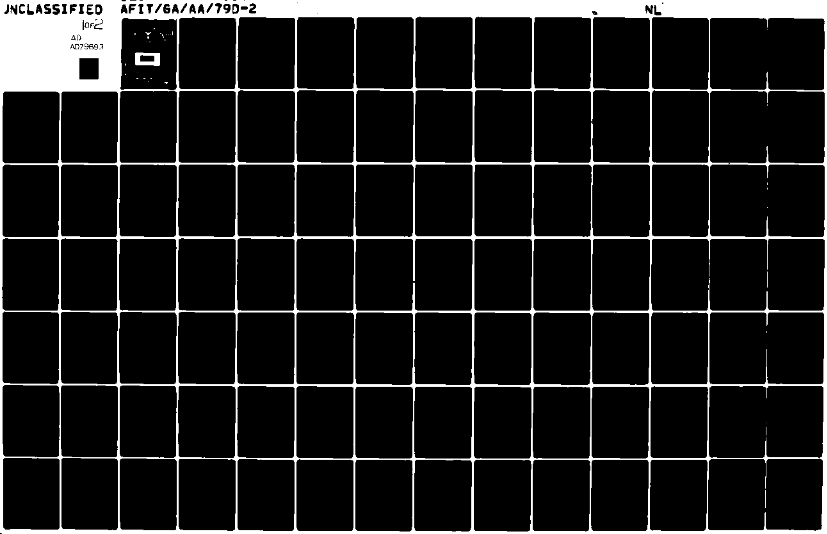
AD-A079 683 AIR FORCE INST OF TECH WRIGHT-PATTERSON AFB OH SCHOO--ETC F/G 22/3
APPLICATION AND COMPARISON OF STABLE PERIODIC ORBITS IN THE VIC--ETC(U)
DEC 79 W D BEEKMAN
UNCLASSIFIED AFIT/GA/AA/79D-2 NL

1 OF 2
AD
A079683



AD A0 76693

LEVEL

AIR FORCE INSTITUTE OF TECHNOLOGY

AIR UNIVERSITY
UNITED STATES AIR FORCE

SCHOOL OF ENGINEERING

WRIGHT-PATTERSON AIR FORCE BASE, OHIO

80 1 25

AFIT/GA/AA/79D-2

1

APPLICATION AND COMPARISON OF STABLE

PERIODIC ORBITS IN THE VICINITY OF

LAGRANGIAN POINTS L4 AND L5 TO

A FOUR-BODY TRUTH MODEL

THESIS

AFIT/GA/AA/79D-2

WILLIAM D. BEEKMAN
CAPTAIN USAF

Approved For Public Release; Distribution Unlimited

14

AFIT/GA/AA/79D-2

6

APPLICATION AND COMPARISON OF STABLE PERIODIC ORBITS IN THE VICINITY OF LAGRANGIAN POINTS L4 AND L5 TO A FOUR-BODY TRUTH MODEL

THESIS

PRESENTED TO THE FACULTY OF THE SCHOOL OF ENGINEERING

OF THE AIR FORCE INSTITUTE OF TECHNOLOGY

AIR UNIVERSITY (ATC)

IN PARTIAL FULFILLMENT OF THE

REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

by

10 William D. Beekman, B.S., M.A.

Captain USAF

Graduate Astronautics

11 Dec[redacted]79

12 127

Approved for public release; distribution unlimited

012 225

Contents

LIST OF FIGURES

Figure		Page

Figure		Page

LIST OF TABLES

LIST OF SYMBOLS

$\underline{a}_1$, $\underline{a}_2$	inertial coordinate frame axes of Earth - centered truth model
a_s	semimajor axis of the Sun
α	angle of rotation of Earth - Moon barycenter about the Sun - Earth - Moon barycenter
$\dot{\alpha}$	rotation rate of B_{em} about B_{sem}
B_{em}	Earth - Moon barycenter
B_{sem}	Sun - Earth - Moon barycenter
$\underline{e}_1$, $\underline{e}_2$	rotating coordinate frame axes about Earth - Moon barycenter of Wheeler system
e_s	eccentricity of the Sun
f_s	true anomoly of the Sun
G	gravitational constant
L4, L5	stable equilibrium libration points of the restricted three-body problem
m_c	mass of the satellite, negligible
m_e	mass of the Earth
m_m	mass of the Moon
m_s	mass of the Sun
μ	m_m, mass of the Moon
$\mu_c{}^2$	geocentric gravitational constant for the satellite
$\mu_m{}^2$	geocentric gravitational constant for the Moon
n_m	mean motion of the Moon
n_s	mean motion of the Sun

$\underline{r}_o$	position vector in a fixed reference ellipse
r_c	Earth - satellite distance
r_{cm}	satellite - Moon distance
r_{cs}	satellite - Sun distance
r_m	Earth - Moon distance
r_{ms}	Moon - Sun distance
rp_s	distance from the Sun to the Earth - Moon barycenter
r_s	Earth - Sun distance
t	time
θ	angle of rotation of Wheeler axis about the Earth - Moon barycenter
$\dot{\theta}$	rotation rate of the Moon about the Earth
v	potential energy
$\underline{\omega}$	velocity vector in a fixed reference ellipse
x_c, y_c	satellite position coordinates of truth model
$\dot{x}_c$, $\dot{y}_c$	satellite velocities of truth model
$\ddot{x}_c$, $\ddot{y}_c$	satellite accelerations of the truth model
x_m, y_m	Moon position coordinates of truth model
$\dot{x}_m$, $\dot{y}_m$	Moon velocities of truth model
$\ddot{x}_m$, $\ddot{y}_m$	Moon accelerations of truth model
x_s, y_s	Sun position coordinates of truth model
$\dot{x}_s$, $\dot{y}_s$	Sun velocities at truth model
$\ddot{x}_s$, $\ddot{y}_s$	Sun accelerations of truth model

ξ, η	satellite position coordinates of B, M & S model
$\dot{\xi}, \dot{\eta}$	satellite velocities of B, M & S model

Abstract

In this study, all previously discovered stable periodic orbits about the triangular libration points are tested on a planar restricted four-body truth model. The truth model is an algorithm developed from equations used by T.A. Heppenheimer for colony location used perturbation theory and the equation of the center. Only one of the orbits, developed by Wheeler using a very restricted four-body problem with Sun - Moon - Earth circular motion, is found to be relatively stable for at least twenty years. It is prograde about L4 having a period in resonance with the lunar synodic month. Two other orbits, one similar to Wheeler's and one 180° out of phase found by Kolenkiewicz and Carpenter are marginally stable.

ACKNOWLEDGMENT

I would like to express my deep gratitude and appreciation to my advisor, Professor William E. Wiesel, for his inspirational guidance, continual availability, and superior knowledge in this area. To Professors Robert Calico and James Rader I extend my appreciation for taking their valuable time to participate as readers. I would also like to extend a special thanks to my typist, Mary Ballmann, whose proficiency and skill were a great asset in bringing this paper to final form. Finally, I would like to express a deep thanks to my wife, Kathy, for her understanding and inspiration throughout this project.

APPLICATION AND COMPARISON OF STABLE PERIODIC ORBITS IN THE VICINITY OF LAGRANGIAN POINTS L4 AND L5 TO A FOUR-BODY TRUTH MODEL

I. INTRODUCTION

Background

Foremost space scientists have been looking in recent years for existence of stable periodic orbits in which to place long-term satellites for weather, observation, colonization, and protection. The subjects of resonance and stability are closely related to the problem of evolution of the solar system. It is a physically involving problem and the methods available to mathematics today seem unsatisfactory to produce pure nonlinear ways of attack. The linearization process in both subjects is clearly of doubtful significance, so that, even if very restrictive, numerical solutions are still the best and more valuable sources of information. It is quite possible that we know now very little more of the entire problem that was known to Poincare', with the advantage that we can now compute much faster and with much more precision.

Hardware developments like the Space Shuttle and Inertial Upper Stage bring the day closer when we will be able to use these orbits. Studies have revealed a number of stable periodic orbits which may exist to replace the unstable near-earth synchronous orbits and give advantages such as a continuous view of the North Pole, which are unobtainable today. The area of of interest for these types of orbits is cislunar space, or the area of space where satellite motion is affected by the gravitational fields of the Sun, Earth and Moon and any other forces caused by the gravitational attraction of Jupiter, Earth oblateness, etc. are negligible. Many orbits have been found in cislunar space but their periodicity and stability in most cases have been determined using three-body or restricted four-body equations rather than a truth model.

Studies in the area of orbital mechanics near L4 and L5 have increased in complexity over the years. Let M_1, M_2 and M_3 denote three point masses such that $M_1 > M_2 > M_3$. The masses move under the influence of their mutual gravitational attractions; the force between any two masses is inversely proportional to the square of their distance and proportional to the product of their masses. It is well known (Ref 27) that there are in this "three body problem" five exact solutions in which the three masses maintain a constant configuration which revolves with constant angular velocity. An important specialization of the three body problem is the restricted three body problem in which M_3 is infinitesimal and M_1 and M_2 move in circular orbits around their barycenter. The smallness of M_3 means that it does not influence the motion of M_1 and M_2. For many purposes it is convenient to describe the motion of M_3 in a coordinate system which is attached to M_1 and M_2. In this rotating coordinate system the five Lagrange solutions show up as five fixed points at which M_3 would be stationary if placed there with zero velocity (i.e., zero velocity in the rotating coordinates). If is further known that, in this rotating coordinate system, M_3 may describe small periodic orbits about the Lagrange solutions. Glyden therefore called the points which correspond to the Lagrange solutions "centres of libration"; they are also often referred to as "libration points" or "Lagrange points."

The libration points are singular points of the differential equations of motion in the restricted problem of three bodies, they are also equilibrium points since the gravitational forces on a mass placed in such a point are balanced by the centrifugal force. Three libration points, the collinear points, are found on the line connecting the two large masses; the other two, the triangular points, form equilateral triangles with the two large masses. By linearizing the equations of motion Charlier (Ref 9) showed that there are two classes of periodic infinitesimal orbits around the triangular libration points, namely those with short period (period very nearly equal to that of the period of the two large masses) and those of long period (the period depending on the mass ratio of the large masses). Each of these classes consists of concentric, coaxial and similar ellipses with semi-major and minor axes in the ratio 2:1 for the short period orbits and a larger ratio, again depending on the mass ratio, for the long period orbits. Plummer in 1911 (Ref 31) considered Charlier's problem in a more general format and from his results some additional conclusions can be drawn. For a mass ratio of the two large masses smaller than 1/27, both classes of orbits around the triangular points can be expressed with trigonometric functions; these points are therefore called stable libration points.

Furthermore, only one of the classes of orbits around the collinear libration points can be expressed in trigonometric functions, the other requiring hyperbolic functions; the collinear points are therefore called unstable libration points.

The discovery in 1906 of the first of a group of asteroids which appear to oscillate (or, in astronomical terms, librate) around the Sun - Jupiter triangular libration points, gave further impetus to the study of these motions. This first discovery was called Achilles and since subsequent discoveries were also called after heroes from the Trojan group. Brown in 1911 (Ref 4) considered the long period orbits around the triangular libration points by supposing finite amplitudes of libration and discussed in some detail the dependence of period and orbit shape on amplitude. In another paper (Ref 5) he discussed libration orbits for mass ratios greater than 1/27. Willard in 1913 (Ref 54) discussed the short period orbits, again of finite amplitude and computed a number of possible orbits. Whereas all this work was based on the restricted problem of three bodies, with the discovery of more Trojans additional theories emerged which attempted to take into consideration the actual physical circumstances. Among the first contributions were those by Linders in 1908 (Ref 28) and Smart in 1918 (Ref 39) and in 1923 (Ref 6) Brown published the explanation for his theory which was accurate enough to compute the position of a Trojan asteroid within a few seconds of arc. This theory was applied numerically to Achilles by Brouwer in 1933 (Ref 2) and to Hector, which has a particularly large libration amplitude, by

Eckert in 1933 (Ref 17). Since this theory was numerical it had to be set up separately for each asteroid. A group theory was outlined by Brown and Shook in 1933 (Ref 8) in which the interesting direct and indirect effects by Saturn were also discussed, Herz in 1943 (Ref 21) carried out some of the details of Brown and Shook's plan. Further work concerning the motion of the Trojans was accomplished by Wilkens in 1917 (Ref 50), 1918 (Ref 51), 1926 (Ref 52) and 1932 (Ref 53).

Thüring in 1930 and 1931 (Ref 46) considered again the problem of the long period motion, in particular the dependence of the period on amplitude. His subsequent contributions in 1950 (Ref 47) were largely based on numerical work and his 1959 paper (Ref 48) was of particular interest because of the application of an electronic digital computer. Thüring's claim of the non-existence of long period orbits through any orbitrary point was refuted by Rabe in 1961 (Ref 32) who made a survey of numerically computed long period libration orbits, expressed in Fourier Series expansions. Rabe also discussed some aspects of the stability of such periodic orbits and extended these studies and his survey in 1962 (Ref 33); similar work in the same year (Ref 34) was done on libration orbits for the Earth - Moon system. Rabe has developed the idea that such periodic orbits should be used as intermediate orbits for computation of real, nonperiodic orbits. Stumpff in 1963 (Ref 41) reconsidered and refined Thüring's theory, in particular with respect to the relations between long period orbits with very large amplitudes around the triangular libration points and nonperiodic orbits in the neighborhood of the collinear libration points.

The study of libration points in the Earth - Moon system was initiated by Klemperer and Benedikt in 1958 (Ref 25). They argued that in analogy with the Trojan asteroids there are to be found in the combined gravitational field of the Earth and the Moon, two areas in which natural or artificial bodies would move, while maintaining a more or less constant configuration with the Earth and the Moon. Again, as was the case with the Trojans, a natural discovery of such a "cloud" near L4 (the libration point 60° ahead of the Moon) was also reported. But since then, the discovery has been refuted although dust particles may remain in the area temporarily according to Roosen (Ref 35) and Wolff (Ref 56) in 1967.

More recently, interest has been shown in the problem of the influence of the sun on motion close to the libration points of the Earth - Moon system as well as motion about the Earth and the Moon itself. One possible model for the Earth - Moon - Sun system was proposed by Su-Shu Huang in 1960 (Ref 22), who called it the "very restricted four-body problem." Here the Earth and Moon describe circular orbits relative to one another, and their center of mass describes a circular orbit around the Sun; all these orbits are Keplerian, lie in a plane, and no perturbations are considered. Using this model Huang studied the motion of a fourth body of an infinitesimal mass in a similar manner as in the restricted four-body problem. He concludes this model gives a general idea of where the fourth body could or could not go under given initial conditions when they are no longer very near the Earth. Columbo in 1961 (Ref 10) considered the motion near L4 and L5 under the influence of the Sun, and the possibility of stabilizing it with a solar sail; in another paper in 1962 (Ref 11) he gave a numerical analysis of the influence of the Moon orbit's eccentricity.

Ellis and Diana in 1960 (Ref 18) on a parellel tack presented some numerically computed libration orbits in the restricted problem. This was extended by deVries and Pauson in 1962 (Ref 14) by adding linearized equations of motion relative to a stable libration point in the restricted problem the principle effects of a fourth body representing the Sun as it is related to the Earth and Moon. Two linear, second order differential equations with time varying coefficients, were obtained which could be solved in powers of the small parameter (mass of Sun divided by the cube of the Earth - Sun distance). The first order solution and the most significant parts of the second order solution were obtained and for a number of different initial conditions this presented a reasonably close agreement with numerically integrated orbits. It appeared to the authors that any so called "stability" was strongly influenced by the Sun but it also appeared possible to choose the initial configuration of Earth - Moon - Sun and initial conditions of the small particle such that this influence was small enough for a usefully long "libration life" to be possible. In a subsequent paper by deVries in 1962 (Ref 15) the influence of the Moon's eccentricity was discussed and it was found that, if the Sun was introduced in the consideration of motion near Earth - Moon libration points, the Moon's eccentricity would have to be considered also. This was to play an important part in the four-body model used in this thesis. Michael in 1963 (Ref 29) discussed orbit envelopes which depend on initial conditions, based on a linearized analysis of the restricted problem.

Using Huang's four-body model Cronin in 1964 (Ref 12) proved that under certain conditions the fourth body has a periodic motion, relative to a rotating coordinate frame, near each of the libration points of the restricted three-body problem. Their proof is based upon assumptions concerning the masses and distances of the bodies which are not satisfied by the Earth - Moon - Sun system.

Siferd in 1965 (Ref 38) used Huang's model for the Earth - Moon - Sun system to generate some periodic orbits. Using a numerical integration procedure, the equations of motion for the very restricted four-body problem were integrated utilizing a digital computer until some periodic orbits were obtained. By this technique eight periodic orbits, in the numerical sense, with a respect to a rotating coordinate system were found. Three orbits were around the Earth, three were around the Moon, and two were around L1. No periodic orbits near the triangular points were obtained.

Danby in 1965 (Ref 13) investigated the influence of the Sun on motion close to the triangular points of the Earth - Moon system. He felt the very restricted four-body model inadequate for his investigation and therefore used a model in which the secular perturbations of the moon due to the Sun were retained. The results may be said to strengthen the hope that stable motion around the triangular points of the Earth - Moon system is possible. Other investigators include Tapley in 1963 (Ref 43) and 1965 (Ref 44) who used a model similar to the very restricted four-body model except the Moon's orbit is inclined with respect to the Earth - Sun plane. The equations of motion for a particle near the triangular points of the Earth - Moon system are numercially integrated on a digital computer for various initial conditions. One result indicates that a particle placed initially at a triangular point (L4) with zero relative velocity has an envelope of motion, centered at L4, going through a mode of expansion to a value of one Earth - Moon distance for the major axis followed by a mode of contraction to a value of 1/8 Earth - Moon distance for the major axis. The envelope repeats this sequence several times during the 2500-day period investigated. Feldt and Shulman in 1966 (Ref 19) extended the investigation of Tapley to 5000 days and found that the expansion - contraction of the envelope of motion did not persist due to a lunar encounter at approximately 4000 days. However, Tapley and Schutz in 1968 (Ref 45) discuss the effect of the constants used in the model and found that if more accurate values were used, an expansion - contraction motion persisted for over 8000 days. Katz in 1975 (Ref 24) investigated numerical orbits of a satellite placed near L5 but all the initial conditions used did not have long-term stability.

Wolaver in 1966 (Ref 55) used a linearized four-body approximation to demonstrate that proper use of initial conditions could aid stability of an orbit in the vicinity of L4. Then Heppenheimer in 1978 (Ref 20) developed realistic models incorporating numerical four-body perturbations in a three-dimensional analysis. Although he worked on resonant orbits about the Earth, his model sufficiently mirrored the real world to be used in this thesis as the truth model starting with the proper initial conditions for stable periodic orbits about the triangular libration points discussed by various authors using more simple models.

In the search for stable periodic orbits about L4 and L5 for space colony candidates, four were found. Schechter in 1968 (Ref 36) concluded a stable, periodic coplanar orbit can exist about the Sun perturbed Earth - Moon triangular point. The model used was a three-dimensional analysis of the long-period features of four-body motion about L4 where short-term period terms are removed from the Hamiltonian via von Ziepel's method resulting in a slowly varying Hamiltonian. He obtained an orbit with a period of 28.6 days with a 1:1 resonance with the Sun and Moon in a rotating coordinate frame. The elliptical motion is clockwise about L4 and has a semimajor axis of approximately 60,000 miles. Schechter demonstrated that out-of-plane motion is not seriously excited by the Sun and has a negligible effect on coplanar motion. It is this coplanar motion which is the dominant factor as far as stability is concerned.

But Kolenkiewicz and Carpenter in 1968 (Ref 26) confirmed Schechter's orbit by numerically producing a somewhat larger orbit having the same essential features of the orbit by Schechter. In addition, a second similar orbit having a phase difference of 180° was calculated. It is believed that the discrepancy in size can be accounted for by differences in the models used for lunar orbit. If Schechter's lunar orbit was perturbed elliptical instead of circular an orbit on the order of 50% larger would result from his equations.

Wheeler in 1978 (Ref 49) found a stable periodic orbit about L4 in the restricted problem of four bodies. It exhibited a 1:1 commensurability with the Moon's synodic month and has a period of 29.5305382 days. The eigenvalues were solved for and the Poincare' exponents were then determined to be pure imaginary. This implied linear stability of the orbit in the restricted four-body problem.

Barkham, Modi, and Soudack in 1975 (Ref 1) found a 2:1 literal solution to the restricted four-body problem about L4 and L5 and numerically generated periodic, four-body solutions that agreed to within 5% of the literal solution.

Last of all, Kamel and Breakwell in 1970 (Ref 23) found similar results to Kolenkiewicz and Carpenter using the von Zeipel technique. More surveys of motion near the triangular Earth - Moon libration points are given by deVries (Ref 16), Steg and deVries (Ref 40), Szebehely (Ref 42), and Schutz (Ref 37).

Problem and Scope

With space colonization on the horizon, it appears that the most fundamental questions about motion near libration points are those about the existence of periodic orbits and the stability of such orbits. If stable periodic solutions exist, solutions of differential equations at or near conditions of commensurability may be used as intermediate orbits for the computation of non-periodic orbits by perturbation analysis. In the restricted problem of three bodies the existence of periodic orbits about the triangular libration points is well established. This result followed from the analysis of the linearized equations of motion and served to exhibit the stability of the triangular configuration, as one of Lagrange's exact solutions of the restricted problem, only in so far as the linearization is valid, that is, only for infinitestimal disturbances. The apparent existence of non-infinitesimal periodic orbits (Brown, Thüring, Rabe) followed either from the analysis of higher order approximations of the differential equations (but still not exact) or from numerical work. It is very difficult to derive meaningful results by qualitative methods and with the problem of libration orbits we may still be in the position of trying to come to specific results by the study of particular analytical or numerical solutions.

Considering the modern trends in the study of nonlinear mechanics toward qualitative methods one may expect that any new work on triangular libration points should concentrate on the establishment of a proof of stability of libration orbits. If then a solution in the form of analytical expressions of the coordinates as functions of time with an exhibition of integration constants would be at all required, one should use periodic orbits (whose existence would first be proved) as intermediate orbits for the perturbation analysis. Two reasons discourage one from the following approach. First of all, even though the past few decades have seen a significant development of methods and theorems in nonlinear mechanics there is still very little known about systems of higher than second order. The methods of the phase plane, so convenient and easily visualized for second order systems, must be transferred to multidimensional phase space which introduces some formidable complications. Secondly, the few qualitative results which are known about the triangular libration points specifically have been derived only for the restricted three-body problem which is really very special since its Hamiltonian does not contain the independent variable explicitly. On the other hand, preliminary studies have shown clearly that in the use of the Earth - Moon libration points the influences of the Sun as the fourth body and of the Moon's orbital eccentricity are quite important. The Hamiltonian of such a problem contains the independent variable in periodic terms of short and long periods, and especially with periods commensurable, or nearly so, with the principle periods of the problem. Very little is known at all about how certain qualitative results derived for constant Hamiltonian could be transferred to a similar problem with time varying Hamiltonian.

So finding a solution to the full analytical equations of motion describing real world forces is not possible to date. The best method to treat the problem of colony location is to first develop reference colony orbits in the restricted three-body problem. Such orbits are then studies in a very restricted four-body problem, wherein the motions of the Earth, Moon, and colony are determined by numerical integration. This author has found four orbits around L4 and L5 claimed to be stable and periodic using very restricted four-body equations. This thesis will attempt to test those orbits using a model more closely resembling the real world to see if the orbits are stable, and similar. If they are not stable, this paper will attempt to explain why they are not.

The Lagrangian points L4 and L5 will be used as reference points. They are the stable equilibrium points in the restricted three-body problem for mass ratios less than .0385. Here the satellite remains fixed relative to the other two bodies, if given the correct initial velocity. However, in cislunar space L4 and L5 are no longer equilibrium points although we will still refer to colony motion about these points. Any truth model used should reduce to the restricted three-body equations of motion. L4 and L5 passes a triangular symmetry with each other in relation to the Earth and Moon so that motion about L4 can be considered identical to motion about L5. Exceptions to this are when perturbative effects of other planetary bodies are taken into account. Then the motion about L4 will exhibit slightly different solutions compared to motion about L5. This author neglects those minute planetary perturbations so when L4 is referred to, it can be considered L5 as well.

II. PROBLEM ANALYSIS

Assumptions

The stable periodic orbits about L4 found to date were computed using very restricted four-body (VRFB) models. The VRFB models neglect the important indirect effect of the Sun, i.e., the gravitational effect on the motion of the Earth and Moon. Hence, the results obtained from the simplified VRFB model cannot be used to infer motion in the real world. That is, based on VRFB results, it is not known whether the expansion - contraction of the envelope of motion which exists in the VRFB models also exists in the real world. Instead the nature of the solar influenced particle motion near L4 will be studied by numercially integrating the equations of motion over a number of years using a model which closely represents the real world.

The model used to test these orbits is one used by T.A. Heppenheimer (Ref 20) in a paper locating space colonies in high Earth orbits. In using this model the assumptions made are:

1) The Sun, Earth and Moon are considered to be point masses.

2) The mass of the colony satellite is negligible compared to the masses of the other three bodies and exerts no forces to affect their motion.

3) The gravitational forces of other planets have a negligible effect on motion and is ignored.

4) The motion of all four bodies is limited to one plane.

5) The motion of the Sun is taken to be an unperturbed ellipse with respect to the Earth - Moon barycenter.

The satellite motion about L4 is far enough away from the three other attracting bodies that the assumption of point masses is supported. The strongest effect would be that of the Earth and the largest term (J_2 term) to have effect acts to the fifth power of the distances from the Earth making this term negligible. Consider Mac Cullagh's Formula (Ref 13) which gives the potential for an attraoting body of any shape at a distance from the attracting body which is large compared with the body's over-all dimensions.

$$V = \frac{G\,m}{R} - \frac{G}{2R^3}(A + B + C - 3I)$$

$$\text{where } I = \ r^2 \sin^2\theta \, dm$$

$$\text{and } A + B + C = 2 \ r^2 \, dm$$

where A, B, and C are the principle moments of inertia and I the moment of inertia of the body about a line drawn from the center of gravity of the body to the satellite. From this equation it can be seen that the satellite does not have to be very many times the radius of the body away before the second term on the right-hand side becomes negligible.

The second assumption is obvious when you consider the tremendous resources needed to orbit a satellite of any great size. Even a large space colony would only have to be ten miles across to meet the needs of the people and one this size would exert no influence upon the attracting bodies.

The effect of the other planets in the solar system on the Sun will be taken into account by the eccentricity and formula used for the Sun's motion. This will account for most of the direct effect since direct effect upon the Earth and Moon will not be as great as upon the Sun. The indirect perturbations are larger and tend to change the origin to the center of mass, or barycenter, of the Sun and planets. Schutz and Tapley studied the effect of other planets using an Ephemeris Model. Numerical integration was performed for L5 with a fixed step size and compared with results of integration neglecting the planets at the same step size. The maximum change in position was 488 km which was small enough to not significantly affect the motion during 2500 days considered but it would have a significant effect over longer periods. The effect is slight enough that a small controller could offset its influence.

Out-of-plane considerations have been dealt with using intrinsic solutions before. It is necessary to consider effects due to the inclination of the Moon and satellite with respect to the ecliptic. The lunar-orbit plane is inclined to the ecliptic by 5.14 degrees. Consequently, if the colony is initially in a coplanar orbit, there are different rates of regression of the lines of nodes of the Moon (due to solar perturbations) and of the satellite (due to lunisolar perturbations). Although both Moon and satellite maintain nearly constant inclinations on the ecliptic, their orbit planes mutually precess and, in time, are mutually inclined by up to 10 degrees. However, as Danby discusses (Ref 13), the angular momentum of the solar system is almost totally coplanar giving the system an invariable plane. Any orbit within the system that leaves this plane will be drawn back in and the terms due to perturbation of the Moon's inclination are periodic. Heppenheimer (Ref 20) estimates the optimal satellite inclination using phase equilibria and finds secular and inclination - type resonances to be small and stable.

Restricted Four-Body Equations of Motion

The general procedure followed to obtain real world orbits is to use current numerical integration schemes on planar restricted four-body equations of motion. In these equations, the motion of the Sun is taken as an unperturbed ellipse with respect to the Earth - Moon barycenter and the lunar motion is given by

$$\ddot{x}_m + \frac{x_m}{r_m^3} = -m_s\left(\frac{x_m - x_s}{r_{ms}^3} + \frac{x_s}{r_s^3}\right) \tag{1}$$

$$\ddot{y}_m + \frac{y_m}{r_m^3} = -m_s\left(\frac{y_m - y_s}{r_{ms}^3} + \frac{y_s}{r_s^3}\right) \tag{2}$$

The motion of the satellite is given by

$$\ddot{x}_c + \frac{(1-\mu)}{r_c^3}\, x_c = -m_s\left(\frac{x_c - x_s}{r_{cs}^3} + \frac{x_s}{r_s^3}\right) - \mu\left(\frac{x_c - x_m}{r_{cm}^3} + \frac{x_m}{r_m^3}\right) \tag{3}$$

$$\ddot{y}_c + \frac{(1-\mu)}{r_c^3}\, y_c = -m_s\left(\frac{y_c - y_s}{r_{cs}^3} + \frac{y_s}{r_s^3}\right) - \mu\left(\frac{y_c - y_m}{r_{cm}^3} + \frac{y_m}{r_m^3}\right) \tag{4}$$

where the subscripts m, s, and c stand for the Moon, Sun, and satellite, respectively. These equations are derived in Appendix A and were used by Heppenheimer (Ref 20). They use a rectangular, nonrotating, Earth - centered coordinate system. These equations reduce to the restricted three-body problem is the perturbation of the fourth body (Sun) is removed. The remaining terms are

$r_{ms}^2 = (x_m - x_s)^2 + (y_m - y_s)^2$ (Sun - Moon distance)

$$r_m^2 = x_m^2 + y_m^2 \qquad \text{(Moon - Earth distance)}$$

$$r_s^2 = x_s^2 + y_s^2 \qquad \text{(Sun - Earth distance)}$$

$$r_c^2 = x_c^2 + y_c^2 \qquad \text{(Satellite - Earth distance)}$$

$$r_{cs}^2 = (x_c - x_s)^2 + (y_c - y_s)^2 \qquad \text{(Satellite - Sun distance)}$$

$$r_{cm}^2 = (x_c - x_m)^2 + (y_c - y_m)^2 \qquad \text{(Satellite - Moon distance)}$$

The following constants were employed by Heppenheimer:

Solar mass = m_s = 329426.3 (5)

Solar semimajor axis = a_s = 389.0548

Solar eccentricity = e_s = 0.0168

Solar mean motion = n_s = 0.0748013

The Sun lies initially at perihelion on the positive x-axis with its true anomaly f_s defined by Brouwer and Clemence's equation of the center (Ref 3)

$$\begin{aligned} f_s = n_s t &+ (2e_s - \frac{1}{4} e_s^3 + \frac{5}{96} e_s^5 + \frac{107}{4608} e_s^7) \sin n_s t \\ &+ (\frac{5}{4} e_s^2 - \frac{11}{24} e_s^4 + \frac{17}{192} e_s^6) \sin 2 n_s t \\ &+ (\frac{13}{12} e_s^3 - \frac{43}{64} e_s^5 + \frac{95}{512} e_s^7) \sin 3 n_s t \\ &+ (\frac{103}{96} e_s^4 - \frac{451}{480} e_s^6) \sin 4 n_s t \\ &+ (\frac{1097}{960} e_s^5 - \frac{5957}{4608} e_s^7) \sin 5 n_s t \\ &+ \frac{1223}{960} e_s^6 \sin 6 n_s t + \frac{47273}{32256} e_s^7 \sin 7 n_s t \end{aligned}$$

(See Appendix B for derivation)

Dropping higher order terms

$$f_s = n_s t + (2e_s - \frac{1}{4} e_s^3) \sin n_s t$$

$$+ \frac{5}{4} e_s^2 \sin 2 n_s t + \frac{13}{12} e_s^3 \sin 3 n_s t \qquad (6)$$

so that the Sun's coordinates are given by

$$x_s = \mu x_m + rp_s \cos f_s$$

$$y_s = \mu y_m + rp_s \sin f_s$$

$$rp_s = \frac{a_s (1 - e_s^2)}{1 + e_s \cos f_s}$$

Lunar motion is initiated also on the positive x-axis. Refer to Figure 1 for a diagram of the initial condition configuration.

These equations of motion are cast into first-order form. The initial conditions of the periodic stable orbits found by the various authors are to be determined and transformed into the preceding four-body coordinate system. The equations of the four-body model are integrated using a fourth-order Adams - Bashforth predictor and Adams - Moulton corrector with a fourth-order Runge - Kutta integrator as a starter. This entire integration package, called ODE, was developed by Shampine and Gordan at Sandia Laboratory. It automatically adjusts the order and step size to control the local error per unit step in a generalized sense. It is the integration package used throughout this report on the CDC 6600 and CYBER tie-in computers at the Air Force Institute of Technology.

