6)$.

Table 5.23 (cont.).

v	k	λ	$B[k,\lambda;v]$
28	14	13	Lemma 4.6, $27 \in B(13,6)$ and $27 \in B(14,7)$ by Lemma 4.5.
29	7	3	Lemma 4.3.
29	8	2	Non-existing by Lemma 1.2.
29	8	6	$X = \mathrm{GF}(4, x^2 = x+1) \times Z(7,3) \cup \{(\infty)\}$. $\mathfrak{B} = \langle(\infty), (\emptyset,\emptyset), (\alpha,0), (\alpha,3), (\alpha+1,2), (\alpha+1,5), (\alpha+2,1), (\alpha+2,4)\rangle$ mod $(-,7)$, $\alpha = 0,1,2$, $\langle(\infty), (\emptyset,\emptyset), (\emptyset,0), (\emptyset,1), (\emptyset,2), (\emptyset,3), (\emptyset,4), (\emptyset,5)\rangle$, 3 times, $\langle(\emptyset,\beta), (\emptyset,\beta+3), (\alpha,0), (\alpha,2), (\alpha,4), (\alpha+\beta,1), (\alpha+\beta,3), (\alpha+\beta,5)\rangle$ mod $(-,7)$, $\alpha = 0,1,2$, $\beta = 1,2$, $\langle(\emptyset,0), (\emptyset,3), (0,\alpha), (0,\alpha+3), (1,\alpha), (1,\alpha+3), (2,\alpha), (2,\alpha+3)\rangle$ mod $(-,7)$, $\alpha = 0,1,2$.
29	8	$\lambda \equiv 0 \pmod 2$, $\lambda > 2$.	Lemma 2.4, $29 \in B(8,4)$ by Lemma 4.4 and $29 \in B(8,6)$.
29	9	18	Lemma 4.2.
29	10	45	Lemma 4.1.
29	11	55	Lemma 4.2.
29	12	33	Lemma 4.1.
29	13	39	Lemma 4.2.
29	14	13	Lemma 4.1.
30	8	28	Lemma 4.6, $29 \in B(7,3)$ and $29 \in B(8,22)$.
30	9	24	Lemma 4.6, $29 \in B(8,6)$ and $29 \in B(9,18)$.
30	10	9	Unknown.
30	10	18	$X = Z(29,2) \cup \{\infty\}$. $\mathfrak{B} = \langle\infty, \emptyset, 0, 1, 7, 8, 14, 15, 21, 22\rangle$ mod 29, $\langle\infty, \emptyset, 0, 2, 7, 9, 14, 16, 21, 23\rangle$ mod 29, $\langle 7\alpha+2, 7\alpha+3, 7\alpha+4, 7\alpha+12, 7\alpha+13, 7\alpha+16, 7\alpha+17, 7\alpha+18, 7\alpha+26, 7\alpha+27\rangle$ mod 29, $\alpha = 0,1$, $\langle 7\alpha+1, 7\alpha+3, 7\alpha+4, 7\alpha+5, 7\alpha+6, 7\alpha+15, 7\alpha+17, 7\alpha+18, 7\alpha+19,$ $7\alpha+20\rangle$ mod 29, $\alpha = 0,1$.
30	10	27	$X = Z(29,2) \cup \{\infty\}$. $\mathfrak{B} = \langle\infty, \emptyset, 2\alpha, 2\alpha+1, 2\alpha+7, 2\alpha+8, 2\alpha+14, 2\alpha+15, 2\alpha+21, 2\alpha+22\rangle$ mod 29, $\alpha = 0,1$, $\langle\infty, \emptyset, 0, 2, 7, 9, 14, 16, 21, 23\rangle$ mod 29, $\langle 7\alpha+2, 7\alpha+5, 7\alpha+6, 7\alpha+10, 7\alpha+11, 7\alpha+16, 7\alpha+19, 7\alpha+20, 7\alpha+24,$ $7\alpha+25\rangle$ mod 29, $\alpha = 0,1$, $\langle 7\alpha+4, 7\alpha+5, 7\alpha+6, 7\alpha+7, 7\alpha+8, 7\alpha+18, 7\alpha+19, 7\alpha+20, 7\alpha+21,$ $7\alpha+22\rangle$ mod 29, $\alpha = 0,1$, $\langle 7\alpha+1, 7\alpha+3, 7\alpha+4, 7\alpha+5, 7\alpha+6, 7\alpha+15, 7\alpha+17, 7\alpha+18, 7\alpha+19, 7\alpha+20\rangle$ mod 29, $\alpha = 0,1$.
30	10	$\lambda \equiv 0 \pmod 9$, $\lambda > 9$.	Lemma 2.4, $30 \in B(10,18)$ and $30 \in B(10,27)$.
30	11	110	$X = Z(29,2) \cup \{\infty\}$. $\mathfrak{B} = \langle\infty, \mu, \mu+2, \mu+4, \mu+6, \mu+8, \mu+14, \mu+16, \mu+18, \mu+20, \mu+22\rangle$ mod 29, once for $\mu = 1$, 2 times for $\mu = 0,3,6,7,8$, $\langle\emptyset, \nu, \nu+2, \nu+4, \nu+6, \nu+8, \nu+14, \nu+16, \nu+18, \nu+20, \nu+22\rangle$ mod 29, once for $\nu = 1$, 2 times for $\nu = 0,2,4,5,9,10,11,12,13$.

Table 5.23 (cont.).

v	k	λ	$B[k, \lambda; v]$
30	12	22	$X\ (Z(5,2) \cup \{\infty\}) \times Z(5,2)$. $\mathcal{B} = \langle(\infty,\emptyset),(\infty,1),(\infty,3),(\emptyset,\emptyset),(0,1),(0,3),(1,0),(1,1),(1,2),(1,3),$ $(2,0),(2,2)\rangle \bmod (5,5)$, $\langle(\infty,\emptyset),(\infty,0),(\infty,2),(\emptyset,\emptyset),(0,0),(0,2),(1,0),(1,2),(2,1),(2,3),$ $(3,1),(3,3)\rangle \bmod (5,5)$, $\langle(\infty,0),(\infty,2),(0,0),(0,2),(1,\emptyset),(1,1),(1,3),(2,0),(2,2),(3,\emptyset),$ $(3,1),(3,3)\rangle \bmod (5,5)$, $\langle(\infty,1),(\infty,3),(0,1),(0,3),(1,\emptyset),(1,0),(1,2),(2,\emptyset),(2,0),(2,2),$ $(3,1),(3,3)\rangle \bmod (5,5)$, $\langle(\infty,\alpha),(\infty,\alpha+2),(\emptyset,\beta),(\emptyset,\beta+2),(0,\beta),(0,\beta+2),(1,\beta),(1,\beta+2),(2,\beta),$ $(2,\beta+2),(3,\beta),(3,\beta+2)\rangle \bmod (-,5)$, $\alpha = 0,1$, $\beta = 0,1$, $\langle(0,\emptyset),(0,1),(0,3),(1,\emptyset),(1,0),(1,2),(2,\emptyset),(2,1),(2,3),(3,\emptyset),$ $(3,0),(3,2)\rangle \bmod (5,5)$.
30	13	156	$X = Z(29,2) \cup \{\infty\}$. $\mathcal{B} = \langle\infty,\mu,\mu+2,\mu+4,\mu+7,\mu+9,\mu+11,\mu+14,\mu+16,\mu+18,\mu+21,\mu+23,$ $\mu+25\rangle \bmod 29$, once for $\mu = 1$, 2 times for $\mu = 3,4,6$, 4 times for $\mu = 0$, $\langle\infty,0,5,6,7,12,13,14,19,20,21,26,27\rangle \bmod 29$, 2 times, $\langle\emptyset,\alpha,\alpha+1,\alpha+6,\alpha+7,\alpha+8,\alpha+13,\alpha+14,\alpha+15,\alpha+20,\alpha+21,\alpha+22,\alpha+27\rangle$ $\bmod 29$, 2 times, $\alpha = 0,1,\ldots,5$, $\langle\emptyset,\nu,\nu+2,\nu+4,\nu+7,\nu+9,\nu+11,\nu+14,\nu+16,\nu+18,\nu+21,$ $\nu+23,\nu+25\rangle \bmod 29$, once for $\nu = 1$, 2 times for $\nu = 2,5$.
30	14	91	Lemma 4.6, $29 \in B(13,39)$ and $29 \in B(14,13)$.
30	15	14	$X = Z(29,2) \cup \{\infty\}$. $\mathcal{B} = \langle\infty,0,2,4,6,8,10,12,14,16,18,20,22,24,26\rangle \bmod 29$, $\langle\emptyset,0,2,4,6,8,10,12,14,16,18,20,22,24,26\rangle \bmod 29$.
31	8	28	Lemma 4.1.
31	9	12	Lemma 4.3.
31	10	3	[12]. $X = (Z(3,2) \cup \{\infty\}) \times Z(7,3) \cup \{(\infty_i): i = 0,1,2\}$. $\mathcal{B} = \langle(\infty_0),(\infty_1),(\infty_2),(\emptyset,\emptyset),(\emptyset,0),(\emptyset,1),(\emptyset,2),(\emptyset,3),(\emptyset,4),(\emptyset,5)\rangle$ $\bmod (3,-)$, $\langle(\infty_\alpha),(\infty,1),(\infty,3),(\infty,5),(\emptyset,\alpha),(\emptyset,\alpha+3),(0,\alpha+1),(0,\alpha+4),(1,\alpha+2),$ $(1,\alpha+5)\rangle \bmod (-,7)$, $\alpha = 0,1,2$, $\langle(\infty,\emptyset),(\emptyset,0),(\emptyset,2),(\emptyset,4),(0,0),(0,2),(0,4),(1,0),(1,2),(1,4)\rangle$ $\bmod (-,7)$.
31	11	11	Lemma 4.2.
31	12	22	Lemma 4.1.
31	13	26	Lemma 4.2.
31	14	91	Lemma 4.1.
31	15	7	Lemma 4.3.
32	8	7	$X = \mathrm{GF}(4, x^2 = x+1) \times (Z(7,3) \cup \{\infty\})$. $\mathcal{B} = \langle(\emptyset,\infty),(0,\infty),(1,\infty),(2,\infty),(\emptyset,\emptyset),(0,\emptyset),(1,\emptyset),(2,\emptyset)\rangle \bmod (-,7)$, $\langle(\emptyset,\infty),(\emptyset,\emptyset),(\alpha,0),(\alpha,3),(\alpha+1,1),(\alpha+1,4),(\alpha+2,2),(\alpha+2,5)\rangle$ $\bmod (4,7)$, $\alpha = 0,1,2$, $\langle(\emptyset,\infty),(\emptyset,\emptyset),(\emptyset,0),(\emptyset,1),(\emptyset,2),(\emptyset,3),(\emptyset,4),(\emptyset,5)\rangle \bmod (4,-)$, 3 times, $\langle(\emptyset,\alpha),(\emptyset,\alpha+3),(0,\alpha),(0,\alpha+3),(1,\alpha),(1,\alpha+3),(2,\alpha),(2,\alpha+3)\rangle$ $\bmod (-,7)$, $\alpha = 0,1,2$.

Table 5.23 (cont.).

v	k	λ	$B[k, \lambda; v]$
32	9	72	Lemma 4.1.
32	10	45	Lemma 4.6, $31 \in B(9, 12)$ and $31 \in B(10, 3)$.
32	11	110	Lemma 4.1.
32	12	33	Lemma 4.6, $31 \in B(11, 11)$ and $31 \in B(12, 22)$.
32	13	156	Lemma 4.1.
32	14	91	$X = Z(31, 3) \cup \{\infty\}$. $\mathcal{B} = \langle \infty, \emptyset, \alpha, \alpha+1, \alpha+5, \alpha+6, \alpha+10, \alpha+11, \alpha+15, \alpha+16, \alpha+20, \alpha+21,$ $\alpha+25, \alpha+26 \rangle \bmod 31, \quad \alpha = 0, 1, 2, 3, 4,$ $\langle \infty, \emptyset, 0, 2, 5, 7, 10, 12, 15, 17, 20, 22, 25, 27 \rangle \bmod 31,$ 2 times, $\langle 0, 1, 5, 6, 10, 11, 15, 16, 20, 21, 25, 26, 5\beta+4, 5\beta+19 \rangle \bmod 31,$ $\beta = 0, 1, 2,$ $\langle 0, 3, 5, 8, 10, 13, 15, 18, 20, 23, 25, 28, 5\beta+4, 5\beta+19 \rangle \bmod 31,$ $\beta = 0, 1, 2,$ $\langle 2, 4, 7, 9, 12, 14, 17, 19, 22, 24, 27, 29, 5\beta, 5\beta+15 \rangle \bmod 31, \quad \beta = 0, 1, 2.$
32	15	210	Lemma 4.1.
32	16	15	Lemma 4.6, $31 \in B(15, 7)$ and $31 \in B(16, 8)$ by Lemma 4.5.
33	7	21	Table 5.22.
33	8	7	$X = Z(3, 2) \times Z(11, 2)$. $\mathcal{B} = \langle (\emptyset, 0), (\emptyset, 2), (\emptyset, 4), (\emptyset, 6), (0, 1), (0, 7), (1, 1), (1, 7) \rangle \bmod (3, 11),$ $\langle (\emptyset, 0), (\emptyset, 4), (\emptyset, 6), (\emptyset, 8), (0, \emptyset), (0, 1), (1, \emptyset), (1, 1) \rangle \bmod (3, 11),$ $\langle (\emptyset, 0), (\emptyset, 2), (\emptyset, 4), (\emptyset, 8), (0, 5), (0, 6), (1, 5), (1, 6) \rangle \bmod (3, 11),$ $\langle (\emptyset, \emptyset), (0, \emptyset), (0, 0), (0, 2), (0, 5), (0, 7), (1, 3), (1, 8) \rangle \bmod (3, 11).$
33	9	3	[31]. Form $B[12, 3; 45]$ with $X = Z(3) \times Z(3) \times Z(5, 2)$ and $\mathcal{B} = \langle (\emptyset, \emptyset, 0), (\emptyset, 0, 0), (\emptyset, 1, 0), (\emptyset, \emptyset, 1), (0, \emptyset, 1), (1, \emptyset, 1), (\emptyset, \emptyset, 2),$ $(0, 1, 2), (1, 0, 2), (\emptyset, \emptyset, 3), (0, 0, 3), (1, 1, 3) \rangle \bmod (3, 3, 5)$ and delete any one block and all its points.
33	10	45	$X = Z(3, 2) \times Z(11, 2)$. $\mathcal{B} = \langle (\emptyset, \emptyset), (\emptyset, \alpha+1), (\emptyset, \alpha+3), (\emptyset, \alpha+5), (0, \alpha), (0, \alpha+2), (0, \alpha+4), (1, \alpha),$ $(1, \alpha+2), (1, \alpha+4) \rangle \bmod (3, 11),$ 3 times, $\alpha = 0, 1, 2, 3, 4,$ $\langle (\emptyset, 0), (\emptyset, 1), (\emptyset, 2), (\emptyset, 3), (\emptyset, 4), (\emptyset, 5), (\emptyset, 6), (\emptyset, 7), (\emptyset, 8), (\emptyset, 9) \rangle$ $\bmod (3, 11).$
33	11	5	Unknown.
33	12	11	$X = Z(3, 2) \times Z(11, 2)$. $\mathcal{B} = \langle (\emptyset, \emptyset), (0, \emptyset), (0, 0), (0, 2), (0, 4), (0, 6), (0, 8), (1, 1), (1, 3), (1, 5), (1, 7),$ $(1, 9) \rangle \bmod (3, 11),$ $\langle (\emptyset, 2\alpha), (\emptyset, 2\alpha+1), (\emptyset, 2\alpha+2), (\emptyset, 2\alpha+3), (0, 2\alpha+2), (0, 2\alpha+3), (0, 2\alpha+4),$ $(0, 2\alpha+5), (1, 2\alpha), (1, 2\alpha+1), (1, 2\alpha+4), (1, 2\alpha+5) \rangle \bmod (-, 11),$ $\alpha = 0, 1, 2, 3, 4.$
33	13	39	$X = Z(3, 2) \times Z(11, 2)$. $\mathcal{B} = \langle (\emptyset, 2\alpha+9), (0, \emptyset), (0, 0), (0, 2), (0, 4), (0, 6), (0, 8), (1, \emptyset), (1, 0), (1, 2),$ $(1, 4), (1, 6), (1, 8) \rangle \bmod (3, 11), \quad \alpha = 0, 1,$ $\langle (\emptyset, \emptyset), (\emptyset, 4\alpha+3), (\emptyset, 4\alpha+5), (0, 0), (0, 2), (0, 4), (0, 6), (0, 8), (1, 0), (1, 2),$ $(1, 4), (1, 6), (1, 8) \rangle \bmod (3, 11), \quad \alpha = 0, 1,$ $\langle (\emptyset, 0), (\emptyset, 2), (\emptyset, 4), (\emptyset, 6), (\emptyset, 8), (0, \emptyset), (0, 4\alpha+1), (0, 4\alpha+3), (\emptyset, 4\alpha+5),$ $(1, \emptyset), (1, 4\alpha+1), (1, 4\alpha+3), (1, 4\alpha+5) \rangle \bmod (3, 11), \quad \alpha = 0, 1,$ $\langle (\emptyset, 1), (\emptyset, 3), (\emptyset, 7), (0, 0), (0, 2), (0, 4), (0, 6), (0, 8), (1, 0), (1, 2), (1, 4),$ $(1, 6), (1, 8) \rangle \bmod (3, 11),$ $\langle (\emptyset, 0), (\emptyset, 2), (\emptyset, 4), (\emptyset, 6), (\emptyset, 8), (0, 1), (0, 3), (0, 5), (0, 9), (1, 1), (1, 3),$ $(1, 5), (1, 9) \rangle \bmod (3, 11).$

Table 5.23 (cont.).

v	k	λ	$B[k, \lambda; v]$
33	14	91	$X = Z(3,2) \times Z(11,2)$. $\mathcal{B} = \langle(\emptyset,\alpha), (\emptyset,\alpha+2), (\emptyset,\alpha+3), (\emptyset,\alpha+5), (\emptyset,\alpha+7), (\emptyset,\alpha+8), (0,\alpha), (0,\alpha+1),$ $(0,\alpha+5), (0,\alpha+6), (1,\alpha), (1,\alpha+4), (1,\alpha+5), (1,\alpha+9)\rangle \bmod (3,11)$, $\alpha = 0,1,2,3,4$, $\langle(\emptyset,2\mu+1), (\emptyset,2\mu+7), (0,\emptyset), (0,0), (0,2), (0,4), (0,6), (0,8), (1,\emptyset)$ $(1,0), (1,2), (1,4), (1,6), (1,8)\rangle \bmod (3,11)$, 2 times for $\mu = 0$, once for $\mu = 1,3,4$, $\langle(\emptyset,\beta+2), (\emptyset,\beta+3), (\emptyset,\beta+4), (\emptyset,\beta+7), (\emptyset,\beta+8), (\emptyset,\beta+9), (0,\beta), (0,\beta+3),$ $(0,\beta+5), (0,\beta+8), (1,\beta+1), (1,\beta+3), (1,\beta+6), (1,\beta+8)\rangle \bmod (3,11)$, $\beta = 0,1$, $\langle(\emptyset,\emptyset), (\emptyset,0), (\emptyset,2), (\emptyset,4), (\emptyset,6), (\emptyset,8), (0,1), (0,5), (0,7), (0,4\beta+9),$ $(1,1), (1,5), (1,7), (1,4\beta+9)\rangle \bmod (3,11)$, $\beta = 0,1$, $\langle(\emptyset,1), (\emptyset,3), (\emptyset,5), (\emptyset,7), (0,0), (0,2), (0,4), (0,6), (0,8), (1,0), (1,2),$ $(1,4), (1,6), (1,8)\rangle \bmod (3,11)$, $\langle(\emptyset,1), (\emptyset,9), (0,\emptyset), (0,0), (0,2), (0,4), (0,6), (0,8), (1,\emptyset), (1,0), (1,2),$ $(1,4), (1,6), (1,8)\rangle \bmod (3,11)$.
33	15	35	$X = Z(3,2) \times Z(11,2)$. $\mathcal{B} = \langle(\emptyset,2\alpha+1), (\emptyset,2\alpha+3), (\emptyset,2\alpha+5), (0,\emptyset), (0,2\alpha), (0,2\alpha+2), (0,2\alpha+4),$ $(0,2\alpha+6), (0,2\alpha+8), (1,\emptyset), (1,2\alpha), (1,2\alpha+2), (1,2\alpha+4), (1,2\alpha+6),$ $(1,2\alpha+8)\rangle \bmod (3,11)$, $\alpha = 0,1,2,3,4$, $\langle(\emptyset,0), (\emptyset,2), (\emptyset,4), (\emptyset,6), (\emptyset,8), (0,0), (0,2), (0,4), (0,6), (0,8), (1,0),$ $(1,2), (1,4), (1,6), (1,8)\rangle \bmod (-,11)$.
33	16	15	Form $B[33,16;67]$ with $X = Z(67,2)$ and $\mathcal{B} = \langle 0,2,4,6,\ldots,64\rangle$ and delete any one block and in other blocks delete the points not included in the deleted block.
34	7	14	Table 5.22.
34	8	28	$X = Z(3,2) \times Z(11,2) \cup \{(\infty)\}$. $\mathcal{B} = \langle(\infty), (\emptyset,\emptyset), (\emptyset,\mu), (\emptyset,\mu+5), (0,\mu), (0,\mu+2), (0,\mu+5), (0,\mu+7)\rangle$ $\bmod (3,11)$, 2 times for $\mu = 1$, once for $\mu = 0,3$, $\langle(\emptyset,\alpha), (\emptyset,\alpha+1), (\emptyset,\alpha+5), (\emptyset,\alpha+6), (0,\alpha), (0,\alpha+5), (1,\alpha+1), (1,\alpha+6)\rangle$ $\bmod (3,11)$, $\alpha = 0,1,2,3,4$, $\langle(\emptyset,\nu+1), (\emptyset,\nu+4), (\emptyset,\nu+6), (\emptyset,\nu+9), (0,\nu), (0,\nu+5), (1,\nu+3), (1,\nu+8)\rangle$ $\bmod (3,11)$, . 2 times for $\nu = 1$, 3 times for $\nu = 0,3$.
34	9	24	$X = Z(3,2) \times Z(11,2) \cup \{(\infty)\}$. $\mathcal{B} = \langle(\infty), (\emptyset,\emptyset), (\emptyset,0), (\emptyset,2), (\emptyset,4), (\emptyset,6), (\emptyset,8), (0,\emptyset), (1,\emptyset)\rangle \bmod (3,11)$, 3 times, $\langle(\emptyset,\emptyset), (\emptyset,2\alpha), (\emptyset,2\alpha+5), (0,\emptyset), (0,2\alpha+1), (0,2\alpha+6), (1,\emptyset), (1,2\alpha+2),$ $(1,2\alpha+7)\rangle \bmod (3,11)$, $\alpha = 0,1,2,3,4$, $\langle(\emptyset,\beta), (\emptyset,\beta+1), (\emptyset,\beta+2), (0,\beta+3), (0,\beta+4), (0,\beta+5), (1,\beta+6), (1,\beta+7),$ $(1,\beta+8)\rangle \bmod (-,11)$, $\beta = 0,1,\ldots,9$.
34	10	15	$X = Z(3,2) \times Z(11,2) \cup \{(\infty)\}$. $\mathcal{B} = \langle(\infty), (\emptyset,\alpha), (\emptyset,\alpha+1), (\emptyset,\alpha+2), (0,\alpha), (0,\alpha+1), (0,\alpha+2), (1,\alpha), (1,\alpha+1),$ $(1,\alpha+2)\rangle \bmod (-,11)$, $\alpha = 0,1,2,3,4$, $\langle(\emptyset,\beta+2), (\emptyset,\beta+7), (0,\beta), (0,\beta+1), (0,\beta+5), (0,\beta+6), (1,\beta+3),$ $(1,\beta+4), (1,\beta+8), (1,\beta+9)\rangle \bmod (3,11)$, $\beta = 0,1$, $\langle(\emptyset,2\beta), (\emptyset,2\beta+1), (\emptyset,2\beta+3), (\emptyset,2\beta+5), (\emptyset,2\beta+6), (\emptyset,2\beta+8), (0,2\beta+2),$ $(0,2\beta+7), (1,2\beta+4), (1,2\beta+9)\rangle \bmod (3,11)$, $\beta = 0,1$.

Table 5.23 (cont.).

v	k	λ	$B[k, \lambda; v]$
34	11	10	$X = Z(3,2) \times Z(11,2) \cup \{(\infty)\}$. $\mathfrak{B} = \langle(\infty), (\emptyset,4), (\emptyset,9), (0,\emptyset), (0,0), (0,7), (0,8), (1,\emptyset), (1,0), (1,7), (1,8)\rangle$ mod $(3,11)$, $\langle(\emptyset,\emptyset), (\emptyset,1), (\emptyset,7), (\emptyset,9), (\emptyset,6\alpha+8), (0,0), (0,5), (0,4\alpha+2), (1,0), (1,5), (1,4\alpha+2)\rangle$ mod $(3,11)$, $\alpha = 0,1$, $\langle(\emptyset,\emptyset), (\emptyset,0), (\emptyset,1), (\emptyset,2), (\emptyset,3), (\emptyset,4), (\emptyset,5), (\emptyset,6), (\emptyset,7), (\emptyset,8), (\emptyset,9)\rangle$ mod $(3,-)$.
34	12	2	Non-existing by Lemma 1.1.
34	12	4	Non-existing by Lemma 1.2.
34	12		$\lambda \equiv 0 \pmod 2$, $\lambda > 4$. Unknown.
34	13	52	$X = Z(2) \times Z(17,3)$. $\mathfrak{B} = \langle(\emptyset,\alpha), (\emptyset,\alpha+1), (\emptyset,\alpha+2), (\emptyset,\alpha+8), (\emptyset,\alpha+9), (\emptyset,\alpha+10), (0,\emptyset), (0,\alpha), (0,\alpha+1), (0,\alpha+3), (0,\alpha+8), (0,\alpha+9), (0,\alpha+11)\rangle$ mod $(-,17)$, $\alpha = 0,1,\dots,7$, $\langle(\emptyset,\beta), (\emptyset,\beta+2), (\emptyset,\beta+4), (\emptyset,\beta+6), (\emptyset,\beta+8), (\emptyset,\beta+10), (\emptyset,\beta+12), (\emptyset,\beta+14), (0,\emptyset), (0,\beta+1), (0,\beta+5), (0,\beta+9), (0,\beta+13)\rangle$ mod $(-,17)$, $\beta = 0,1,2,3$, $\langle(\emptyset,\emptyset), (\emptyset,2\beta+2), (\emptyset,2\beta+6), (\emptyset,2\beta+7), (\emptyset,2\beta+10), (\emptyset,2\beta+14), (\emptyset,2\beta+15), (0,2\beta+1), (0,2\beta+2), (0,2\beta+3), (0,2\beta+9), (0,2\beta+10), (0,2\beta+11)\rangle$ mod $(-,17)$, $\beta = 0,1,2,3$, $\langle(\emptyset,2\gamma+1), (\emptyset,2\gamma+2), (\emptyset,2\gamma+5), (\emptyset,2\gamma+6), (\emptyset,2\gamma+9), (\emptyset,2\gamma+10), (\emptyset,2\gamma+13), (\emptyset,2\gamma+14), (0,\emptyset), (0,2\gamma+2), (0,2\gamma+6), (0,2\gamma+10), (0,2\gamma+14)\rangle$ mod $(-,17)$, $\gamma = 0,1$, $\langle(\emptyset,\emptyset), (\emptyset,4\gamma+3), (\emptyset,4\gamma+11), (0,4\gamma+2), (0,4\gamma+10), (0,0), (0,1), (0,4), (0,5), (0,8), (0,9), (0,12), (0,13)\rangle$ mod $(-,17)$, $\gamma = 0,1$, $\langle(\emptyset,0), (\emptyset,1), (\emptyset,4), (\emptyset,5), (\emptyset,8), (\emptyset,9), (\emptyset,12), (\emptyset,13), (0,\emptyset), (0,1), (0,5), (0,9), (0,13)\rangle$ mod $(-,17)$, $\langle(\emptyset,\emptyset), (\emptyset,0), (\emptyset,4), (\emptyset,8), (\emptyset,12), (0,1), (0,2), (0,5), (0,6), (0,9), (0,10), (0,13), (0,14)\rangle$ mod $(-,17)$.
34	14	91	$X = Z(3,2) \times Z(11,2) \cup \{(\infty)\}$. $\mathfrak{B} = \langle(\infty), (\emptyset,2\alpha+1), (\emptyset,2\alpha+3), (\emptyset,2\alpha+5), (0,0), (0,2), (0,4), (0,6), (0,8), (1,0), (1,2), (1,4), (1,6), (1,8)\rangle$ mod $(3,11)$, $\alpha = 0,1,2,3,4$, $\langle(\infty), (\emptyset,\emptyset), (0,\emptyset), (0,0), (0,2), (0,4), (0,6), (0,8), (1,\emptyset), (1,0), (1,2), (1,4), (1,6), (1,8)\rangle$ mod $(3,11)$, 2 times, $\langle(\emptyset,2\alpha+1), (\emptyset,2\alpha+3), (\emptyset,2\alpha+5), (0,0), (0,2), (0,4), (0,6), (0,8), (1,0), (1,2), (1,4), (1,6), (1,8), (\beta,\emptyset)\rangle$ mod $(3,11)$, $\alpha = 0,1,2,3,4$, $\beta = 0,1$.
34	15	70	$X = Z(3,2) \times Z(11,2) \cup \{(\infty)\}$. $\mathfrak{B} = \langle(\infty), (\emptyset,2\alpha+1), (\emptyset,2\alpha+3), (0,\emptyset), (0,0), (0,2), (0,4), (0,6), (0,8), (1,\emptyset), (1,0), (1,2), (1,4), (1,6), (1,8)\rangle$ mod $(3,11)$, $\alpha = 0,1,2,3,4$, $\langle(\emptyset,2\alpha+2), (\emptyset,2\alpha+3), (\emptyset,2\alpha+4), (\emptyset,2\alpha+7), (\emptyset,2\alpha+8), (\emptyset,2\alpha+9), (0,\emptyset), (0,2\alpha), (0,2\alpha+1), (0,2\alpha+5), (0,2\alpha+6), (1,2\alpha), (1,2\alpha+1), (1,2\alpha+5), (1,2\alpha+6)\rangle$ mod $(3,11)$, $\alpha = 0,1,2,3,4$, $\langle(\emptyset,0), (\emptyset,2), (\emptyset,4), (\emptyset,6), (\emptyset,8), (0,0), (0,2), (0,4), (0,6), (0,8), (1,0), (1,2), (1,4), (1,6), (1,8)\rangle$ mod $(-,11)$, 4 times.

Table 5.23 (cont.).

v	k	λ	$B[k,\lambda;v]$
34	16	40	$X = Z(3,2)\times Z(11,2)\cup\{(\infty)\}$. $\mathcal{B} = \langle(\infty),(\emptyset,0),(\emptyset,2),(\emptyset,4),(\emptyset,6),(\emptyset,8),(0,1),(0,3),(0,5),(0,7),(0,9),$ $(1,1),(1,3),(1,5),(1,7),(1,9)\rangle \bmod (3,11)$, 2 times, $\langle(\infty),(\emptyset,2\alpha),(\emptyset,2\alpha+1),(\emptyset,2\alpha+2),(\emptyset,2\alpha+3),(\emptyset,4\alpha+5),(0,2\alpha),(0,2\alpha+1),$ $(0,2\alpha+2),(0,2\alpha+3),(0,4\alpha+5),(1,2\alpha),(1,2\alpha+1),(1,2\alpha+2)\ (1,2\alpha+3),$ $(1,4\alpha+5)\rangle \bmod (-,11)$, $\alpha = 0,1$, $\langle(\emptyset,\alpha),(\emptyset,\alpha+1),(\emptyset,\alpha+2),(\emptyset,\alpha+4),(\emptyset,\alpha+5),(\emptyset,\alpha+6),(\emptyset,\alpha+7),(\emptyset,\alpha+9),$ $(0,\alpha),(0,\alpha+1),(0,\alpha+5),(0,\alpha+6),(1,\alpha+2),(1,\alpha+3),(1,\alpha+7),$ $(1,\alpha+8)\rangle \bmod (3,11)$, $\alpha = 0,1$, $\langle(\emptyset,1),(\emptyset,2),(\emptyset,3),(\emptyset,4),(\emptyset,6),(\emptyset,7),(\emptyset,8),(\emptyset,9),(0,0),(0,4),(0,5),$ $(0,9),(1,1),(1,4),(1,6),(1,9)\rangle \bmod (3,11)$.
34	17	16	Form $B[33,16;67]$ with $X = (67,2)$ and $\mathcal{B} = \langle 0,2,4,6,\ldots,64\rangle$ and delete any one block and all its points.
35	7	3	Unknown.
35	7	9	$X = Z(5,2)\times Z(7,3)$. $\mathcal{B} = \langle(\emptyset,0),(\emptyset,2),(\emptyset,4),(0,2\alpha+1),(1,2\alpha+3),(2,2\alpha+1),(3,2\alpha+3)\rangle \bmod (5,7)$, 2 times, $\alpha = 0,1,2$, $\langle(\emptyset,0),(\emptyset,2),(\emptyset,4),(0,\emptyset),(1,\emptyset),(2,\emptyset),(3,\emptyset)\rangle \bmod (5,7)$, $\langle(\emptyset,\emptyset),(\emptyset,0),(\emptyset,1),(\emptyset,2),(\emptyset,3),(\emptyset,4),(\emptyset,5)\rangle \bmod (5,-)$, 2 times.
35	7	$\lambda\equiv 0 \pmod 3$, $\lambda > 3$.	Lemma 2.4, $35\in B(7,6)$ by Lemma 5.39 and $35\in B(7,9)$.
35	8	28	$X = Z(5,2)\times Z(7,3)$. $\mathcal{B} = \langle(\beta,0),(\beta,2),(\beta,4),(\beta+1,\emptyset),(\beta+2,0),(\beta+2,2),(\beta+2,4),(\beta+3,\emptyset)\rangle$ $\bmod (5,7)$, 4 times, $\beta = 0,1$, $\langle(\beta,\alpha),(\beta,\alpha+3),(\beta+1,\alpha+1),(\beta+1,\alpha+4),(\beta+2,\alpha),(\beta+2,\alpha+3),$ $(\beta+3,\alpha+2),(\beta+3,\alpha+5)\rangle \bmod (5,7)$, $\alpha = 0,1,2$, $\beta = 0,1$, $\langle(0,\alpha),(0,\alpha+3),(1,\alpha+1),(1,\alpha+4),(2,\alpha),(2,\alpha+3),(3,\alpha+1),(3,\alpha+4)\rangle$ $\bmod (5,7)$, $\alpha = 0,1,2$.
35	9	36	$X = Z(5,2)\times Z(7,3)$. $\mathcal{B} = \langle(\emptyset,\gamma),(\emptyset,\gamma+2),(\emptyset,\gamma+4),(\beta,0),(\beta,2),(\beta,4),(\beta+2,0),(\beta+2,2),(\beta+2,4)\rangle$ $\bmod (5,7)$, 2 times, $\beta = 0,1$, $\gamma = 0,1$, $\langle(\emptyset,\emptyset),(\beta,\alpha),(\beta,\alpha+3),(\beta+1,\alpha+1),(\beta+1,\alpha+4),(\beta+2,\alpha),(\beta+2,\alpha+3),$ $(\beta+3,\alpha+2),(\beta+3,\alpha+5)\rangle \bmod (5,7)$, $\alpha = 0,1,2$, $\beta = 0,1$, $\langle(\emptyset,\emptyset),(0,\alpha),(0,\alpha+3),(1,\alpha+1),(1,\alpha+4),(2,\alpha),(2,\alpha+3),(3,\alpha+1),$ $(3,\alpha+4)\rangle \bmod (5,7)$, $\alpha = 0,1,2$.
35	10	9	$(GF(4, x^2 = x+1)\cup\{\infty\})\times Z(7,3)$. $\mathcal{B} = \langle(\infty,0),(\infty,1),(\infty,2),(\infty,3),(\infty,4),(\infty,5),(\emptyset,\emptyset),(0,\emptyset),(1,\emptyset),(2,\emptyset)\rangle$ $\bmod (-,7)$, $\langle(\infty,2\alpha),(\infty,2\alpha+3),(\emptyset,2\alpha+1),(\emptyset,2\alpha+5),(0,2\alpha),(0,2\alpha+2),(1,2\alpha+2),$ $(1,2\alpha+4),(2,2\alpha),(2,2\alpha+4)\rangle \bmod (4,7)$, $\alpha = 0,1,2$, $\langle(\infty,\emptyset),(\emptyset,0),(\emptyset,1),(\emptyset,2),(\emptyset,3),(\emptyset,4),(\emptyset,5),(0,\emptyset),(1,\emptyset),(2,\emptyset)\rangle$ $\bmod (4,7)$.
35	11	55	$X = Z(5,2)\times Z(7,3)$. $\mathcal{B} = \langle(\emptyset,1),(\emptyset,3),(\emptyset,5),(\alpha,0),(\alpha,2),(\alpha,4),(\alpha+1,\emptyset),(\alpha+2,0),(\alpha+2,2),$ $(\alpha+2,4),(\alpha+3,\emptyset)\rangle \bmod (5,7)$, 5 times, $\alpha = 0,1$, $\langle(\emptyset,0),(\emptyset,2),(\emptyset,4),(\alpha,0),(\alpha,2),(\alpha,4),(\alpha+1,\emptyset),(\alpha+2,0),(\alpha+2,2),$ $(\alpha+2,4),(\alpha+3,\emptyset)\rangle \bmod (5,7)$, 3 times, $\alpha = 0,1$, $\langle(\emptyset,\emptyset),(\emptyset,0),(\emptyset,1),(\emptyset,2),(\emptyset,3),(\emptyset,4),(\emptyset,5),(0,\emptyset),(1,\emptyset),(2,\emptyset),(3,\emptyset)\rangle$ $\bmod (5,7)$.

Table 5.23 (cont.).

v	k	λ	$B[k, \lambda; v]$
35	12	66	$X = Z(5,2) \times Z(7,3)$. $\mathfrak{B} = \langle(\emptyset,\emptyset), (\emptyset,1), (\emptyset,3), (\emptyset,5), (\alpha,0), (\alpha,2), (\alpha,4), (\alpha+1,\emptyset), (\alpha+2,0),$ $(\alpha+2,2), (\alpha+2,4), (\alpha+3,\emptyset)\rangle \bmod (5,7)$, 6 times, $\alpha = 0, 1$, $\langle(\emptyset,\emptyset), (\emptyset,0), (\emptyset,2), (\emptyset,4), (\alpha,0), (\alpha,2), (\alpha,4), (\alpha+1,\emptyset), (\alpha+2,0),$ $(\alpha+2,2), (\alpha+2,4), (\alpha+3,\emptyset)\rangle \bmod (5,7)$, $\alpha = 0, 1$, $\langle(\emptyset,\beta), (\emptyset,\beta+1), (\emptyset,\beta+3), (\emptyset,\beta+4), (0,\beta), (0,\beta+3), (1,\beta+1), (1,\beta+4),$ $(2,\beta), (2,\beta+3), (3,\beta+1), (3,\beta+4)\rangle \bmod (5,7)$, $\beta = 0, 1, 2$.
35	13	78	$X = Z(5,2) \times Z(7,3)$. $\mathfrak{B} = \langle(\emptyset,1), (\emptyset,3), (\emptyset,5), (\alpha,\emptyset), (\alpha,0), (\alpha,2), (\alpha,4), (\alpha+1,\emptyset), (\alpha+2,\emptyset),$ $(\alpha+2,0), (\alpha+2,2), (\alpha+2,4), (\alpha+3,\emptyset)\rangle \bmod (5,7)$, 5 times, $\alpha = 0, 1$, $\langle(\emptyset,\emptyset), (0,0), (0,2), (0,4), (1,1), (1,3), (1,5), (2,0), (2,2), (2,4),$ $(3,1), (3,3), (3,5)\rangle \bmod (5,7)$, 5 times, $\langle(\emptyset,\emptyset), (0,0), (0,2), (0,4), (1,0), (1,2), (1,4), (2,0), (2,2), (2,4),$ $(3,0), (3,2), (3,4)\rangle \bmod (5,7)$, 2 times.
35	14	13	Unknown.
35	14	26	$X = (Z(3,2) \cup \{(\infty_i): i = 0, 1\}) \times Z(7,3)$. $\mathfrak{B} = \langle(\infty_0,\emptyset), (\infty_0,0), (\infty_0,1), (\infty_0,2), (\infty_0,3), (\infty_0,4), (\infty_0,5), (\infty_1,\emptyset),$ $(\infty_1,0), (\infty_1,1), (\infty_1,2), (\infty_1,3), (\infty_1,4), (\infty_1,5)\rangle$, 2 times, $\langle(\infty_0,\emptyset), (\infty_1,\alpha), (\infty_1,\alpha+2), (\infty_1,\alpha+4), (\emptyset,\emptyset), (\emptyset,\alpha+1), (\emptyset,\alpha+3),$ $(\emptyset,\alpha+5), (0,\alpha), (0,\alpha+2), (0,\alpha+4), (1,\alpha), (1,\alpha+2), (1,\alpha+4)\rangle$ $\bmod (3,7)$, $\alpha = 0, 1$, $\langle(\infty_0,\emptyset), (\infty_0,\alpha), (\infty_0,\alpha+2), (\infty_0,\alpha+4), (\infty_1,\emptyset), (\emptyset,\emptyset), (0,\emptyset), (0,\alpha+1),$ $(0,\alpha+3), (0,\alpha+5), (1,\emptyset), (1,\alpha+1), (1,\alpha+3), (1,\alpha+5)\rangle \bmod (3,7)$, $\alpha = 0, 1$, $\langle(\infty_0,\alpha), (\infty_0,\alpha+2), (\infty_0,\alpha+4), (\infty_1,\alpha), (\infty_1,\alpha+2), (\infty_1,\alpha+4), (\emptyset,\emptyset),$ $(0,\alpha+1), (0,\alpha+3), (0,\alpha+5), (1,\emptyset), (1,\alpha), (1,\alpha+2), (1,\alpha+4)\rangle \bmod (3,7)$, $\alpha = 0, 1$, $\langle(\infty_0,\alpha), (\infty_0,\alpha+2), (\infty_0,\alpha+4), (\infty_1,\emptyset), (\infty_1,\alpha+1), (\infty_1,\alpha+3), (\infty_1,\alpha+5),$ $(\emptyset,\emptyset), (0,\alpha), (0,\alpha+2), (0,\alpha+4), (1,\alpha), (1,\alpha+2), (1,\alpha+4)\rangle \bmod (3,7)$, $\alpha = 0, 1$.
35	14	39	$X = (Z(3,2) \cup \{(\infty_i): i = 0, 1\}) \times Z(7,3)$. $\mathfrak{B} = \langle(\infty_0,\emptyset), (\infty_0,0), (\infty_0,1), (\infty_0,2), (\infty_0,3), (\infty_0,4), (\infty_0,5), (\infty_1,\emptyset),$ $(\infty_1,0), (\infty_1,1), (\infty_1,2), (\infty_1,3), (\infty_1,4), (\infty_1,5)\rangle$, 3 times, $\langle(\infty_0,\emptyset), (\infty_1,\emptyset), (\infty_1,\alpha), (\infty_1,\alpha+3), (\emptyset,\alpha), (\emptyset,\alpha+2), (\emptyset,\alpha+3), (\emptyset,\alpha+5),$ $(0,\emptyset), (0,\alpha+2), (0,\alpha+5), (1,\emptyset), (1,\alpha), (1,\alpha+3)\rangle \bmod (3,7)$, $\alpha = 0, 1, 2$, $\langle(\infty_0,\alpha+1), (\infty_0,\alpha+2), (\infty_0,\alpha+4), (\infty_0,\alpha+5), (\infty_1,\emptyset), (\emptyset,\emptyset), (0,\alpha),$ $(0,\alpha+1), (0,\alpha+3), (0,\alpha+4), (1,\alpha), (1,\alpha+2), (1,\alpha+3), (1,\alpha+5)\rangle$ $\bmod (3,7)$, $\alpha = 0, 1, 2$, $\langle(\infty_0,\emptyset), (\infty_0,\alpha+1), (\infty_0,\alpha+4), (\infty_1,\emptyset), (\infty_1,\alpha+2), (\infty_1,\alpha+5), (\emptyset,\emptyset),$ $(0,\emptyset), (0,\alpha), (0,\alpha+3), (1,\alpha), (1,\alpha+2), (1,\alpha+3), (1,\alpha+5)\rangle \bmod (3,7)$, $\alpha = 0, 1, 2$, $\langle(\infty_0,\emptyset), (\infty_0,\alpha+2), (\infty_0,\alpha+5), (\infty_1,\alpha), (\infty_1,\alpha+2), (\infty_1,\alpha+3), (\infty_1,\alpha+5),$ $(\emptyset,\emptyset), (0,\emptyset), (0,\alpha+1), (0,\alpha+4), (1,\emptyset), (1,\alpha+2), (1,\alpha+5)\rangle \bmod (3,7)$, $\alpha = 0, 1, 2$.
35	14	$\lambda \equiv 0 \pmod{13}$, $\lambda > 13$.	Lemma 2.4, $35 \in B(14, 26)$ and $35 \in B(14, 39)$.

Table 5.23 (cont.).

v	k	λ	$B[k, \lambda; v]$
35	15	21	$X = (Z(4) \cup \{\infty\}) \times Z(7,3)$. $\mathcal{B} = \langle(\infty,\emptyset),(\infty,0),(\infty,2),(\infty,4),(0',\emptyset),(0',1),(0',3),(0',5),(1',0),(1',2),$ $(1',4),(2',1),(2',3),(2',5),(3',\emptyset)\rangle \bmod (4,7),$ $\langle(\infty,\emptyset),(\infty,0),(\infty,2),(\infty,4),(0',\emptyset),(0',0),(0',2),(0',4),(1',1),(1',3),$ $(1',5),(2',\emptyset),(3',1),(3',3),(3',5)\rangle \bmod (4,7),$ $\langle(\infty,1),(\infty,3),(\infty,5),(0',\emptyset),(0',1),(0',3),(0',5),(1',\emptyset),(1',1),(1',3),$ $(1',5),(2',\emptyset),(2',1),(2',3),(2',5)\rangle \bmod (4,7),$ $\langle(\infty,\emptyset),(0',\emptyset),(0',0),(0',2),(0',4),(1',0),(1',2),(1',4),(2',\emptyset),(2',1),$ $(2',3),(2',5),(3',1),(3',3),(3',5)\rangle \bmod (4,7),$ $\langle(\infty,1),(\infty,3),(\infty,5),(0',0),(0',2),(0',4),(1',0),(1',2),(1',4),(2',0),$ $(2',2),(2',4),(3',0),(3',2),(3',4)\rangle \bmod (-,7).$
35	16	120	Lemma 2.6, $35 \in B(17,8)$ as below and $17 \in B(16,15)$ trivially.
35	17	8	[29]. $X = Z(5,2) \times Z(7,3)$. $\mathcal{B} = \langle(\emptyset,\emptyset),(0,\emptyset),(0,0),(0,2),(0,4),(1,\emptyset),(1,1),(1,3),(1,5),(2,\emptyset),(2,0),$ $(2,2),(2,4),(3,\emptyset),(3,1),(3,3),(3,5)\rangle \bmod (5,7).$
36	8	2	Non-existing by Lemma 1.3.
36	8	$\lambda \equiv 0$	(mod 2), $\lambda > 2$. Unknown.
36	9	8	$X = Z(5,2) \times Z(7,3) \cup \{(\infty)\}$. $\mathcal{B} = \langle(\infty),(\emptyset,0),(\emptyset,2),(\emptyset,3),(\emptyset,5),(0,1),(1,4),(2,1),(3,4)\rangle \bmod (5,7),$ $\langle(\emptyset,\emptyset),(\emptyset,0),(\emptyset,2),(\emptyset,3),(\emptyset,5),(0,1),(1,4),(2,1),(3,4)\rangle \bmod (5,7),$ $\langle(\emptyset,\emptyset),(0,2),(0,4),(1,\emptyset),(1,0),(2,2),(2,4),(3,\emptyset),(3,0)\rangle \bmod (5,7),$ $\langle(\emptyset,\emptyset),(0,\emptyset),(0,4),(1,1),(1,5),(2,\emptyset),(2,4),(3,1),(3,5)\rangle \bmod (5,7).$
36	10	9	$X = (Z(3,2) \cup \{\infty\} \times GF(9, x^2 = 2x+1)$. $\mathcal{B} = \langle(\infty,\alpha+3),(\infty,\alpha+7),(\emptyset,0),(\emptyset,3),(\emptyset,4),(\emptyset,7),(0,1),(0,5),(1,2),(1,6)\rangle$ $\bmod (3,9), \quad \alpha = 0,1,$ $\langle(\infty,\emptyset),(\emptyset,\emptyset),(0,1),(0,2),(0,5),(0,6),(1,1),(1,2),(1,5),(1,6)\rangle$ $\bmod (3,9),$ $\langle(\infty,0),(\infty,3),(\infty,4),(\infty,7),(0,\emptyset),(0,0),(0,4),(1,\emptyset),(1,3),(1,7)\rangle$ $\bmod (3,9),$ $\langle(\infty,\alpha),(\infty,\alpha+2),(\infty,\alpha+4),(\infty,\alpha+6),(\emptyset,\alpha+1),(\emptyset,\alpha+5),(0,\alpha+1),$ $(0,\alpha+5),(1,\alpha+1),(1,\alpha+5)\rangle \bmod (-,9), \quad \alpha = 0,1.$
36	11	22	$X = Z(5,2) \times Z(7,3) \cup \{(\infty)\}$. $\mathcal{B} = \langle(\infty),(\emptyset,0),(\emptyset,2),(\emptyset,3),(\emptyset,5),(\beta,1),(\beta,4),(\beta+1,\emptyset),(\beta+2,1),$ $(\beta+2,4),(\beta+3,\emptyset)\rangle \bmod (5,7), \quad \beta = 0,1,$ $\langle(\infty),(\emptyset,2),(\emptyset,5),(0,2),(0,5),(1,2),(1,5),(2,2),(2,5),(3,2),(3,5)\rangle$ $\bmod (-,7),$ $\langle(\emptyset,\emptyset),(\emptyset,\alpha),(\emptyset,\alpha+3),(0,\alpha+1),(0,\alpha+4),(1,\alpha+2),(1,\alpha+5),(2,\alpha+1),$ $(2,\alpha+4),(3,\alpha+2),(3,\alpha+5)\rangle \bmod (5,7), \quad \alpha = 0,1,2,$ $\langle(\emptyset,\emptyset),(\emptyset,2),(\emptyset,5),(0,0),(0,3),(1,0),(1,3),(2,0),(2,3),(3,0),(3,3)\rangle$ $\bmod (5,7),$ $\langle(\emptyset,\emptyset),(\emptyset,0),(\emptyset,1),(\emptyset,2),(\emptyset,3),(\emptyset,4),(\emptyset,5),(0,\emptyset),(1,\emptyset),(2,\emptyset),(3,\emptyset)\rangle$ $\bmod (5,7).$
36	12	11	$X = Z(5,2) \times Z(7,3) \cup \{(\infty)\}$. $\mathcal{B} = \langle(\infty),(\emptyset,\emptyset),(0,0),(0,3),(0,4),(0,5),(1,\emptyset),(2,0),(2,3),(2,4)$ $(2,5),(3,\emptyset)\rangle \bmod (5,7),$ $\langle(\emptyset,0),(\emptyset,3),(0,0),(0,3),(1,\emptyset),(1,1),(1,4),(2,0),(2,3),(3,\emptyset),$ $(3,1),(3,4)\rangle \bmod (5,7),$ $\langle(\emptyset,1),(\emptyset,3),(\emptyset,4),(\emptyset,5),(0,0),(0,2),(0,4),(1,\emptyset),(2,0),(2,2),$ $(2,4),(3,\emptyset)\rangle \bmod (5,7).$

Table 5.23 (cont.).

v	k	λ	$B[k, \lambda; v]$
36	13	156	$X = Z(5,2) \times Z(7,3) \cup \{(\infty)\}$. $\mathfrak{B} = \langle(\infty), (0,0), (0,2), (0,4), (1,1), (1,3), (1,5), (2,0), (2,2), (2,4), (3,1),$ $(3,3), (3,5)\rangle \bmod (5,7)$, 10 times, $\langle(\infty), (0,0), (0,2), (0,4), (1,0), (1,2), (1,4), (2,0), (2,2), (2,4), (3,0),$ $(3,2), (3,4)\rangle \bmod (5,7)$, 3 times, $\langle(\emptyset,\emptyset), (\emptyset,2\alpha), (\emptyset,2\alpha+1), (\emptyset,2\alpha+3), (\emptyset,2\alpha+4), (\beta,2\alpha), (\beta,2\alpha+3),$ $(\beta+1,2\alpha+2), (\beta+1,2\alpha+5), (\beta+2,2\alpha), (\beta+2,2\alpha+3), (\beta+3,2\alpha+2),$ $(\beta+3,2\alpha+5)\rangle \bmod (5,7)$, 2 times, $\alpha = 0,1,2$, $\beta = 0,1$, $\langle(\emptyset,\emptyset), (\emptyset,2\alpha), (\emptyset,2\alpha+1), (\emptyset,2\alpha+3), (\emptyset,2\alpha+4), (0,2\alpha+2), (0,2\alpha+5),$ $(1,2\alpha+2), (1,2\alpha+5), (2,2\alpha+2), (2,2\alpha+5), (3,2\alpha+2), (3,2\alpha+5)\rangle$ $\bmod (5,7)$, 2 times, $\alpha = 0,1,2$, $\langle(\emptyset,\emptyset), (0,0), (0,2), (0,4), (1,1), (1,3), (1,5), (2,0), (2,2), (2,4), (3,1),$ $(3,3), (3,5)\rangle \bmod (5,7)$, 5 times.
36	14	13	$X = (Z(3,2) \cup \{\infty\}) \times GF(9, x^2 = 2x+1)$. $\mathfrak{B} = \langle(\infty,\emptyset), (\infty,1), (\infty,3), (\infty,5), (\infty,7), (\emptyset,\emptyset), (0,0), (0,2), (0,4), (0,6),$ $(1,0), (1,2), (1,4), (1,6)\rangle \bmod (3,9)$, $\langle(\infty,1), (\infty,3), (\infty,5), (\infty,7), (\emptyset,\emptyset), (0,\emptyset), (0,0), (0,2), (0,4), (0,6),$ $(1,1), (1,3), (1,5), (1,7)\rangle \bmod (3,9)$, $\langle(\infty,\alpha), (\infty,\alpha+4), (\emptyset,\alpha), (\emptyset,\alpha+1), (\emptyset,\alpha+4), (\emptyset,\alpha+5), (0,\alpha), (0,\alpha+2),$ $(0,\alpha+4), (0,\alpha+6), (1,\alpha), (1,\alpha+3), (1,\alpha+4), (1,\alpha+7)\rangle \bmod (-,9)$, $\alpha = 1,2,3$.
36	15	2	Non-existing by Lemma 1.1.
36	15	4	Non-existing by Lemma 1.1.
36	15	6	[31]. $X = GF(4, x^2 = x+1) \times GF(9, x^2 = 2x+1)$. $\mathfrak{B} = \langle(\emptyset,0), (\emptyset,1), (\emptyset,2), (\emptyset,4), (\emptyset,5), (\emptyset,6), (0,\emptyset), (0,0), (0,4), (1,\emptyset),$ $(1,1), (1,5), (2,\emptyset), (2,2), (2,6)\rangle \bmod (4,9)$.
36	15	8	[29]. $X = GF(4, x^2 = x+1) \times Z(3,2) \times Z(3,2)$. $\mathfrak{B} = \langle(\emptyset,\emptyset,\emptyset), (\emptyset,\emptyset,0), (\emptyset,0,\emptyset), (\emptyset,0,0), (\emptyset,1,0), (\emptyset,1,1), (0,\emptyset,\emptyset),$ $(0,0,\emptyset), (0,1,0), (1,\emptyset,\emptyset), (1,0,\emptyset), (1,1,0), (2,\emptyset,1), (2,0,1),$ $(2,1,\emptyset)\rangle \bmod (4,3,3)$, $\langle(\emptyset,\emptyset,\emptyset), (\emptyset,\emptyset,0), (\emptyset,\emptyset,1), (\emptyset,0,\emptyset), (\emptyset,0,0), (\emptyset,0,1), (0,\emptyset,\emptyset),$ $(0,\emptyset,0), (0,\emptyset,1), (1,\emptyset,\emptyset), (1,\emptyset,0), (1,\emptyset;1), (2,1,\emptyset), (2,1,0),$ $(2,1,1)\rangle \bmod (4,3,-)$.
36	15	10	$X = (Z(5,2) \cup \{(\infty_i): i = 0,1\}) \times Z(5,2) \cup \{(\infty,\infty)\}$. $\mathfrak{B} = \langle(\infty,\infty), (\infty_0,1), (\infty_0,3), (\infty_1,0), (\infty_1,2), (0,1), (0,3), (1,\emptyset), (1,0),$ $(1,2), (2,1), (2,3), (3,\emptyset), (3,0), (3,2)\rangle \bmod (5,5)$, $\langle(\infty_0,\emptyset), (\infty_0,1), (\infty_0,3), (\infty_1,1), (\infty_1,3), (0,\emptyset), (0,1), (0,3), (1,0),$ $(1,2), (2,\emptyset), (2,1), (2,3), (3,0), (3,2)\rangle \bmod (5,5)$, $\langle(\infty_1,\emptyset), (\infty_1,0), (\infty_1,1), (\infty_1,2), (\infty_1,3), (\emptyset,1), (\emptyset,3), (0,1), (0,3),$ $(1,1), (1,3), (2,1), (2,3), (3,1), (3,3)\rangle \bmod (-,5)$, $\langle(\emptyset,\emptyset, (\emptyset,1), (\emptyset,3), (0,\emptyset), (0,1), (0,3), (1,\emptyset), (1,1), (1,3), (2,\emptyset),$ $(2,1), (2,3), (3,\emptyset), (3,1), (3,3)\rangle \bmod (-,5)$,
36	15		$\lambda \equiv 0 \pmod 2$, $\lambda > 4$. Lemma 2.4, $36 \in B(15,6)$, $36 \in B(15,8)$ and $36 \in B(15,10)$.
36	16	12	$X = (GF(4, x^2 = x+1) \cup \{\infty\}) \times Z(7,3) \cup \{(\infty,\infty)\}$. $\mathfrak{B} = \langle(\infty,\infty), (\infty,1), (\infty,3), (\infty,5), (\emptyset,1), (\emptyset,3), (\emptyset,5), (0,\emptyset), (0,0), (0,5),$ $(1,\emptyset), (1,2), (1,1), (2,\emptyset), (2,4), (2,3)\rangle \bmod (4,7)$, $\langle(\infty,\emptyset), (\infty,0), (\infty,2), (\infty,4), (\emptyset,\emptyset), (\emptyset,0), (\emptyset,1), (\emptyset,3), (0,\emptyset), (0,2),$ $(0,3), (0,5), (2,\emptyset), (2,1), (2,4), (2,5)\rangle \bmod (4,7)$, $\langle(\emptyset,\emptyset), (\emptyset,0), (\emptyset,2), (\emptyset,4), (0,\emptyset); (0,0), (0,2), (0,4), (1,\emptyset), (1,0), (1,2),$ $(1,4), (2,\emptyset), (2,0), (2,2), (2,4)\rangle \bmod (-,7)$.

