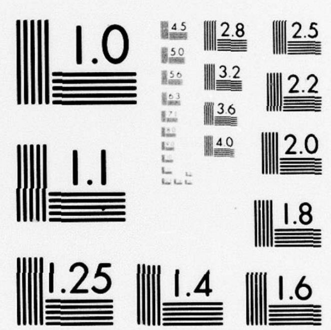
AD-A079 708 WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER F/G 20/11
THE INVERSE REFLECTION PROBLEM FOR A SMOOTHLY STRATIFIED ELASTI--ETC(U)
OCT 79 W SYMES DAAG29-75-C-0024
UNCLASSIFIED MRC-TSR-2007 NL

| OF |

AD
A079708

END
DATE
FILMED
3-80
DDC



MICROCOPY RESOLUTION TEST CHART

ADA079708

MRC Technical Summary Report # 2007

THE INVERSE REFLECTION PROBLEM FOR A SMOOTHLY STRATIFIED ELASTIC MEDIUM.

W. Symes

MRC-TSR-2007

LEVEL

Mathematics Research Center
University of Wisconsin–Madison
610 Walnut Street
Madison, Wisconsin 53706

Oct [redacted] 79 43

(Received July 25, 1979)

DAAG29-75-C-0024, NSF-MCS78-09525

Sponsored by

U.S. Army Research Office
P.O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, D.C. 20550

221 200

80 1 15 051

UNIVERSITY OF WISCONSIN-MADISON
MATHEMATICS RESEARCH CENTER

THE INVERSE REFLECTION PROBLEM FOR A SMOOTHLY STRATIFIED ELASTIC MEDIUM

W. Symes

Technical Summary Report # 2007
October 1979

ABSTRACT

The subject of this paper is a version of the inverse reflection problem for a smoothly stratified elastic medium. The same mathematics describes the inverse problem of the vibrating string. This problem is solved in a constructive way. Also, *a priori* estimates are derived which exhibit the continuous dependence of the solution (index of refraction, relative sound speed) on the data (scattering or reflection measurements).

AMS (MOS) Subject Classification - 35R30

Key Words - Inverse scattering, Nonlinear Volterra systems, Stratified elastic medium

Work Unit Number 1 - Applied Analysis

Sponsored by United States Army under Contract No. DAAG29-75-C-0024. This material is based on work supported by the National Science Foundation under Grant No. MCS78-09525.

SIGNIFICANCE AND EXPLANATION

In an inverse reflection or scattering problem, the mechanical (say, elastic) properties of a medium are to be determined from knowledge of incident and reflected waves. Such problems arise in exploration geophysics, materials testing, plasma diagnostics, physical chemistry, ultrasound radiology, and many other contexts. A simplified problem of great practical importance is the inverse acoustic reflection problem for a smoothly stratified medium in which the density varies as a function of depth only.

This report gives a constructive, exact solution of this problem, using only techniques available also for the three-dimensional inverse acoustic problem for unstratified elastic media. Additionally, the methods used here provide <u>stability estimates</u> which are extremely important for applications and for numerical implementation.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

THE INVERSE REFLECTION PROBLEM FOR A
SMOOTHLY STRATIFIED ELASTIC MEDIUM

W. Symes

§1. Introduction

The subject of this paper is a version of the inverse reflection problem for a smoothly stratified elastic medium. The same mathematics describes the inverse problem of the vibrating string. This problem is solved in a constructive way. Also, a priori estimates are derived which exhibit the continuous dependence of the solution (index of refraction, relative sound speed) on the data (scattering or reflection measurements).

We take particular care to use only constructions which apply in principle to higher-dimensional problems of a similar sort. Most other treatments of this problem proceed via a reduction to an inverse Sturm-Liouville problem (see e.g. [1] and [2]). This coordinate transformation is no longer available in higher-dimensional problems, so we avoid it, except where it can be interpreted as a coordinate transformation along the rays of geometric optics: this is the case for the a priori estimates of c at the end of Section 4. (See §4 for a discussion of this point).