Numerical experiments will not have to be performed to determine a proper step size for integration of the equations over a given range. ODE will optimize the step size preventing large round-off error due to a small step size over a large number of integration steps or truncation error due to a large

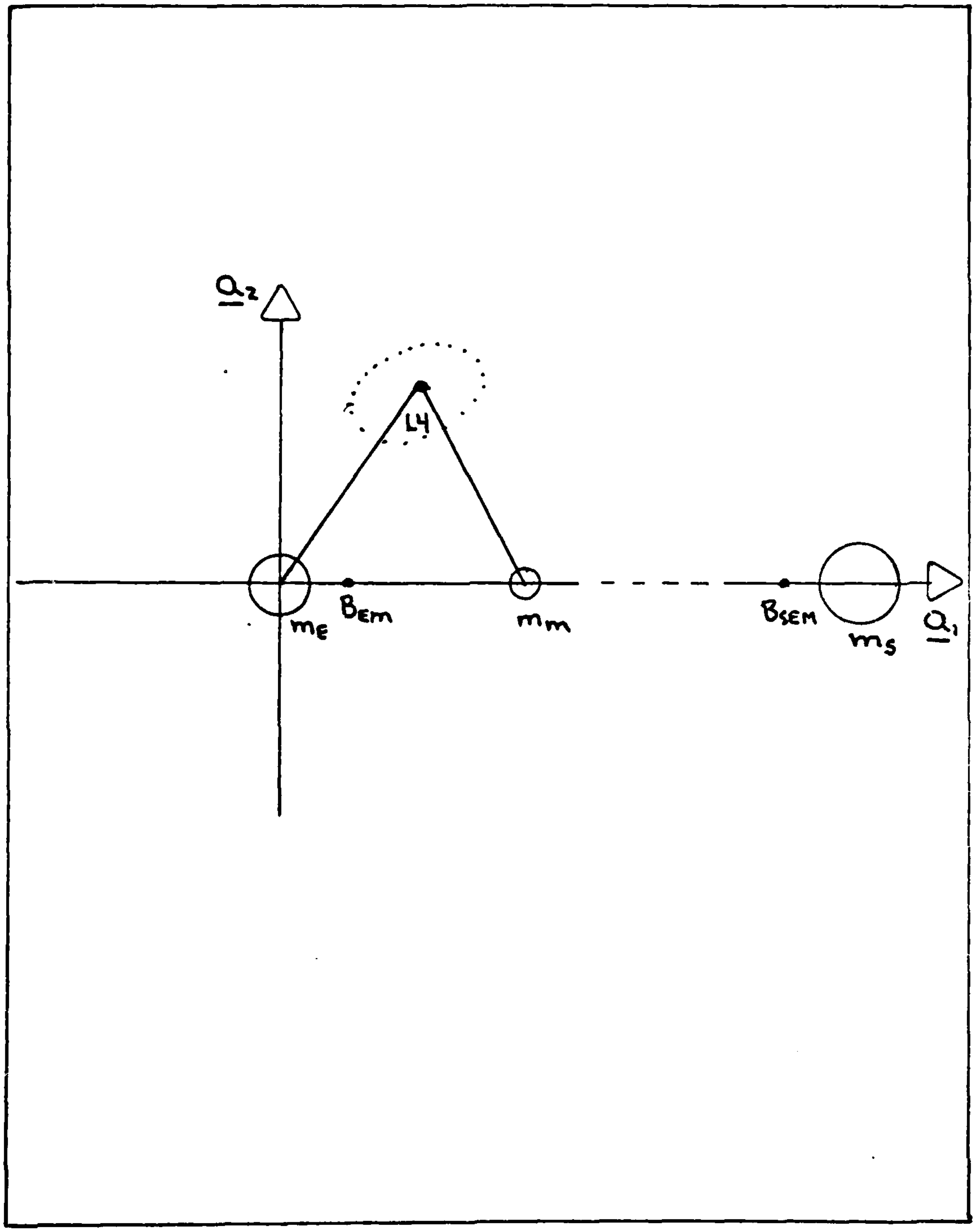


Fig 1. Initial Condition Configuration for Truth Model

step size over many steps.

Other constants used by Heppenheimer which will be compared to the other authors' constants

Unit mass = 1 (Earth + Moon) (7)

Lunar mass = μ = 0.01215

Unit distance = 3.8441×10^8 m (Earth - Moon distance)

Unit time = 4.3484167 $\frac{\text{days}}{\text{T.U.}}$

Unit velocity = 1023.17 m/sec

Unit acceleration = 0.00273 m/sec^2

The unit time corresponds to one radian per time unit mean motion of the Moon for a 27.321661 day sidereal period and 0.07480133 radian per time unit mean motion of the Sun for a 365.256365 day sidereal year.

III. WHEELER'S ORBIT

Overview of Wheeler's Work

In Wheeler's study (Ref 49) the equations of motion for a satellite near L4 in a planar restricted four-body problem are derived and tested. A computer algorithm for finding periodic motion is formulated and the initial starting point of the Sun, Moon and Earth in line is used. Rabe's periodic orbits in the three-body problem are used as starting conditions to begin searching for four-body motion. He employs a linearization of small displacements about the non-linear periodic orbit. The periodic orbit found has a period in resonance with the lunar synodic month, 29.5305882 days. Wheeler again calculates this same orbit by slowly increasing the mass of the Sun from zero to its real value and presents a synodic period orbit about L4. He proves this orbit is generated mathematically from the L4 point and not from Rabe's orbits. A successful stability analysis is performed on the orbit with the orbit found to be stable.

Assumptions and Coordinate System

The assumptions made using Wheeler's very restricted four-body model (Ref 49) include:

1) The Sun, Earth and Moon are considered to be point masses.

2) The mass of the satellite is negligible when compared to the other three bodies and it, therefore does not affect their motion.

3) The gravitational effects of the other planets in the solar system can be ignored.

4) The motion of all four bodies is limited to one plane.

5) The Earth and Moon move in circular orbits about their barycenter at a constant rotation rate.

6) The Earth - Moon barycenter moves in a circular orbit about the Sun at a constant rotation rate.

The difference of Wheeler's model is obviously the assumption of circular orbits instead of the actual perturbed orbits for the Earth, Moon and Sun. Later in this report, Wheeler's orbit will be reproduced using the truth model by removing the perturbative forces from the Moon and Earth orbit. These forces need to be accounted for to correctly model orbital motion in cislunar space.

Refer to Figure 2 for Wheeler's rotating coordinate system. The satellite orbiting about L4 remains in a rotating coordinate system with its center at the Earth - Moon barycenter and the Moon on the negative x-axis. So the coordinate system rotates with the period of the Moon's synodic period and the entire system rotates in turn about the barycenter between the Earth - Moon barycenter and the Sun. Its rotational rate is that of one sidereal year.

Constants

Pertinent constants used in Wheeler's paper are

$$\text{Solar mass} = m_s = 328900.12 \qquad (8)$$

$$\text{Mean solar semimajor axis} = a_s = 388.82028$$

$$\text{Solar eccentricity} = e_s = 0.0$$

Solar eccentricity is equivalent to the movement of the Earth - Moon barycenter about the Sun.

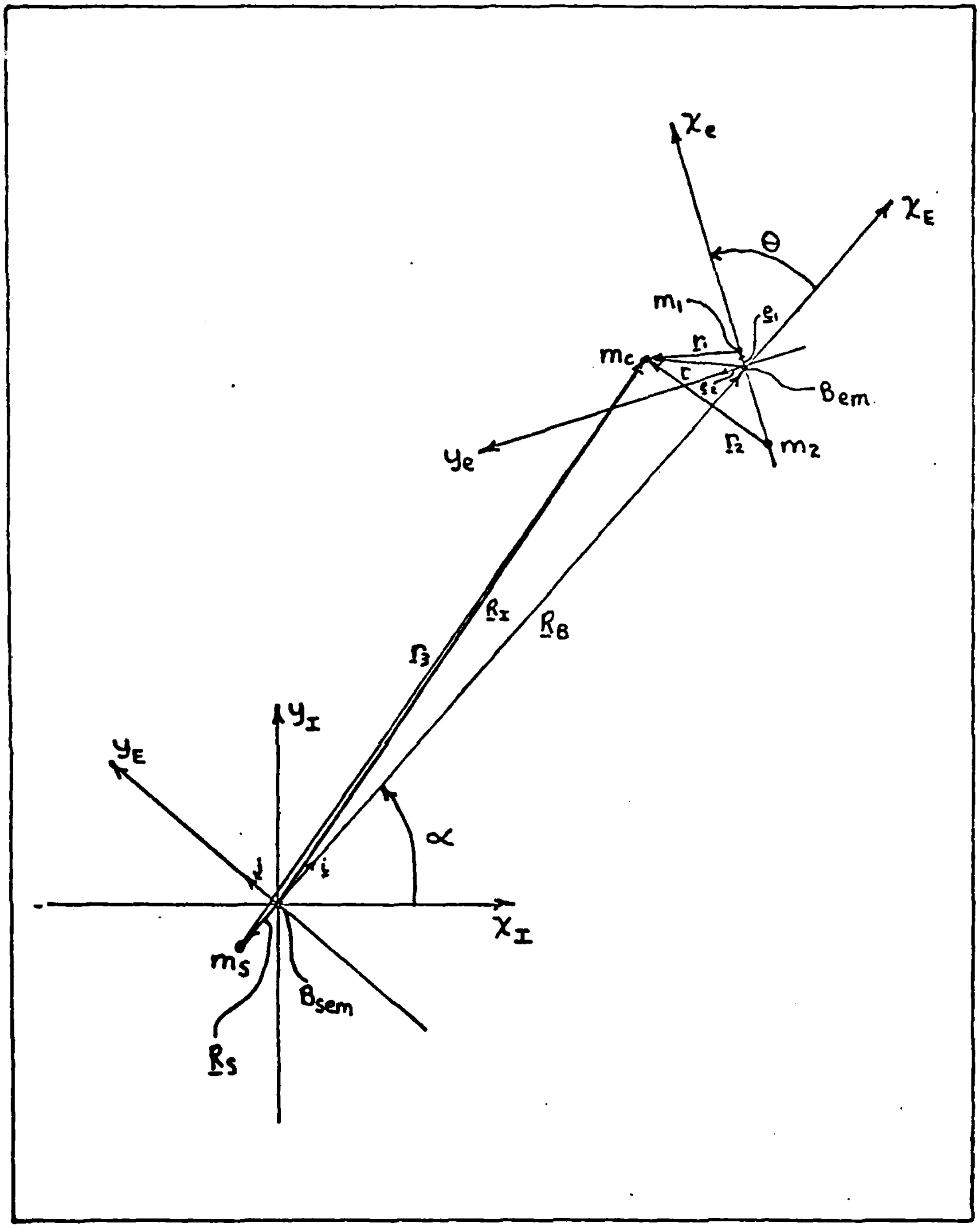


Fig 2. Wheeler's Four-Body Configuration

$$\text{Solar mean motion} = n_s = .0808489351$$

From Figure 2, $\dot{\theta}$ is normalized to 1.0 and so the entire rotation of the Earth - Moon system with respect to an inertial frame

$$w = \dot{\theta} + \dot{\alpha}$$

$$w = 1.0 + .0808489351 = 1.0808489351 \tag{9}$$

which is equivalent to lunar mean motion.

If the mass of the Moon is set to some arbitrary valve, μ, and the sum of the masses of the Earth and Moon is set equal to 1, then the masses of the Earth equals $1 - \mu$. These values also correspond to the normalized Earth - Moon distance between their barycenter. In Wheeler's study

$$\mu = .0121396054$$

<u>Conversion to Truth Model</u>

The constants need to be converted to the truth model situation and the initial conditions of Wheeler's orbit which will be used to generate the truth model orbit must be transformed into the coordinate system used for the truth model.

The initial conditions of Wheeler's 1/1 resonance orbit are (Ref 49)

$$x = -\ 0.72418782459 \qquad \dot{x} = 0.07948061949 \tag{10}$$

$$y = 0.81[illegible]68639689 \qquad \dot{y} = 0.22438007788$$

They were generated from the initial conditions at the L4 point

$$x = -\ 0.4878603946 \qquad \dot{x} = 0.0 \tag{11}$$

$$y = 0.8660254038 \qquad \dot{y} = 0.0$$

for $m_s = 0.0$ (three-body problem) and increasing up to its present value in a very restricted four-body problem. The initial conditions (10) provide periodic motion in the restricted four-body model for periods which are integer multiples of the lunar synodic month. Figure 3 describes Wheeler's initial condition configuration with the Sun and Moon starting on the negative x-axis of the rotating B_{cm} coordinate system with angles θ and α equal zero.

Wheeler's initial conditions must be transformed from a rotating Earth - Moon barycenter frame to an Earth - centered inertial frame. Referring to Figure 4, the frames are initially lined up and the distance from the origin of the truth model E to the initial starting point is

$$\underline{r}^{e}_{P/E} = \underline{r}^{e}_{P/B} + \underline{r}^{e}_{B/E}$$

where

$$\underline{r}^{e}_{P/B} = x\ \underline{e}_1 + y\ \underline{e}_2 \text{ and } \underline{r}^{e}_{B/E} = -\ \mu\ \underline{e}_1$$

combining

$$\underline{r}^{e}_{P/E} = (x - \mu)\ \underline{e}_1 + y\ \underline{e}_2 \tag{12}$$

in terms of Wheeler's coordinates. For the velocity vector

$$\begin{aligned} \underline{v}^{e}_{P/E} &= \underline{v}^{e}_{P/B} + \underline{v}^{e}_{B/E} \\ &= \underline{v}^{e}_{P/B} + \underline{w}^{ea} \times \underline{r}^{e}_{P/B} + \underline{v}^{e}_{B/E} + \underline{w}^{ea} \times \underline{r}^{e}_{B/E} \end{aligned}$$

where

$$\underline{v}^{e}_{P/B} = \dot{x}\ \underline{e}_1 + \dot{y}\ \underline{e}_2$$

$$\underline{v}^{e}_{B/E} = \dot{\mu}\ \underline{e}_1 = 0$$

$$\underline{w}^{ea} = \dot{\theta}\ \underline{e}_3$$

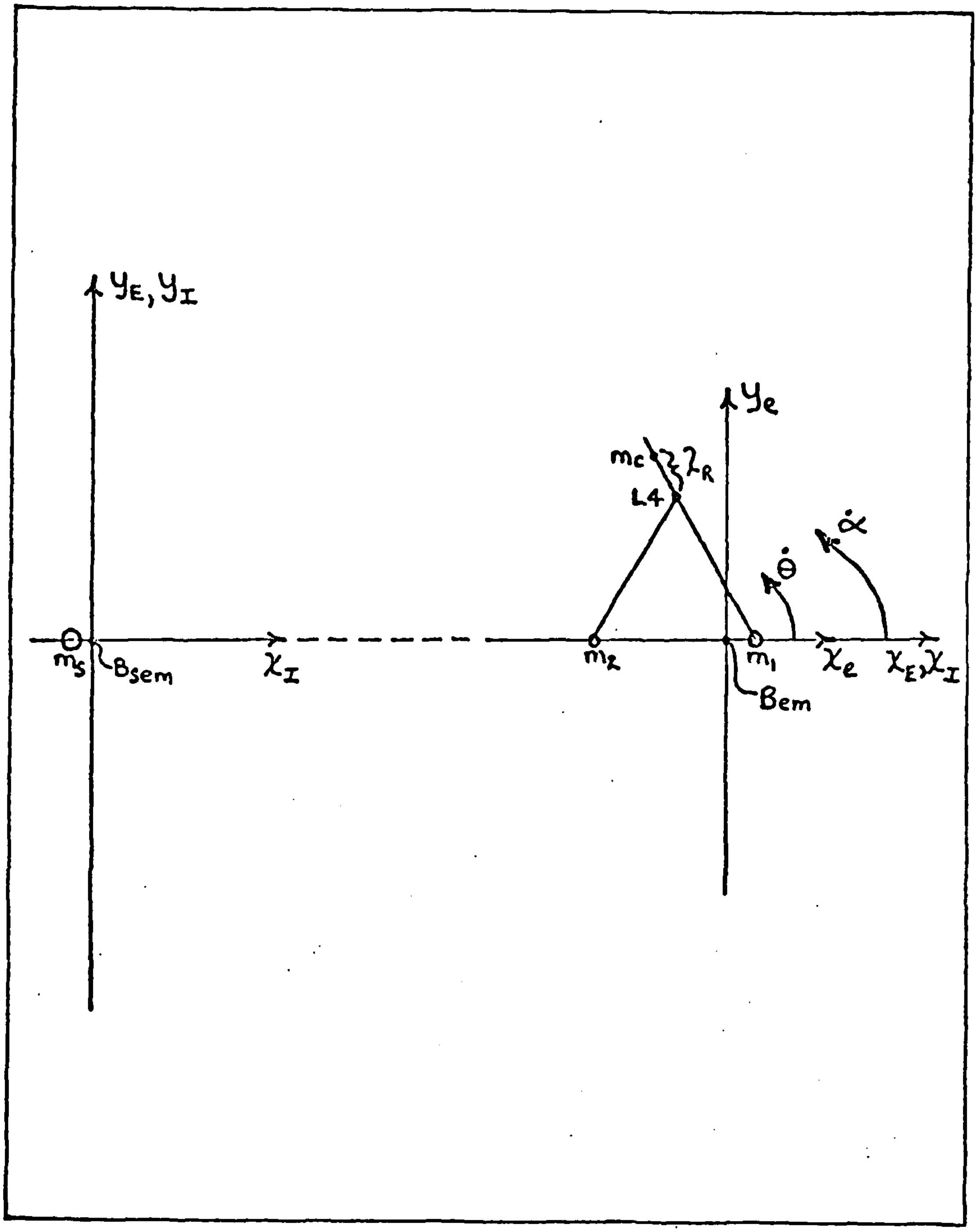


Fig 3. Wheeler's Initial Condition Configuration

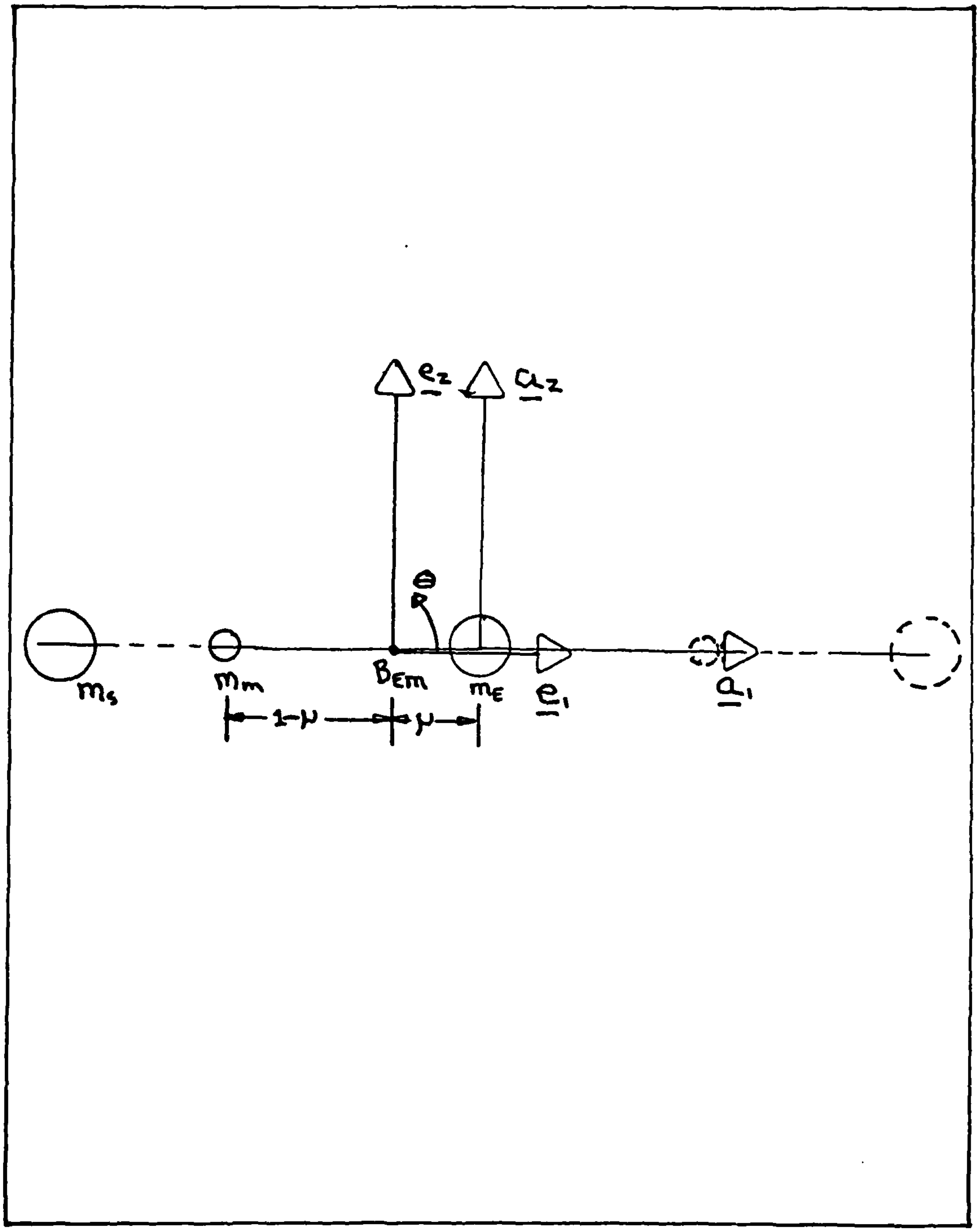


Fig 4. Transformation From Wheeler Frame To Truth Frame

substituting

$$\underline{v}_{P/E}^{e} = \dot{\theta}\ \underline{e}_3 \times (-\mu\ \underline{e}_1) + \dot{x}\ \underline{e}_1 + \dot{y}\ \underline{e}_2 + \dot{\theta}\ \underline{e}_3 \times (x\ \underline{e}_1 + y\ \underline{e}_2)$$

$$\underline{v}_{P/E}^{e} = (\dot{x} - \dot{\theta}\ y)\ \underline{e}_1 + (\dot{y} + \dot{\theta}\ x - \dot{\theta}\ \mu)\ \underline{e}_2$$

Equations (12) and (13) are in terms of Wheeler's coordinates and must be rotated into truth model coordinates. The rotation matrix is

$$\underline{r}_{P/E}^{a} = [C^{ae}]\ \underline{r}_{P/E}^{e}$$

$$\underline{r}_{P/E}^{a} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \underline{r}_{P/E}^{e}$$

Multiply the rotation matrix times equations (12) and (13) to get the coordinates in terms of the truth model frame.

$$x_a = (x - \mu) \cos\theta - y \sin\theta \tag{14}$$

$$y_a = (x - \mu) \sin\theta + y \cos\theta$$

$$\dot{x}_a = (x - \dot{\theta} y) \cos\theta - (\dot{y} + \dot{\theta}\ x - \dot{\theta}\ \mu) \sin\theta$$

$$\dot{y}_a = (\dot{x} - \dot{\theta} y) \sin\theta + (\dot{y} + \theta\ x - \dot{\theta}\ \mu) \cos\theta$$

Since Heppenheimer's model initially positions the Sun and Moon on the positive x-axis, the Sun and Moon starting positions will be rotated 180° to the negative x-axis and $\theta=0°$. In Wheeler's frame $\dot{\theta} = 1$ but there is a time difference between the two frames which will affect the transformation of the velocity initial conditions. In Wheeler's frame the period of rotation is one lunar synodic month, 29.5305882 days, whereas in the truth model frame the period of rotation is one lunar sidereal month, 27.32166101 days. So the slower truth model frame must have the Wheeler velocities scaled down by a factor

$$\frac{27.32166101}{29.5305882} = \frac{1}{1.0808489351}$$

So the transformation equations (14) are now

$$x_a = x - \mu \tag{15}$$

$$y_a = y$$

$$\dot{x}_a = - y + \dot{x}/1.0808489351$$

$$\dot{y}_a = x - \mu + \dot{y}/1.0808489351$$

Substituting Wheeler's initial conditions (10) we obtained initial conditions for the truth model

$$x = - 0.73632742999 \qquad \dot{x} = -0.74215104336 \tag{15}$$

$$y = 0.81568639689 \qquad \dot{y} = -0.52873127999$$

Wheeler's constant (8) will be used in the truth model along with his value of μ and the transformed initial conditions. The solar mean motion of Wheeler's model must be converted to the slower truth model frame.

$$\frac{n_s}{w} = \frac{0.0808489351}{1.0808489351} = 0.7480133 \tag{17}$$

The true anomoly of the equation of the center (6) will be rotated to the negative x-axis by having ¶ or 180° added to it. The Moon initial conditions will also begin on the negative x-axis to correspond with Wheeler's scenario.

Verification of the Truth Model

The algorithm for integration of the equations of motion and plotting the orbits formulated is given in Appendix C. Wheeler's assumptions are applied to the truth model to verify that the same orbit is reproduced. Wheeler's orbit of 1/1 resonance is shown in Figure 5. Since Wheeler's assumptions involve circular orbits the eccentricity of the Sun is zero ($e_s = 0.0$) and the Moon's initial conditions are

$$x_m = 1.0 \qquad \dot{x}_m = 0.0 \tag{18}$$

$$y_m = 0.0 \qquad \dot{y}_m = \sqrt{\frac{\mu}{r}} = -1.0 \text{ where } \mu = (Gm_e + m_m) = 1.0$$

The truth model equations of motion for lunar motion are simplified by eliminating the perturbative effect of the Sun. Equations (1) and (2) now become

$$\ddot{x}_m + \frac{x_m}{r_m^3} = 0 \tag{19}$$

$$\ddot{y}_m + \frac{y_m}{r_m^3} = 0$$

The equations for satellite motion (3) and (4) remain the same.

A period of rotation of the truth model is 2¶ or 6.2831853 radians or time units. But Wheeler's period occurs when the bodies (Sun, Earth, Moon) are all lined up again which is a longer period of time

$$2¶ \times \frac{29.5305882 \text{ days}}{27.32166101 \text{ days}} = 6.791174148 \text{ time units} \tag{20}$$

so the orbit period will be considered this number of time units and the orbit found will be transformed in Wheeler's frame for comparison and plotting purposes.

The equations of motion were integrated for over fifty years and the orbit found (Figure 5) is identical to Wheeler's. Figure 6 and 7 show three and 20 orbits respectively. Points

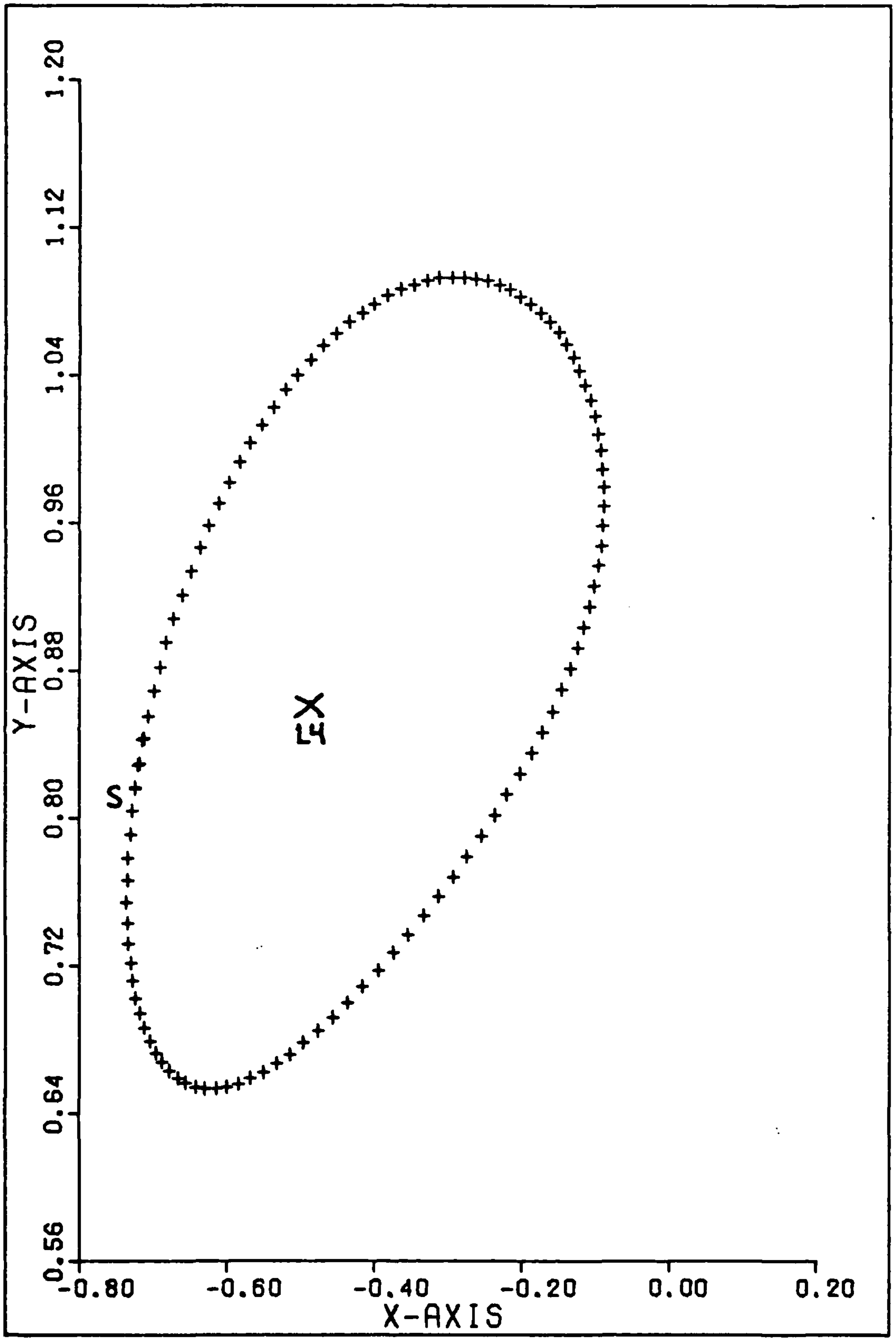


Fig 5. Wheeler's Very Restricted Four-Body

1/1 Periodic Orbit, 1 Orbit

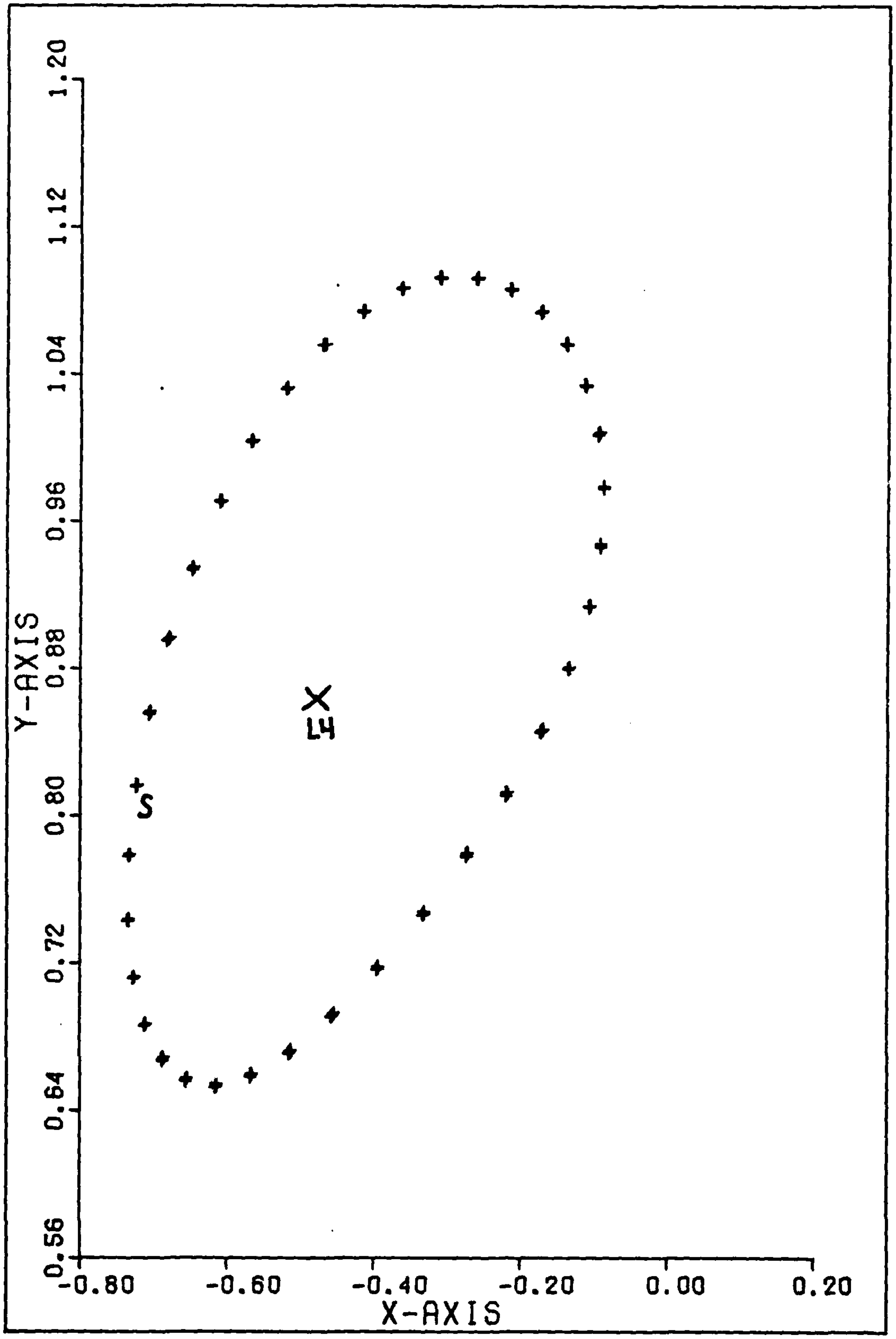


Fig 6. Wheeler's Very Restricted Four-Body

1/1 Periodic Orbit, 3 Orbits

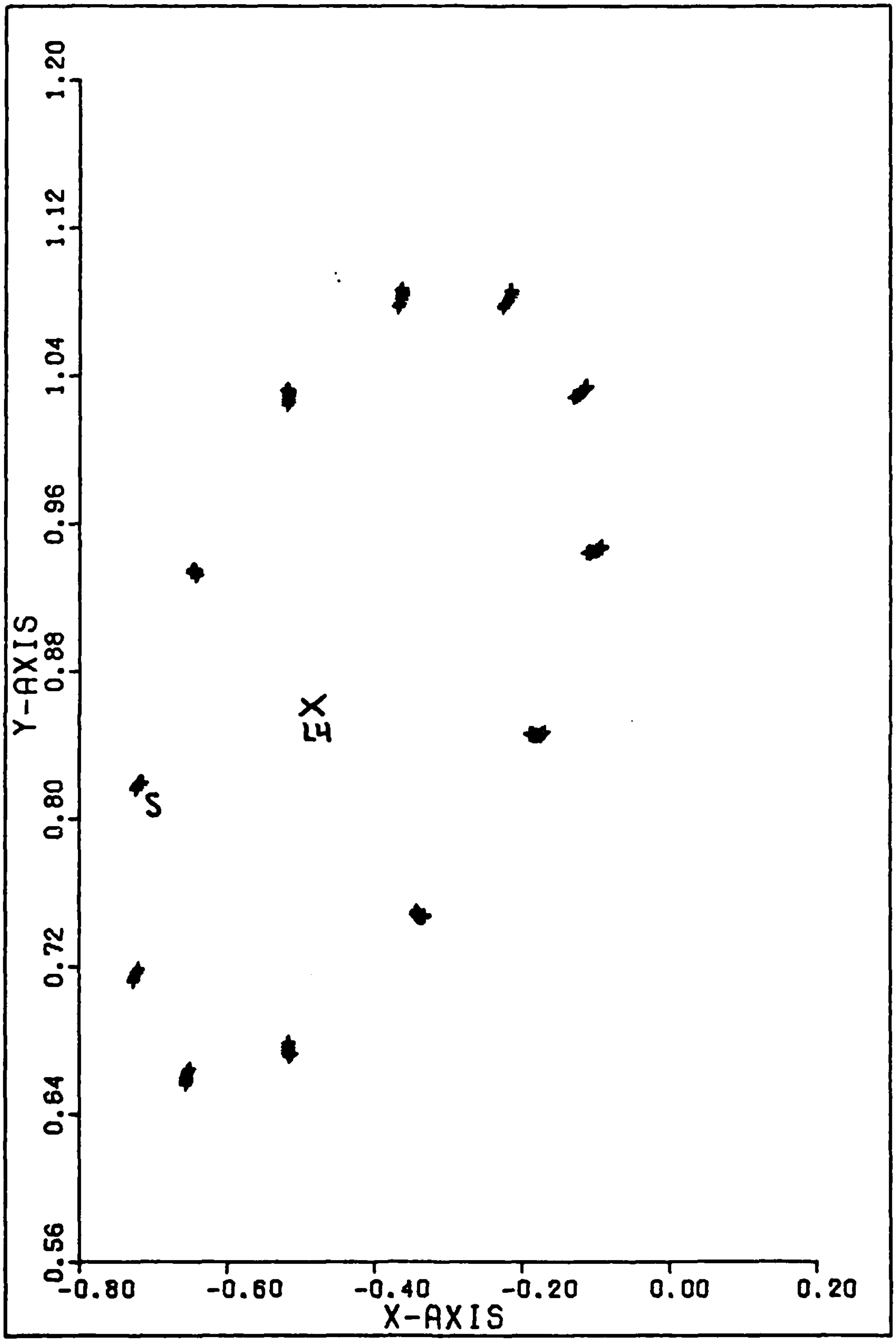


Fig 7. Wheeler's Very Restricted Four-Body

1/1 Periodic Orbit,20 Orbits

along the periodic stable orbit were matched identically over the same time period less computer round-off in the fifth and smaller digits. The orbit is in the same orientation about L4 with an approximate semimajor axis of 80,000 miles and semiminor axis of 40,000 miles. This is strong evidence that Wheeler's equations for the very restricted four-body model are correct and that the truth model from Heppenheimer is correct and can be reduced to the three-body problem. The next step is to apply the full model to Wheeler's orbit.