Table 5.23 (cont.).

v	k	λ	$B[k, \lambda; v]$
36	17	272	Lemma 4.6, $35 \in B(16, 120)$ and $35 \in B(17, 8)$.
36	18	17	$X = Z(5, 2) \times Z(7, 3) \cup \{(\infty)\}$. $\mathcal{B} = \langle(\infty), (\emptyset, \emptyset), (0, \emptyset), (0, 0), (0, 2), (0, 4), (1, \emptyset), (1, 1), (1, 3), (1, 5), (2, \emptyset),$ $(2, 0), (2, 2), (2, 4), (3, \emptyset), (3, 1), (3, 3), (3, 5)\rangle \bmod (5, 7),$ $\langle(\emptyset, 0), (\emptyset, 1), (\emptyset, 2), (\emptyset, 3), (\emptyset, 4), (\emptyset, 5), (0, 0), (0, 2), (0, 4), (1, 1),$ $(1, 3), (1, 5), (2, 0), (2, 2), (2, 4), (3, 1), (3, 3), (3, 5)\rangle \bmod (5, 7)$.
37	8	14	Lemma 4.1.
37	9	2	[12]. $X = Z(37, 2)$. $\mathcal{B} = \langle 0, 4, 8, 12, 16, 20, 24, 28, 32\rangle \bmod 37$.
37	10	5	Lemma 4.4.
37	11	55	Lemma 4.2.
37	12	11	Lemma 4.1.
37	13	13	Lemma 4.2.
37	14	91	Lemma 4.1.
37	15	35	Lemma 4.3.
37	16	20	$X = Z(37, 2)$. $\mathcal{B} = \langle\emptyset, 4\alpha, 4\alpha+1, 4\alpha+2, 4\alpha+3, 4\alpha+4, 4\alpha+12, 4\alpha+13, 4\alpha+14, 4\alpha+15, 4\alpha+16,$ $4\alpha+24, 4\alpha+25, 4\alpha+26, 4\alpha+27, 4\alpha+28\rangle \bmod 37, \quad \alpha = 0, 1, 2$.
37	17	68	Lemma 4.2.
37	18	17	Lemma 4.1.
38	8	28	$X = Z(37, 2) \cup \{\infty\}$. $\mathcal{B} = \langle\infty, \emptyset, 3\alpha+\beta+1, 3\alpha+\beta+7, 3\alpha+\beta+13, 3\alpha+\beta+19, 3\alpha+\beta+25, 3\alpha+\beta+31\rangle$ $\bmod 37, \quad \alpha = 0, 1, \quad \beta = 0, 1,$ $\langle 2\gamma, 2\gamma+2, 2\gamma+6, 2\gamma+8, 2\gamma+18, 2\gamma+20, 2\gamma+24, 2\gamma+26\rangle \bmod 37,$ $\gamma = 0, 1, \ldots, 8,$ $\langle 3\delta, 3\delta+2, 3\delta+6, 3\delta+10, 3\delta+18, 3\delta+20, 3\delta+24, 3\delta+28\rangle \bmod 37,$ $\delta = 0, 1, \ldots, 5$.
38	9	72	Lemma 4.6, $37 \in B(8, 14)$ and $37 \in B(9, 2)$.
38	10	45	Lemma 4.6, $37 \in B(9, 2)$ and $37 \in B(10, 5)$.
38	11	110	$X = Z(37, 2) \cup \{\infty\}$. $\mathcal{B}$ = Blocks of $B[10, 5; 37]$ on $Z(37)$ with ∞ adjoint to each block, $\langle\infty, \alpha, \alpha+1, \alpha+2, \alpha+9, \alpha+10, \alpha+18, \alpha+19, \alpha+27, \alpha+28, \alpha+29\rangle \bmod 37,$ $\alpha = 0, 1, \ldots, 8,$ $\langle\emptyset, \alpha, \alpha+1, \alpha+9, \alpha+10, \alpha+18, \alpha+19, \alpha+27, \alpha+28, \alpha+9\beta+2, \alpha+9\beta+11\rangle$ $\bmod 37, \quad \alpha = 0, 1, \ldots, 8, \quad \beta = 0, 1, 2$.
38	12	66	$X = Z(37, 2) \cup \{\infty\}$. $\mathcal{B} = \langle\infty, 3\alpha, 3\alpha+3, 3\alpha+6, 3\alpha+9, 3\alpha+12, 3\alpha+15, 3\alpha+18, 3\alpha+21, 3\alpha+24,$ $3\alpha+27, 3\alpha+30\rangle \bmod 37, \quad \alpha = 0, 1, \ldots, 5,$ $\langle\mu, \mu+3, \mu+6, \mu+9, \mu+12, \mu+15, \mu+18, \mu+21, \mu+24, \mu+27, \mu+30, \mu+33\rangle$ $\bmod 37$, once for $\mu = 0$, 6 times for $\mu = 1, 2$.
38	13	156	$X = Z(37, 2) \cup \{\infty\}$. $\mathcal{B} = \langle\infty, \mu, \mu+3, \mu+6, \mu+9, \mu+12, \mu+15, \mu+18, \mu+21, \mu+24, \mu+27, \mu+30,$ $\mu+33\rangle \bmod 37$, 4 times for $\mu = 0, 2$, 5 times for $\mu = 1$, $\langle\emptyset, \nu, \nu+3, \nu+6, \nu+9, \nu+12, \nu+15, \nu+18, \nu+21, \nu+24, \nu+27, \nu+30, \nu+33\rangle$ $\bmod 37$, 10 times for $\nu = 0$, 7 times for $\nu = 1$, 8 times for $\nu = 2$.
38	14	91	$X = Z(37, 2) \cup \{\infty\}$. $\mathcal{B}$ = Blocks of $B[13, 13; 37]$ on $Z(37)$ with ∞ adjoint to each block, 2 times, $\langle\infty, \emptyset, 0, 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33\rangle \bmod 37,$ $\langle 3\alpha+\beta+2, 3\alpha+\beta+3, 3\alpha+\beta+4, 3\alpha+\beta+5, 3\alpha+\beta+6, 3\alpha+\beta+7, 3\alpha+\beta+16,$ $3\alpha+\beta+20, 3\alpha+\beta+21, 3\alpha+\beta+22, 3\alpha+\beta+23, 3\alpha+\beta+24, 3\alpha+\beta+25,$ $3\alpha+\beta+34\rangle \bmod 37, \quad \alpha = 0, 1, \ldots, 5, \quad \beta = 0, 1$.

Table 5.23 (cont.).

v	k	λ	$B[k, \lambda; v]$
38	15	210	$X = Z(37,2) \cup \{\infty\}$. $\mathfrak{B}$ = Blocks of $B[15,35;37]$ on $Z(37)$, $\langle \infty, 2\alpha, 2\alpha+2, 2\alpha+3, 2\alpha+6, 2\alpha+9, 2\alpha+12, 2\alpha+15, 2\alpha+18, 2\alpha+20, 2\alpha+21, 2\alpha+24, 2\alpha+27, 2\alpha+30, 2\alpha+33 \rangle$ mod 37, $\alpha = 0, 1, \ldots, 8$, $\langle \infty, \emptyset, 3\beta, 3\beta+1, 3\beta+2, 3\beta+3, 3\beta+4, 3\beta+12, 3\beta+13, 3\beta+14, 3\beta+15, 3\beta+24, 3\beta+25, 3\beta+26, 3\beta+27 \rangle$ mod 37, $\beta = 0, 1, \ldots, 5$, $\langle \mu, \mu+1, \mu+2, \mu+3, \mu+4, \mu+12, \mu+13, \mu+14, \mu+15, \mu+16, \mu+24, \mu+25, \mu+26, \mu+27, \mu+28 \rangle$ mod 37, 3 times for $\mu = 1, 3$, 2 times for $\mu = 0, 4$, once for $\mu = 5$, $\langle \nu, \nu+1, \nu+2, \nu+3, \nu+5, \nu+12, \nu+13, \nu+14, \nu+15, \nu+17, \nu+24, \nu+25, \nu+26, \nu+27, \nu+29 \rangle$ mod 37, once for $\nu = 0, 3$, 2 times for $\nu = 1, 4$.
38	16	120	$X = Z(37,2) \cup \{\infty\}$. $\mathfrak{B} = \langle \infty, \mu, \mu+1, \mu+2, \mu+3, \mu+4, \mu+12, \mu+13, \mu+14, \mu+15, \mu+16, \mu+24, \mu+25, \mu+26, \mu+27, \mu+28 \rangle$ mod 37, $\mu = 0, 5, 6, 11, 12, 13, 14, 17$, $\langle \emptyset, \nu, \nu+1, \nu+2, \nu+3, \nu+4, \nu+12, \nu+13, \nu+14, \nu+15, \nu+16, \nu+24, \nu+25, \nu+26, \nu+27, \nu+28 \rangle$ mod 37, $\nu = 0, 1, 2, 3, 4, 7, 8, 9, 10, 15, 16$.
38	17	272	$X = Z(37,2) \cup \{\infty\}$. $\mathfrak{B}$ = Blocks of $B[16,20;37]$ on $Z(37)$ with ∞ adjoint to each block, 5 times, Blocks of $B[17,68;37]$ on $Z(37)$, $\langle \infty, \emptyset, 3\alpha, 3\alpha+1, 3\alpha+2, 3\alpha+3, 3\alpha+4, 3\alpha+12, 3\alpha+13, 3\alpha+14, 3\alpha+15, 3\alpha+16, 3\alpha+24, 3\alpha+25, 3\alpha+26, 3\alpha+27, 3\alpha+28 \rangle$ mod 37, $\alpha = 0, 1$, $\langle \emptyset, 3\beta+\mu+1, 3\beta+\mu+2, 3\beta+\mu+3, 3\beta+\mu+4, 3\beta+\mu+10, 3\beta+\mu+11, 3\beta+\mu+12, 3\beta+\mu+13, 3\beta+\mu+19, 3\beta+\mu+20, 3\beta+\mu+21, 3\beta+\mu+22, 3\beta+\mu+28, 3\beta+\mu+29, 3\beta+\mu+30, 3\beta+\mu+31 \rangle$ mod 37, $\beta = 0, 1, 2$, once for $\mu = 0$, 3 times for $\mu = 1$.
38	18	153	Lemma 4.6, $37 \in B(17,68)$ and $37 \in B(18,17)$.
38	19	18	$X = Z(37,2) \cup \{\infty\}$. $\mathfrak{B} = \langle \infty, 0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34 \rangle$ mod 37, $\langle \emptyset, 0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34 \rangle$ mod 37.
39	7	21	Table 5.22.
39	8	28	$X = Z(3,2) \times Z(13,2)$. $\mathfrak{B} = \langle (\emptyset,\alpha), (\emptyset,\alpha+6), (0,\alpha+1), (0,\alpha+2), (0,\alpha+3), (1,\alpha+7), (1,\alpha+8), (1,\alpha+9) \rangle$ mod (3, 13), $\alpha = 0, 1, \ldots, 11$, $\langle (0,\emptyset), (0,0), (0,4), (0,8), (1,\emptyset), (1,0), (1,4), (1,8) \rangle$ mod (3, 13) 7 times.
39	9	12	$X = Z(3,2) \times Z(13,2)$. $\mathfrak{B} = \langle (\emptyset,\emptyset), (0,\emptyset), (0,2\alpha+3), (0,2\alpha+9), (1,\emptyset), (1,2\alpha), (1,2\alpha+1), (1,2\alpha+6), (1,2\alpha+7) \rangle$ mod (3, 13), $\alpha = 0, 1, 2$, $\langle (\emptyset,2\alpha+3), (\emptyset,2\alpha+9), (0,\emptyset), (0,2\alpha), (0,2\alpha+6), (1,2\alpha+1), (1,2\alpha+2), (1,2\alpha+7), (1,2\alpha+8) \rangle$ mod (3, 13), $\alpha = 0, 1, 2$, $\langle (\emptyset,0), (\emptyset,4), (\emptyset,8), (0,0), (0,4), (0,8), (1,0), (1,4), (1,8) \rangle$ mod (–, 13).
39	10	45	Lemma 2.6, $39 \in B(19,9)$ as below and $19 \in B(10,5)$ by Lemma 4.5.
39	11	55	$X = Z(3,2) \times Z(13,2)$. $\mathfrak{B} = \langle (\emptyset,\emptyset), (0,\emptyset), (0,2\alpha), (0,2\alpha+3), (0,2\alpha+6), (0,2\alpha+9), (1,\emptyset), (1,2\alpha+1), (1,2\alpha+4), (1,2\alpha+7), (1,2\alpha+10) \rangle$ mod (3,13), 3 times, $\alpha = 0, 1, 2$, $\langle (\emptyset,\emptyset), (\emptyset,\mu+2), (\emptyset,\mu+8), (0,\mu), (0,\mu+3), (0,\mu+6), (0,\mu+9), (1,\mu+2), (1,\mu+5), (1,\mu+8), (1,\mu+11) \rangle$ mod (3,13), 2 times for $\mu = 0, 2, 4$, once for $\mu = 1, 3, 5$, $\langle (\emptyset,0), (\emptyset,4), (\emptyset,8), (0,\emptyset), (0,0), (0,4), (0,8), (1,\emptyset), (1,0), (1,4), (1,8) \rangle$ mod (3, 13).

Table 5.23 (cont.).

v	k	λ	B[k, λ; v]
39	12	22	X = Z(3, 2) × Z(13, 2). 𝔅 = ⟨(∅, α), (∅, α+6), (0, ∅), (0, α), (0, α+3), (0, α+6), (0, α+9), (1, ∅), (1, α+1), (1, α+4), (1, α+7), (1, α+10)⟩ mod (3, 13), α = 0, 1, ..., 5, ⟨(∅, ∅), (∅, 0), (∅, 4), (∅, 8), (0, ∅), (0, 0), (0, 4), (0, 8), (1, ∅), (1, 0), (1, 4), (1, 8)⟩ mod (−, 13).
39	13	6	Unknown
39	13	12	X = Z(2) × Z(19, 2) ∪ {(∞)}. 𝔅 = ⟨(∞), (∅, 0), (∅, 1), (∅, 6), (∅, 7), (∅, 12), (∅, 13), (0, 1), (0, 2), (0, 7), (0, 8), (0, 13), (0, 14)⟩ mod (2, 19), ⟨(∅, ∅), (∅, 0), (∅, 4), (∅, 6), (∅, 10), (∅, 12), (∅, 16), (0, 0), (0, 2), (0, 6), (0, 8), (0, 12), (0, 14)⟩ mod (2, 19), ⟨(∅, ∅), (∅, 4), (∅, 10), (∅, 16), (0, 0), (0, 1), (0, 3), (0, 6), (0, 7), (0, 9), (0, 12), (0, 13), (0, 15)⟩ mod (2, 19).
39	13	18	X = Z(2) × Z(19, 2) ∪ {(∞)}. 𝔅 = ⟨(∞), (∅, 4), (∅, 10), (∅, 16), (0, 1), (0, 2), (0, 3), (0, 7), (0, 8) (0, 9), (0, 13), (0, 14), (0, 15)⟩ mod (2, 19), ⟨(∞), (∅, 1), (∅, 4), (∅, 7), (∅, 10), (∅, 13), (∅, 16), (0, 1), (0, 4), (0, 7), (0, 10), (0, 13), (0, 16)⟩ mod (−, 19), ⟨(∅, ∅), (∅, 0), (∅, 1), (∅, 6), (∅, 7), (∅, 12), (∅, 13), (0, 0), (0, 4), (0, 6), (0, 10), (0, 12), (0, 16)⟩ mod (2, 19), ⟨(∅, ∅), (∅, 1), (∅, 2), (∅, 7), (∅, 8), (∅, 13), (∅, 14), (0, 0), (0, 1), (0, 6), (0, 7), (0, 12), (0, 13)⟩ mod (2, 19), ⟨(∅, ∅), (∅, 1), (∅, 5), (∅, 7), (∅, 11), (∅, 13), (∅, 17), (0, 0), (0, 2), (0, 6), (0, 8), (0, 12), (0, 14)⟩ mod (2, 19).
39	13		λ ≡ 0 (mod 6), λ > 6. Lemma 2.4, 39 ∈ B(13, 12) and 39 ∈ B(13, 18).
39	14	91	X = Z(3, 2) × Z(13, 2). 𝔅 = ⟨(∅, ∅), (∅, α), (∅, α+6), (0, ∅), (0, α), (0, α+3), (0, α+6), (0, α+9), (1, α+1), (1, α+3), (1, α+5), (1, α+7), (1, α+9), (1, α+11)⟩ mod (3, 13), 3 times, α = 0, 1, ..., 5, ⟨(0, ∅), (0, 0), (0, 2), (0, 4), (0, 6), (0, 8), (0, 10), (1, ∅), (1, 1), (1, 3), (1, 5), (1, 7), (1, 9), (1, 11)⟩ mod (3, 13).
39	15	35	X = Z(3, 2) × Z(13, 3). 𝔅 = ⟨(∅, ∅), (∅, 3α), (∅, 3α+2), (∅, 3α+6), (∅, 3α+8), (0, ∅), (0, 3α), (0, 3α+4), (0, 3α+6), (0, 3α+10), (1, ∅), (1, 3α+1), (1, 3α+5), (1, 3α+7), (1, 3α+11)⟩ mod (3, 13), α = 0, 1, ⟨(∅, ∅), (∅, 3α+5), (∅, 3α+11), (0, ∅), (0, 3α+1), (0, 3α+5), (0, 3α+7), (0, 3α+11), (1, ∅), (1, 3α), (1, 3α+1), (1, 3α+4), (1, 3α+6), (1, 3α+7), (1, 3α+10)⟩ mod (3, 13), α = 0, 1, ⟨(∅, ∅), (0, ∅), (0, 0), (0, 2), (0, 4), (0, 6), (0, 8), (0, 10), (1, ∅), (1, 1), (1, 3), (1, 5), (1, 7), (1, 9), (1, 11)⟩ mod (3, 13), ⟨(∅, ∅), (∅, 0), (∅, 3), (∅, 6), (∅, 9), (0, ∅), (0, 1), (0, 4), (0, 7), (0, 10), (1, ∅), (1, 2), (1, 5), (1, 8), (1, 11)⟩ mod (3, 13), ⟨(∅, ∅), (∅, 2), (∅, 5), (∅, 8), (∅, 11), (0, ∅), (0, 2), (0, 5), (0, 8), (0, 11), (1, ∅), (1, 2), (1, 5), (1, 8), (1, 11)⟩ mod (−, 13).

Table 5.23 (cont.).

v	k	λ	$B[k, \lambda; v]$
39	16	120	$X = Z(3,2) \times Z(13,2)$. $\mathfrak{B} = \langle(\emptyset,\emptyset),(\emptyset,\alpha+1),(\emptyset,\alpha+7),(0,\alpha),(0,\alpha+2),(0,\alpha+4),(0,\alpha+6),(0,\alpha+8),$ $(0,\alpha+10),(1,\emptyset),(1,\alpha),(1,\alpha+2),(1,\alpha+4),(1,\alpha+6),(1,\alpha+8),$ $(1,\alpha+10)\rangle \bmod (3,13)$, 2 times, $\alpha = 0,1,\ldots,5$, $\langle(\emptyset,\emptyset),(\emptyset,\beta+1),(\emptyset,\beta+5),(\emptyset,\beta+9),(0,\beta),(0,\beta+2),(0,\beta+4),(0,\beta+6),$ $(0,\beta+8),(0,\beta+10),(1,\beta),(1,\beta+2),(1,\beta+4),(1,\beta+6),(1,\beta+8),$ $(1,\beta+10)\rangle \bmod (3,13)$, $\beta = 0,1,2,3$, $\langle(\emptyset,\gamma),(\emptyset,\gamma+3),(\emptyset,\gamma+6),(\emptyset,\gamma+9),(0,\gamma),(0,\gamma+2),(0,\gamma+4),$ $(0,\gamma+6),(0,\gamma+8),(0,\gamma+10),(1,\gamma+1),(1,\gamma+3),(1,\gamma+5),(1,\gamma+7),$ $(1,\gamma+9),(1,\gamma+11)\rangle \bmod (3,13)$, $\gamma = 0,1,2$.
39	17	136	$X = Z(3,2) \times Z(13,2)$. $\mathfrak{B} = \langle(\emptyset,\alpha),(\emptyset,\alpha+3),(\emptyset,\alpha+4),(\emptyset,\alpha+7),(\emptyset,\alpha+8),(0,\alpha),(0,\alpha+1),(0,\alpha+4),$ $(0,\alpha+5),(0,\alpha+8),(0,\alpha+9),(1,\alpha),(1,\alpha+2),(1,\alpha+4),(1,\alpha+6),$ $(1,\alpha+8),(1,\alpha+10)\rangle \bmod (3,13)$, $\alpha = 0,1,\ldots,11$, $\langle(\emptyset,1),(\emptyset,2),(\emptyset,3),(\emptyset,5),(\emptyset,6),(\emptyset,7),(\emptyset,9),(\emptyset.10),(\emptyset,11),(0,\emptyset),$ $(0,0),(0,4),(0,8),(1,\emptyset),(1,0),(1,4),(1,8)\rangle \bmod (3,13)$, 7 times.
39	18	51	$X = Z(3,2) \times Z(13,2)$. $\mathfrak{B} = \langle(\emptyset,2\alpha),(\emptyset,2\alpha+3),(\emptyset,2\alpha+6),(\emptyset,2\alpha+9),(0,\emptyset),(0,2\alpha),(0,2\alpha+2),$ $(0,2\alpha+4),(0,2\alpha+6),(0,2\alpha+8),(0,2\alpha+10),(1,\emptyset),(1,2\alpha),$ $(1,2\alpha+1),(1,2\alpha+2),(1,2\alpha+6),(1,2\alpha+7),(1,2\alpha+8)\rangle \bmod (3,13)$, $\alpha = 0,1,2$, $\langle(\emptyset,2\alpha+2),(\emptyset,2\alpha+5),(\emptyset,2\alpha+8),(\emptyset,2\alpha+11),(0,2\alpha),(0,2\alpha+1),(0,2\alpha+3),$ $(0,2\alpha+4),(0,2\alpha+6),(0,2\alpha+7),(0,2\alpha+9),(0,2\alpha+10),(1,2\alpha),$ $(1,2\alpha+2),(1,2\alpha+4),(1,2\alpha+6),(1,2\alpha+8),(1,2\alpha+10)\rangle \bmod (3,13)$, $\alpha = 0,1,2$, $\langle(\emptyset,0),(\emptyset,2),(\emptyset,4),(\emptyset,6),(\emptyset,8),(\emptyset,10),(0,0),(0,2),(0,4),(0,6),(0,8),$ $(0,10),(1,0),(1,2),(1,4),(1,6),(1,8),(1,10)\rangle \bmod (-,13)$.
39	19	9	[29]. $X = Z(2) \times Z(19,2) \cup \{(\infty)\}$. $\mathfrak{B} = \langle(\infty),(\emptyset,0),(\emptyset,2),(\emptyset,4),(\emptyset,6),(\emptyset,8),(\emptyset,10),(\emptyset,12),(\emptyset,14),(\emptyset,16),$ $(0,0),(0,2),(0,4),(0,6),(0,8),(0,10),(0,12),(0,14),(0,16)\rangle$ $\bmod (-,19)$, $\langle(\emptyset,\emptyset),(\emptyset,1),(\emptyset,3),(\emptyset,5),(\emptyset,7),(\emptyset,9),(\emptyset,11),(\emptyset,13),(\emptyset,15),(\emptyset,17),$ $(0,0),(0,2),(0,4),(0,6),(0,8),(0,10),(0,12),(0,14),(0,16)\rangle \bmod (-,19)$, $\langle(0,\emptyset),(0,0),(0,1),(0,2),(0,3),(0,4),(0,5),(0,6),(0,7),(0,8),(0,9),$ $(0,10),(0,11),(0,12),(0,13),(0,14),(0,15),(0,16),(0,17)\rangle$.
40	7	14	Table 5.22.
40	8	7	$X = Z(3,2) \times Z(13,2) \cup \{(\infty)\}$. $\mathfrak{B} = \langle(\infty),(\emptyset,\emptyset),(0,0),(0,4),(0,8),(1,0),(1,4),(1,8)\rangle \bmod (3,13)$, $\langle(\emptyset,2\alpha),(\emptyset,2\alpha+3),(\emptyset,2\alpha+6),(\emptyset,2\alpha+9),(0,2\alpha+1),(0,2\alpha+7),$ $(1,2\alpha+5),(1,2\alpha+11)\rangle \bmod (3,13)$, $\alpha = 0,1,2$, $\langle(0,\emptyset),(0,0),(0,4),(0,8),(1,\emptyset),(1,0),(1,4),(1,8)\rangle \bmod (3,13)$.
40	9	24	Lemma 2.6, $40 \in B(13,4)$ as below and $13 \in B(9,6)$ by Lemma 4.5.
40	10	3	Unknown
40	10	6	Unknown

Table 5.23 (cont.).

v	k	λ	B[k, λ; v]
40	10	9	X = Z(3, 2) × Z(13, 2) ∪ {(∞)}. 𝔅 = ⟨(∞), (0, 0), (0, 4), (0, 8), (1, 1), (1, 3), (1, 5), (1, 7), (1, 9), (1, 11)⟩ mod (3, 13), ⟨(∅, ∅), (∅, 2), (∅, 6), (∅, 10), (0, 5), (0, 7), (0, 11), (1, 2), (1, 6), (1, 10)⟩ mod (3, 13), 2 times, ⟨(∅, ∅), (∅, 0), (∅, 4), (∅, 8), (0, 1), (0, 5), (0, 9), (1, 0), (1, 4), (1, 8)⟩ mod (3, 13).
40	10	12	X = Z(3, 2) × Z(13, 2) ∪ {(∞)}. 𝔅 = ⟨(∞), (∅, α), (∅, α+4), (∅, α+8), (0, α), (0, α+4), (0, α+8), (1, α), (1, α+4), (1, α+8)⟩ mod (−, 13), α = 0, 1, 2, 3, ⟨(∅, ∅), (∅, 2β), (∅, 2β+4), (∅, 2β+8), (0, 2β+1), (0, 2β+5), (0, 2β+9), (1, 2β+2), (1, 2β+6), (1, 2β+10)⟩ mod (3, 13), β = 0, 1, ⟨(∅, ∅), (0, 2β+1), (0, 2β+5), (0, 2β+9), (1, 2β), (1, 2β+3), (1, 2β+4), (1, 2β+7), (1, 2β+8), (1, 2β+11)⟩ mod (3, 13), β = 0, 1.
40	10	15	X = Z(3, 2) × Z(13, 2) ∪ {(∞)}. 𝔅 = ⟨(∞), (∅, ∅), (0, ∅), (0, 0), (0, 4), (0, 8), (1, ∅), (1, 1), (1, 5), (1, 9)⟩ mod (3, 13), ⟨(∞), (∅, α), (∅, α+4), (∅, α+8), (0, α), (0, α+8), (1, α), (1, α+4), (1, α+8)⟩ mod (−, 13), α = 0, 1, ⟨(∅, ∅), (0, 0), (0, 4), (0, 8), (1, 0), (1, 1), (1, 4), (1, 5), (1, 8), (1, 9)⟩ mod (3, 13), 2 times, ⟨(∅, ∅), (∅, α), (∅, α+4), (∅, α+8), (0, 2), (0, 6), (0, 10), (1, 3), (1, 7), (1, 11)⟩ mod (3, 13), α = 0, 1, ⟨(∅, ∅), (∅, 1), (∅, 5), (∅, 9), (0, 3), (0, 7), (0, 11), (1, 0), (1, 4), (1, 8)⟩ mod (3, 13).
40	10	λ ≡ 0 (mod 3), λ > 6.	Lemma 2.4, 40 ∈ B(10, 9), 40 ∈ B(10, 12) and 40 ∈ B(10, 15).
40	11	110	X = Z(3, 2) × Z(13, 2) ∪ {(∞)}. 𝔅 = ⟨(∞), (0, ∅), (0, α), (0, α+1), (0, α+6), (0, α+7), (1, ∅), (1, α), (1, α+2), (1, α+6), (1, α+8)⟩ mod (3, 13), α = 0, 1, ..., 5, ⟨(∞), (∅, ∅), (0, ∅), (0, β), (0, β+3), (0, β+6), (0, β+9), (1, β+1), (1, β+4), (1, β+7), (1, β+10)⟩ mod (3, 13), β = 0, 1, 2, ⟨(∞), (∅, 3γ+3), (∅, 3γ+9), (0, 3γ), (0, 3γ+1), (0, 3γ+6), (0, 3γ+7), (1, 3γ), (1, 3γ+5), (1, 3γ+6), (1, 3γ+11)⟩ mod (3, 13), γ = 0, 1, ⟨(∅, ∅), (∅, δ), (∅, δ+1), (0, ∅), (0, δ+2), (0, δ+3), (0, δ+4), (1, ∅), (1, δ+5), (1, δ+6), (1, δ+7)⟩ mod (3, 13), 2 times, δ = 0, 1, ..., 11, ⟨(∅, ∅), (0, ∅), (0, β), (0, β+3), (0, β+6), (0, β+9), (1, ∅), (1, β+1), (1, β+4), (1, β+7), (1, β+10)⟩ mod (3, 13), β = 0, 1, 2, ⟨(∅, ∅), (∅, 3γ), (∅, 3γ+2), (∅, 3γ+4), (∅, 3γ+6), (∅, 3γ+8), (∅, 3γ+10), (0, 3γ+1), (0, 3γ+7), (1, 3γ+1), (1, 3γ+7)⟩ mod (3, 13), γ = 0, 1,
40	12	11	X = Z(3, 2) × Z(13, 2) ∪ {(∞)}. 𝔅 = ⟨(∞), (∅, ∅), (0, ∅), (0, 1), (0, 5), (0, 9), (1, 0), (1, 1), (1, 4), (1, 5), (1, 8) (1, 9)⟩ mod (3, 13), ⟨(∅, 1), (∅, 3), (∅, 5), (∅, 7), (∅, 9), (∅, 11), (0, 0), (0, 4), (0, 8), (1, 0), (1, 4), (1, 8)⟩ mod (3, 13), ⟨(∅, 0), (∅, 3), (∅, 6), (∅, 9), (0, 1), (0, 4), (0, 7), (0, 10), (1, 2), (1, 5), (1, 8), (1, 11)⟩ mod (3, 13), ⟨(∅, ∅), (∅, 0), (∅, 4), (∅, 8), (0, ∅), (0, 0), (0, 4), (0, 8), (1, ∅), (1, 0), (1, 4), (1, 8)⟩ mod (−, 13).

Table 5.23 (cont.).

v	k	λ	$B[k, \lambda, v]$
40	13	4	[24]. $X = Z(40)$. $\mathcal{B} = \langle 0', 1', 2', 4', 5', 8', 13', 14', 17', 19', 24', 26', 34' \rangle \bmod 40$.
40	14	7	Unknown
40	14	14	$X = Z(3,2) \times Z(13,2) \cup \{(\infty)\}$. $\mathcal{B} = \langle (\infty), (\emptyset,\emptyset), (\emptyset,2), (\emptyset,5), (\emptyset,8), (\emptyset,11), (0,1), (0,4), (0,7), (0,10), (1,1),$ $(1,4), (1,7), (1,10) \rangle \bmod (3,13)$, $\langle (\infty), (\emptyset,\emptyset), (\emptyset,0), (\emptyset,1), (\emptyset,2), (\emptyset,3), (\emptyset,4), (\emptyset,5), (\emptyset,6), (\emptyset,7), (\emptyset,8),$ $(\emptyset,9), (\emptyset,10), (\emptyset,11) \rangle \bmod (3,-)$, $\langle (\emptyset,3\alpha), (\emptyset,3\alpha+1), (\emptyset,3\alpha+2), (\emptyset,3\alpha+6), (\emptyset,3\alpha+7), (\emptyset,3\alpha+8), (0,\emptyset),$ $(0,3\alpha+2), (0,3\alpha+4), (0,3\alpha+8), (0,3\alpha+10), (1,\emptyset), (1,3\alpha+1),$ $(1,3\alpha+7) \rangle \bmod (3,13), \quad \alpha = 0,1$.
40	14	21	$X = Z(5,2) \times (Z(7,3) \cup \{\infty\})$. $\mathcal{B} = \langle (\emptyset,\infty), (\emptyset,\emptyset), (\emptyset,0), (\emptyset,1), (\emptyset,3), (\emptyset,4), (0,0), (0,3), (1,\infty), (1,\emptyset),$ $(2,1), (2,4), (3,\infty), (3,\emptyset) \rangle \bmod (5,7)$, $\langle (\emptyset,0), (\emptyset,3), (\emptyset,\alpha+1), (\emptyset,\alpha+4), (0,1), (0,4), (1,0), (1,3), (2,1),$ $(2,4), (3,2-\alpha), (3,5-\alpha), (\alpha,\infty), (\alpha+2,\infty) \rangle \bmod (5,7), \quad \alpha = 0,1$, $\langle (0,\emptyset), (0,1), (0,4), (1,\infty), (1,\emptyset), (1,1), (1,4), (2,\emptyset), (2,2), (2,5),$ $(3,\infty), (3,\emptyset), (3,0), (3,3) \rangle \bmod (5,7)$, $\langle (\emptyset,0), (\emptyset,3), (0,\emptyset), (0,0), (0,3), (1,\emptyset), (1,2), (1,5), (2,\emptyset), (2,2),$ $(2,5), (3,\emptyset), (3,1), (3,4) \rangle \bmod (5,7)$, $\langle (0,\emptyset), (0,0), (0,1), (0,2), (0,3), (0,4), (0,5), (2,\emptyset), (2,0), (2,1),$ $(2,2), (2,3), (2,4), (2,5) \rangle \bmod (5,-)$.
40	14		$\lambda \equiv 0 \pmod 7$, $\lambda > 7$. Lemma 2.4, $40 \in B(14,14)$ and $40 \in B(14,21)$.
40	15	14	$X = (Z(2) \cup \{\infty\}) \times Z(13,2) \cup \{(\infty,\infty)\}$. $\mathcal{B} = \langle (\infty,\infty), (\infty,\emptyset), (\infty,2), (\infty,6), (\infty,10), (\emptyset,2), (\emptyset,6), (\emptyset,10), (0,\emptyset), (0,0),$ $(0,2), (0,4), (0,6), (0,8), (0,10) \rangle \bmod (2,13)$, $\langle (\infty,\infty), (\infty,0), (\infty,2), (\infty,4), (\infty,6), (\infty,8), (\infty,10), (\emptyset,\emptyset), (\emptyset,1), (\emptyset,5),$ $(\emptyset,9), (0,\emptyset), (0,1), (0,5), (0,9) \rangle \bmod (-,13)$, $\langle (\infty,0), (\infty,4), (\infty,8), (\emptyset,0), (\emptyset,2), (\emptyset,4), (\emptyset,6), (\emptyset,8), (\emptyset,10), (0,1),$ $(0,3), (0,5), (0,7), (0,9), (0,11) \rangle \bmod (2,13)$, $\langle (\infty,0), (\infty,1), (\infty,4), (\infty,5), (\infty,8), (\infty,9), (\emptyset,0), (\emptyset,4), (\emptyset,8), (0,2),$ $(0,3), (0,6), (0,7), (0,10), (0,11) \rangle \bmod (2,13)$, $\langle (\infty,\emptyset), (\infty,0), (\infty,2), (\infty,4), (\infty,6), (\infty,8), (\infty,10), (\emptyset,\emptyset), (\emptyset,3), (\emptyset,7),$ $(\emptyset,11), (0,\emptyset), (0,3, (0,7), (0,11) \rangle \bmod (-,13)$.
40	16	10	Unknown
40	16	20	$X = Z(3,2) \times Z(13,2) \cup \{(\infty)\}$. $\mathcal{B} = \langle (\infty), (\emptyset,\emptyset), (\emptyset,0), (\emptyset,3), (\emptyset,6), (\emptyset,9), (0,\emptyset), (0,1), (0,4), (0,7),$ $(0,10), (1,\emptyset), (1,2), (1,5), (1,8), (1,11) \rangle \bmod (3,13)$, $\langle (\infty), (\emptyset,\emptyset), (\emptyset,1), (\emptyset,4), (\emptyset,7), (\emptyset,10), (0,\emptyset), (0,1), (0,4), (0,7),$ $(0,10), (1,\emptyset), (1,1), (1,4), (1,7), (1,10) \rangle \bmod (-,13)$, $\langle (\emptyset,0), (\emptyset,1), (\emptyset,3), (\emptyset,4), (\emptyset,6), (\emptyset,7), (\emptyset,9), (\emptyset,10), (0,1), (0,4),$ $(0,7), (0,10), (1,2\alpha), (1,2\alpha+3), (1,2\alpha+6), (1,2\alpha+9) \rangle \bmod (3,13)$, $\alpha = 0,1$.
40	16	30	$X = Z(3,2) \times Z(13,2) \cup \{(\infty)\}$. $\mathcal{B} = \langle (\infty), (\emptyset,\emptyset), (\emptyset,0), (\emptyset,3), (\emptyset,6), (\emptyset,9), (0,\emptyset), (0,1), (0,4), (0,7),$ $(0,10), (1,\emptyset), (1,2), (1,5), (1,8), (1,11) \rangle \bmod (3,13)$, 2 times, $\langle (\emptyset,\alpha), (\emptyset,\alpha+1), (\emptyset,\alpha+3), (\emptyset,\alpha+4), (\emptyset,\alpha+6), (\emptyset,\alpha+7), (\emptyset,\alpha+9),$ $(\emptyset,\alpha+10), (0,\alpha), (0,\alpha+3), (0,\alpha+6), (0,\alpha+9), (1,\alpha+1), (1,\alpha+4),$ $(1,\alpha+7), (1,\alpha+10) \rangle \bmod (3,13), \quad \alpha = 0,1,2$.

Table 5.23 (cont.).

v	k	λ	$B[k, \lambda, v]$
40	16		$\lambda \equiv 0 \pmod{10}$, $\lambda > 10$. Lemma 2.4, $40 \in B(16, 20)$ and $40 \in B(16, 30)$.
40	17	272	$X = Z(3,2) \times Z(13,2) \cup \{(\infty)\}$. $\mathfrak{B} = \langle(\infty), (\emptyset,\emptyset), (\emptyset,\alpha), (\emptyset,\alpha+8), (\emptyset,\alpha+9), (0,\emptyset), (0,\alpha), (0,\alpha+1), (0,\alpha+2),$ $(0,\alpha+3), (0,\alpha+4), (1,\emptyset), (1,\alpha), (1,\alpha+1), (1,\alpha+5), (1,\alpha+6), (1,\alpha+7)\rangle$ $\text{mod } (3,13), \quad \alpha = 0, 1, \dots, 11,$ $\langle(\infty), (0,\mu), (0,\mu+1), (0,\mu+3), (0,\mu+4), (0,\mu+6), (0,\mu+7), (0,\mu+9),$ $(0,\mu+10), (1,\mu), (1,\mu+1), (1,\mu+3), (1,\mu+4), (1,\mu+6), (1,\mu+7),$ $(1,\mu+9), (1,\mu+10)\rangle \text{ mod } (3,13), \quad$ 3 times for $\mu = 2$, once for $\mu = 0, 1,$ $\langle(\emptyset,\emptyset), (\emptyset,\beta+1), (\emptyset,\beta+3), (\emptyset,\beta+7), (\emptyset,\beta+9), (0,\beta), (0,\beta+2), (0,\beta+4),$ $(0,\beta+6), (0,\beta+8), (0,\beta+10), (1,\beta), (1,\beta+2), (1,\beta+4), (1,\beta+6),$ $(1,\beta+8), (1,\beta+10)\rangle \text{ mod } (3,13), \quad$ 3 times, $\beta = 0, 1, \dots, 5,$ $\langle(\emptyset,\emptyset), (\emptyset,\gamma+2), (\emptyset,\gamma+5), (\emptyset,\gamma+8), (\emptyset,\gamma+11), (0,\gamma), (0,\gamma+2), (0,\gamma+3),$ $(0,\gamma+6), (0,\gamma+8), (0,\gamma+9), (1,\gamma), (1,\gamma+3), (1,\gamma+5), (1,\gamma+6),$ $(1,\gamma+9), (1,\gamma+11)\rangle \text{ mod } (3,13), \quad \gamma = 0, 1, 2,$ $\langle(\emptyset,\emptyset), (\emptyset,3\delta+4), (\emptyset,3\delta+5), (\emptyset,3\delta+10), (\emptyset,3\delta+11), (0,3\delta+1), (0,3\delta+2),$ $(0,3\delta+4), (0,3\delta+7), (0,3\delta+8), (0,3\delta+10), (1,3\delta+1), (1,3\delta+2),$ $(1,3\delta+5), (1,3\delta+7), (1,3\delta+8), (1,3\delta+11)\rangle \text{ mod } (3,13), \quad \delta = 0, 1.$
40	18	51	$X = Z(3,2) \times Z(13,2) \cup \{(\infty)\}$. $\mathfrak{B} = \langle(\infty), (\emptyset,\emptyset), (\emptyset,2\alpha), (\emptyset,2\alpha+1), (\emptyset,2\alpha+6), (\emptyset,2\alpha+7), (0,0), (0,2), (0,4),$ $(0,6), (0,8), (0,10), (1,1), (1,3), (1,5), (1,7), (1,9), (1,11)\rangle$ $\text{mod } (3,13), \quad \alpha = 0, 1, 2,$ $\langle(\emptyset,\emptyset), (\emptyset,2\alpha+1), (\emptyset,2\alpha+7), (0,2\alpha), (0,2\alpha+1), (0,2\alpha+2), (0,2\alpha+4),$ $(0,2\alpha+6), (0,2\alpha+7), (0,2\alpha+8), (0,2\alpha+10), (1,\emptyset), (1,2\alpha+1),$ $(1,2\alpha+2), (1,2\alpha+3), (1,2\alpha+7), (1,2\alpha+8), (1,2\alpha+9)\rangle \text{ mod } (3,13),$ $\alpha = 0, 1, 2,$ $\langle(\emptyset,\beta), (\emptyset,\beta+2), (\emptyset,\beta+4), (\emptyset,\beta+6), (\emptyset,\beta+8), (\emptyset,\beta+10), (0,\beta), (0,\beta+2),$ $(0,\beta+4), (0,\beta+6), (0,\beta+8), (0,\beta+10), (1,\beta), (1,\beta+2), (1,\beta+4),$ $(1,\beta+6), (1,\beta+8), (1,\beta+10)\rangle \text{ mod } (-,13), \quad \beta = 0, 1.$
40	19	114	Lemma 4.6, $39 \in B(18, 51)$ and $39 \in B(19, 9)$.
40	20	19	Lemma 4.6, $39 \in B(19, 9)$ and $39 \in B(20, 10)$ by Lemma 4.5.
41	7	21	Lemma 4.2.
41	8	7	Lemma 4.1.
41	9	9	Lemma 4.2.
41	10	9	Lemma 4.1.
41	11	11	Lemma 4.2.
41	12	33	Lemma 4.1.
41	13	39	Lemma 4.2.
41	14	91	Lemma 4.1.
41	15	21	Lemma 4.3.
41	16	6	Unknown.
41	16	12	$X = Z(41, 6)$. $\mathfrak{B} = \langle\emptyset, \alpha, \alpha+1, \alpha+5, \alpha+8, \alpha+9, \alpha+13, \alpha+16, \alpha+17, \alpha+21, \alpha+24, \alpha+25,$ $\alpha+29, \alpha+32, \alpha+33, \alpha+37\rangle \text{ mod } 41, \quad \alpha = 0, 1.$
41	16	18	$X = Z(41, 6)$. $\mathfrak{B} = \langle\emptyset, \alpha, \alpha+1, \alpha+3, \alpha+8, \alpha+9, \alpha+11, \alpha+16, \alpha+17, \alpha+19, \alpha+24, \alpha+25,$ $\alpha+27, \alpha+32, \alpha+33, \alpha+35\rangle \text{ mod } 41, \quad \alpha = 0, 1,$ $\langle\emptyset, 2, 3, 6, 10, 11, 14, 18, 19, 22, 26, 27, 30, 34, 35, 38\rangle \text{ mod } 41.$
41	16		$\lambda \equiv 0 \pmod 6$, $\lambda > 6$. Lemma 2.4, $41 \in B(16, 12)$ and $41 \in B(16, 18)$.

Table 5.23 (cont.).

v	k	λ	$B[k, \lambda, v]$
41	17	34	Lemma 4.2.
41	18	153	Lemma 4.1.
41	19	171	Lemma 4.2.
41	20	19	Lemma 4.1.
42	8	28	$X = Z(41,6) \cup \{\infty\}$. $\mathcal{B} = \langle \infty, \emptyset, 5\alpha+4, 5\alpha+5, 5\alpha+14, 5\alpha+24, 5\alpha+25, 5\alpha+34 \rangle$ mod 41, $\alpha = 0,1,2,3$, $\langle 5\alpha+\beta, 5\alpha+\beta+5, 5\alpha+\beta+10, 5\alpha+\beta+16, 5\alpha+\beta+20, 5\alpha+\beta+25, 5\alpha+\beta+30,$ $5\alpha+\beta+36 \rangle$ mod 41, $\alpha = 0,1,2,3$, $\beta = 0,1,2,3$, $\langle 4,9,14,19,24,29,34,39 \rangle$ mod 41.
42	9	24	$X = Z(41,6) \cup \{\infty\}$. $\mathcal{B} = \langle \infty, \mu, \mu+5, \mu+10, \mu+15, \mu+20, \mu+25, \mu+30, \mu+35 \rangle$ mod 41, $\mu = 1,2,4$, $\langle \emptyset, 5\alpha+3\beta, 5\alpha+3\beta+1, 5\alpha+3\beta+2, 5\alpha+3\beta+3, 5\alpha+3\beta+20, 5\alpha+3\beta+21,$ $5\alpha+3\beta+22, 5\alpha+3\beta+23 \rangle$ mod 41, $\alpha = 0,1,2,3$, $\beta = 0,1$, $\langle \emptyset, \nu, \nu+5, \nu+10, \nu+15, \nu+20, \nu+25, \nu+30, \nu+35 \rangle$ mod 41, once for $\nu = 1$, 2 times for $\nu = 3$.
42	10	45	Lemma 4.6, $41 \in B(9,9)$ and $41 \in B(10,9)$.
42	11	110	$X = Z(41,6) \cup \{\infty\}$. $\mathcal{B} = \langle \infty, \mu, \mu+4, \mu+8, \mu+12, \mu+16, \mu+20, \mu+24, \mu+28, \mu+32, \mu+36 \rangle$ mod 41, 2 times for $\mu = 0, 3$, 3 times for $\mu = 2$, 4 times for $\mu = 1$, $\langle \emptyset, \alpha, \alpha+4, \alpha+8, \alpha+12, \alpha+16, \alpha+20, \alpha+24, \alpha+28, \alpha+32, \alpha+36 \rangle$ mod 41, 6 times, $\alpha = 0,1,2,3$, $\langle \emptyset, \nu, \nu+4, \nu+8, \nu+12, \nu+16, \nu+20, \nu+24, \nu+28, \nu+32, \nu+36 \rangle$ mod 41, once for $\nu = 2$, 2 times for $\nu = 3$, 4 times for $\nu = 0$.
42	12	22	$X = Z(41,6) \cup \{\infty\}$. $\mathcal{B} = \langle \infty, \emptyset, 2\alpha, 2\alpha+4, 2\alpha+8, 2\alpha+12, 2\alpha+16, 2\alpha+20, 2\alpha+24, 2\alpha+28, 2\alpha+32,$ $2\alpha+36 \rangle$ mod 41, $\alpha = 0,1$, $\langle 2\beta, 2\beta+3, 2\beta+5, 2\beta+10, 2\beta+13, 2\beta+15, 2\beta+20, 2\beta+23, 2\beta+25,$ $2\beta+30, 2\beta+33, 2\beta+35 \rangle$ mod 41, $\beta = 0,1,2,3,4$.
42	13	156	$X = Z(41,6) \cup \{\infty\}$. $\mathcal{B} = \langle \infty, \mu, \mu+1, \mu+5, \mu+10, \mu+11, \mu+15, \mu+20, \mu+21, \mu+25, \mu+30, \mu+31,$ $\mu+35 \rangle$ mod 41, once for $\mu = 5$, 2 times for $\mu = 3,6,7,9$, $\langle \infty, 5\alpha+4, 5\alpha+5, 5\alpha+6, 5\alpha+14, 5\alpha+15, 5\alpha+16, 5\alpha+24, 5\alpha+25, 5\alpha+26,$ $5\alpha+34, 5\alpha+35, 5\alpha+36 \rangle$ mod 41, 2 times, $\alpha = 0,1$, $\langle \emptyset, 5\alpha+\beta, 5\alpha+\beta+1, 5\alpha+\beta+2, 5\alpha+\beta+10, 5\alpha+\beta+11, 5\alpha+\beta+12, 5\alpha+\beta+20,$ $5\alpha+\beta+21, 5\alpha+\beta+22, 5\alpha+\beta+30, 5\alpha+\beta+31, 5\alpha+\beta+32 \rangle$ mod 41, 2 times, $\alpha = 0,1$, $\beta = 0,1,2,3$, $\langle \emptyset, \nu, \nu+1, \nu+5, \nu+10, \nu+11, \nu+15, \nu+20, \nu+21, \nu+25, \nu+30, \nu+31,$ $\nu+35 \rangle$ mod 41, once for $\nu = 5$, 2 times for $\nu = 0,1,2,4,8,9$.
42	14	13	Unknown
42	14	26	$X = Z(3,2) \times (Z(13,2) \cup \{\infty\})$. $\mathcal{B} = \langle (\emptyset, \alpha+1), (\emptyset, \alpha+4), (\emptyset, \alpha+7), (\emptyset, \alpha+10), (0, \infty), (0,0), (0,3), (0,6),$ $(0,9), (1,\infty), (1,0), (1,3), (1,6), (1,9) \rangle$ mod (3, 13), $\alpha = 0,1$, $\langle (\emptyset, \infty), (\emptyset, \emptyset), (\emptyset, 0), (\emptyset, 3), (\emptyset, 6), (\emptyset, 9), (0,2), (0,5), (0,8), (0,11),$ $(1,2), (1,5), (1,8), (1,11) \rangle$ mod (3, 13), 2 times, $\langle (\emptyset, \infty), (\alpha, \emptyset), (\alpha, 0), (\alpha, 1), (\alpha, 2), (\alpha, 3), (\alpha, 4), (\alpha, 5), (\alpha, 6), (\alpha, 7),$ $(\alpha, 8), (\alpha, 9), (\alpha, 10), (\alpha, 11) \rangle$ mod (3, –), 2 times, $\alpha = 0,1$, $\langle (\emptyset, \alpha+1), (\emptyset, \alpha+4), (\emptyset, \alpha+7), (\emptyset, \alpha+10), (0, \emptyset), (0,0), (0,3), (0,6),$ $(0,9), (1,\emptyset), (1,0), (1,3), (1,6), (1,9) \rangle$ mod (3, 13), $\alpha = 0,1$.