Truth Model Application

It greatly simplifies the four-body problem to start out with the Sun, Moon and Earth lying on a straight line. Thus, in order for periodic motion to occur, the equations must be integrated forward to the exact time when these initial conditions occur again and when the satellite has, of course, returned again to its original position about L4. This orbit being examined returns after one lunar synodic month in the very restricted four-body case. But as the perturbations increase by adding more segments to the truth model the orbit will no longer return to exactly the same spot or at exactly the proper time. The synodic period will continue to be used as a basis for looking at orbit positions to see how they are changed by the more realistic forces of the four-body problem.

Perturbations were added one at a time to observe the effects upon stability and periodicity noting that one effect could offset part of another one when all the forces are inter-

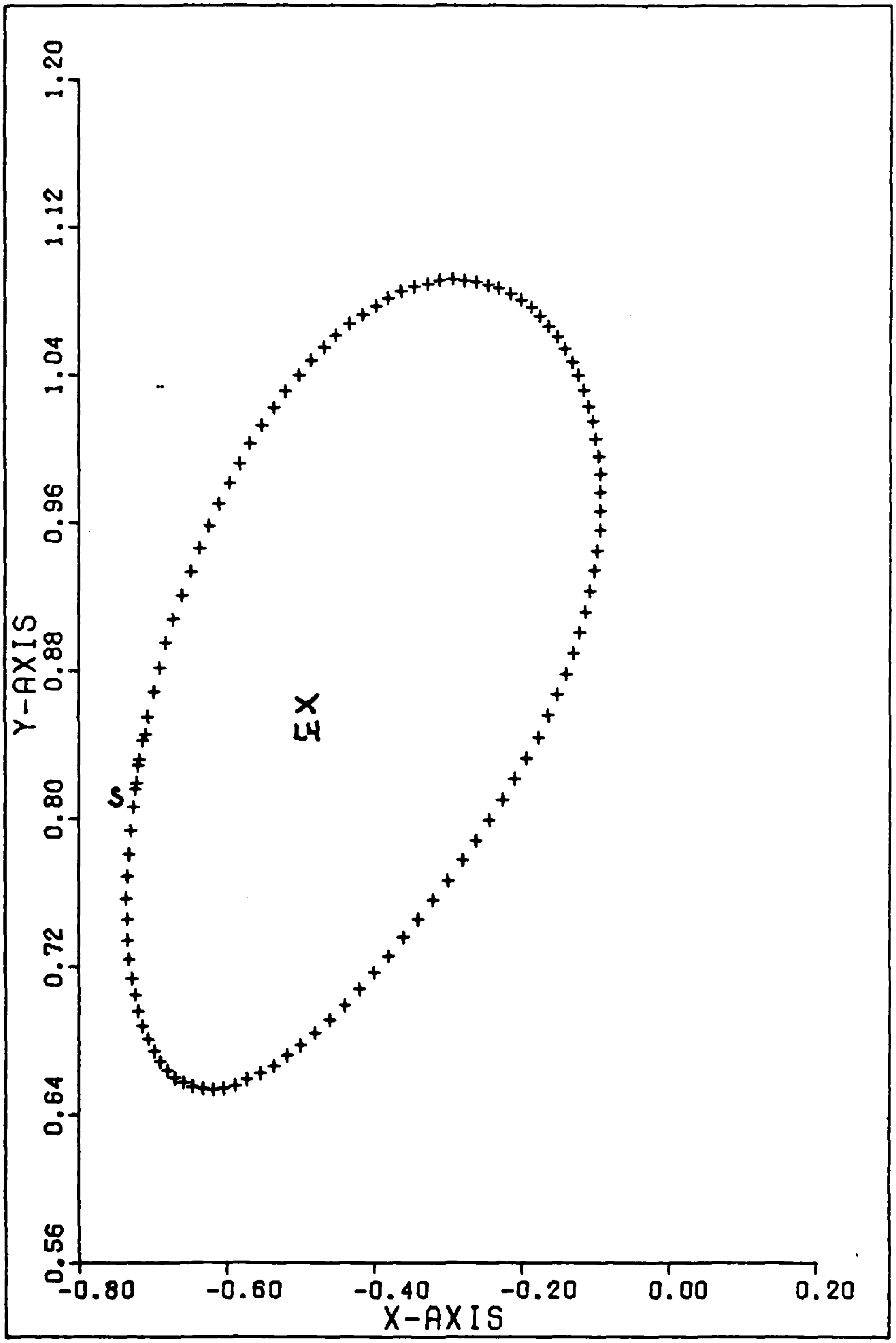


Fig 8. Wheeler Orbit With Solar Eccentricity, 1 Orbit

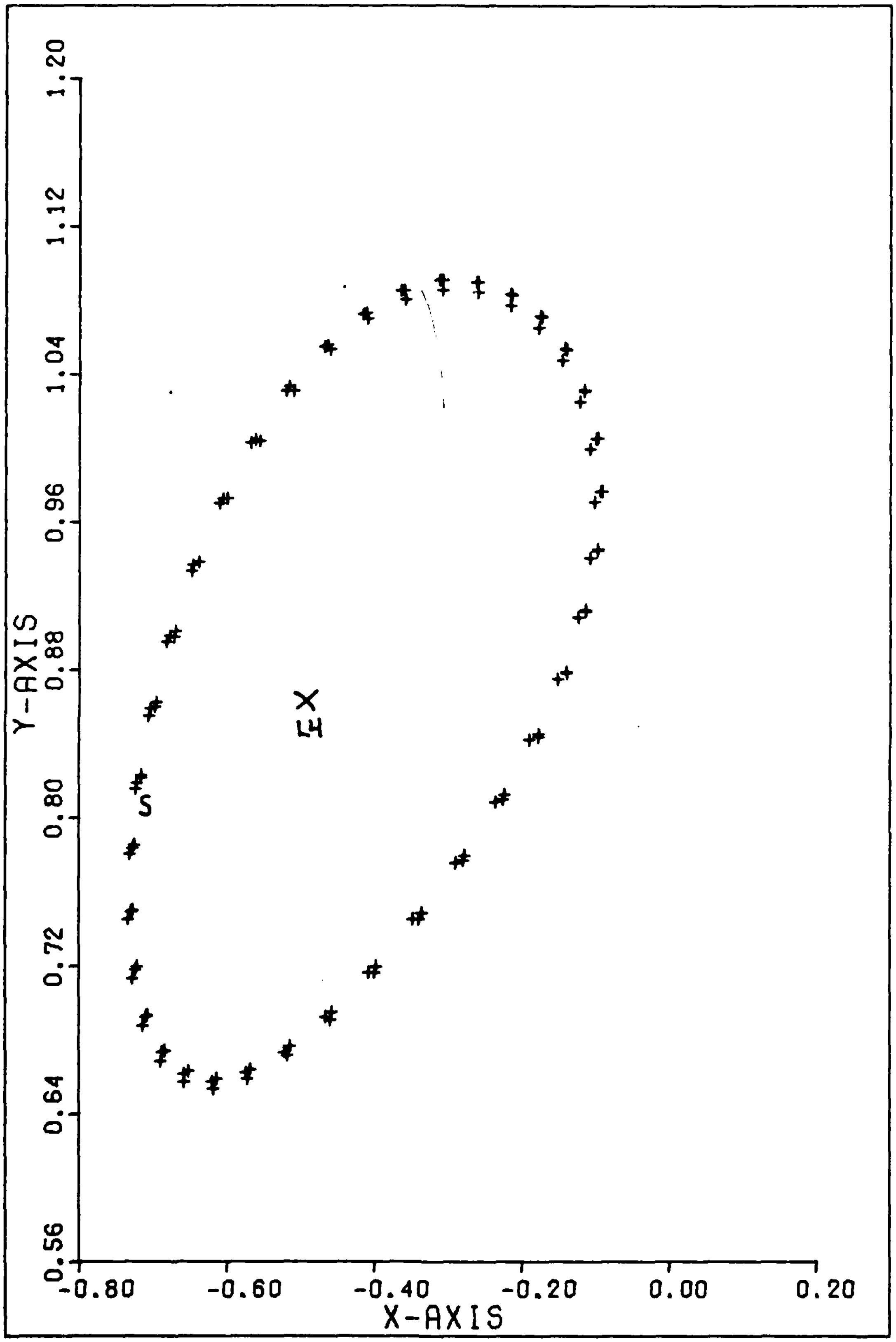


Fig 9. Wheeler Orbit With Solar Eccentricity,

3 Orbits

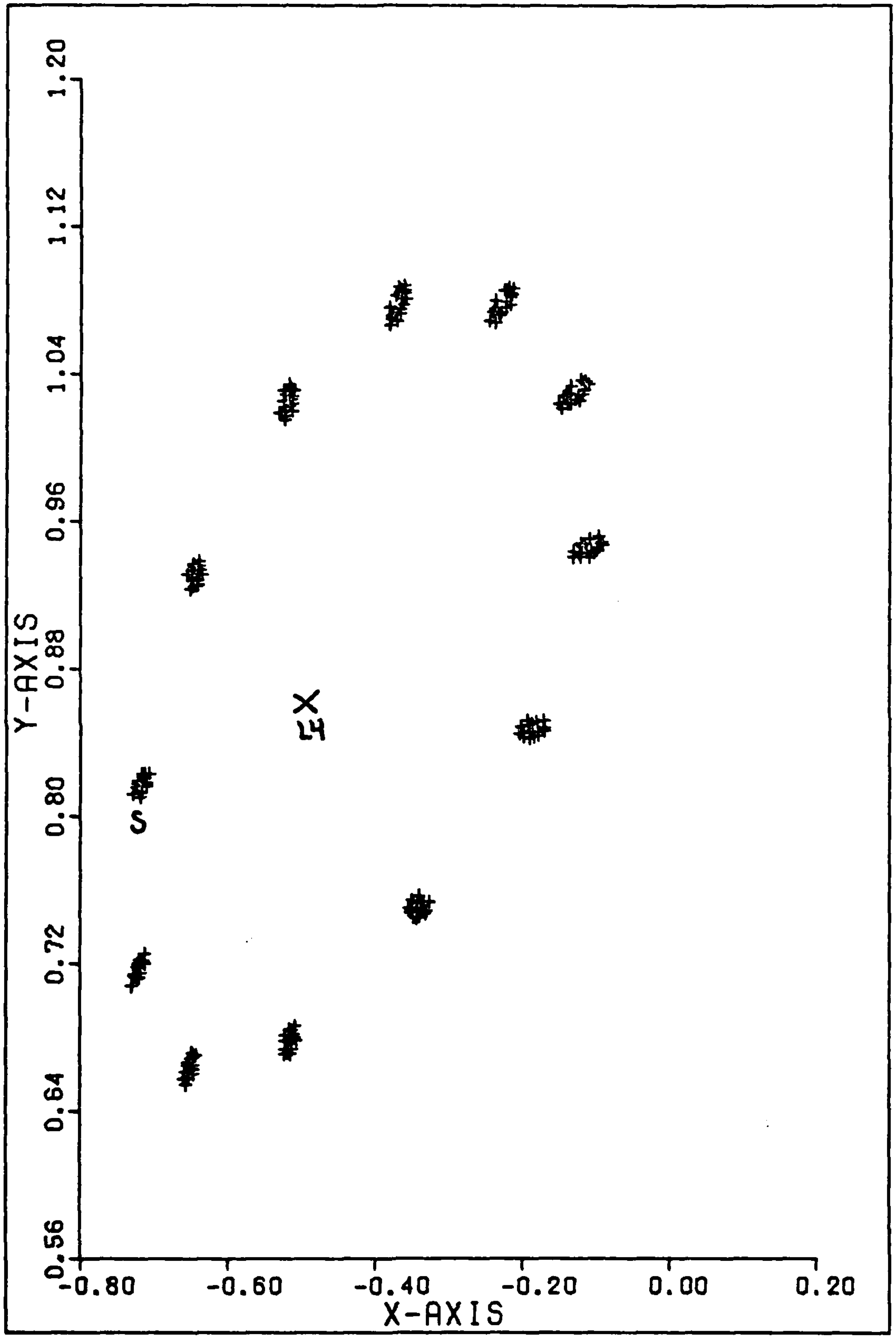


Fig 10. Wheeler Orbit With Solar Eccentricity, 20 Orbits

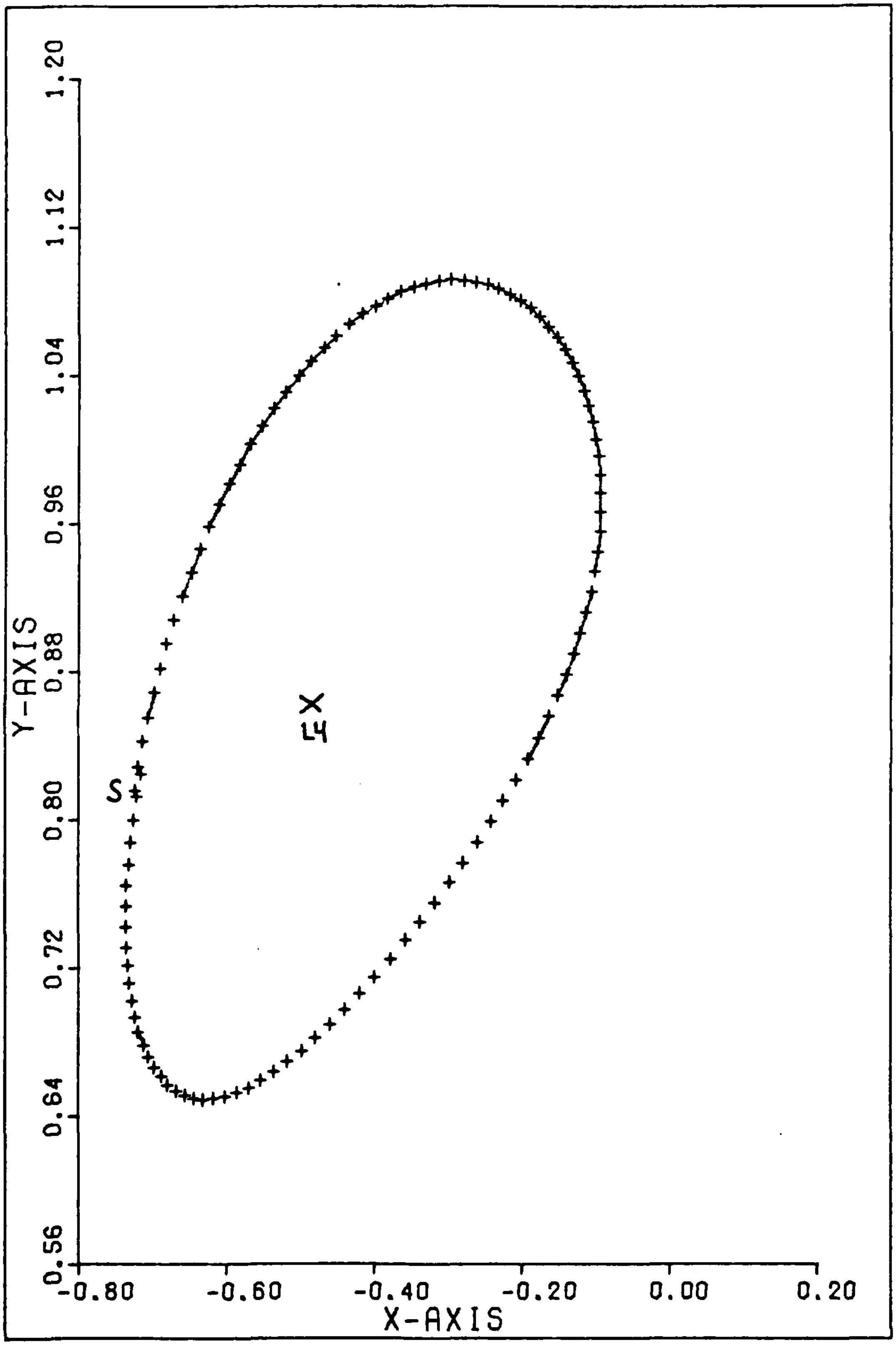


Fig 11. Wheeler Orbit With Sun-Perturbed Lunar Motion, 1 Orbit

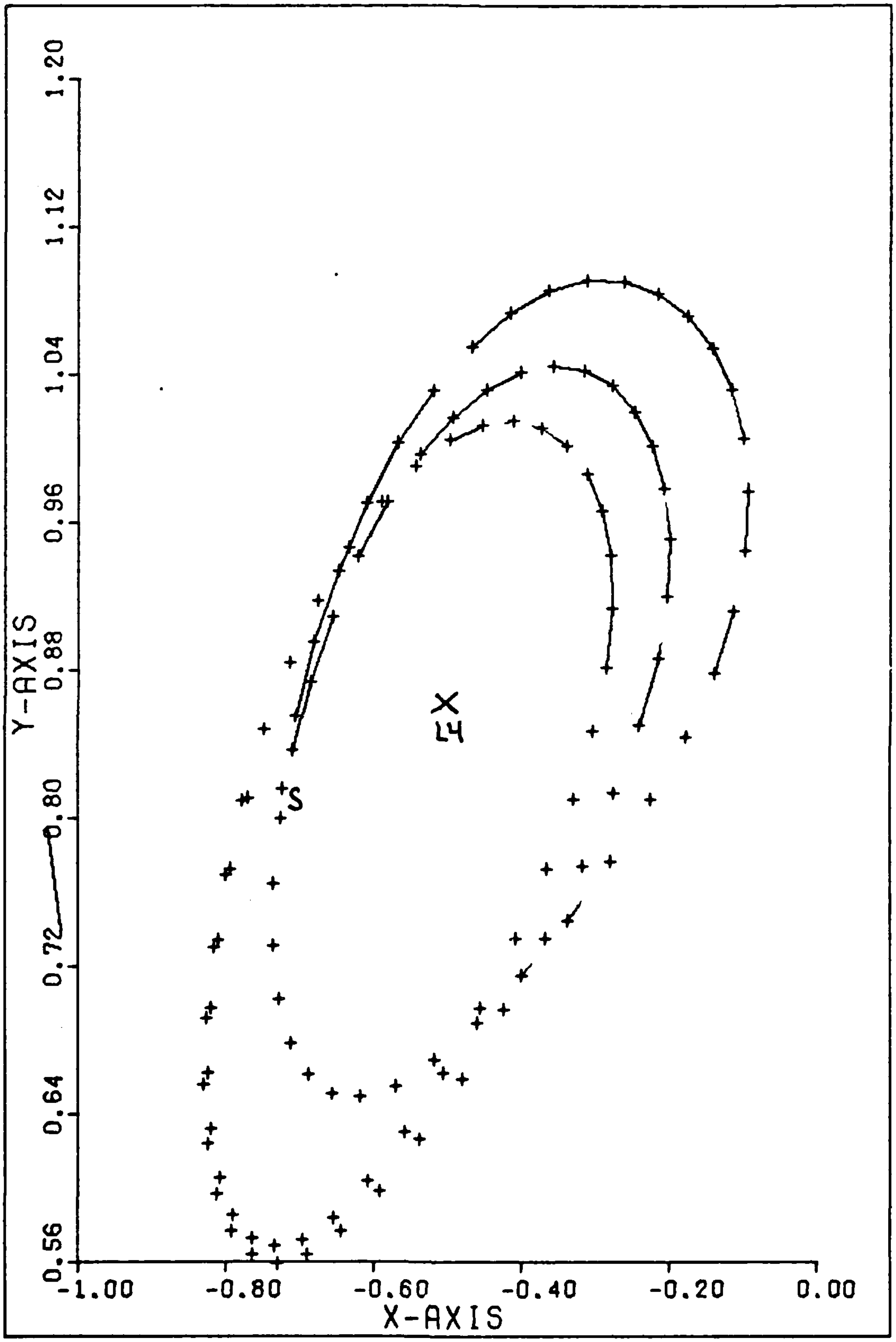


Fig 12. Wheeler Orbit With Sun-Perturbed Lunar Motion, 3 Orbits

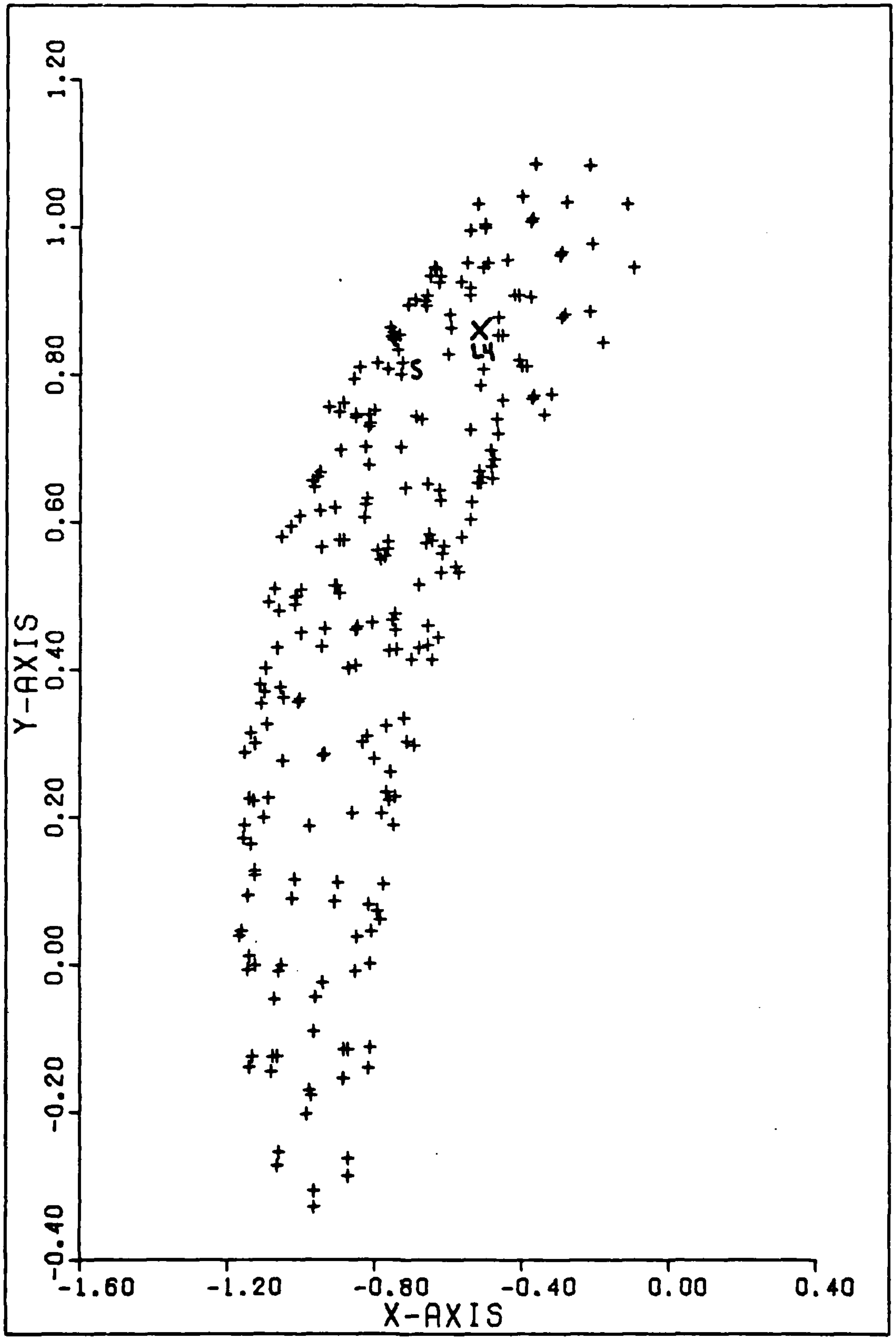


Fig 13. Wheeler Orbit With Sun-Perturbed Lunar Motion, 20 Orbits

acting. The eccentricity of the Sun was input and the orbit integrated for ten years. Figures 8, 9 and 10 depict the orbit for one, three and twenty periods respectively. Note there is only a slight perturbation away at each position compared to the previous orbit. The change from one orbit to the next is at most 650 km anywhere in the orbit and the basic orbit shape or position about L4 does not change after ten years. Orbits after ten years are no more than 1000 km away from the first orbit and tend to return to the starting orbit.

Next the perturbative effect of the Sun upon lunar motion is returned to the problem by using equations (1) and (2). The orbit is now highly perturbed changing shape somewhat and up to 4000 km off initial orbit position the second time around. The orbit slowly begins to drift out of the quadrant of Wheeler's frame after only thirteen months. This is not so damaging, but the changing shape of the orbit shows little consistency in the orbit, as Figures 11, 12 and, especially, 13 will attest. After ten months the orbit approaches the Moon close enough to be thrown out of the Earth - Moon system completely.

The truth model appeared to prove the Wheeler orbit unstable. Analysis was made of the data to try to determine the cause of the instability. The eccentricity of the Sun is removed with little effect on the orbit, showing that the direct influence of the perturbing Sun does not change the stability of a periodic orbit with its small deviations. The two terms of the Sun's effect on lunar motion were analyzed and found to offset each other to such a degree that removing one or the other cau s the orbit to deteriorate even more rapidly. Finally, the phase difference between the Sun and Moon is examined in the truth model. This is thought to be the primary cause for the loss of stability. A program is written to minimize the difference in the phase of the two bodies by adjusting the solar mean motion. If the Sun and Moon were held in phase, this author believes the stability would return. The optimization was not very efficient but after three cycles had improved stability 50%. The orbit is indeed sensitive to the Moon getting out of phase with the Sun.

During this phase analysis an oversight came to light. Although the lunar motion now includes perturbing forces of the Sun to force the Moon out of circular motion, the lunar initial conditions are still the circular orbit conditions determined earlier (18). Elliptical initial conditions must be found. Kolenkiewicz and Carpenter (Ref 26) developed a model of the Moon's orbit using a three-body system which is explained in detail in the Chapter Four. Since a perturbed orbit was obtained using the two primary bodies, Earth and Sun, which affect it, this orbit is considered accurate enough to supply initial conditions.

$$x_m = -\,0.99220573479 \qquad \dot{x}_m = 0.0 \tag{21}$$

$$y_m = 0.0 \qquad \dot{y}_m = 1.00990709043$$

When these perturbed lunar initial conditions were first input, calculator accuracy to eight places were used.

$$x_m = 0.99220574 \qquad \dot{x}_m = 0.0 \tag{22}$$

$$y_m = 0.0 \qquad \dot{y}_m = -1.01055416$$

Now the Moon and Sun are more in phase and the orbit is reasonably stable. The orbit varies by no more than 5% over the first 24 months and the orbit shape or size does not change. It varies by no more than 2% from orbit to orbit. Refer the Figures 14, 15 and 16, for plots of the orbit. The equations of motion are integrated for over 100 years and the orbit moves slowly around the Earth in a clockwise direction away from the Moon, returning to the original orbit about L4 after 94 years with agreement to within three decimal places. Refer to Figure 20 which plots one point on every fourth orbit for 100 years. It is sixteen years before the orbit begins to leave this quadrant.