Table 5.23 (cont.)

v	k	λ	$B[k,\lambda;v]$
42	14	39	$X = Z(3,2) \times (Z(13,2) \cup \{\infty\})$. $\mathfrak{B} = \langle(\emptyset,\alpha+1),(\emptyset,\alpha+4),(\emptyset,\alpha+7),(\emptyset,\alpha+10),(0,\infty),(0,\alpha),(0,\alpha+3),$ $(0,\alpha+6),(0,\alpha+9),(1,\infty),(1,\alpha),(1,\alpha+3),(1,\alpha+6),(1,\alpha+9)\rangle$ $\bmod (3,13)$, $\alpha = 0,1,2$, $\langle(\emptyset,\infty),(\emptyset,\emptyset),(\emptyset,\alpha),(\emptyset,\alpha+3),(\emptyset,\alpha+6),(\emptyset,\alpha+9),(0,\alpha+2),(0,\alpha+5),$ $(0,\alpha+8),(0,\alpha+11),(1,\alpha+2),(1,\alpha+5),(1,\alpha+8),(1,\alpha+11)\rangle$ $\bmod (3,13)$, $\alpha = 0,1,2$, $\langle(\emptyset,\infty),(\beta,\emptyset),(\beta,0),(\beta,1),(\beta,2),(\beta,3),(\beta,4),(\beta,5),(\beta,6),(\beta,7),(\beta,8),$ $(\beta,9),(\beta,10),(\beta,11)\rangle \bmod (3,-)$, 3 times, $\beta = 0,1$, $\langle(\emptyset,\alpha+2),(\emptyset,\alpha+5),(\emptyset,\alpha+8),(\emptyset,\alpha+11),(0,\emptyset),(0,\alpha),(0,\alpha+3),(0,\alpha+6),$ $(0,\alpha+9),(1,\emptyset),(1,\alpha),(1,\alpha+3),(1,\alpha+6),(1,\alpha+9)\rangle \bmod (3,13)$, $\alpha = 0,1,2$.
42	14	$\lambda \equiv 0 \pmod{13}$, $\lambda > 13$.	Lemma 2.4, $42 \in B(14,26)$ and $42 \in B(14,39)$.
42	15	70	$X = Z(41,6) \cup \{\infty\}$. $\mathfrak{B} = \langle\infty,4\alpha,4\alpha+3,4\alpha+5,4\alpha+10,4\alpha+13,4\alpha+15,4\alpha+18,4\alpha+20,4\alpha+23,$ $4\alpha+25,4\alpha+30,4\alpha+33,4\alpha+35,4\alpha+38\rangle \bmod 41$, $\alpha = 0,1,2,3,4$, $\langle\emptyset,4\alpha,4\alpha+1,4\alpha+5,4\alpha+8,4\alpha+10,4\alpha+15,4\alpha+17,4\alpha+20,4\alpha+21,$ $4\alpha+25,4\alpha+28,4\alpha+30,4\alpha+35,4\alpha+37\rangle \bmod 41$, $\alpha = 0,1,2,3,4$.
42	16	120	Lemma 4.6, $41 \in B(15,21)$ and $41 \in B(16,78)$.
42	17	272	Lemma 4.6, $41 \in B(16,102)$ and $41 \in B(17,34)$.
42	18	51	$X = Z(41,6) \cup \{\infty\}$. $\mathfrak{B} = \langle\infty,\emptyset,\mu,\mu+2,\mu+5,\mu+7,\mu+10,\mu+12,\mu+15,\mu+17,\mu+20,\mu+22,$ $\mu+25,\mu+27,\mu+30,\mu+32,\mu+35,\mu+37\rangle \bmod 41$, 2 times for $\mu = 0$, once for $\mu = 3$, $\langle 5\alpha,5\alpha+1,5\alpha+2,5\alpha+3,5\alpha+4,5\alpha+5,5\alpha+6,5\alpha+13,5\alpha+16,5\alpha+20,$ $5\alpha+21,5\alpha+22,5\alpha+23,5\alpha+24,5\alpha+25,5\alpha+26,5\alpha+33,5\alpha+36\rangle$ $\bmod 41$, $\alpha = 0,1,2,3$.
42	19	342	$X = (Z(5,2) \cup \{\infty\}) \times Z(7,3)$. $\mathfrak{B} = \langle(\infty,\emptyset),(\infty,\alpha),(\infty,\alpha+2),(\infty,\alpha+4),(\emptyset,\emptyset),(\alpha,0),(\alpha,2),(\alpha,4),(\alpha+1,\emptyset),$ $(\alpha+1,1),(\alpha+1,3),(\alpha+1,5),(\alpha+2,0),(\alpha+2,2),(\alpha+2,4),(\alpha+3,\emptyset),$ $(\alpha+3,1),(\alpha+3,3),(\alpha+3,5)\rangle \bmod (5,7)$, 15 times, $\alpha = 0,1$, $\langle(\infty,\beta+1),(\infty,\beta+4),(\emptyset,\emptyset),(\emptyset,\beta),(\emptyset,\beta+3),(\alpha,\beta),(\alpha,\beta+1),(\alpha,\beta+3),$ $(\alpha,\beta+4),(\alpha+1,\emptyset),(\alpha+1,\beta),(\alpha+1,\beta+3),(\alpha+2,\beta),(\alpha+2,\beta+1),$ $(\alpha+2,\beta+3),(\alpha+2,\beta+4),(\alpha+3,\emptyset),(\alpha+3,\beta),(\alpha+3,\beta+3)\rangle \bmod (5,7)$, $\alpha = 0,1$, $\beta = 0,1,2$, $\langle(\infty,\emptyset),(\emptyset,\beta+1),(\emptyset,\beta+4),(\alpha,0),(\alpha,1),(\alpha,2),(\alpha,3),(\alpha,4),(\alpha,5),$ $(\alpha+1,\beta),(\alpha+1,\beta+3),(\alpha+2,0),(\alpha+2,1),(\alpha+2,2),(\alpha+2,3),(\alpha+2,4),$ $(\alpha+2,5),(\alpha+3,\beta),(\alpha+3,\beta+3)\rangle \bmod (5,7)$, $\alpha = 0,1$, $\beta = 0,1,2$. $\langle(\infty,\emptyset),(\infty,\alpha),(\infty,\alpha+2),(\infty,\alpha+4),(\emptyset,\alpha+1),(\emptyset,\alpha+3),(\emptyset,\alpha+5),(0,\alpha),$ $(0,\alpha+2),(0,\alpha+4),(1,\alpha),(1,\alpha+2),(1,\alpha+4),(2,\alpha),(2,\alpha+2),(2,\alpha+4),$ $(3,\alpha),(3,\alpha+2),(3,\alpha+4)\rangle \bmod (5,7)$, $\alpha = 0,1$, $\langle(\infty,\alpha),(\infty,\alpha+2),(\infty,\alpha+4),(0,\emptyset),(0,\alpha),(0,\alpha+2),(0,\alpha+4),(1,\emptyset),(1,\alpha),$ $(1,\alpha+2),(1,\alpha+4),(2,\emptyset),(2,\alpha),(2,\alpha+2),(2,\alpha+4),(3,\emptyset),(3,\alpha),$ $(3,\alpha+2),(3,\alpha+4)\rangle \bmod (5,7)$, $\alpha = 0,1$, $\langle(\infty,\emptyset),(\emptyset,0),(\emptyset,1),(\emptyset,2),(\emptyset,3),(\emptyset,4),(\emptyset,5),(0,0),(0,2),(0,4),(1,1),$ $(1,3),(1,5),(2,0),(2,2),(2,4),(3,1),(3,3),(3,5)\rangle \bmod (5,7)$, 3 times, $\langle(\infty,\emptyset),(\infty,1),(\infty,3),(\infty,5),(\emptyset,0),(\emptyset,2),(\emptyset,4),(0,0),(0,2),(0,4),$ $(1,0),(1,2),(1,4),(2,0),(2,2),(2,4),(3,0),(3,2),(3,4)\rangle \bmod (-,7)$.

Table 5.23 (cont.).

v	k	λ	$B[k, \lambda; v]$
42	20	190	$X = Z(41,6) \cup \{\infty\}$. $\mathfrak{B} = \langle \infty, \emptyset, 2\alpha, 2\alpha+2, 2\alpha+4, 2\alpha+6, 2\alpha+8, 2\alpha+10, 2\alpha+12, 2\alpha+14, 2\alpha+16, 2\alpha+20, 2\alpha+22, 2\alpha+24, 2\alpha+26, 2\alpha+28, 2\alpha+30, 2\alpha+32, 2\alpha+34, 2\alpha+36\rangle$ mod 41, $\alpha = 0, 1, \ldots, 9$, $\langle 0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34, 36, 38\rangle$ mod 41, 11 times.
42	21	20	$X = Z(41,6) \cup \{\infty\}$. $\mathfrak{B} = \langle \infty, 0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34, 36, 38\rangle$ mod 41, $\langle \emptyset, 0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34, 36, 38\rangle$ mod 41.
43	7	1	Non-existing by Lemma 1.2.
43	7		$\lambda > 1$. Lemma 2.4, $43 \in B(7,2)$ as in Table 5.22 and $43 \in B(7,3)$ by Lemma 4.3.
43	8	4	Lemma 4.4.
43	9	12	Lemma 4.3.
43	10	15	Lemma 4.4.
43	11	55	Lemma 4.2.
43	12	22	Lemma 4.1.
43	13	26	Lemma 4.2.
43	14	13	Lemma 4.1.
43	15	5	Non-existing by Lemma 1.2.
43	15	10	$X = Z(43,3)$. $\mathfrak{B} = \langle 0, 1, 3, 4, 8, 14, 15, 17, 18, 22, 28, 29, 31, 32, 36\rangle$ mod 43, $\langle 2, 3, 4, 5, 10, 16, 17, 18, 19, 24, 30, 31, 32, 33, 38\rangle$ mod 43.
43	15		$\lambda \equiv 0 \pmod 5$, $\lambda > 5$. Lemma 2.4, $43 \in B(15,10)$ and $43 \in B(15,15)$ by Lemma 4.2.
43	16	40	Lemma 4.4.
43	17	136	Lemma 4.2.
43	18	51	Lemma 4.1.
43	19	57	Lemma 4.2.
43	20	190	Lemma 4.1.
43	21	10	Lemma 4.3.

6. Necessary and sufficient conditions for group divisible designs

6.1. Introduction

The group divisible designs were introduced in Section 2.3 as a mean for construction of BIBD's. However, the group divisible designs became so important in the combinatorial reasearch that the study of them for their own sake is advisable. We start with a theorem stating a necessary condition for the existence of group divisible designs, a theorem which is similar to Theorem 1.1.

Theorem 6.1. *A necessary condition for the existence of a group divisible design* GD$[k, \lambda, m; v]$ *is that*

$$v \equiv 0 \pmod{m},\ v \geqslant km,\ \lambda(v-m) \equiv 0 \pmod{(k-1)}$$

$$\textit{and}\quad \lambda v(v-m) \equiv 0 \pmod{k(k-1)}.$$

Proof. $v \equiv 0 \pmod m$ and $v \geqslant km$ follow from the definition of a group divisible design. Further, $\lambda(v-m)/(k-1)$ is the replication number of every point and $\lambda v(v-m)/(k(k-1))$ is the total number of blocks.

The condition of Theorem 6.1 is clearly not sufficient for the existence of GD$[k, \lambda, m; v]$. It has been already proved by Tarry [32] that GD$[4, 1, 6; 24]$ does not exist. Further, it follows from Lemma 3.14, that if $1 < m < k-1$, then GD$[k, 1, m; km]$ does not exist. However, we shall prove that the condition of Theorem 6.1 is sufficient for the existence of group divisible designs with block-size $k = 3$.

6.2. Group divisible designs with block-size 3

Lemma 6.1. *If* $v \in$ GD$(3, \lambda, m)$ *holds and if* r *is a positive integer, then* $rv \in$ GD$(3, \lambda, rm)$ *holds.*

Proof. Let $X = (I(r) \times I(m)) \times I(n)$, where $mn = v$. By the hypothesis of the lemma, there exists GD$[3, \lambda, m; v]$ on $I(m) \times I(n)$. For each block B of this design form a transversal design T$[3, 1; r]$ on $I(r) \times B$; such a design exists by Theorem 3.1. The blocks of these transversal designs form the required GD$[3, \lambda, rm; rv]$.

Lemma 6.2. *If* $n \equiv 0$ *or* $1 \pmod 3$, *then* $n \in B(\{3, 4, 6\}, 1)$.

Proof. By Lemma 5.1, $n \in$ GD$(\{3, 4\}, 1, M_3)$ and by Lemma 2.23, $n \in B(M_3, 1)$. Further, $\{7, 19\} \subset B(3, 1)$ by Lemma 4.3.

Lemma 6.3. *If* $n \equiv 0$ *or* $1 \pmod 3$, *then* $2n \in$ GD$(3, 1, 2)$ *holds.*

Proof. By Lemmas 6.2 and 2.16, it suffices to show that $2n \in$ GD$(3, 1, 2)$

for $n \in \{3, 4, 6\}$. For $n = 3$ this follows from Theorem 3.1. For $n \in \{4, 6\}$ we have

$n = 4$; $8 \in GD(3, 1, 2)$. $X = Z(2) \times (Z(3, 2) \cup \{\infty\})$.

$\mathcal{P} = \langle(\emptyset;\infty), (\emptyset;\emptyset), (0;0) \bmod (2;3),$
$\quad\langle(0;\emptyset), (0;0), (0;1)\rangle \bmod (2;-)$.

$n = 6$; $12 \in GD(3, 1, 2)$. $X = Z(2) \times (Z(2) \times Z(3, 2))$.

$\mathcal{P} = \langle(\emptyset;\emptyset,\emptyset), (\emptyset;\emptyset,0), (\emptyset;\emptyset,1)\rangle \bmod (2;-,-),$
$\quad\langle(\emptyset;\emptyset,\emptyset), (\emptyset;0,0), (\emptyset;0,1)\rangle \bmod (-;-,3),$
$\quad\langle(\emptyset;\emptyset,\emptyset), (0;0,\emptyset), (0;0,0)\rangle \bmod (-;-,3),$
$\quad\langle(\emptyset;\emptyset,\emptyset), (0;\emptyset,0), (0;0,1)\rangle \bmod (-;-,3),$
$\quad\langle(\emptyset;\emptyset,\emptyset), (\emptyset;0,\emptyset), (0;\emptyset,1)\rangle \bmod (-;-,3),$
$\quad\langle(\emptyset;0,\emptyset), (0;\emptyset,\emptyset), (0;0,1)\rangle \bmod (-;-,3),$
$\quad\langle(\emptyset;0,\emptyset), (0;\emptyset,0), (0;0,0)\rangle \bmod (-;-,3)$.

Lemma 6.4. *If* $n \equiv 1 \pmod 2$, *then* $3n \in GD(3, 1, 3)$ *holds.*

Proof. Let $X = Z(3, 2) \times Z(n)$.

$\mathcal{P} = \langle(\emptyset;\emptyset), (0;\alpha'), (0;-\alpha')\rangle \bmod (3;n), \quad \alpha = 1, 2, \ldots, \frac{1}{2}(n-1)$.

Lemma 6.5. *For every* $n \geq 3$, $6n \in GD(3, 1, 6)$ *holds.*

Proof. By Lemmas 5.3 and 2.16, it suffices to show that $6n \in GD(3, 1, 6)$ for every $n \in K_3$. For $n \in \{3, 4, 6\}$ this follows from Lemmas 6.3 and 6.1; for $n = 5$ this follows from Lemmas 6.4 and 6.1; for $n = 8$ we prove:

$48 \in GD(3, 1, 6)$. $X = Z(6) \times (Z(7, 3) \cup \{\infty\})$.

$\mathcal{P} = \langle(\alpha';\infty), (\beta';\emptyset), ((\beta+3)';\alpha+1)\rangle \bmod (-;7), \quad \alpha = 0, 1, \ldots, 5, \ \beta = 0, 1, 2,$
$\quad\langle(0';3\gamma), (2';3\gamma+2), (4';3\gamma+4)\rangle \bmod (6;7), \quad \gamma = 0, 1,$
$\quad\langle(0';\emptyset), (1';\beta), (1';\beta+3)\rangle \bmod (6;7), \quad \beta = 0, 1, 2.$

Lemma 6.6. *For every* $n \geq 3$, $3n \in GD(3, 2, 3)$ *holds.*

Proof. By Lemmas 5.3 and 2.16, it suffices to show that $3n \in GD(3, 2, 3)$ for every $n \in K_3$. For $n \in \{3, 4, 6\}$ this follows from Lemmas 5.6 and 6.1; for $n = 5$ this follows Lemma 6.4; and for $n = 8$ we prove:

$24 \in GD(3, 2, 3)$. $X = Z(3, 2) \times (Z(7, 3) \cup \{\infty\})$.

$\mathcal{P}$ = Blocks of GD[3, 1, 3; 21], (exists by Lemma 6.4) on $Z(3) \times Z(7)$,
$\quad\langle(\emptyset;\infty), (0;\emptyset), (1;3\alpha)\rangle \bmod (3;7), \quad \alpha = 0, 1,$

$\langle(\emptyset;\infty),(\emptyset;0),(\emptyset;3)\rangle \bmod (3;7)$,
$\langle(\emptyset;\emptyset),(0;\alpha+1),(0;\alpha+4)\rangle \bmod (3;7)$, $\alpha = 0, 1$.

Lemma 6.7. *For every $n \geqslant 3$, $2n \in GD(3,3,2)$ holds.*

Proof. By Lemmas 5.3 and 2.16, it suffices to show that $2n \in GD(3,3,2)$ for every $n \in K_3$. For $n \in \{3,4,6\}$ this follows from Lemma 6.3; for $n = 5$ this follows from Lemmas 5.5 and 6.1; for $n = 8$ we prove:
$16 \in GD(3,3,2)$. $X = Z(2) \times (Z(7,3) \cup \{\infty\})$.
$\mathcal{B}$ = Blocks of GD[3, 1, 2; 14], (exists by Lemma 6.3) on $Z(2) \times Z(7)$, 2 times,
$\langle(\emptyset;\infty),(\emptyset;\emptyset),(0;2\alpha)\rangle \bmod (2;7)$, $\alpha = 0, 1, 2$,
$\langle(\emptyset;0),(\emptyset;2),(\emptyset;4)\rangle \bmod (2;7)$.

Theorem 6.2. *Let m, λ and v be positive integers. A necessary and sufficient condition for the existence of a group divisible design* $GD[3,\lambda,m;v]$ *is that*

$$v \equiv 0 \pmod{m},\ v \geqslant 3m,\ \lambda(v-m) \equiv 0 \pmod{2}$$

$$\textit{and} \quad \lambda v(v-m) \equiv 0 \pmod{6}.$$

Proof. The necessity follows from Theorem 6.1. In order to prove sufficiency we note that every pair of values of λ and m determines the values of v for which the condition of the theorem is satisfied. By Lem-

Table 6.1.

λ	m	v	$n = v/m$	Proof
1	1	1 or 3 (mod 6)	1 or 3 (mod 6)	Lemma 5.4.
1	2	0 or 2 (mod 6)	0 or 1 (mod 3)	Lemma 6.3.
1	3	3 (mod 6)	1 (mod 2)	Lemma 6.4.
1	6	0 (mod 6), $v \geqslant 18$	every $n \geqslant 3$	Lemma 6.5.
2	1	0 or 1 (mod 3)	0 or 1 (mod 3)	Lemma 5.6.
2	2	0 or 2 (mod 6)	0 or 1 (mod 3)	Lemma 6.3.
2	3	0 (mod 3), $v \geqslant 9$	every $n \geqslant 3$	Lemma 6.6.
3	1	1 (mod 2)	1 (mod 2)	Lemma 5.5.
3	2	0 (mod 2), $v \geqslant 6$	every $n \geqslant 3$	Lemma 6.7.
3	3	3 (mod 6)	1 (mod 2)	Lemma 6.4.
6	1	every $v \geqslant 3$	every $n \geqslant 3$	Lemma 5.7.

mas 2.3 and 6.1 it suffices to consider only those values of λ and of m which are factors of 6. In all these cases the existence of the relevant group divisible designs is proved in the lemmas listed in Table 6.1.

7. Covering and packing designs

7.1. Introduction

So far we have been dealing with exact designs, i.e., designs in which every pair of points is contained in a fixed number (λ) of blocks. Such designs exist only if the total number of points satisfies certain conditions as proved in the foregoing sections. In the general case – when the total number of points does not necessarily satisfy the prescribed conditions – an exact design not always exists and in order to deal with such cases we introduce coverings or ample designs in which every pair of points is included in at least λ blocks and packings or scarce designs in which every pair of points is included in at most λ blocks. Naturally we are interested in ample designs having a minimal number of blocks and – conversely – in scarce designs having a maximal number of blocks. More formally: A design $(X, \mathcal{B})$ is called an *ample design* AD$[k, \lambda; v, b]$ (or – respectively – a *scarce design* SD$[k, \lambda; v, b]$) if

(i) $|X| = v$;

(ii) the blocks are of size k;

(iii) $|\mathcal{B}| = b$;

(iv) every pairset $\{x, y\} \subset X$ is included in at least (at most) λ blocks of $\mathcal{B}$.

Denote by $\alpha(k, \lambda; v)$ the smallest number b of blocks for which AD$[k, \lambda; v, b]$ exists and by $\sigma(k, \lambda; v)$ the greatest value of b for which SD$[k, \lambda; v, b]$ exists. Clearly

$$\sigma(k, \lambda; v) \leqslant \lambda v(v-1)/(k(k-1)) \leqslant \alpha(k, \lambda; v)$$

and the equality sign on both sides holds if and only if a BIBD $B[k, \lambda; v]$ exists.

For the general case, Schonheim [27,28] introduced the notation

$$\phi(k, \lambda; v) = \left\lceil \frac{v}{k} \left\lceil \frac{v-1}{k-1} \lambda \right\rceil \right\rceil, \qquad \psi(k, \lambda; v) = \left\lfloor \frac{v}{k} \left\lfloor \frac{v-1}{k-1} \lambda \right\rfloor \right\rfloor,$$

where $\lceil x \rceil$ is the smallest and $\lfloor x \rfloor$ the largest integer satisfying $\lfloor x \rfloor \leq x \leq \lceil x \rceil$, and proved:

Theorem 7.1. *For every positive integers k, λ and $v \geq k$.*

$$\sigma(k,\lambda;v) \leq \psi(k,\lambda;v) \leq \lambda v(v-1)/(k(k-1)) \leq \phi(k,\lambda;v) \leq \alpha(k,\lambda;v).$$

Proof. In the case of ample design the replication number of every point is clearly not smaller than $\lceil \lambda(v-1)/(k-1) \rceil$ and accordingly the total number of blocks is not smaller than $\phi(k,\lambda;v)$. The converse goes for scarce designs.

In the case $k = 3$, $\lambda = 1$, Fort and Hedlund proved [13] that $\alpha(3,1;v) = \phi(3,1;v)$ for every $v \geq 3$. Later, Schönheim proved [28] that $\sigma(3,1;v) = \psi(3,1;v)$ if $v \not\equiv 5 \pmod 6$ and $\sigma(3,1;v) = \psi(3,1;v) - 1$ if $v \equiv 5 \pmod 6$.

In this section we shall determine $\alpha(3,\lambda;v)$ and $\sigma(3,\lambda;v)$ for $k = 3$, every λ and every $v \geq 3$.

7.2. Basic lemmas on covering and packing

A design $(X, \mathcal{B})$ with $X' \subset X$ is called an *almost ample design* $AD^*[k,\lambda;v(t),b]$ (or – respectively – an *almost scarce design* $SD^*[k,\lambda;v(t),b]$) if

(i) $|X| = v$;

(ii) the blocks are of size k;

(iii) $|\mathcal{B}| = b$;

(iv) $|X'| = t$;

(v) every pairset $\{x,y\} \subset X$ such that $\{x,y\} \not\subset X'$ is included in at least (at most) λ blocks of $\mathcal{B}$;

(vi) no pairset $\{x,y\} \subset X'$ is included in any block of $\mathcal{B}$.

Let $v \equiv t \bmod (k(k-1))$; it is easily seen that for given values of k, λ and t, both $\phi(k,\lambda;v)$ and $\psi(k,\lambda;v)$ are polynomials in v of the form $P(v) = (\lambda v^2 + cv + d)/(k(k-1))$, where c and d are integers. Later we shall prove that for $k = 3$ also $\alpha(3,\lambda;v)$ and $\sigma(3,\lambda;v)$ are such polynomials. The following lemma will be most helpful for this purpose.

Lemma 7.1. *For given integers k, λ and m, let $mn \in \mathrm{GD}(k,\lambda,m)$ for* *every integer* $n \geq k$. *If in addition for* $u = m + t$ $(0 \leq t \leq m)$, *both designs*

$$\mathrm{AD}[k, \lambda; u, (\lambda u^2 + a_1 u + a_0)/(k(k-1))]$$

and

$$\mathrm{AD}^*[k, \lambda; u(t), m(\lambda(u+t) + a_1)/(k(k-1))]$$

exist, then

$$\alpha(k, \lambda; v) \leq (\lambda v^2 + a_1 v + a_0)/(k(k-1))$$

for every $v = mn + t$.

Similarly if both designs

$$\mathrm{SD}[k, \lambda; u, (\lambda u^2 + s_1 u + s_0)/(k(k-1))]$$

and

$$\mathrm{SD}^*[k, \lambda; u(t), m(\lambda(u+t) + s_1)/(k(k-1))]$$

exist, then

$$\sigma(k, \lambda; v) \geq (\lambda v^2 + s_1 v + s_0)/(k(k-1))$$

for every $v = mn + t$.

Proof. We shall prove the first part of the lemma, the proof of the second part being analogous. We have to prove the existence of an ample design

$$\mathrm{AD}[k, \lambda; v, (\lambda v^2 + a_1 v + a_0)/(k(k-1))]$$

for every $v = mn + t$ $(n \geq k)$. Let $X = I(m) \times I(n) \cup I(t)$. Form a group divisible design $\mathrm{GD}[k, \lambda, m; mn]$ on $I(m) \times I(n)$. As mentioned in the proof of Theorem 6.1, this design has $m^2 n(n-1)\lambda/(k(k-1))$ blocks. Further form

$$\mathrm{AD}[k, \lambda; u, (\lambda u^2 + a_1 u + a_0)/(k(k-1))]$$

on $I(m) \times \{0\} \cup I(t)$ and

$$\mathrm{AD}^*[k, \lambda; u(t), m(\lambda(u+t) + a_1)/(k(k-1))]$$

on $I(m) \times \{i\} \cup I(t)$, where $i = 1, 2, ..., n-1$ and $X' = I(t)$. It is easily checked that the total number of blocks is $(\lambda v^2 + a_1 v + a_0)/(k-1))$.

Lemma 7.2. *If* $\lambda(v-1) \equiv 0 \pmod{k-1}$ *and* $\lambda v(v-1)/(k-1) \equiv -1 \pmod{k}$ *hold, then* $\alpha(k, \lambda; v) \geqslant \phi(k, \lambda; v) + 1$. *Similarly, if* $\lambda(v-1) \equiv 0 \pmod{k-1}$ *and* $\lambda v(v-1)/(k-1) \equiv 1 \pmod{k}$ *hold, then* $\sigma(k, \lambda; v) \leqslant \psi(k, \lambda; v) - 1$.

Proof. We shall prove the first part of the lemma, the proof of the second part being analogous. Suppose that $\alpha(k, \lambda; v) < \phi(k, \lambda; v) + 1$, then by Theorem 7.1, $\alpha(k, \lambda; v) = \phi(k, \lambda; v)$ and

$$\mathrm{AD}[k, \lambda; v, (\lambda v(v-1) + k - 1)/(k(k-1))]$$

exists. This means that each pair of points occurs in λ blocks, but some pairs occur in more than λ blocks and that the total number of these additional occurrences is $\frac{1}{2}(k-1)$. Let p_0 be a point belonging to at least one pair which occurs in more than λ blocks, then p_0 occurs in $\pi_0 = (\lambda(v-1) + d)/(k-1)$ blocks, where d is an integer $1 \leqslant d \leqslant \frac{1}{2}(k-1)$. However, this is impossible, because π_0 is not an integer.

7.3. Designs with block-size 3

In the following lemmas we shall prove that for $v \equiv t \pmod{m}$ and for given values of λ, m and t, $\alpha(3, \lambda; v) \leqslant \phi(3, \lambda; v) + \epsilon$ and $\sigma(3, \lambda; v) \geqslant \psi(3, \lambda; v) - \epsilon'$, where $\epsilon \in \{0, 1\}$ and $\epsilon' \in \{0, 1\}$.

Whenever λ is odd we shall take $m = 6$ and whenever λ is even we shall take $m = 3$. Accordingly – by Theorem 6.2 and Table 6.1 – $mn \in \mathrm{GD}(3, \lambda, m)$ for every $n \geqslant 3$. By Lemma 7.1, it will be sufficient to prove the existence of the designs $\mathrm{AD}[3, \lambda; v, b]$ and $\mathrm{SD}[3, \lambda; v, b]$ – for the relevant values of b – for $v = m + t$, $v = 2m + t$ and if $t \geqslant 3$, also for $v = t$. Moreover – whenever $t \geqslant 2$ – we have to prove the existence of $\mathrm{AD}^*[3, \lambda; m + t(t), b]$ and $\mathrm{SD}^*[3, \lambda; m + t(t), b]$ with the relevant values of b.

The proofs of $\alpha(3, \lambda; v) \leqslant \phi(3, \lambda; v) + \epsilon$ and $\sigma(3, \lambda; v) \geqslant \psi(3, \lambda; v) - \epsilon'$ will be given by construction of designs with number of blocks satisfying the equalities in the above formulas. Evidently this proves the relevant conjectures.

Lemma 7.3. *If* $v \equiv 0 \pmod{6}$, *then* $\alpha(3, 1; v) \leqslant \phi(3, 1; v) = \frac{1}{6}v^2$.

Table 7.1.

Design	Construction
$SD^*[3,1;10(4),12]$	$X' = \{(\infty_i): i = 0,1,2,3\}$. $X = Z(2) \times Z(3) \cup X'$. $\mathcal{B} = \langle(\emptyset,\beta'),(\emptyset,(\beta+1)'),(\infty_\beta)\rangle \bmod (2,-)$, $\beta = 0,1,2$, $\langle(\emptyset,(\beta+2)'),(0,(\beta+2)'),(\infty_\beta)\rangle$, $\beta = 0,1,2$, $\langle(\emptyset,0'),(0,1'),(\infty_3)\rangle \bmod (-,3)$.
$SD^*[3,1;16(4),36]$ [‡]	$X' = \{(\infty_i): i = 0,1,2,3\}$. $X = Z(2) \times Z(6) \cup X'$. $\mathcal{B} = \langle(\emptyset,0'),(0,\beta'),(\infty_\beta)\rangle \bmod (-,6)$, $\beta = 0,1,2,3$, $\langle(\emptyset,0'),(0,4'),(0,5')\rangle \bmod (-,6)$, $\langle(\emptyset,(2\gamma)'),(\emptyset,(2\gamma+1)'),(\emptyset,(2\gamma+3)')\rangle$, $\gamma = 0,1,2$, $\langle(0,\delta'),(0,(\delta+2)'),(0,(\delta+4)')\rangle$, $\delta = 0,1$, $\langle(\emptyset,0'),(\emptyset,2'),(\emptyset,4')\rangle$.
$SD[3,1;4,1]$	$X = I(4)$. $\mathcal{B} = \langle 0,1,2\rangle$.
$SD[3,1;10,13]$	$SD^*[3,1;10(4),12]$ and $SD[3,1;4,1]$ on X'.
$SD[3,1;16,37]$	$SD^*[3,1;16(4),36]$ and $SD[3,1;4,1]$ on X'.

‡ For further reference.

Proof. Here $t = 0$, $m = 6$, $\lambda = 1$. We prove the existence of:
$AD[3,1;6,6]$. $X = Z(6)$. $\mathcal{B} = \langle 0', 1', 3'\rangle \bmod 6$.
$AD[3,1;12,24]$. $X = Z(12)$.

$$\begin{aligned}\mathcal{B} = {} & \langle\beta',(\beta+2)',(\beta+6)'\rangle, \quad \beta = 0,1,\dots,5,\\ & \langle\beta',(\beta+8)',(\beta+9)'\rangle, \quad \beta = 0,1,\dots,5,\\ & \langle(\beta+3)',(\beta+6)',(\beta+8)'\rangle, \quad \beta = 0,1,\dots,5,\\ & \langle(\beta+2)',(\beta+3)',(\beta+9)'\rangle, \quad \beta = 0,1,\dots,5.\end{aligned}$$

Lemma 7.4. *If* $v \equiv 2$ *or* 4 (mod 6), *then* $\alpha(3,1;v) \leqslant \phi(3,1;v) = \frac{1}{6}(v^2+2)$.

Proof. We prove this lemma directly – without making use of Lemma 7.1. Let $X = I(v-1) \cup \{\infty\}$.

$$\begin{aligned}\mathcal{B} = {} & \text{Blocks of } B[3,1;v-1], \quad \text{(exists by Lemma 5.4) on } I(v-1),\\ & \langle(2\beta)',(2\beta+1)',\infty\rangle, \quad \beta = 0,1,\dots,\tfrac{1}{2}(v-4),\\ & \langle 0',(v-2)',\infty\rangle.\end{aligned}$$

Lemma 7.5. *If* $v \equiv 0$ *or* 2 (mod 6), *then* $\sigma(3,1;v) \geqslant \psi(3,1;v) = \frac{1}{6}v(v-2)$.

Proof. As above. In the BIBD $B[3,1;v+1]$ delete one point and all the blocks in which it occurs.

Lemma 7.6. *If* $v \equiv 4$ (mod 6), *then* $\sigma(3,1;v) \geqslant \psi(3,1;v) = \frac{1}{6}(v^2-2v-2)$.

Table 7.2.

Design	Construction
AD*[3, 1; 11(5), 15] SD*[3, 1; 11(5), 15]	$X' = \{(\infty_i): i = 0, 1, 2, 3, 4\}$, $X = Z(2) \times Z(3) \cup X'$. $\mathcal{B} = \langle(\emptyset, \beta'), (\emptyset, (\beta+1)'), (\infty_\beta)\rangle \bmod (2, -)$, $\beta = 0, 1, 2$, $\langle(\emptyset, (\beta+2)'), (0, (\beta+2)'), (\infty_\beta)\rangle$, $\beta = 0, 1, 2$, $\langle(\emptyset, 2'), (0, \gamma'), (\infty_{\gamma+3})\rangle \bmod (-, 3)$, $\gamma = 0, 1$.
AD*[3, 1; 17(5), 42][‡] SD*[3, 1; 17(5), 42][‡]	$X' = \{(\infty_i): i = 0, 1, 2, 3, 4\}$, $X = Z(3) \times GF(4, x^2 = x+1) \cup X'$. $\mathcal{B} = \langle(\beta', \emptyset), ((\beta+1)', \emptyset), (\infty_\beta)\rangle \bmod (-, 4)$, $\beta = 0, 1, 2$, $\langle((\beta+2)', \emptyset), ((\beta+2)', 2), (\infty_\beta)\rangle$, $\beta = 0, 1, 2$, $\langle((\beta+2)', 0), ((\beta+2)', 1), (\infty_\beta)\rangle$, $\beta = 0, 1, 2$, $\langle(0', \emptyset), (0', \gamma), (\infty_{\gamma+3})\rangle \bmod (3, -)$, $\gamma = 0, 1$, $\langle(0', 2), (0', 1-\gamma), (\infty_{\gamma+3})\rangle \bmod (3, -)$, $\gamma = 0, 1$, $\langle(0', \emptyset), (1', \beta), (2', \beta+1)\rangle \bmod (-, 4)$, $\beta = 0, 1, 2$.
AD[3, 1; 5, 4]	$X = I(5)$. $\mathcal{B} = \langle \beta', 3', 4' \rangle$, $\beta = 0, 1, 2$, $\langle 0', 1', 2' \rangle$.
AD[3, 1; 11, 19]	AD*[3, 1; 11(5), 15] and AD[3, 1; 5, 4] on X'.
AD[3, 1; 17, 46]	AD*[3, 1; 17(5), 42] and AD[3, 1; 5, 4] on X'.
SD[3, 1; 5, 2]	$X \doteq I(5)$. $\mathcal{B} = \langle 0', 1', 2' \rangle$, $\langle 0', 3', 4' \rangle$.
SD[3, 1; 11, 17]	SD*[3, 1; 11(5), 15] and SD[3, 1; 5, 2] on X'.
SD[3, 1; 17, 44]	SD*[3, 1; 17(5), 42] and SD[3, 1; 5, 2] on X'.

‡ For further reference.

Proof. In this case $t = 4$, $m = 6$, $\lambda = 1$. The construction of the required designs is given in Table 7.1.

Lemma 7.7. *If* $v \equiv 5 \pmod 6$, *then* $\alpha(3, 1; v) \leqslant \phi(3, 1; v) = \frac{1}{6}(v^2 - v + 4)$ *and* $\sigma(3, 1; v) \geqslant \psi(3, 1; v) - 1 = \frac{1}{6}(v^2 - v - 8)$.

Table 7.3

Design	Construction
AD*[3, 2; 5(2), 6] SD*[3, 2; 5(2), 6] SD[3, 2; 5, 6]	$X' = \{\infty_i: i = 0, 1\}$, $X = Z(3) \cup X'$. $\mathcal{B} = \langle 0', 1', \infty_\beta \rangle \bmod 3$, $\beta = 0, 1$.
AD*[3, 2; 8(2), 18][‡] SD[3, 2; 8, 18]	$X' = \{\infty_i: i = 0, 1\}$, $X = Z(6) \cup X'$. $\mathcal{B} = \langle 0', (\beta+1)', \infty_\beta \rangle \bmod 6$, $\beta = 0, 1$, $\langle 0', 1', 3' \rangle \bmod 6$.
AD[3, 2; 5, 8]	AD*[3, 2; 5(2), 6] and $\langle 0', \infty_0, \infty_1 \rangle$, 2 times.
AD[3, 2; 8, 20]	AD*[3, 2; 8(2), 18] and $\langle 0', \infty_0, \infty_1 \rangle$, 2 times.

‡ For further reference.

Table 7.4.

Design	Construction
$AD^*[3, 3; 8(2), 28]$	$X' = \{\infty_i : i = 0, 1\}$, $X = Z(6) \cup X'$. $\mathfrak{B} = \langle 0', (\beta+1)', \infty_\beta \rangle \bmod 6$, $\beta = 0, 1$, $\langle \gamma', (\gamma+3)', \infty_\beta \rangle$, $\beta = 0, 1$, $\gamma = 0, 1, 2$, $\langle 0', 1', 3' \rangle \bmod 6$, $\langle (2\gamma)', (2\gamma+1)', (2\gamma+2)' \rangle$, $\gamma = 0, 1, 2$, $\langle 1', 3', 5' \rangle$.
$AD[3, 3; 8, 30]$	$X = Z(6) \cup \{\infty_i : i = 0, 1\}$. $\mathfrak{B} = \langle 0', \infty_0, \infty_1 \rangle \bmod 6$, $\langle 0', (\beta+1)', \infty_\beta \rangle \bmod 6$, $\beta = 0, 1$, $\langle 0', 1', 3' \rangle \bmod 6$, 2 times.
$AD[3, 3; 14, 94]$	$X = Z(2) \times Z(6) \cup \{(\infty_i) : i = 0, 1\}$. $\mathfrak{B} = \langle (\emptyset, 0'), (\infty_0), (\infty_1) \rangle \bmod (-, 6)$, $\langle (\emptyset, 0'), (0, (\beta+3\gamma)'), (\infty_\beta) \rangle \bmod (-, 6)$, $\beta = 0, 1$, $\gamma = 0, 1$, $\langle (0, \delta'), (0, (\delta+3)'), (\infty_\beta) \rangle$, $\beta = 0, 1$, $\delta = 0, 1, 2$, $\langle (\emptyset, 0'), (0, (2\gamma)'), (0, (2\gamma+1)') \rangle \bmod (2, 6)$, $\gamma = 0, 1$, $\langle (\emptyset, 0'), (\emptyset, 1'), (\emptyset, 3') \rangle \bmod (2, 6)$, $\langle (0, 0'), (\emptyset, (3\gamma+2)'), (\emptyset, (3\gamma+4)') \rangle \bmod (-, 6)$, $\gamma = 0, 1$, $\langle (0, 0'), (\emptyset, 1'), (\emptyset, 4') \rangle \bmod (-, 6)$, $\langle (0, \gamma'), (0, (\gamma+2)'), (0, (\gamma+4)') \rangle$, 2 times, $\gamma = 0, 1$.

Proof. Here $t = 5$, $m = 6$, $\lambda = 1$. The needed constructions are given in Table 7.2.

Lemma 7.8. *If* $v \equiv 2 \pmod 3$, *then* $\alpha(3, 2; v) \leq \phi(3, 2; v) + 1 = \frac{1}{3}(v^2 - v + 4)$ *and* $\sigma(3, 2; v) \geq \psi(3, 2; v) = \frac{1}{3}(v^2 - v - 2)$.

Proof. $t = 2$, $m = 3$, $\lambda = 2$. The necessary constructions are in Table 7.3.

Lemma 7.9. *If* $v \equiv 0 \pmod 6$, *then* $\alpha(3, 3; v) \leq \phi(3, 3; v) = \frac{1}{6}v(3v - 2)$ *and* $\sigma(3, 3; v) \geq \psi(3, 3; v) = \frac{1}{6}v(3v - 4)$.

Proof. Form $B[3, 2; v]$ by Lemma 5.6 and $AD[3, 1; v, \frac{1}{6}v^2]$ by Lemma 7.3, or $SD[3, 1; v, \frac{1}{6}v(v-2)]$ by Lemma 7.5, respectively.

Lemma 7.10. *If* $v \equiv 2 \pmod 6$, *then* $\alpha(3, 3; v) \leq \phi(3, 3; v) = \frac{1}{6}(3v^2 - 2v + 4)$.

Proof. $t = 2$, $m = 6$, $\lambda = 3$. For the necessary constructions see Table 7.4.

Lemma 7.11. *If* $v \equiv 2 \pmod 6$, *then* $\sigma(3, 3; v) \geq \psi(3, 3; v) = \frac{1}{6}(3v^2 - 4v - 4)$.

Table 7.5.

Design	Construction
$AD^*[3,4;5(2),12]$	$X' = \{\infty_i : i = 0,1\}$, $X = Z(3) \cup X'$.
	$\mathfrak{B} = \langle 0', 1', \infty_\beta \rangle$ mod 3, 2 times, $\beta = 0, 1$.
$AD[3,4;5,14]$	$B[3,3;5]$ by Lemma 5.5 and $AD[3,1;5,4]$ by Lemma 7.7.
$AD[3,4;8,38]$	$X = Z(2) \times Z(3) \cup \{(\infty_i): i = 0,1\}$.
	$\mathfrak{B} = \langle (\emptyset, 0'), (\infty_0), (\infty_1) \rangle$ mod (2, 3),
	$\langle (\emptyset, 0'), (\emptyset, 1'), (\infty_0) \rangle$ mod (2, 3),
	$\langle (\emptyset, 0'), (0, 0'), (\infty_0) \rangle$ mod (–, 3),
	$\langle (\emptyset, 0'), (0, \beta'), (\infty_1) \rangle$ mod (–, 3), $\beta = 0, 1, 2$,
	$\langle (\emptyset, 0'), (0, \gamma'), (0, (\gamma+1)') \rangle$ mod (2, 3), $\gamma = 0, 1$,
	$\langle (\emptyset, 0'), (\emptyset, 1'), (0, 2') \rangle$ mod (2, –).

Proof. Form $SD[3,2;v,\frac{1}{3}(v^2-v-2)]$ by Lemma 7.8, and $SD[3,1;v,\frac{1}{6}v(v-2)]$ by Lemma 7.5.

Lemma 7.12. *If* $v \equiv 4 \pmod 6$, *then* $\alpha(3,3;v) \leqslant \phi(3,3;v) = \frac{1}{6}(3v^2-2v+2)$ *and* $\sigma(3,3;v) \geqslant \psi(3,3;v) = \frac{1}{6}(3v^2-4v-2)$.

Proof. Form $B[3,2;v]$ by Lemma 5.6 and $AD[3,1;v,\frac{1}{6}(v^2+2)]$ by Lemma 7.4 or $SD[3,1;v,\frac{1}{6}(v^2-2v-2)]$ by Lemma 7.6, respectively.

Lemma 7.13. *If* $v \equiv 2 \pmod 3$, *then* $\alpha(3,4;v) \leqslant \phi(3,4;v) = \frac{2}{3}(v^2-v+1)$.

Proof. $t = 2$, $m = 3$, $\lambda = 4$. For constructions see Table 7.5.

Lemma 7.14. *If* $v \equiv 2 \pmod 3$, *then* $\sigma(3,4;v) \geqslant \psi(3,4;v) - 1 = \frac{2}{3}(v^2-v-2)$.

Proof. Form $SD[3,2;v,\frac{1}{3}(v^2-v-2)]$ as by Lemma 7.8, 2 times.

Lemma 7.15. *If* $v \equiv 0 \pmod 6$, *then* $\alpha(3,5;v) \leqslant \phi(3,5;v) = \frac{1}{6}v(5v-4)$ *and* $\sigma(3,5;v) \geqslant \psi(3,5;v) = \frac{1}{6}v(5v-6)$.

Proof. Form $B[3,2;v]$ by Lemma 5.6, 2 times, and $AD[3,1;v,\frac{1}{6}v^2]$ by Lemma 7.3 or – respectively – $SD[3,1;v,\frac{1}{6}v(v-2)]$ by Lemma 7.5.

Lemma 7.16. *If* $v \equiv 2 \pmod 6$, *then* $\alpha(3,5;v) \leqslant \phi(3,5;v) = \frac{1}{6}v(5v-4)$ *and* $\sigma(3,5;v) \geqslant \psi(3,5;v) = \frac{1}{6}(5v^2-6v-2)$.

Table 7.6.

Design	Construction
$AD^*[3,5;8(2),46]$	$X' = \{(\infty_i): i = 0,1\}$, $X = Z(2) \times Z(3) \cup X'$. $\mathfrak{B}$ = Blocks of $B[3,2;6]$ on $Z(2) \times Z(3)$, Blocks of $AD[3,1;6,6]$ on $Z(2) \times Z(3)$, $\langle(\emptyset,0'),(\emptyset,1'),(\infty_\beta)\rangle \bmod (2,3)$, $\beta = 0,1$, $\langle(\emptyset,0'),(0,\gamma'),(\infty_\beta)\rangle \bmod (-,3)$, $\beta = 0,1$, $\gamma = 0,1,2$.
$SD^*[3,5;8(2),44]$	$X' = \{(\infty_i): i = 0,1\}$, $X = Z(2) \times Z(3) \cup X'$. $\mathfrak{B}$ = Blocks of $B[3,2;6]$ on $Z(2) \times Z(3)$, Blocks of $SD[3,1;6,4]$ on $Z(2) \times Z(3)$, $\langle(\emptyset,0'),(\emptyset,1'),(\infty_\beta)\rangle \bmod (2,3)$, $\beta = 0,1$, $\langle(\emptyset,0'),(0,\gamma'),(\infty_\beta)\rangle \bmod (-,3)$, $\beta = 0,1$, $\gamma = 0,1,2$.
$AD[3,5;8,48]$	$X = Z(2) \times Z(3) \cup \{(\infty_i): i = 0,1\}$. $\mathfrak{B} = \langle(\emptyset,0'),(\infty_0),(\infty_1)\rangle \bmod (2,3)$, $\langle(\emptyset,0'),(\emptyset,1'),(\infty_\beta)\rangle \bmod (2,3)$, $\beta = 0,1$, $\langle(\emptyset,0'),(0,1'),(\infty_\beta)\rangle \bmod (2,3)$, $\beta = 0,1$, $\langle(\emptyset,0'),(0,0'),(0,1')\rangle \bmod (2,3)$, 3 times.
$AD[3,5;14,154]$	$X = Z(2) \times Z(7,3)$. $\mathfrak{B} = \langle(\emptyset,\emptyset),(0,\beta),(0,\beta+3)\rangle \bmod (2,7)$, 2 times, $\beta = 0,1,2$, $\langle(\emptyset,\emptyset),(0,\emptyset),(0,\gamma)\rangle \bmod (-,7)$, $\gamma = 0,1,\ldots,5$, $\langle(\emptyset,0),(\emptyset,2),(\emptyset,4)\rangle \bmod (2,7)$, $\langle(\emptyset,0),(\emptyset,2),(\emptyset,4)\rangle \bmod (-,7)$, 2 times.
$SD[3,5;8,45]$	$X = Z(7,3) \cup \{\infty\}$. $\mathfrak{B} = \langle\gamma,\gamma+3,\infty\rangle \bmod 7$, $\gamma = 0,1$, $\langle\beta',(\beta+3)',\infty\rangle$, $\beta = 0,1,2$, $\langle 0,2,4\rangle \bmod 7$, 4 times.
$SD[3,5;14,149]$	$X = Z(13,2) \cup \{\infty\}$. $\mathfrak{B} = \langle 2\beta,2\beta+6,\infty\rangle \bmod 13$, $\beta = 0,1$, $\langle\gamma',(\gamma+6)',\infty\rangle$, $\gamma = 0,1,\ldots,5$, $\langle 0,4,8\rangle \bmod 13$, 4 times, $\langle 1,5,9\rangle \bmod 13$, 5 times.

Proof. Here $t = 2$, $m = 6$, $\lambda = 5$. The construction of the designs needed by Lemma 7.1 is given in Table 7.6.

Lemma 7.17. *If* $v \equiv 4 \pmod 6$, *then* $\alpha(3,5;v) \leqslant \phi(3,5;v) = \frac{1}{6}(5v^2 - 4v + 2)$ *and* $\sigma(3,5;v) \geqslant \psi(3,5;v) = \frac{1}{6}(5v^2 - 6v - 2)$.

Proof. Form $B[3,2;v]$ by Lemma 5.6, 2 times, and $AD[3,1;v,\frac{1}{6}(v^2+2)]$ by Lemma 7.4, or – respectively – $SD[3,1;v,\frac{1}{6}(v^2-2v-2)]$ by Lemma 7.6.

Lemma 7.18. *If* $v \equiv 5 \pmod 6$, *then* $\alpha(3,5;v) \leqslant \phi(3,5;v) + 1 = \frac{1}{6}(5v^2 - 5v + 8)$ *and* $\sigma(3,5;v) \geqslant \psi(3,5;v) = \frac{1}{6}(5v^2 - 5v - 4)$.

Table 7.7.
$\alpha(3,\lambda;v) \leqslant \phi(3,\lambda;v)+\epsilon$, $\sigma(3,\lambda;v) \geqslant \psi(3,\lambda;v)-\epsilon'$.

λ	v	ϵ	Proof	ϵ'	Proof
1	0 (mod 6)	0	Lemma 7.3.	0	Lemma 7.5.
1	1 (mod 6)	0	Lemma 5.4.	0	Lemma 5.4.
1	2 (mod 6)	0	Lemma 7.4.	0	Lemma 7.5.
1	3 (mod 6)	0	Lemma 5.4.	0	Lemma 5.4.
1	4 (mod 6)	0	Lemma 7.4.	0	Lemma 7.6.
1	5 (mod 6)	0	Lemma 7.7.	1	Lemma 7.7.
2	0 (mod 3)	0	Lemma 5.6.	0	Lemma 5.6.
2	1 (mod 3)	0	Lemma 5.6.	0	Lemma 5.6.
2	2 (mod 3)	1	Lemma 7.8.	0	Lemma 7.8.
3	1 (mod 2)	0	Lemma 5.5.	0	Lemma 5.5.
3	0 (mod 6)	0	Lemma 7.9.	0	Lemma 7.9.
3	2 (mod 6)	0	Lemma 7.10.	0	Lemma 7.11.
3	4 (mod 6)	0	Lemma 7.12.	0	Lemma 7.12.
4	0 (mod 3)	0	Lemma 5.6.	0	Lemma 5.6.
4	1 (mod 3)	0	Lemma 5.6.	0	Lemma 5.6.
4	2 (mod 3)	0	Lemma 7.13.	1	Lemma 7.14.
5	0 (mod 6)	0	Lemma 7.15.	0	Lemma 7.15.
5	1 (mod 6)	0	Lemma 5.4.	0	Lemma 5.4.
5	2 (mod 6)	0	Lemma 7.16.	0	Lemma 7.16.
5	3 (mod 6)	0	Lemma 5.4.	0	Lemma 5.4.
5	4 (mod 6)	0	Lemma 7.17.	0	Lemma 7.17.
5	5 (mod 6)	1	Lemma 7.18.	0	Lemma 7.18.