Even more surprising results occured when the lunar initial conditions from the computer (21) are input. Although these initial conditions differ only in the fourth decimal place, the orbit (Figure 17, 18 and 19) now moves slowly in a counter-clockwise direction toward the Moon and would again return to the same orbit about L4 after 57 years. The orbit begins to leave the quadrant after ten years but both of these orbits will move into the Moon's vicinity and be thrown out of cislunar space. The apparent drift of the orbit is extremely consistent. The Wheeler frame for plotting the orbits is investigated to discover some reason for the apparent drift. The Wheeler frame rotates at a constant rotation rate to coincide with the lunar synodic month. The elliptical motion of the Moon and the variations in its rotation rate is causing the reference frame to drift rather than the orbit. A new algorithm is formulated

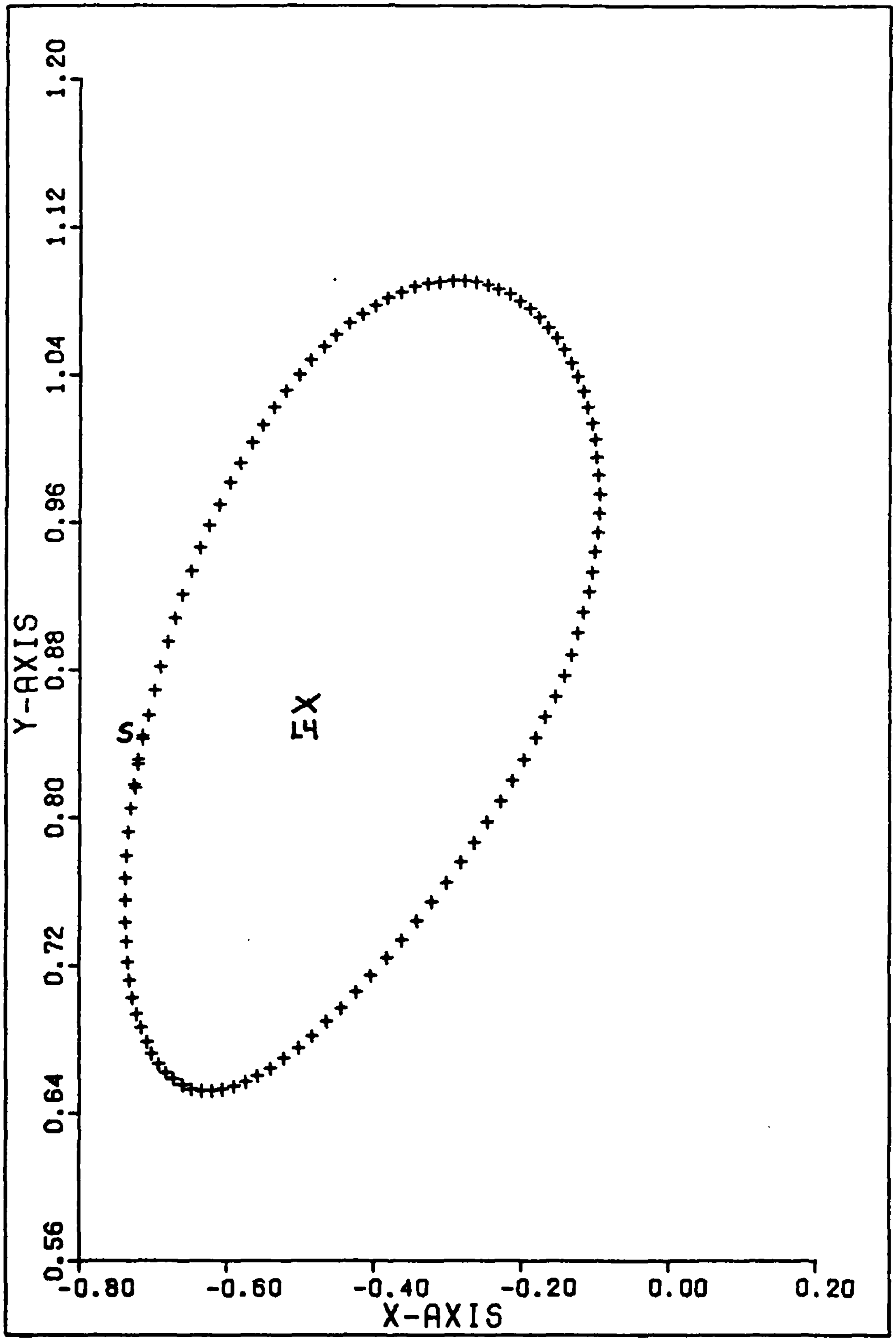


Fig 14. Truth Model of Wheeler Orbit, Calculator Accurate Lunar Init. Cond., 1 Orbit

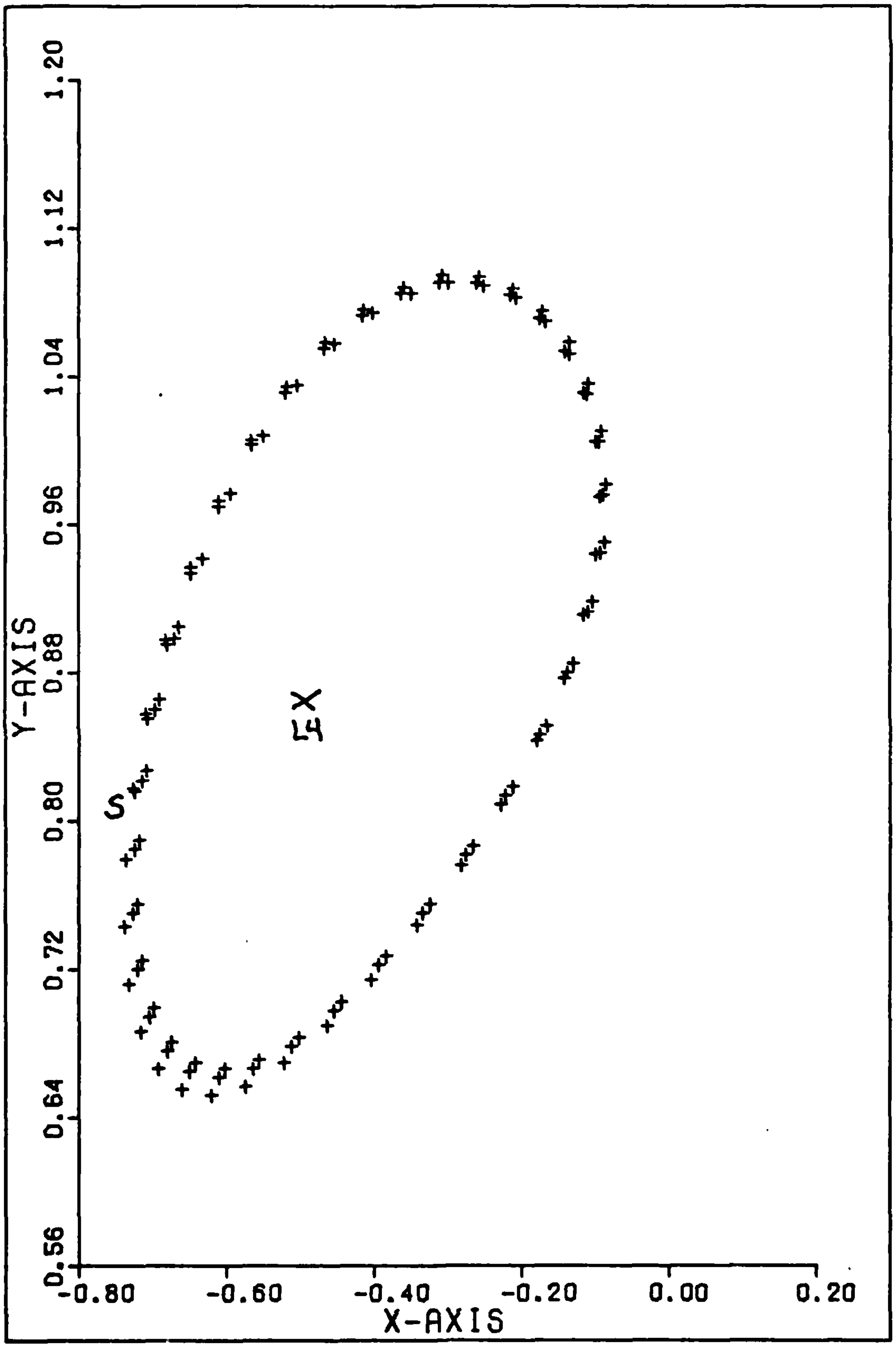


Fig 15. Truth Model of Wheeler Orbit, Calculator Accurate Lunar Init. Cond., 3 Orbits

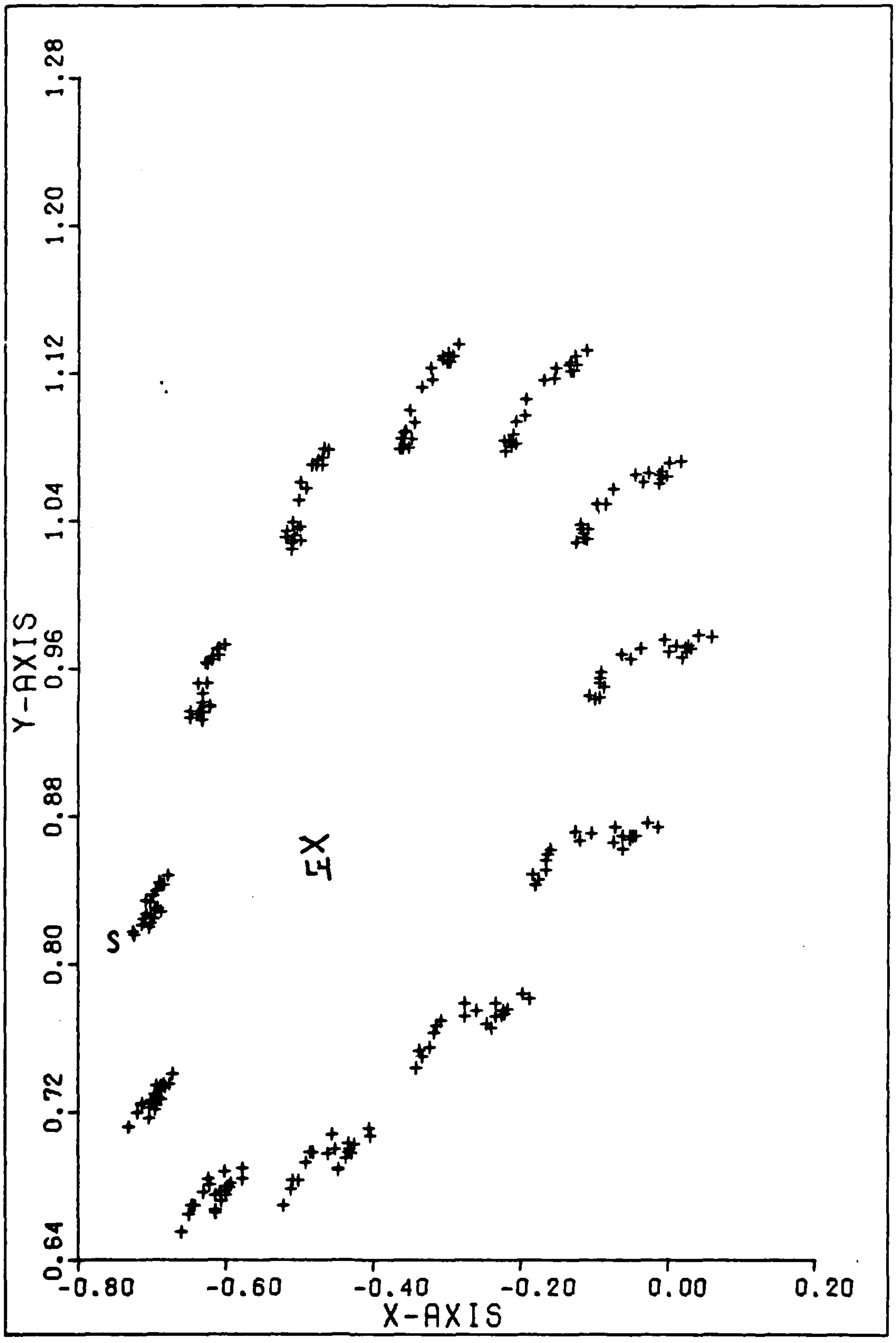


Fig 16. Truth Model Of Wheeler Orbit, Calculator Accurate Lunar Init. Cond., 20 Orbits

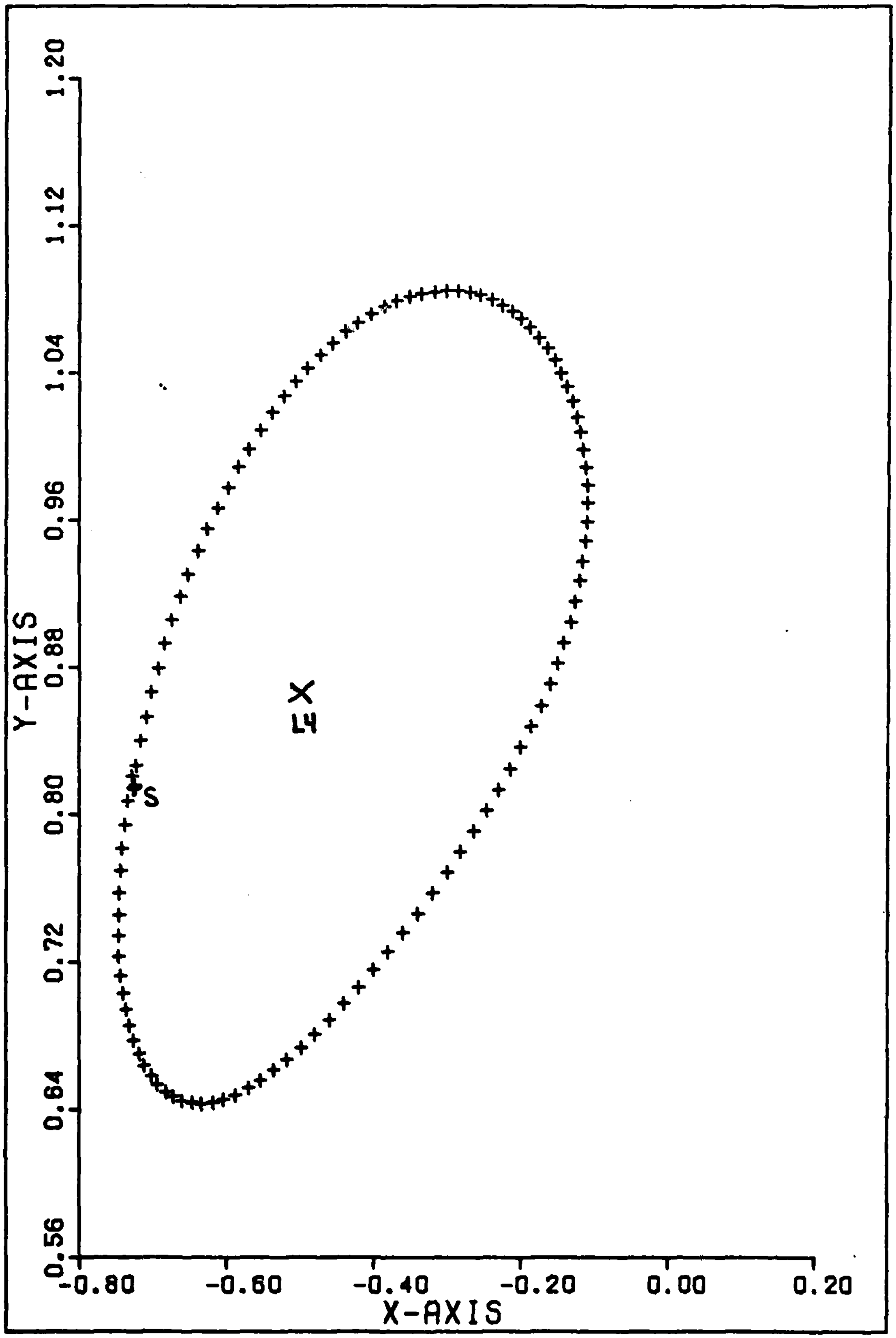


Fig 17. Truth Model of Wheeler Orbit, Computer Accurate Lunar Init. Cond., 1 Orbit

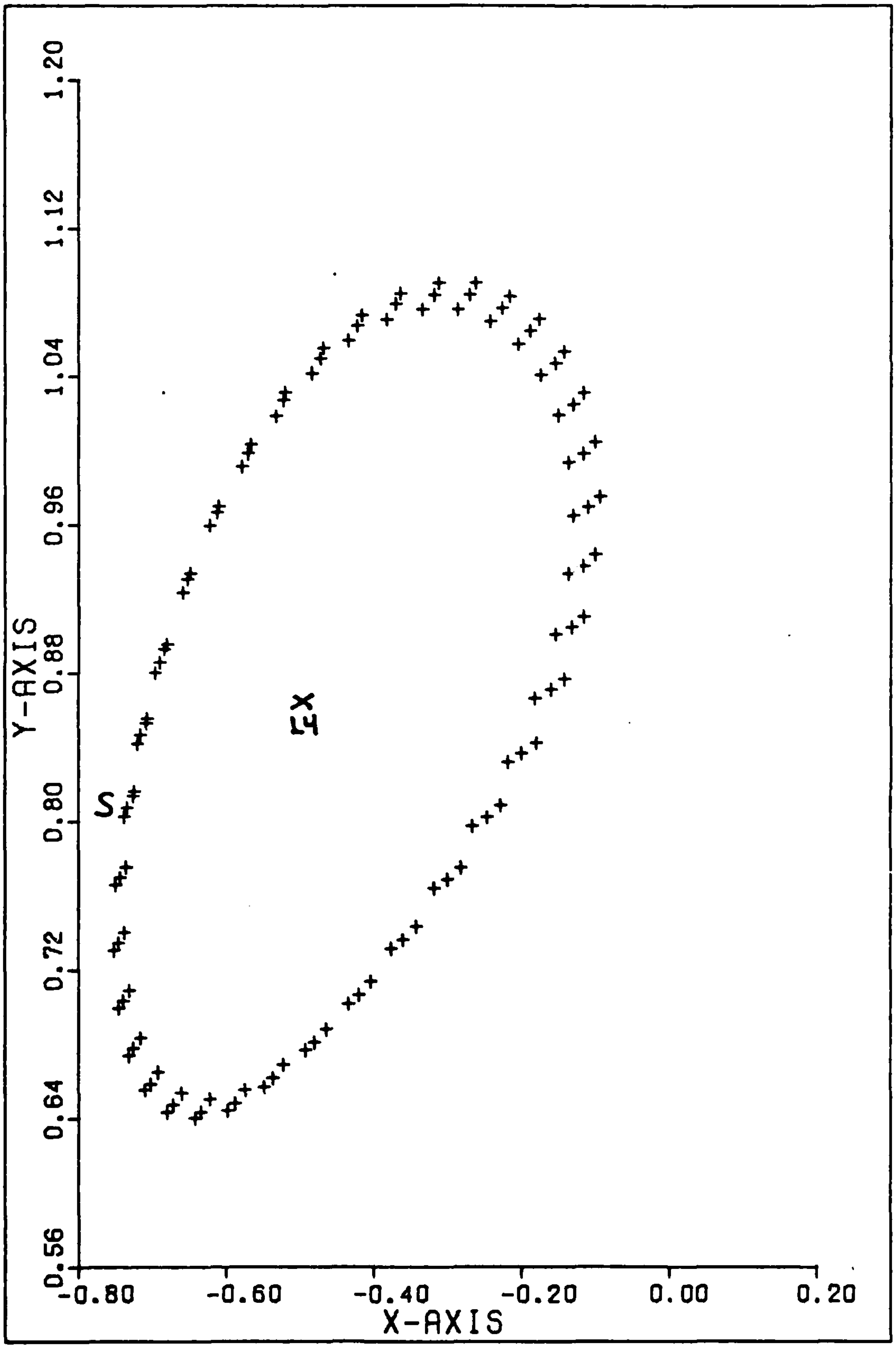


Fig 18. Truth Model of Wheeler Orbit, Computer Accurate Lunar Init. Cond., 3 Orbits

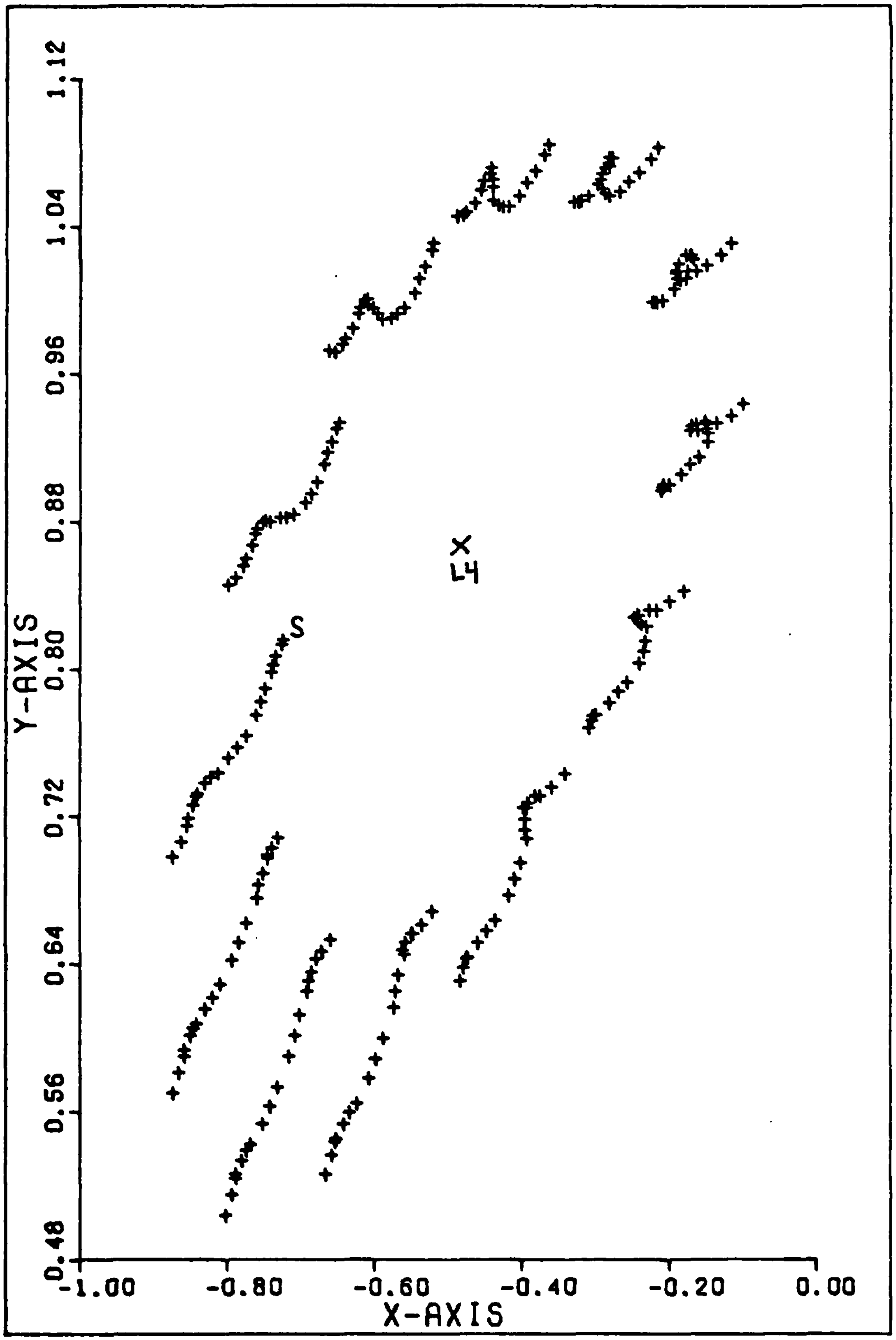


Fig 19. Truth Model of Wheeler Orbit, Computer Accurate Lunar Init. Cond., 20 Orbits

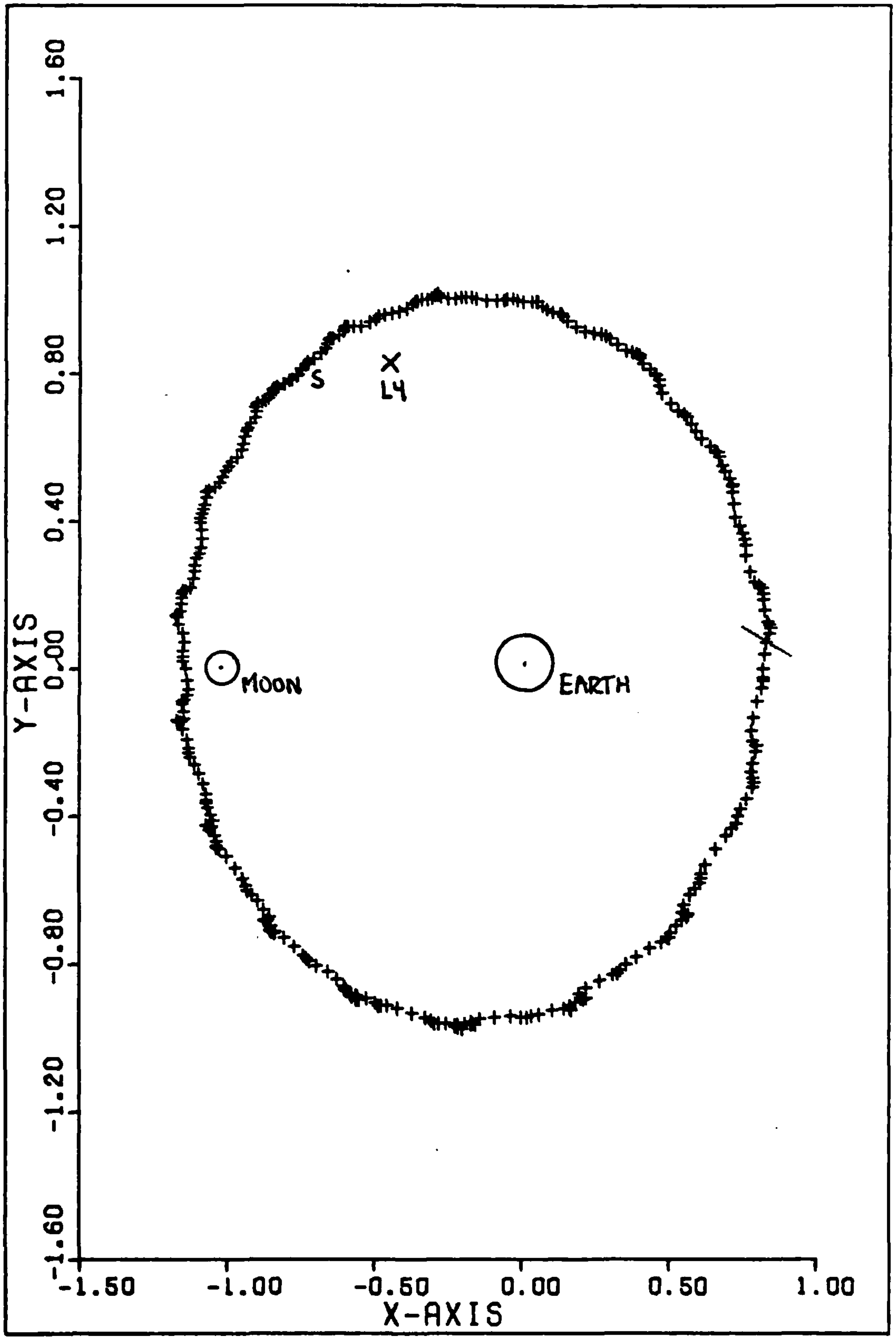


Fig 20. Drift of Wheeler Orbit Over 60 Years

to lock the reference frame to the Moon wherever it is and keep it on the negative x-axis. When this is done the Wheeler truth model is found to be very stable with less than a 5% change in position over 45 months and 10% over seven years. It changes from orbit to orbit no more than 4000 km. The orbit slowly wanders back and forth for over 100 years and never leaves the L4 quadrant. The orbit slowly changes shape over the years depending on its position in the quadrant in relation to the other bodies.

IV. KOLENKIEWICZ AND CARPENTER'S ORBIT

Overview

Kolenkiewicz and Carpenter (hereafter termed K & C) use a restricted four-body model in which the three principle bodies are periodic, coplanar, and obey the equations of motion and have no mean orbital eccentricities (Ref 26). They investigated by trigonometric series the possibility of a coplanar monthly periodic motion in the general vicinity of L4. This investigation of the Sun perturbed Earth - Moon triangular point yields, in addition to a small unstable orbit, two similar but not identical stable periodic orbits about 50% larger than Schechter's stable orbits, one synchronized with the Sun in its motion around L4; the other 180° out-of-phase. The two stable orbits are periodic with respect to the synodic system, make one loop about the triangular point, and are elliptical in shape (see Figure 21). The orbits have an approximate semimajor axis of 90,000 miles and a semiminor axis of 44,000 miles. The major axis is perpendicular to the line joining the Earth and L4. The particle describing the orbits is synchronized with the Sun so their angular positions almost coincide when the particle crosses one of the axes of the ellipse.

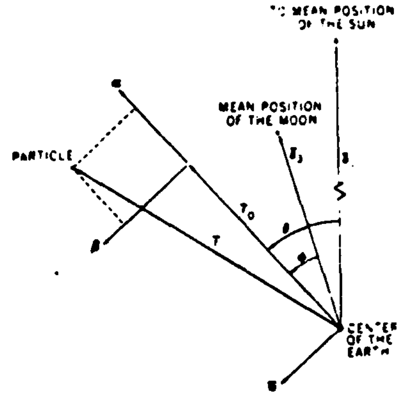
Assumptions and Coordinate System

The assumptions made using K & C's restricted four-body model include:

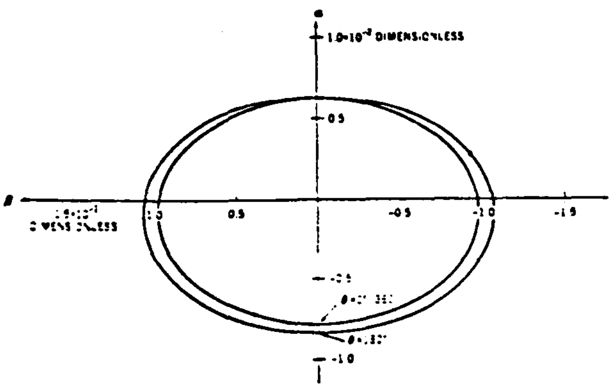
1) Sun, Earth and Moon are considered point masses.

2) The gravitational effects of other planets are ignored.

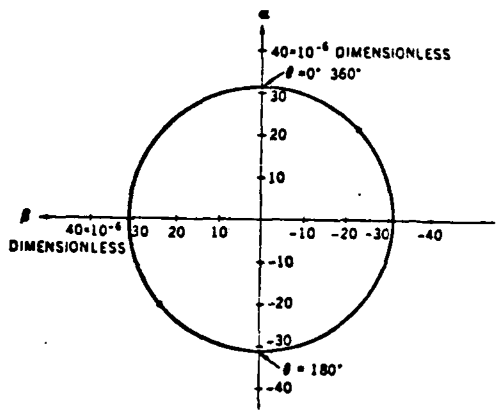
3) Only gravitational forces are considered.



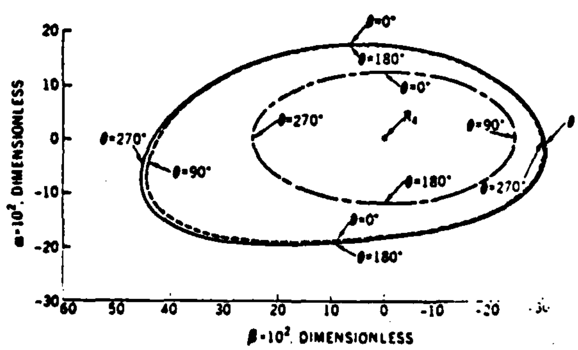
Coordinate system.



Three-body moon solution.



Three-body sun solution.



Comparison of stable triangular point orbit
—·—, orbit from Ref. 1; ——, Orbit I; --- Orbit II.

Fig 21. Kolenkiewicz and Carpenter's Results

4) The motion of all four bodies is limited to one plane.

5) The Moon's orbit is perturbed by the Sun and Earth alone.

6) The Sun's orbit is perturbed by the Moon and Earth alone.

The only difference from the truth model is the unrealistic orbit obtained for the Sun which gives it an approximate eccentricity of only .0007313 instead of the actual value of .0168. This should have little effect since removing the solar eccentricity from the truth model on Wheeler's orbit gives a slightly larger orbit and a bit less stability.

The coordinate system (Figure 21) is the same as the truth model, an Earth - centered inertial system with the Moon and Sun lying along the positive x-axis initially and the triangular point lying 60° off that axis. There is another internal coordinate system for locating the satellite in relation to the triangular point rotating in advance of the Moon. Its origin is located at

$$\phi = 60°; \; \underline{r}_o = a = 3.831841237 \times 10^8 \text{m} \qquad (23)$$

from

$$n_3{}^2 \, a^3 = Gm_e$$

Constants

Many of the constants are in the SI system and will remain or be converted into SI until calculated by the computer but values corresponding to units previously used will appear in parenthesis where applicable.

mass ratios: $\frac{m_m}{m_e} = \frac{1}{81.30}$

$$\frac{m_s}{m_e} = 332958.087932061$$

therefore

$$\text{Solar mass} = m_s = 328912.42 \tag{24}$$

$$\mu = .01215067$$

also

$$\text{Mean solar semimajor axis} = a_s = 149600 \times 10^6 \text{ m}$$

(= 388.8235 mean E - M distances)

$$\text{Mean lunar semimajor axis} - a_m - 3.847487965 \times 10^8 \text{ m}$$

(= 1.0 mean E - M distance)

$$\text{Solar mean motion} = n_s = \frac{129597742''.38}{\text{Julian Century}}$$

$$= 2.0 \times 10^{-7} \frac{\text{radians}}{\text{sec}}$$

$$(= .08084893 \frac{\text{radians}}{\text{time unit}})$$

$$\text{Solar mean motion} = n_m = \frac{1732559353.''56}{\text{Julian Century}}$$

$$= 2.66 \times 10^{-6} \frac{\text{radians}}{\text{sec}}$$

$$(= 1.0808489 \frac{\text{radians}}{\text{time unit}})$$

The mean motions correspond to the lunar synodic month.

Geocentric gravitational constant =

$$Gm_e = 398603 \times 10^9 \text{ m}^3/\text{sec}^2$$

since

$$m_e = .98784933$$

then

$$G = 4.0350283 \times 10^{14}$$

Conversion to Truth Model

K & C's procedure used in obtaining a solution for the Earth - Moon - Sun model as well as for the periodic triangular point orbits is based on Musen's (Ref 30) method with the perturbations represented in trigonometric series with numerical coefficients. The solutions of the equations are given in the following form (see Figure 21)

$$\underline{r} = (1 = \alpha)\ \underline{r}_o + \beta\ \underline{\omega} \tag{25}$$

where α and β are the components of the perturbations, $\underline{r}_o$ is the position vector in a fixed reference ellipse, and

$$\underline{\omega} = (1/n)\ (d\underline{r}_o/dt)$$

The mean motion, n, is given but can be calculated as a check from Kepler's law which is $n^2\ a^2 = \mu^2$ where a is the semimajor axis of the reference ellipse. All the reference ellipses used have zero eccentricity (circles) so $r_o = a$.

The functions α and β are represented by the trigonometric series

$$\alpha = \sum_{k=0}^{\infty} (\alpha_k{}^c \cos k\theta + \alpha_k{}^s \sin k\theta) \tag{26}$$

$$\beta = \sum_{k=0}^{\infty} (\beta_k{}^c \cos k\theta + \beta_k{}^s \sin k\theta) \tag{27}$$

where

$$\theta = (n_m - n_s)\ t.$$

The basic inertial coordinate system matches the truth model and needs no transformation. However, the initial conditions need to be determined from the orbits of the Sun, Moon and satellite and these are described using constant coefficients of a trigonometric series (Tables 1 - 4). The initial conditions of each orbit can be determined from the coefficients at t = o. Expanding on equation (25) and stating the velocity equation, $\dot{\underline{r}}$

$$\underline{r} = (1 + \alpha)\, \underline{r}_o + \frac{\beta}{n} \sqrt{\frac{\mu^2}{r}}\, \underline{a}_2 \tag{28}$$

$$\dot{\underline{r}} = [\alpha - \frac{\beta}{n} \frac{\mu^2}{r_o^{\,3}}]\, \underline{r}_o + [\frac{\dot{\beta}}{n} + 1 + \alpha] \sqrt{\frac{\mu^2}{r_o}}\, \underline{a}_2 \tag{29}$$

where $\underline{r}_o$ is in the $\underline{a}_1$ direction in the truth model frame. The values of α and β can be found from equations (26) and (27) for $t = o$ and $\theta = 0°$ for the Sun and Moon and $\theta=60°$ for the satellite. The equations for $\dot{\alpha}$ and $\dot{\beta}$ are

$$\dot{\alpha} = (n_m - n_s) \sum_{k=0}^{\infty} (- k\, \alpha_k^{\,c} \sin k\theta + k\, \alpha_k^{\,s} \cos k\theta) \tag{30}$$

$$\dot{\beta} = (n_m - n_s) \sum_{k=0}^{\infty} (- k\, \beta_k^{\,c} \sin k\theta + k\, \alpha_k^{\,s} \cos k\theta) \tag{31}$$

Solutions for the motion of the Moon and Sun are found using K & C's same four-body equations. Starting with the Sun constrained to move in a circular, coplanar, Keplerian orbit with respect to the Earth, the equations of motion are solved. Trigonometric coefficients α and β describing the Moon's perturbed orbit are thus obtained. The role of the bodies is reversed, the Moon's motion is constrained to move in the perturbed orbit defined by α and β, and the equations of motion for the Sun are solved. The α and β, coefficients describing the Sun's perturbed orbit are thus obtained. The roles of the bodies are reversed again and again, each time using the latest acquired α and β coefficients of each. Ultimately the values of α and β converge for each body. The

Table 1 Three-body sun solution, $\varphi = 0°$

k	$\alpha_k^{(e)} \times 10^6$	$\beta_k^{(e)} \times 10^6$
0	0.045105	0.000000
1	30.949868	31.492851
2	−0.004947	−0.005012
3	0.047329	0.047319
4	−0.000005	−0.000005
5	0.000184	0.000183
6	0.000000	0.000000
7	0.000001	0.000001

Table 2 Three-body moon solution, $\varphi = 0°$

k	$\alpha_k^{(e)} \times 10^6$	$\beta_k^{(e)} \times 10^6$
0	−906.915740	0.000000
1	287.606767	−609.076345
2	−7173.506863	10202.254541
3	−7.507078	7.212259
4	6.028443	5.719334
5	−0.003392	0.005816
6	0.032454	0.027566
7	0.000011	0.000025
8	0.000187	0.000163
9	0.000000	0.000000
10	0.000001	0.000001

Table 3 Periodic orbit I, $\varphi = 60°$

k	$\alpha_k^{(e)} \times 10^6$	$\alpha_k^{(o)} \times 10^6$	$\beta_k^{(e)} \times 10^6$	$\beta_k^{(o)} \times 10^6$
0	−19171.568123	0.000000	74753.542768	0.000000
1	187801.135978	17178.314916	−13120.769748	−377986.165218
2	11131.030603	−3722.872058	2352.545921	18027.013465
3	−2874.472418	737.568028	−637.564775	−2521.769547
4	582.134781	−176.988257	173.574850	518.179570
5	−123.327707	47.337357	−46.872029	−110.607813
6	26.570848	−12.692022	12.244230	24.010789
7	−5.830085	3.271408	−3.092692	−5.355760
8	1.327555	−0.827212	0.781757	1.245872
9	−0.314593	0.214693	−0.205945	−0.299764
10	0.075554	−0.058253	0.056669	0.072270
11	−0.017829	0.016064	−0.015680	−0.017002
12	0.004114	−0.004335	0.004214	0.003918
13	−0.000955	0.001136	−0.001100	−0.000914
14	0.000230	−0.000296	0.000288	0.000222
15	−0.000057	0.000080	−0.000078	−0.000055
16	0.000014	−0.000022	0.000022	0.000013
17	−0.000003	0.000006	−0.000006	−0.000003
18	0.000001	−0.000002	0.000002	0.000001

Table 4 Periodic orbit II, $\varphi = 60°$

k	$\alpha_k^{(e)} \times 10^6$	$\alpha_k^{(o)} \times 10^6$	$\beta_k^{(e)} \times 10^6$	$\beta_k^{(o)} \times 10^6$
0	−18160.912624	0.000000	72212.688988	0.000000
1	−183627.357659	−16818.088205	11392.917179	370250.263893
2	10460.900708	−3371.790401	2116.289582	17696.741894
3	2715.813738	−648.117571	563.594410	2418.999900
4	544.831763	−152.630065	151.103055	487.952665
5	114.117818	−40.353414	40.270023	102.700732
6	24.309814	−10.697742	10.365541	22.014409
7	5.274431	−2.716444	2.571492	4.851307
8	1.187793	−0.674485	0.637277	1.115552
9	0.278619	−0.171971	0.164972	0.265686
10	0.066360	−0.046021	0.044817	0.063541
11	0.015564	−0.012554	0.012271	0.014861
12	0.003576	−0.003350	0.003259	0.003409
13	0.000826	−0.000865	0.000838	0.000792
14	0.000198	−0.000222	0.000215	0.000192
15	0.000049	−0.000059	0.000057	0.000048
16	0.000012	−0.000016	0.000016	0.000012
17	0.000003	−0.000004	0.000004	0.000003
18	0.000001	−0.000001	0.000001	0.000001

initial conditions of the Sun formed here will not be used because the computed approximate eccentricity is .0007313 which is not close to the actual .0168 (Ref 20) since other forces obviously act on the Sun to perturb its orbit besides the Earth and Moon.

The lunar initial conditions (21) are computed using the coefficients from Table 2, equations (26) and (31), $r_o = a_m$ and

$$\mu_m^2 = n_m^2 a_m^3 = 4.0299 \times 10^{14} \text{ m}^3/\text{sec}^2$$

Since $\mu_m^2 = G(m_m + m_e)$ also, this was used as a check using values from (24).

The satellite initial conditions for both orbits are computed in a similar manner using the coefficients from Tables 3 and 4, equations (26), (27), (30) and (31), $r_o = a$ from (23), θ=60°, and $\mu_c^2 = Gm_e = G(m_e + m_c)$ from (24) and the direction of $\underline{r}_o$ and $\underline{\omega}$ are

$$\underline{r}_o = a(\cos\theta\ \underline{a}_1 + \sin\theta\ \underline{a}_2)$$

$$\underline{\omega} = (1/n_m)\sqrt{\frac{\mu^2}{r_o}}\ (-\sin\theta\ \underline{a}_1 + \cos\theta\ \underline{a}_2)$$

The initial conditions of Orbit I which is synchronized with the Sun in its orbit about L4 are

$$x = .7504475430226 \qquad \dot{x} = -.7370404960431 \qquad (32)$$
$$y = .8127310078686 \qquad \dot{y} = .5259745433478$$

which is the same as Wheeler's transformed conditions (16) to two places except the x value which is about one-hundreth off.

The initial conditions of Orbit II which is 180° out of phase are

$$x = .07542634277321 \qquad \dot{x} = -1.003173415668 \qquad (33)$$

$$y = .9507111102549 \qquad \dot{y} = .254983014774$$

Figure 21 shows the plotted coefficients of the Sun, Moon, and the two orbits compared to Schechter's Reference I orbit (Ref 36).

Truth Model Application

Both orbits are run with the initial conditions and constants stated for over 100 years. Orbit I (Figures 22, 23 and 24) begins with an orbit the same shape but about 5% larger than the Wheeler orbit. The orbit once again changes shape very slowly and to a slight degree as it moves around the Earth due to its orientation to the body. But this orbit drifts faster than Wheeler's orbit. The orbit moves approximately 35,000 km per revolution (month) and begins to leave to quadrant after 17 months. Figure 25 shows a particular point on the orbit each revolution for 34 years if the satellite is not thrown out of the system by a close approach to the Moon. Once again this apparent orbit drift is deceiving because part of it is caused by the Moon drifting off the x-axis of the Wheeler reference frame. When the problem is corrected, the orbit continues to drift but much more slowly, averaging less than 20,000 km per revolution and remaining in the quadrant for seven years. Within a year after leaving the quadrant the satellite is thrown out of the system by a close approach to the Moon.

Orbit II is integrated for over 100 years (Figures 26, 27 and 28). The first revolution is the same size as Wheeler's truth model orbit but it does not close on itself, missing by 10,000 km. The orbit begins to depart the Wheeler frame L4 quadrant at about the same time as Orbit I, but unlike the first

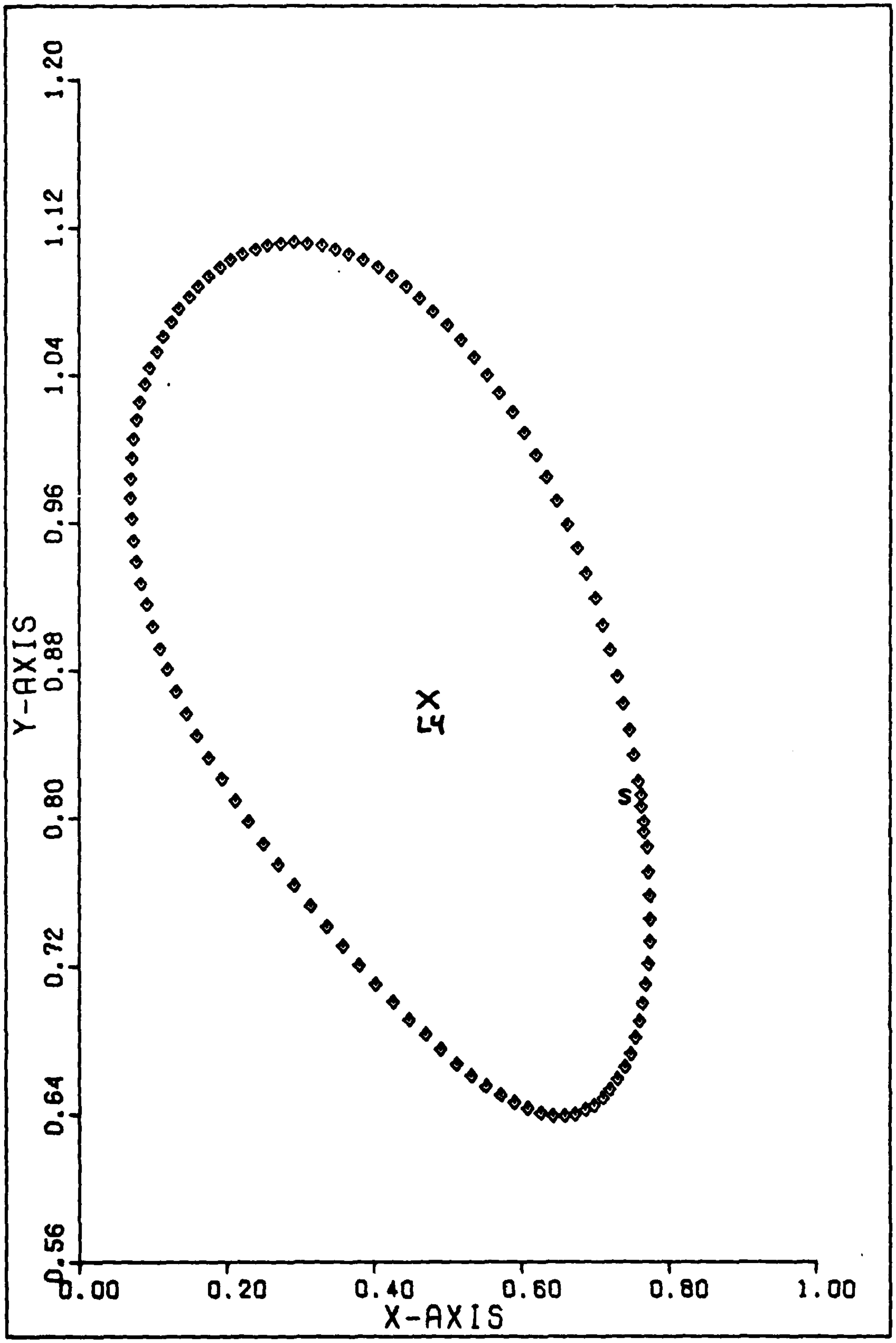


Fig 22. Truth Model of K&C Orbit I, 1 Orbit

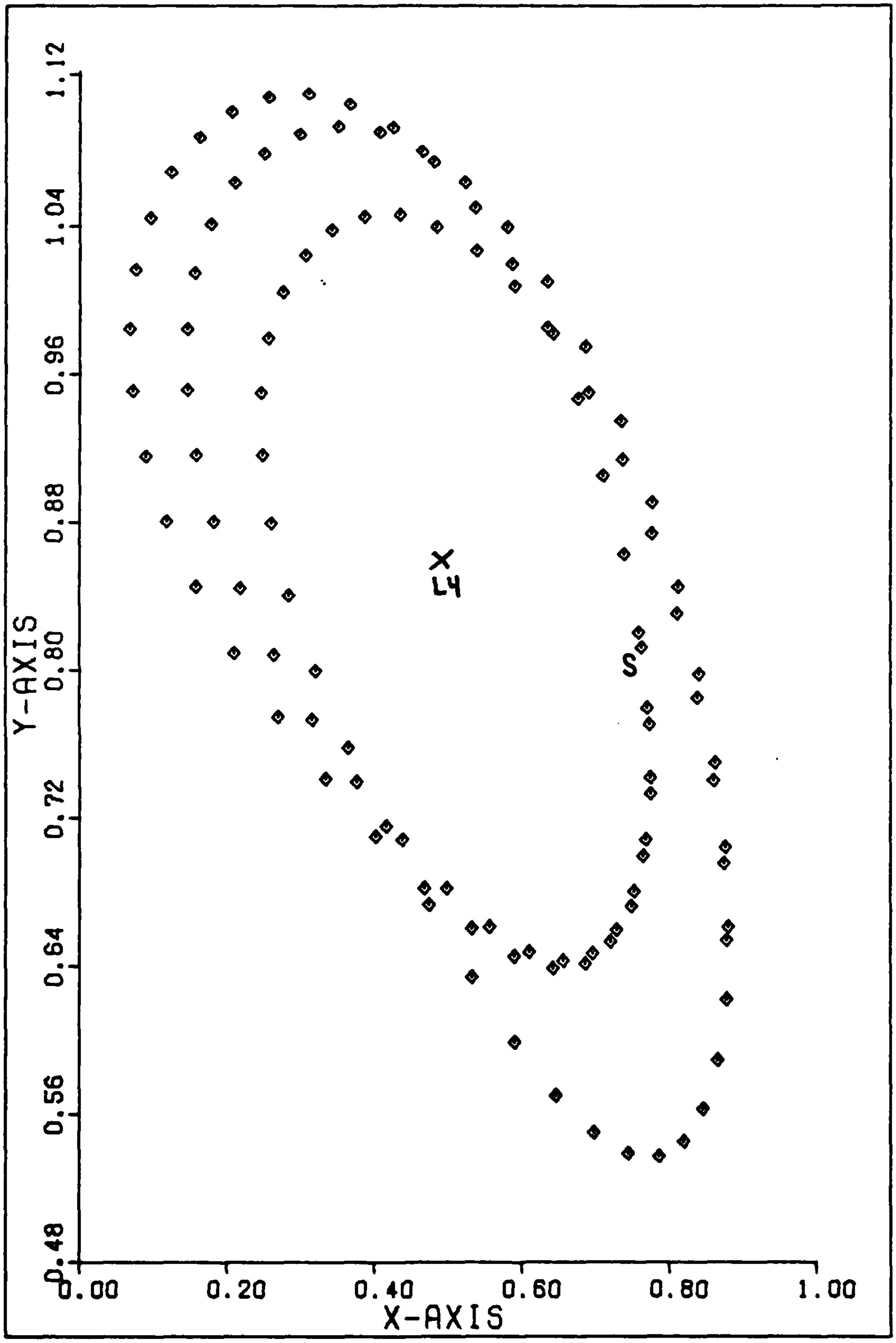


Fig 23. Truth Model of K&C Orbit I, 3 Orbits

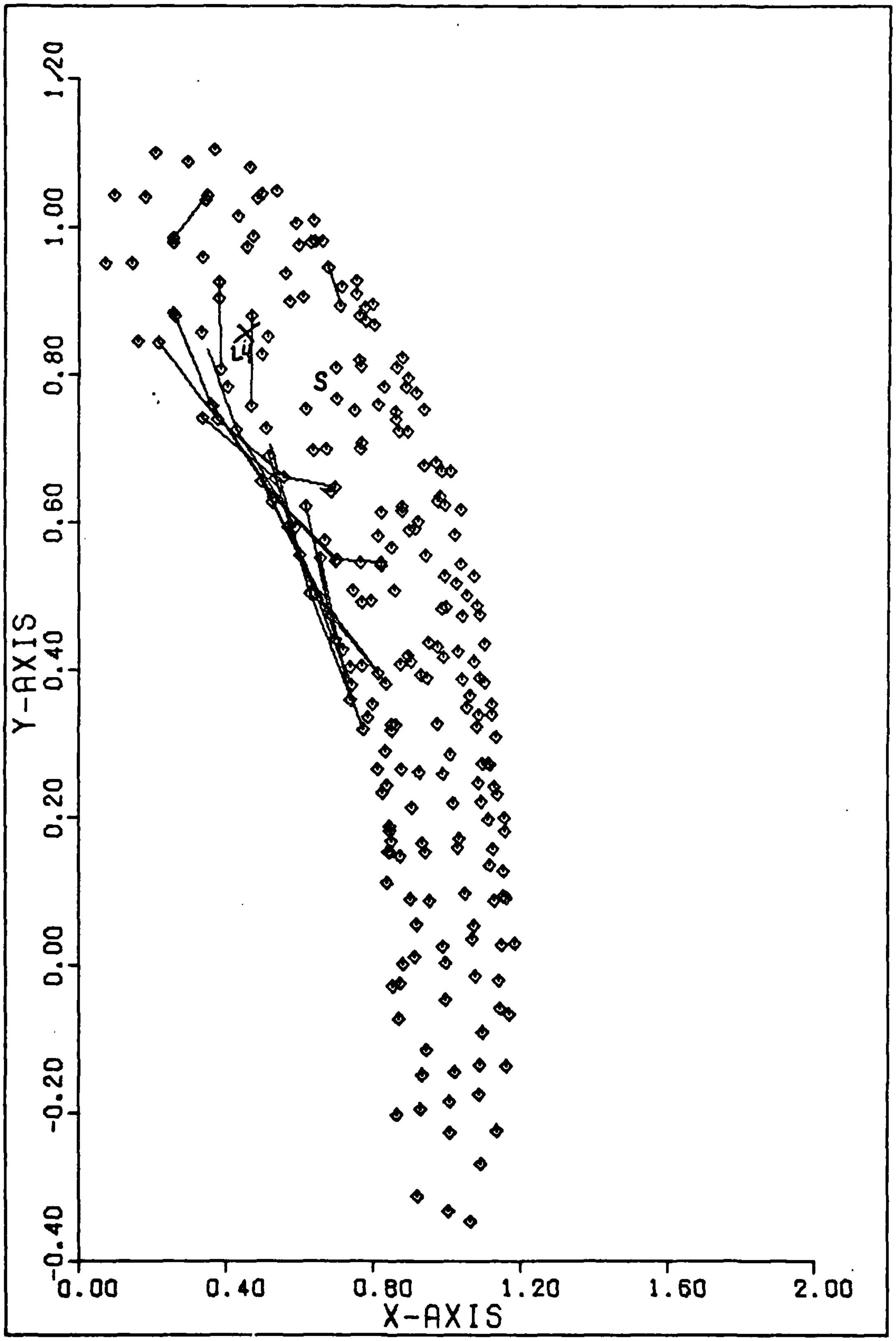


Fig 24. Truth Model of K&C Orbit I, 20 Orbits

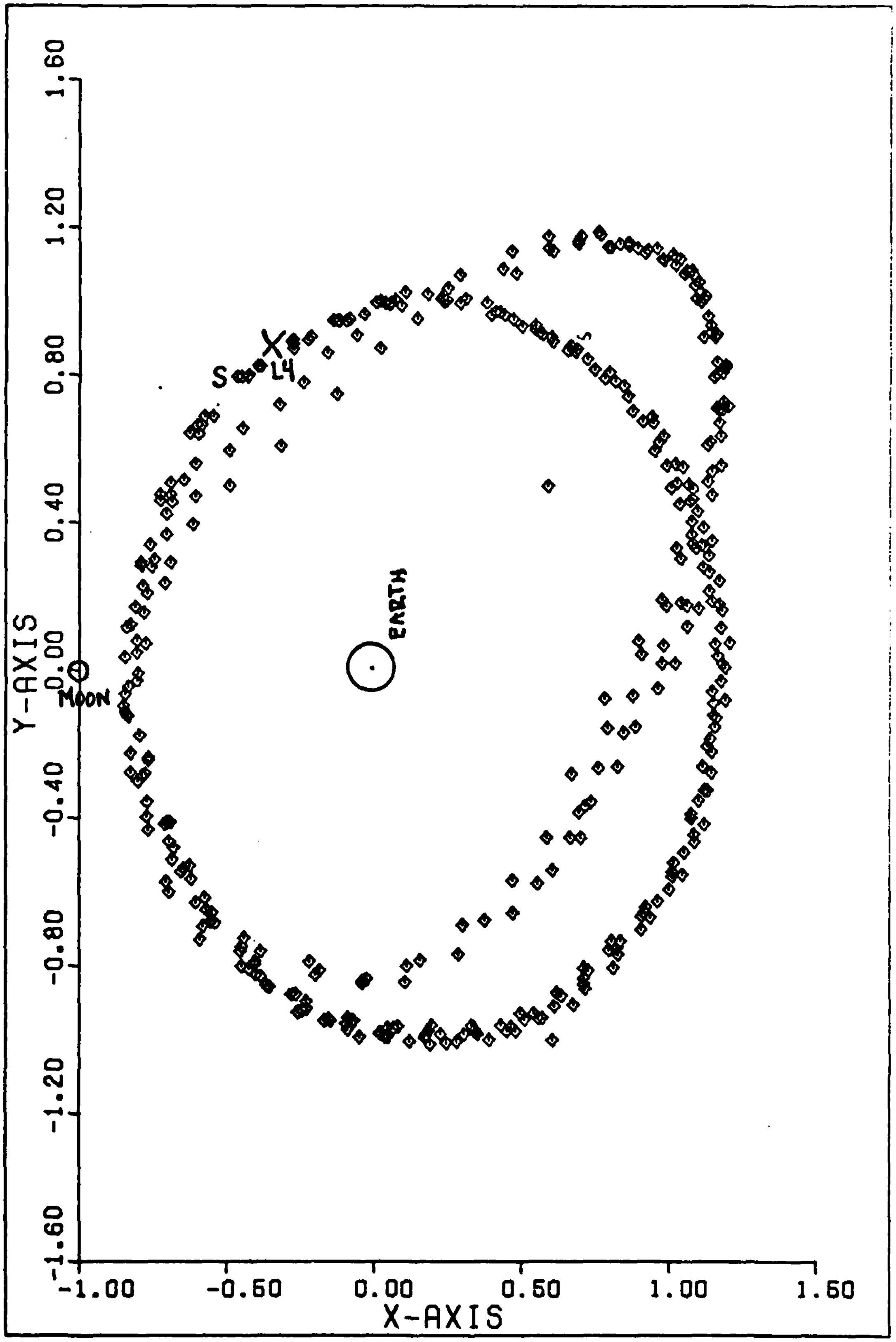


Fig 25. Drift of K&C Orbit I Over 60 Years

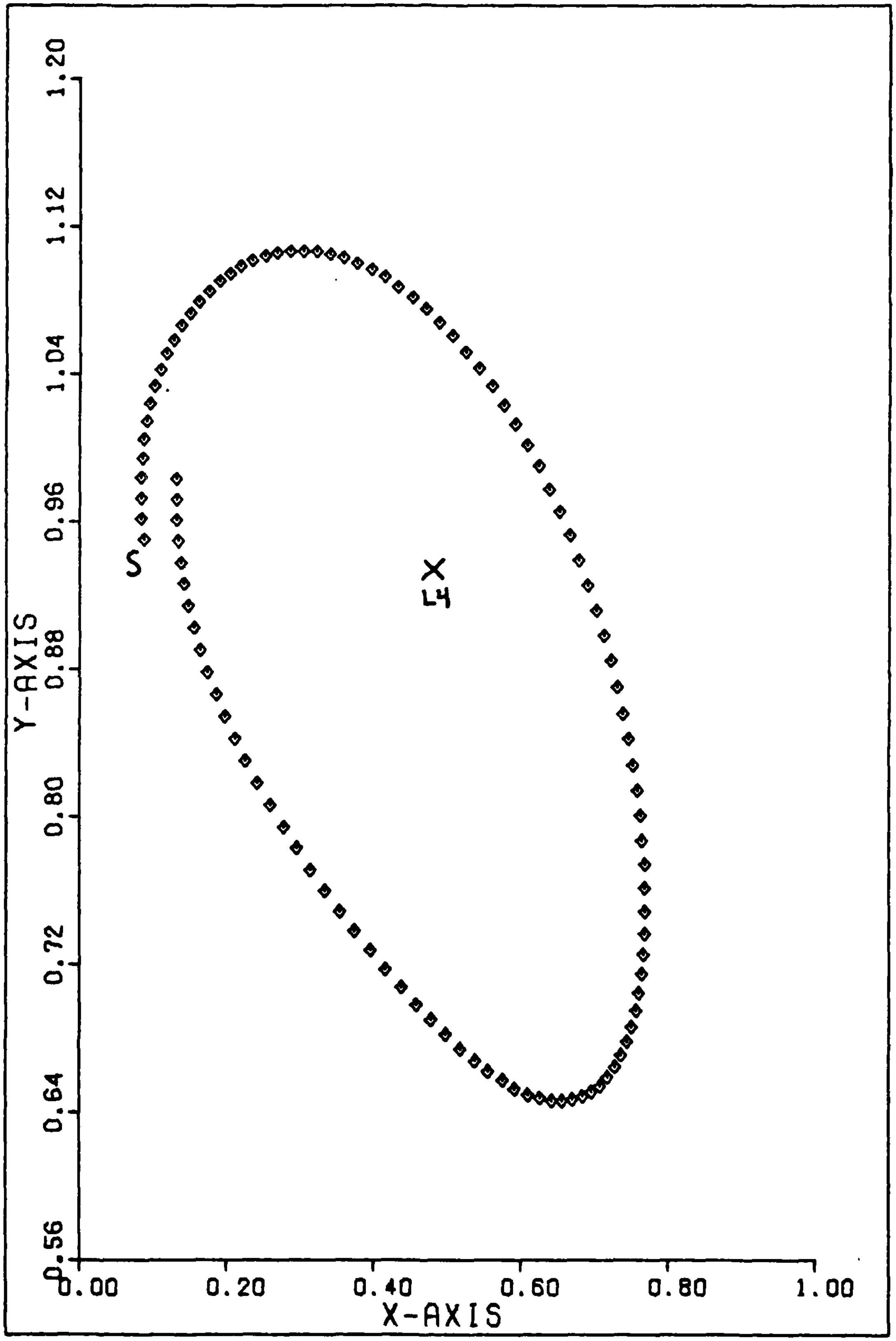


Fig 26. Truth Model of K&C Orbit II, 1 Orbit

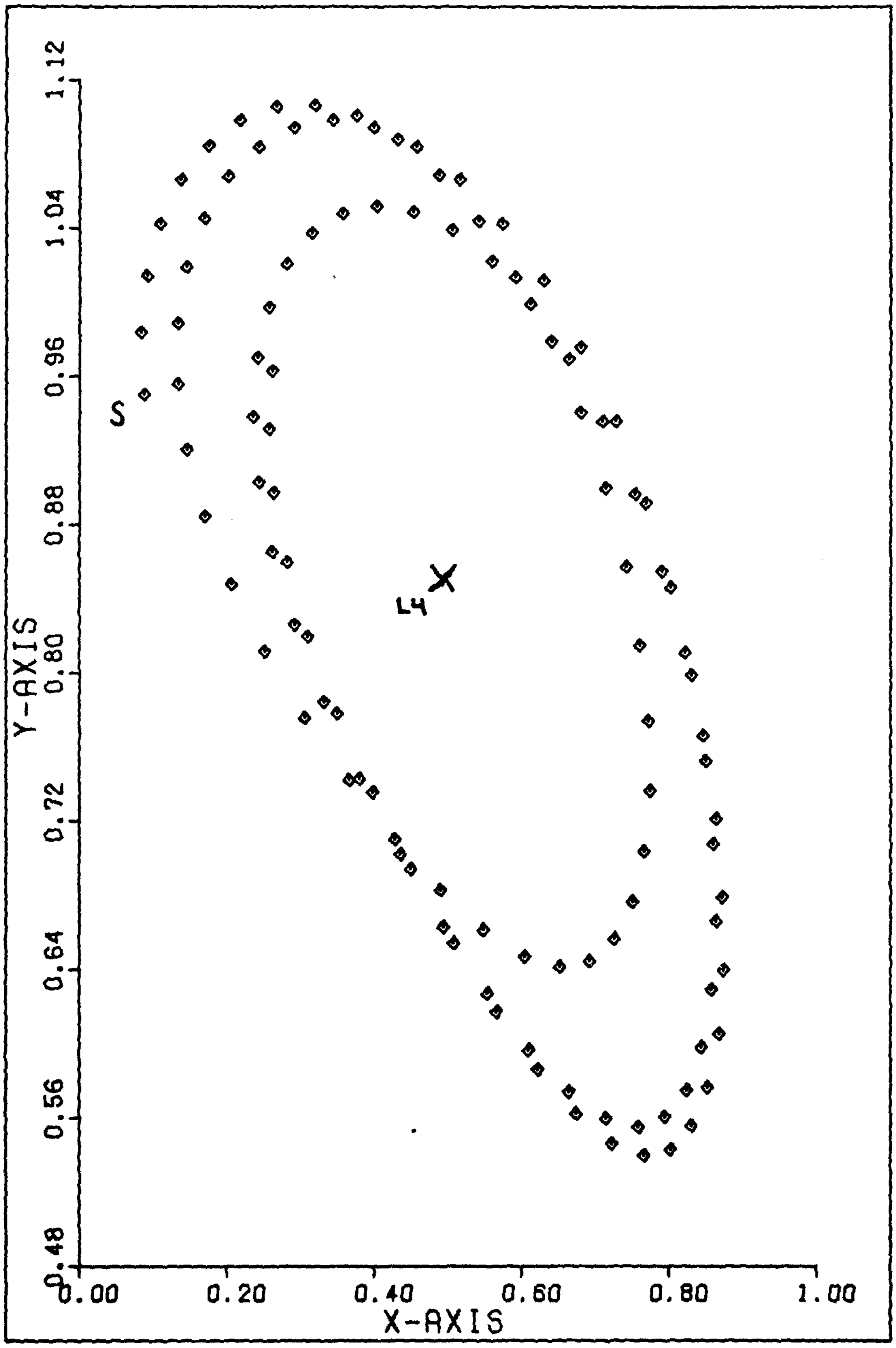


Fig 27. Truth Model of K&C Orbit II, 3 Orbits

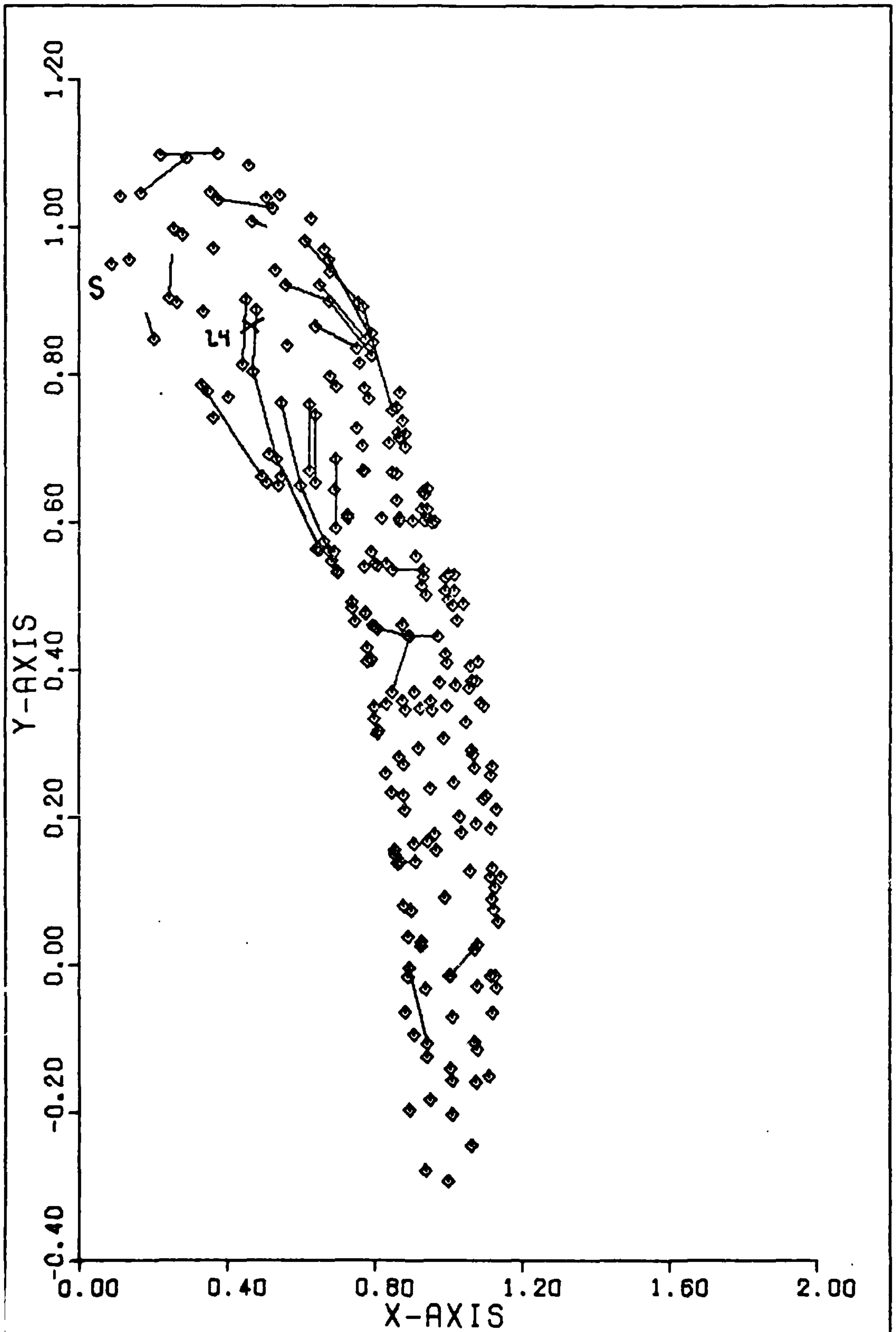


Fig 28. Truth Model of K&C Orbit II, 20 Orbits

orbit, Orbit II rapidly leaves cislunar space after that. The same correction for the Wheeler frame drift was made and the drift rate of the orbit once again is found to be slower than anticipated. The orbit drifts an average of 30,000 km per revolution and now doesn't leave the quadrant for over 10 years instead of an apparent 19 months. Orbit II drifts faster than Orbit I but still has the property of being relatively stable. The shape of the orbit indicates the possibility that with slightly different initial conditions the orbit might come much closer to closing upon itself and, thus, more stable. It is interesting to note that when both orbits initial conditions are entered into the VRFB problem that neither is periodic or very stable.