Proof. Form $B[3,3;v]$ by Lemma 5.5 and $\mathrm{AD}[3,2;v,\frac{1}{3}(v^2-v+4)]$ or $\mathrm{SD}[3,2;v,\frac{1}{3}(v^2-v-2)]$ by Lemma 7.8.

Theorem 7.2. *For every positive integers λ and $v \geqslant 3$,*

$$\alpha(3,\lambda;v)=\phi(3,\lambda;v)+\epsilon$$

and

$$\sigma(3,\lambda;v)=\psi(3,\lambda;v)-\epsilon',$$

where $\epsilon=1$ if both $v\equiv\lambda\equiv 2 \pmod 3$ and $\lambda(v-1)\equiv 0 \pmod 2$, and $\epsilon'=1$ if both $v\equiv\lambda+1\equiv 2 \pmod 3$ and $\lambda(v-1)\equiv 0 \pmod 2$, and $\epsilon=\epsilon'=0$ otherwise.

Proof. If $v\in B(3,\lambda)$, then clearly

$$\sigma(k,\lambda;v)=\psi(k,\lambda;v)=\lambda v(v-1)/(k(k-1))=\phi(k,\lambda;v)=\alpha(k,\lambda;v).$$

Further by Theorem 7.1 and Lemma 7.2, $\alpha(3,\lambda;v)$ is not smaller and $\sigma(3,\lambda;v)$ is not greater than the values indicated in the theorem. From the Lemmas 7.3–7.18 it follows that for $0 < \lambda < 6$, $\alpha(3, \lambda; v)$ is not greater and $\sigma(3, \lambda; v)$ is not smaller than the values indicated in the theorem as specified in Table 7.7. For $\lambda \geq 6$, let $\lambda = 6l + \lambda'$. In this case $AD[3, \lambda; v, \alpha(3, \lambda; v)]$ is obtained by taking l times $B[3, 6; v]$ by Lemma 5.7 and $AD[3, \lambda'; v, \alpha(3, \lambda'; v)]$ and similarly $SD[3, \lambda; v, \sigma(3, \lambda; v)]$ is obtained by taking l times $B[3, 6; v]$ and $SD[3, \lambda'; v, \sigma(3, \lambda'; v)]$.

Added in proof. With regard to the opening remarks of Section 5.4, we produce here a construction of a BIBD $B[6, 1; 106]$ found by W.H. Mills ("A new block design" to appear in Proc. Sixth Southeastern Conference on Combinatorics, Graph Theory and Computing, Florida Atlantic University, Boca Rayon, Fla. (1975)) in January 1975.

$B[6, 1; 106]$. $X = Z(2) \times Z(53, 2)$.

$$\begin{aligned}\mathcal{B} = {} & \langle(\emptyset, \emptyset), (\emptyset, 0), (\emptyset, 6), (\emptyset, 17), (\emptyset, 38), (0, \emptyset)\rangle \bmod (-, 53),\\ & \langle(\emptyset, \emptyset), (\emptyset, 37), (\emptyset, 42), (\emptyset, 47), (0, 36), (0, 41)\rangle \bmod (-, 53),\\ & \langle(\emptyset, \emptyset), (\emptyset, 14), (\emptyset, 46), (0, 3), (0, 4), (0, 21)\rangle \bmod (-, 53),\\ & \langle(\emptyset, \emptyset), (\emptyset, 31), (\emptyset, 34), (0, 19), (0, 24), (0, 51)\rangle \bmod (-, 53),\\ & \langle(\emptyset, \emptyset), (\emptyset, 13), (\emptyset, 24), (0, 9), (0, 27), (0, 39)\rangle \bmod (-, 53),\\ & \langle(\emptyset, \emptyset), (\emptyset, 2), (0, 13), (0, 16), (0, 30), (0, 44)\rangle \bmod (-, 53),\\ & \langle(\emptyset, \emptyset), (0, 1), (0, 14), (0, 28), (0, 42), (0, 46)\rangle \bmod (-, 53).\end{aligned}$$

It follows from Lemma 2.17 that for every non-negative integer n, if $v = 106.5^n + \frac{1}{4}(5^n - 1)$, then $v \in B(6, 1)$ holds.

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Characterization of Projective Graphs

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We denote the distance between vertices x and y of a graph by $d(x, y)$, and $p_{ij}(x, y) = |\{z : d(x, z) = i, d(y, z) = j\}|$. The (s, q, d)-projective graph is the graph having the s-dimensional subspaces of a d-dimensional vector space over $GF(q)$ as vertex set, and two vertices x, y adjacent iff $\dim(x \cap y) = s - 1$. These graphs are regular graphs. Also, there exist integers λ and $\mu > 4$ so that μ is a perfect square, $p_{11}(x, y) = \lambda$ whenever $d(x, y) = 1$, and $p_{11}(x, y) = \mu$ whenever $d(x, y) = 2$. The (s, q, d)-projective graphs where $2d/3 < s < d - 2$ and $(s, q, d) \neq (2d/3, 2, d)$, are characterized by the above conditions together with the property that there exists an integer r satisfying certain inequalities.

1. INTRODUCTION

Graphs considered here have at least one vertex, and have no loops or multiple edges. The vertex set of a graph G is denoted $V(G)$. If some edge joins vertices x and y we say that x and y are *adjacent*. Graphs G and H are *isomorphic* iff there is a bijection σ from $V(G)$ to $V(H)$ such that $\sigma(x)$ and $\sigma(y)$ are adjacent iff x and y are adjacent. We will say that the *distance* between vertices x and y of a graph is n (written $d(x, y) = n$) if there is a sequence $x = x_0, x_1, \ldots, x_n = y$ so that x_i is adjacent to x_{i-1} $(1 \leqslant i \leqslant n)$ and n is the least integer so that such a sequence exists. For a vertex x let $\Delta_i(x)$ be the set of vertices at distance i from x. We define (for $i, j \geqslant 0$ and $x, y \in V(G)$) $p_{ij}(x, y) = |\Delta_i(x) \cap \Delta_j(y)|$. We say that G is *regular* of *valence* n_1 if $p_{11}(x, x) = n_1$ for all vertices x. A *clique* is a set of mutually adjacent vertices.

Let V be a d-dimensional vector space over $GF(q)$. For $0 \leqslant i \leqslant d$ let W_i be the set of i-dimensional subspaces of V. Let G be the graph having W_s as vertex set and two vertices x and y adjacent iff $x \cap y \in W_{s-1}$. Then x and y are adjacent iff the span of x and y is in W_{s+1}. An *(s, q, d)-projective graph* is any graph isomorphic to G. A $(d - s, q, d)$-projective graph is isomorphic to an (s, q, d)-projective graph.

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THEOREM 1. *Let r and $q > 1$ be positive integers, and k, α be real numbers. Let G be a connected graph with the following properties.*

(g1) *G is regular of valence $r(k-1)$.*
(g2) $p_{11}(x, y) = k - 2 + \alpha$ *if* $d(x, y) = 1$.
(g3) $p_{11}(x, y) = (q + 1)^2$ *if* $d(x, y) = 2$.
(g4) $p_{31}(x, y) > (k - 1 + \alpha)(r - q - 2)$ *if* $d(x, y) = 2$.
(g5) $k > \max(1 + \frac{1}{2}(r + 1)(rq^2 + 2rq - 2\alpha), (q + 1)^2 + (2r - 1)\alpha)$.
(g6) $r > q + 1$.

Let s and d be real numbers defined by $k = (q^{s+1} - 1)/(q - 1)$ and $r = (q^{d-s} - 1)/(q - 1)$. Then s and d are integers, q is a prime power, and G is an (s, q, d)-projective graph.

The necessity of hypothesis (g6) is demonstrated in Section 4. In Section 3 we will sketch the proof of a converse to Theorem 1:

THEOREM 2. *Let G be an (s, q, d)-projective graph, $r = (q^{d-s} - 1)/(q - 1)$, $k = (q^{s+1} - 1)/(q - 1)$, and $\alpha = (r - 1)q$. Then G is connected and satisfies* (g1)–(g3). *If $s < d - 2$ then* (g6) *is satisfied. If $s \geqslant 2d/3$ and $(q, s) \neq (2, 2d/3)$ then* (g4) *and* (g5) *are satisfied.*

Since an (s, q, d)-projective graph is isomorphic to a $(d - s, q, d)$-projective graph, these theorems characterize (s, q, d)-projective graphs for which $2 < s \leqslant d/3$ or $2d/3 \leqslant s < d - 2$ (and if $q = 2$, then $s \neq d/3, 2d/3$).

The hypothesis that G be connected in Theorem 1 is not essential: if G satisfies all hypothesis of Theorem 1 except the hypothesis that G be connected, then each component of G satisfies all hypotheses of Theorem 1, so each component of G is an (s, q, d)-projective graph.

2. TOOLS

An *incidence structure* is an ordered triple (P, L, I) such that P and L are finite sets and $I \subseteq P \times L$. We will call elements of P *points* and elements of L *lines*. If $(p, m) \in I$ we say that p and m *are incident*, p *lies on* m, or m *contains* p. Incidence structures (P, L, I) and $(P'L'I')$ are *isomorphic* iff there are bijections $\sigma: P \to P'$ and $\tau: L \to L'$ so that for $(p, m) \in P \times L$, $(p, m) \in I$ iff $(\sigma(p), \tau(m)) \in I'$. Where P and L are both sets of subsets of some set X the incidence structure $(P, L, \subseteq)$ is the incidence structure in which p and m are incident iff $p \subseteq m$. Similarly, $(P, L, \in)$ is the incidence structure in which each line is a set of points and p is incident with m iff $p \in m$.

An *(r, k)-incidence structure* is an incidence structure (P, L, I) such that (1) for all $p, p' \in P$ $(p \neq p')$ there is at most one line incident with both p

and p'; (2) every point is incident with exactly r lines; (3) every line is incident with exactly k points.

Let $\pi = (P, L, I)$ be an incidence structure. We define $r(\pi)$ to be the integer so that some point is incident with exactly $r(\pi)$ lines and no point is incident with fewer than $r(\pi)$ lines. We let $k(\pi)$ be the integer so that some line is incident with $k(\pi)$ points and no line is incident with fewer than $k(\pi)$ points. The *adjacency graph* of π is the graph having P as vertex set and two vertices adjacent iff some line is incident with both. The *line graph* of π is the graph having L as vertex set and two vertices adjacent iff some point is incident with both. The *distance* between two points of π equals the distance between them in the adjacency graph of π. The *distance* between a point p and a line m is $\min\{d(p, q): (q, m) \in I\}$. The *dual* of π is the incidence structure $\pi^* = (L, P, I')$ where for any point p and line m, $(m, p) \in I'$ iff $(p, m) \in I$.

Let V be a d-dimensional vector space over $GF(q)$ and W_i be the set of i-dimensional subspaces of V. For $1 \leqslant s \leqslant d$ any incidence structure isomorphic to $(W_{s-1}, W_s, \subsetneq)$ is called an *(s, q, d)-projective incidence structure*. Note that the adjacency graph of an (s, q, d)-projective incidence structure is an $(s - 1, q, d)$-projective graph.

In [3] the following was shown.

THEOREM 3. *Let $q \geqslant 2$ be an integer and π be an incidence structure satisfying*

(f1) $3 \leqslant s < d - 1$, *where s and q are defined by* $k(\pi) = (q^s - 1)/(q - 1)$ *and* $r(\pi) = (q^{d-s+1} - 1)/(q - 1)$.

(f2) *There exists at most one line joining two distinct points.*

(f3) *If p is a point and m is a line such that $d(p, m) = 1$ then there are exactly $q + 1$ lines which contain p and intersect m.*

(f4) *If p and p' are points such that $d(p, p') = 2$, then there are exactly $q + 1$ lines m such that m contains p' and $d(p, m) = 1$.*

(f5) *The adjacency graph of π is connected.*

Then s and d are integers, q is a prime power, and π is an (s, q, d)-projective incidence structure. Conversely, for $3 \leqslant s < d - 1$ any (s, q, d)-projective incidence structure satisfies (f1)–(f5).

THEOREM 4 (Bose, Lasker [2]). *Let r be a positive integer, k, α, β be real numbers, and $\beta \geqslant 0$. Let G be a graph (not edgeless) which has the following properties.*

(b1) *G is regular of valence $r(k - 1)$.*

(b2) $p_{11}(x, y) = k - 2 + \alpha$ *for all* $x, y \in V(G)$, $d(x, y) = 1$.

(b3) $p_{11}(x, y) \leqslant \beta + 1$ *for all* $x, y \in V(G)$, $d(x, y) = 2$.

Define a grand clique to be a clique which is maximal and has at least $k - (r-1)\alpha$ vertices. Let

$$k > \max(1 + \tfrac{1}{2}(r+1)(r\beta - 2\alpha), \beta + 1 + (2r-1)\alpha).$$

Then any two adjacent vertices are in a unique grand clique, and each vertex is in exactly r grand cliques.

We have weakened the hypotheses of the theorem as stated in [2] from the requirement that k, α, β be nonnegative integers and $r\beta - 2\alpha \geqslant 0$ to the requirement that $\beta \geqslant 0$ and G be not edgeless. The proof in [2] is valid without modification for the theorem as stated here.

3. Proofs of Theorems

First we sketch the proof of Theorem 2. Let G be an (s, q, d)-projective graph. Let $r = (q^{d-s} - 1)/(q - 1)$ and $k = (q^{s+1} - 1)/(q - 1)$. We assume $2 \leqslant s \leqslant d - 2$ since in the contrary case G is a complete graph satisfying (g1)–(g3). Let V be a d-dimensional vector space and W_i be the set of i-dimensional subspaces of V. We may assume G is the adjacency graph of the $(s + 1, q, d)$-projective incidence structure $\pi = (W_s, W_{s+1}, \subseteq)$. As explained in [3], π is an (r, k)-incidence structure satisfying (f3) and (f4). Consequently G satisfies (g1)–(g3) where $\alpha = (r - 1)q$. If $s < d - 2$ then $r > q + 1$.

Let $x, y \in W_s$ such that $d(x, y) = 2$. We show $p_{31}(x, y) = (r - q - 1) \times (k - q^2 - q - 1)$. By (f4), $r - q - 1$ lines contain y and are at distance 2 from x. Let $m \in W_{s+1}$ and m contain y; then $d(x, m) = 2$ in π iff $x \cap m$ is an $(s - 2)$-dimensional space. For each such line m, $q^2 + q + 1$ points of m are at distance 2 from x. Therefore $p_{31}(x, y)$ has the value claimed. It may be computed that if $q = 2$ and $s \geqslant (2d + 1)/3$ then (g4) and (g5) are satisfied. If $q > 2$ and $s \geqslant 2d/3$ then (g4) and (g5) are satisfied.

In the remainder of this section our goal is the proof of Theorem 1. Toward this, let G be a connected graph satisfying (g1)–(g6). Define a grand clique to be a maximal clique which contains at least $k - (r - 1)\alpha$ vertices. Let L be the set of grand cliques of G, and $P = V(G)$. Vertices of G will be called points.

Lemma 1. *G is the adjacency graph of the incidence structure $\pi = (P, L, \in)$. Every point is contained in exactly r lines. Every two lines contain at most one point in common.*

Proof. By Theorem 4, $(P, L, \in)$ is an incidence structure in which lines are cliques of G, any two adjacent points of G are contained in a unique line, and each point is contained in exactly r lines. The conclusions of the lemma follow immediately from this.

Let $\lambda = k - 2 + \alpha$.

LEMMA 2. *Every line contains at most $\lambda + 2$ points.*

Proof. If m is a line containing points x, y then every point of $m - \{x, y\}$ is adjacent to both x and y.

Then $|m - \{x, y\}| \leqslant p_{11}(x, y) = \lambda$ by (g2).

LEMMA 3. *Let $x, y \in P$ and $d(x, y) = 2$. Let the lines containing x be $m_1, m_2, \ldots, m_r$ and the lines containing y be $n_1, n_2, \ldots, n_r$. Then (after re-indexing the lines if necessary) m_i and n_j intersect iff $i \leqslant q + 1$ and $j \leqslant q + 1$.*

Proof. Let $n_1, n_2, \ldots, n_t$ be the lines containing y which intersect at least one line m_i. No point of $n_1, n_2, \ldots, n_t$ is at distance 3 from x. Therefore

$$p_{31}(x, y) \leqslant \left| \bigcup_{j=t+1}^{r} n_j - y \right| \leqslant (r - t)(\lambda + 1).$$

From (g4) we see that $t < q + 2$. Hence at most $n_1, n_2, \ldots, n_{q+1}$ intersect any lines m_i; similarly (after reindexing) at most $m_1, m_2, \ldots, m_{q+1}$ intersect any lines n_j. Since each intersection contains exactly one point and $p_{11}(x, y) = (q + 1)^2$, this implies that m_i and n_j intersect whenever $i \leqslant q + 1$ and $j \leqslant q + 1$.

LEMMA 4. *For all points x and lines m so that $d(x, m) = 1$, exactly $q + 1$ lines containing x intersect m.*

Proof. Since m is a grand clique of G, some point y of m is not adjacent to x. Then exactly $q + 1$ lines containing x intersect m by Lemma 3.

LEMMA 5. *All lines have k points.*

Proof. Let $m \in L$, $x \in m$. Let n be a line distinct from m and containing x. By Lemma 4,

$$|\{(y, z): y \in m - x, z \in n - x, d(y, z) = 1\}| = |n - x| q = |m - x| q.$$

Therefore $|m| = |n|$. Therefore all lines containing x contain the same number of points. Since the valence of x is $r(k - 1)$, every line containing x contains k points.

Proof of Theorem 1. We show π is an $(s + 1, q, d)$-projective incidence structure. We have already shown π satisfies (f2)–(f5). Only (f1) remains. $r(\pi) = r$ and $k(\pi) = k$ by Lemmas 1 and 5. Let s and d be defined as in the statement of Theorem 1. By (g5) and (g6) $3 \leqslant s + 1 < d - 1$. Then s and d are integers, q is a prime power, and π is an $(s + 1, q, d)$-projective incidence structure. Its adjacency graph, G, then is an (s, q, d)-projective graph.

4. Necessity of Hypothesis (g6)

An (r, k, t)-*partial geometry* is an (r, k)-incidence structure in which for any point x and any line m not containing x, exactly t lines contain x and intersect m. The dual of an (r, k, t)-partial geometry is a (k, r, t)-partial geometry [1, p. 396]. For an (r, k, t)-partial geometry π, $t = k$ iff the adjacency graph of π is complete. The notion of (r, k, k)-partial geometry is the same as the notion of (v, k, λ)-BIBD where $\lambda = 1$.

Let $\mathscr{D} = (P, B, I)$ be a $(v, k, 1)$-BIBD. Then $\mathscr{D}$ is also an (r, k, k)-partial geometry where $r(k - 1) = v - 1$. The dual π of $\mathscr{D}$ is a (k, r, k)-partial geometry. Let $R = k$ and $K = r$. Then π is an (R, K, R)-partial geometry. The adjacency graph G of π has the following properties [1, p. 396]:

(1) G is regular of valence $R(K - 1)$;

(2) $p_{11}(x, y) = K - 2 + (R - 1)^2$ if $d(x, y) = 1$;

(3) $p_{11}(x, y) = R^2$ if $d(x, y) = 2$.

Then G satisfies properties (g1)–(g3) where $q = R - 1$ and $\alpha = (R - 1)^2$. G satisfies (g4) because the right side of the inequality in (g4) is negative.

If

$$K > \max(2R^3 - 3R^2 + 4R - 1, \tfrac{1}{2}(R^4 - R^3 + R^2 + R))$$

then (g5) is satisfied.

We produce examples showing the necessity of hypothesis (g6) in Theorem 1 as follows. Let $\mathscr{D}$ be a $(v, k, 1)$-BIBD with r blocks containing each point, and let

$$r > \max(2k^3 - 3k^2 + 4k - 1, \tfrac{1}{2}(k^4 - k^3 + k^2 + k)).$$

Then the line graph G of $\mathscr{D}$ satisfies (g1)–(g5) where $q = k - 1$ and $\alpha = (k - 1)^2$ but G is not normally a projective graph. For example, let $\mathscr{D}$ be a $(v, 3, 1)$-BIBD where at least 39 blocks contain each point (hence $v - 79$). Such designs exist for all values of $v \equiv 1, 3 \bmod 6$. Then the line graph G of $\mathscr{D}$ satisfies (g1)–(g5) where $q = 2$ and $\alpha = 4$. However G cannot be a projective graph unless $r = 2^{s+1} - 1$ for some integer s. Since $v = r(k - 1) + 1$, G cannot be a projective graph unless $v = 2^{s+2} - 1$ for some integer s.

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A COMBINATORIAL CHARACTERIZATION OF ATTENUATED SPACES

D. K. Ray-Chaudhuri and Alan P. Sprague

1. *Introduction.*

In this paper we consider graphs without multiple edges. Such a graph consists of a set of vertices and a symmetric relation called adjacency. For any x, $\Delta(x)$ is the set of vertices adjacent to x, and

$$\Delta(x_1, x_2,\dots,x_n) = \bigcap_{i=1}^{n} \Delta(x_i) .$$

A *clique* is a set of adjacent vertices. A *path* of length n is a sequence $P = (x_0, x_1,\dots,x_n)$ of vertices such that x_{i-1} and x_i are adjacent $(1 \le i \le n)$. The path P is said to *join* the vertices x_0 and x_n. The *distance* between vertices x and y (written $d(x,y)$) is the minimum length of any path joining them. If S and T are sets of vertices, $d(S,T)$ is defined as $\min\{d(x,y): x \in S, y \in T\}$.

An *incidence structure* is a triple $\pi = (P,L,I)$, where P and L are nonempty finite sets and $I \subseteq P \times L$. Elements of P and L are called *points* and *lines*, respectively. Usual geometric terminology will be used. For a point x, $r(x)$ is the number of lines containing x, and for a line m, $k(m)$ is the number of points on the line. Lines m and n will be said to *intersect*, written $m \# n$, if $m \ne n$ and some point lies on both. An incidence structure π is said to be *semilinear* if two points are joined by at most one line. Let m and n be lines of semilinear incidence structure π. If m and n intersect at a point x, then a line t will be called a *transversal* of m and n if t intersects m and n but x does not lie on it. Let x and y be points of a semilinear incidence structure and m be a line containing x and y; then we also denote m by $\langle x,y \rangle$. An incidence structure is called *linear* if two points are joined by exactly one line.

The dual of $\pi = (P,L,I)$ is the incidence structure $\pi^* = (L,P,I')$, where for $p \in P$ and $m \in L$, p and m are incident in π^* if they are incident in π. $(P,L,\in)$ or (P,L) will be used to represent an incidence structure where for $p \in P$ and $m \in L$, p lies on m if and only if $p \in m$. The *adjacency graph* of an incidence structure $(P,L,I) = \pi$ is the graph, denoted $G(\pi)$, having P as vertex set and two points adjacent if and only if some line contains both. We will use terminology borrowed from the adjacency graph of π when dealing with π. For example, two points are called adjacent if and only if some line contains both. Also, for $x, y \in P$ and $m, n \in L$, $d(x,y)$, $d(x,m)$, and $d(m,n)$ refer to distance in $G(\pi)$. An incidence structure π is called *connected* if both $G(\pi)$ and $G(\pi^*)$ are connected, where π^* is the dual of π. Equivalently, π is connected if and only if $G(\pi)$ is connected and every line of π is incident with at least one point.

Let $(P,L,I) = \pi$ be an incidence structure and $S \subseteq P$. We let $L(S)$ be the set of lines of π which contains at least two points of S. Let $I' = I\cap(S \times L(S))$. If $L(S) \neq \emptyset$, then $(S,L(S), I') = \pi'$ is an incidence structure and two points of S are adjacent in π if and only if they are adjacent to π'.

An (r,k)-incidence structure is a semilinear incidence structure in which every point lies on r lines and every line contains k points. The dual of an (r,k)-incidence structure is a (k,r)-incidence structure. An (r,k,t)-partial geometry is an (r,k)-incidence structure in which for every point x and every line m not containing x, x is adjacent to exactly t points of m.

We mention a simple result here.

LEMMA 1.1. *Let* $t > 1$. *Let* π *be a connected semilinear incidence structure having the property that for any point* x *and line* m *such that* $d(x,m) = 1$, *exactly* t *points of* m *are adjacent to* x. *Then there exist integers* r *and* k *such that every point lies on* r *lines and every line contains* k *points.*

Proof. Since π is connected, to show that all lines have the same number of points it is sufficient to show that two lines containing a point in common have the same number of points. Let m and n be intersecting lines. The number of transversals of m and n equals $(k(m)-1)(t-1)$, and also equals $(k(n)-1)(t-1)$. Since $t > 1$, $k(m) = k(n)$. That all points lie on r lines is shown by a dual argument.

Let (X, A, I) be a semilinear incidence structure. The following statement is called *Pasch's Axiom*.

(1) Let $x \in X$, $a_j, b_i, \in A$ $(i=1,2;\ j=1,2)$ and $a_1 \neq a_2$. Let x be incident with a_1 and a_2, and not incident with b_1 or b_2. If some $y_{ij} \in X$ is incident with both a_i and b_j $(i=1,2;\ j=1,2)$ then some element of X is incident with both b_1 and b_2.

Rephrased in the language of (X,A,I), this becomes

(2) Let lines a_1 and a_2 intersect, and b_1, b_2 be transversals of a_1, a_2. Then $b_1 \neq b_2$.

If we phrase this in the language of the incidence structure (A,X,I') which is dual to (X,A,I) (so that elements of A are now called points and elements of X are called lines) we obtain

(3) Let x be a line and $a_1 \neq a_2$ be points of x. If b_1,b_2 are points but not on x, and b_i is adjacent to $a_j (i=1,2;\ j=1,2)$ then b_1 is adjacent to b_2.

Where $(A, X, I') = (A, X, \epsilon)$ we may simplify (3) to

(4) Let x be a line and a_1, a_2 distinct points of x. Then $\Delta(a_1,a_2)$-m is a clique.

We call property (3) or (4) the *dual of Pasch's Axiom*. For any incidence structure π and its dual π', π satisfies the dual of Pasch's Axiom if and only if π' satisfies Pasch's Axiom.

Let $d > s > 1$ be integers and q a prime power. Let V be a d-dimensional vector space over $GF(q)$ and W a $(d-s)$-dimensional subspace. Let P be the set of s-dimensional subspaces of V whose intersection with W is 0, the 0-dimensional subspace, and L be the set of $(s-1)$-dimensional subspaces of V whose intersection with W is 0. Any incidence structure isomorphic to $(P,L,\supseteq)$ (that is, for $p \in P$ and $m \in L$, p and m are incident if and only if $p \supseteq m$) is called the (d,q,s)-attenuated space.

Charles Sims' proof of a rather different theorem ([4],[1]) contains the proof of Theorem 1 below. This result has recently been improved by Thas and DeClerck; they show that the hypothesis $k > q^2$ is unnecessary [6].

THEOREM 1. *Let* (P,L) *be a* $(q+1, k,q)$ *partial geometry with* $k > q^2$ *and such that the incidence structure dual to* (P,L) *satisfies Pasch's Axiom. Then* q *is a prime power,* $k=q^{d-2}$ *for some integer* d, *and*

(P,L) *is isomorphic to the* (d,q,2)*-attenuated space.*

THEOREM 2. *Let* (X,L) *be a connected semilinear incidence structure satisfying the following conditions, where* $q > 1$ *is some integer.*

(a1) If $x \in X$, $m \in L$, *and* $d(x,m) = 1$ *then exactly* q *points of* m *are adjacent to* x.

(a2) If $x,z \in X$ *and* $d(x,z) = 2$ *then* $|\{m \in L : d(x,m)=1,\ z \in m\}| = q + 1$.

(a3) The dual of Pasch's Axiom.

(a4) If $x,z_1,z_2 \in X$ *and* $d(x,z_1) = d(x,z_2) = 2$, $d(z_1,z_2) = 1$, *then* $\Delta(x,z_1,z_2) \neq \emptyset$.

(a5) For some $x \in X$, $q + 2 \le r(x) \le q^2 + q + 1$.

(a6) For some $x \in X$ *and* $m \in L$, $k(m) > r(x)(q-1) + 1$.

Then q *is a prime power and* (X,L) *is isomorphic to the* (d,q,3)*-attenuated space for some integer* $d > 6$. *Conversely the* (d,q,3)*-attenuated space,* $d > 6$, *satisfies the above conditions.*

The proof of the first part of this theorem occupies Sections 3 to 5. The converse is proven quickly in Section 2. A connected semilinear incidence structure is called an *abstract attenuated space* if it satisfies the six axioms stated above.

2. *An Algebraic Characterization.*

In this section we first establish a second characterization of attenuated spaces and look at their automorphism groups and finally sketch the proof of the converse part of Theorem 2. Where S is a set of vectors or subspaces of a vector space V, we use <S> to denote the subspace of V spanned by S.

Let $d > s > 1$. Let V be a d-dimensional vector space over GF(q) and W a (d-s)-dimensional subspace. Let $\pi = (P,L)$ be a (d,q,s)-attenuated space. We may take P (respectively, L) to be the set of s-dimensional ((s-1)-dimensional) subspaces whose intersection with W is 0. Let U be an s-dimensional subspace of V so that $V = U \times W$, and $\{e_1,e_2,\dots,e_s\}$ be a basis for U. If Φ is the projection map onto U so that $\ker \Phi = W$, then for each $x \in P$, $\Phi(x)$ is an s-dimensional subspace of U, so $\Phi(x) = U$. Then there are vectors $w_i \in W$ so that $x = \langle \{e_i + w_i : 1 \le i \le s\} \rangle$. Hence we may define a map $\sigma^* : P \to W^s$ so that $\sigma^*(x) = (w_1,w_2,\dots,w_s)$, where $x = \langle \{e_i + w_i : 1 \le i \le s\} \rangle$. This map is clearly a bijection. Let σ be the inverse of σ^*. Then σ is a bijection from W^s to P.

Let $w_i \in W$ and $\delta_i \in GF(q)(1 \le i \le s)$ and not all the δ_i be zero. Let $T = (w_1,w_2,\dots,w_s) + (\delta_1,\delta_2,\dots,\delta_s)W$ be the subset

$$\{(w_1 + \delta_1 v,\ w_2 + \delta_2 v, \dots, w_s + \delta_s v) : v \in W\}.$$

Then

$$\sigma(T) = \left\{ x \in P : x = \left\langle \{e_i + w_i + \delta_i v : 1 \le i \le s\} \right\rangle \right\}.$$

Note that for any $\alpha \in GF(q) - 0$,

$$(w_1,w_2,\dots,w_s) + (\delta_1,\delta_2,\dots,\delta_s)W = (w_1,w_2,\dots,w_s) + (\alpha\delta_1,\alpha\delta_2,\dots,\alpha\delta_s)W .$$

Choose j so that $\delta_j \neq 0$ and for all $i < j$, $\delta_i = 0$. By the above remark we may presume $\delta_j = 1$. For any $v \in W$,

$$\left\langle \{e_i + w_i + \delta_i v : 1 \le i \le s\} \right\rangle = \left\langle e_j + w_j + v, \{f_i : 1 \le i \le s,\ i \neq j\} \right\rangle,$$

where $f_i = e_i - \delta_i e_j + w_i - \delta_i w_j$, so every $x \in \sigma(T)$ is a point of π containing the (s-1)-dimensional space $\langle \{f_i : 1 \le i \le s,\ i \neq j\} \rangle$. This space has null intersection with W, so it is a line of π. Also every point of π containing this (s-1)-dimensional space is contained in $\sigma(T)$. Therefore for some $m \in L$, $\sigma(T)$ is the set of points lying on m. Let $\mathcal{T}$ be the class of all sets $(w_1,w_2,\dots,w_s) + (\delta_1,\delta_2,\dots\delta_s)W$, where $w_i \in W$, $\delta_i \in GF(q)$, and not all δ_i are zero (and the first nonzero δ_i may be presumed to be 1). For all $T \in \mathcal{T}$, $\sigma(T)$ is the set of points on some line, so we may define a function $\tau : \mathcal{T} \to L$ by $\tau(T)$ as the line which contains all points of $\sigma(T)$. Certainly τ is an injective map. It may be shown that every line of π may be written in the form $\langle \{f_i : 1 \le i \le s,\ i \neq j\} \rangle$ for vectors f_i defined analogously to those above. Hence τ is a bijection from $\mathcal{T}$ to L. Then the pair of maps σ, τ establish an isomorphism from $(W^s, \mathcal{T}, \epsilon)$ to $(P, L, \supseteq)$.

We continue this discussion, shifting our focus to the automorphism group of $(P,L,\supseteq)$. Let Γ be the subgroup of the general linear group on V which fixes W. It may be seen that Γ is transitive on P, and that the stabilizer in Γ of a point x of P is 2-transitive on lines through x. (Indeed, this is true even if Γ is the subgroup of the general linear group on V which fixes every vector of W.)

Let $s = 2$. Let $x \in P$ and $m,n \in L$ so that m and n intersect at x. Since Γ is transitive on P and Γ_x is 2-transitive on the lines containing x, we may choose $x = U$, $m = \langle e_1 \rangle$, $n = \langle e_2 \rangle$. Then $\sigma^* : P \to W^2$ satisfies $\sigma^*(x) = (0,0)$. And τ satisfies $\tau^{-1}(m) = (0,0) + (0,1)W = 0 \times W$ and $\tau^{-1}(n) = (0,0) + (1,0)W = W \times 0$. A $(d,q,2)$-attenuated space is a $(q+1, k, q)$ - partial geometry in which L may be partitioned into $q+1$ parallel classes, with two lines intersecting if and only if they belong to different parallel classes. Let L_1 be the parallel class containing m and L_2 the parallel class containing n. For each $w \in W$, $w \times W = (w,0) + (0,1)W$ is a line of (W^2,T,ϵ) with $\tau(w \times W) = \langle e_1 + w \rangle$. Clearly $w_1 \times W$ and $w_2 \times W$ are disjoint if $w_1 \neq w_2$. Hence τ^{-1} maps the class of lines of $(P,L,\supseteq)$ parallel to m to the class $\{w \times W : w \in W\}$ of lines of (W^2,T,ϵ). Similarly, for $n' \in L$, $\tau^{-1}(n') = W \times w$ for some $w \in W$ if and only if $n' \in L_2$.

We now sketch briefly the proof of the converse part of Theorem 2; we show that for $d > 6$ a $(d,q,3)$-attenuated space satisfies the hypotheses of Theorem 2. For subspaces x,y of a vector space V, we use $x \wedge y$ to represent the intersection of x and y. Recall that $d(\, , \,)$ is used for distance in a graph or incidence structure; dim ( ) will be used for vector space dimension.

Let V be a d-dimensional vector space over $GF(q)$. Let P^+ (respectively, L^+) be the set of all 3-dimensional (2-dimensional) subspaces of V. Let $\pi^+ = (P^+,L^+,\supseteq)$; π^+ is an (r^+,k^+)-incidence structure where $r^+ = q^2 + q + 1$. Both it and the incidence structure dual to it satisfy Pasch's Axiom. Let W be a $(d-3)$-dimensional subspace of V. Let $(P,L,\supseteq) = \pi$ be a $(d,q,3)$-attenuated space. We may take $P = \{x \in P^+ : \dim (x \wedge W) = 0\}$ and $L = \{m \in L^+ : \dim (m \wedge W) = 0\}$. π and π^+ are closely related: for $x \in P$ and $m \in L$, x and m are incident in π if and only if they are incident in π^+. Furthermore, for any $x \in P$, and $m \in L^+$, if m and x are incident in π^+ then m is a subspace of x, and so m is a line of π. Then

(1) x is on the same number of lines in π^+ as it is on in π, and

(2) for any $y \in P$, x and y are adjacent in π if and only if they are adjacent in π^+.

Several properties of π are easily deduced from properties of π^+. Since π^+ is semilinear, so is π. Let $x \in P$; then x is incident with r^+ elements of L. Hence π satisfies (a5). Since π^+ satisfies the dual of Pasch's Axiom and two points of π are adjacent in π if and only if they are adjacent in π^+, π satisfies the dual of Pasch's Axiom. That each line of π has q^{d-3} points is most easily seen by the characterization given earlier in this section. Then π satisfies (a6) if and only if $d > 6$.

Let $x = \langle e_1,e_2,e_3 \rangle$ be a point of π, where e_1,e_2,e_3 are vectors of V. Let y be another point of π. Then $y = \langle e_1 + w_1, e_2 + w_2, e_3 + w_3 \rangle$ for some vectors $w_i \in W$. Then $z_1 = \langle e_1,e_2,e_3 + w_3 \rangle$ and $z_2 = \langle e_1,e_2 + w_2,e_3 + w_3 \rangle$ are points of π, and (x,z_1,z_2,y) is a path in π from x to y. Thus π is connected. A similar method will show that the distance between x and y in π equals the distance between them in π^+, and hence equals $\dim x - \dim x \wedge y$.

Let $x \in P$ and $m \in L$. If $\dim x \wedge m = 0$ then *the distance* between x and m in π^+ is 2, so the distance between them in π is at least 2. If $\dim x \wedge m = 1$ then $\langle x,m \rangle$ is a 4-dimensional space, so $\langle x,m \rangle \wedge W$ is a 1-dimensional space. In this case the points of π^+ which lie on m and are adjacent to x are just the $q + 1$ 3-dimensional spaces y such that $m \subseteq y \subseteq \langle x,m \rangle$. Of these, one equals $\langle m,\langle x,m \rangle \wedge W \rangle$ and the other q have nil intersection with W. Thus (a1) is valid for π. Another result obtained from this same argument is that for $x \in P$ and $m \in L$, the distance between x and m in π is 1 if and only if $\dim x \wedge m = 1$. (a2) is verified similarly.

We verify (a4). Let $x,z_1,z_2 \in P$. Then x,z_1,z_2 are 3-dimensional subspaces of V having nil intersection with W. Let $d(x,z) = d(x,z_2) = 2$, $d(z_1,z_2) = 1$. Then $x \wedge z_1 = \langle e_1 \rangle$, $x \wedge z_2 = \langle e_2 \rangle$ for some nonzero vectors $e_1,e_2 \in V$.

Case 1. $\langle e_1 \rangle = \langle e_2 \rangle$. $z_1 \wedge z_2 \in L$ and $\dim x \wedge (z_1 \wedge z_2) = 1$, so the distance between x and the line $z_1 \wedge z_2$ in π is 1. Then there is a point y of π on the line $z_1 \wedge z_2$ and adjacent to x, so (a4) is valid in this case.

Case 2. $\langle e_1\rangle \neq \langle e_2\rangle$. Let $z_1 \wedge z_2 = \langle e_3, e_4\rangle$, where e_3, e_4 are vectors of V. $\{e_1,e_2,e_3,e_4\}$ is a set of independent vectors because $\langle e_1\rangle \neq \langle e_2\rangle$, $\dim \langle e_3,e_4\rangle = 2$, and

$$\langle e_1,e_2\rangle \wedge \langle e_3,e_4\rangle = \langle e_1,e_2\rangle \wedge z_1 \wedge z_2 \subseteq x \wedge z_1 \wedge z_2 = \{0\}.$$

$\langle z_1,z_2\rangle = \langle e_1,e_2,e_3,e_4\rangle$. $W \wedge \langle e_1,e_2,e_3,e_4\rangle$ is a 1-dimensional space; let $\sum_{i=1}^{4} \alpha_i e_i$ be a vector of this space. Since $W \wedge x = \{0\}$ and $x \supseteq \langle e_1,e_2\rangle$, it is not possible for $\alpha_3 = \alpha_4 = 0$. Let $\beta_3, \beta_4 \in GF(q)$ so that $\langle \alpha_3 e_3 + \alpha_4 e_4\rangle = \langle \beta_3 e_3 + \beta_4 e_4\rangle$. Let $y = \langle e_1, e_2, \beta_3 e_3 + \beta_4 e_4\rangle$. Then y is a 3-dimensional space whose intersection with W is nil, so $y \in P$. Since $4 = \dim\langle y,z_1\rangle \neq \dim\langle x,z_1\rangle = 5$, $y \neq x$. Now y is adjacent to x in π because $\dim x \wedge y = \dim\langle e_1,e_2\rangle = 2$ and y is adjacent to z_1 and z_2 because $\dim\langle y,z_1,z_2\rangle = 4$. This completes the proof of (a4).

3. *Start of the Proof.*

We start the proof of Theorem 2. Let (X,L) be a connected semi-linear incidence structure satisfying (a1) to (a6). By Lemma 1.1 all points are on the same number of lines, say r, and every line has the same number of points (say k). Then $k > r(q-1) + 1$. The symbols u,v,w,x,y,z will be reserved for points.

LEMMA 3.1. *Let* $d(x,z) = 2$. *Then the lines containing* x *(respectively,* z*) may be labelled* $m_1,m_2,\ldots,m_r$ $(n_1,n_2,\ldots,n_r)$ *so that* $m_i \nparallel n_j$ *if and only if* $i \le q+1$, $j \le q+1$, $i \neq j$.

Proof. Let $S = \{m \in L : x \in m,\ d(z,m) = 1\}$ and $T = \{n \in L : z \in n,\ d(x,n) = 1\}$. $|S| = |T| = q + 1$ by (a2). For each $m \in S$ and $n \in T$, m intersects q members of T and n intersects q members of S, by (a1). The completion of the proof is clear.

COROLLARY. *If* $d(x,z) = 2$ *then* $|\Delta(x,z)| = q^2 + q$.

Proof. Let x and y be adjacent, and $m = \langle x,y\rangle$. Let $S_1 = \Delta(x,y) - m$ and S_2 be the set of points of m which are adjacent to all points of S_1. We define the *assembly* $A_{xy} = S_1 \cup S_2$. Clearly $x,y \in A_{xy}$. S_1 is a clique by the dual of Pasch's Axiom, so A_{xy} is a clique.

LEMMA 3.2. *Assembly* A_{xy} *is a clique of* $r(q-1)+1$ *points, and every line containing at least one point of* A_{xy} *contains* q *points of* A_{xy}.

Proof. Let S_1, S_2, and m be defined as above. We show that $|S_2| = q$. Toward this, let $w \in S_1$; clearly $S_2 \subseteq \Delta(w) \cap m$, so it is sufficient to show that for all $v \in S_1 - w$, $\Delta(v) \supseteq \Delta(w) \cap m$. Now $\langle v,w\rangle$ does not contain both x and y; say $y \notin \langle v,w\rangle$. Let $z \in \Delta(w) \cap m$, $z \neq y$. Now $z,v \in \Delta(w,y) - \langle w,y\rangle$, so z is adjacent to v. Then $\Delta(v) \supseteq \Delta(w) \cap m$.

For any line n containing x, $n \neq m$, S_1 contains $q-1$ points of $n-x$, and every point of S_1 is in one such line. Hence $|S_1| = (r-1)(q-1)$ so $|A_{xy}| = r(q-1) + 1$. It was already remarked that A_{xy} is a clique.

Let $z \in A_{xy}$ and the lines containing z be $n_1, n_2, \ldots, n_r$. Every point of $A_{xy}-z$ is contained in exactly one such n_i. The points of A_{xy} are not collinear so not all are on the same n_i. Then for each n_i, $|n_i \cap A_{xy}| \le q$. (Let $w \in A_{xy} - n_i$. $A_{xy} \cap n_i \subseteq \Delta(w) \cap n_i$, and $|\Delta(w) \cap n_i| \le q$.) Therefore $|n_i \cap A_{xy} - z| \le q-1$. If for some i $|\Delta n_i \cap A_{xy} - z| < q-1$ then the lines n_i would contain among them fewer than $r(q-1)$ points of $A_{xy} - z$, which is impossible.

LEMMA 3.3. *If* A *and* A' *are assemblies and* $|A \cap A'| \ge 2$ *then* $A = A'$

Proof. A_{xy} is the unique maximal clique containing the adjacent points x and y, and not contained in $\langle x,y\rangle$. It is immediate than an assembly is the unique maximal clique (not contained in a line) containing any two of its points.

LEMMA 3.4. *If* x *is not contained in the assembly* A, *then at most one line through* x *contains points of* A.

Proof. Assume there exist at least two lines m and n which contain both x and points of A. Let $u \in m \cap A$ and $v \in n \cap A$. Then

$x \in \Delta(u) \cap \Delta(v) - \langle u,v\rangle$ so $x \in A_{uv} = A$, a contradiction.

For any assembly A and point u so that $d(u,A) = 2$, we define $B(u,A) = \{x \in A : d(u,x)) = 2\}$. We will say that $B(u,A)$ is a *plane* of A.

LEMMA 3.5. *Let* A *be an assembly*, $d(u,A) = 2$ *and* $B = B(u,A)$. *Then*

(i) $(B,L(B))$ *is an affine plane of order* q;

(ii) *for any line* m *which is not disjoint from* A, $m \in L(B)$ *if and only if* $d(u,m) = 1$;

(iii) *if* $m \in L(B)$ *then* $m \cap B = m \cap A$.

Proof. Let $m \in L$ and $d(u,m) = 2$. If m contains distinct points z, z' of B then by (a4) there is a point y such that $y \in \Delta(u,z,z')$. Since $d(u,m) = 2$, $y \notin m$. Then $y \in A_{z,z'} = A$, so $d(u,A) = 1$, contrary to hypothesis. Hence if $d(u,m) = 2$ then m contains at most one point of B, so $m \notin L(B)$. If $d(u,m) = 1$ then clearly $m \cap A \subseteq B$. Then $|m \cap B| = |m \cap A| \in \{0,q\}$ by Lemma 3.2. Assertions (ii) and (iii) are proved.

To prove assertion (i) we show that $(B,L(B))$ is a $(q+1, q)$-incidence structure. Each point of B is on $q + 1$ lines of B by (a2). By assertion (iii) and Lemma 3.2 each line of B contains q points of B. Since A is a clique, $(B,L(B))$ is a linear incidence structure. It is easily seen that a linear $(q+1, q)$-incidence structure is an affine plane.

LEMMA 3.6. *Let* $m \in L$, A *be an assembly, and* $d(m,A) = 1$. *Then for some* $u \in m$, $d(u,A) = 2$.

Proof. Let $S = \{y \in m : d(y,A) = 1\}$ and $T = \{x \in A : d(x,m) = 1\}$. $|T| \le A = r(q-1) + 1$. By Lemmas 3.2 and 3.4 each point of S is adjacent to q points of T, and each point of T is adjacent to q points of S. Hence $|S| = |T| \le r(q-1) + 1 < k$. Then $|S| < |m|$, so some point of m is at distance 2 from A.

LEMMA 3.7. *Let* x,y,z *be noncollinear points of assembly* A. *Then some plane of* A *contains* x,y,z.

Proof. Let $m = \langle x,y\rangle$, and $n = \langle y,z\rangle$. Let $w \in m - A$. The points of A to which w is adjacent are all on m, so w is adjacent to only one point (namely y) of $n \cap A$. Then w is adjacent to $q - 1 > 0$ points of $n - A$.

Let $v \in \Delta(w) \cap (n - A)$, and $m^* = \langle v,w\rangle$. Since only one line through w contains points of A, $d(m^*,A) = 1$. Let $u \in m^*$ so that $d(u,A) = 2$. Then $x, y, z \in B(u,A)$.

We define the relation $\sim$ between lines : $m \sim n$ if and only if for all $x \in m$, $d(x,n) = 1$. Clearly, if $m \sim n$ then m and n do not intersect. If $m \sim n$, then the number of lines intersecting both m and n is $q|m| = q|n|$, so $n \sim m$. Thus $\sim$ is a symmetric relation.

LEMMA 3.8. *Let* A *be an assembly*, $m \in L$, *and* $d(m,A) = 1$. Let $S_A = \{x \in A : d(x,m) = 1\}$ *and* $S_m = \{y \in m : d(y,A) = 1\}$. *Then*

(i) $|S_A| = q^2$;

(ii) $|S_m| = q^2$.

Proof. We will first show

(s1) $|S_A| = |S_m|$,

(s2) $|S_m| \le q^2$.

Then (i) follows from (ii) and (s1). Let $E = \{(x,y) : x \in S_A, y \in S_m, d(x,y) = 1\}$. $|E| = q|S_A| = q|S_m|$ by (a1) and Lemma 3.4. This shows (s1). Choose $u \in m$ so that $d(u,A) = 2$. $S_A \subseteq B(u,A)$, so $|S_m| = |S_A| \le q^2$, and (s2) is verified.

Let w be a point such that $d(w,A) = 1$, and n be the line containing w and points of A. We compute the cardinality of some sets: let

$$T = \{z : d(z,w) = d(z,A) = 1, z \notin n\},$$

$$R = \{(y,z) : y \in A, z \in T, d(y,z) = 1\},$$

$$R_1 = \{(y,z) : (y,z) \in R, y \in n\},$$

$$R_2 = \{(y,z) : (y,z) \in R, y \notin n\}.$$

Then $|R| = |R_1| + |R_2|$, and $|R| = q|T|$.

$|A \cap n| = q$; for every $y \in A \cap n$, there are $|\Delta(w,y)-n| = (r-1)(q-1)$ points z so that $(y,z) \in R_1$. Therefore $|R_1| = (r-1)(q-1)q$. $|A - n| = (r-1)(q-1)$; for each $y \in A - n$ there are $q^2 + q$ points z in $\Delta(w,y)$ of which q are in A, so $|R_2| = (r-1)(q-1)q^2$.

Then $|R| = |R_1| + |R_2| = (r-1)(q-1)(q^2 + q)$, and $|T| = |R|/q = (r-1)(q^2-1)$.

The r-1 lines distinct from n containing w partition T. Each such line contains at most $q^2 - 1$ points of T, by (s2). Hence each line contains $q^2 - 1$ points of T. This proves (ii).

LEMMA 3.9. *Let* m *be a line and* $d(x,m) = 1$. *Then*

(i) x *is contained in a unique line* n *so that* $m \sim n$;

(ii) if $h \in L$ *contains* x *then either* $h \sim m$, $h \neq m$, *or exactly* q *points of* m *are at distance* 1 *from* h.

Proof. Let A be an assembly so that $x \in A$ and $m \cap A = \emptyset$. (Such an assembly exists. Let $w \in m \cap \Delta(x)$; then any assembly which contains x and is distinct from A_{xw} is disjoint from m.) Then $d(m,A) = 1$. Let $S_A = \{z \in A : d(z,m) = 1\}$. Then $|S_A| = q^2$.

Let u be any point of m so that $d(u,A) = 2$. $S_A \subseteq B(u,A)$ and $|B(u,A)| = q^2$ so $S_A = B(u,A)$. For all points z of the affine plane S_A, $q + 1$ lines of S_A contain z; of these q intersect m by Lemma 3.5(ii) and (a1), and one does not. Then the set of lines of S_A which do not intersect m is a parallel class of lines of S_A; we call this set of lines L.

Let $n \in L$ so that $x \in n$. We show that $m \sim n$. For any $u \in m$ so that $d(u,A) = 2$, we have $d(u,n) = 1$ by Lemma 3.5(ii). Let $y \in m$ so that $d(y,A) = 1$. $d(y,n) \geq 1$ since n does not intersect m. Let h be a line so that $y \in h$ and $h \cap A \neq \emptyset$. Then h is a line of S_A, but is not in L because $h \neq m$. Then h intersects every line of the parallel class L, so $d(y,n) = 1$. Then $m \sim n$.

Now x is contained in $q + 1$ lines of S_A; of these q intersect m, and one, which we will denote n, has the property that $n \sim m$. Let h contain x, and not be a line of S_A. To show (ii) and the uniquenss of the line n in (i) it is sufficient to show that

exactly q points of m are at distance 1 from h. Now $|\Delta(x) \cap m| = q$, and for every $y \in \Delta(x) \cap m$, $d(y,h) = 1$. Thus at least q points of m are at distance 1 from h. Let $y \in m - \Delta(x)$. Then $d(y,n) = 1$ since $m \sim n$, and $d(y,t) = 1$ for each of the q lines t containing x and intersecting m. Then by (a2), $d(y,h) > 1$. This completes the proof.

LEMMA 3.10. *Let* m *and* n *be parallel lines of* $B(u,A)$. *Then* $m \sim n$.

Proof. We first show that m and n do not intersect. Suppose $m \cap n = \{x\}$. Then $x \notin B(u,A)$. Since $m \cap A = m \cap B(u,A)$, $x \notin A$ and is contained in at least two lines of A, contradicting Lemma 3.4. In the light of Lemma 3.9 (ii), to show $m \sim n$ it is sufficient to find $q + 1$ points of m which are at distance 1 from n. Let $T = \{z : d(z,u) = 1, d(z,A) = 1\}$. For every point x of $B(u,A)$, x is adjacent to $q^2 + q$ points of T, by the corollary to Lemma 3.1. For every point z of T, z is adjacent to q points of $B(u,A)$. $|B(u,A)| = q^2$. Then $|T| = q^3 + q^2$.

Every line containing u which contains at least one point of T contains q^2 points of T, by Lemma 3.8 (ii). Therefore $q + 1$ lines, say $h_0,h_1,\ldots,h_q$ contain u and points of T. Since m, $n \in L(B(u,A))$, $d(u,m) = d(u,n) = 1$. Therefore q lines among $h_0,h_1,\ldots,h_q$ intersect m, and q of these lines intersect n, so at least $q - 1 > 0$ intersect m and n. We may suppose h_1 intersects both m and n. Let $h_1 \cap m = \{y\}$; $y \notin A$. Then $\{y\} \cup (m \cap A)$ is a set of $q + 1$ points of m which are at distance 1 from n.

LEMMA 3.11. *Let* A *be an assembly. Any three noncollinear points of* A *are contained in a unique plane of* A.

Proof. Let x,y,z be noncollinear points of A. By Lemma 3.7, some plane of A contains x,y,z. We show that such a plane is unique. Let B_1 and B_2 be planes of A which contain x,y,z. Let $m = \langle x,y \rangle$. For $i = 1,2$ let n_i be the line such that $z \in n_i$ and n_i and m are parallel lines of the affine plane B_i. Then $n_1 \sim m$ and $n_2 \sim m$, so $n_1 = n_2$. Let $m \cap A = \{w_i : 1 \leq i \leq q\}$ and $h_i = \langle z,w_i \rangle$ for $1 \leq i \leq q$. Then $B_1 = \left(n_1 \cup \bigcup_{i=1}^{q} h_i\right) \cap A = B_2$.

LEMMA 3.12. *(i) If two planes of an assembly* A *contain a point in common, then they have a line in common.*

(ii) $r = q^2 + q + 1$.

Proof. It is clear that if $B_1 \neq B_2$ are planes of A then $B_1 \cap B_2$ is either empty, a point, or a line.