V. BARKHAM, MODI, AND SOUDACK'S ORBIT

Overview

Barkham, Modi and Soudack (hereafter B, M and S) offer a theoretical solution to a four-body problem (Ref 1). In their model a small particle moves in the vicinity of two masses, forming a close binary, in orbit about a distant mass. Unique, uniformly valid solutions of this four-body problem are found for motion near both equilateral triangle points of the binary system in terms of a small parameter ξ, where the primaries move in accordance with a uniformly - valid three-body solution. Accuracy is maintained within a constant error $0(\xi^8)$ and the solutions are uniformly valid as ξ tends to zero for time intervals $0(\xi^{-3})$. Orbital position errors near L4 and L5 of the Earth - Moon system are found to be less than 5% when numerically - generated periodic solutions are used as a standard of comparison. Once again the lunar synodic month is used and a 2π periodic stable orbit is found which makes two cycles per month. It is a small orbit with a semimajor axis slightly over 6000 km. The theoretical orbit was checked with numerical solutions of four-body perturbation equations of motion, from which the theoretical solution is derived, simplified to three-body and with numerical solutions of the theoretical three-body equations where three-body solutions of the Earth and Moon orbits are used to simplify the problem. The two orbits computed have very little difference and they differ from the theoretical solution by no more than 5%.

Assumptions and Coordinate System

The theoretical model assumed

1) Sun, Earth and Moon are considered point masses.

2) The gravitational effects of other planets are dropped.

3) The motion of all four bodies lies in one plane.

4) All perturbations on the satellite except those caused by the Sun neglected.

5) The Sun orbits in circular motion.

6) Distance to Sun is decreased from infinity to present distance.

As the Sun's distance increases to infinity the problem reduces to the three-body problem and the particle contracts to the libration point. The checks to the solution try to show that the Sun causes the primary perturbations and the satellite is relatively insensitive to small variations in the orbits of the Earth and Moon. This may be true but indirect effects of the Sun upon the Earth and Moon may cause large enough variations in the orbit to affect long term stability. A circular Sun has only a slight effect on the orbit.

The coordinate system is identical to Wheeler's system (see Figures 2 and 29) with a rotating system with the lunar synodic month about the Earth - Moon barycenter. The Moon and Sun start initially on the negative ξ-axis or x-axis.

Constants

For the Earth - Moon - Sun system the following constants are given (Ref 1)

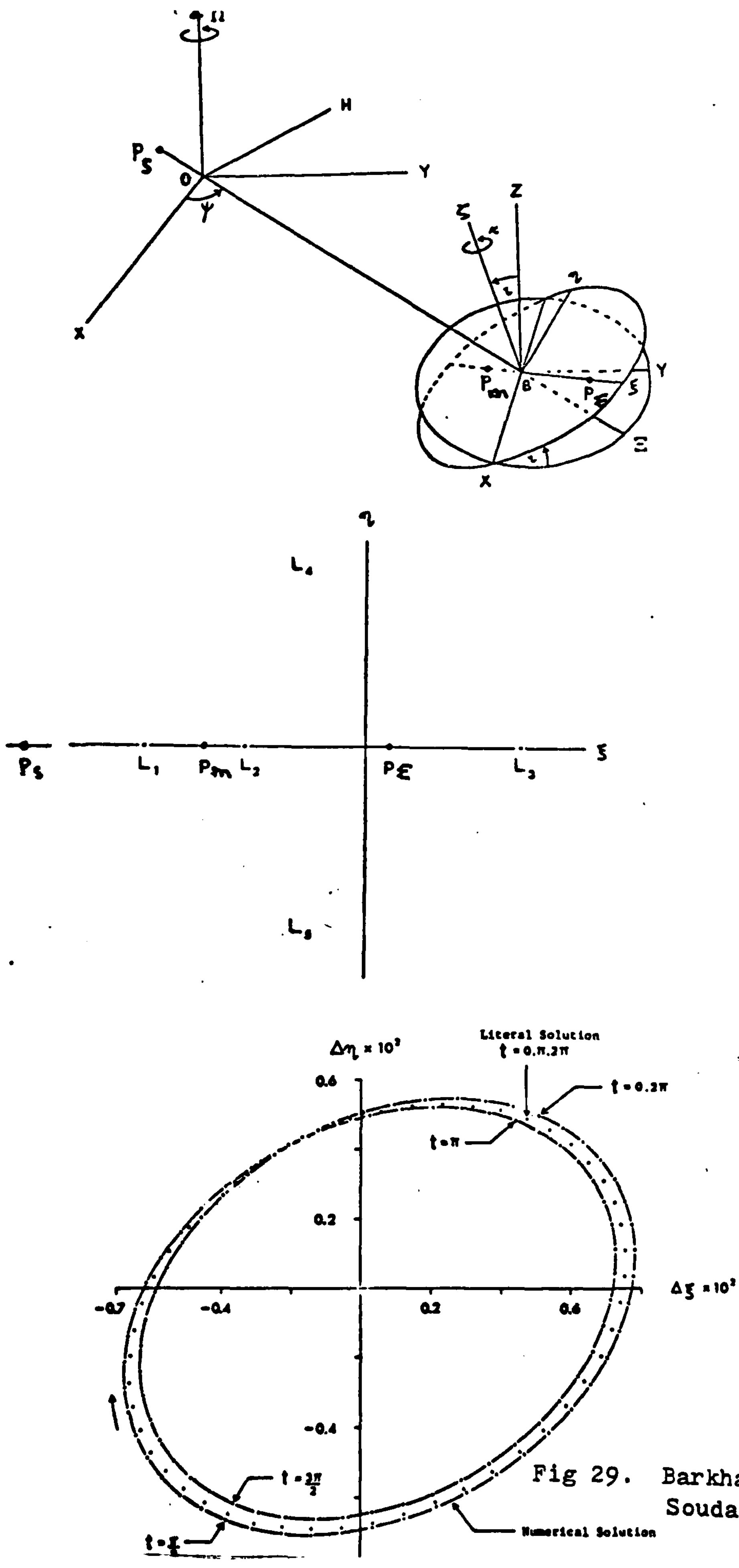


Fig 29. Barkham, Modi, and Soudack Results

Solar mass - m_s = 329794.6384

Mean solar semimajor axis = a_s = 389.17242

Solar eccentricity = 0.0

Solar mean motion = n_s = .0808489

Lunar mean motion = n_m = 1.0808489

μ_m = .01215032

Conversion to Truth Model

The solutions to the satellite and Earth orbits are given by trigonometric series and formulas are given to convert Earth to Moon coordinates. To find the initial conditions substitute t=o into the series solutions. Initial conditions for the satellite are

$$\xi = -.48313894 \qquad \dot{\xi} = .01133797 \tag{34}$$

$$\eta = .87089898 \qquad \dot{\eta} = .0044541$$

and for the Earth are

$$\xi = .01205487 \qquad \dot{\xi} = 0$$

$$\eta = 0 \qquad \dot{\eta} = .00023703$$

converting to the initial conditions for the Moon (Ref 1)

$$\xi = -.98008937 \qquad \dot{\xi} = 0 \tag{35}$$

$$\eta = 0 \qquad \dot{\eta} = -.019272$$

The same translation is used on coordinates (34) and (35) as for Wheeler's system

$$\underline{r} = (\xi - \mu)\ \underline{a}_1 + \eta\ \underline{a}_2$$

$$\underline{\dot{r}} = (\dot{\xi} - \dot{\theta}\eta)\ \underline{a}_1 + (\dot{\eta} + \dot{\theta}\ \xi - \dot{\theta}\mu)\ \underline{a}_2$$

And the velocity components have to be scaled down for the time difference in the periods of rotation used so divide by 1.0808489351. The satellite initial conditions are

$$x = .49528926 \qquad \dot{x} = .86040910 \tag{36}$$

$$y = .87089898 \qquad \dot{y} = .49116833$$

and the Moon initial conditions are

$$x = -.99223969 \qquad \dot{x} = 0 \tag{37}$$

$$y = 0 \qquad y = 1.0100701$$

Truth Model Application

The satellite is unstable when the truth model is applied to it. It slowly begins to orbit L4 and after half a period wanders toward the Moon until being thrown out of cislunar space (Figure 30). The satellite travels 50,000 km in the first lunar synodic month. When the Moon is locked to the x-axis to eliminate drift the orbit is found to be more stable, only increasing its orbit size 10% over the first 25 months. But the orbit shape is much different making a small 3/4 orbit in half of a synodic month and a larger 3/4 orbit in the next half of a synodic orbit. This makes a very irregular orbit shape that would not lend itself to an elementary rendevous with a satellite in this tight orbit. The smaller 3/4 orbit has a semimajor axis of less than 4000 km. It is over seven years before the orbit of the satellite drifts out of the quadrant and is thrown out of the system shortly thereafter. Note that the orbit is unstable in VRFB system. The initial lunar conditions computed by B, M and S are very close to those calculated by Kolenkiewicz and Carpenter.

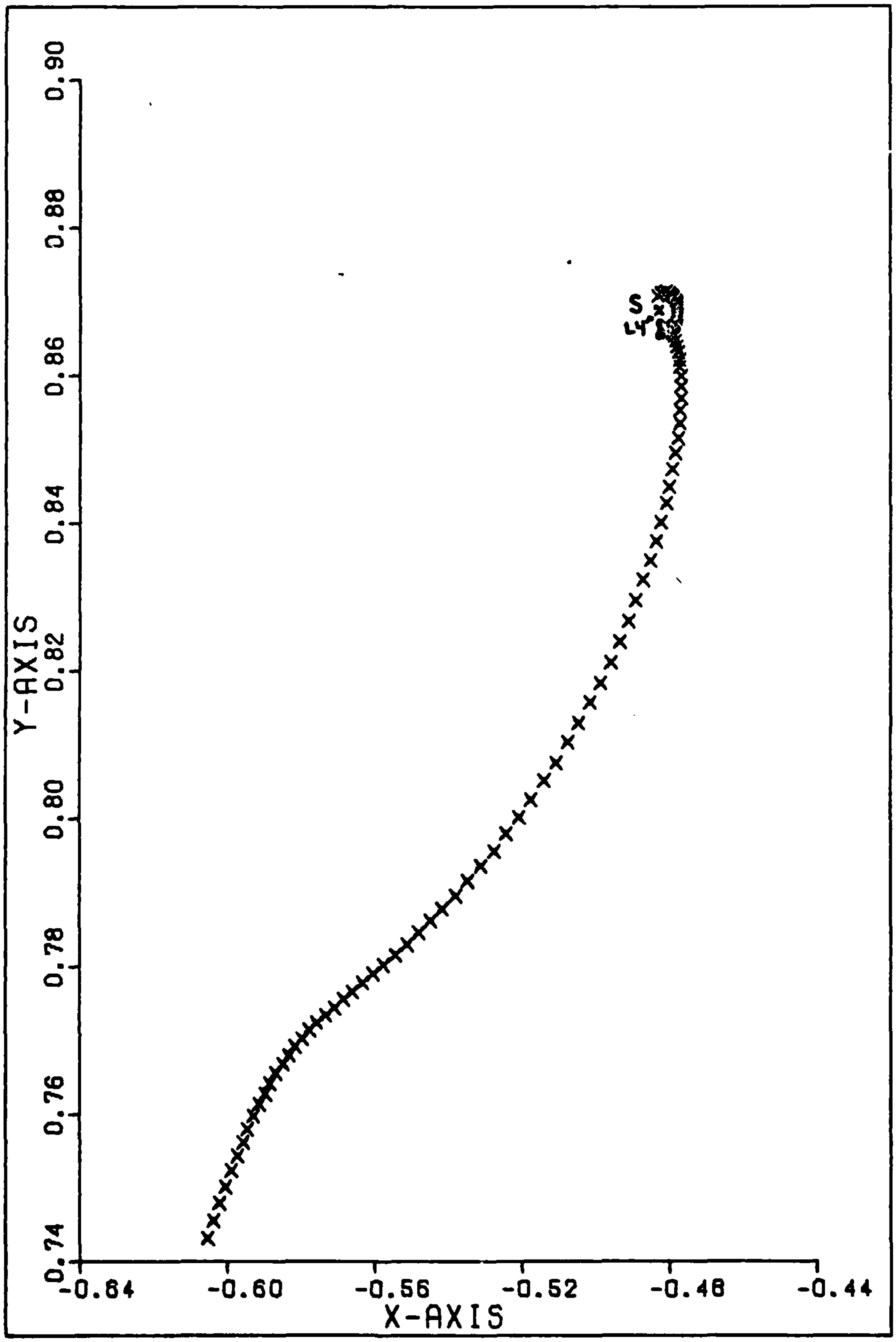


Fig 30. Truth Model of BM&S Orbit,

One Orbit Period

VI. RESULTS

Discussion

With the four-body truth model, none of the orbits found are periodic but some do nearly close on themselves and maintain a period of one lunar synodic month. Wheeler's orbit is stable enough to not require a control system to maintain it in the orbit for at least 20 years (Figures 31, 32 and 33). It will stay in the vicinity of L4 for over a century and may never leave. The transients are small enough for space colonies but may be large enough to affect pointing in a particular direction for military applications. This would have to be investigated; if the minor perturbations could not be accurately predicted or there are fast variations, then a small controller might be needed to maintain an exact orbit but it certainly would not have to be large in size or expensive in fuel consumption.

Kolenkiewicz Orbit I was close to Wheeler's but a one percent error in initial conditions is enough to cause it to be marginally stable (Figure 34). The reason they didn't derive the same initial conditions as Wheeler is probably due to the solar orbit used which is unrealistically perturbed by the Earth and Moon alone.

The same holds for Orbit II, which is also marginally stable (Figure 35). It the initial conditions were derived using a more accurate model, a Wheeler orbit beginning on the opposite side of the orbit might be found.

Barkham, Modi, and Soudack's orbit is marginally stable and not suitable for military or civilian use. It is started too close to L4 and it has been proved by Schechter and many other

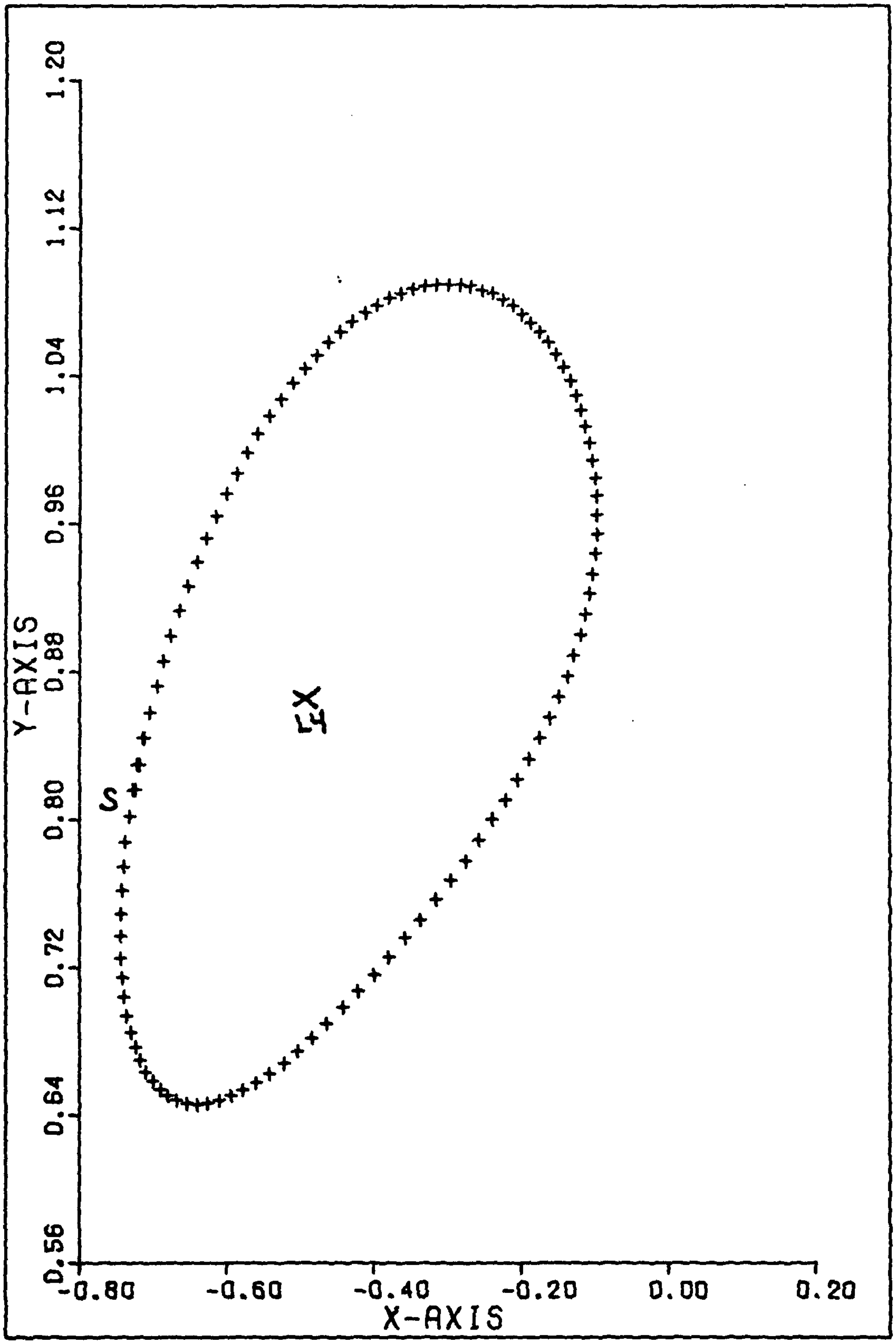


Fig 31. No Drift Display of Wheeler Truth Model Orbit, 1 Orbit

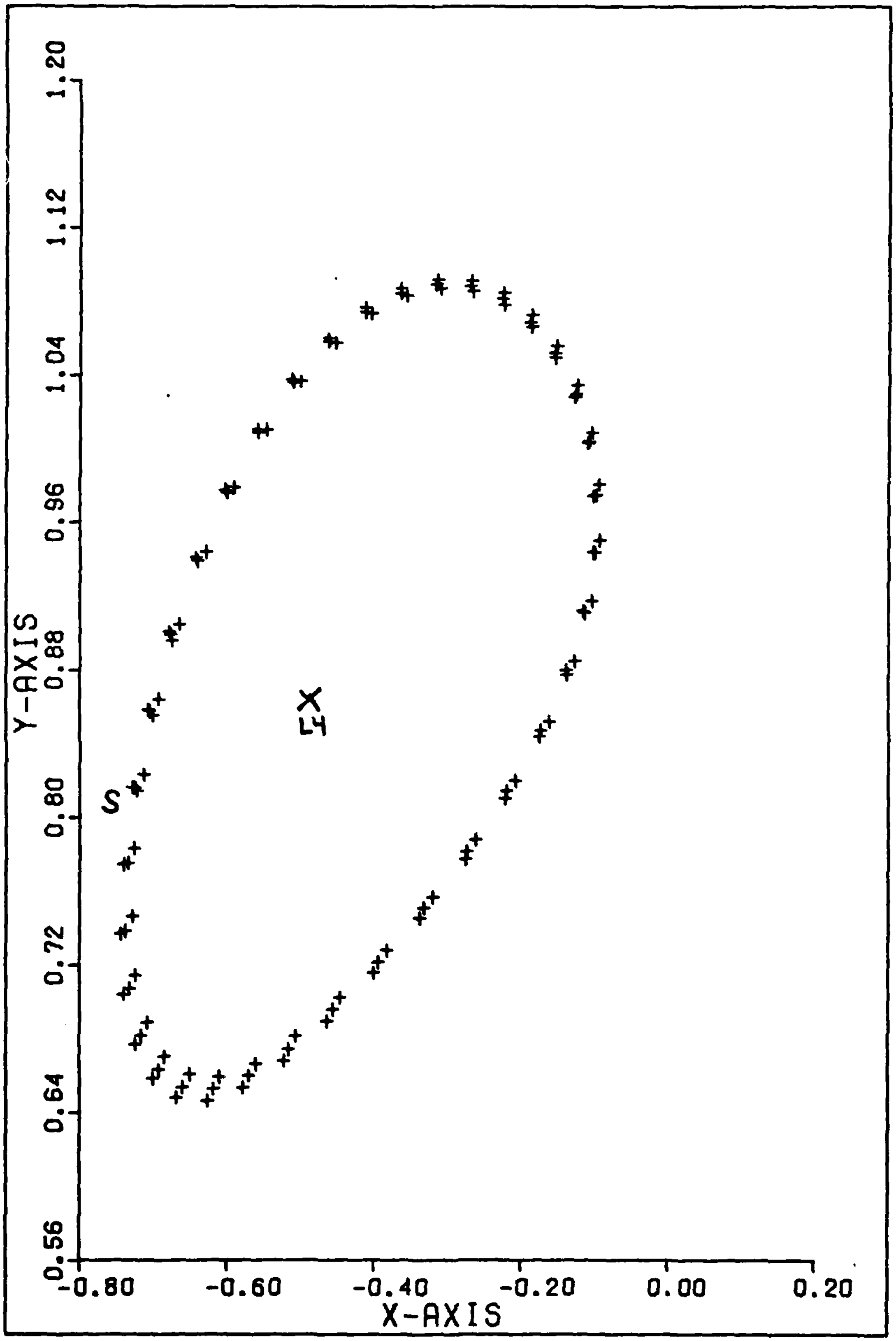


Fig 32. No Drift Display of Wheeler Truth Model Orbit, 3 Orbits

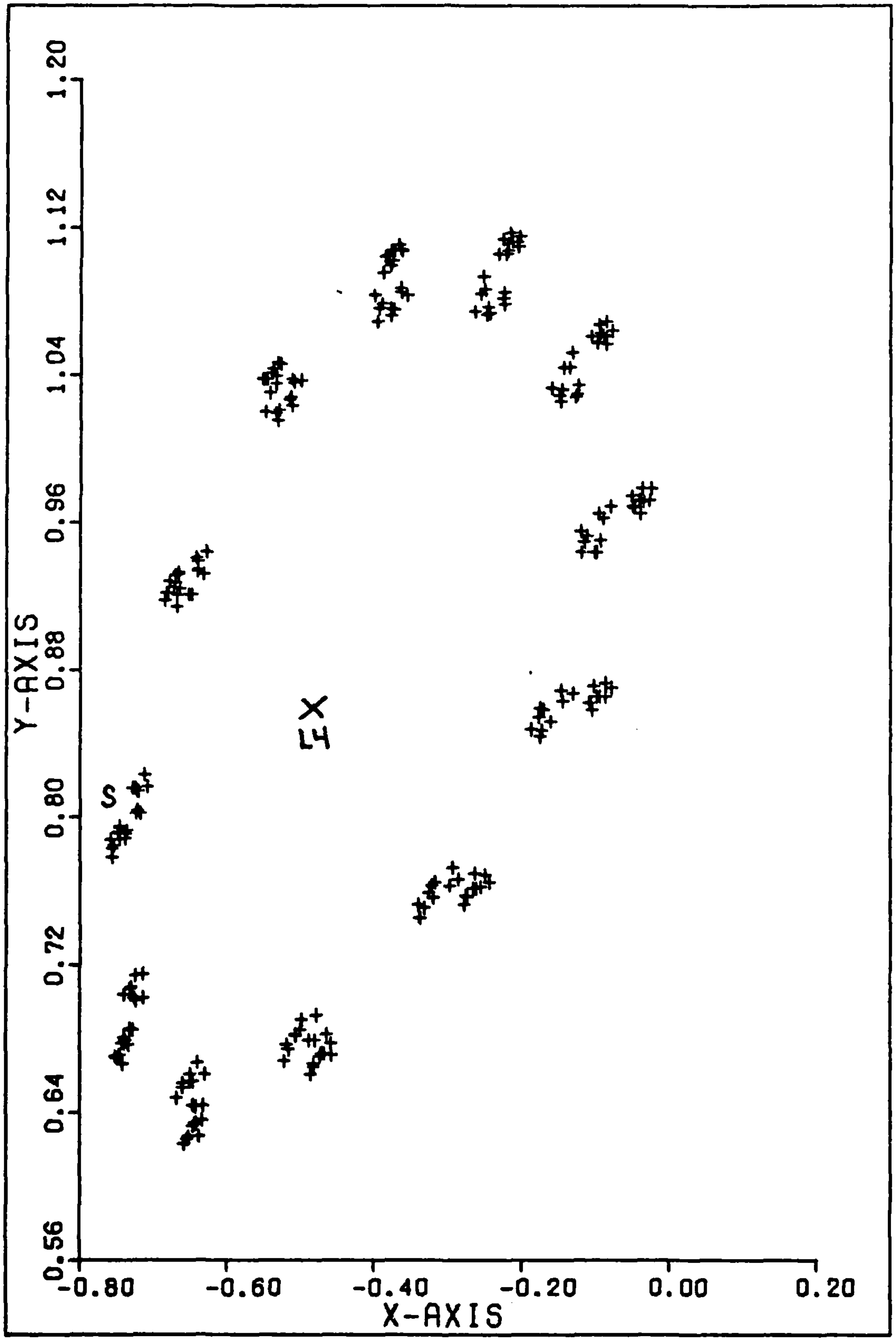


Fig 33. No Drift Display of Wheeler Truth Model Orbit, 20 Orbits

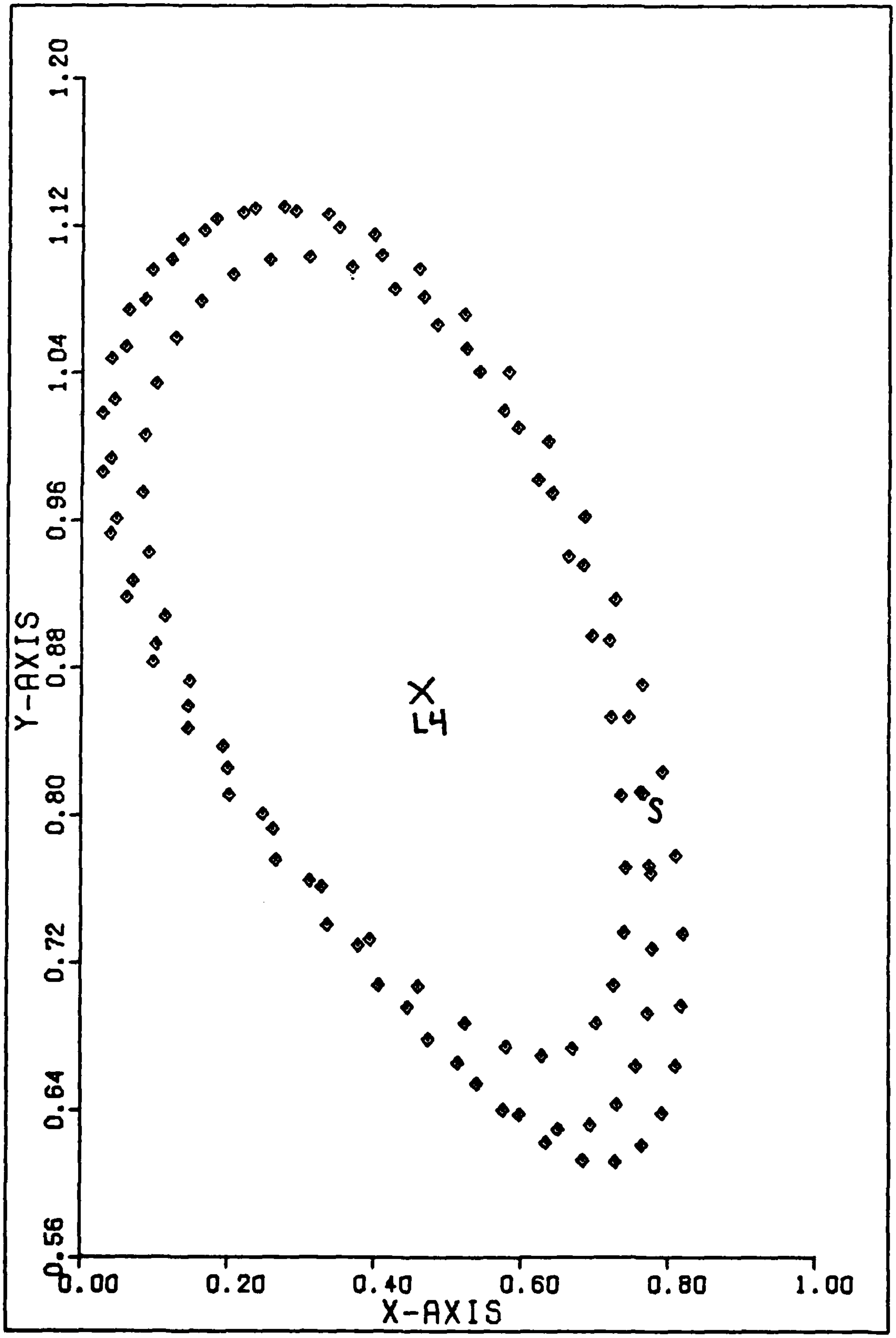


Fig 34. No Drift Display of K&C Orbit I Truth Model, 3 Orbits

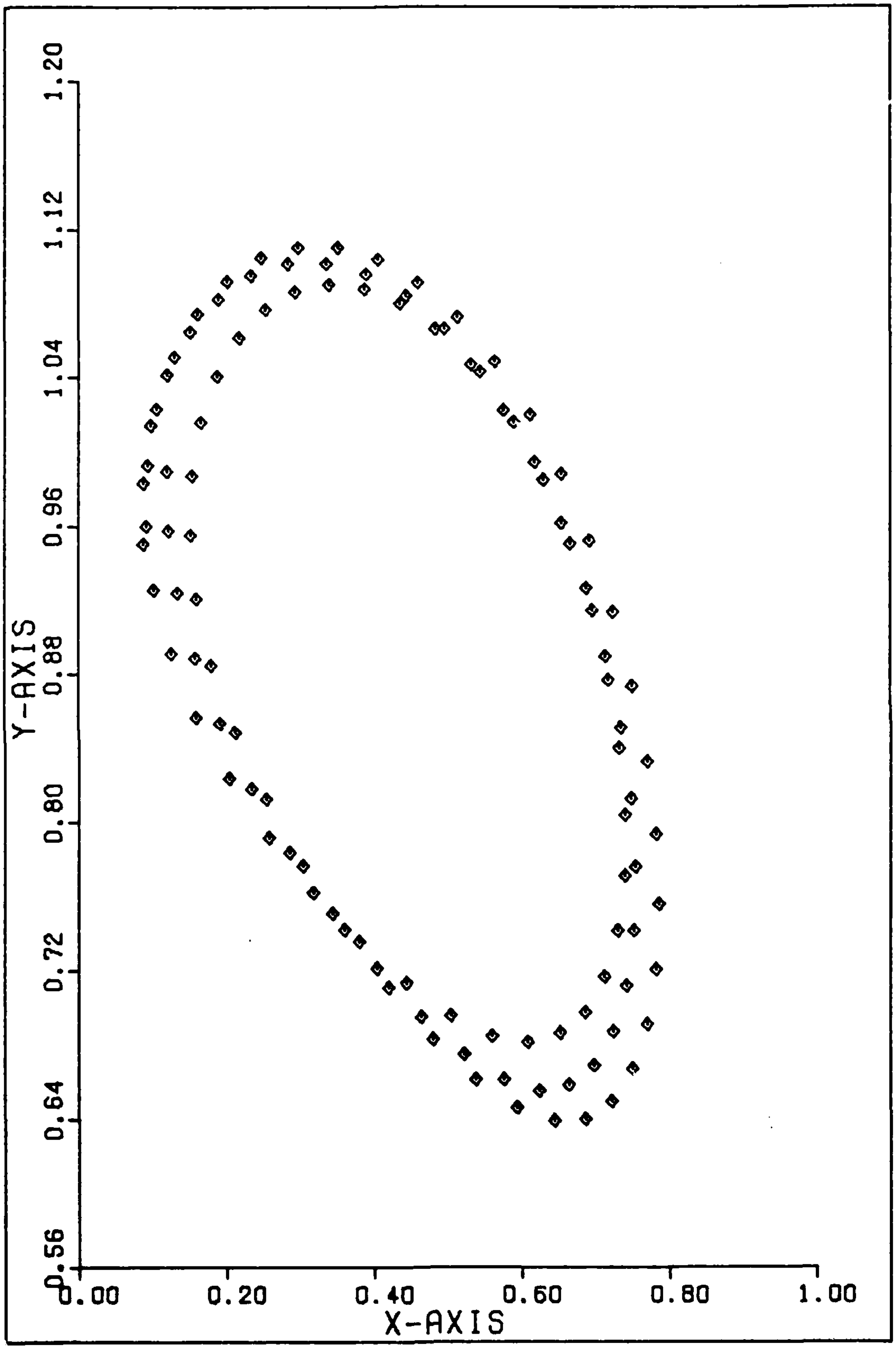


Fig 35. No Drift Display of K&C Orbit II Truth Model, 3 Orbits

AD-A079 683 AIR FORCE INST OF TECH WRIGHT-PATTERSON AFB OH SCHOO--ETC F/G 22/3
APPLICATION AND COMPARISON OF STABLE PERIODIC ORBITS IN THE VIC--ETC(U)
DEC 79 W D BEEKMAN

UNCLASSIFIED AFIT/GA/AA/79D-2 NL

AD
A079683

END
DATE
FILMED
2-80
DDC

noted individuals that any particle near L4 will slowly be ejected from the vicinity (Figures 36, 37 and 38). Both of the numerical orbits derived to check B, M and S's orbit were not truth model checks. The first was from the four-body perturbation equations but they were simplified by driving the distance to the Sun to infinity. The other numerical solution was using a set of three body equations formed in the same manner as the four-body which came up with the less than stable B, M and S orbit. A stability analysis should have been performed.