Let x be a point of A. Let b be the number of planes of A containing x. Counting the number of elements of the set $\{(m, n, B) : \{x\} = m \cap n,\ m, n \in L(B),\ B \text{ is a plane of } A\}$ in two ways shows that $r(r-1) = b(q+1)q$. Therefore $b = r(r-1)/(q+1)q$.

Let m be a line of A. The planes of A containing m partition A − m, so m is contained in $(|A| - q)/(q^2 - q) = (r - 1)/q$ planes of A.

Let B be a plane of A so that $x \in B$. Let $m_0, m_1, \ldots, m_q$ be the lines of B containing x. Each line m_i is contained in $((r-1)/q) - 1$ planes of A distinct from B. Therefore for $(q + 1)[((r - 1)/q) - 1] = c$ (say) planes B', $B' \cap B$ is a line containing x.

Then $c + 1 \leq b$, and $c + 1 = b$ if and only if for all planes B' containing x, either $B' = B$ or $B' \cap B$ is a line. The equation $c + 1 = b$ is a quadratic equation in r, and has solutions $r = q + 1$ and $r = q^2 + q + 1$. The inequality $c + 1 \leq b$ has no solutions for $q + 1 < r < q^2 + q + 1$. By assumption, $q + 2 \leq r \leq q^2 + q + 1$. Then $r = q^2 + q + 1$, and for all planes B' containing x, $B' = B$ or $B' \cap B$ is a line. Since B and $x \in B$ are arbitrary, the proof is complete.

LEMMA 3.13. *An assembly is an affine 3-space of order* q.

Proof. Let A be an assembly. We list some properties of A.

(1) Any two points of A are on a unique line of A, and every line of A contains at least two points of A.

(2) Any three noncollinear points of A are on a unique plane of A, and every plane of A contains at least three noncollinear points.

(3) For any $x \in A$, $m \in L(A)$ there is a unique line n containing x such that m and n are parallel lines of some plane of A. (Since each plane of A is an affine plane, this follows from (2).)

(4) If two planes of A contain a point in common then they have a line in common.

(5) There are three noncollinear points.

Since A has finitely many points and satisfies (1) to (5), A satisfies the hypotheses of Sasaki's characterization of affine spaces [3], [2, p.107]. Clearly, A is a 3-space, and has order q.

4. *Parallelism.*

We continue the logical development initiated in Section 3. The hypotheses of Section 3 still apply.

Let $m, n \in L$. We say that m is parallel to n (written $m||n$) if and only if $m = n$, $m \sim n$, or $d(m,n) = 2$. Since $m \sim n$ if and only if $n \sim m$, so also $m||n$ if and only if $n||m$. If $d(x,m) = 1$ then there exists a unique line n such that $x \in n$ and $n||m$, by Lemma 3.9(i).

LEMMA 4.1. *Let* $m, n \in L$, $d(m,n) \leq 1$. *Then the following statements are equivalent.*

(i) $m||n$.

(ii) For all assemblies A, *if* $m, n \in L(A)$ *then* m *and* n *are parallel lines of the affine space* A.

(iii) m *and* n *are parallel lines of some assembly.*

Proof. If $d(m,n) = 0$ then the assertion of the lemma is clearly valid, so we prove (i) ⇒ (ii) ⇒ (iii) ⇒ (i) under the assumption that $d(m,n) = 1$.

To prove (i) ⇒ (ii), let $m||n$, and $m,n \in L(A)$ for some assembly A. Let $w \in m - A$. Then $d(w,n) = 1$; let $h \in L$ such that $w \in h$ and $h \nparallel n$. Since $w \notin A$, $h \cap A = \emptyset$ by Lemma 3.4. Let $u \in h$ so that $d(u,A) = 2$. Then $m,n \in L(B(u,A))$ and m and n do not intersect, so m and n are parallel lines of the affine plane B(u,A), and consequently of A.

To prove (ii) ⇒ (iii), all we need to show is that for some assembly A, $m, n \in L(A)$. Let $x \in m$ and $y \in n$ so that $d(x,y) = 1$. Then $m, n \in L(A_{xy})$.

To prove (iii) $\Rightarrow$ (i), let m and n be parallel lines of some assembly A. Then for some plane B of A, m and n are parallel lines of B. By Lemma 3.10 $m \sim n$, so $m || n$.

LEMMA 4.2. *Let* $d(x,z) = 2$. *Let the lines containing* x *and* z *be labelled* m_i *and* n_i *respectively* $(0 \leq i \leq q^2 + q)$ *so that* $d(x,n_i) = 1$ *if and only if* $i \leq q$, $d(z,m_i) = 1$ *if and only if* $i \leq q$, *and* $m_i \cap n_i = \emptyset$ *for* $i \leq q$. *Then* $m_i || n_i$ *for* $i \leq q$.

Proof. This follows from Lemma 3.9(ii), since m_i does not intersect n_i, and $\{x\} \cup (m_i \cap \Delta(z))$ is a set of $q + 1$ points of m_i which are at distance 1 from n_i.

LEMMA 4.3. *Let* n *intersect parallel lines* m *and* m', *and* $n' || n$. *Then* n' *intersects* m *if and only if* n' *intersects* m'.

Proof. Let $n' \nparallel m'$. Let $\{x\} = m \cap n$ and $\{z\} = m' \cap n'$. If $d(x,z) = 1$ then by considering the affine space A_{xz} it is clear that $n' \nparallel m$.

Let $d(x,z) = 2$. By Lemma 4.2 some line n_i contains z, is parallel to n, and intersects m. Since the line containing z and parallel to n is unique, $n' = n_i$. Therefore $n' \nparallel m$.

LEMMA 4.4. *Let* $n || n'$, m *intersect both* n *and* n', *and* $z \in m - n - n'$. *Then a line through* z *is parallel to* n *if and only if it is parallel to* n', *and intersects* n *if and only if it intersects* n'.

Proof. Let $\{x\} = m \cap n$ and $\{y\} = m \cap n'$. Let h be a line containing z $(h \neq m)$.

It certainly cannot be that h is parallel to one of n, n' and intersects the other: if $h || n$ and $h \nparallel n'$ (say at u) then h and n' are two lines both containing u and parallel to n. In light of this and the symmetry of the hypothesis, to prove the lemma it is sufficient to show that if h and n are either parallel or intersecting then h and n' are either parallel or intersecting.

Case 1. $h \nparallel n$. Let $\{w\} = h \cap n$. We consider two cases: $d(w,y) = 1$ and $d(w,y) = 2$.

Suppose $d(w,y) = 1$. Then A_{wy} exists, and $x, z \in A_{wy}$. Then $m, n, n', h \subset L(A_{wy})$, and clearly as lines of the affine space A_{xy} they are coplanar. Then $h \nparallel n'$.

Suppose $d(w, y) = 2$. Since $n \sim n'$, $d(w,n') = 1$. Therefore $h || n'$ or $h \nparallel n'$ by Lemma 4.2.

Case 2. $h || n$. Let $y' \in n' - \Delta(z)$, and m' be a line containing y' and parallel to m. By applying Lemma 4.3 twice we see that $m' \nparallel n$, and $m' \nparallel h$. Since $d(z,y') = 2$, by Lemma 4.2 $h || n'$ or $h \nparallel n'$.

LEMMA 4.5. *Let* n *be a line and* $d(x,n) \leq 2$. *Then there is a unique line which contains* x *and is parallel to* n.

Proof. We have already established that if $d(x,n) \leq 1$ then a unique line contains x and is parallel to n. Let $d(x,n) = 2$. Let $S = \{z \in n : d(x,z) = 2\}$ and $T = \{y : d(x,y) = d(y,n) = 1\}$. For any $z \in S$ and $y \in T$ we declare z and y to be incident if and only if $d(z,y) = 1$; let I be the set of incident pairs. Then (S,T,I) is a $(q^2 + q, q)$-incidence structure.

Let $z, z' \in S$. The set of elements of T incident with both z and z' equals $\Delta(x,z,z')$ and so equals $A_{zz'} \cap \Delta(x)$. By (a4) this set is not empty; then $|A_{zz'} \cap \Delta(x)| = q$. Therefore (S,T,I) is a $(v^*,k^*,\lambda^*,)$-BIBD with $v^* = |S|$, $k^* = q$, $\lambda^* = q$, $r^* = q^2 + q$. A balanced incomplete block design with parameters r^*,k^*,λ^*,v^*, and b^* satisfies the equations $r^*(k^*-1) = \lambda^*(v^*-1)$ and $v^* r^* = b^*k^*$. From these equations we obtain $|S| = q^2$ and $|T| = q^2(q+1)$.

Let $h_1, h_2, \ldots, h_j$ be the lines which contain x and are at distance 1 from n. Since for each i, $d(h_i, n) = 1$ but h_i contains x, and $d(x,n) = 2$, h_i is not parallel to n. The lines h_i partition T, and each contains q points of T. Then $j = |T|/q = q^2 + q$, so exactly one line contains x and is at distance 2 from n. Thus a unique line contains x and is parallel to n.

LEMMA 4.6. *Let* $m || n$, $m || n'$, *and* $d(n,n') \leq 1$. *Then* $n || n'$.

Proof. Since $m || n$, $d(m,n) \leq 2$. The case $d(m,n) = 0$ implies $m = n$, and is trivial. Likewise we may assume $1 \leq d(m,n') \leq 2$. We may also assume $n \neq n'$.

Case 1. $d(m,n) = d(m,n') = 1$. In this case it is clear that n does not intersect n', by Lemma 4.5. Let $u \in n$ and $v \in n'$ so that $d(u,v) = 1$. Both u and v are at distance 1 from m. $|\Delta(u) \cap m| = |\Delta(v) \cap m| = q$. We have two cases, according to whether $\Delta(u) \cap m$ and $\Delta(v) \cap m$ are equal or not.

Case 1.1. $\Delta(u) \cap m = \Delta(v) \cap m$. Let x and y be distinct points of $\Delta(u) \cap m$. Let $A = A_{xy}$; then $m \in L(A)$. Some line h containing u has the property that $h \in L(A)$ and $h||m$. Since the line containing u and parallel to m is unique, then $h = n$. Then $n \in L(A)$. Similarly $n' \in L(A)$. Since parallelism is transitive among lines of A, $n||n'$.

Case 1.2. $\Delta(u) \cap m \neq \Delta(v) \cap m$. Let $x \in \Delta(u) \cap m - \Delta(v)$ and $y \in \Delta(v) \cap m - \Delta(u)$. Then $d(u,y) = 2$, so $\langle u, v\rangle$ and m are either parallel or intersecting, by Lemma 4.2. By Lemma 4.5 $\langle u, v\rangle$ and m are not parallel. Therefore $\langle u, v\rangle$ intersects m, n, and n'. By Lemma 4.4, $n||n'$.

Case 2. $d(m,n) = 1$, $d(m,n') = 2$. Let $z \in n'$ and $y \in n$ so that $d(y,z) = 1$. Then $d(m,z) = 2$. Let h be a line containing z and parallel to n. Assume n and n' are not parallel, so $h \neq n'$. Since $m||n'$, m is not parallel to h. Since h contains z, $1 \le d(m,h) \le 2$. But if $d(m,h) = 2$ then $m||h$ by definition, and if $d(m,h) = 1$ then $m||h$ by Case 1. This is a contradiction. Therefore $n||n'$.

Case 3. $d(m,n) = 2$. Let (x,y,z) be a path so that $x \in m$ and $z \in n$. Let h be a line containing y and parallel to m. Then $h||n$ by Case 2. Let $v \in n'$ and $w \in n$ so that $d(v,w) = 1$. Then $d(w,h) = 1$, so $d(n',h) \le 2$. If $d(n',h) = 2$ then $n'||h$. If $d(n',h) \le 1$ then $n'||h$ by Case 1. Using Case 1 or 2 again, on lines h,n,n', we see that $n||n'$.

LEMMA 4.7. *For all points* x *and lines* m, $d(x,m) \le 2$.

Proof. We first show that if $m, n \in L$ and $d(m,n) = 2$, then $d(w,m) = 2$ for all $w \in n$. Let $d(m,n) = 2$, and (x,y,z) be a path such that $x \in m$ and $z \in n$. Let h be a line parallel to m and containing y. Then $h||n$. Since for all $w \in n$, $d(w,h) = 1$ and for all $v \in h$, $d(v,m) = 1$ then for all $w \in n$, $d(w,m) \le 2$. This and $d(m,n) = 2$ imply that $d(w,m) = 2$ for all $w \in n$.

Now suppose that u is a point such that $d(u,m) = 3$, and let (u,u_1,u_2,u_3) be a path with $u_3 \in m$. Then $d(\langle u\,u_1\rangle,m) = 2$ and $\langle uu_1\rangle$ contains a point at distance 3 from m. This contradicts the result of the preceding paragraph.

COROLLARY. *For all points* x *and lines* m, *there is a unique line containing* x *and parallel to* m.

Proof. This follows from Lemmas 4.5 and 4.7.

LEMMA 4.8. *Parallelism is an equivalence relation.*

Proof. We only need show that parallelism is transitive. Let $m||n$ and $m||n'$. Certainly $d(n,n') \le 2$. If $d(n,n') = 2$ then $n||n'$ by the definition of parallelism. If $d(n,n') < 2$ then Lemma 4.6 shows $n||n'$.

5. *End of the Proof.*

We conclude the development of Sections 3 and 4. The hypotheses of Section 3 still apply. Let m and n be intersecting lines. We define the *plane* $H(m,n)$ to be the union of all lines which intersect m and are parallel to n:

$$H(m,n) = \{x : \text{for some line } n',\ n'||n,\ n' \# m,\ x \in n'\}.$$

Clearly, if t is a line and $t||n$, then either $t \subseteq H(m,n)$ or $t \cap H(m,n) = \emptyset$.

LEMMA 5.1. *Let* $t \in L(H(m,n))$. *Then* $t \subseteq H(m,n)$.

Proof. Let $H = H(m,n)$. By hypothesis, $|t \cap H| \ge 2$. If $t||n$ then $t \subseteq H$. Let $S = \{n' : n'||n,\ n' \# m\}$. If $t||m$ then t intersects every line n' in S, by Lemma 4.3. Then $|S|$ points of t are in H, so $t \subseteq H$.

Let t be parallel to neither m nor n. Let $x,y \in t \cap H$, and let $x \in n_1$, $y \in n_2$, where $n_1, n_2 \in S$. Let m' contain y and be parallel to m. Then $m' \# n_1$ so by Lemma 4.4 $t \# m$. Not both x and y are on m; say $x \notin m$. By Lemma 4.4, since $t \# m$ and $t \# n_1$, t intersects all lines of S, so $t \subseteq H$.

LEMMA 5.2. *Every plane is a* $(q+1, k, q)$-*partial geometry.*

Proof. let $x \in H = H(m,n)$. Let $S = \{n' : n' || n, n' \nparallel m\}$. We may assume $x \notin n$, for $H(m,n) = H(m,n')$ for all $n' \in S$. Let $n^* \in S$ such that $x \in n^*$. Since $n^* || n$ and $d(n,n^*) = 1$, $d(x,n) = 1$. Then at least $q+1$ lines of H contain x, namely n^* and the q lines which contain x and intersect n. Also, if $x \in h \in L(H)$ and h is not parallel to n, then $|h \cap n'| \le 1$ for all $n' \in s$. Since $h \subseteq H$, h intersects $|h| = |S|$ members of S; in particular h intersects n. Hence every point of H is contained in $q+1$ lines of H.

By Lemma 5.1, every line of H contains k points of H. To show, for all $y \in H$, $h \in L(H)$, $y \notin h$, that q lines of H contain y and intersect h it is sufficient to show that $d(y,h) = 1$. Let $n^* \in S$ such that $y \in n^*$. If $h \in S$ then $d(h,n^*) = 1$, so $d(y,h) = 1$. If $h \notin S$ then h intersects all lines of S. In particular, $h \nparallel n^*$, so $d(y,h) = 1$. Then H is a $(q+1, k, q)$-partial geometry.

COROLLARY. *Each plane is isomorphic to a* $(d,q,2)$-*attenuated space for some prime power* q *and integer* d.

Proof. Let H be a plane. H is a $(q+1, k, q)$-partial geometry, and $k > q^2$. Since (P,L) satisfies the dual of Pasch's Axiom, so does $(H,L(H))$. Theorem 1 completes the proof.

LEMMA 5.3. *Let* H *be a plane,* $m,n, \in L(H)$, *and* $m \nparallel n$. *Then* $H = H(m,n)$.

Proof. Since H and $H(m,n)$ are both $(q+1, k, q)$-partial geometries, $|H| = |H(m,n)|$. Therefore it is sufficient to show $H \subseteq H(m,n)$. Let $x \in H$. If $x \in m \cup n$, then $x \in H(m,n)$, so for the remainder of the proof we may presume $x \notin m \cup n$. Let $z = m \cap n$.

Case 1. $d(x,z) > 1$. x is on $q+1$ lines of H, q of which intersect m and q of which intersect n, so at least $q - 1 \ge 1$ intersect both m and n. Since $d(x,z) > 1$, each of these is a transversal. Since x is on a transversal h of m and n, $x \in h \subseteq H(m,n)$ by Lemma 5.1.

Case 2. $d(x,z) = 1$. Let $h \in L(H)$ such that $x \in h$, $z \notin h$. For $k - q$ points y of h, $d(y,z) > 1$, so, by Case 1, h contains at least $k - q \ge 2$ points of $H(m,n)$. Then $h \in L(H(m,n))$ so $x \in H(m,n)$.

COROLLARY. *Two lines of a plane either intersect or are parallel.*

LEMMA 5.4. *For any two planes* H_1 *and* H_2, $H_1 \cap H_2$ *is either empty or a line.*

Proof. We first show that if $m \subseteq H_1 \cap H_2$ then $m = H_1 \cap H_2$. Otherwise we have $x \notin m$ such that $m \cup x \subseteq H_1 \cap H_2$. Then some line n contains x and intersects m. Since $|n \cap H_i| \ge 2$, $n \in L(H_i)$ $(i = 1,2)$. Therefore $H_1 = H(m,n) = H_2$.

We show that a line is contained in $q+1$ planes. Let m be a line, and $x \in m$. Let R be the set of lines containing x and distinct from m. The planes containing m partition R, and each contains q lines of R. $|R| = q^2 + q$. Therefore exactly $q+1$ planes contain m.

We show that each point is on $q^2 + q + 1$ planes. Let $y \in X$. Counting the number of elements of the set $\{(m,n,H) : \{y\} = m \cap n, H = H(m,n)\}$ in two ways shows that $(q^2+q+1)(q^2+q) = |\{H : y \in H\}|(q+1)q$. Therefore y is contained in $q^2 + q + 1$ planes.

Let H be one of the planes containing the point y. H contains $q+1$ lines which contain y. Each of these is contained in q planes distinct from H. Therefore there are $(q+1)q$ planes H' such that $y \in H'$ and $H \cap H'$ is a line. Since y is contained in $q^2 + q + 1$ planes, $H \cap H'$ is a line for every plane H' which contains y and is distinct from H. Since H and $y \in H$ are arbitrary, the proof is complete.

We will say that a line m *intersects* a plane H if $|m \cap H| = 1$.

LEMMA 5.5. *Let* L^* *be a parallel class of lines and* H *be a plane. If some line of* L^* *intersects* H *then every line of* L^* *intersects* H.

Proof. Let $n,n' \in L^*$, and n' intersect H at the point x. If $d(n,n') = 1$, then let H' be a plane containing n and n'. H and H' contain a point in common, so their intersection is a line h which intersects n' at x. Since h and n are nonparallel lines of H', $h \neq n$. Then H contains exactly one point of n.

If $d(n,n') = 2$, let $h \in L^*$ so that $d(n',h) = d(n,h) = 1$. Then by the preceding paragraph, h intersects H, and also n intersects H. This completes the proof of Lemma 5.5.

We will now complete the proof of Theorem 2. We show that (X,L) is an attenuated space by coordinatizing X and using the characterization of attenuated spaces of Section 2.

Let x_0 be a point and m_0,n_1,n_2 be three lines containing x_0 such that $H_1 = H(m_0,n_1)$ and $H_2 = H(m_0,n_2)$ are distinct. Let $y_1 \in n_1 - x_0$ and $y_2 \in n_2 - x_0$ so that $d(y_1,y_2) = 1$, and let L^* be the parallel class of lines which contains $\langle y_1,y_2\rangle$. Every line of L^* intersects H_1 and intersects H_2. $H_1 \cap H_2 = m_0$.

For any $x \in H_1 - m_0$, x is in a unique line of L^*, which in turn intersects H_2. Therefore we may define a mapping σ from $H_1 - m_0$ to $H_2 - m_0$ as follows: for each $x \in H_1 - m_0$ we let $\sigma(x)$ be the point in H_2 such that $\langle x,\sigma(x)\rangle \in L^*$. σ may be extended to a mapping from H_1 to H_2 by defining $\sigma(x) = x$ for all $x \in m_0$. Clearly, σ is a bijection from H_1 to H_2.

Let $h \in L(H_1)$, $h \neq m_0$, and $x \in h - m_0$. Let $H^* = H(h,\langle x,\sigma(x)\rangle)$, $H^* \cap H_1 = h$. For all $y \in h$, $\sigma(y) \in H^*$ so $\sigma(h) = H^* \cap H_2$. Since $H^* \cap H_2$ is a line, $\sigma(h)$ is a line. Also, $\sigma(m_0)$ is the line m_0. Then σ maps $L(H_1)$ to $L(H_2)$. σ, acting on the points and lines of H_1, is an isomorphism from H_1 to H_2.

H_1 is a $(d,q,2)$-attenuated space for some prime power q and integer $d > 2$. Let W be a $(d-2)$-dimensional vector space over $F = GF(q)$. Let $T = \{(w_1,w_2) + (\delta_1,\delta_2)W : w_1,w_2 \in W,\ \delta_1,\delta_2 \in F, (\delta_1,\delta_2) \neq (0,0)\}$. We denote $(W \times W, T)$ by π. $(H_1,L(H_1),\epsilon)$ and $(W \times W, T, \epsilon)$ are both $(d,q,2)$-attenuated spaces, so they are isomorphic. Let $\phi_1' : H_1 \to W \times W$ and $\chi : L(H_1) \to T$ be an isomorphism. Then ϕ_1', acting on $L(H_1)$, is the same map as χ. We henceforth disregard χ.

Let $\mu : W \times W \to W \times W \times W$ be defined by $\mu((w_1,w_2)) = (w_1,w_2,0)$, and let $\phi_1 = \mu\phi_1'$. Then ϕ_1 is a bijection from H_1 to $W \times W \times 0$, and for each line m of H_1, $\phi_1(m) = (w_1,w_2,0) + (\delta_1,\delta_2,0)W$ for some $w_1,w_2 \in W$ and $\delta_1,\delta_2 \in F$. Since the automorphism group of a $(d,q,2)$-attenuated space is transitive on points and the stabilizer of a point

is 2-transitive on the lines containing that point, we may choose ϕ_1' so that

$$\phi_1(x_0) = (0,0,0)\ ,$$
$$\phi_1(m_0) = 0 \times W \times 0\ ,$$
$$\phi_1(n_1) = W \times 0 \times 0\ .$$

For any $w \in W$, $W \times w = \{(w',w) : w' \in W\}$ is a line of π. $\phi_1'^{-1}(W \times w)$ is a line of H_1, and is disjoint from n_1 or equal to n_1. Hence $\phi_1'^{-1}(W \times w)$ is parallel to n_1. It is easily seen that ϕ_1 maps the set of lines of H_1 which are parallel to n_1 to $\{W \times w \times 0 : w \in W\}$.

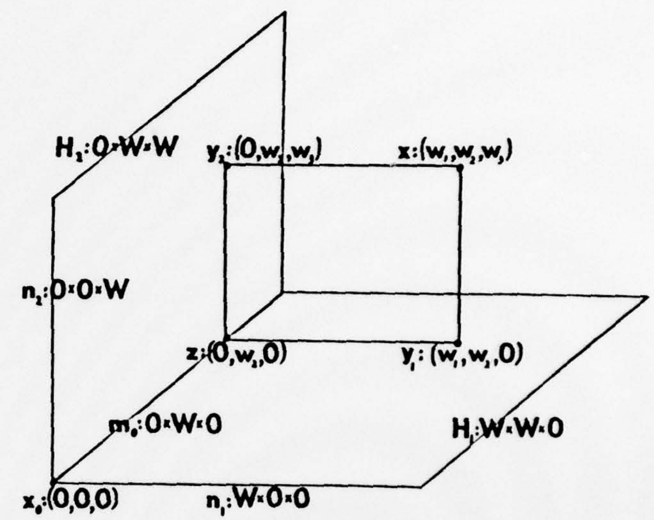

Figure 1

We coordinatize H_2, using ϕ_1' and σ. Let $\nu : W \times W \to 0 \times W \times W$ be defined by $\nu(w_1,w_2) = (0,w_2,w_1)$. Let $\phi_2 = \nu\,\phi_1'\,\sigma^{-1}$. Since σ, ϕ_1', and ν are bijections, ϕ_2 is a bijection from H_2 to $0 \times W \times W$. Since σ and ϕ_1', acting on the points and lines of H_1,H_2, and π, are incidence structure isomorphisms, ϕ_2 maps the set of points of a line of H_2 to the set of vectors $(0,w_2,w_1) + (0,\delta_2,\delta_1)W$ for some $w_1,w_2 \in W$ and $\delta_1,\delta_2 \in F$. We have

$$\phi_2(x) = \phi_1(x) \quad \text{for} \quad x \in m_0 ,$$
$$\phi_2(n_2) = 0 \times 0 \times W .$$

Figure 1 shows points and lines of H_1, followed by a colon and their co-ordinates by the mapping ϕ_1, and likewise shows points and lines of H_2 and their coordinates by ϕ_2.

Let L_0,L_1,L_2 be the parallel classes of lines containing respectively m_0,n_1,n_2. We coordinatize X. Let $x \in X$. Let $y_1 \in H_1$ so that $\langle x,y_1\rangle \in L_2$, and let $y_2 \in H_2$ so that $\langle x,y_2\rangle \in L_1$ (if $x \in H_1$ then $y_1 = x$, and if $x \in H_2$, then $y_2 = x$). Let h_i be a line containing y_i and parallel to n_i $(i = 1,2)$. Then $h_1 || \langle x,y_2\rangle$ and $h_2 || \langle x,y_1\rangle$, so $h_1 \nparallel h_2$. Let $\{z\} = h_1 \cap h_2$. Since $h_i \subseteq H_i$, $z \in m_0$. For some $w_2 \in W$, $\phi_1(z) = \phi_2(z) = (0,w_2,0)$. Then $\phi_1(h_1) = W \times w_2 \times 0$ so $\phi_1(y_1) = (w_1,w_2,0)$ for some $w_1 \in W$. Similarly $\phi_2(y_2) = (0,w_2,w_3)$ for some $w_3 \in W$. Define $\phi(x) = (w_1,w_2,w_3)$. Then ϕ is a bijection from X to W^3, and its restriction to H_i equals ϕ_i $(i = 1,2)$. See Figure 1 again.

Let $T^* = \{(w_1,w_2,w_3) + (\delta_1,\delta_2,\delta_3)W : w_i \in W, \delta_i \in F, 1 \le i \le 3,$ some δ_i is nonzero$\}$. To show that (X,L) and (W^3,T^*) are isomorphic it is sufficient to show that $\phi(h) \in T^*$ for all $h \in L$. This we do by stages.

(1) Let $h \in L_1$. Let $\{y\} = h \cap H_2$ and $\phi_2(y) = (0,w_2,w_3)$. Then $\phi(h) = W \times w_2 \times w_3 = (0,w_2,w_3) + (1,0,0)W$. Similarly, if $h \in L_2$ then $\phi(h) = (w_1,w_2,0) + (0,0,1)W$ for some $w_1,w_2 \in W$.

(2) Let H be a plane, $n^* \in L(H)$, and $n^* || n_2$. Then $H \cap H_1$ is a line; let $t = H \cap H_1$.

Since ϕ_1' is an isomorphism of $(H_1,L(H_1))$ to the $(d,q,2)$-attenuated space π, then $\phi(t) = \phi_1(t) = (w_1,w_2,0) + (\delta_1,\delta_2,0)W$ for some $w_1,w_2 \in W$ and $\delta_1,\delta_2 \in F$. $H = H(t,n^*)$ is the set of points contained in lines parallel to n_2 and intersecting t, so $\phi(h) = (w_1,w_2,0) + (\delta_1,\delta_2,0) + (0,0,1)W$. Similarly, if H is a plane, $n^* \in L(H)$ and $n^* || n_1$, then $\phi(h) = (0,w_2,w_3) + (0,\delta_2,\delta_3)W + (1,0,0)W$. The converses of these statements are also valid.

(3) Let H be a plane. Then $\phi(H) = W \times w \times W$ for some $w \in W$ if and only if H contains a line parallel to n_1 and also contains a line parallel to n_2, by the results of paragraph (2).

(4) Let h be a line such that no plane containing h contains both lines parallel to n_1 and lines parallel to n_2.

Let $x \in h$, and t_i be a line containing x and parallel to $n_i (i = 1,2)$. Let $H_1^* = H(h,t_1)$ and $H_2^* = H(h,t_2)$. $H_1^* \neq H_2^*$. Then $h = H_1^* \cap H_2^*$ so $\phi(h) = \phi(H_1^*) \cap \phi(H_2^*)$. Let $\phi(x) = (v_1,v_2,v_3)$, and $\phi(H_1^*) = (w_1,w_2,0) + (\delta_1,\delta_2,0)W + (0,0,1)W$. Now $\phi(x) \in \phi(H_1^*)$, so $(v_1,v_2,v_3) - (w_1,w_2,0) \in (\delta_1,\delta_2,0)W + (0,0,1)W$. Therefore we may write $\phi(H_1^*)$ as

$$\phi(H_1^*) = (v_1,v_2,v_3) + (\delta_1,\delta_2,0)W + (0,0,1)W .$$

Similarly,

$$\phi(H_2^*) = (v_1,v_2,v_3) + (0,\epsilon_2,\epsilon_3)W + (1,0,0)W .$$

If $\delta_2 = 0$ then $H_1^* = W \times v_2 \times W$, contrary to the hypothesis that no plane containing h (in particular H_1^*) contains both lines parallel to n_1 and lines parallel to n_2. Hence $\delta_2 \neq 0$, and similarly $\epsilon_2 \neq 0$. Then we may assume that $\delta_2 = \epsilon_2 = 1$. Then $\phi(h) = \phi(H_1^*) \cap \phi(H_2^*) = (v_1,v_2,v_3) + (\delta_1,1,\epsilon_3)W$.

(5) If $h || m_0$ then, by the above, $\phi(h) = (v_1,v_2,v_3) + (\delta_1,1,\epsilon_3)W$ for some $v_1,v_2,v_3 \in W$, $\delta_1,\epsilon_3 \in F$. It is easily shown that $\delta_1 = \epsilon_3 = 0$.

(6) Let H be a plane which contains a line parallel to m_0, and let $x \in H$. Let t_i be the line containing x and parallel to $n_i(i = 1,2)$. $H \neq H(t_1,t_2)$ (otherwise m_0, n_1, and n_2 would all belong to the same plane). $H \cap H(t_1,t_2)$ is a line. Of the $q + 1 \ge 3$ lines of H containing x, one is parallel to m_0 and one equals $H \cap H(t_1,t_2)$; let h be any other line of H containing x. It is easily seen that $H \cap H(t_1,t_2)$ is the only line of H which contains x and whose image under ϕ is not yet established (that is, the only line not dealt with in paragraph (4)). Then $\phi(h) = (v_1,v_2,v_3) + (\delta_1,1,\epsilon_3)W$ for some $v_1,v_2,v_3 \in W$, $\delta_1,\epsilon_3, \in F$. For any $y \in h$ let m_y be the line containing y and parallel to m_0. For any $y \in h$,

$\phi(y) = (v_1,v_2,v_3) + (\delta_1,1,\varepsilon_3)w$ for some $w \in W$, so

$\phi(m_y) = (v_1,v_2,v_3) + (\delta_1,1,\varepsilon_3)w + (0,1,0)W$. $H = \bigcup_{y \in h} m_y$

so $\phi(h) = \bigcup_{y \in h} \phi(m_y) = (v_1,v_2,v_3) + (\delta_1,1,\varepsilon_3)W + (0,1,0)W$.

(7) Let h be a line of the plane H^* where $\phi(H^*) = W \times w \times W$. Let $x \in h$, and m be a line parallel to m_0 and containing x. Let $H = H(h,m)$. Then

$\phi(H) = (v_1,v_2,v_3) + (\delta_1,1,\varepsilon_3)W + (0,1,0)W$ for some $v_1,v_2,v_3 \in W$, $\delta_1,\varepsilon_3 \in F$. $\phi(h) = \phi(H) \cap \phi(H^*) = (v_1,w,v_3) + (\delta_1,0,\varepsilon_3)W$.

By the result of paragraph (3), every line has had its image by ϕ computed in either (4) or (7). Since $\phi(h) \in T^*$ for all $h \in L$, ϕ is an incidence structure isomorphism. Therefore (X,L) is isomorphic to a $(d,q,3)$-attenuated space. Since $k > q^3$ by (a6), $d > 6$. This completes the proof of Theorem 2.

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CHARACTERIZATION OF PROJECTIVE INCIDENCE STRUCTURES

1. Introduction and Statement of Theorem

For a finite set X, $|X|$ will denote the number of elements of X. An incidence structure is an ordered triple (P, L, I) where P and L are disjoint sets and $I \subseteq P \times L$. Elements of P will be called points or vertices and elements of L lines. A line l and a point p are called incident iff $(p, l) \in I$. We also say in this case that l contains p or p lines on l. Two lines l and m are said to intersect iff they have a common incident point. With any incidence structure (P, L, I) is associated its dual incidence structure (L, P, I^*) where $I^* = \{(l, p) : (p, l) \in I\}$. If L is a set of subsets of P and $(p, l) \in I$ iff $p \in l$, we will refer to (P, L, I) as $(P, L, \in)$ or (P, L). The dual of $(P, L, \in)$ will be written as $(L, P, \ni)$. If each element of L and P is a set and $(p, l) \in I$ iff $p \subseteq l$, we write (P, L, I) as $(P, L, \subseteq)$ and its dual as $(L, P, \supseteq)$. For a line l, P_l will denote the set of points incident with line l. If P_l is a finite set, we write $k(l)$ for the cardinality of P_l. Similarly, for a point p, L_p denotes the set of lines l incident with the point p and we write $r(p)$ for $|L_p|$. An incidence structure is said to be simple iff for any two distinct lines l and l', $P_l \neq P_{l'}$. Incidence structures (P, L, I) and (P', L', I') will be called isomorphic iff there exist bijections $\sigma: P \to P'$ and $\tau: L \to L'$ such that $(p, l) \in I$ iff $(\sigma(p), \tau(l)) \in I'$.

An incidence structure $\pi = (P, L, I)$ is said to be finite iff both P and L are finite sets. All incidence structures in this paper are finite. For a finite incidence structure, we will set $r(\pi) = \min\{r(p) : p \in P\}$ and $k(\pi) = \min\{k(l) : l \in L\}$. Let q be a positive integer. If $q = 1$, we define $s(\pi, q)$ to be equal to $k(\pi)$. If $q \geqslant 2$, we define $s(\pi, q)$ to be the unique real number s which satisfies $q^s - 1 = k(\pi)(q - 1)$. If $q = 1$, we define $d(\pi, q)$ to be equal to $r(\pi) + s(\pi, q) - 1$. If $q \geqslant 2$, we define $d(\pi, q)$ to be the unique real number d which satisfies $q^{d - s(\pi, q) + 1} - 1 = (q - 1)r(\pi)$. We normally write $s(\pi, q)$ as $s(\pi)$ and $d(\pi, q)$ as $d(\pi)$. The incidence structure π is said to be semilinear iff $\forall p, p' \in P, p \neq p'$, $\exists$ at most one line l incident with both p and p'. Let r and k be positive integers. A semilinear incidence structure π is said to be an (r, k) incidence structure iff for every point p, $r(p) = r$ and every line l, $k(l) = k$. Let π be a semilinear incidence structure and l and m be two lines. A line n will be called a transversal of l and m iff n intersects both l and m and $P_n \cap P_l \neq P_n \cap P_m$. A semilinear incidence structure π is said to satisfy Pasch's axiom iff for any pair of intersecting lines m_1 and m_2 and any pair of transversals n_1 and n_2 of m_1 and m_2, n_1 intersects n_2. A subset $F \subseteq P$ is called

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a flat iff $\forall l \in L$, $|P_l \cap F| \geqslant 2$ implies $P_l \subseteq F$. Clearly, any intersection of flats is a flat. For $S \subseteq P$, the flat $\langle S \rangle = \bigcap_{F \supseteq S} F$ is said to be the flat generated by S. For any flat F, rank F is the smallest integer n such that there exists a set $S \subseteq P$, $|S| = n$ and $\langle S \rangle = F$. The rank of the flat P is called rank π.

A simple graph is a simple incidence structure in which every line is incident with exactly two points. Points and lines of a simple graph will usually be called vertices and edges, respectively. Two vertices p and p' will be called adjacent iff there exists an edge l incident with p and p'. Adjacency is a symmetric relation on the set of vertices of a graph and determines a simple graph completely. All graphs considered in this paper will be finite and simple. Let G be a simple graph with vertex set V and edge set E. Let n be a nonnegative integer. A path of length n from u to v is a sequence $(u = v_0, l_1, v_1, l_2, v_2, \ldots, l_n, v_n = v)$ where l_i is an edge incident with v_{i-1} and v_i, $i = 1, 2, \ldots, n$. If for any two vertices u and v there exists a path from u to v, then the graph G is said to be connected. In a connected graph G the distance $d(u, v)$ between two vertices u and v is the smallest nonnegative integer n such that a path of length n from u to v in G exists.

Let $\pi = (P, L, I)$ be an incidence structure. The adjacency graph $G(\pi)$ of π is a graph having vertex set P and two vertices adjacent iff some line of π contains both. The graph $G(\pi^*)$ of the dual incidence structure π^* will be called the line graph of π. Distance between two points p and p' of π will be same as the distance between them in $G(\pi)$. For $S \subseteq P$ and $l, m \in L$, we will set $d(l, S) = \min\{d(p, p') : p' \in S,\ p \text{ incident with } l\}$ and $d(l, m) = \min\{d(l, p) : p \text{ incident with } m\}$ where $d(p, p')$ is the distance between the vertices p and p' in $G(\pi)$. Sometimes the points of π will be called vertices.

Let $q \geqslant 2$ be a prime power and $1 \leqslant s \leqslant d$ be integers. Let V be a d-dimensional vector space over a finite field of order q. Let W_i be the set of i-dimensional subspaces of V, $0 \leqslant i \leqslant d$. Let $(W_{s-1}, W_s, \subseteq)$ be the incidence structure whose points are $(s-1)$-dimensional subspaces, lines are s-dimensional subspaces and incidence is set inclusion. Any incidence structure π isomorphic to $(W_{s-1}, W_s, \subseteq)$ will be called an (s, q, d) projective incidence structure (p.i.s.). For $q = 1$, also we define an $(s, 1, d)$-projective incidence structure. Let Y be a finite set with $|Y| = d$. A subset $Y' \subseteq Y$ is called an i-subset of Y iff $|Y'| = i$. Let Z_i be the set of i-subsets of Y. Any incidence structure isomorphic to $(Z_{s-1}, Z_s, \subseteq)$ will be called an $(s, 1, d)$-projective incidence structure. The incidence structure $(W_{d-s+1}, W_{d-s}, \supseteq)$ is isomorphic to $(W_{s-1}, W_s, \subseteq)$. Also, $(Z_{d-s+1}, Z_{d-s}, \supseteq)$ is isomorphic to $(Z_{s-1}, Z_s, \subseteq)$.

The following classical theorem about finite projective spaces characterizes $(2, q, d)$-projective incidence structures for $d \geqslant 4$.

THEOREM. *Let π be a finite incidence structure satisfying*

(p1) *There exists exactly one line joining two distinct points.*

(p2) *Every line contains at least three points.*

(p3) *Pasch's axiom.*

(p4) *Rank of* $\pi \geqslant 4$.

Then there exists a prime power $q \geqslant 2$ *and an integer* $d \geqslant 4$ *such that* π *is a* $(2, q, d)$*-projective incidence structure. Conversely, any* $(2, q, d)$*-projective incidence structure with* $d \geqslant 4$, $q \geqslant 2$ *satisfies* (*p1*)–(*p4*).

Extending this classical theorem, we prove a characterization of (s, q, d)-projective incidence structures when $3 \leqslant s < d - 1$.

THEOREM 1. *Let* $q \geqslant 1$ *be an integer and* π *be a finite incidence structure satisfying*

(f1) $3 \leqslant s(\pi, q) < d(\pi, q) - 1$.

(f2) *There exists at most one line joining two distinct points.*

(f3) *If* p *is a point and* l *is a line such that* $d(p, l) = 1$, *then there are exactly* $(q + 1)$ *lines which pass through* p *and intersect* l.

(f4) *If* p *and* p' *are two distinct points such that* $d(p, p') = 2$, *then there are exactly* $(q + 1)$ *lines* l *such that* l *passes through* p' *and* $d(p, l) = 1$.

(f5) $G(\pi)$ *is connected.*

Then $s = s(\pi, q)$ *and* $d = d(\pi, q)$ *are integers,* $q = 1$ *or a prime power and* π *is an* (s, q, d)*-projective incidence structure. Conversely, for* $3 \leqslant s < d - 1$, *any* (s, q, d)*-projective incidence structure satisfies* (*f1*)–(*f5*).

We also show that the Axioms (f1)–(f5) are minimal for the purpose of characterizing (s, q, d)-p.i.s., $3 \leqslant s < d - 1$. For any choice of $j \in \{1, 2, 3, 4, 5\}$, there exists incidence structures π' which satisfy the four axioms other than (f_j) and is not an (s, q, d)-p.i.s. with $3 \leqslant s < d - 1$. A finite incidence structure π satisfying (f2)–(f5) is called an (s, q, d)-pseudo projective incidence structure where $s(\pi, q) = s$ and $d(\pi, q) = d$. The Axiom (f5) in the statement of Theorem 1 is not an essential axiom. Let $\pi_i = (P_i, L_i, I_i)$, $i = 1, 2$ be two incidence structures such that $P_1 \cap P_2 = L_1 \cap L_2 = \varnothing$. We define the direct sum $\pi = \pi_1 + \pi_2$ by $\pi = (P_1 \cup P_2, L_1 \cup L_2, I)$ where $(p, l) \in I$ iff $\exists i$, $1 \leqslant i \leqslant 2$, $p \in P_i$, $l \in L_i$, and $(p, l) \in I_i$.

THEOREM 2. *Let* $q \geqslant 1$ *be an integer and* π *be a finite incidence structure satisfying the Axioms* (*f1*)–(*f4*). *Then* $q = 1$ *or a prime power and* π *is isomorphic to the direct sum of one or more projective incidence structures. Conversely, if* $q = 1$ *or a prime power and* π *is the direct sum of several* (s_i, q, d_i)-p.i.s. *where* $3 \leqslant s_i < d_i - 1$, *then* π *satisfies Axioms* (*f1*)–(*f4*).

Outline of the Proof of Theorem 1. Let π be an (s, q, d)-pseudo projective incidence structure. Let m and n be two lines containing a common point O. Let $C(m, n)$ be the set of lines containing the transversals of m and n and all lines l which contain O and intersect at least one transversal of m and n. $C(m, n)$ is called the plane generated by m and n. Let $\mathscr{C}$ be the set of all planes. One of the important steps in the proof is to show that the incidence structure $(L, \mathscr{C}, \in)$ is an $(s + 1, q, d)$-pseudo projective incidence structure. One starts with an (s, q, d)-pseudo p.i.s. and finally obtains an $(d - 1, q, d)$-pseudo p.i.s. which is then shown to be dual of a projective space.

2. Preliminary Propositions

LEMMA 1. *Let $q \geqslant 1$ be an integer, π be a finite incidence structure such that $r(p)$ and $k(l)$ are positive for all points p and line l. Let π satisfy the Axioms (f2), (f3) and (f5) and $r = r(\pi)$, $k = k(\pi)$. Then π is an (r, k)-incidence structure.*

Proof. Let $\pi = (P, L, I)$. To show that $\forall l \in L$, $k(l) = k$, it is sufficient to show that $\forall l' \in L$, $k(l) = k(l')$. Let l and l' be two intersecting lines and z be the common point. We calculate

$$b = |\{(p, p') : (p, l) \in I, (p', l') \in I, p, p' \neq z, d(p, p') = 1\}|.$$

For every point $p \neq z$ of l, $d(p, l') = 1$. So there are q points p' of l' such that $d(p, p') = 1$ and $p' \neq z$. Hence, $b = (k(l) - 1)(q)$. By symmetry $b = (k(l') - 1)(q)$. Since $q \geqslant 1$, $k(l) = k(l')$. Let l and l' be any two lines. Since $G(\pi)$ is connected, we can find a sequence $l_0 = l, l_1, l_2, \ldots, l_i = l'$ such that l_{j-1} and l_j intersect for $j = 1, 2, \ldots, i$. Since $k(l_{j-1}) = k(l_j)$ for $j = 1, 2, \ldots, i$, it follows that $k(l') = k(l)$. It is easily checked that the dual incidence structure π^* satisfies (f2), (f3) and (f5). Therefore, we get $r(p) = r(p')$, $\forall p, p' \in P$ and hence, $r(p) = r$, $\forall p \in P$.

LEMMA 2. *Let $q = 1$ or a prime power and $3 \leqslant s < d - 1$ be integers. Then any (s, q, d)-projective incidence structure is an (s, q, d)-pseudo projective incidence structure.*

Proof. First we consider the case q a prime power, $q \geqslant 2$. Let $\pi = (W_{s-1}, W_s, \subseteq)$ be an (s, q, d)-projective incidence structure where $3 \leqslant s \leqslant d - 2$ and W_i is the set of i-dimensional subspaces of a vector space V of dimension d over GF(q), $0 \leqslant i \leqslant d$. The number of $(s - 1)$-dimensional subspaces contained in an s-dimensional subspace is $(q^s - 1)/(q - 1)$ and hence, $k(\pi) = (q^s - 1)/(q - 1)$ and $s(\pi) = s$. Similarly, the number of s-dimensional subspaces containing a given $(s - 1)$-dimensional subspace is $(q^{d-s+1} - 1)/(q - 1)$. Therefore, $r(\pi) = (q^{d-s+1} - 1)/(q - 1)$ and $d(\pi) = d$. The Axiom (f1) holds since $3 \leqslant s \leqslant d - 2$. Let p and p' be two $(s - 1)$-dimensional subspaces and l be an s-dimensional subspace such that $p, p' \subseteq l$. Then l is the subspace spanned by p and p'. Hence, there exist at most one line joining p and p' and π is semilinear. Let p and p' be two $(s - 1)$-dimensional subspaces such that $\{u_1, u_2, \ldots, u_i, v_1, \ldots, v_{s-i}\}$ and $\{u_1, u_2, \ldots, u_i, w_1, w_2, \ldots, w_{s-i}\}$ are respectively bases of p and p', $0 \leqslant i \leqslant s - 1$. Let p_j be the subspace spanned by $\{u_1, u_2, \ldots, u_i, w_1, w_2, \ldots, w_j, v_{j+1}, \ldots, v_{s-i}\}$, $j = 0, 1, \ldots, s - i$. Then $p_0 = p$ and $p_{s-i} = p'$ and p_j and p_{j+1} are adjacent in $G(\pi)$. Hence, there exists a path joining p and p' in $G(\pi)$. This establishes that $G(\pi)$ is connected. Let $p \in W_{s-1}$ and $l \in W_s$ such that $d(p, l) = 1$. Then $p \nsubseteq l$ and there exists an $l' \in W_s$ such that $p \subseteq l'$ and $l \cap l' \in W_{s-1}$. It follows that $p \cap l = u$ is an $(s - 2)$-dimensional subspace. There are $(q + 1)$ $(s - 1)$-dimensional subspaces p_i, $1 \leqslant i \leqslant q + 1$ such that $u \subseteq p_i \subseteq l$. Let $l_i = \langle p, p_i \rangle$. Then l_i,

$1 \leq i \leq q+1$ are the only lines of π which contain p and intersect l in a point. It follows that π satisfies (f3). Let $p, p' \in W_{s-1}$ such that $d(p, p') = 2$. This implies that $p \cap p' = v \in W_{s-3}$. Let $u_1, u_2, \ldots, u_{q+1}$ be the $(s-2)$-dimensional subspaces such that $v \subseteq u_i \subseteq p$, $1 \leq i \leq q+1$. Let $l_i = \langle u_i, p' \rangle$, $1 \leq i \leq q+1$. Then $l_1, l_2, \ldots, l_{q+1}$ are the only lines of π which pass through p' and have distance 1 from p. Therefore, π satisfies (f4). This establishes the lemma when $q \geq 2$. For $q = 1$, we take $\pi = (Z_{s-1}, Z_s, \subseteq)$ where Z_i is the set of i-element subsets of a d-set Y, $0 \leq i \leq d$. It is easily checked that π satisfies the Axioms (f1)–(f5).

In the sequel we will assume without loss of generality (wlog) that lines are subsets of points. We assume that q is a fixed positive integer and s and d real numbers satisfying $3 \leq s < d - 1$ and π is a pseudo projective incidence structure and $s(\pi) = s$, $d(\pi) = d$, $r(\pi) = r$, $k(\pi) = k$.

LEMMA 3. *Let p and p' be two distinct points of π such that $d(p, p') = 2$. Let L_1 be the set of lines containing p and at distance 1 from p' and let L_2 be the set of lines containing p' and at distance 1 from p. Then each line of L_1 intersects each line of L_2.*

Proof. Let $n \in L_2$ and $n^* = \{z \in n : d(z, p) = 1\}$. Then $|n^*|$ equals the number of lines of L_1 which intersect n. By (f3), $|n^*| = (q+1)$ and by (f4), $|L_1| = q+1$. Hence, each line of L_1 intersects n.

For a pair of lines m and n, $T(m, n)$ denotes the set of transversals of m and n.

LEMMA 4. *Let m and n be two distinct lines of π such that $d(m, n) = 1$. Then* (i), *$d(p, m) = 1$ for exactly $(q+1)$ points p of n and* (ii) $|T(m, n)| \leq (q+1)^2$.

Proof. Since $d(m, n) = 1$, there exists points x and y such that $x \in m$, $y \in n$ and $d(x, y) = 1$. Since $d(x, n) = 1$, by (f3) there exists $(q+1)$ points $y_0 = y, y_1, \ldots, y_q$ such that $d(x, y_i) = 1$, $y_i \in n$ and $d(y_i, m) = 1$, $0 \leq i \leq q$. If possible, let $y \in n$, $y \neq y_i$, $0 \leq i \leq q$ and $d(y, m) = 1$. Then $d(x, y) = 2$, $d(x, n) = d(y, m) = 1$ and $x \in m$, $y \in n$. By Lemma 3, m and n must intersect whence $d(m, n) \neq 1$. This completes the proof of (i) and (ii) follows easily.

LEMMA 5. *Pasch's axiom is valid in (P, L). For any pair of intersecting lines m_1 and m_2, $|T(m_1, m_2)| = (k-1)q$.*

Proof. Let $\{x\} = m_1 \cap m_2$. For each $y \in m_1 - x$, y is adjacent to q vertices of $m_2 - x$. So, q transversals of m_1 and m_2 contain y. Therefore, $|T(m_1, m_2)| = (k-1)q$. Let $n \in T(m_1, m_2)$. Let $a \in n \cap m_1$, $b \in n \cap m_2$, S_1 be the set of $(q-1)$ vertices of $m_1 - \{x, a\}$ adjacent to b and S_2 be the set of $(q-1)$ vertices of $m_2 - \{x, b\}$ adjacent to a. Let $h \in T(m_1, m_2)$ such that $\{c\} = h \cap m_1 \not\subset S_1$. Then b and c are not adjacent. We get $d(b, c) = 2$, $b \in n$, $d(c, n) = 1$, $c \in h$, $d(b, h) = 1$. By Lemma 3, n and h intersect. It follows that if $h \in T(m_1, m_2)$ and h and n do not intersect, then $h \cap m_1 \in S_1$. Similarly, $h \cap m_2 \in S_2$. Therefore the number of lines of $T(m_1, m_2)$ not intersecting n

is at most $(q-1)^2 = |S_1|\,|S_2|$. If $q = 1$, $(q-1)^2 = 0$. Then all lines of $T(m_1, m_2)$ intersect n, so Pasch's axiom is valid.

Let $q \geqslant 2$. If possible, let n and h be two non-intersecting lines of $T(m_1, m_2)$. There are at least $|T(m_1, m_2)| - 2(q-1)^2 = (k-1)q - 2(q-1)^2$ lines of $T(m_1, m_2)$ which intersect both n and h. Also, m_1 and m_2 intersect both n and h. Hence, the number of lines intersecting both n and h is at least $kq - 2q^2 + 3q$. On the other hand, since $d(n, h) = 1$, by Lemma 5 there are at most $(q+1)^2$ lines intersecting both n and h. This gives us $(q+1)^2 \geqslant kq - 2q^2 + 3q$. Since $s \geqslant 3$, $k \geqslant q^2 + q + 1 \geqslant 3q$ and $(q+1)^2 \geqslant 3q^2 - 2q^2 + 3q$. Simplifying the inequality we get $1 \geqslant q$ which contradicts the assumption.

If S is a set of lines such that any two lines of S intersect each other, then S is a clique in the line graph of (P, L); we refer to such a set S as a *clique of lines.*

Let m and n be intersecting lines and x be the point of intersection. We let

$$C(m, n) = T(m, n) \cup \{h : h \in L,\ x \in h,\ h \cap n' \neq \varnothing \text{ for some } n' \in T(m, n)\}.$$

LEMMA 6. *Let m_1 and m_2 be intersecting lines. Then $C(m_1, m_2)$ is a maximal clique of lines.*

Proof. We denote $T(m_1, m_2)$ by T and $C(m_1, m_2)$ by C. Let $\{x\} = m_1 \cap m_2$. T is a clique of lines, and so is $C - T$ since x belongs to each line of $C - T$. We show that if $h \in C - T$ and $n' \in T$, then h intersects n'. Since $h \in C - T$, $x \in h$ and h intersects n for some transversal n of m_1 and m_2. We may assume (by exchanging m_1 and m_2 if necessary) that $n \cap m_2 \neq n' \cap m_2$. Then $h, n' \in T(n, m_2)$. So, h and n' intersect. Hence C is a clique of lines. It is clear from the definition of C that no proper superset of C is a clique of lines.

We call each $C(m_1, m_2)$ a plane, and let $\mathscr{C}$ be the set of planes.

COROLLARY 1. *Each plane contains $qk + 1$ lines.*

Proof. Let m and n be lines which intersect at x. Let $h \in T(m, n)$. By Lemma 6 every line of $C(m, n) - T(m, n)$ intersects h, so $C(m, n) - T(m, n)$ is the set of $q + 1$ lines which contain x and intersect h. Then $|C(m, n)| = |T(m, n)| + q + 1 = qk + 1$.

LEMMA 7. *Let K be a clique of lines, $m, n \in K$ $(m \neq n)$, and $\{x\} = m \cap n$. Then either all lines of K contain x or $K \subseteq C(m, n)$.*

Proof. We assume that some line n' of K does not contain x and show that $K \subseteq C(m, n)$. Let $h \in K$. Then h intersects m, n, and n'. If $x \notin h$, then $h \in T(m, n)$. So $h \in C(m, n)$. Next suppose $x \in h$. Since h intersects n', $h \in C(m, n)$. Therefore, $K \subseteq C(m, n)$.

LEMMA 8. (i) *Each pair of intersecting lines is in a unique plane.* (ii) *If the plane C contains at least 1 line containing x, then C contains exactly $q + 1$ lines containing x.* (iii) *Each line is contained in $(r-1)/q$ planes.*

Proof. (i) Let m and n be intersecting lines and the plane C contain m and n. By Lemma 7 $C \subseteq C(m, n)$. But all planes have the same cardinality, so $C = C(m, n)$. (ii) Let $x \in m \in C$. Let $n \in C$ so that $x \notin n$. Then $C = C(m, n)$. Every line of C which contains x also intersects n. There are $q + 1$ lines which contain x and intersect n. One of these lines is m, and the remaining q lines are transversals of m and n, so $q + 1$ lines of C contain x. (iii) Let m be a line. Choose $x \in m$ and let $m_2, m_3, \ldots, m_r$ be the lines containing x which are distinct from m. Each plane which contains m contains exactly q lines among $m_2, m_3, \ldots, m_r$. By part (i), each line m_i is contained in a unique plane containing m. Hence exactly $(r - 1)/q$ planes contain m.

From Lemma 8 and Corollary 1, the following statement is immediate.

COROLLARY 2. *$(L, \mathscr{C})$ is a semilinear $((r - 1)/q, qk + 1)$-incidence structure.*

For any plane C we define $\bar{C} = \bigcup_{m \in C} m$.

LEMMA 9. *Let $m \in L$ and $C \in \mathscr{C}$. If $|m \cap \bar{C}| \geqslant 2$, then $m \in C$.*

Proof. Let $x, y \in m \cap \bar{C}$. Then for some n_1 and n_2 $(n_1, n_2 \neq m)$ $x \in n_1 \in C$ and $y \in n_2 \in C$. Lines n_1 and n_2 intersect since all lines of C intersect, so $C = C(n_1, n_2)$. Since m is a transversal of n_1 and n_2, $m \in C$.

Since each pair of intersecting lines is contained in a plane, and each plane is a clique of lines, two lines contain a point in common iff they are both contained in some plane. Therefore the adjacency graph of $(L, \mathscr{C})$ is identical to the line graph of (P, L). Let H be the adjacency graph of $(L, \mathscr{C})$.

LEMMA 10. *If m and n are distinct lines then $d_H(m, n) = d_G(m, n) + 1$. If the line m is not contained in the plane C then $d_H(m, C) = d_G(m, \bar{C}) + 1$.*

Proof. Let $d_G(m, n) = i - 1$ where $m \neq n$. Denote m by m_0 and n by m_i. Let $(m_0, x_1, m_1, x_2, \ldots, x_i, m_i)$ be a sequence of points and lines such that x_j is contained in m_{j-1} and m_j $(1 \leqslant j \leqslant i)$. Let $C_j = C(m_{j-1}, m_j)$ for $1 \leqslant j \leqslant i$. Then $(m_0, C_1, m_1, C_2, \ldots, C_i, m_i)$ is a sequence of lines and planes so that C_j contains m_{j-1} and m_j $(1 \leqslant j \leqslant i)$, so $d_H(m, n) \leqslant i$. Since the direction of this argument is reversible, we may conclude that $d_H(m, n) = d_G(m, n) + 1$.

Let $m \notin C$. Now $d_G(m, \bar{C}) = \min\{d_G(m, n) : n \in C\}$ and $d_H(m, C) = \min\{d_H(m, n) : n \in C\}$. Since $d_H(m, n) = d_G(m, n) + 1$ for distinct lines m and n, $d_H(m, C) = d_G(m, \bar{C}) + 1$.

LEMMA 11. *$(L, \mathscr{C})$ is an $(s + 1, q, d)$-pseudo projective incidence structure.*

Proof. We have already established that $(L, \mathscr{C})$ is a semilinear (r^*, k^*)-incidence structure where $r^* = (r - 1)/q = (q^{d-s} - 1)/(q - 1)$ and $k^* = qk + 1 = (q^{s+1} - 1)/(q - 1)$. (If $q = 1$ then $r^* = (r - 1)/q = d - s$ and $k^* = qk + 1 = s + 1$.) The graph H is connected since G is. We prove (f4).

Let m and n be lines and $d_H(m, n) = 2$ (so $d_G(m, n) = 1$). Let $S = \{C : C \in \mathscr{C}, n \in C, d_H(m, C) = 1\}$. We are to show $|S| = q + 1$. Now $S = \{C : C \in \mathscr{C}, n \in C, m \cap \bar{C} \neq \varnothing\}$.

If h is a line and z a vertex so that $d_G(z, h) = 1$ then there exist at least two lines h_1 and h_2 so that $z \in h_i$ and h_i intersects h $(i = 1, 2)$. The plane $C(h_1, h_2)$ contains both h and z. For any plane C containing both z and h we have $|h_i \cap \bar{C}| \geqslant 2$ so $h_i \in C$ $(i = 1, 2)$, and consequently $C = C(h_1, h_2)$. Therefore for any line h and vertex z so that $d_G(z, h) = 1$, a unique plane contains both z and h.

Lines m and n do not intersect. So, no plane contains both. Every plane in S contains n and at least one point of m. Let $x_0, x_1, \ldots, x_q$ be the points of m satisfying $d_G(x_i, n) = 1$ $(0 \leqslant i \leqslant q)$. Let C_i be the unique plane containing x_i and n $(0 \leqslant i \leqslant q)$. If for some i and j $(i \neq j)$ $C_i = C_j$ then $|m \cap \bar{C}_i| \geqslant 2$. By Lemma 9 this would imply that $m \in C_i$, which is false. Then $S = \{C_0, C_1, \ldots, C_q\}$, so $|S| = q + 1$.

To prove (f3), let $d_H(m, C) = 1$. Then $d_G(m, C) = 0$. So, $m \cap \bar{C} \neq \varnothing$. By Lemma 9, $|m \cap \bar{C}| = 1$. Let $\{x\} = m \cap \bar{C}$.

We are to show that $d_H(m, n) = 1$ for exactly $q + 1$ lines n of C, in other words $d_G(m, n) = 0$ for exactly $q + 1$ lines n of C. But this is clear, since exactly $q + 1$ lines of C contain x.

LEMMA 12. *Pasch's axiom is valid in* $(\mathscr{C}, L, \ni)$.

Proof. We first state Pasch's axiom for $(\mathscr{C}, L, \ni)$, recalling that two lines intersect (i.e. contain a vertex in common) iff they are both incident with some plane. Pasch's axiom for $(\mathscr{C}, L, \ni)$ states that if lines m and n intersect, and lines h_1 and h_2 intersect both m and n but no plane contains h_1, m, and n and no plane contains h_2, m, and n, then h_1 and h_2 intersect.

Let $\{x\} = m \cap n$. Now $h_1 \notin C(m, n)$, so $h_1 \notin T(m, n)$. Since $h_1 \notin T(m, n)$ but h_1 intersects both m and n, $x \in h_1$. Similarly $x \in h_2$. Therefore h_1 and h_2 intersect, and Pasch's axiom is valid.

Let $\bar{P} = \{\bar{p} : p \in P\}$ where $\bar{p} = \{m : m \in L, p \in m\}$.

LEMMA 13. *The mapping* $\alpha : P \to \bar{P}$ *defined by* $\alpha(p) = \bar{p}$ *is a bijection.*

Proof. The mapping α is clearly surjective. We show that α is injective. (P, L) is a semilinear (r, k)-incidence structure, therefore $|\bar{x}| = r > 1$ and $|\bar{x} \cap \bar{y}| \leqslant 1$ for all $x, y \in P$. It follows that $\bar{x} \neq \bar{y}$ for all distinct $x, y \in P$.

LEMMA 14. $\bar{P} \cup \mathscr{C}$ *is a partition of the set of maximal cliques of* H.

Proof. It is clear from Lemma 7 that every maximal clique of lines is contained in $\bar{P} \cup \mathscr{C}$. We have shown that every plane is a maximal clique of lines. Therefore it is sufficient to show that $\bar{x}$ is a maximal clique of lines for every $x \in P$, and that $\bar{X}$ and $\mathscr{C}$ are disjoint. $\bar{X}$ and $\mathscr{C}$ are disjoint because the lines of a plane are not concurrent.

Let $x \in P$. Clearly $\bar{x}$ is a clique of lines. Let K be a maximal clique containing $\bar{x}$. If possible, let $K \supsetneq \bar{x}$. Let $m \in K - \bar{x}$. Then $x \notin m$. By (f3), the number of lines of $\bar{x}$ intersecting m is at most $q + 1$. Since K is a clique of lines, every line of $\bar{x}$ intersects m. Therefore $q + 1 \geqslant |\bar{x}| = r > q + 1$ which is a contradiction.

In Lemmas 15–17 we examine (s, q, d)-pseudo projective incidence structures where $s = d - 1$.

LEMMA 15. *Let (P, L) be a $(d - 1, q, d)$-pseudo projective incidence structure. Then any two lines intersect and if $q = 1$, $|L| = d$.*

Proof. (P, L) is an (r, k)-incidence structure where $r = q + 1$ and $k = (q^{d-1} - 1)/(q - 1)$ (if $q = 1$ then $k = d - 1$).

Let G be the adjacency graph of (P, L). Since G is connected, the distance between any two lines is finite. If not all lines intersect then there are lines m and n so that $d(m, n) = 1$. Assume that $d(m, n) = 1$. Then for some $x \in m$, $d(x, n) = 1$. By (f3) $q + 1$ lines contain x and intersect n. Then these lines together with m constitute $q + 2$ lines containing x, which violates the condition $r = q + 1$. Therefore any two lines intersect.

Let $m \in L$. Since $k(r - 1)$ lines intersect m and all lines intersect, $|L| = k(r - 1) + 1$. If $q = 1$, $k = d - 1$ and $|L| = d$.

LEMMA 16. *Let (P, L) be a $(d - 1, q, d)$-pseudo projective incidence structure where $d > 3$ and $q \geqslant 2$, and let the incidence structure dual to (P, L) satisfy Pasch's axiom. Then*

(i) *q is a prime power and d is an integer,*

(ii) *the incidence structure dual to (P, L) is a $(2, q, d)$-projective incidence structure, and*

(iii) *(P, L) is a $(d - 1, q, d)$-projective incidence structure.*

Proof. For $r = q + 1$ and some k, (P, L) is an (r, k)-incidence structure.

We show that $(L, P, \ni)$ satisfies the axioms (p1)–(p4) of Section 1. Now elements of L will be called points and elements of P will be called lines. By Lemma 15 any two points are incident with some line. Therefore $(L, P, \ni)$ satisfies (p1). By hypothesis (p3) is satisfied. Every element of P is incident with $q + 1 \geqslant 3$ elements of L. Since $d > 3$ every element of L is incident with more than $q + 1$ elements of P. It easily follows that rank of (P, L) is at least 4. Therefore by the theorem about finite projective spaces $(L, P, \in)$ is a $(2, q', d')$-projective incidence structure. Clearly we must have $q' = q$ and $d' = d$. This establishes (ii) and (i). Since a $(d - 1, q, d)$-projective incidence structure is dual to a $(2, q, d)$-projective incidence structure (iii) follows.

LEMMA 17. *Let (P, L) be a $(d - 1, 1, d)$-pseudo projective incidence structure where $d > 2$. Then d is an integer and (P, L) is a $(d - 1, 1, d)$-projective incidence structure.*

Proof. (P, L) is an (r, k)-incidence structure with $r = 2$ and $k = d - 1$. Since k is an integer, d is an integer. We examine the dual incidence structure $(L, P, \ni)$. Elements of L will be called dual points and elements of P dual lines. Each dual line is incident with exactly 2 dual points. Therefore dual lines are equivalent to the edges of the adjacency graph of $(L, P, \ni)$. By Lemma 15, each pair of dual points is incident with some dual line. So, the adjacency graph of $(L, P, \ni)$ is the complete graph on $|L| = d$ vertices. Let Y be a d-set and Z_i be the set of i-subsets of Y_i, $1 \leqslant i \leqslant d - 1$. We have proved that $(L, P, \ni)$ is isomorphic to (Y, Z_2). Therefore (P, L) is isomorphic to $(Z_2, Y, \ni)$ and hence to $(Z_{d-2}, Z_{d-1}, \subseteq)$.

LEMMA 18. *There is no (s, q, d)-pseudo projective incidence structure where $3 \leqslant s$ and $d - 2 < s < d - 1$.*

Proof. Assume $\pi = (P, L)$ is an (s, q, d)-pseudo projective incidence structure where $3 \leqslant s$ and $d - 2 < s < d - 1$. If $q = 1$ then $r(\pi) = d - s + 1$ is not an integer. Therefore $q > 1$. Define $\mathscr{C}$ as in Lemmas 6–11. By Lemma 11 $\pi^* = (L, \mathscr{C})$ is an $(s + 1, q, d)$-pseudo projective incidence structure $r(\pi^*) = (q^{d-s} - 1)/(q - 1)$ so $1 < r(\pi^*) < q + 1$. Since $r(\pi^*) \geqslant 2$ and $k(\pi^*) \geqslant 2$ there exist $m \in L$ and $C \in \mathscr{C}$ so that in the adjacency graph of π^* $d(m, C) = 1$. By (f3), $r(m) \geqslant q + 1$. Since $r(m) = r(\pi^*)$ the impossibility of the assumed incidence structure is established.

3. Proof of the Theorems

The heart of the inductive procedure for Theorem 1 is contained in the next lemma.

LEMMA 19. *For $j = 1, 2$ let*

(i) *B_j be a set,*

(ii) *A_j and C_j be sets of subsets of B_j,*

(iii) *the incidence structures (B_j, A_j) and (B_j, C_j) have the same adjacency graph H_j,*

(iv) *$A_j \cup C_j$ be the set of maximal cliques of H_j,*

(v) *$A_j \cap C_j = \varnothing$.*

Let (B_1, C_1) and (B_1, C_2) be isomorphic. Then $(A_1, B_1, \ni)$ and $(A_2, B_2, \ni)$ are isomorphic.

Proof. By hypothesis (B_1, C_1) and (B_2, C_2) are isomorphic; let $\sigma: B_1 \to B_2$ and $\tau: C_1 \to C_2$ be bijections which preserve incidence. For any $B' \subseteq B_1$ we let $\sigma(B') = \{\sigma(b) : b \in B'\}$; in particular, for $c \in C_1$, $\sigma(c) = \{\sigma(b) : b \in c\}$. Then $\sigma(c) = \tau(c)$ for all $c \in C_1$.

σ is an isomorphism between the adjacency graph H_1 of (B_1, C_1) and the adjacency graph H_2 of (B_2, C_2). Therefore σ induces a bijection between the maximal cliques of H_1 and the maximal cliques of H_2. The set of maximal cliques of H_1 is $A_1 \cup C_1$ and the set of maximal cliques of H_2 is $A_2 \cup C_2$. Since $A_1 \cap C_1 = \varnothing = A_2 \cap C_2$ and σ induces a bijection from C_1 to C_2, σ induces a bijection from A_1 to A_2. Then the bijection $\sigma: B_1 \to B_2$ and the bijection from A_1 to A_2 induced by σ show that the incidence structures (B_1, A_1) and (B_2, A_2) are isomorphic, and also that $(A_1, B_1, \ni)$ and $(A_2, B_2, \ni)$ are isomorphic.

In order to shorten the proof of Theorem 1, we introduce some terminology. For $q = 1$ and a positive integer d, $V_{d,q}$ will denote a finite d-element set. For q a prime power $V_{d,q}$ will denote a d-dimensional vectorspace over GF(q). For $q = 1$ an i-dimensional object of $V_{d,q}$ will mean an i-element subset of $V_{d,q}$. For q a prime power, an i-dimensional object of $V_{d,q}$ will mean an i-dimensional subspace of $V_{d,q}$. For $0 \leqslant i \leqslant d$, W_i will denote the set of i-dimensional objects of $V_{d,q}$.

Proof of Theorem 1. Assume that there exists a counter example to the statement of Theorem 1. Among all such counter examples we choose an incidence structure $\pi = (P, L, I)$ for which $r(\pi)$ is as small as possible. Wlog we assume that lines are subsets of points. We write s for $s(\pi)$ and d for $d(\pi)$. Let $\mathscr{C}$ be as in Section 2. By Lemma 11, $\pi^* = (L, \mathscr{C})$ is an $(s + 1, q, d)$-pseudo projective incidence structure. Note that $r(\pi^*) < r(\pi)$ and that the dual of π^* satisfies Pasch's axiom by Lemma 12. By Lemma 18, $s \leqslant d - 2$. If $s < d - 2$, then π^* satisfies the hypotheses of the theorem and $r(\pi^*) < r(\pi)$. Therefore π^* is an $(s + 1, q, d)$-projective incidence structure. If $s = d - 2$, then by Lemmas 16 and 17 π^* is an $(d - 1, q, d)$-projective incidence structure. So in either case d is an integer, $q = 1$ or is a prime power and π^* is isomorphic to $(W_s, W_{s+1}, \subseteq)$ where W_i is the class of i dimensional objects of a $V_{d,q}$ where $i = s, s + 1$. For $w \in W_{s+1}$, let $\overline{w} = \{u : u \in W_s \text{ and } u \subseteq w\}$ and $\overline{W}_{s+1} = \{\overline{w} : w \in W_{s+1}\}$. For $w \in W_{s-1}$, let $w' = \{u : u \in W_s, u \supseteq w\}$ and $W'_{s-1} = \{w' : w \in W_{s-1}\}$. It is easily seen that $(W_s, W_{s+1}, \subseteq)$ is isomorphic to $(W_s, \overline{W}_{s+1})$ and $(W_{s-1}, W_s, \subseteq)$ is isomorphic to $(W'_{s-1}, W_s, \ni)$. We now apply Lemma 19 with $B_1 = W_s$, $C_1 = \overline{W}_{s+1}$ and $A_1 = W'_{s-1}$, $B_2 = L$, $C_2 = \mathscr{C}$ and $A_2 = \overline{P}$. $(W_s, \overline{W}_{s+1})$ and (W_s, W'_{s-1}) have the same adjacency graph H_1. $W'_{s-1} \cup \overline{W}_{s+1}$ is a partition of the set of maximal cliques of H_1. By the remark after Lemma 9, $(L, \mathscr{C})$ and $(L, \overline{P})$ have the same adjacency graph H_2. By Lemma 14, $\overline{P} \cup \mathscr{C}$ is a partition of the set of maximal cliques of H_2. Finally $(L, \mathscr{C})$ and $(W_s, \overline{W}_{s+1})$ are isomorphic. Therefore by Lemma 19 $(\overline{P}, L, \ni)$ and $(W'_{s-1}, W_s, \ni)$ are isomorphic and hence (P, L, I) and $(W_{s-1}, W_s, \subseteq)$ are isomorphic. Hence there is no counter example to the statement of Theorem 1.

Proof of Theorem 2. Wlog assume that lines of π are subsets of points. Consider the connected components of $G(\pi)$. Let P_i be the vertex set of the ith component, $1 \leqslant i \leqslant t$. Let L_i be the set of lines of π which contain at least one point of P_i, $1 \leqslant i \leqslant t$. Then $P = P_1 \cup P_2 \cup \cdots \cup P_t$ and $L = L_1 \cup L_2 \cup \cdots \cup L_t$ are partitions and each line of L_i is a subset of P_i,

$1 \leqslant i \leqslant t$. It is easily checked that for $1 \leqslant i \leqslant t$, (P_i, L_i) satisfy the Axioms (f1)–(f5) with respect to the integer q. Therefore for some integers s_i and d_i (P_i, L_i) is an (s_i, q, d_i)-projective incidence structure and π is the direct sum of these incidence structure. The converse follows from Lemma 2.

4. Minimality of the axioms

Let $\mathscr{P}$ be the class of (s, q, d)-projective incidence structures with $3 \leqslant s < d-1$. The Axioms (f1)–(f5) form a minimal set of axioms for the purpose of characterization of the class $\mathscr{P}$. We now demonstrate the minimality of the axiom set (f1)–(f5). For $j \in \{1, 2, 3, 4, 5\}$ we choose q and construct an incidence structure π' which satisfies the four axioms other than (fj) and is not a member of $\mathscr{P}$. For $j = 5$, we saw that the direct sum of two (s, q, d)-projective incidence structures $(3 \leqslant s < d - 1)$ satisfy the four axioms other than (f5). For $j = 1$, our example is a nondesarguesian finite projective plane π of order q. It is easy to see that π satisfies the Axioms (f2)–(f5) and that $s(\pi) = 2$. Since π is nondesarguesian, π is not an (s, q, d)-projective incidence structure. For $j = 2$, we construct an example as follows. Let q be a given prime power. We choose positive integers s and d satisfying $3 \leqslant s \leqslant d - 2$ and

$$(2q + 1)^2 + (2q + 1) + 1 \leqslant \operatorname{Min}\left(\frac{q^s - 1}{q - 1}, 2\,\frac{q^{d-s+1} - 1}{q - 1}\right).$$

Let π be an (s, q, d)-projective incidence structure. The point set of the incidence structure π' will be the same as that of π and for each line l of π, π' will have two lines l and l' with $P_l = P_{l'}$. It is easily checked $k(\pi') = (q^s - 1)/(q - 1)$ and $r(\pi') = 2(q^{d-s+1} - 1)/(q-1)$. Therefore with respect to $(2q + 1)$, $3 \leqslant s(\pi') \leqslant d(\pi') - 2$ and also π' satisfies (f3), (f4) and (f5) w.r.t. $2q + 1$. Clearly π' is not a member of $\mathscr{P}$, and is not an (s, q, d)-projective incidence structure. For $j = 3$, we proceed to construct an example as follows. Consider an affine space Aff(n, q) where q is a prime power, and $q^2 + q + 1 \leqslant (q^{n-2} - 1)/(q - 1)$. Let π' be an incidence structure whose points are the planes of the affine space and lines are the 3-spaces of the affine space and incidence is containment. Points and lines of π will be respectively called ideal points and ideal lines. It will be helpful to view the affine space as a projective space PG(n, q) minus a hyperplane Σ. The number of planes contained in an affine 3-space is $q^3 + q^2 + q$. Therefore $k(\pi) = q^3 + q^2 + q$ and $r(\pi) = (q^{n-2} - 1)/(q - 1)$. For an ideal point p and an ideal line l, p' and l' will respectively denote the corresponding projective plane and projective 3-space. The Axiom (f1) is satisfied by π'. Clearly (f2) and (f5) hold for π'. Let p_1 and p_2 be two ideal points such that $d(p_1, p_2) = 2$. Then p_1 and p_2 are affine planes (Figure 1). Since $d(p_1, p_2) = 2$, there exist an ideal point p_3 such that $d(p_i, p_3) = 1$, $i = 1, 2$. Hence $\langle p_i', p_3'\rangle$ is a 3-space for $i = 1, 2$. Therefore $p_i' \cap p_3'$ is a line for $i = 1, 2$. Therefore $p_1' \cap p_2'$ is a point O. Let l be an ideal line such that p_1 is incident with l and

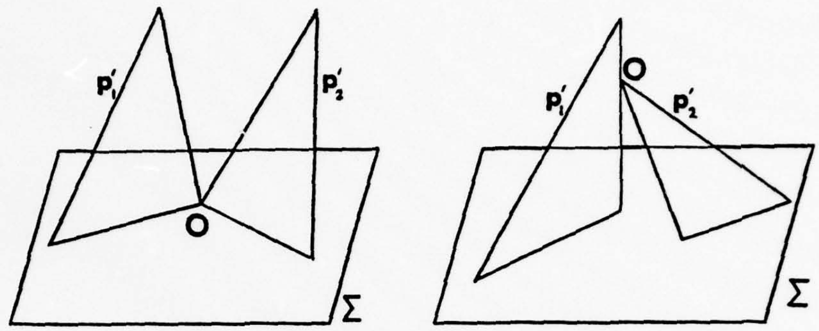

Fig. 1. Two possibilities: O on or off Σ.

$d(p_2, l) = 1$. Then l' is a projective 3-space such that $p'_1 \subseteq l'$ and $p'_2 \cap l'$ is a projective line passing through O. Let x_i $(i = 0, 1, \ldots, q)$ be the projective lines of p'_2 passing through O. Then letting $l'_i = \langle p'_1, x_0 \rangle$, $0 \leqslant i \leqslant q$, l'_i, $0 \leqslant i \leqslant q$ are all the projective 3-spaces satisfying $p'_1 \subseteq l'$, $p'_2 \cap l' =$ a projective line. The corresponding affine 3-spaces l_i, $0 \leqslant i \leqslant q$ are all the ideal lines satisfying $d(p_1, l) = 0$ and $d(p_2, l) = 1$. We proved that π' satisfies (f4) w.r.t. q.

We now show that (f3) does not hold in π'. Let l be an ideal line and p be an ideal point such that $d(p, l) = 1$. Let p' and l' be the corresponding projective plane and 3-space respectively. Since $d(p, l) = 1$, $p' \cap l'$ must be a projective line. Case 1 (Figure 2). $p' \cap l' = y$ is a projective line contained in Σ. There are $(q + 1)$ planes of l' which contain y. Of these one is $l' \cap \Sigma$ which does not correspond to an ideal point of π. Therefore in Case 1 there are q ideal points p_i such that p_i is incident with l and $d(p, p_i) = 1$, $1 \leqslant i \leqslant q$. Case 2 (Figure 3). $p' \cap l' = y$ is a projective line not contained in Σ. In this case there will be $(q + 1)$ ideal points p_i such that p_i is incident with l and

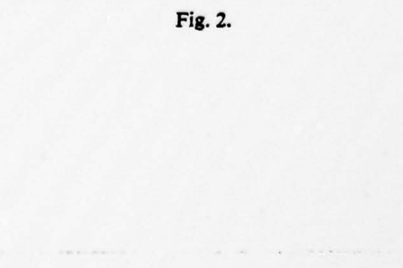

Fig. 2.

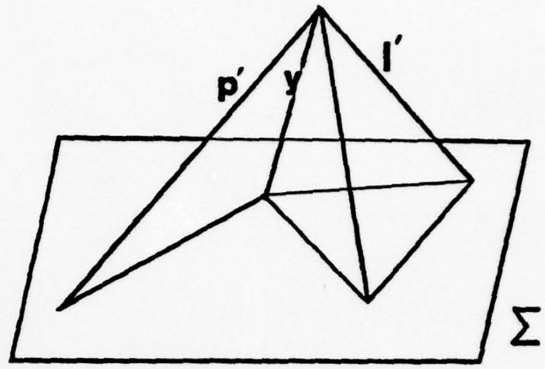

Fig. 3.

$d(p, p_i) = 1$, $0 \leqslant i \leqslant q$. With respect to q, π' satisfies all the four axioms except (f3) and $\pi' \notin \mathscr{P}$.

We now consider $j = 4$. Let q be a prime power, $n \geqslant 4$, PG(n, q) be an n-dimensional projective space over GF(q) and Σ_{n-3} be an $(n - 3)$-flat of PG(n, q). Let π' be an incidence structure whose points are the lines of PG(n, q) not intersecting Σ_{n-3} and lines are the planes of PG(n, q) not intersecting Σ_{n-3}. As before points and lines of π will be referred to as ideal points and ideal lines respectively. Lines and planes of PG(n, q) will be called projective lines and projective planes. Clearly every ideal line is incident with $q^2 + q + 1$ ideal point and hence $k(\pi) = q^2 + q + 1$. The number of projective planes of PG(n, q) containing a given projective line is $(q^{n-1} - 1)/(q - 1)$. Of these projective planes $(q^{n-2} - 1)/(q - 1)$ will intersect Σ_{n-3}. Hence the number of ideal lines passing through a given ideal point is q^{n-2}. Since $n \geqslant 4$, the Axiom (f1) holds for π' with respect to $(q - 1)$.

Clearly (f2) and (f5) hold for π'. We now check (f3) for π. Let p and l respectively be an ideal point and an ideal line such that $d(p, l) = 1$. Then the projective line p intersects the projective plane l in a projective point O (Figure 4). Let $\Sigma_{n-1} = \langle p, \Sigma_{n-3}\rangle$ be the span of p and Σ_{n-3} and $p_0 = l \cap \Sigma_{n-1}$. There are $q + 1$ projective lines of l passing through O. Let $p_0, p_1, \ldots, p_q$ be these lines. The projective plane $\langle p, p_0\rangle$ is contained in Σ_{n-1}, intersects Σ_{n-3} and hence is not an ideal line of π. Therefore the distance between the ideal points p and p_0 is greater than 1. The ideal points $p_1, p_2, \ldots, p_q$ are the only ideal points p satisfying $p_i \subset l$ and $d(p, p_i) = 1$, $1 \leqslant i \leqslant q$. Therefore π satisfies (f3) with respect to $(q - 1)$.

We now show that (f4) does not hold in π'. Let p_1 be an ideal point and $\langle \Sigma_{n-3}, p_1\rangle = \Sigma_{n-1}$. Let p_2 be an ideal point such that $d(p_1, p_2) = 2$. Case 1. p_2 is a projective line not intersecting p_1 and Σ_{n-3} and not contained in Σ_{n-1}

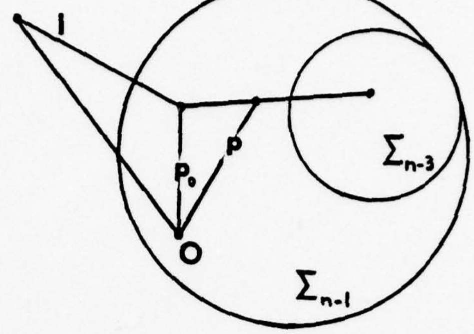

Fig. 4.

(Figure 5). Let x_i, $0 \leqslant i \leqslant q$ be the $q + 1$ points of p_2 where $x_0 \in \Sigma_{n-1}$, and $l_i = \langle p_1, x_i \rangle$, $0 \leqslant i \leqslant q$. The projective plane l_0 intersects Σ_{n-3}. In this case $l_1, \ldots, l_q$ are the only ideal lines which contain p_1 and have distance one from p_2.

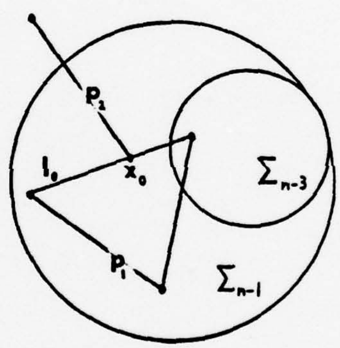

Fig. 5.

Case 2. The projective line p_2 intersects p_1 and is contained in Σ_{n-1} (Figure 6). The projective plane $\langle p_1, p_2 \rangle$ intersects Σ_{n-3} and hence is not an ideal line. Therefore $d(p_1, p_2) = 2$. Let l_1 be any projective plane which contains p_1 and is not contained in Σ_{n-1}. Then it is easily seen that $p_1 \subseteq l_1$ and $d(p_2, l_1) = 1$. In case 2 the number of ideal lines l satisfying $p_1 \subseteq l$, $d(p_2, l) = 1$ is q^{n-2}. Therefore (f4) does not hold in π with respect to $(q - 1)$. Obviously π' is not a $(s, q - 1, d)$-projective incidence structure for any choice of s and d.

This completes the proof of the minimality of the system of Axioms (f1)–(f5).

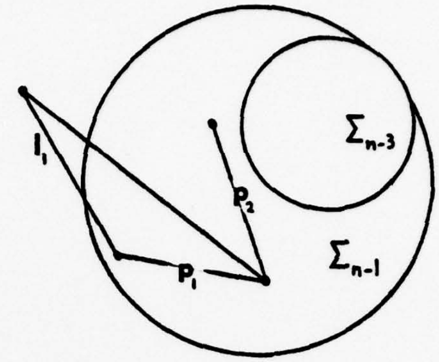

Fig. 6.

Concluding Remarks. Consider a simple graph whose vertices are s-dimensional subspaces of a d-dimensional vector space V over GF(q). Two vertices in this graph are adjacent iff the corresponding s-dimensional subspaces intersect in an $(s - 1)$-dimensional subspace. This graph will be called an (s, q, d)-projective graph. The Theorem 1 of this paper can be used to obtain a characterization of the (s, q, d)-projective graphs provided d is larger than some function of s and q. We are also considering characterization problems of Affine spaces and Polar spaces in terms of flats of higher dimensions. These results will be communicated in a subsequent communication.

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Characterization of "Linegraph of an Affine Space"*

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1. INTRODUCTION AND STATEMENT OF THEOREM

An incidence structure is a triple (X, B, I) where X and B are disjoint sets and $I \subseteq X \times B$. Members of X are called *points* or *vertices*, and elements of B are called *blocks* or *lines*. A simple, undirected graph is an incidence structure (V, E, I) where for each $e \in E$ e is incident with exactly two elements of V. Members of E are commonly known as *edges*. We shall only consider finite, undirected, simple graphs and as such, we will drop the adjectives. In order to specify the graph G, we often write $V(G)$ as the set of vertices and $E(G)$ as the set of edges. An edge e will also be denoted by (x, y) where x and y are the vertices incident with e.

Let G be a graph. Two vertices x and y are said to be *adjacent* if and only if $(x, y) \in E(G)$. If $x_1, ..., x_k$ are vertices of G, then $\Delta(x_1, ..., x_k)$ denotes the set of vertices adjacent to all of the x_i's, $i = 1, ..., k$, and $d(x_1, ..., x_k)$ denotes the cardinality of $\Delta(x_1, ..., x_k)$. A graph G is said to be *regular* if and only if every vertex of G has the same vertex degree. Three distinct vertices are said to form a *triangle* if they are pairwise adjacent. If x, y and z form a triangle in G, then $d(x, y, z)$ is called the *triangle-degree* of (x, y, z). A 2-*claw* $(x; y, z)$ is a triple of vertices in G such that x is adjacent to both y and z, but y and z are nonadjacent; hence $d(x, y, z)$ is the 2-*claw-degree* of $(x; y, z)$.

A (v, n, λ, μ) *strongly regular graph* G is a graph satisfying the following:

(1) $|V(G)| = v$,

(2) G is regular with vertex degree n,

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(3) G is edge-regular with edge-degree λ and

(4) G is nonedge-regular with nonedge-degree μ.

A strongly regular graph is an (r, k, t)-strongly regular graph if $v = (1/t)(r-1)(k-1)(k-t) + r(k-1) + 1$, $n = r(k-1)$, $\lambda = (t-1)(r-1) + k - 2$ and $\mu = rt$. The *linegraph* $L(D)$ of an incidence structure D is a graph whose vertex set consists of the blocks of D, and two vertices are adjacent if and only if there exists a point $x \in X$ such that x is incident with both blocks in D. The linegraph of a projective plane has been characterized by Dowling and Laskar [5]. We prove a theorem characterizing the linegraph of an affine space.

THEOREM. *If G is a (q, k, q)-strongly regular graph satisfying the following*:

(A1) $q \geqslant 4$,

(A2) $k > \frac{1}{2}(q(q-1) + q(q+1)(q^2 - 2q + 2))$,

(A3) *for every triangle (A, B, C) in G, $d(A, B, C)$ either equals $q(q-2)$ or is at least $k-3$,*

(A4) *for every 2-claw $(A; B, C)$ in G, $d(A, B, C)$ equals either $2(q-1)$ or $q(q-1)$,*

then G is isomorphic to the linegraph of an affine space $AF(q, n)$. Furthermore, q is a power of prime and $k = (q^n - 1)/(q-1)$.

It can be easily checked that the linegraph of an affine space $AF(q, n)$ is a (q, k, q) strongly regular graph satisfying axioms (A1)-(A4). However, Sims [8] has shown that the linegraph of a projective space $PG(d-1, q)$ is also a $(q+1, k, q+1)$ strongly regular graph satisfying axioms (A1) and (A2). Therefore in order to characterize the linegraph of an affine space, we have to study the triangle-degrees and the 2-claw-degrees of the graph. In fact, axioms (A3) and (A4) form the bases for defining "parallelism" among the lines in $AF(q, n)$. We shall also discuss the cases where $q = 1, 2$ and 3.

2. Outline of the proof

An (r, k, t) partial geometry is an incidence structure that has the following properties:

(1) every point is contained in exactly r lines,

(2) every line contains exactly k points,

(3) every two distinct points are contained in at most one line,

(4) if p is a point not on the line ℓ, then p is incident with exactly t points on ℓ.

To every partial geometry, there is associated a natural graph defined on the point set of the geometry: two vertices are adjacent if and only if their corresponding points are incident with each other in the geometry. It can be easily verified that the graph thus defined is an (r, k, t) strongly regular graph. Conversely, an (r, k, t) strongly regular graph is said to be *geometrizable* if there exists an (r, k, t) partial geometry whose incidence graph is isomorphic to G. Not every (r, k, t) strongly regular graph is geometrizable. Bose [1] has given the following sufficient condition for the geometrizability of an (r, k, t) strongly regular graph.

PROPOSITION 1. *An (r, k, t) strongly regular graph is geometrizable if* $k > \frac{1}{2}(r(r-1)+1) + t(r+1)(r^2-2r+2))$.

Based on Bose's result, we see that if $k > \frac{1}{2}(q(q-1) + q(q+1)(q^2-2q+2))$ then a (q, k, q) strongly regular graph is geometrizable. Let G be a (q, k, q) strongly regular graph satisfying axioms (A1)-(A4) stated in the theorem. Let $\pi(G)$ denote the (q, k, q) partial geometry associated with G. If $S = \{x_1, ..., x_n\}$ is a subset of $V(G)$ such that the vertices in S are pairwise adjacent, then S is called a *clique*. S is *maximal* if it is not a proper subset of any other clique in G. A clique in G is called a *grand clique* if it is both maximal and of size at least $k - (q-1)^2(q-1)$. From Bose's Geometrization Theorem, we observe that every two adjacent vertices A and B in G are contained in a unique grand clique, denoted by $C(A, B)$. Since each grand clique in G corresponds to a line in $\pi(G)$, we sometimes call a grand clique a *line*. Hence, if A and B are contained in a line, A and B are called *collinear*. The set of lines in G has the following property:

PROPOSITION 2. *If G is a geometrizable (q, k, q) strongly regular graph, then any two distinct grand cliques in G intersect each other at a unique vertex.*

Proof. Suppose there exist two nonintersecting grand cliques in G, say x and y. Let A be any vertex in x. A is a vertex not in y; hence A is adjacent to exactly q vertices $B_1, ..., B_q$ in y. For each i, $1 \leqslant i \leqslant q$, B_i and A determine a unique grand clique $C(A, B_i)$ containing A; thus, there are q distinct grand cliques containing A. Now x is also a grand clique containing the vertex A, but is distinct from the q grand cliques $C(A, B_i)$, $1 \leqslant i \leqslant q$. This contradicts the fact that there are exactly q grand cliques containing A. Hence any two distinct grand cliques intersect at a unique point. ∎

By virtue of this proposition, we are able to adopt the notation $x \wedge y$

for the unique point of intersection of the two grand cliques x and y. It is also clear that if we define the dual of the (q, k, q) partial geometry $\pi(G)$ to be an incidence structure P whose point set and line set are respectively the sets of lines and points of $\pi(G)$, then every two distinct points in P determine a unique line and every two distinct lines intersect at at most one point. Buekenhout [3] showed that

PROPOSITION 3. *If $\mathscr{P}$ is an incidence structure with points, lines, and planes such that* (1) *every three noncollinear points determine a unique plane,* (2) *every plane of $\mathscr{P}$ is an affine plane,* (3) *there exist three noncollinear points, and* (4) *every line contains at least four points, then $\mathscr{P}$ is an affine space.*

In view of this theorem, if we can establish the parallel lines and affine planes in P, then we will have shown that the dual of the (q, k, q) partial geometry $\pi(G)$ is isomorphic to an affine space; hence, the (q, k, q) strongly regular graph is isomorphic to the linegraph of an affine space. To this end, we study the geometries of the set of vertices which are adjacent to two other vertices in G. Throughout the rest of this paper, we shall assume that G satisfies axioms (A1)-(A4) in the main theorem, unless otherwise stated.

3. TRANSVERSALS

DEFINITION. *If A and B are two nonadjacent vertices in G, then a transversal of A and B is an element in $\Delta(A, B)$. If A and B are two distinct adjacent vertices in G, then a transversal of A and B is a vertex in $\Delta(A, B)$ which is not contained in $C(A, B)$.*

The set of transversals of A and B is denoted by $T(A, B)$. We shall first study transversals of two nonadjacent vertices. Let A and B be two nonadjacent vertices. For every vertex $C \in T(A, B)$, the triple $(C; A, B)$ is a 2-claw in G. It is clear that every grand clique containing either A or B contains exactly q transversals of A and B. Hence if $C \in T(A, B)$, then there are least $2(q-1)$ other transversals of A and B that are adjacent to C, namely the transversals contained in $C(A, C)$ and $C(B, C)$. Thus, for any 2-claw $(C; A, B)$, $d(C, A, B) \geqslant 2(q-1)$.

LEMMA 4. *Let A and B be any two nonadjacent vertices in G. If C and D are two distinct transversals of A and B, such that C is adjacent to D and (A, C, D) and (B, C, D) are noncollinear triples, then $d(A, B, D) = d(A, B, C) = q(q-1)$.*

Proof. Since (A, C, D) and (B, C, D) are noncollinear triples, $D \notin C(A, C)$ and $D \notin C(B, C)$. From the remark above, there are $2(q-1)$ transversals of A and B that are adjacent to C and are contained in either $C(A, C)$ or $C(B, C)$. Now, D is a transversal of A and B which is adjacent to C and is distinct from these $2(q-1)$ transversals; hence $d(A, B, C) > 2(q-1)$. By Axiom (A4), $d(A, B, C) = q(q-1)$. Similarly, $d(A, B, D) = q(q-1)$. ∎

We shall now partition the transversals of two non-adjacent vertices A and B in accordance to the different cliques containing either A or B. Let us first introduce the following notations. If A is a vertex in G, then $\ell(A)$ denotes the set of grand cliques containing A. Let A and B be two distinct vertices in G, and $C \in T(A, B)$. If $x \in \ell(A) \cup \ell(B)$, but $x \neq C(A, B)$, then

$$\mathcal{M}(A, B, C, x) = \{D \in T(A, B) \mid D \text{ is adjacent to } C \text{ and is contained in } x\}.$$

and $m(A, B, C, x) = |\mathcal{M}(A, B, C, x)|$. (See Figure 1)

Lemma 5. *Let A and B be two nonadjacent vertices and $C \in T(A, B)$. If $x \in \ell(A) \cup \ell(B)$, then $q - 1 \geqslant m(A, B, C, x) \geqslant 1$.*

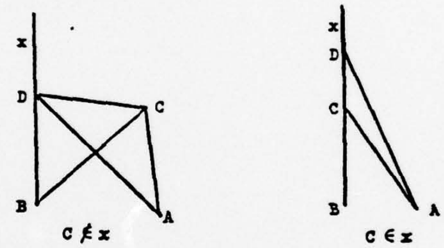

Figure 1. $\mathcal{M}(A,B,C,x)$

Proof. Without loss of generality we may assume that B is contained in x. Since $C(A, C)$ and x are two distinct grand cliques, they intersect each other at a unique point D. If $D \neq C$, then clearly $D \in \mathcal{M}(A, B, C, x)$ and $m(A, B, C, x) \geqslant 1$. If $D = C$, then C is contained in x; A is adjacent to exactly $q - 1$ vertices of x other than C and we have $m(A, B, C, x) = q - 1$.

To show $q - 1 \geqslant m(A, B, x)$, we only have to consider C as a vertex not contained in x. Since C is not in x, C is adjacent to exactly q vertices in x, one of which is B; hence $m(A, B, C, x) \leqslant q - 1$. ∎

Lemma 6. *Let $(C; A, B)$ be a 2-claw. If there exists a grand clique x_0 in $\ell(A) \cup \ell(B)$ such that $x_0 \neq C(B, C)$, $x_0 \neq C(A, C)$ and $m(A, B, C, x_0) > 1$, then*

$$\text{ave } m(A, B, C, \cdot) = q - 1$$

where the average runs over all grand cliques in $\ell(A) \cup \ell(B)$.

Proof. Without loss of generality, let B be contained in x_0. Since $m(A, B, C, x_0) > 1$, there exists a transversal D of A and B contained in x_0 which is adjacent to C. But both (B, C, D) and (A, C, D) are noncllollinear triples; furthermore, $x_0 \neq C(B, C)$ and $x_0 \neq C(A, C)$; hence by Lemma 4, $d(A, B, C) = q(q - 1)$.

Next, we count the number of ordered pairs (x, E) where $x \in \ell(A) \cup \ell(B)$ such that E is a transversal of A and B, is contained in x, and is adjacent to C. Fixing a grand clique x, $x \in \ell(A) \cup \ell(B)$, there are $m(A, B, C, x)$ choices of E. On the other hand, if we fix a transversal E of A and B such that E is adjacent to C, then there are exactly 2 choices of x, namely $C(A, E)$ and $C(B, E)$. Since there are $d(A, B, C)$ choices of E, we have

$$\sum_x m(A, B, C, x) = 2q(q - 1).$$

The vertices A and B are nonadjacent, so the cliques containing either A or B are all distinct, and we have $|\ell(A) \cup \ell(B)| = 2q$. Thus

$$ave\ m(A, B, C, \cdot) = q - 1. \quad \blacksquare$$

From these two lemmas, we obtain

PROPOSITION 7. *Let A and B be two nonadjacent vertices in G. If $C \in T(A, B)$ and $x \in \ell(A) \cup \ell(B)$, then $m(A, B, C, x) \in \{1, q - 1\}$.*

Proof. Case 1. $x = C(A, C)$ or $x = C(B, C)$: From the proof of Lemma 5, we have $m(A, B, C, x) = q - 1$. Case 2. $x \neq C(A, C)$ and $x \neq C'(B, C)$: If $m(A, B, C, x) > 1$, then from the previous lemma, $ave\ m(A, B, C, \cdot) = q - 1$ where the average runs over all grand cliques in $\ell(A) \cup \ell(B)$. By Lemma 5, $m(A, B, C, x) \leqslant q - 1$; hence $m(A, B, C, x) = q - 1$. $\blacksquare$

Similar to nonadjacent vertices, we would like to study the transversals of two adjacent vertices A and B which are also adjacent to a fixed transversal C of A and B.

LEMMA 8. *Let A and B be two adjacent vertices. If $C \in T(A, B)$, then C is adjacent to exactly $(q - 1)(q - 2)$ other transversals of A and B.*

Proof. Since G is a (q, k, q)-strongly regular graph and A and B are two adjacent vertices, $d(A, B) = k - 2 + (q - 1)^2$. But $C(A, B)$ is a grand clique containing $k - 2$ vertices other than A and B, which are not transversals of A and B; hence $|T(A, B)| = (q - 1)^2$.

If C is a transversal of A and B, then $C \notin C(A, B)$. Therefore, C is adjacent to exactly $q-2$ vertices in $C(A, B)$ other than A and B. Hence, for sufficiently large k

$$\begin{aligned} d(A, B, C) &\leqslant |T(A, B)| - 1 + q - 2 \\ &= (q-1)^2 + q - 3 \\ &< k - 3. \end{aligned}$$

By Axiom (A3) in the theorem, $d(A, B, C) = (q-2)q$. However, we have seen that the set $\Delta(A, B, C) \cap C(A, B)$ consists of $q-2$ vertices which are not transversals of A and B. Hence C is adjacent to exactly $(q-2)(q-1)$ transversals of A and B. ∎

A direct consequence of the above lemma is

LEMMA 9. *Let A and B be two adjacent vertices. If $C \in T(A, B)$, then C is nonadjacent to exactly $q-2$ transversals of A and B.*

Again, we would like to know how these $(q-1)(q-2)$ transversals of A and B which are adjacent to C are partitioned in accordance to the various cliques containing A or B.

LEMMA 10. *Let A and B be two adjacent vertices in G. If $C \in T(A, B)$, then ave $m(A, B, C, \cdot) = q - 2$ where the average runs over all grand cliques in $\ell(A) \cup \ell(B) - C(A, B)$.*

Proof. We shall count the number of ordered pairs (x, D) where x is a grand clique in $\ell(A) \cup \ell(B) - C(A, B)$ and $D \in \mathscr{M}(A, B, C, x)$. Fixing x, there are $m(A, B, C, x)$ choices of D. On the other hand, if we fix a vertex D such that $D \in \Delta(A, B, C)$ and $D \in T(A, B)$, then there are exactly two choices of x, namely $C(A, D)$ and $C(B, D)$. From Lemma 8, there are $(q-1)(q-2)$ choices of D; hence

$$\sum_x m(A, B, C, x) = 2(q-1)(q-2).$$

Since $|\ell(A) \cup \ell(B) - C(A, B)| = 2(q-1)$ we have *ave* $m(A, B, C, \cdot) = q-2$.

LEMMA 11. *Let A and B be two adjacent vertices and $C \in T(A, B)$. If x is a grand clique in $\ell(A) \cup \ell(B) - C(A, B)$, then $m(A, B, C, x) \geqslant q - 2$.*

Proof. Without loss of generality, we may assume x to be a clique containing A.

Case 1. $x = C(A, C)$: Every vertex in x different from A and C, which is adjacent to B, is contained in $\mathscr{M}(A, B, C, x)$; hence $m(A, B, C, x) = q - 2$.

Case 2. $x \neq C(A, C)$: Since B is not contained in x, B is adjacent to q vertices in x, namely $A, D_2, \ldots, D_q$. If C is adjacent to each D_i, $2 \leqslant i \leqslant q$, then clearly, $m(A, B, C, x) \geqslant q - 2$. Suppose C is nonadjacent to D_2; let us consider the 2-claw $(B; C, D_2)$. The grand clique x contains D_2; hence by Proposition 7, $m(C, D_2, B, x)$ equals either 1 or $q - 1$. Since A and $C(B, C) \wedge x$ are two distinct vertices in $\mathcal{M}(C, D_2, B, x)$, we have $m(C, D_2, B, x) = q - 1$. This means that for every i, $3 \leqslant i \leqslant q$, D_i is adjacent to C. Consequently, $m(A, B, C, x) \geqslant q - 2$. ▌

From the results of Lemmas 10 and 11, we arrive at the following:

Proposition 12. *Let A and B be two adjacent vertices and $C \in T(A, B)$. If x is a grand clique in $\ell(A) \cup \ell(B) - C(A, B)$, then C is adjacent to exactly $q - 2$ transversals of A and B which are contained in x.*

In the next proposition, we shall see how the transversals of two adjacent vertices A and B can be partitioned in accordance to the various grand cliques in G.

Proposition 13. *Let A and B be two adjacent vertices. If x is a grand clique in G that contains neither A nor B, then there exists either none or exactly $q - 2$ transversals of A and B contained in x.*

Proof. Let C be a transversal of A and B which is contained in x. Since A, B and C are noncollinear, B is a transversal of A and C. Hence, by the previous proposition, we have $m(A, B, C, x) = q - 2$; that is, there are exactly $q - 1$ vertices in x including C which are adjacent to both A and B. But one of these $q - 1$ vertices is the vertex $x \wedge C(A, B)$, which is not a transversal of A and B. Thus there are exactly $q - 2$ transversals of A and B which are contained in x. ▌

Using the results concerning transversals developed in this section, we shall differentiate between two types of nonadjacencies in G and define the relation 'parallelism' accordingly.

4. Parallelism

In this section we shall establish two types of nonadjacencies in G. Let A and B be two nonadjacent vertices in G. If C is a transversal of A and B and x is a grand clique in $\ell(A) \cup \ell(B)$, then by Proposition 7, $m(A, B, C, x)$ equals either 1 or $q - 1$. Furthermore, for every transversal C of A and B, if $x \neq C(A, C)$, $x \neq C(B, C)$ and $m(A, B, C, x) = q - 1$, then $m(A, B, C, y) = q - 1$ for all y in $\ell(A) \cup \ell(B)$. Thus, in differentiating the two types of nonadjacencies, we may assume that the function $m(A, B, C, x)$ depends only on the 2-claw $(C; A, B)$.

Definition. *Let A and B be two nonadjacent vertices. A is parallel to B if and only if either $A = B$ or there exists a transversal C of A and B such that $m(A, B, C, x) = q - 1$, where $x \in \ell(A) \cup \ell(B)$ and $x \neq C(A, C)$, $x \neq C(B, C)$. We shall denote the parallelism by $A//B$.*

First, we have to show that the relation of parallelism is independent of the choice of the transversal C of A and B.

Lemma 14. *Let A and B be two nonadjacent vertices and let $C \in T(A, B)$. If $m(A, B, C, x) = q - 1$ for some grand clique $x \in \ell(A) \cup \ell(B)$ and $x \neq C(A, C)$, $x \neq C(B, C)$, then B is adjacent to every vertex in x, which is distinct from A and is adjacent to C.*

Proof. The vertex C is not contained in x, so C is adjacent to exactly q vertices in x, namely $A, D_2, ..., D_q$. Since $m(A, B, C, x) = q - 1$, B must be adjacent to each D_i, $2 \leqslant i \leqslant q$. ∎

Proposition 15. *If $m(A, B, C, x) = q - 1$ for some grand clique $x \in \ell(A) \cup \ell(B)$, $x \neq C(A, C)$ and $x \neq C(B, C)$, then for any transversal C' of A and B, $m(A, B, C', x) = q - 1$.*

Proof. Without loss of generality, we may assume that $x \in \ell(A)$. For clarity, we shall denote $C(B, C)$ by y. Let us consider the vertex C'. If C' is contained in x, then clearly $m(A, B, C', x) = q - 1$. Henceforth, we shall assume that C' is not contained in x.