The fact that Wheeler's orbit was stable in the very restricted four-body model (circular Sun - Earth - Moon orbits) and K & C and B, M & S's orbits were unstable gives evidence that the VRFB model is a valid simplified model to test or search for orbits which would be relatively stable in the real world. Wheeler's orbit tends to support this. The truth model also decreases the VRFB orbit by 25%. Wheeler's orbit now has a semimajor axis of 60,000 mi and semiminor axis of 30,000 mi. Figure 39 shows the movement of one particular point on each orbit for a period of ten years showing that the orbit moves very little in the quadrant.

It was also found during the analysis of forces on a satellite about L4 that the phase between the Moon and Sun is the most important factor in the stability of the satellite's orbit. The Moon having its orbit perturbed by the Sun is the driving force. Remember that Wheeler's orbit will exist only if the satellite is initially injected into the orbit with the Moon and Sun in phase on the x-axis.

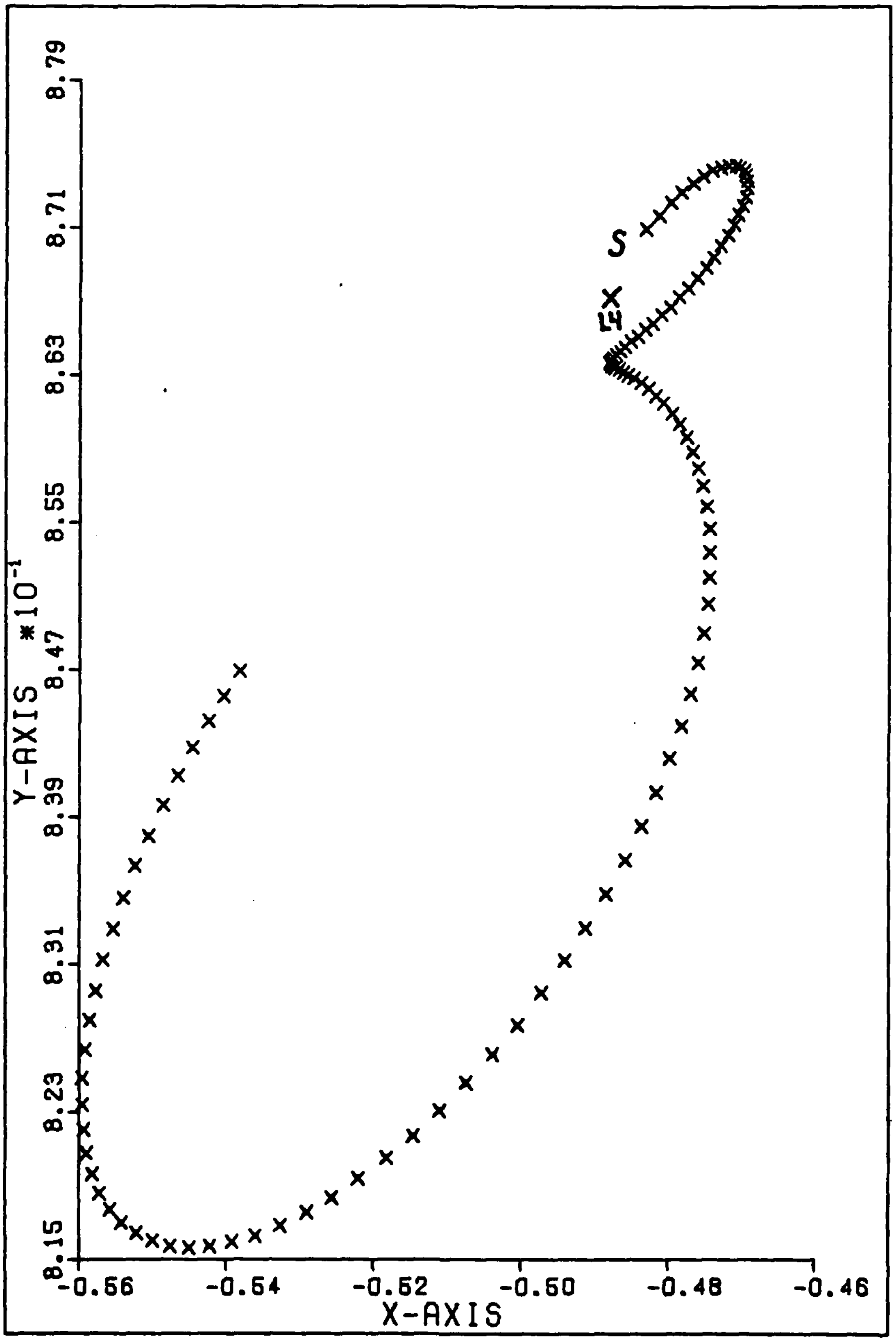


Fig 36. No Drift Display of BM&S Truth Model Orbit, 1 Orbit

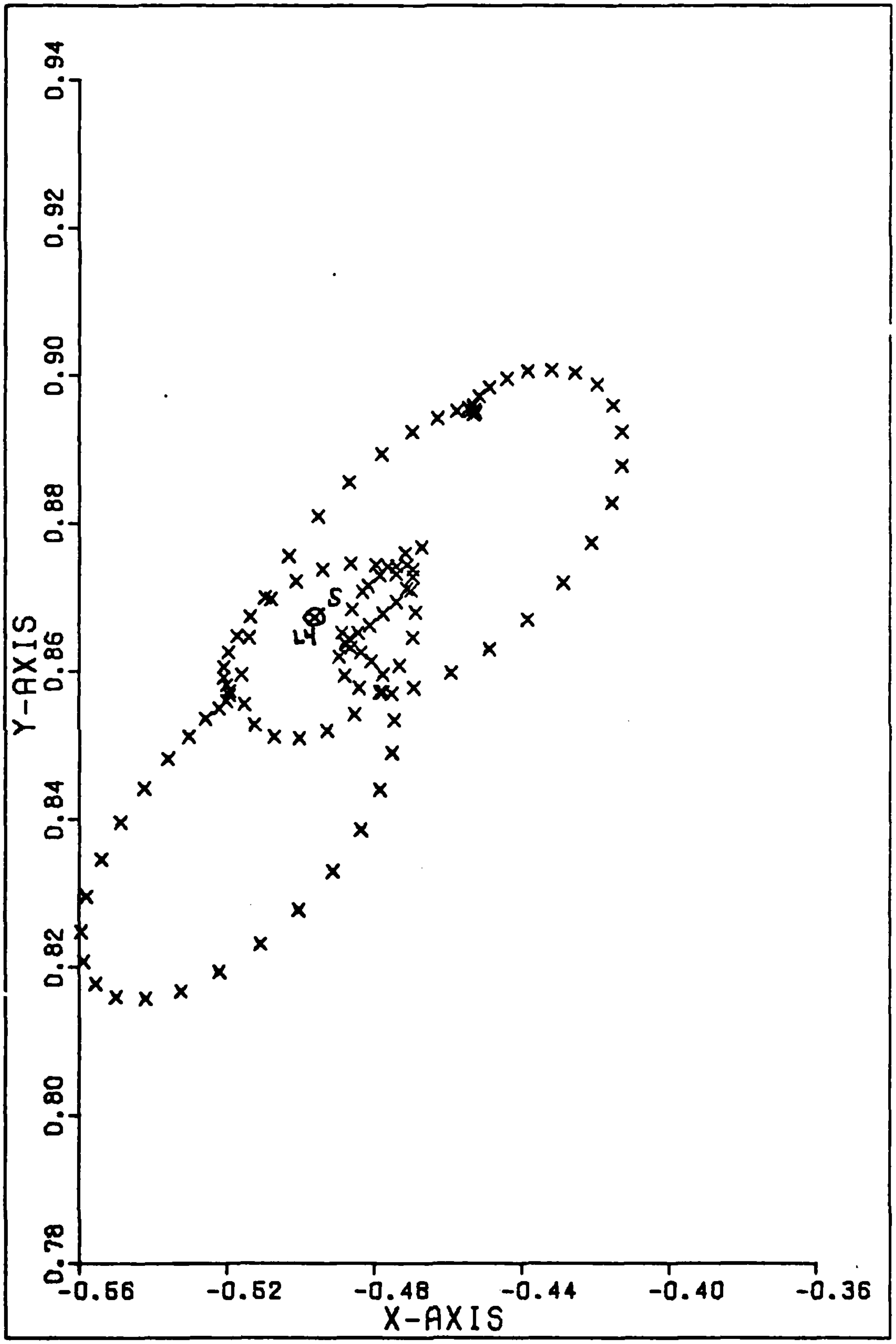


Fig 37. No Drift Display of BM&S Truth Model Orbit, 3 Orbits

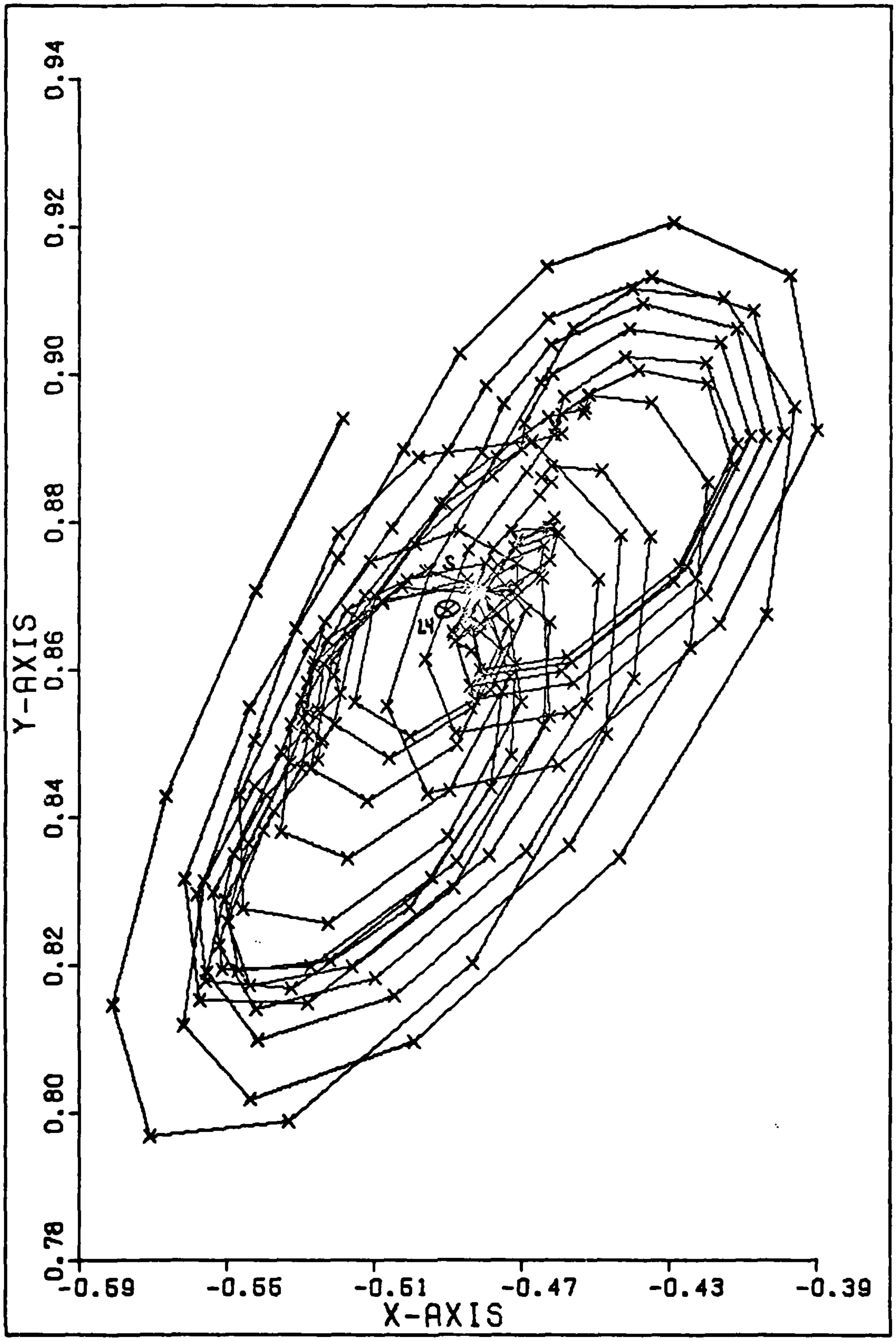


Fig 38. No Drift Display of BM&S Truth Model Orbit, 20 Orbits

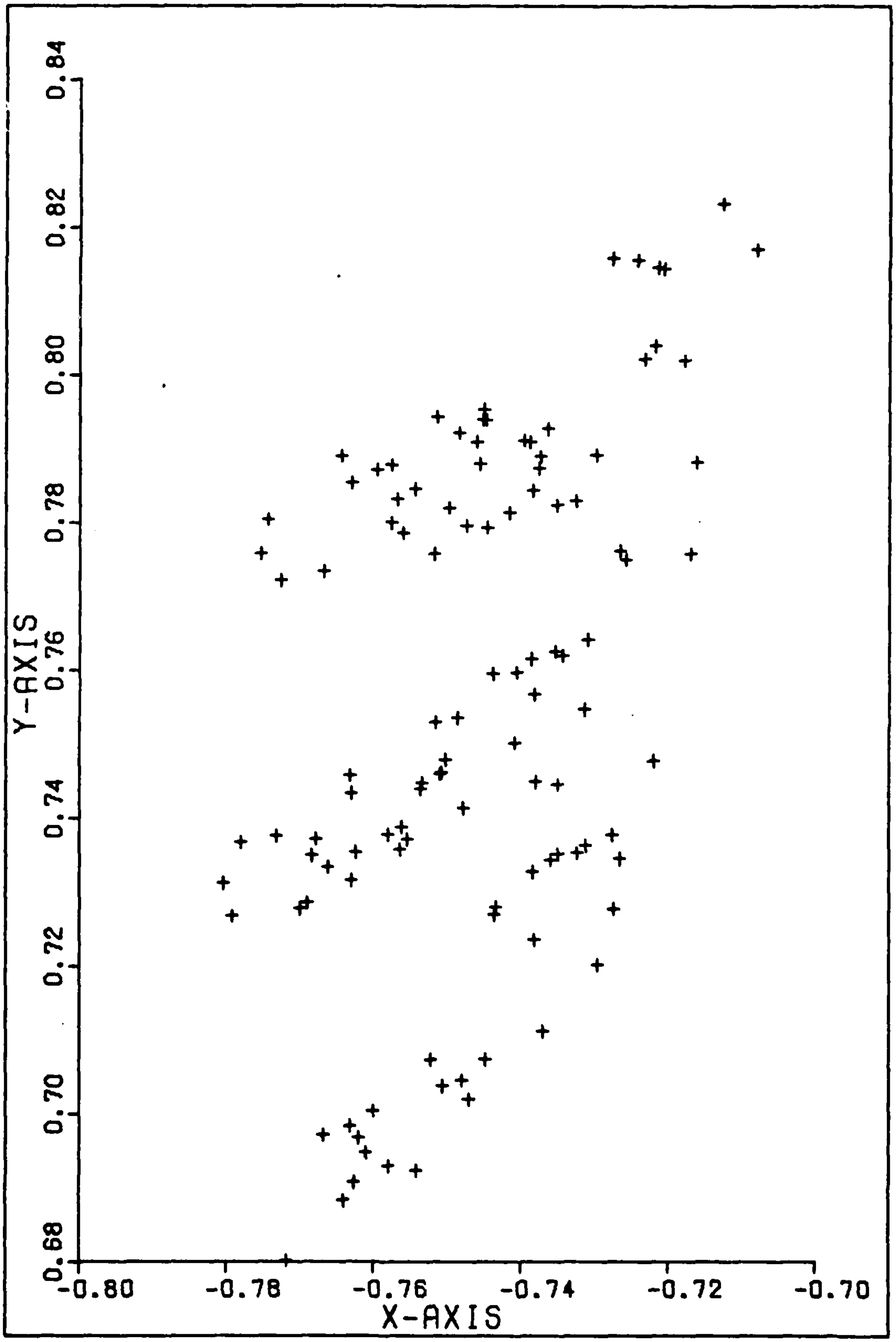


Fig 39. No Drift Display of Wheeler Orbit Over 10 Years

Recommendations

The truth model itself could be improved in accuracy using Ephemeris Tables for accurate lunar and solar initial conditions, using the most accurate constants to date from the various authors, and using more terms to higher powers for the equation of center which determines the solar coordinates.

The next step would be to investigate out-of-plane considerations and apply a third dimension to the truth model. If the small transients need to be kept to a minimum for the nearly stable Wheeler orbit, then an optimal plane of rotation for the orbit needs to be found. It could be in phase with the plane of the Moon's or with the the plane of the Sun or somewhere in between.

The 180° phase orbit could be investigated using a Monte Carlo analysis in a very restricted four-body model to see if a stable orbit exists and then apply it to the truth model.

Last of all, look at attitute problems in the Wheeler orbit due to the small perturbations of the nearly stable orbit. The initial conditions could be varied slightly and expanded to see if there is a small family of orbits that exhibit stability in that region.

Bibliography

1. Barkham, P.G.D., et al. "Asymptotic Solutions In the Many-Body Problem," Part I and II, Boston: D, Reidel Publishing Company, 1975.

2. Brouwer, D. "Theory and Tables of the Motion of (588) Achilles", Transactions of the Yale Observatory, 6 (Pt. VII):177-88 (1933).

3. Brouwer, D. and Clemence, C.M., Methods of Celestial Mechanics, New York: Academic Press, 1961.

4. Brown, E.W. "On a New Family of Periodic Orbits in the Problem of Three Bodies", Monthly Notices of the Royal Astronomical Society, 71:438-54 (1911).

5. Brown, E.W. "On the Oscillating Orbits About the Triangular Libration Points in the Problem of Three Bodies", Monthly Notices of the Royal Astronomical Society, 71:492-502 (1911).

6. Brown, E.W. "The General Orbits of the Asteroids of the Trojan Group", Astronomical Journal, 35:69-80, 124 (Aug 1923).

7. Brown, E.W. "Theory of the Trojan Group Asteroids", Transactions of the Yale Observatory, 3:1-47, 87-133 (1926).

8. Brown, E.W. and Shook C.A. Planetary Theory, Cambridge: University Press, 1933.

9. Charlier, Meddelanden Från Lunds Astronomika Observatoricum, No. 18.

10. Columbo, G. "The Stabilization of an Artifical Satellite at the Inferior Conjunction Point of the Earth - Moon System", Smithsonian Institution, SAO Special Report, No. 80, Nov 1, 1961.

11. Columbo, G. et al. "On the Librations of a Particle Near the Triangular Point in the Semi-Restricted Three Body Problem," Smithsonian Institution, SAO Special Report, No. 108, Nov 20, 1962.

12. Cronin, J.P., et al. "Periodic Motion Near Three-Body Libration Points", Icarus, 3:423 (1964).

13. Danby, J.M.A. "Influence of the Sun on Motion Close to the Triangular Points", Astronomical Journal, 70:181 (1965).

14. deVries, J.P. and Pauson W.M. "A Study of Motion Near Libration Points in the Earth - Moon System", General Electric Company, TIS R62SD5 (Jan 1962).

15. deVries, J.P. "The Sun's Perturbing Effect on Motion Near a Triangular Lagrange Point", XIII International Astronautical Congress, Sep 1962.

16. deVries, J.P. "Motion of a Particle in the Vicinity of a Triangular Libration Point in the Earth - Moon System", General Electric Company, Space Sciences Lab R63SD99.

17. Eckert, W.J. "The General Orbit of Hector", Transactions of the Yale Observatory, 6 (Pt VI):161-72 (1933).

18. Ellis, G.A. and Diana A.C. "Motion of a Satellite About the Earth -.Moon Sextile Points", Rome Air Development Commands, Technical Memorandum RCLS-TM-60-1 (Sep 1960).

19. Feldt, W.T. and Shulman Y. AIAA Journal, 4:1501 (1966).

20. Heppendeimer, T.A. "Steps Toward Space Colonization: Colony Location and Transfer Trajectories." Journal of Spacecraft and Rockets, 15:305-12 (1975).

21. Herz, H.G. "On the Theory of Trojan Asteroids", Astronomical Journal, 50:121-25 (1943).

22. Huang, S.S. "Very Restricted Four-Body Problem", NASA, TN D-501, 1960.

23. Kamel, A.A. "Perturbation Theory Based on Lie Transforms and its Application to the Stability of Motion Near Sun-Perturbed Earth - Moon Triangular Libration Points", NASA, CR-1622 (1970).

24. Katz, J.I. "Numerical Orbits Near the Triangular Lunar Libration Points." Incarus, 25:356-59 (1975).

25. Klemperer W.B. and Benedikt E.T. "Selenoid Satellites", Astronautica Acta, 4:25-30.

26. Kolenkiewicz, R. and Carpenter L. "Stable Periodic Orbits About the Sun-Perturbed Earth - Moon Triangular Points", AIAA Journal, 6:13C2 (1968).

27. LaGrange, Oeuvres Completes, Paris: Gauthiers - Villars, Vol VI:277 (1867).

28. Linders, L.J. Arkiv för Mat. Astronom., OCH FYS., Stockholm: SO. VET. AK., Vol. 4, No. 20 (1908).

29. Michael, W.H., Jr. "Considerations of the Motion of a Small Body in the Vicinity of the Stable Libration Points of the Earth - Moon System", NASA, TR R-160, 1963.

30. Musen, P. and Carpenter L. "On the General Planetary Perturbations in Rectangular Coordinates", Journal of Geophysical Research, 68:2727-34 (1963).

31. Plummer, H.C. "On Periodic Orbits in the Neighborhood of Centers of Libration," Monthly Notices of the Royal Astronomical Society, 62:6-17 (Nov 1901).

32. Rabe, E. "Determination and Survey of Periodic Trojan Orbits in the Restricted Problem of Three Bodies", Astronomical Journal, 66 (No. 9):500-13 (Nov 1961).

33. Rabe, E. "Additional Periodic Trojan Orbits and Further Studies of their Stability Features", Astronomical Journal, 67:389-90 (Aug 1962).

34. Rabe, E. and Schanzle A. "Periodic Librations About the Triangular Solutions of the Restricted Earth - Moon Problem and their Orbital Stabilities", Astronomical Journal, 67 (No. 10):732-9 (Dec 1962).

35. Roosen, R.G. "Objects in the Vicinity of L4 and L5", Icarus, 9:429 (1968).

36. Schechter, H.B. "Three-Dimensional Nonlinear Stability Analysis of the Sun - Perturbed Earth - Moon Equilateral Points", AIAA, Paper 67-566 (1967).

37. Schutz, B.E. "Numerical Studies of Solar Influenced Particle Near L4", Unpublished Dissertation. University of Texas.

38. Siferd, R.E. "Generating Periodic Orbits", Unpublished Thesis. Wright-Patterson Air Force Base, Ohio: Air Force Institute Of Technology, December 1965.

39. Smart, W. "Libration of the Trojan Planets", Paper I, Memoirs of the Royal Astronomical Society, 62:79-112 (1918).

40. Steg, L. and deVries, J.P. "Survey of Motion Near Libration Points", Space Science Review, 5:210 (1966).

41. Stumpff, K. "On LaGrange's Theory of the Three-Body Problem", NASA, TN D-1417 (Jan 1963).

42. Szebehely, V.G. Theory of Orbits, New York: Academic Press, 1967.

43. Tapley, B.D. and Lewallen, J.M. AIAA Journal, Vol. 2 No. 4 (1963).

44. Tapley, B.D. and Schultz, B.E. AIAA Journal, Vol. 3 No. 10 (1965).

45. Tapley, B.D. and Schutz, B.E. AIAA Journal, 6:1405 (1968).

46. Thüring, B. "The Trojan Libration Periods of Orbits of Various Amplitudes", Astronomische Nachrichten, 243:181 (1931).

47. Thüring, B. "Seeking Solutions to the Theory of Movement of the Sun - Jupiter Group", Astronomische Nachrichten, 279:217-30 (Sep 1950).

48. Thüring, B. "Program Formation of Libration Station Keeping", Astronomische Nachrichten, 285:71-80 (1959).

49. Wheeler, J.E. "Determination of Periodic Orbits and their Stability in the Very Restricted Four-Body Model in the Vicinity of the Lagrangian Points L4 and L5", Unpublished Thesis. Dayton: Air Force Institute of Technology, Dec 1978.

50. Wilkens, A. "Methoden Zur Ernittlung Der Speziellen Und Der Absoluten Koordinatenstörungen Der Planeten Der Jupitergruppe", Astronomische Nachrichten, 206:149-56 (Apr 1918).

51. Wilkens, A. "Integration of the Differential Equations of the Jupiter Problem," Astronomische Nachrichten, 206:149-56 (Apr, 1918).

52. Wilkens, A. "Concerning the Libration Theory in the Sun System", Astronomische Nachrichten, 227:337-48 (May 1926).

53. Wilkens, A. "Understanding of the Sun - Jupiter Theory", Astronomische Nachrichten, 245:229-64 (May 1932).

54. Willard, H.R. "On a Family of Oscillating Orbits of Short Periods", Monthly Notices of the Royal Astronomical Society, 73-471-84 (1913).

55. Wolaver, L.E. "Effects of Initial Configurations on Libration Point Motion." Progress in Astronautics, 17:75:99 (1966).

56. Wolff, C. et al. "Cloudlike Objects at Triangular Points", Science, 157:427 (1967).

APPENDIX A

DERIVATION OF HEPPENHEIMER'S EQUATIONS OF MOTION

Let a system on n bodies consist of point masses m_i at $\underline{r}_i$, where i = 1,2,...n and the $\underline{r}_i$ are expressed with respect to an inertial frame of reference.

$$\text{Let } r_{ji} = \left| \underline{r}_i - \underline{r}_j \right|$$

Then the equation of motion of m_i is

$$m_i \ddot{\underline{r}}_i = - G m_i \sum_{j=1}^{n} m_j \frac{(\underline{r}_i - \underline{r}_j)}{\underline{r}_{ij}^{\,3}} \qquad (1)$$

Applying Newton's Law of Universal Gravitation, the force $\underline{F}$ exerted on m_i by m_j is

$$\underline{F}_g = \frac{-G\, m_i\, m_j}{r_{ji}^{\,3}} \; \underline{r}_{ji}$$

The vector sum of all such gravitational forces acting on the ith body is

$$\underline{F}_g = - \frac{G\, m_i\, m_1}{r_{1i}^{\,3}} (\underline{r}_{1i}) - \frac{G\, m_i\, m_2}{r_{2i}^{\,3}} (\underline{r}_{2i}) - \ldots - \frac{G\, m_i\, m_n}{r_{ni}^{\,3}} (\underline{r}_{ni})$$

This equation does not contain the j = i term since the body does not exert a force on itself.

Simplifying

$$\underline{F}_g = - G\, m_i \sum_{\substack{j=1 \\ j \neq 1}}^{n} \frac{m_j}{r_{ji}^{\,3}} (\underline{r}_{ji}) \qquad (2)$$

Comparing equation (2) to equation (1) with the assumptions that the mass of the ith body remains constant and drag and other external forces are not present, we obtain

$$\ddot{\underline{r}}_i = -G \sum_{\substack{j=1 \\ j\neq 1}}^{n} \frac{m_j}{r_{ji}^3} (\underline{r}_{ji}) \tag{3}$$

If m_2 is a satellite and m_1 is a planet, writing equation (3) for both .

$$\ddot{\underline{r}}_1 = -G \sum_{j=2}^{n} \frac{m_j}{r_{ji}^3} (\underline{r}_{ji}) \tag{4}$$

$$\ddot{\underline{r}}_2 = -G \sum_{\substack{j=1 \\ j\neq 2}}^{n} \frac{m_j}{r_{j2}^3} (\underline{r}_{j2}) \tag{5}$$

And since $\underline{r}_{12} = \underline{r}_2 - \underline{r}_1$ so that $\ddot{\underline{r}}_{12} = \ddot{\underline{r}}_2 - \ddot{\underline{r}}_1$. Substituting equations (4) and (5) into this last equation

$$\ddot{\underline{r}}_{12} = -G \sum_{\substack{j=1 \\ j\neq 2}}^{n} \frac{m_j}{r_{j2}^3} (\underline{r}_{j2}) + G \sum_{j=2}^{n} \frac{m_j}{r_{j1}^3} (\underline{r}_{j1})$$

Expanding

$$\ddot{\underline{r}}_{12} = -\left[\frac{G\, m_1}{r_{12}^3} (\underline{r}_{12}) + G \sum_{j=3}^{n} \frac{m_j}{r_{j2}^3} (\underline{r}_{j2})\right]$$

$$- \left[-\frac{G\, m_2}{r_{21}^3} (\underline{r}_{21}) - G \sum_{j=3}^{n} \frac{m_j}{r_{j1}^3} (\underline{r}_{j1})\right]$$

Combining the first terms in each bracket since $\underline{r}_{12} = -\underline{r}_{21}$

$$\ddot{\underline{r}}_{12} = -G \frac{(m_1 + m_2)}{r_{12}^3} (\underline{r}_{12}) - \sum_{j=3}^{n} G\, m_j \left(\frac{\underline{r}_{j2}}{r_{j2}^3} - \frac{\underline{r}_{j1}}{r_{j1}^3} \right)$$

$$\ddot{\underline{r}}_{12} = - \frac{G\,(m_1 + m_2)}{r_{12}^{\;3}} \quad (\underline{r}_{12}) - \sum_{j=3}^{n} G\, m_j \left(\frac{\underline{r}_2 - \underline{r}_j}{r_{2j}^{\;3}} - \frac{\underline{r}_1 - \underline{r}_j}{r_{1j}^{\;3}} \right)$$

Consider an Earth - Moon - Sun system with a colony satellite in the vicinity of the lagrangian point L4 or L5. Refer to Figure 1. Assume the following:

1) Coplanar Orbits

2) Mass of the satellite is negligible compared to the masses of the other bodies.

3) The Sun, Earth, and Moon are considered point masses.

4) Ignore effects of other planets, upon the satellite.