Case 1. C' is contained in y. Since both vertices C and C' are contained in y and A is not in y, C is not contained in the clique determined by A and C', so C is a transversal of A and C'. By Proposition 12, C is adjacent to exactly $q - 2$ transversals of A and C' which are contained in x. But by the previous lemma, the vertex B is also adjacent to all of these $q - 2$ transversals of A and C'. Hence

$$m(A, B, C', x) \geqslant q - 2 > 1.$$

This implies that $m(A, B, C', x) = q - 1$.

Case 2. C' is not contained in y.

Subcase 2.1. C' is adjacent to C. Let z denote the grand clique containing C and C', and let D be the point $x \wedge z$. The vertex D is contained in x and is adjacent to C, by Lemma 14, B is adjacent to D. Furthermore, $x \wedge C(B, C')$ is another point contained in x and is adjacent to A, B and C'. Thus, $m(A, B, C', x) \geqslant 2$ and hence $m(A, B, C', x) = q - 1$.

Subcase 2.2. C' is nonadjacent to C. Let z denote the grand clique containing B and C'. If we can show that there exists a transversal C'' of A and B contained in z such that C'' is adjacent to C, then C'' is not contained in either x or y, and by the previous subcase, $m(A, B, C'', x) = q - 1$. But C' is a vertex contained in z and not in x, by case 1, we have $m(A, B, C', x) = q - 1$. Hence, we are left to show the existence of C''.

Since $m(A, B, C, x) = q - 1$ and $z \in \ell(B)$, $m(A, B, C, z) = q - 1$; this means that there are $q - 1$ transversals of A and B which are contained in z and are adjacent to C. Thus, C'' exists and the proof is complete. ▌

It is clear from this proposition that 'parallelism' is a well-defined relation on pairs of nonadjacent vertices. We still have to construct the affine planes in the structure P defined in the previous section. Before that, let us establish the next two theorems concerning parallelism.

THEOREM 16. *Let x be a grand clique in G. If B is a vertex in G, not contained in x, then there exists a unique vertex A contained in x such that A is parallel to B.*

Proof. Let $n(x, B)$ denote the number of vertices contained in x which are parallel to B. We shall count the number of triples (A, C, D) where A is a vertex in x nonadjacent to B, C is a transversal of A and B which is not contained in x, and D is a vertex in x adjacent to both B and C. Fixing a vertex A in x nonadjacent to B, there are $q^2 - q$ transversals C of A and B which are not contained in x. If A is parallel to B, then for every transversal C of A and B, $m(A, B, C, x) = q - 1$, that is, there are $q - 1$ choices of D. If A is not parallel to B, then $m(A, B, C, x) = 1$ and there exists a unique choice of D. Since there are $n(x, B)$ vertices in x which are parallel to B, there are $k - q - n(x, B)$ vertices in x which are neither adjacent nor parallel to B. Hence, the number of triples (A, C, D)

$$= n(x, B)(q^2 - q)(q - 1) + (k - q - n(x, B))(q^2 - q) \cdot 1. \quad (1)$$

Next, we consider a vertex C which is adjacent to B but is not contained in x, and count the number of vertices D in x which is adjacent to both B and C. If $D \neq C(B, C) \wedge x$, then D is a transversal of B and C. But by Proposition 13, there exists either one or exactly $q - 1$ choices of D. In the former case, any vertex A which is contained in x, distinct from D and adjacent to C, is nonadjacent to B and there are exactly $q - 1$ such vertices A. In the latter case, since there are $q - 1$ vertices D in x which are adjacent to both B and C, there exists a unique vertex A in x, which is adjacent ot C but is nonadjacent to B. Thus, we see that in both cases there are $q - 1$ choices of the pairs (A, D). Since there are $q(k - 1) - q$ vertices C which are adjacent

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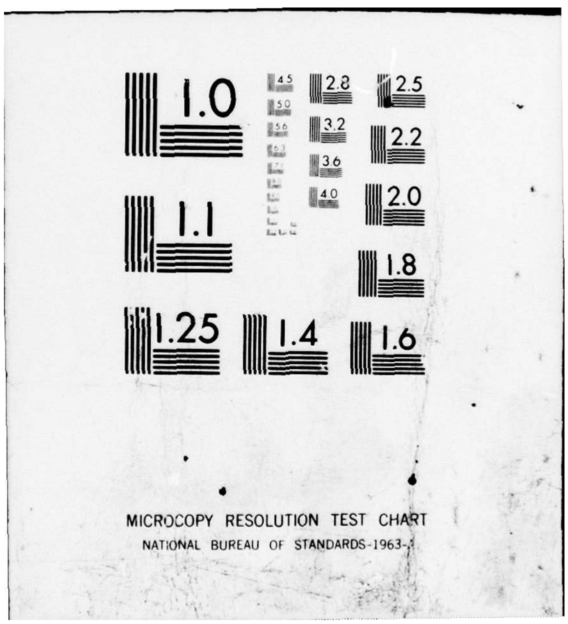
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to B and not contained in x, we obtain that the number of triples (A, C, D)

$$= (q(k-1) - q)(q-1) = q(k-2)(q-1). \qquad (2)$$

Equating Equations (1) and (2) we have $n(x, B) = 1$. Consequently, there exists a unique vertex in x which is parallel to the vertex B. ∎

THEOREM 17. *If A and B are two distinct vertices which are parallel to C, then A is nonadjacent to B.*

Proof. Suppose A is adjacent to B. Since C is parallel to both A and B, C is a vertex not in the grand clique $C(A, B)$. By the previous theorem, there exists a unique vertex in $C(A, B)$ which is parallel to C, but this contradicts the fact that both A and B are parallel to C. Hence, A is nonadjacent to B. ∎

COROLLARY 18. *Let x be a grand clique in G. If A is a vertex contained in x, then A itself is the unique vertex contained in x, which is parallel to A.*

Proof. By definition, A is parallel to itself. If there exists another vertex B contained in x which is parallel to A, then A and B are nonadjacent; this contradicts that A and B are both contained in x. ∎

5. AFFINE PLANES

In this section we shall define affine planes on P, based on the notion of parallel lines developed in the previous section. Henceforth, two parallel vertices will be called *p-reltaed*, two adjacent vertices will be called *a-related*. For every distinct pair of vertices A and B, $\Delta_{pa}(A, B)$ will denote the set of vertices that are parallel to A and are adjacent to B. Clearly, if A and B are parallel, then $\Delta_{pa}(A, B)$ is empty. Hence, we are only interested in two adjacent vertices A and B. First, we shall compute the number of vertices in $\Delta_{pa}(A, B)$.

LEMMA 19. *If A and C are two adjacent vertices, then* $|\Delta_{pa}(A, C)| = q$.

Proof. By Theorem 16 and Corollary 18, for every grand clique x in $\ell(C)$, there exists a unique vertex D in x which is parallel to A. Since there are q grand cliques in $\ell(C)$, $|\Delta_{pa}(A, C)| = q$. ∎

Let us now define a subincidence structure (A, B, C) in P based on a 2-claw $(C; A, B)$ in G such that A is parallel to B. Let θ be a set of vertices in G, we define

$$\ell(\theta) = \{x \mid x \text{ is a grand clique containing a vertex } D \text{ in } \theta\}.$$

Let $\mathscr{L}(A, B, C) = T(A, B) \cup \Delta_{pa}(A, C)$, and $\mathscr{P}(A, B, C) = \ell(\mathscr{L}(A, B, C))$. We shall show that $\mathscr{P}(A, B, C)$ is an affine plane with $\mathscr{L}(A, B, C)$ as the set of lines of the plane and $\mathscr{P}(A, B, C)$ as the set of points with the obvious incidence relation. First, we show that $\mathscr{L}(A, B, C)$ contains the correct number of lines in an affine plane.

PROPOSITION 20. $|\mathscr{L}(A, B, C)| = q^2 + q$.

Proof. Since every transversal of A and B cannot be parallel to A. the two sets $T(A, B)$ and $\Delta_{pa}(A, C)$ are disjoint, thus

$$|\mathscr{L}(A, B, C)| = |T(A, B)| + |\Delta_{pa}(A, C)| = q^2 + q. \quad \blacksquare$$

Next, we shall show that $\mathscr{P}(A, B, C)$ has the correct number of points in an affine plane.

LEMMA 21. *Let A and B be two adjacent vertices and let $C \in T(A, B)$. If x is a grand clique containing either A or B but not both, and x does not contain C, then there exists a unique vertex D in x such that $D \in T(A, B)$ and D is nonadjacent to C. In fact, D is the unique vertex parallel to C.*

Proof. Without loss of generality, we may assume $x \in \ell(A)$. By Proposition 12, $m(A, B, C, x) = q - 2$, but there exist $q - 1$ transversals of A and B in x. Hence there exists a unique transversal D in x which is not adjacent to C. Consider the 2-claw $(B; C, D)$. Since x is a grand clique containing D but not B and there at least $q - 2$ vertices in x which are adjacent to both B and C, $m(C, D, B, x) \geqslant q - 2$. Thus, $m(C, D, B, x) = q - 1$ and C is therefore parallel to D. $\blacksquare$

LEMMA 22. *Let A be parallel to B. If $C \in T(A, B)$, then $\Delta_{pa}(A, C) = \Delta_{pa}(B, C)$.*

Proof. It suffices to show that for any $E \in \Delta_{pa}(A, C)$, E is parallel to B. Let x be a grand clique in $\ell(A)$ such that x does not contain C. Let D be a transversal of A and B in x such that D is adjacent to C. Furthermore, let D be such that D is not contained in $C(B, C)$. (Note that at least two such D's exist, because $m(A, B, C, x) = q - 1$ and $q \geqslant 4$). Since neither one of the triples (A, C, D) and (B, C, D) is collinear, both A and B are transversals of the adjacent vertices C and D.

Using the previous lemma, we see that for every grand clique y containing C which does not contain A or D (that is, $y \neq C(A, C)$ and $y \neq C(D, C)$), there exists a unique vertex A_y such that $A_y // A$ and $A_y \in T(C, D)$. If y contains the vertex B, then clearly $A_y = B$ and A_y is parallel to B. If y does not contain the vertex B and both A_y and B are parallel to A, then by Theorem 17,

A_v and B are nonadjacent. But by the previous lemma the unique transversal A_v of C, D which is nonadjacent to B is parallel to B. Thus, for all vertices E in $\Delta_{pa}(A, C)$ such that $C(E, C) \neq C(A, C)$ and $C(E, C) \neq C(D, C)$, E is parallel to B. Hence, $E \in \Delta_{pa}(B, C)$.

If $E \in \Delta_{pa}(A, C)$ and $C(E, C) = C(A, C)$, then E is the unique vertex parallel to A and contained in $C(A, C)$; hence $E = A$. Obviously, $A // B$; so $E // B$ and $E \in \Delta_{pa}(A, C)$.

It remains to show that if $E' \in \Delta_{pa}(A, C)$ such that $C(E', C) = C(D, C)$, then $E' \in \Delta_{pa}(B, C)$. At the beginning of the proof we have observed that there exists another vertex D' which has the same properties as those of D. But $C(D', C) \neq C(D, C)$; hence $C(E', C) \neq C(D', C)$ and using the same arguments as above, $E' \in \Delta_{pa}(B, C)$. ∎

From this lemma, we observe that $\mathscr{P}(A, B, C) = \mathscr{P}(B, A, C)$. So A and B are 'equivalent' in the sense that they can be interchanged without affecting the definition of $\mathscr{P}(A, B, C)$.

LEMMA 23. *Let E and E' be two vertices in $\mathscr{P}(A, B, C)$. If $E // A$ and $E' // A$, then $E // E'$.*

Proof. Since $E // A$ and $E \in \mathscr{P}(A, B, C)$, E is adjacent to C. Thus, $C \in T(A, E)$. By Lemma 22, $\Delta_{pa}(A, C) = \Delta_{pa}(E, C)$. But $E' // A$ and $E' \in \mathscr{P}(A, B, C)$, $E' \in \Delta_{pa}(A, C)$. This implies that $E' // E$. ∎

THEOREM 24. $|\mathscr{P}(A, B, C)| = q^2$.

Proof. Let us first compute the number of grand cliques contained in $\ell(\Delta_{pa}(A, C))$. Since the vertices in $\Delta_{pa}(A, C)$ are pairwise nonadjacent, we have

$$|\ell(\Delta_{pa}(A, C))| = \sum_{E \in \Delta_{pa}(A, C)} |\ell(E)| = q^2.$$

Thus, if we can show that $\ell(\Delta_{pa}(A, C)) = \mathscr{P}(A, B, C)$, then we are done. But $\Delta_{pa}(A, C) \subseteq \mathscr{L}(A, B, C)$; hence $\ell(\Delta_{pa}(A, C)) \subseteq \mathscr{P}(A, B, C)$. So we only have to show that $\mathscr{P}(A, B, C) \subseteq \ell(\Delta_{pa}(A, C))$.

Let $x \in \mathscr{P}(A, B, C)$. If $x \in \ell(\Delta_{pa}(A, C))$, then we have nothing else to prove. If $x \in \ell(T(A, B))$, then there exists a transversal D of A and B such that D is contained in x. If x contains either A or B, then clearly $x \in \ell(\Delta_{pa}(A, C))$. Henceforth we shall assume that x does not contain A or B.

Case 1. D is adjacent to C. Since A and B are nonadjacent and both are adjacent to C and D, at least one of the two vertices A and B is a transversal of C and D. By the remark that A and B are 'equivalent', without loss of generality, we may assume that $A \in T(C, D)$. By Lemma 21, there exists a unique transversal E of C and D in x such that $E//A$; that is $E \in \Delta_{pa}(A, C)$ and E is contained in x. Thus, $x \in \ell(\Delta_{pa}(A, C))$.

Case 2. D is nonadjacent to C. Let E denote the vertex $x \wedge C(B, C)$. Clearly, E is adjacent to both B and C. If E is adjacent to A, then $E \in T(A, B)$ and E is contained in x; hence using Case 1 of the proof, $x \in \ell(\Delta_{pa}(A, C))$. So we only need to show that E is adjacent to A.

Both the vertices D and E are contained in x, so D and E are adjacent. But B is not contained in x; hence $B \in T(D, E)$. Let z denote the clique $C(A, D)$. Clearly z does not contain B. Thus, there exists a unique transversal T of D and E in z such that $T//B$. Since A is the unique vertex in z which is parallel to B, $A = T$. T is adjacent to E and the proof is complete. ∎

Next, we shall show that $\mathscr{P}(A, B, C)$ is an affine plane.

THEOREM 25. *$\mathscr{P}(A, B, C)$ is an affine plane.*

Proof. We shall first show that every two distinct points in $\mathscr{P}(A, B, C)$ determine a unique line in $\mathscr{L}(A, B, C)$. Let us count the number of triples (x, y, L) where x and y are distinct points in $\mathscr{P}(A, B, C)$ and L is a line in $\mathscr{L}(A, B, C)$ that contains both x and y. For every line L in $\mathscr{L}(A, B, C)$, L contains q points; hence there are $q(q - 1)$ choices of (x, y). Since $|\mathscr{L}(A, B, C)| = q(q + 1)$, we have

$$\text{Number of triples } (x, y, L) = q(q + 1)\, q(q - 1). \quad (3)$$

On the other hand, if we fix a pair (x, y) in $\mathscr{P}(A, B, C)$ and let $f(x, y)$ denote the number of lines in $\mathscr{L}(A, B, C)$ that contain x and y, then there are $\sum f(x, y)$ such triples (x, y, L) where the sum runs over all ordered pairs of distinct points in $\mathscr{P}(A, B, C)$. Thus,

$$\text{Number of triples } (x, y, L) = q^2(q^2 - 1)\, ave\, f(\cdots) \quad (4)$$

Equating Equations (3) and (4) we obtain $ave\, f(\cdots) = 1$.

Since every two distinct grand cliques x and y in G intersect at *a* unique point in G, $f(x, y) \leqslant 1$. Hence, $f(x, y) = 1$, that is, every two distinct points in $\mathscr{P}(A, B, C)$ determine a unique line in $\mathscr{L}(A, B, C)$.

Next, we will show that for every line in $\mathscr{L}(A, B, C)$ and a point x not in L, there exists a unique line L_x in $\mathscr{L}(A, B, C)$ such that L_x contains x and is parallel to L. Let us consider $\Delta_{pa}(A, C)$. The set of parallel lines in $\Delta_{pa}(A, C)$ partitions the points in $\mathscr{P}(A, B, C)$. Hence, for every line L in $\Delta_{pa}(A, C)$ and every point x not in L, there exists a unique line L_x containing x which is parallel to A. By Lemma 23 L_x is parallel to L and $L_x \in \mathscr{L}(A, B, C)$.

Consider a line $L \in T(A, B)$ and a point x in $\mathscr{P}(A, B, C)$ such that x is not in L. If x is a point in A, then let C' be the line determined by x and $C(B, L)$; otherwise, let C' be the line containing x and the point $C(A, L)$. Since A and B are 'equivalent', without loss of generality, we may assume C' to be the latter. Consider the pair (B, C'). L is a transversal of B and C', and x is a grand clique containing C' but neither B nor L. By Lemma 21, there exists a unique transversal L_x of B and C' such that L_x is contained in x and $L_x // L$. If L_x is adjacent to A, then $L_x \in T(A, B)$ and L_x is contained in $\mathscr{L}(A, B, C)$. So we only need to show that L_x is adjacent to A. Since $A // B$, $m(A, B, C, x) = q - 1$, that is, every vertex in x which is adjacent to B is adjacent to A. Thus, L_x is adjacent to A.

Since $\mathscr{P}(A, B, C)$ possesses the two properties we have derived above, and clearly $\mathscr{P}(A, B, C)$ contains 3 noncollinear points, $\mathscr{P}(A, B, C)$ is an affine plane and the proof is complete. ∎

Thus far we have defined affine planes $\mathscr{P}(A, B, C)$ on P based on a 2-claw $(C; A, B)$. But in order to show that these affine planes are well-defined, we have to show that they are independent of the choice of the transversal C of A and B, and are also independent of the choice of the pair of parallel lines A and B in the plane.

LEMMA 26. *Let A and B be two distinct parallel vertices in G. If C and D are both in $T(A, B)$, then $\Delta_{pa}(A, C) = \Delta_{pa}(A, D)$.*

Proof. Let $E \in \Delta_{pa}(A, C)$. If E is adjacent to D, then clearly $E \in \Delta_{pa}(A, D)$. If E is not adjacent to D, then E and D are parallel lines in $\mathscr{P}(A, B, C)$. Since E is also parallel to A, D is parallel to A. But this contradicts the fact that $D \in T(A, B)$. Hence, $E \in \Delta_{pa}(A, D)$ and $\Delta_{pa}(A, C) \subseteq \Delta_{pa}(A, D)$. By symmetry, $\Delta_{pa}(A, D) \subseteq \Delta_{pa}(A, C)$ and the proof is complete. ∎

By virtue of the above lemma, we may simplify the notation $\mathscr{P}(A, B, C)$ to $\mathscr{P}(A, B)$ where A and B are two distinct parallel lines.

LEMMA 27. *Let A and B be two distinct parallel lines in P. If M and N are two distinct parallel lines in $\mathscr{L}(A, B)$, then $\mathscr{P}(A, B) = \mathscr{P}(M, N)$.*

Proof. Since $\mathscr{P}(A, B)$ and $\mathscr{P}(M, N)$ are both determined by the lines in $\mathscr{L}(A, B)$ and $\mathscr{L}(M, N)$ respectively, we shall show that $\mathscr{L}(A, B) = \mathscr{L}(M, N)$.

Case 1. $M // A$. By Lemma 23, any line that is parallel to A must be parallel to M. Similarly, any transversal of A and B must be a transversal of M and N. Thus, $T(A, B) \subseteq T(M, N)$. Let $C \in T(A, B)$, then $C \in T(M, N)$. If $E \in \Delta_{pa}(A, C)$, then clearly $E \in \Delta_{pa}(M, C)$. Therefore, we have $\mathscr{L}(A, B) \subseteq \mathscr{L}(M, N)$. Since both A and B are in $\mathscr{L}(M, N)$, by similar arguments, we have $\mathscr{L}(M, N) \subseteq \mathscr{L}(A, B)$. Hence, $\mathscr{L}(A, B) = \mathscr{L}(M, N)$.

Case 2. M is adjacent to A. Clearly $M \in T(A, B)$ and $A \in T(M, N)$. Let

$E \in T(A, B)$. If E is adjacent to M, then E is adjacent to N; hence $E \in T(M, N)$ and E is also contained in $\mathscr{L}(M, N)$. If E is parallel to M, then $E \in \Delta_{pa}(M, A)$ and E is also contained in $\mathscr{L}(M, N)$. Thus, $T(A, B) \subseteq \mathscr{L}(M, N)$.

If $E \in \Delta_{pa}(A, M)$, then E is adjacent to M. Hence E is adjacent to N. Thus, $E \in T(M, N)$ and $T(M, N) \subseteq \mathscr{L}(M, N)$. So $\Delta_{pa}(A, M) \subseteq \mathscr{L}(M, N)$.

Consequently, we have $\mathscr{L}(A, B) \subseteq \mathscr{L}(M, N)$. By Proposition 20, $|\mathscr{L}(A, B)| = |\mathscr{L}(M, N)|$. Hence, $\mathscr{L}(A, B) = \mathscr{L}(M, N)$.

Based on these two lemmas, one easily sees that the affine planes $\mathscr{P}(A, B)$ defined on a pair of distinct parallel lines A and B are well-defined planes.

6. Proof of the Theorem

Let G be a geometrizable (q, k, q)-strongly regular graph and let $\pi(G)$ denote its corresponding (q, k, q)-partial geometry. If $\pi^*(G)$ is the dual geometry of $\pi(G)$, then $\pi^*(G)$ is a (k, q, q)-partial geometry. Let P be an incidence structure with the point set as the set of points in $\pi^*(G)$ and the line set as the set of lines in $\pi^*(G)$, then every two distinct points determine a unique line and every two intersecting lines intersect at a unique point.

If, in addition, $q \geqslant 4$ and G has its triangle degrees equal to either $q(q-2)$ or to be at least $k-3$, and its 2-claw degrees equal to $q(q-1)$ or $2(q-1)$, then affine planes $\mathscr{P}(A, B)$ can be defined on P based on any two distinct parallel lines A and B in P. Thus P is an incidence structure with points, lines and planes such that each plane is an affine plane.

Let x, y and z be 3 noncollinear points in P. There exists a unique line L containing z and parallel to the line $\langle x, y\rangle$. Since $z \in L$ and $z \notin \langle x, y\rangle$, $L \neq \langle x, y\rangle$; hence, L and $\langle x, y\rangle$ determine a unique affine plane which contains x, y and z. So every 3 noncollinear points in P determine a unique plane.

From the above discussion, we see that P satisfies the hypotheses in Buekenhout's theorem, P is therefore an affine space. Clearly the linegraph of P is isomorphic to the (q, k, q) strongly regular graph G. Therefore, we have characterized the linegraph of an affine space.

7. Cases where $q = 1, 2$ and 3.

For the case where $q = 1$, the dual of a $(1, k, 1)$ strongly regular graph consists of a singleton point, and is a trivial graph. For $q = 2$, the dual of a $(2, k, 2)$ strongly regular graph is simply a complete graph on $k+1$ vertices. If a $(2, k, 2)$ strongly regular graph is isomorphic to the linegraph of an affine space $AF(2, n)$, then it is easily checked that $k = 2^n - 1$. Since complete graphs of arbitrary size k exist, not every $(2, k, 2)$ strongly regular graph is isomorphic to the linegraph of an affine space $AF(2, n)$. For $q = 3$, the Hall matroid on 81 points, constructed by Hall [6], has the properties that the linegraph of the matroid is a $(3, k, 3)$-strongly regular graph, and each plane is isomorphic to an affine plane $AF(3, 2)$, but the matroid is not isomorphic to an affine sapce $AF(3, 3)$. Thus, we see that for $q = 2$ and 3, the answer to our characterization question is negative.

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A Characterization of the Line–Hyperplane Design of a Projective Space and Some Extremal Theorems for Matroid Designs†

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Combinatorial characterizations of some incidence structures obtained from projective spaces and affine spaces over finite fields are given. Also some extremal theorems for matroid designs are proved.

1. Introduction

An incidence structure is a triple $(X, \mathscr{B}, I)$, X and $\mathscr{B}$ are finite sets, and $I \subseteq X \times \mathscr{B}$. Elements of X are called treatments, while elements of $\mathscr{B}$ are called blocks. If $(x, B) \in I$, we say that x is incident with B, and denote it by $x \in B$. We shall follow the usual notations of incidence structures, and often consider a block as the set of treatments incident with it. If D and D' are isomorphic incidence structures, we write $D \simeq D'$. For any $B \in \mathscr{B}$, we shall

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denote by $|B|$ the number of treatments incident with it. Incidence structures are also called designs; for various definitions connected with incidence structures, see Dembowski [4].

An incidence structure $D = (X, \mathscr{B}, I)$ is said to be BIBD (v, b, r, k, λ) iff (if and only if) it satisfies following conditions:

(a) $|X| = v$, $|\mathscr{B}| = b$.
(b) $|B| = k$ for all $B \in \mathscr{B}$.
(c) Each treatment is incident with exactly r blocks.
(d) Every pair of treatments is incident with exactly λ blocks.

Parameters (v, b, r, k, λ) satisfy the equations

$$\lambda(v-1) = r(k-1) \tag{1.1}$$

$$bk = rv \tag{1.2}$$

$$b \geq v \text{ (Fisher's inequality)}. \tag{1.3}$$

A BIBD (v, b, r, k, λ) with $v = b$ and hence $r = k$ is called an SBIBD (v, k, λ). For further results on BIBDs, see [4].

An incidence structure $G = (X, \mathscr{B}, I)$ is called a simple graph iff $|B| = 2$ for each $B \in \mathscr{B}$, and if $B_1, B_2 \in \mathscr{B}$, $B_1 \neq B_2$, B_1 and B_2 are distinct treatment sets. Treatments of a graph are also called vertices, while blocks are called edges. Two vertices x and y are said to be adjacent iff there exists an edge incident with both x and y. A simple graph $G = (X, \mathscr{B}, I)$ is said to be a strongly regular graph $G(v, n_1, p_{11}^1, p_{11}^2)$ [2], iff it satisfies the following conditions:

(A) $|X| = v$.
(B) Each vertex is incident with exactly n_1 vertices.
(C) If $x, y \in X$, $x \neq y$, there are exactly p_{11}^1 or p_{11}^2 vertices z such that z is adjacent to both x and y according as x and y are adjacent or not adjacent.

Let X be a set of v elements and $G = G(v, n_1, p_{11}^1, p_{11}^2)$ be a strongly regular graph with vertex set X. An incidence structure $D = (X, \mathscr{B}, I)$ is called a PBIBD $(v, b, r, k, \lambda_1, \lambda_2)$, with association graph $G(v, n_1, p_{11}^1, p_{11}^2)$ iff following conditions are satisfied:

(i) $|X| = v$, $|\mathscr{B}| = b$.
(ii) $|B| = k$ for each $B \in \mathscr{B}$.
(iii) Each treatment is incident with exactly r blocks.
(iv) Any two distinct treatments x and y occur together in exactly λ_1 or λ_2 blocks of D according as x and y are adjacent or not adjacent in G.

We shall denote the association graph of a PBIBD D by $G(D)$. For various examples and results on PBIBDs, see [4]. We shall denote by

$PG(d, q)$, a projective space of dimension d over a finite field $GF(q)$, while by $AG(d, q)$ we shall denote a d-dimensional affine space over $GF(q)$.

For $d > m > l \geq 0$, define an incidence structure $P_{l,m}(d, q)$ as follows. Treatments of $P_{l,m}(d, q)$ are all l-dimensional subspaces of $PG(d, q)$. An incidence relation is given by containment. $A_{l,m}(d, q)$ is similarly defined by taking $AG(d, q)$ instead of $PG(d, q)$. When no confusion arises we shall also write $P_{l,m}$ for $P_{l,m}(d, q)$ and $A_{l,m}$ for $A_{l,m}(d, q)$. The following results are well known and can be proved easily by using common properties of projective and affine spaces:

(a) $P_{0,m}(d, q)$ and $A_{0,m}(d, q)$ are BIBDs for all $d > m > 0$.

(b) If $d \geq 3$, $P_{1,d-1}(d, q)$ is a PBIBD $(v, b, r, k, \lambda_1, \lambda_2)$, where (1.4)

$$v = \frac{(q^{d+1} - 1)(q^d - 1)}{(q^2 - 1)(q - 1)}, \quad b = \frac{q^{d+1} - 1}{q - 1}, \quad r = \frac{q^{d-1} - 1}{q - 1}$$

$$k = \frac{(q^d - 1)(q^{d-1} - 1)}{(q^2 - 1)(q - 1)}, \quad \lambda_1 = \frac{q^{d-2} - 1}{q - 1}, \quad \lambda_2 = \frac{q^{d-3} - 1}{q - 1} \tag{1.5}$$

$G(P_{1,d-1})$ is a strongly regular graph $(v, n_1, p_{11}^1, p_{11}^2)$ where

$$n_1 = (q + 1)q\left(\frac{q^{d-1} - 1}{q - 1}\right), \quad p_{11}^1 = \frac{q^d - 1}{q - 1} - 2 + q^2$$

$$p_{11}^2 = (q + 1)^2. \tag{1.6}$$

Any two vertices in $G(P_{1,d-1})$ are joined by an edge iff they are intersecting lines in $PG(d, q)$.

By "replacing" a desarguesian plane by nondesarguesian plane in $PG(d, q)$ or $AG(d, q)$ one can construct BIBDs, with the same parameters as those of $P_{0,m}$ or $A_{0,m}$, but nonisomorphic to $P_{0,m}$ or $A_{0,m}$, respectively (see, for example, Mavron [8]). Dembowski and Wagner [5] proved a characterization theorem for $P_{0,d-1}(d, q)$ for $d \geq 3$. In Section 3 we prove the following characterization theorem for $P_{1,d-1}(d, q)$ for $d \geq 6$.

Theorem 1 *Let d and q be positive integers, $d \geq 6, q > 1$. Suppose D_1 is a PBIBD $(v, b, r, k, \lambda_1, \lambda_2)$, where parameters are given by* (1.5), *and $G(D_1)$ has parameters $(v, n_1, p_{11}^1, p_{11}^2)$, given by Eqs.* (1.6), *then q is a prime power,*

$$D_1 \simeq P_{1,d-1}(d, q) \quad \textit{and} \quad G(D_1) \simeq G(P_{1,d-1}).$$

We define a combinatorial geometry (or matroid) by the "hyperplane axioms." An incidence structure $D = (X, \mathscr{B}, I)$, is said to be a combinatorial geometry iff it satisfies following conditions:

(a) Given $H_1, H_2 \in \mathscr{B}$, $H_1 \neq H_2$, H_1 is not a proper subset of H_2.

(b) Given $H_1, H_2 \in \mathscr{B}$, $H_1 \neq H_2$, and $x \in X$, there exists a block $H_3 \in \mathscr{B}$, s.t. (such that) $H_1 \cap H_2 \cup \{x\} \subseteq H_3$.

(c) For each $x \in X$, $\cap H = \{x\}$, where intersection is taken over all blocks H s.t. $x \in H$.

Treatments of a combinatorial geometry are called points, while blocks are called hyperplanes. Combinatorial geometries have been studied in detail, and for various results on combinatorial geometry, we refer to Crapo and Rota [3]. We shall follow the notation of Crapo and Rota [3].

A combinatorial geometry $D = (X, \mathscr{B}, I)$ is called a geometric design iff D is also a BIBD. A regular geometric design is a combinatorial geometry in which all flats of rank i have the same cardinality m_i. It can be easily seen that a regular geometric design is indeed a geometric design [12]. Geometric designs have been studied by Edmonds, Young, and Murti [12], in particular they have given many examples of such designs. We list a few well-known examples of regular geometric designs.

(a) A $t-(v, k, \lambda)$ design [4] with $\lambda = 1$ is a regular geometric design of rank $(t+1)$ s.t.

$$m_i = i \quad \text{for} \quad 0 \leq i < t \quad \text{and} \quad m_t = k.$$

(b) $P_{0,m}$ is a regular geometric design of rank $m+1$ s.t.

$$m_i = \frac{q^i - 1}{q - 1} \quad \text{for} \quad 0 < i \leq m < d.$$

(c) $A_{0,m}$ is a regular geometric design of rank $m+1$ s.t.

$$m_i = q^{i-1} \quad \text{for} \quad 0 < i \leq m < d.$$

We note that $m_0 = 0$, $m_1 = 1$; and m_2 is the size of a line in any regular geometric design. Clearly $m_2 \geq 2$.

In Section 4 we shall prove the following results.

Theorem 2 *If $D = (X, \mathscr{B}, I)$ is a regular geometric design of rank $n \geq 4$, then:*

(a) $m_i - m_{i-1} \geq (m_2 - 1)(m_{i-1} - m_{i-2})$ *for* $n \geq i \geq 3$.

(b) *Equality holds in* (a) *for any i iff $D \simeq P_{0,n}(d, q)$ for some prime power q and $d \geq n+1$ or $m_2 = 2$ and D is a 3-design.*

Geometric designs of rank 3 are precisely BIBDs with $\lambda = 1$. We also study geometric designs of rank 4 in Section 4, and show that all such designs are regular and examples (a)–(c) given above are the extreme cases of certain inequalities to be satisfied.

Theorem 3 *If $D = (X, \mathscr{B}, I)$ is a regular geometric design of rank $n \geq 4$, then:*

(a) $m_i - m_j \geq (m_2 - 1)^{i-j}(m_{i-1} - m_{j-1})$ *for* $n \geq i > j \geq 2$.

(b) *Equality holds in* (a) *for any i iff* $D \simeq P_{0,n}(d, q)$ *for some prime power q and* $d \geq n + 1$ *or* $m_2 = 2$ *and D is a* 3*-design.*

Theorem 4 *Let $D = (X, \mathscr{B}, I)$ be a geometric design (v, b, r, k, λ) of rank 4, then:*

(a) $v \equiv k \bmod(\lambda - 1)$.

(b) $\lambda k - 2(\lambda - 1) \geq v \geq \lambda k - \frac{1}{2}(\lambda - 1) - \frac{1}{2}(\lambda - 1)\sqrt{4k - 3}$.

(c) $v = \lambda k - 2(\lambda - 1)$ *iff D is a* 3-$(v, k, 1)$*-design.*

(d) $v = \lambda k - \frac{1}{2}(\lambda - 1) - \frac{1}{2}(\lambda - 1)\sqrt{4k - 3}$ *iff* $k = q^2 + q + 1$ *for some power q and* $D \simeq P_{0,2}(n, q)$ *where n is given by* $v = q^{n+1}/(q - 1)$ *or* $k \leq 7$ *and D is a* 3-$(v, k, 1)$*-design.*

(e) *If* $v > \lambda k - \frac{1}{2}(\lambda - 1) - \frac{1}{2}(\lambda - 1)\sqrt{4k - 3}$, *then* $v \geq \lambda k - (\lambda - 1)\sqrt{k}$ *and for* $k \geq 16$ *equality holds iff* $D \simeq A_{0,2}(n, q)$ *where* $k = q^2$ *for some power q and* $v = q^n$.

2. Preliminaries

In this section we shall state various known results on characterizing geometries, which we shall be using in the next section.

Let D be a BIBD (v, b, r, k, λ) with $\lambda = 1$. It is well known that the dual D' is a PBIBD $(b, v, k, r, \lambda_1, \lambda_2)$ with $\lambda_1 = 1$, $\lambda_2 = 0$ s.t. $G(D')$ is a strongly regular graph with parameters $(b, r(k - 1), r - 2 + (k - 1)^2, k^2)$. Any two vertices B_1, B_2 are joined in $G(D')$ if and only if $|B_1 \cap B_2| = 1$ in D.

The following result is a particular case of a well-known theorem of Bose on partial geometries [2].

Proposition 5 *If $G = (V, E, I)$ is any strongly regular graph with parameters $(b, r(k - 1), r - 2 + (k - 1)^2, k^2)$, where b, r, k are integers s.t.*

$$r > \tfrac{1}{2}[k(k - 1) + k(k + 1)(k^2 - 2k + 2)]$$

then there exists a unique BIBD $D(v, b, r, k, \lambda)$ with $\lambda = 1$, $v = r(k - 1) + 1$ s.t. V is the set of blocks of D and $G(D') = G$.

Let $D = (X, \mathscr{B}, I)$ be an incidence structure. For $x, y \in X$, $x \neq y$, line xy is defined by

$$xy = \bigcap B$$

where intersection is taken over all blocks B s.t. $x, y \in B$.

The following well-known theorem due to Dembowski and Wagner [5] gives a characterization of the incidence structure $P_{0,d-1}(d, q)$ obtained from $PG(d, q)$.

Proposition 6 *If D is a symmetric BIBD (v, b, r, k, λ), $\lambda > 1$ s.t. its dual D' is a combinatorial geometry, then $D \simeq P_{0,d-1}(d, q)$ for some positive integer d and prime power q.*

We note that the dual D' of a BIBD $D(v, b, r, k, \lambda)$ is a combinatorial geometry iff every line of D meets every block of D.

Let $D = (X, \mathscr{L}, I)$ be an incidence structure. For each block B of D we define a new incidence structure D_B as follows. Treatments of D_B are treatments of D incident with B. Blocks of D_B are the lines of D contained in B. The following results on regular geometric designs of rank 4 are well known. The first one is essentially the definition of a projective space. The second is due to Buekenhout [1].

Proposition 7 *If D is a regular geometric design of rank 4 s.t. $m_2 \geq 3$ and D_B is a projective plane for all blocks B of D, then $D \simeq P_{0,2}(d, q)$ for some d and prime power q.*

Proposition 8 *If D is a regular geometric design of rank 4 s.t. $m_2 \geq 4$ and D_B is an affine plane for all blocks B of D, then $D \simeq A_{0,2}(d, q)$ for some d and prime power q.*

3. Proof of Theorem 1

Let d and q be positive integers, $d \geq 6$. Throughout this section we will assume that $D_1 = (X, \mathscr{B}, I)$ is a PBIBD $(v, b, r, k, \lambda_1, \lambda_2)$ with association graph $G_1(v, n_1, p^1_{11}, p^2_{11})$, where these parameters are given by Eqs. (1.5) and (1.6). The proof of Theorem 1 is essentially based on the Propositions 5 and 6. We first prove the following simple lemma.

Lemma 9 *The dual of D_1 is a BIBD with parameters (b, v, k, r, λ) where*

$$\lambda = \frac{(q^{d-1} - 1)(q^{d-2} - 1)}{(q - 1)(q^2 - 1)}.$$

Proof We have only to prove that any two blocks of D_1 intersect in λ treatments. The following two equations are obtained easily by counting the occurrences of treatments and pairs of treatments in blocks of D_1:

$$\sum_{B_1, B_2 \in \mathscr{B}} |B_1 \cap B_2| = b(r - 1)k = vr(r - 1) \tag{3.1}$$

and

$$\sum_{B_1, B_2 \in \mathscr{B}} |B_1 \cap B_2|(|B_1 \cap B_2| - 1) = vn_1\lambda_1(\lambda_1 - 1) + v(v - 1 - n_1)\lambda_2(\lambda_2 - 1) \tag{3.2}$$

where the summations in (3.2) and (3.3) are over all pairs of distinct blocks B_1 and B_2 of D_1. Using (3.2) and (3.3) we can derive the following equation easily:

$$\sum_{B_1, B_2 \in \mathscr{B}} (|B_1 \cap B_2| - \lambda)^2 = 0. \tag{3.3}$$

From (3.3) it follows that $|B_1 \cap B_2| = \lambda$ for all blocks B_1 and B_2 of D_1, $B_1 \neq B_2$. This completes the proof.

Consider the PBIBD $D_1 = (X, \mathscr{B}, I)$. Let $l \in X$ and $B \in \mathscr{B}$. Let t_l denote the number of treatments $l_1 \in X$ s.t. $l_1 \in B$ and the vertices l and l_1 are joined in G_1. Suppose $l \in B$, then counting the occurrences of treatments of B in the remaining blocks of D_1, containing l and using Lemma 9 we have

$$(\lambda_1 - 1)t_l + (\lambda_2 - 1)(k - 1 - t_l) = (\lambda - 1)(r - 1)$$

since $\lambda_1 - \lambda_2 = q^{d-3} \neq 0$, solving the above equation for t_l, we get

$$t_l = (q + 1)q\left(\frac{q^{d-2} - 1}{q - 1}\right) \qquad \text{for} \quad l \in B.$$

Similarly, if $l \in B$, we can obtain the following equation

$$\lambda_1 t_l + \lambda_2(k - t_l) = \lambda r$$

and hence

$$t_l = q\left(\frac{q^{d-2} - 1}{q - 1}\right) + 1 = \frac{q^{d-1} - 1}{q - 1} \qquad \text{for} \quad l \notin B.$$

Thus we have proved

Lemma 10 *For $l \in X$ and $B \in \mathscr{B}$, if t_l is as defined above, then t_l is independent of the choice of B and is given by*

$$t_l = (q + 1)q\left(\frac{q^{d-2} - 1}{q - 1}\right) \qquad \textit{for} \quad l \in B$$

$$\textit{and} \qquad t_l = \frac{q^{d-1} - 1}{q - 1} \qquad \textit{for} \quad l \notin B.$$

Lemma 11 *Let $d \geq 6$. There exists a unique BIBD $D_2 = (X_2, X, I_2)$ with parameters*

$$\left(\frac{q^{d+1} - 1}{q - 1}, \frac{(q^{d+1} - 1)(q^d - 1)}{(q^2 - 1)(q - 1)}, \frac{q^d - 1}{q - 1}, q + 1, 1\right), \qquad \textit{s.t. } G(D_2') \simeq G_1.$$

Proof Since $d \geq 6$, the conditions of Proposition 5 are satisfied for the graph G_1 and the result follows from the same.

From now on we shall assume that $d \geq 6$ and D_2 is the design defined by the above lemma. We note that blocks of D_2 are the treatments of D_1. We define a new incidence structure $D_3 = (X_2, \mathscr{B}, I_3)$ as follows. Treatments of D_2 are the treatments of D_3, while the blocks of D_1 are the blocks of D_3. A treatment x of D_3 is incident with the block B iff there exists $l \in X$ s.t. $x \in l$ in D_2 and $l \in B$ in D_1.

We proceed to compute the block size k_3 in D_3. Let B be a block of D_3. For each treatment $x \in B$ in D_3, define α_x to be the number of treatments l of D_1 s.t. $l \in B$ in D_1 and $x \in l$ in D_2. We first show that $\alpha_x = (q^{d-1} - 1)|q - 1$ for each $x \in B$ in D_3. Let l be any treatment of D_1 s.t. $l \in B$ in D_1. Then using Lemma 10 and the definition of D_2, etc., we have

$$\sum \alpha_x = (q+1)\left(q\left(\frac{q^{d-2}-1}{q-1}\right)+1\right) \tag{3.4}$$

where the summation is over all the $q + 1$ treatments $x \in l$ in D_2.

Also if for each $x \in l$ in D_2 there is some treatment l' of D_1, $l' \notin B$ in D_1 s.t. $x \in l'$ in D_2, then again using Lemma 10, etc., we have

$$\alpha_x \leq t_{l'} = \frac{q^{d-1}-1}{q-1} = \frac{q(q^{d-2}-1)}{q-1} + 1. \tag{3.5}$$

Using (3.4) and (3.5) it follows that

$$\alpha_x = q\left(\frac{q^{d-2}-1}{q-1}\right) + 1 \qquad \text{for all} \quad x \in l$$

$$\text{or there is some } x \in l \text{ in } D_2 \text{ s.t. } \alpha_x = \frac{q^d - 1}{q-1}. \tag{3.6}$$

Suppose there exists $x \in B$ in D_3 s.t.

$$\alpha_x = \frac{q^d-1}{q-1} \tag{3.7}$$

We next show that for all $z \neq x$, $z \in B$, $\alpha_z < q(q^{d-2} - 1)/(q - 1)$.

Now if z is any treatment of D_3, $z \in B$, $z \neq x$, then using (3.7), if l' is the unique block of D_2 containing x and z, $l' \in B$ in D_1. Hence using (3.4) for l' we have

$$\alpha_z \leq (q+1)q\left(\frac{q^{d-1}-1}{q-1}\right) - \frac{q^d-1}{q-1} < q\left(\frac{q^{d-2}-1}{q-1}\right) \tag{3.8}$$

for all $z \in B$ in D_3, $z \neq x$.

Now since the block size in D_1 is $(q^d - 1)(q^{d-1} - 1)/(q^2 - 1)(q - 1)$, there is at least one treatment l' of D_1 s.t. $l' \in B$ in D_1 but $x \notin l'$ in D_2.

Equation (3.6) for l' and Eq. (3.8) give a contradiction. Hence

$$\alpha_x = q\left(\frac{q^{d-2}-1}{q-1}\right) + 1 = \frac{q^{d-1}-1}{q-1} \tag{3.9}$$

for all $x \in B$ in D_3.

Now using (3.9), by counting the number of elements in the set $\{(x, l) \mid x \in l \text{ in } D_2, l \in B \text{ in } D_1\}$ we have

$$\left(\frac{q^{d-1}-1}{q-1}\right)k_3 = \frac{(q^d-1)(q^{d-1}-1)}{(q-1)(q^2-1)}(q+1).$$

Hence

$$k_3 = (q+1)\left(\frac{q^d-1}{q^2-1}\right) = \frac{q^d-1}{q-1} \tag{3.10}$$

for all blocks B of D_3. We can now prove the following lemma.

Lemma 12 *Let $d \geq 6$. Then D_3 is an SBIBD*

$$\left(\frac{q^{d+1}-1}{q-1}, \frac{q^d-1}{q-1}, \frac{q^{d-1}-1}{q-1}\right).$$

Proof It is obvious that the number of treatments or the number of blocks in D_3 is

$$v_3 = \frac{q^{d+1}-1}{q-1}.$$

Using (3.10) it follows that each block of D_3 is of size k_3, where

$$k_3 = \frac{q^d-1}{q-1}.$$

Hence we have only to prove that any two treatments of D_3 occur in exactly $(q^{d-1}-1)/(q-1)$ blocks. Suppose x_1 and x_2 are two treatments of D_3. Then there is a unique treatment l of D_1 s.t. $x_1, x_2 \in l$ in D_2. Now l occurs in exactly $r = (q^{d-1}-1)/(q-1)$ blocks of D_1. Hence x_1, x_2 occur in at least $(q^{d-1}-1)/(q-1)$ blocks of D_3. Counting the elements of the set

$$\{(y, z, B) \mid y, z \in X_2, y, z \in B \text{ in } D_3, B \in \mathscr{B}\},$$

we easily derive that in D_3 any two distinct treatments are incident with exactly $(q^d-1)/(q-1)$ blocks. This completes the proof.

We can now complete the proof of Theorem 1. We first note that the blocks of the BIBD D_2 correspond to the lines of D_3. For $l \in X$, let (l) denote the elements of X_2 incident with l in D_2. It is easily proved that for $x, y \in (l)$,

(l) is the line generated by x and y in D_3. For the sake of brevity, l will also denote the line of D_3. Using Lemma 10 given any block B of D_3 and a line l of D_3, there exists at least one treatment m of D_1 s.t. $m \in B$ and l and m are incident with a common block of D_1. Therefore any line and any block of D_3 intersect. Thus the conditions of Proposition 6 are satisfied. Hence $D_3 \simeq P_{0,d-1}(d, q)$. It follows that $D_1 \simeq P_{1,d-1}(d, q)$ and $G_1 \simeq G(P_{1,d-1})$ and q is a prime power.

4. Geometric Designs

In this section we shall study regular geometric designs. We shall also study geometric designs of rank 4. We shall first prove Theorem 2.

Let $D = (X, \mathscr{B}, I)$ be a regular geometric design of rank $d \geq 3$. Let $d \geq j \geq 3$. Let $P \subseteq X$ be any $(j-3)$-flat of D. And $Q \supseteq P$ be a j-flat of D. We define an incidence structure $D_{P,Q}$ as follows. Treatments of $D_{P,Q}$ are all $(j-2)$-flats that contain P and are contained in Q. Blocks are all $(j-1)$-flats that contain P and are contained in Q. Incidence is given by containment. The following result can be proved easily. In fact the usual proof for projective spaces can be extended to regular geometric designs [12].

Proposition 13 *$D_{P,Q}$ is a BIBD with parameters*

$$\left(\frac{m_j - m_{j-3}}{m_{j-2} - m_{j-3}}, \frac{(m_j - m_{j-3})(m_j - m_{j-2})}{(m_{j-1} - m_{j-3})(m_{j-1} - m_{j-2})}, \frac{m_j - m_{j-2}}{m_{j-1} - m_{j-2}}, \frac{m_{j-1} - m_{j-3}}{m_{j-2} - m_{j-3}}, 1\right)$$

Proof of Theorem 2 Let $D = (X, \mathscr{B}, I)$ be a regular geometric design as defined above. Using (1.2) and (1.3) for $D_{P,Q}$ we have

$$\frac{m_j - m_{j-2}}{m_{j-1} - m_{j-2}} \geq \frac{m_{j-1} - m_{j-3}}{m_{j-2} - m_{j-3}}.$$

Hence

$$\frac{m_j - m_{j-1}}{m_{j-1} - m_{j-2}} \geq \frac{m_{j-1} - m_{j-2}}{m_{j-2} - m_{j-3}} \qquad \text{for all} \quad d \geq j \geq 3. \tag{4.1}$$

Using (4.1) it follows that

$$\frac{m_i - m_{i-1}}{m_{i-1} - m_{i-2}} \geq m_2 - 1 \qquad \text{for all} \quad i \geq 3$$

hence $m_i - m_{i-1} \geq (m_2 - 1)(m_{i-1} - m_{i-2})$ for $d \geq i \geq 3$. This proves statement (a). Now suppose

$$m_i - m_{i-1} = (m_2 - 1)(m_{i-1} - m_{i-2}) \qquad \text{for some } i.$$

Then we will have equality in (4.1) for all j, $i \geq j \geq 3$. In particular

$$m_3 - m_2 = (m_2 - 1)^2.$$

Hence

$$m_3 = (m_2 - 1)^2 + (m_2 - 1) + 1.$$

Thus if Q is a rank 3 flat of D and 0 denotes the 0-flat, then $D_{Q,0}$ will be a projective plane for all Q. Now let D_1 be an incidence structure defined as follows. Treatments of D_1 are the treatments of D. Blocks of D_1 are all the 3-flats of D. It is obvious that D_1 is a regular geometric design of rank 4. If $m_2 \geq 3$, then D_1 satisfies the conditions of Proposition 7 and hence $D_1 \simeq P_{0,2}(d, q)$ for some d and prime power q. Now it follows easily that $D \simeq P_{0,n}(d, q)$ and $d \geq n + 1$. If $m_2 = 2$, then any three points determine a unique 3-flat of D, and hence clearly they are incident with exactly

$$\frac{(m_n - m_3)(m_n - m_4) \cdots (m_n - m_{n-2})}{(m_{n-1} - m_3) \cdots (m_{n-1} - m_{n-2})}$$

blocks of D. Thus D is a 3-design. This completes the proof of the theorem. Theorem 3 is the obvious inductive extension of Theorem 2.

For the rest of this section we shall assume that $D = (X, \mathscr{B}, I)$ is a geometric design of rank 4. Let the parameters of D be (v, b, r, k, λ). Since rank $D = 4$, $\lambda > 1$.

Lemma 14 *All geometric design of rank 4 are regular.*

Proof Let D be a geometric design of rank 4, with parameters (v, b, r, k, λ). Let $x, y \in X$, $x \neq y$. Let l be the unique 2-flat containing x, y, i.e., line xy. Now from the properties of combinatorial geometries it follows that given $z \in X$, $z \notin l$, there is a unique block B of D s.t. $l \cup \{z\} \subseteq B$. Hence counting the number of elements in the set $\{(z, B) \mid z \in X - l, z \in B, l \subseteq B\}$, we have

$$\lambda(k - |l|) = v - |l|.$$

Hence

$$|l| = \frac{\lambda k - v}{\lambda - 1}.$$

Thus the cardinality of a line l is independent of the choice of the line l. Proof of Lemma 14 is now complete.

We now prove the final result of this paper.

Proof of Theorem 4 Using Lemma 14, D is a regular geometric design with

$$m_2 = \frac{\lambda k - v}{\lambda - 1} \quad \text{and} \quad m_3 = k. \tag{4.2}$$

Since m_2 is an integer, $v \equiv k \bmod (\lambda - 1)$. Also since $m_2 \geq 2$,

$$\lambda k - v \geq 2(\lambda - 1).$$

Hence

$$v \leq \lambda k - 2(\lambda - 1).$$

Suppose $v = \lambda k - 2(\lambda - 1)$. Then $m_2 = 2$ and clearly D is a 3-$(v, k, 1)$-design. Thus we have proved (c). We now prove the remaining parts of Theorem 4. Using Theorem 2(a) for $i = 3$, we have

$$k \geq m_2(m_2 - 1) + 1. \tag{4.3}$$

Hence using (4.1) we have

$$v^2 - v(2\lambda k - (\lambda - 1)) + \lambda^2 k^2 - (\lambda - 1)^2 k \leq 0.$$

Hence

$$v \geq \lambda k - \frac{\lambda - 1}{2} - \frac{1}{2}\sqrt{(2\lambda k - (\lambda - 1))^2 + 4(\lambda^2 k^2 - (\lambda - 1)^2 k)}$$

i.e.,

$$v \geq \lambda k - \frac{\lambda - 1}{2} - \frac{\lambda - 1}{2}\sqrt{4k - 3}.$$

This proves (b). Now suppose $v = \lambda k - \frac{1}{2}(\lambda - 1) - \frac{1}{2}(\lambda - 1)\sqrt{4k - 3}$; then we will have equality in (4.3) and the statement (d) follows from Theorem 2(b).

Now suppose $v > \lambda k - \frac{1}{2}(\lambda - 1) - \frac{1}{2}(\lambda - 1)\sqrt{4k - 3}$. Hence

$$k > m_2(m_2 - 1) + 1.$$

Thus if P is any 3-flat of D and 0 denotes the 0-flat, then $D_{P,0}$ is not an SBIBD. Parameters of $D_{P,0}$ are

$$\left(m_3, \frac{m_3(m_3 - 1)}{m_2(m_2 - 1)}, \frac{m_3 - 1}{m_2 - 1}, m_2, 1\right).$$

Hence

$$\frac{m_3 - 1}{m_2 - 1} \geq m_2 + 1 \tag{4.4}$$

i.e., $k \geq m_2^2$. Hence using (4.2) we have

$$v^2 - 2\lambda k v - (\lambda - 1)^2 k + \lambda^2 k^2 \leq 0.$$

Hence

$$v \geq \lambda k - \tfrac{1}{2}\sqrt{4\lambda^2 k^2 - 4[(\lambda - 1)^2 k + \lambda^2 k^2]}$$

i.e.,

$$v \geq \lambda k - (\lambda - 1)\sqrt{k}.$$

If $v = \lambda k - (\lambda - 1)\sqrt{k}$ and $k \geq 16$, then we will have equality in (4.4) and $m_2 \geq 4$. Hence all designs $D_{P,0}$ are affine planes, and the statement (e) follows from Proposition 8. This completes the proof of the theorem.

A natural question arises. Are there geometric designs of rank 4 different from those given by cases (c)–(e) of Theorem 4. We note that using examples of Steiner triple systems due to Hall [6], one can construct one such geometric designs of rank 4 (see also Teirlinck [11]). This example also shows that the hypothesis $k \geq 16$ in statement (e) is necessary. An interesting problem will be to classify all geometric designs of rank 4.

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STRONGLY REGULAR GRAPHS *

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In this paper we have tried to summarize the known results on strongly regular graphs. Both groupal and combinatorial aspects of the theory have been included. We give the list of all known strongly regular graphs and a large bibliography of this subject.

0. Introduction

The theory of strongly regular graphs (s.r. graphs) was introduced by Bose [7] in 1963, in connection with partial geometries and 2-class association schemes. One year later, Higman [34] initiated the study of the rank 3 permutation groups using the strongly regular graphs. Both combinatorial and groupal aspects have been developed in recent years. Moreover, the interest in strongly regular graphs has been stimulated by the discovery of new simple groups. In this paper we have tried to summarize the main results on this subject, and to include some new ones. We have also included an extensive bibliography on s.r. graphs.

Throughout this paper we use the notation of Seidel [65] that arises from the study of equiangular lines (see Van Lint and Seidel [80]). This notation is well adapted to the concept of complementation, and to the complementary graph; in the first section we mention the connection with other notation used by Bose and Higman that arises respectively from the *n*-class association schemes and from the centralizer ring.

We have included as an appendix a table of all the s.r. graphs (to the best of our knowledge) related to classical groups, to sporadic groups and arising from combinatorial constructions.

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1. Basic definitions and notations

The graphs considered here are undirected, without loops and multiple edges.

Let $\mathcal{G}$ be such a graph and let v be the number of vertices. $\mathcal{G}$ is said to be *regular* if each vertex is adjacent to the same number k of vertices; k is the valency of $\mathcal{G}$. $\mathcal{G}$ is said to be *strong* if:

(i) given any two adjacent vertices x, y, the sum of the number of vertices adjacent to both x and y and the number of vertices nonadjacent to x and y is constant;

(ii) given any two nonadjacent vertices, the same sum is constant.

$\mathcal{G}$ is *strongly regular* if it is strong and regular; it implies that:

(i) the number of vertices adjacent to both endpoints of an edge is constant and equal to λ;

(ii) the number of vertices adjacent to two nonadjacent vertices is constant and equal to μ.

We shall denote by $\Delta(p)$ (resp. $\Gamma(p)$) the set of vertices adjacent (resp. nonadjacent) to a vertex p.

$\bar{\mathcal{G}}$ is the *complement* of $\mathcal{G}$. The set of vertices of $\bar{\mathcal{G}}$ is the set of vertices of $\mathcal{G}$ and two vertices in $\bar{\mathcal{G}}$ are adjacent if and only if they were not adjacent in $\mathcal{G}$. If $\mathcal{G}$ is regular, then $\bar{\mathcal{G}}$ is regular of valency $l = v-k-1$. Moreover if $\mathcal{G}$ is strong, $\bar{\mathcal{G}}$ is strong.

Let $\mathcal{X}_1 \cup \mathcal{X}_2$ be a partition of $\mathcal{G}$. The graph $\mathcal{G}'$ obtained by *complementation* or *switching* with respect to $\{\mathcal{X}_1, \mathcal{X}_2\}$ is the graph whose vertices are the vertices of $\mathcal{G}$; all the adjacencies in $\mathcal{X}_1$ and $\mathcal{X}_2$ are preserved, but a vertex of $\mathcal{X}_1$ is joined to a vertex of $\mathcal{X}_2$ if and only if they were not joined in $\mathcal{G}$.

To any graph one may associate an *adjacency matrix A*. Let the vertices of $\mathcal{G}$ be labelled by $1, 2, \ldots, v$. Now construct the $v \times v$ matrix A whose entries are defined in the following way:

$a_{ij} = \alpha$ if $i = j$,
$a_{ij} = \beta$ if i and j are adjacent,
$a_{ij} = \gamma$ if i and j are not adjacent.

If $\mathcal{G}$ is a strongly regular graph, it can be easily shown that the matrix algebra generated by A, I and J (the matrix with "1" in each entry) is of dimension 3. A straightforward calculation gives

$$A^2 = [2(\alpha-\gamma) + (\lambda-\mu)(\beta-\gamma)]A + [k(\beta-\gamma)^2 + \lambda(\gamma-\alpha)(\beta-\gamma) + \mu(\alpha-\beta)(\beta-\gamma) - (\alpha-\gamma)^2]I + [k\gamma(2\beta-\gamma) + l\gamma^2 - \lambda\gamma(\beta-\gamma) + \mu\beta(\beta-\gamma) + \gamma^2]J,$$

$$AJ = (\alpha + k\beta + l\gamma)J.$$

In particular, if $\alpha = 0, \beta = -1, \gamma = 1$, A is the $(0, -1, 1)$ adjacency matrix of $\mathcal{G}$ and it satisfies:

$$A^2 + 2(\lambda-\mu+1)A - (4k-2\lambda-2\mu-1)I = (v-4k+2\lambda+2\mu)J,$$

$$AJ = (l-k)J.$$

The eigenvalues of A are $\rho_0 = l-k$ and ρ_1, ρ_2 which are roots of the quadratic equation

$$x^2 + 2(\lambda-\mu+1)x - (4k-2\lambda-2\mu-1) = 0 \qquad (\rho_1 > 0 > \rho_2).$$

The multiplicities m_1, m_2 of ρ_1, ρ_2 are respectively,

$$[-\rho_0 - \rho_2(v-1)](\rho_1-\rho_2)^{-1}, \quad [\rho_0 + \rho_1(v-1)](\rho_1-\rho_2)^{-1}.$$

Looking at these multiplicities, one finds necessary conditions for the existence of a strongly regular graph; either

(I) $k = l$, $\mu = \lambda + 1 = \frac{1}{2}k$, or

(II) $d = (\lambda-\mu)^2 + 4(k-\mu)$ is a square and $\sqrt{d}$ divides $2k + (\lambda-\mu)(v-1)$.

Moreover if v is even, $2\sqrt{d}$ does not divide this quantity; but if v is odd, it does.

In the second case, the eigenvalues ρ_1, ρ_2 are odd integers. In the first case $\rho_0 = 0$, $\rho_1 = -\rho_2 = \sqrt{v}$ and it is known that a necessary condition for the existence of such graphs is that $v = a^2 + b^2$, where a and b are integers of different parity. Graphs of this type have been constructed for all admissible $v = p^\alpha$, p prime, and for some other values.

Let us mention that the $(0, -1, 1)$-adjacency matrix A of a strong graph satisfies the equation $(A - \rho_1 I)(A - \rho_2 I) = (v - 1 + \rho_1\rho_2)J$. If $v - 1 + \rho_1\rho_2 \neq 0$, Seidel has shown [65] that the graph is strongly regular.

Complementation with respect to $(\mathcal{X}_1, \mathcal{X}_2)$ transforms the matrix A to a matrix A' such that

$a'_{ij} = a_{ij}$ if both i and j are in $\mathcal{X}_1$ or $\mathcal{X}_2$,
$a'_{ij} = -a_{ij}$ if i and j are not in the same $\mathcal{X}_k$.

The matrix A of the complementary graph $\bar{\mathcal{G}}$ of $\mathcal{G}$ is equal to $-A$; we have

$$\bar{v} = v, \quad \bar{k} = l, \quad \bar{l} = k, \quad \bar{\lambda} = l-k+\mu-1, \quad \bar{\mu} = l-k+\lambda+1,$$

$$\rho_0 + \bar{\rho}_0 = \rho_1 + \bar{\rho}_2 = \rho_2 + \bar{\rho}_1 = 0.$$

Other notations used

(A) Higman [34] considers the (0, 1) adjacency matrix of a graph defined by $\alpha = 0$, $\beta = 1$, $\gamma = 0$; this matrix satisfies the equation $A^2 - (\lambda - \mu)A - (k - \mu)I = \mu J$ and $AJ = kJ$. The eigenvalues s and t are related to ρ_1 and ρ_2 by $\rho_1 = -(2s+1)$, $\rho_2 = -(2t+1)$.

(B) With the terminology of the 2-class association schemes (see Bose [7]) the vertices of the graph are the varieties; two varieties are first (resp. second) associates if the two corresponding vertices are adjacent (resp. nonadjacent). The correspondences between the notations are:

$$k = n_1, \quad l = n_2, \quad \lambda = p^1_{11}, \quad \mu = p^2_{11}.$$

2. Partial geometries

A partial geometry is a set of elements called points, together with a set of subsets called lines such that

(i) every pair of points lies on at most one line,

(ii) every line contains K points, $K \geq 2$,

(iii) every point is on R lines, $R \geq 2$,

(iv) given a nonincident point-line pair, (p, L), there are exactly T lines on p incident with L, $K \geq T \geq 1$.

Any partial geometry yields a strongly regular graph $\mathcal{G}$ defined as follows: the vertices of $\mathcal{G}$ are the points of the geometry; and two vertices are adjacent if and only if the two corresponding points are on a line; in other words $\mathcal{G}$ is the point-graph of the geometry. The parameters of the graph may be computed from the parameters (R, K, T) of the partial geometry. We have:

$$v = KT^{-1}[(R-1)(K-1)+T],$$
$$k = R(K-1),$$
$$l = (K-1)(R-1)(K-T)T^{-1},$$
$$\lambda = K-2+(R-1)(T-1),$$
$$\mu = RT.$$

Obviously these numbers are integers, hence T divides $K(K-1)(R-1)$. Moreover the graph $\mathcal{G}$ is strongly regular; the necessary conditions for the existence of a s.r. graph imply that $T(R+K-T-1)$ divides $RK(K-1)(R-1)$.

A graph $\mathcal{G}$ having a set of parameters (v, k, l, λ, μ) such that there exist integers (RKT) satisfying the above relations is called *pseudo-geometrizable*; it is *geometrizable* if such a geometry exists.

A pseudo-geometric graph does not always arise from a partial geometry; for instance there exists a graph with parameters $v = 16$, $k = 6$, $\lambda = 2$ which corresponds to $R = 2$, $K = 4$, $T = 1$ and this graph is not isomorphic to the point-graph of the unique partial geometry (2, 4, 1).

A theorem of Bose [7] gives a sufficient condition for a pseudo-geometric graph to be geometrizable: there exists in $\mathcal{G}$ a set Σ of cliques (complete subgraphs) such that (1) two adjacent vertices are in exactly one clique of Σ, (2) every vertex belongs to exactly R cliques of Σ, (3) K, the common size of all Σ-cliques, in greater than R. Another result of Bose gives a sufficient condition; a pseudo-geometric graph $\mathcal{G}$ is geometrizable if

$$K > \tfrac{1}{2}[R(R-1) + T(R+1)(R^2 - 2R + 2)].$$

Given any partial geometry with parameters (R, K, T), the dual, i.e. the geometry whose points and lines are the lines and the points of the first one, is again a partial geometry with parameters (K, R, T).

Those partial geometries have been investigated by Bose [7], Sims [73], Higman [11] and Ahrens and Szekeres [1].

3. t-designs and symmetric 2-designs

A *t-design* $S_\lambda(t, k, v)$ is a set of v *points* with subsets of size k, called *blocks*, such that every t distinct points belong to exactly λ blocks; the number b of blocks is given by $b = \lambda\binom{v}{t}/\binom{k}{t}$.

If $t = 2$, it may happen that the number of blocks equals the number of points. In this case $\lambda(v-1) = k(k-1)$. The simplest examples are given by the projective spaces (the blocks are the hyperplanes). Let us mention that in a symmetric 2-design, two blocks intersect in λ points and there are k blocks containing each point.

In some particular cases it is possible to derive a symmetric design from a strongly regular graph.

(A) If $\lambda = \mu$, take as blocks the sets $\Delta(p)$ for every vertex p of $\mathcal{G}$. This yields a $S_\lambda(2, k, v)$ symmetric design.

(B) If $\lambda = \mu - 2$, take for blocks the sets $\{p\} \cup \Delta(p)$ for every vertex p of $\mathcal{G}$. One obtains the symmetric design $S_\mu(2, k+1, v)$.

Let us notice that if $\lambda = \mu - 2$, $\bar{\lambda} = \bar{\mu}$ in the complementary graph $\bar{\mathcal{G}}$ and the symmetric design is the complement of the design obtained from the complementary graph.

In case (A), $\lambda = \mu$, the necessary conditions for the existence of a strongly regular graph reduce to:

$4(k-\lambda) = d$ is a square,

$\sqrt{d}$ divides $2k$.

Hence $k - \lambda = m^2$ and m divides k; this implies m divides λ. Thus for a given λ there are finitely many strongly regular graphs with $\lambda = \mu$.

These graphs, sometimes called (v, k, λ)-graphs, have been studied by Bose and Shrikhande [10], Rudvalis [62] and Wallis [83]. Rudvalis [62] proved that the existence of a (v, k, λ) graph is equivalent to the existence of a symmetric $S_\lambda(2, k, v)$ admitting a polarity without absolute points. A necessary (but not sufficient) condition for the existence of such graphs (or designs) is that $(v,k,\lambda) = (s\{(s+a)^2-1\}/a, s(s+a), sa)$, where a is a divisor of $s(s^2-1)$ and if a is even, s and $s(s^2-1)/a$ must be odd integers.

The connection between (v, k, λ) graphs and some symmetric designs has been studied by Hall et al. [30] and Hubaut [49].

4. Equiangular lines

A set of v lines (through one point) in an r-dimensional euclidean space is called equiangular if the angle between every pair of lines is the same. An interesting problem is to determine the maximum number $v(r)$ of such lines in an r-dimensional space. Seidel [67] proved that it is possible to derive from any symmetric $(0, -1, 1)$-$(v \times v)$ matrix A a set of equiangular lines in an r-dimensional space. If ρ is the smallest eigenvalue (necessarily negative) with multiplicity $v - r$, then the matrix $(A - \rho I)/\rho$ may be interpreted as the grammian matrix of v vectors in an r-space; the angle between two lines satisfies $\cos\alpha = 1/\rho$. In the case of strongly regular and strong graphs, the multiplicity of the smallest eigenvalue is often very large and therefore the number v of equiangular lines is large with respect to r. Moreover there exists a connection between these sets of lines and a regular polyhedron, so that for some sets of equiangular lines it happens that the automorphism group of the corresponding graph is very large. More details may be found in [52].

5. Rank 3 graphs

Let G be a transitive permutation group on a set Ω. If G_p, the sub-

group of G fixing $p \in \Omega$, has r orbits, then G is said to be a rank r group. In the case where $r = 3$ let the 3 orbits be $\{p\}$, $\Delta(p)$, $\Gamma(p)$. It is obvious that $q \in \Delta(p) \Rightarrow p \in \Delta(q)$ holds if and only if G is of even order. In this case it is possible to derive from G a strongly regular graph $\mathcal{G}$ whose set of vertices is Ω; two vertices p and q are adjacent in $\mathcal{G}$ iff $p \in \Delta(q)$.

Higman [34–42] has developed the theory of rank 3 groups; they act as an automorphism group on $\mathcal{G}$, transitive on the vertices and on the edges.

Infinite classes of strongly regular graphs arise in the study of representations of classical groups, especially simple groups. Most of them have rank 3 representations and are a normal subgroup of Aut($\mathcal{G}$). In some cases there are higher rank representations but it may happen that they yield strongly regular graphs.

A result of Seitz [68] gives important information about the rank 3 representations of Chevalley groups. There exists for each class of Chevalley group $G(q)$ an integer N such that if $q > N$, the only rank 3 representations of $G(q)$ are representations on the cosets of a parabolic subgroup. (In fact Seitz's result applies to all ranks.) Such rank 3 representations do not occur for $G_2(q)$, $E_7(q)$, $E_8(q)$, $F_4(q)$ and twisted $E_6'(q)$, $F_4'(q)$ and $D_4''(q)$.

Most of the representations have been discovered by geometrical means. We would mention the papers of Primrose [58] and Ray-Chaudhuri [59] for the orthogonal groups, Bose and Chakravarti [8] and Chakravarti [12,13] for unitary groups, Higman and McLaughlin [43] for symplectic and unitarian groups and a series of papers by Wan-Zhe Xian, Yang Ben-Fu, Dai-Zong Duo and Fen-Xuning [19,23,85,86,87,88] on classical groups.

Exceptional representations of $PO_{2n+1}(3)$ have been discovered by Rudvalis [62]. Other exceptional graphs and designs which appear to be related to $V_{2n} \cdot O^{\pm}_{2n}(2)$ have been combinatorially constructed by Mann [54] and also, in connection with coding theory, by Delsarte and Goethals [20]. Taylor [76] has constructed strong graphs with $PSU_n(q^2)$ as an automorphism group. For sporadic groups having rank 3 representations, we refer the reader to Tits [78].

6. Some results about rank 3 graphs

6.1. General results

Foulser [24] and Dornhoff [21] have determined the primitive rank

3 solvable groups G. The corresponding graphs have parameters as follows:

(i) $(v, k, \lambda) = (n^2, g(n-1), (g-1)(g-2)+n-2)$, i.e., the graph is of type $L_g(n)$ (K.1). The only possible values for (n,g) are $(3^2, 4)$, $(13,6)$ $(17,6)$ $(19,8)$ $(3^3, 4)$ $(29,6)$ $(31,8)$ $(47,24)$ or $(3^2,4)$ $(7^2,10)$.

(ii) $(v, k, \lambda) = (64, 27, 10)$.

Moreover there exist two other classes when G is a subgroup of the affine group of the line or when G acts on a vector space V such that $V = V_1 \oplus V_2$, $V_1 \cup V_2$ being a set of imprimitivity for G_0.

When the rank 3 group acts on an affine plane, Kallaher [51] and Liebler [53] proved that the plane is a translation plane.

If a rank 3 group possesses a normal regular subgroup, it is isomorphic to a subgroup of automorphisms of the affine space AG(n,g) containing the translations.

6.2. *Characterization by the constituents*

Tsuzuku [79] has determined the primitive rank 3 extensions of Sym(k) acting in its natural representation on k points. The following extensions occur if $k > 1$:

(i) $k = 2$, $v = 5$ and $G \simeq D_5$ dihedral of order 10,

(ii) $k = 3$, $v = 10$, $\mathcal{G}$ is the Petersen graph $G \simeq$ Sym(5),

(iii) $k = 5$, $v = 16$, $\mathcal{G}$ is the Clebsch graph $G \simeq 2^4 \cdot$ Sym(5),

(iv) $k = 7$, $v = 50$, $\mathcal{G}$ is the Hoffman-Singleton graph and $G \simeq \mathrm{PSU}_3^*(5^2)$.

Iwasaki [50] proved that the same result holds for $G_{0|\Delta} = \mathrm{Alt}(k)$ in its natural representation. Montague [57] has shown the non-existence of most extensions of $\mathrm{PSL}_2(q)$, $\mathrm{PSU}_3(q^2)$, $R(q)$, $S_z(q)$ in their natural representation on $k = q+1$, q^3+1, q^3+1 points, with $|\Delta| = k$, $|\Gamma| = \frac{1}{2}(k(k-1))$. The only exceptional cases are

(i) $\mathrm{PSL}_2(2) \simeq$ Sym(3): the extension is the Petersen graph,

(ii) $\mathrm{PSL}_2(4) \simeq$ Alt(5) which extends in the Clebsch graph,

(iii) $\mathrm{PSL}_2(9)$ which gives $\mathrm{PSL}_3(4)$ acting on the graph (56, 10, 0) (S.1).

If the groups $G_{0|\Delta}$ and $G_{0|\Gamma}$ are 2-transitive on both Δ and Γ, the graph is isomorphic either to a pentagon, or to the union of two complete graphs of order $n = \frac{1}{2}v$. In the last case the normal subgroup fixing each component must be 3-fold transitive on K_n.

Bannai [3] proved that if $G_{0|\Delta} \simeq \mathrm{PSL}_n(2^f)$ in its natural representation, then $n = 2$ and $f = 1$ or 2. These cases are covered by the result of Montague. In another paper Bannai [4] studied the case where $G_{0|\Delta}$ is 4-fold transitive; he showed that $|\Delta| = 5, 7$ and $G_{0|\Delta} \simeq$ Sym(5), Sym(7) or Alt(7).

6.3. *Characterization by subdegrees*

In some cases, the knowledge of the subdegrees k and l, together with the rank 3 assumption is sufficient to classify the rank 3 graphs. Higman proved [39]:

(i) $v = m^2$, $k = 2(m-1)$, $m \geq 2$, then $G \simeq \mathrm{Sym}(m) \setminus \mathrm{Sym}(2)$ and $\mathcal{G}$ is of type $L_2(m)$.

(ii) $v = \binom{m}{2}$, $k = 2(m-2)$, $m \geq 5$, then G is a 4-fold transitive subgroup of Sym(m) and $\mathcal{G}$ is of type $T(m)$, or

(a) $G \simeq \mathrm{P\Gamma L}_2(8)$ and $\mathcal{G}$ is of type $T(g)$,

(b) $\mu = 6$, $m = 9, 17, 27, 57$,

(c) $\mu = 7$, $m = 51$,

(d) $\mu = 8$, $m = 28, 36, 325, 903, 8128$.

The only known case is $G_2(2)$ on 36 points (S.9).

(iii) $v = Q_n Q_{n-1}/Q_2$, $k = qQ_2$ with $Q_n = (q^n-1)/(q-1)$, then G is a subgroup of $\mathrm{P\Gamma L}_m(q)$ acting on the lines of $P_{m-1}(q)$ transitive of the 4 simplices or possibly $m = 4$ or 5 either $m = 2\alpha+1$ and $17 \geq m \geq 7$ with $\mu \neq (q+1)^2$. An analogous result has been proved by Enomoto [22]: If $v = m^2$ and $k = 3(m-1)$, then $\mu = 6$ unless $\mu = 4$, $m = 14$ or 352. Furthermore if one makes the rank 3 assumption then if $m > 23$, beside the two exceptional cases, G is the automorphism group of a graph of type $L_3(n)$ with $n = 2^f$.

Let us mention that in these results the condition on the parameters is independent of the rank 3 assumption.

Another result of Higman [35] for graphs with $\lambda = 0$, $\mu = 1$, i.e. with $v = k^2+1$, is that k must be 2, 3, 7, 57. These graphs exist and are rank 3 for $k = 2, 3, 7$. Ashbacher [2] proved that if $k = 57$, there is no such rank 3 graph.

If $\lambda = 0$ and $\mu \neq 2, 4, 6$, there are at most finitely many such graphs for fixed μ (Biggs [6]).

Higman [41] also studied the pseudo-geometric graphs with $T = 1$ and $v < 100$; he proved, among other results, the nonexistence of rank 3 graphs with parameters (76, 21, 2) and (96, 20, 4) and the uniqueness of the rank 3 graph (64, 18, 2).

Wales [81] has shown that the knowledge of $G_{0|\Delta}$ and $G_{0|\Gamma}$ determines uniquely the graph $\mathcal{G}$ as soon as some suborbits are known.

6.4. *Characterization of s.r. graphs by the eigenvalues*

Seidel [66] has determined all s.r. graphs with smallest eigenvalue

$\rho_2 = -3$. They are:

(i) $L_2(n)$,

(ii) $T(n)$,

(iii) the nongeometric graph $v = 16$, $k = 6$, $\lambda = 2$,

(iv) the three Chang graphs [14],

(v) the graphs of Petersen, Clebsch and Schäfli.

It is interesting to note that the last three form a rank 3 tower. Sims [73] proved, using a result of Ray-Chaudhuri [59] on line graphs, that a rank 3 graph with smallest eigenvalue $\rho_2 = -3$ is of type (i), (ii) or (v). Moreover he conjectured that if there exist infinitely many rank 3 graphs with smallest eigenvalue ρ_2, then $\rho_2 = 2q+1$ ($q = p^\alpha$) and with finitely many exceptions they are graphs of type (C.1 and C.11). Another formulation is that the parameter μ is bounded by a function of the smallest eigenvalue. This conjecture is proved using a result of Hoffman.

7. Rank 3 towers

Let $\mathcal{G}$ be a rank 3 graph and $\Delta(p)$, $\Gamma(p)$ the two non-trivial orbits of G_p. It may happen that $G_{p|\Delta}$ is also a rank 3 representation of the same group. This process may occur several times and yield a so-called *rank 3 tower.* The groups involved in the known tower are generally sporadic groups and some other "exceptional" simple group. We give the parameters of the known towers in Table 1. For the first four the reader should refer to Tits [77] for more explanation.

In the last case, it should be noticed that in the first graph the sub-orbit $\Gamma(p)$ is again an orthogonal tower.

8. Strongly regular graph related to Chevalley groups *

1. $PSL_n(q)$ ($A_{n-1}(q)$) acting in the lines of $PG(n-1, q)$, $n \geq 4$, adjacent vertices = intersecting lines.
2. $P\Omega_{2n+1}(q)$ ($B_n(q)$) acting on the points of a quadric in $PG(2n, q)$ adjacent vertices = point on a generator.
3. $PSp_{2n}(q)$ ($C_n(q)$) acting on the points of $PG(2n-1, q)$ with a symplectic polarity; adjacent vertices = conjugate points.

* The full automorphism group of the graph is usually the automorphism group of the defining space except, sometimes, for small values of q and n.

Table 1

	v	k	l	λ	μ	G	G_0
Higman–Sims tower	77	60	16	47	45	M_{22}	2^4A_6
	100	77	22	60	56	HS	M_{22}
McLaughlin tower	162	105	56	72	60	$PSU_4(3)$	$PSL_3(4)$
	275	162	112	105	81	McL	$PSU_4(3)$
Suzuki tower	36	14	21	4	6	$G_2(2) \simeq PSU_3(3^2)$	$PSL_3(2) \simeq PSL_2(7)$
	100	36	63	14	12	HJ	$G_2(2)$
	416	100	315	36	20	$G_2(4)$	HJ
	1782	416	1365	100	96	Suz	$G_2(4)$
Fisher tower	693	180	512	51	45	$PSU_6(2^2)$	$2^9 \cdot PSU_4(2^2)$
	3510	693	2816	180	126	Fi_{22}	$PSU_6(2^2)$
	31 671	3510	28 160	693	351	Fi_{23}	Fi_{22}
	306 936	31 671	275 263	3510	3240	Fi_{24}	Fi_{23}
Conway tower	1408	840	567	488	520	$PSU_6(2^2)$	$PSU_4(3^2)$
	2300	1408	891	840	896	$Co \cdot 2$	$2 \cdot PSU_6(2^2)$
Mathieu tower	1288	792	495	476	504	M_{24}	$2 \cdot M_{12}$
	2048	1288	759	792	840	$2^{11} \cdot M_{24}$	M_{24}
Orthogonal towers	2^{2n}	$2^{2n-1} \mp 2^{n-1}$	$2^{2n-1} \pm 2^{n-1} - 1$	$2^{2n-2} \mp 2^{n-1}$	$2^{2n-2} \mp 2^{n-1}$	$2^{2n} \cdot O^{\pm}_{2n}(2)$	$O^{\pm}_{2n}(2)$
	$2^{2n+1} \pm 2^n - 1$	2^{2n}	$2^{2n} \pm 2^n - 2$	$2^{2n-1} \mp 2^{n-1}$	2^{2n-1}	$O^{\pm}_{2n+2}(2)$	$2^{2n} \cdot O_{2n}(2)$

4⁺. $P\Omega^+_{2n}(q)$ ($D_n(q)$) acting on the points of an hyperbolic quadric of PG($2n-1$, q); adjacent vertices = points on a generator.

4⁻. $P\Omega^-_{2n}(q)$ ($^1D_n(q^2)$) acting on the points of an elliptic quadric of PG($2n-1$, q); adjacent vertices = points on a generator.

5. $PSU_n(q^2)$ ($^1A_{n-1}(q^2)$) acting in the points of an hermitian variety of PG($n-1$, q^2); adjacent vertices = points on a generator.

6. $E_6(q)$ acting on the points of the 26-dimensional projective representation; adjacent vertices = points on a generator.

7. $P\Omega^+_{10}(q)$ acting on one family of isotropic 4-planes of an hyperbolic quadric in PG($9,q$); adjacent vertices = 4-planes intersecting along a 2-plane.

8. $PSU_5(q^2)$ acting on the lines of an hermitian variety of PG($4,q^2$); adjacent vertices = intersecting lines.

Note: the same representation of $PSU_4(q^2)$ on the lines of an hermitian variety is isomorphic to (4⁻) with $n = 3$.

9^±. $P\Omega^\pm_{2n}(2)$ acting on the points of a quadric in PG($2n-1$, 2); adjacent vertices = points on a tangent.

10^±. $P\Omega_{2n+1}(3)$ acting on the points inside (resp. outside) a quadric of PG($2n$, 3); adjacent vertices = points on a nonintersecting line.

Other graphs related to classical groups

11. The stabilizer of a coline in PG(n, q), $n \geq 3$, acting on the nonintersecting lines; adjacent vertices = intersecting lines.

12^±. $V_{2n}(q)\cdot O^\pm_{2n}(q)$ acting on the points of a euclidean space ($V_{2n}(q)$) with a quadratic form; adjacent vertices = points on an isotropic line

13. The subgroup $x' = a^2x + b$ of the affine line over GF(q) with $q = 1$ (mod 4) acting on the points; adjacent vertices = points whose coordinates differ by a square.

Note: 11, 12^±, 13 are rank 3 representations.

14. $PSU_n(q^2)$ ($^1A_{n-1}(q^2)$) acting on the points of an hermitian variety of PG($n-1$, q^2); adjacent vertices = points on a tangent.

15. Automorphism group of AG(nq) acting on the lines; adjacent vertices = intersecting lines (rank 4 representation).

16. Automorphism group of AG($3, q^2$) with a Baer subplane at infinity; adjacent vertices = points on an "isotropic" line (rank 3 representation).

17. Automorphism group of AG($3, 2^r$) with a complete conic at infinity (union of a conic and its knot), acting on the points; adjacent vertices = points on an "isotropic line" (rank 4 representation (except for $r = 2$, rank 3)).

18. Same group acting on the isotropic lines; adjacent vertices = intersecting lines (nontransitive representation except for $r = 2$ rank 4).

19. Subgroup of automorphism of AG($2, q$) preserving m isotropic directions acting on the points; adjacent vertices = points on an isotropic line.

20. Automorphism group of an hermitian parabola P in AG($2, q^2$) (q odd). If the equation of P is $x\bar{x} + (y + \bar{y}) = 0$, two vertices are adjacent iff the corresponding points of coordinates ($x_1 y_1$) and ($x_2 y_2$) satisfy $x_1\bar{x}_2 + y_1 + \bar{y}_2$ is a square (resp. a non-square) in GF(q) when $-1 \notin GF^{\times 2}(q)$ (resp. $-1 \in GF^{\times 2}(q)$).

S. *Strongly regular graphs related to sporadic groups*

1. $PSL_3(4)$ acting on an orbit of 56 complete conics of PG(2,4); adjacent vertices = disjoint conics (see Gewirtz [25], Goethals and Seidel [26] and Montague [57]). Rank 3 representation of $PSL_3(4)$ over Alt(6).

2. M_{22} acting on the 77 blocks of S(3, 6, 22); adjacent vertices = disjoint blocks. Rank 3 representation of M_{22} over $2^4\cdot$Alt(6).

3. $PSU_3(5^2)$ acting over subsets of autoconjugate triangles in PG($2, 5^2$) with an hermitian conic. A simple construction is obtained in the following way. Let $p = A_7$, $\Delta(p)$ be the set of 7 subgroups of A_7 isomorphic to A_6 (fixing one-letter). Let $\Gamma(p)$ be the set of 7×6 subgroups of each A_6 isomorphic to A_5 but transitive on the 6 letters. Two points of Γ are joined if the two A_5 intersect in D_5. See Hoffman and Singleton [48], Benson and Losey [5] and Schult [64]. Rank 3 representation of $PSU_3(5^2)$ over Alt(7).

4. $PSL_3(4)$ acting on the 105 flags of PG(2, 4). Two vertices are adjacent if the corresponding flags have distinct centers and axis and if the center of one belongs to the axis of the other (see also Seidel [67] and Goethals and Seidel [26]). Rank 6 representation.

5. $PSL_3(4)$ acting on an orbit of 120 Baer subplanes of PG(2, 4); adjacent vertices = planes intersecting in a single point (see Goethals and Seidel [26]). Rank 5 representation.

6. M_{23} acting on the 253 blocks of S(4, 7, 23); adjacent vertices = blocks intersecting in a single point. Rank 3 representation of M_{23} over $2^4\cdot$Alt(7).

7. M_{22} acting on the 176 blocks of M_{23} avoiding one point; adjacent

vertices = blocks intersecting in a single point [26]. Rank 3 representation of M_{22} over Alt(7).

8. $PSU_3(5^2)$ acting on the 175 edges of the Hoffman–Singleton graph (S.3). Another description may be given using graph 7; in this graph, given a point p, take the subgraph $\Delta(p) \cup \Gamma(p)$ and switch with respect to (Δ, Γ). Rank 4 representation of $PSU_3(5^2)$ over $2 \cdot Alt(6)$.
9. Rank 3 representation of $G_2(2)$ on 36 points [68].
10. Rank 3 representation of HS in 100 points [42, 78, 26].
11. Rank 3 representation of HJ in 100 points [30, 77, 78].
12. Rank 3 representation of $PSU_4(3)$ on 162 points [78].
13. Rank 3 representation of McL on 275 points. A simple description may be given using graph 6. Given a point of S(4, 7, 23) the 253 blocks fall into two classes $\mathcal{H}_1 \mathcal{H}_2$, the blocks through the point (77) and the others (176). Now switch graph 6 with respect to $(\mathcal{H}_1 \mathcal{H}_2)$, take $\bar{\mathcal{H}}_1$, and add the 22 points of S(4, 7, 23) with the following adjacencies: a point is adjacent to each nonincident block of $\mathcal{H}_1$, and to every incident block of $\mathcal{H}_2$ (see Conway [17] and also [55, 78]).
14. Rank 3 representation of $G_2(4)$ on 416 vertices [78].
15. Rank 3 representation of Suz on 1782 vertices [78, 79, 81].
16. Rank 3 representation of Fi_{22} on 3510 vertices [78].
17. Rank 3 representation of Fi_{23} on 31671 vertices [78].
18. Rank 3 representation of Fi_{24} on 306,936 vertices [78].
19. Rank 3 representation of $2^{11} \cdot M_{24}$ on 2048 vertices. This is related to Golay code; see [26].
20. Rank 3 representation of M_{24} over $2 \cdot M_{12}$. The 1288 vertices of the graph are the 1288 partitions of the 24 points of S(5, 8, 24) in two subsets of length 12 on which M_{12} acts in nonequivalent ways. Two vertices are adjacent if the 2 pairs of dodecads intersect in (4, 8, 4, 8) points.
21. Rank 3 representation of $Co \cdot 2$ over $2 \cdot PSU_6(2)$. In the Leech lattice take two points at distance $4\sqrt{2}$ and the two spheres, centered at these points, of some radius. $Co \cdot 2$ acts on the 2300 pairs of opposite points of the lattice which belong to the intersection sphere (see [17]).
22. Rank 3 representation of $PSU_6(2)$ over $PSU_4(3)$. Again in the Leech lattice, take a triangle of type 222; the stabilizer of the vertices of the triangle acts on the 408 points which complete the triangle in a tetrahedron of type 222222.
23. Rank 3 representation of Rudvalis group on $'F_4(2)$ (see [18]).

9. Combinatorial strongly regular graphs

The combinatorial point of view leads also to some interesting classes of s.r. graphs. Moreover some graphs are characterized only by combinatorial relations. We have already mentioned the result of Seidel [66] about graphs having −3 as smallest eigenvalue. In his proof he uses the previous results of Chang [14] (also a result of Connor [16]), of Hoffman [46] concerning the triangular association schemes $T(m)$ and the result of Shrikhande [69] on the L_2-association schemes $L_2(n)$. In fact the parameters determine the two classes of graphs with 2 exceptions. For $v = 28$, $k = 12$, $\lambda = 6$, beside $T(8)$ there exist 3 other nonisomorphic graphs [15] obtained from $T(8)$ by the switching process [65]. For $v = 16$, $k = 6$, $\lambda = 2$ there exists a nongeometric graph with those parameters. In the same direction Bussemaker and Seidel [11] have proved the existence of more than 80 nonisomorphic graphs $L_2(6)$, and more than 23 graphs of type $NL_2(6)$.

Other classes of s.r. graphs have been constructed by various means.

Mesner [56], using a result of Ray-Chaudhuri [60], constructed two classes (in fact only one) of s.r. graphs of negative Latin square type (i.e., having the same parameters as $L_{-g}(n)$).

Bose and Shrikhande [10] have constructed a large number of s.r. graphs with $\lambda = \mu$ of type $L_2(2r)$, $NL_2(2r)$ and with parameters $(vk\lambda) = (4r^2 - 1, 2r^2, r^2)$. Also Wallis [83, 84] has studied graphs with $\lambda = \mu$ and constructed other types of graphs using affine resolvable designs; he also showed the existence of at least 2 nonisomorphic graphs with parameters $(n^2(n+2), n(n+1), n)$ for $n = p^\alpha$.

These graphs were first constructed by Hall [29] when $p = 2$ and Ahrens and Szekeres [1] in the general case. The construction of Ahrens and Szekeres yields another class of s.r. graphs (K.3). If $p \neq 2$, this construction is closely related to the construction of a graph on an hermitian parabola in an affine plane.

Other constructions by means of Hadamard matrices may be found in Hall [28] and in Goethals and Seidel [26]. In the last paper some other interesting graphs, related to quasi-symmetric designs (designs which have only two types of intersection) are also constructed. In particular there is a very neat study of the S(5, 8, 24) and its derived graphs. Let us mention that the conjecture about the existence of a s.r. graph with $v = 1288$ (see [26, p. 613]) is true (see rank 3 towers).

K. *Combinatorial rank 3 graphs*

1. Square lattice graphs $L_g(n)$. Given $g-2$ orthogonal lattice squares of order n, the graph is constructed on the n^2 cells of the square. Two vertices are adjacent if the cells are in the same row, or column, or if they contain the same letter.
2. Triangular graphs. They correspond to a rank 3 representation of Sym(n) acting on the pairs. Two vertices are adjacent if the pairs contain a common index (see also C.1).
3. Graph of a partial geometry of type $(\lambda+2, \lambda, 1)$, $\lambda = p^\alpha$ (see Ahrens and Szekeres [11]).
4. Line graph of the same geometry.
5. Negative square lattice graphs (see [56,10]).
6. Line graph of an $S(2, \tilde{k}, \tilde{v})$; adjacent vertices = intersecting blocks.

Appendix

	v	k	l	λ	μ	ρ_1	ρ_2
S.1.	56	10	45	0	2	7	−5
S.2.	77	16	60	0	4	11	−5
S.3.	50	7	42	0	1	5	−5
S.4.	105	32	72	4	12	19	−5
S.5.	120	42	77	8	18	23	−5
S.6.	253	112	140	36	60	51	−5
S.7.	176	70	105	18	34	35	−5
S.8.	175	72	102	20	36	35	−5
S.9.	36	14	21	4	6	7	−5
S.10.	100	22	77	0	6	15	−5
S.11.	100	36	63	14	12	7	−13
S.12.	162	56	105	10	24	31	−5
S.13.	275	112	162	30	56	55	−5
S.14.	416	100	315	36	20	7	−41
S.15.	1782	416	1365	100	96	31	−41
S.16.	3510	693	2816	180	126	17	−127
S.17.	31 671	3510	28 160	693	351	17	−703
S.18.	306 936	31 671	275 264	3510	3240	161	−703
S.19.	2048	759	1288	310	264	17	−111
S.20.	1288	495	792	206	180	17	−71
S.21.	2300	891	1408	378	324	17	−127
S.22.	1408	567	840	246	216	17	−79
S.23.	4060	1755	2304	730	780	129	−31

	v	k	l
K.1.	n^2	$g(n-1)$	$(n-g+1)(n-1)$
K.2.	$\binom{n}{2}$	$2(n-2)$	$\binom{n-2}{2}$
K.3.	q^3	$(q-1)(q+2)$	$(q-1)^2(q+1)$
K.4.	$q^2(q+2)$	$q(q+1)$	$(q+1)^2(q-1)$
K.5.	n^2	$g(n+1)$	$(n-g-1)(n+1)$
K.6.	$\frac{\tilde{v}(\tilde{v}-1)}{\tilde{k}(\tilde{k}-1)}$	$\tilde{k}\frac{\tilde{v}-\tilde{k}}{\tilde{k}-1}$	$\frac{(\tilde{v}-\tilde{k})(\tilde{v}-\tilde{k}^2+\tilde{k}-1)}{\tilde{k}(\tilde{k}-1)}$

	λ	μ	$\{\rho_1, \rho_2\}$
K.1.	$(g-1)(g-2)+n-2$	$g(g-1)$	$2g-1, 2g-2n-1$
K.2.	$n-2$	4	$2n-7, -3$
K.3.	$q-2$	$q+2$	$2q+3, 3-2q$
K.4.	q	q	$2q-1, -2q-1$
K.5.	$(g+1)(g+2)-n-2$	$g(g+1)$	$-2g+1, -2g+2n-1$
K.6.	$(\tilde{k}-1)^2+\frac{\tilde{v}-1}{\tilde{k}-1}-2$	$\tilde{k}^2$	$2\tilde{k}-1, 2\tilde{k}-1-2\frac{\tilde{v}-\tilde{k}}{\tilde{k}-1}$

	v	k	l
C.1.	$\frac{(q^{n+1}-1)(q^n-1)}{(q+1)(q-1)^2}$	$\frac{q(q+1)(q^{n-1}-1)}{q-1}$	$\frac{q^4(q^{n-1}-1)(q^{n-2}-1)}{(q+1)(q-1)^2}$
C.2.3.	$\frac{q^{2n}-1}{q-1}$	$q\frac{q^{2n-2}-1}{q-1}$	q^{2n-1}
C.4$^\pm$.	$\frac{(q^n\mp 1)(q^{n-1}\pm 1)}{q-1}$	$\frac{q(q^{n-1}\mp 1)(q^{n-2}\pm 1)}{q-1}$	q^{2n-2}
C.5.*	$\frac{(q^{n-1}\mp 1)(q^n\pm 1)}{q^2-1}$	$\frac{q^2(q^{n-3}\mp 1)(q^{n-2}\pm 1)}{q^2-1}$	q^{2n-3}
C.6.	$\frac{(q^{12}-1)(q^9-1)}{(q^4-1)(q-1)}$	$\frac{q(q^8-1)(q^3+1)}{q-1}$	$\frac{q^8(q^5-1)(q^4+1)}{q-1}$
C.7.	$\frac{(q^8-1)(q^3+1)}{q-1}$	$\frac{q(q^5-1)(q^2+1)}{q-1}$	$\frac{q^6(q^5-1)}{q-1}$

* Upper sign if n is odd, lower sign if n is even.

	v	k	l
C.8.	$(q^5+1)(q^3+1)$	$q^3(q^2+1)$	q^8
C.9.$^{\pm}$	$2^{2n-1} \pm 2^{n-1}$	$2^{2n-2}-1$	$2^{2n-2} \pm 2^{n-1}$
C.10.	$\frac{3^n(3^n \pm 1)}{2}$	$\frac{3^{n-1}(3^n \mp 1)}{2}$	$3^{2n-1} \pm 3^{n-1} \cdot 2 - 1$
C.11.	q^{2n-2}	$(q+1)(q^{n-1}-1)$	$q(q^{n-1}-1)(q^{n-2}-1)$
C.12$^{\pm}$.	q^{2n}	$(q^n \mp 1)(q^{n-1} \pm 1)$	$q^{n-1}(q-1)(q^n \mp 1)$
C.13.	$4\alpha+1=q$	2α	2α
C.14.*	$\frac{q^{n-1}(q^n \pm 1)}{q+1}$	$(q^{n-1} \mp 1)(q^{n-2} \pm 1)$	$\frac{q^{n-2}(q^{n-1} \mp 1)(q^2-q-1)}{q+1}$
C.15.	$\frac{q^{n-1}(q^n-1)}{q-1}$	$\frac{q^2(q^{n-1}-1)}{q-1}$	$\frac{q(q^n-1)(q^{n-2}-1)}{q-1}+q-1$
C.16.	q^6	$(q+1)(q^3-1)$	$q(q^2-1)(q^3-1)$
C.17.	2^{3r}	$2^{2r}+2^r-2$	$(2^{2r}-1)(2^r-1)$
C.18.	$2^{2r}(2^r+2)$	$2^r(2^r+1)$	$(2^{2r}-1)(2^r+1)$
C.19.	q^2	$m(q-1)$	$(q+1-m)(q-1)$
C.20.	q^3	$\frac{(q-1)(q^2+1)}{2}$	$\frac{(q+1)(q^2-1)}{2}$

	λ	μ	$\{\rho_1, \rho_2\}$
C.1.	$\frac{q^n-1}{q-1}+q^2-2$	$(q+1)^2$	$2q+1, -2\frac{q^{n-1}-1}{q-1}+2q+3$
C.2.3.	$\frac{q^{2n-2}-1}{q-1}-2$	$\frac{q^{2n-2}-1}{q-1}$	$2q^{n-1}+1, -2q^{n-1}+1$
C.4.$^{\pm}$	$\frac{q(q^{n-2} \mp 1)(q^{n-2} \pm q)}{q-1}+q-1$	$\frac{(q^{n-1} \mp 1)(q^{n-2} \pm 1)}{q-1}$	$\pm 2q^{n-2}+1, \mp 2q^{n-1}+1$
C.5.	$\frac{q^3(q^{n-4} \pm 1)(q^{n-4} \mp q)}{q^2-1}+q^2-1$	$\frac{(q^{n-3} \mp 1)(q^{n-2} \pm 1)}{q^2-1}$	$\pm 2q^{n-2}+1, \mp 2q^{n-3}+1$
C.6.	$\frac{q^2(q^2+1)(q^5-1)}{q-1}+q-1$	$\frac{(q^4-1)(q^3+1)}{q-1}$	$2q^3+1, -2\frac{q^4(q^5-1)}{q-1}+1$

* Upper sign if n is odd, lower sign if n is even.

	λ	μ	$\{\rho_1, \rho_2\}$
C.7.	$\frac{q^2(q^3-1)(q+1)}{q-1}+q-1$	$\frac{(q^3-1)(q^2+1)}{q-1}$	$2q^2+1, \frac{-2q^3(q^3-1)}{q-1}+1$
C.8.	q^3-1	q^2+1	$2q^2+1, -2q^3+1$
C.9.$^{\pm}$	$2^{2n-3}-2$	$2^{2n-3} \mp 2^{n-2}$	$1 \pm 2^{n-1}, -1 \mp 2^n$
C.10.	$\frac{3^{n-1}(3^{n-1} \mp 1)}{2}$	$\frac{3^{n-1}(3^{n-1} \mp 1)}{2}$	$3^{n-1} \cdot 2-1, -3^{n-1} \cdot 2-1$
C.11.	$q^{n-1}+q^2-q-2$	$q(q+1)$	$2q+1, -2q^{n-1}+2q+1$
C.12.$^{\pm}$	$q(q^{n-1} \mp 1)(q^{n-2} \pm 1)+q-2$	$q^{n-1}(q^{n-1} \pm 1)$	$\pm 2q^{n-1}+1, \pm 2q^{n-1} \mp 2q^n+1$
C.13.	$\alpha-1$	α	$\sqrt{q}, -\sqrt{q}$
C.14.*	$q^{2n-5}(q+1) \pm q^{n-2}(q-1)-q$	$q^{n-3}(q+1)(q^{n-2} \pm 1)$	$\pm q^{n-2}+1, \mp 2q^{n-3}(q^2-q-1)+1$
C.15.	$\frac{q^n-1}{q-1}+q^2-2q-1$	q^2	$2q-1, -2\frac{q^n-1}{q-1}+2q+1$
C.16.	q^3+q^2-q-2	$q(q+1)$	$2q+1, -2q^3+2q+1$
C.17.	2^r-2	2^r+2	$2^{r-1}+3, -2^{r-1}+3$
C.18.	2^r	2^r	$2^{r-1}-1, -2^{r-1}-1$
C.19.	$(m-1)(m-2)+q-2$	$m(m-1)$	$2m-1, 2m-2q-1$
C.20.	$\frac{(q-1)^3}{4}-1$	$\frac{(q-1)(q^2+1)}{4}$	$q^2, -q$

* Upper sign if n is odd, lower sign if n is even.

Notes added in proof

Since 1973, new results have been obtained. We would like to include most of them.

2. Partial geometries. J. Thas (Simon Stevin 46 (1973) 95–98) has constructed partial geometries which are never balanced nor meshed.

6.2. Characterization by the constituents. The author has determined extensions of Sym(n) or Alt(n) acting on the $\binom{n}{2}$ unordered pairs of elements of a set of cardinality n; the only exceptional rank 3 extensions occur for $n = 5$ and 8. In these cases $(v, k, \lambda) = (16, 10, 5)$ and (64, 28, 35); otherwise one gets a triangular graph $T(n+2)$.

P.J. Cameron (Proc. London Math. Soc. 25 (1972) 427–440) has shown that if $G_{01\Delta}$ is t-transitive ($t \geq 3$), then one of the following holds: (i) $|\Gamma| = \frac{1}{2}(k(k-1))$, (ii) $|\Gamma| = k(k-1)$, or (iii) $v = (\alpha+1)^2(\alpha+4)^2$, $k = (\alpha+1)(\alpha^2+5\alpha+5)$, $\mu = (\alpha+1)(\alpha+2)$ (the only known example of (iii) is the HiS graph S.10).

F. Buekenhout and the author have determined all extensions when $G_{01\Delta}$ contains a classical group PSp, PΩ or PSU acting in its natural rank 3 representation. The rank 3 extensions are: (i) C.12$^{\pm}$, extension of C.4$^{\pm}$ when $q = 2$; (ii) C$q^{\pm}$, extension of C.2 with $q = 2$; (iii) $(v, k, \lambda) =$ (35, 16, 6), (176, 40, 12), (275, 112, 30), (126, 25, 8), (126, 45, 12) and (3510, 693, 180) whose automorphism groups contain Alt(8), $PSU_5(4)$, McL, Alt(10), $P\Omega_6^-(3)$ and Fi_{22} (to appear).

6.3. Characterization by subdegrees. If $\mathcal{G}$ is a rank 3 graph with

$$(v, k, l) = \left(\frac{q^r - 1}{q - 1},\ q\frac{q^{r-2} - 1}{q - 1},\ q^{r-1}\right),$$

then under some assumptions, $\mathcal{G}$ is a graph of type C.3. The strongest result has been obtained by W.M. Kantor (Rank 3 characterization of classical geometries) when $r \geq 6$. Other results with more hypotheses may be found in [34] and in a paper by A. Yanushka (a characterization of the symplectic groups PS$p(2m, q)$ as rank 3 permutation groups).

Concerning rank 3 extensions, let us mention a result of M.S. Smith (Bull. London Math. Soc. 6 (1974) 1–3); given a rank 3 graph $\mathcal{G}$, under assumptions on the characters of the representation, there are at most 4 rank 3 extensions $\mathcal{G}'$ of $\mathcal{G}$. Combinatorially, the hypothesis is equivalent to the requirement that $\mathcal{G}'$ and $\mathcal{G}$ have a common eigenvalue.

7. Rank 3 towers. Two infinite towers occur as representations of orthogonal groups over GF(3). The groups are $P\Omega_4^{\pm}(3) \subset P\Omega_5(3) \subset P\Omega_6^{\mp}(3) \subset P\Omega_7(3) \ldots$ in the representations C.10′ and complementary to C.10$^{\pm}$ (for C.10′ see below).

Unitary groups over GF(4) also yield an infinite tower; the graphs are the complements of the graphs C.14, with $q = 2$.

8. S.R. graphs. Let us add the following ones: C.10′$^{\pm}$. $P\Omega_{2n}^{\pm}(3)$ acting on a set of imprimitivity of the points off a quadric in PG$(2n-1, 3)$; adjacent vertices = points on a non-intersecting line,

$$v = 3^{n-1}\frac{3^n \mp 1}{2},\quad k = 3^{n-1}\frac{3^{n-1} \mp 1}{2},\quad l = 3^{2n-2} - 1,$$

$$\lambda = 3^{n-2}\frac{3^{n-1} \pm 1}{2},\quad \mu = 3^{n-1}.$$

C.11, $V_{10}(q) \cdot PSL(5, q)$ acting on the skew-symmetric tensors; adjacent vertices = tensors whose difference is a bivector.

$$v = q^{10},\quad k = (q^5 - 1)(q^2 + 1),$$

$$l = q^2(q^3 - 1)(q^5 - 1),\quad \lambda = q(q+1)(q^3 - 2) + q - 2,$$

$$\mu = q^2(q^2 + 1);$$

P.J. Cameron (communicated by M.S. Smith).

S.24. $2^{11}M_{24}$ acting on M_{24}. The vertices of Δ are the 276 unordered pairs of points of the S(5, 8, 24); the vertices of Γ are the 1771 decompositions into 6 4-tuples. A pair is adjacent to a set of 6 4-tuples iff it is contained in one of the 4-tuples. Given one set of 6 4-tuples, there are 240 other sets in which one of the 4-tuples has an intersection (3, 1) with 2 4-tuples of the other set.

$$v = 2048,\quad k = 276,\quad l = 1771,\quad \lambda = 44,\quad \mu = 36,$$

$$\rho_1 = -41,\quad \rho_2 = 23$$

(J.H. Conway and M.S. Smith).

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