From Figure 1

$\underline{r}_s = x_s\, \underline{a}_1 + y_s\, \underline{a}_2$

$\underline{r}_e = o$

$\underline{r}_m = x_m\, \underline{a}_1 + y_m\, \underline{a}_2$

$\underline{r}_c = x_c\, \underline{a}_1 + y_c\, \underline{a}_2$

$\underline{r}_{ms} = (x_m - x_s)\, \underline{a}_1 + (y_m - y_s)\, \underline{a}_2$

$\underline{r}_{cs} = (x_c - x_s)\, \underline{a}_1 + (y_c - y_s)\, \underline{a}_2$

$\underline{r}_{cm} = (x_c - x_m)\, \underline{a}_1 + (y_c - y_m)\, \underline{a}_2$

and

$r_m^{\;2} = x_m^{\;2} + y_m^{\;2}$

$r_s^{\;2} = x_s^{\;2} + y_s^{\;2}$

$r_c^{\;2} = x_c^{\;2} + y_c^{\;2}$

$r_{ms}^{\;2} = (x_m - x_s)^2 + (y_m - y_s)^2$

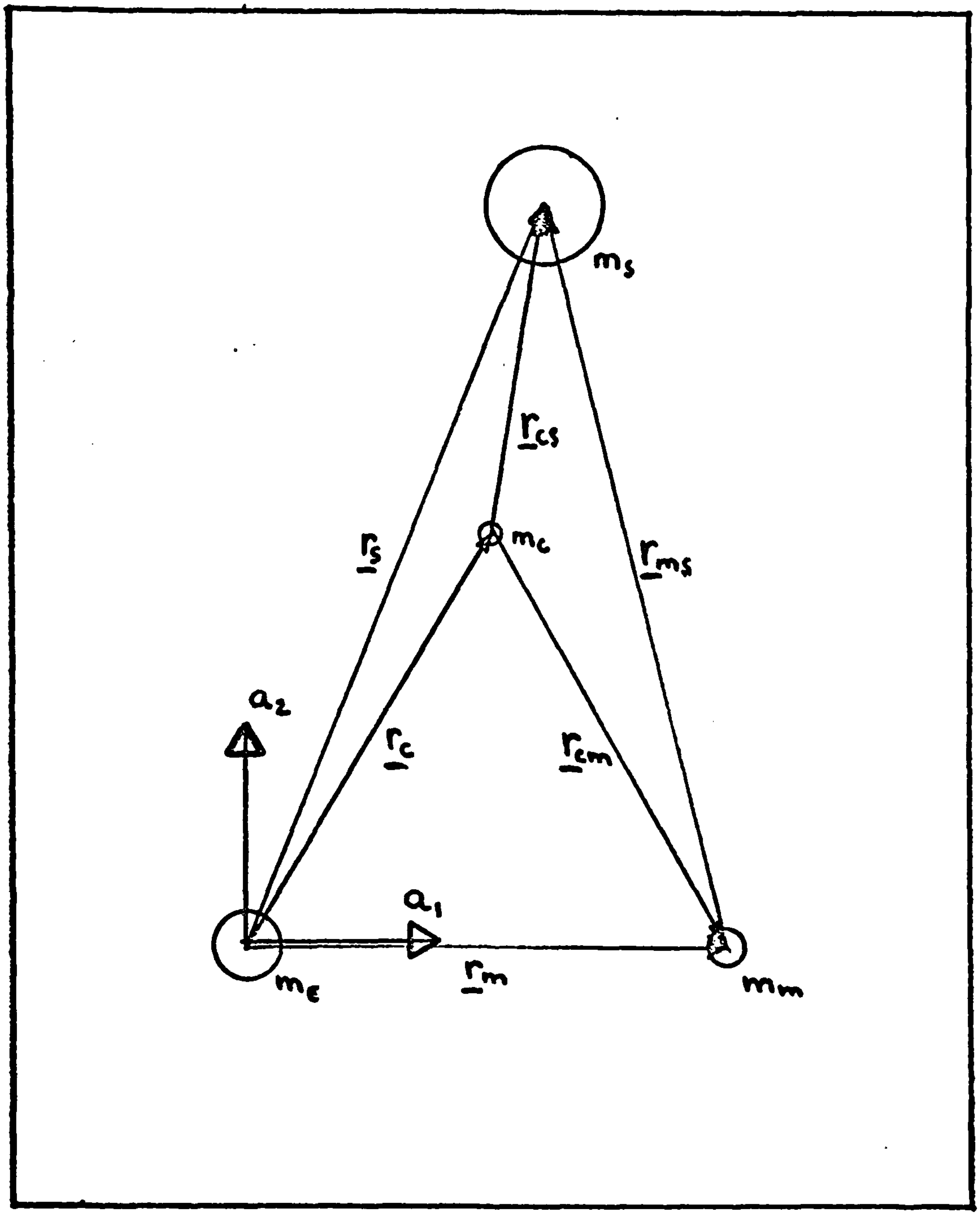


Fig 1. Heppenheimer's Four-Body Configuration

$$r_{cs}^2 = (x_c - x_s)^2 + (y_c - y_s)^2$$

$$r_{cm}^2 = (x_c - x_m)^2 + (y_c - y_m)^2$$

Using the perturbative function equation (6) for lunar motion where $\ddot{\underline{r}}_{12} = \ddot{\underline{r}}_m$

$$\ddot{\underline{r}}_m = -\frac{G\,(m_e + m_m)}{r_m^3}\,\underline{r}_m - G\,m_s\left(\frac{\underline{r}_m - \underline{r}_s}{r_{ms}^3} - \frac{\cancelto{0}{\underline{r}_e} - \underline{r}_s}{r_s^3}\right)$$

Where constants $m_m = \mu$

$m_e = 1 - \mu$

$G = 1$

So equations for lunar motion

$$\text{X-DIR:}\quad \ddot{x}_m + \frac{x_m}{r_m^3} = -m_s\left(\frac{x_m - x_s}{r_{ms}^3} + \frac{x_s}{r_s^3}\right) \qquad (7)$$

$$\text{Y-DIR:}\quad \ddot{y}_m + \frac{y_m}{r_m^3} = -m_s\left(\frac{y_m - y_s}{r_{ms}^3} + \frac{x_s}{r_s^3}\right) \qquad (8)$$

Using equation (6) for the colony satellite motion

$$\ddot{\underline{r}}_c = -\frac{G\,(m_e + m_c)}{r_c^3}\,\underline{r}_c - G\,m_s\left(\frac{\underline{r}_c - \underline{r}_s}{r_{cs}^3} - \frac{\cancelto{0}{\underline{r}_e} - \underline{r}_s}{r_s^3}\right)$$

$$- G\,m_m\left(\frac{\underline{r}_c - \underline{r}_m}{r_{cm}^3} + \frac{\cancelto{0}{\underline{r}_e} - \underline{r}_m}{r_m^3}\right)$$

Using the same constants and m_c negligible compared to m_e

$$\ddot{\underline{r}}_c + \frac{(1-\mu)\,\underline{r}_c}{r_c^3} = -m_s\left(\frac{\underline{r}_c - \underline{r}_s}{r_{cs}^3} + \frac{\underline{r}_s}{r_s^3}\right) - \mu\left(\frac{\underline{r}_c - \underline{r}_m}{r_{cm}^3} + \frac{\underline{r}_m}{r_m^3}\right)$$

$$\text{X-DIR:}\quad \ddot{x}_c + \frac{1-\mu}{r_c^3}\,x_c = -m_s\left(\frac{x_c - x_s}{r_{cs}^3} + \frac{x_s}{r_s^3}\right) - \mu\left(\frac{x_c - x_m}{r_{cm}^3} + \frac{x_m}{r_m^3}\right)$$

(9)

$$\text{Y-DIR:} \quad \ddot{y}_c + \frac{1-\mu}{r_c^{\,3}}\, y_c = -\, m_s \left(\frac{y_c - y_s}{r_{cs}^{\,3}} + \frac{y_s}{r_s^{\,3}} \right) - \mu \left(\frac{y_c - y_m}{r_{cm}^{\,3}} + \frac{y_m}{r_m^{\,3}} \right) \tag{10}$$

Cast these four second order differential equations (7, 8 9 and 10) into eight first order state equations

$$\underline{x} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{Bmatrix} = \begin{Bmatrix} x_m \\ y_m \\ \dot{x}_m \\ \dot{y}_m \\ x_c \\ y_c \\ \dot{x}_c \\ \dot{y}_c \end{Bmatrix} \tag{11}$$

So that the following differential equations result

$$\begin{aligned}
\dot{x}_1 &= \dot{x}_m \\
\dot{x}_2 &= \dot{y}_m \\
\dot{x}_3 &= \ddot{x}_m \\
\dot{x}_4 &= \ddot{y}_m \\
\dot{x}_5 &= \dot{x}_c \\
\dot{x}_6 &= \dot{y}_c \\
\dot{x}_7 &= \ddot{x}_c \\
\dot{x}_8 &= \ddot{y}_c
\end{aligned}$$

In terms of the state variables, the equations of motion become

$$\dot{x}_1 = x_3 \tag{12}$$

$$\dot{x}_2 = x_4 \tag{13}$$

$$\dot{x}_3 = \frac{x_i}{r_m^3} - m_s \left(\frac{x_i - x_s}{r_{ms}^3} + \frac{x_s}{r_s^3} \right) \tag{14}$$

$$\dot{x}_4 = \frac{x_2}{r_m^3} - m_s \left(\frac{x_2 - y_s}{r_{ms}^3} + \frac{y_s}{r_s^3} \right) \tag{15}$$

$$\dot{x}_5 = x_7 \tag{16}$$

$$\dot{x}_6 = x_8 \tag{17}$$

$$\dot{x}_7 = - \frac{(1 - \mu)}{r_c^3} x_5 - m_s \left(\frac{x_5 - x_s}{r_{cs}^3} + \frac{x_s}{r_s^3} \right) - \mu \left(\frac{x_5 - x_1}{r_{cm}^3} + \frac{x_1}{r_m^3} \right) \tag{18}$$

$$\dot{x}_s = - \frac{(1 - \mu)}{r_c^3} x_6 - m_s \left(\frac{x_6 - y_s}{r_{cs}^3} + \frac{y_s}{r_c^3} \right) - \mu \left(\frac{x_6 - x_2}{r_{cm}^3} + \frac{x_2}{r_m^3} \right) \tag{19}$$

Where

$$r_c^3 = (x_5^2 + x_6^2)^{3/2} \tag{20}$$

$$r_{cs}^3 = [(x_5 - x_s)^2 + (x_6 - y_s)^2]^{3/2} \tag{21}$$

$$r_{cm}^3 = [(x_5 - x_1)^2 + (x_6 - x_2)^2]^{3/2} \tag{22}$$

$$r_s^3 = (x_s^2 + y_s^2)^{3/2} \tag{23}$$

$$r_m^3 = (x_1^2 + x_2^2)^{3/2} \tag{24}$$

$$r_{ms}^3 = [(x_1 - x_s)^2 + (x_2 - y_s)^2]^{3/2} \tag{25}$$

APPENDIX B

DERIVATION OF THE EQUATION OF CENTER

Sun coordinates are needed for the satellite equations of motion in the truth model. Motion of the Sun is defined by the equation of center.

A fixed system of rectangular coordinates with the origin in the center of the Sun has equations (Ref 3) of motion

$$\frac{d^2x}{dt^2} + \mu \frac{x}{r^3} = \frac{\partial R}{\partial x} \qquad (1)$$

$$\frac{d^2y}{dt^2} + \mu \frac{y}{r^3} = \frac{\partial R}{\partial y}$$

$$\frac{d^2z}{dt^2} + \mu \frac{z}{r^3} = \frac{\partial R}{\partial z}$$

where $m_s = 1$, m is the disturbed planet, m' is the disturbing planet, $\mu = G\ (1 + m)$ and

$$R = [(x' - x)^2 + (y' - y)^2 + (z' - z)^2]^{1/2} - \frac{x'x + y'y + z'z}{r'^3}$$

Perturbed coordinates are

$$x = x_o + \delta_x \qquad y = y_o + \delta_y \qquad z = \delta_z$$

$$\dot{x} = \dot{x}_o + \delta_{\dot{x}} \qquad \dot{y} = \dot{y}_o + \delta_{\dot{y}} \qquad \dot{z} = \delta_{\dot{z}}$$

with the x-axis directed toward the perihelion. Substracting the equations of elliptic motion from (1) yields

$$\frac{d^2}{dt^2} (\delta x) + \frac{\mu}{r^3} (x_o + \delta x) - \mu \frac{x_o}{r_o^3} = \frac{\delta R}{\delta x}$$

(same for y and x components)

Develop the second item in the left-hand members in powers of

$\delta_x, \delta_y, \delta_z$

$$\text{Ex:} \quad \frac{d^2}{dt^2}(\delta x) + \frac{\mu}{r_o^3}(\delta x) - \frac{3\mu x o}{r_o^5}(x_o \ \delta x + \delta y) = G_x$$

and expand the partical derivatives of the disturbing function in Taylor's series in terms of R_o

$$\text{Ex:} \quad G_x = \frac{\partial R}{\partial x_o} + \frac{\partial^2 Ro}{\partial x_o}\delta x + \frac{\partial^2 Ro}{\partial x_o \ \partial y_o}\delta y + \frac{\partial^2 Ro}{\partial x_o \ \partial z_o}\delta z$$

$$+ \frac{\partial^2 Ro}{\partial x_o \ \partial x_o'}\delta x' + \frac{\partial^2 Ro}{\partial x_o \ \partial y_o'}\delta y' + \frac{\partial^2 Ro}{\partial x_o \ \partial z_o'}\delta z'$$

$$+ \mu \left[\left(\frac{9}{2}\frac{x_o}{r_o^5} - \frac{15}{2}\frac{x_o^3}{r_o^7} \right) \delta x^2 + \left(3\frac{y_o}{r_o^5} - 15\frac{x_o^2 y_o}{r_o^7} \right) \delta x \, \delta y \right.$$

$$+ \left(\frac{3}{2}\frac{x_o}{r_o^5} - \frac{15}{2}\frac{x_o y_o^2}{r_o^7} \right) \delta y^2 + \frac{3}{2}\frac{x_o}{r_o^5}\delta z^2$$

Integrating the expanded equations (first order approximation to solve second order of disturbing forces, etc.) to find a formal, explicit solution yields expressions containing the terms

$$\delta x = C_x + C_{ox}t + C_{1x}\cos L + C_{2x}\cos 2L + C_{3x}\cos 3L + \dots$$

$$+ S_{1x}\sin L + S_{2x}\text{Sin } 2L + S_{3x}\text{Sin } 3L + \dots$$

(δy is similar)

where C and S are numbers, secular and terms factored by t^2 or t^3 are ignored. Time is the independent variable. Converting to a true anomoly

$$\delta v = \frac{1}{r_o^2} (x_o \, \delta y - y_o \, \delta x)$$

$$= C + C_o t + C_1 \cos L + C_2 \cos 2 L + C_3 \cos 3 L + \ldots$$

$$+ S_1 \sin L + S_2 \sin 2 L + S_3 \sin 3 L + \ldots$$

Expressing a finite increment of v as a function or arbitrary increments of the four elements ω, n, e, $\tilde{\omega}$ and $\omega_o - \tilde{\omega} = L = nt$ for the Sun as a disturbing body

$$v = nt + (2 e + \frac{1}{4} e^3) \sin nt + (\frac{5}{4} e^2 - \frac{11}{24} e^4) \sin 2 nt$$

$$+ \frac{13}{12} e^3 \sin 3 nt$$ for 3rd order eccentricity

Use this to calculate the Sun coordinates starting at perihelion.

$$\delta x = r_o \cos v \qquad \delta y = r_o \sin v$$

therefore

$$x_s = \mu \, x_m + r_s \cos v$$

$$y_s = \mu \, y_m + r_s \sin v$$

where

$$r_s = A_s (1 - e_s^2)/(1 + e_s \cos v)$$

APPENDIX C

COMPUTER PROGRAM TO CALCULATE ORBITS IN THE RESTRICTED FOUR-BODY TRUTH MODEL

This appendix contains the computer routines used applying Wheeler's and Kolenkiewicz and Carpenter's orbital data to the truth model. Barkham, Modi, and Soudack's routine is not displayed because of its similarity to Wheeler's routine. The computer language utilized was Fortran Extended Version IV, and all work was accomplished on the AFIT CDC 6613 and CYBER 74 computers. Several comment statements have been employed to aid the reader and smooth the flow of the program. ODE was the integration package utilized in the program. For the integration steps, one time unit corresponds to 4.3483774 days. Data cards were input into the Kolenkiewicz and Carpenter routine containing the α and β coefficients from Tables 3 and 4.

Wheeler Truth Model with Subroutines

```
      PROGRAM GRADU(INPUT=/80,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT,PL
      EXTERNAL F1
      COMMON MS,AS,ES,NS,MU,PI
      REAL MS,NS,MU
      DIMENSION X(8),XM(1300),YM(1300),VXM(1300),VYM(1300),XC(13
     -YC(1300),VXC(1300),VYC(1300),TM(1300)
      DIMENSION IWORK(500),WORK(500)
      DIMENSION XW(1300),YW(1300),VXW(1300),VYW(1300)
      DIMENSION AA(450),BB(450)
      MS=328900.12
      AS=388.8202843
      ES=.0168
      NS=.9748013
      MU=.01215060554
      PI=3.1415926536
      XMIN=10.
      YMIN=10.
      XMAX=0.
      YMAX=0.
      X(1)=-.9922057379
      X(2)=0
      X(3)=0
      X(4)=-1.010561513436
      X(5)=-.7363274299
      X(6)=.8130853389
      X(7)=-X(6)+.7794831949/1.0808483351
      X(8)=X(5)+.2243307738/1.0808483351
      NEQN=8
      T=0
      TOUT=2.0*PI*1.0808483351/130.0
      DELT=TOUT
      RELERR=.000000001
      ABSERR=.000000001
      IFLAG=1
      KOUNT=1
    1 TM(KOUNT)=T
      XM(KOUNT)=X(1)
      YM(KOUNT)=X(2)
      VXM(KOUNT)=X(3)
      VYM(KOUNT)=X(4)
      XC(KOUNT)=X(5)
      YC(KOUNT)=X(6)
      VXC(KOUNT)=X(7)
      VYC(KOUNT)=X(8)
      A=ATAN(X(2)/X(1))
      IF(X(1).GT.0.0) A=A+PI
      IF(X(1).LT.0.0.AND.X(2).GT.0.0) A=A+2.*PI
      FS=NS*T+(2*ES-.25*ES**3)*SIN(NS*T)+1.25*ES**2*SIN(2*NS*T)+(1
     -*ES**3*SIN(3*NS*T)
      FS=FS+PI
      RSP=AS*(1.0-ES**2)/(1.0+ES*COS(FS))
      XS=MU*X(1)+RSP*COS(FS)
      YS=MU*X(2)+RSP*SIN(FS)
      PRINT*,"T= ",T,"  XS= ",XS,"  YS= ",YS
      XW(KOUNT)=X(5)*COS(A)+X(6)*SIN(A)+MU
      YW(KOUNT)=-X(5)*SIN(A)+X(6)*COS(A)
      VXW(KOUNT)=(X(7)+YW(KOUNT))*1.0808483351
```

```
      VYW(KOUNT)=(X(8)-XW(KOUNT))*1.0838435351
    5 CALL ODE(F1,NEQN,X,T,TOUT,RELERR,ABSERR,IFLAG,WORK,IWORK)
      IF(IFLAG.EQ.1) GO TO 5
      IF(IFLAG.NE.2) GO TO 2
      KOUNT=KOUNT+1
      IF(KOUNT.EQ.250) GO TO 4
      TOUT=TOUT+DELT
      GO TO 1
    4 CONTINUE
      PRINT*,"MOON ORBIT"
      DO 6 I=1,KOUNT
      WRITE(6,100) TM(I),XM(I),YM(I),VXM(I),VYM(I)
  100 FORMAT(10X,"T= ",F14.9,4X,"X= ",G18.11,5X,"Y= ",G18.11,5X,"VX= ",
     -G18.11,5X,"VY= ",G18.11)
    6 CONTINUE
      PRINT*,"SATELLITE ORBIT"
      DO 7 I=1,KOUNT
      WRITE(6,300) TM(I),XC(I),YC(I),VXC(I),VYC(I)
  300 FORMAT(10X,"T= ",F14.9,4X,"X= ",G18.11,5X,"Y= ",G18.11,5X,"VX= ",
     -G18.11,5X,"VY= ",G18.11)
    7 CONTINUE
      PRINT*,"WHEELER ORBIT"
      DO 9 I=1,KOUNT
      WRITE(6,500) TM(I),XW(I),YW(I),VXW(I),VYW(I)
  500 FORMAT(10X,"T= ",F14.9,4X,"X= ",G18.11,5X,"Y= ",G18.11,5X,"VX= ",
     -G18.11,5X,"VY= ",G18.11)
    9 CONTINUE
      N=111
      DO 12 I=1,N
      AA(I)=XW(I)
      BB(I)=YW(I)
   12 CONTINUE
      MM=N
      CALL PLOTSC(AA,BB,MM)
    2 PRINT*,"IFLAG= ",IFLAG
    3 CONTINUE
      STOP
      END
```

SUBROUTINE PLOTXY 74/74 OPT=1 FTN

```
      SUBROUTINE PLOTXY(X,Y,N,XMIN,XMAX,YMIN,YMAX,IST)
      DIMENSION X(N),Y(N)
      CALL PLOT(0.0,-1.0,-3)
      CALL PLOT(0.0,0.1,-3)
      XFAC=10.0/(XMAX-XMIN)
      YFAC=10.0/(YMAX-YMIN)
      DO 999 I=1,N
      XX=(X(I)-XMIN)*XFAC
      YY=(Y(I)-YMIN)*YFAC
      CALL SYMBOL(XX,YY,0.075,3,0.0,-1)
  999 CONTINUE
      CALL PLOT(10.,0.0,-3)
      IF (IST.EQ.0) RETURN
      JJ=0
      CALL PLOTE(JJ)
      RETURN
      END
```

SUBROUTINE F1   74/74   OPT=1   FTN 4.7+475

```
      SUBROUTINE F1(T,X,DX)
      COMMON MS,AS,ES,NS,MU,PI
      REAL MS,NS,MU
      DIMENSION X(8),DX(8)
      FS=NS*T+(2*ES-.25*ES**3)*SIN(NS*T)+1.25*ES**2*SIN(2*NS*T)+(13/
     -*ES**3*SIN(3*NS*T)
      FS=FS+PI
      RSP=AS*(1.0-ES**2)/(1.0+ES*COS(FS))
      XS=MU*X(1)+RSP*COS(FS)
      YS=MU*X(2)+RSP*SIN(FS)
      RMS=((X(1)-XS)**2+(X(2)-YS)**2)**.5
      RM=(X(1)**2+X(2)**2)**.5
      RS=(XS**2+YS**2)**.5
      RC=(X(5)**2+X(6)**2)**.5
      RCS=((X(5)-XS)**2+(X(6)-YS)**2)**.5
      RCM=((X(5)-X(1))**2+(X(6)-X(2))**2)**.5
      DX(1)=X(3)
      DX(2)=X(4)
      DX(3)=-X(1)/RM**3-MS*((X(1)-XS)/RMS**3+XS/RS**3)
      DX(4)=-X(2)/RM**3-MS*((X(2)-YS)/RMS**3+YS/RS**3)
      DX(5)=X(7)
      DX(6)=X(8)
      DX(7)=-(1.0-MU)*X(5)/RC**3-MS*(((X(5)-MU*X(1)-RSP*COS(FS))/RCS*
     -+((MU*X(1)+RSP*COS(FS))/RS**3))-MU*(((X(5)-X(1))/RCM**3)
     -+X(1)/RM**3)
      DX(8)=-(1.0-MU)*X(6)/RC**3-MS*(((X(6)-MU*X(2)-RSP*SIN(FS))/RCS
     -+((MU*X(2)+RSP*SIN(FS))/RS**3))-MU*(((X(6)-X(2))/RCM**3)
     -+X(2)/RM**3)
      RETURN
      END
```

SUBROUTINE PLOTSC   74/74   OPT=1   FTN

```
      SUBROUTINE PLOTSC(X,Y,M)
      DIMENSION X(M),Y(M)
      CALL PLOTS(30)
      CALL PLOT(0.,-3.,-3)
      CALL PLOT(0.,.5,-3)
      CALL PLOT(1.25,.5,3)
      CALL PLOT(1.25,9.5,2)
      CALL PLOT(7.25,9.5,2)
      CALL PLOT(7.25,.5,2)
      CALL PLOT(1.25,.5,2)
      CALL PLOT(1.75,1.0,-3)
      CALL SCALE(X,5.,M,1)
      CALL SCALE(Y,8.,M,1)
      CALL AXIS(0.,0.,6HX-AXIS,-6,5.,0.,X(M+1),X(M+2))
      CALL AXIS(0.,0.,6HY-AXIS,6,8.,90.,Y(M+1),Y(M+2))
      CALL LINE(X,Y,M,1,-1,3)
      CALL PLOTE(0)
      RETURN
      END
```

```
      PROGRAM GRADU(INPUT=/80,OUTPUT,TAPE5=INPUT,T
      EXTERNAL F1
      COMMON AS,NS,ES,MS,MU,PI,AK(19),AO(19),BC(19),
      REAL MS,NS,MU,N3,N1,MMU
      DIMENSION X(8),XM(1300),YM(1300),VXM(1300),VYM
     -YC(1300),VXC(1300),VYC(1300),TM(1300)
      DIMENSION IWORK(500),WORK(500)
      DIMENSION XW(1300),YW(1300),VXW(1300),VYW(13
      DIMENSION AA(200),BB(200)
   11 READ(5,*) AK,AO,BC,BS
      IF(EOF(5).NE.0) GO TO 3
      CALL ERXSET(10,0)
      MS=328912.42
      AS=3(0.8233
      ES=.01(8
      NS=.0748(13
      MU=.0121507
      PI=3.141592653(
      AC=3(318.123.7
      AM=3(4743796.5
      N1=(1299.7762.33*PI)/(7652F.*24.*3600.*180.
      NN=(29.53059382/(2*PI))*24.*60.*6).
      N3=.0(002661(+357)88
      ALPHA=AK(1)
      BETA=BC(1)
      ADOT=0
      BDOT=0
      DO 8 I=1,18
      ALPHA=ALPHA+AK(I+1)*COS(I*1.0471976)+AO(I+1)*S
      BETA=BETA+BC(I+1)*COS(I*1.0471976)+BS(I+1)*SIN
      ADOT=ADOT-AK(I+1)*I*(N3-N1)*SIN(I*1.0471976)+A
     -COS(I*1.0471975)
      BDOT=BDOT-BC(I+1)*I*(N3-N1)*SIN(I*1.0471976)+B(
     -COS(I*1.0471975)
    8 CONTINUE
      ALPHA=ALPHA*.000001
      BETA=BETA*.000001
      ADOT=ADOT*.000001*NN
      BDOT=BDOT*.000001*NN
      PRINT*,ALPHA,"  ",BETA,"  ",ADOT,"  ",BDOT
      X(1)=1.0-.0077942(5(1
      X(2)=0.
      X(3)=0.
      MMU=(N3**2)*(AM**3)
      X(4)=(4.3857596E-8/N3+X(1))*(MMU/AM)**.5*(NN/(
      CMU=(N3**2)*(AC**3)
      X(5)=(1+ALPHA)*(AC/AM)*COS(1.0471976)+BETA*(1./(
     -((CMU/AC)**.5)*(NN/AM)*(-SIN(1.0471976))
      X(6)=(1+ALPHA)*(AC/AM)*SIN(1.0471975)+BETA*(1./(
     -((CMU/AC)**.5)*(NN/AM)*COS(1.0471975)
      X(7)=(ADOT-(BETA/N3)*(CMU/AC**3)*NN)*(AC/AM)*
     -COS(1.0471975)+(BDOT/(N3*NN)+1.+ALPHA)*((CMU/AC)
     -(-SIN(1.0471976))
      X(7)=X(7)/1.0303(89351
      X(8)=(ADOT-(BETA/N3)*(CMU/AC**3)*NN)*(AC/AM)*
     -SIN(1.0471976)+(BDOT/(N3*NN)+1.+ALPHA)*((CMU/AC)
     -COS(1.0471976)
```

```
      X(8)=X(8)/1.0368-88331
      PRINT*,"X4   ",X(4),"   X5   ",X(5),"   X6   ",X(6)
      PRINT*,"X7   ",X(7),"   X8   ",X(8)
      NEQN=8
      T=0
      TOUT=2.0*PI*1.03-88331/30.0
      DELT=TOUT
      RELERR=.0000000001
      ABSERR=.0000000001
      IFLAG=1
      KOUNT=1
1     TM(KOUNT)=T
      XM(KOUNT)=X(1)
      YM(KOUNT)=X(2)
      VXM(KOUNT)=X(3)
      VYM(KOUNT)=X(4)
      XC(KOUNT)=X(5)
      YC(KOUNT)=X(6)
      VXC(KOUNT)=X(7)
      VYC(KOUNT)=X(8)
      A=ATAN(X(2)/X(1))
      IF(X(1).LT.0.0) A=A+PI
      IF(X(1).GT.0.0.AND.X(2).LT.0.0) A=A+2.*PI
      XW(KOUNT)=X(5)*COS(A)+X(6)*SIN(A)+MU
      YW(KOUNT)=-X(5)*SIN(A)+X(6)*COS(A)
      VXW(KOUNT)=(X(7)-X(6))*COS(A)+(X(8)+X(5)-MU)*SIN(A)
      VYW(KOUNT)=-(X(7)-X(6))*SIN(A)+(X(8)+X(5)-MU)*COS(A)
5     CALL ODE(F1,NEQN,X,T,TOUT,RELERR,ABSERR,IFLAG,WORK,IWORK)
      IF(IFLAG.EQ.1) GO TO 5
      KOUNT=KOUNT+1
      IF(KOUNT.EQ.250) GO TO 4
      TOUT=TOUT+DELT
      GO TO 1
4     CONTINUE
      PRINT*,"MOON ORBIT"
      DO 6 I=1,KOUNT
      WRITE(6,100) TM(I),XM(I),YM(I),VXM(I),VYM(I)
100   FORMAT(10X,"T= ",F14.9,4X,"X= ",G18.11,5X,"Y= ",G18.11,5X,"VX= "
     -G18.11,5X,"VY= ",G18.11)
6     CONTINUE
      PRINT*,"SATELLITE ORBIT"
      DO 7 I=1,KOUNT
      WRITE(6,300) TM(I),XC(I),YC(I),VXC(I),VYC(I)
300   FORMAT(10X,"T= ",F14.9,4X,"X= ",G18.11,5X,"Y= ",G18.11,5X,"VX= "
     -G18.11,5X,"VY= ",G18.11)
7     CONTINUE
      PRINT*,"WHEELER ORBIT"
      DO 9 I=1,KOUNT
      WRITE(6,500) TM(I),XW(I),YW(I),VXW(I),VYW(I)
500   FORMAT(10X,"T= ",F14.9,4X,"X= ",G18.11,5X,"Y= ",G18.11,5X,"VX= "
     -G18.11,5X,"VY= ",G18.11)
9     CONTINUE
      N=111
      DO 12 I=1,N
      AA(I)=XW(I)
      BB(I)=YW(I)
12    CONTINUE
```

```
      MM=N
      CALL PLOTSC(AA,BB,MM)
    2 PRINT*,"IFLAG= ",IFLAG
      GO TO 11
    3 CONTINUE
      STOP
      END
```

VITA

William D. Beekman was born on August 19, 1946, in Marion, Ohio and was raised in Toledo, Ohio where he graduated from Roy C. Start High School in June of 1964. The next Fall he entered the Air Force Academy in Colorado. He graduated from the Academy in June 1968 with a Bachelor of Science degree in Astronautics and Engineering Sciences.

He then complete Navigator Training at Mather Air Force Base, California and served as Navigator in the F-4 at MacDill Air Force Base, Florida and in South East Asia. After completing 175 combat missions, he was shot down and spent ten months as a prisoner of war in Hanoi. After his repatriation, he attended undergraduate Pilot Training at Williams Air Force Base, Arizona and then served as a C-9A Pilot at Scott Air Force Base, Illinois. During this time he earned a Master of Arts degree in Management.

In June of 1978, he was assigned to the Air Force Institute of Technology resident School of Engineering at Wright-Patterson Air Force Base, Ohio and began his studies toward a Master of Science degree in Astronautical Engineering. After graduation, he will be assigned to the Air Force Weapons Lab at Kirtland Air Force Base, New Mexico as a LASER Systems Analyst.

Permanent Address: 4909 Burnham Ave
Toledo, Ohio 43612

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER AFIT/GA/AA/79D-11	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) APPLICATION AND COMPARISON OF STABLE PERIODIC ORBITS IN THE VICINITY OF LAGRANGIAN POINTS L4 AND L5 TO A FOUR-BODY TRUTH MODEL		5. TYPE OF REPORT & PERIOD COVERED MS Thesis
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) WILLIAM D. BEEKMAN CAPTAIN		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS AIR FORCE INSTITUTE OF TECHNOLOGY WRIGHT-PATTERSON AFB, OHIO 45433		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS		12. REPORT DATE December 1979
		13. NUMBER OF PAGES 112
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION DOWNGRADING SCHEDULE

16 DISTRIBUTION STATEMENT (of this Report)

Approved for Public Release; Distribution Unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES Approved for Public Release; IAW AFR 190-17

Joseph P. Hipps, Major, USAF
Director of Public Affairs, A.F.I.T

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Periodic Orbits
Libration Points
L4 and L5

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In this report, all previously discovered stable periodic orbits about the triangular libration points are tested on a planar restricted four-body truth model. The truth model is an algorithm developed from equations used by T.A. Heppenheimer for colony location used perturbation theory and the equation of the center. Only one of the orbits, developed by Wheeler using a very restricted four-body problem with Sun - Moon - Earth circular motion, is found to be relatively stable for at least twenty years. It is prograde about L4 having a period in resonance with the lunar synodic month. Two other orbits, one

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) similar to Wheeler's and one 180°, out of phase found by Kolenkiewicz and Carpenter are marginally stable.