Other authors who avoid dependence on peculiarly one-dimensional tricks have invariably used approximate methods (JWKB, Born series: see [3],[4], Ch. XIII, and practically any article in the literature of inverse scattering in exploration geophysics, physical chemistry, ultrasound radiology, etc.) In contrast, our methods are exact: we construct the index of refraction exactly by an iterative approximation procedure.

Our formulation of the inverse reflection problem is time-dependent, which also contrasts with many other treatments (e.g. [2],[5]). The solution propounded here can also be adapted to steady-state (frequency-domain) problem formulations, other boundary conditions, etc.

We abjure further discussion of the literature and proceed to describe our results.

Sponsored by United States Army under Contract No. DAAG29-75-C-0024
This material is based on work supported by the National Science Foundation under Grant No. MCS78-09525.

The physical setting of our problem is as follows. Consider a stratified elastic medium with unit elastic moduli (Lamé constants). (This latter restriction may be removed, in which case we recover the elastic impedence rather than the acoustic index of refraction). The density ρ varies as a function of only one of the coordinates, say x. We assume that ρ is known and constant in the half-space $\{x \leq 0\}$. In fact, for the sake of simplicity $\rho \equiv 1$ for $x \leq 0$, and smooth (of class C^3) otherwise. The methods can be modified to cope with discontinuous ρ -- that is easy. The hard part is smooth ρ. We set up a measuring device at $x = 0$, and record the (infinitesimal) reflected wave $F(t)$, $t > 0$, resulting from an (infinitesimal) impulsive incident wave of the form $\delta(x - t)$, $t < 0$, impinging on the unknown medium $\{x \geq 0\}$.

Our results are phrased in terms of the index of refraction $c = \rho^{-1/2}$. We obtain sufficient conditions on F (which are very close to necessary -- see [12]), as in:

Theorem 1. Suppose $T > 0$, $F : [0,2T] \to \mathbb{R}$ is of class C^2, $F(0) \subset 0$, and the kernel

$$H(s,t) \equiv \frac{1}{2} F(s+t) - \int_0^s d\tau\, F(s-\tau)F(t-\tau)$$

$$0 \leq s \leq t \leq T,$$

$$H(t,s) \equiv H(s,t)$$

defines a self-adjoint Hilbert-Schmidt operator $\mathbb{H}$ on $L^2[0,T]$ with the property

$$\mathbb{I} + 4\mathbb{H} \geq \epsilon > 0$$

Then there exists a unique $x > 0$ and a unique positive function $c : [0,x] \to \mathbb{R}^+ = \{c > 0\}$ of class C^3 so that $\int_0^x c^{-1} = T$ and $F(t) = u(0,t)$, $t > 0$ for the solution u of the initial value problem (for which define $c \equiv 1$ for $x < 0$):

$$\left(\frac{\partial^2}{\partial t^2} - c^2(x)\frac{\partial^2}{\partial x^2}\right)u(x,t) \equiv 0 \qquad (1.1a)$$

$$u(x,0) = \delta(x)$$

$$\frac{\partial u}{\partial t}(x,0) \equiv 0 \qquad (1.1b)$$

Denote by $\mathcal{J} \subset \mathbb{R}^+ \times C^3(\mathbb{R}^+)$ the set

$$\mathcal{J} = \{(X,c) : X > 0,\ c \in C^3[0,X]\}.$$

Then the map $F \mapsto (X,c)$ whose existence is implicit in the above statement is continuous (even locally Lipschitz) as a map

$$\mathcal{F}_T = \{F \in C^2[0,2T] : F(0) = 0\} \to \mathcal{J}.$$

The proof proceeds along the lines laid down in [6], where a similar problem was solved with the differential equation (1.1a) replaced by

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + q(x)\right)u(x,t) = 0$$

In reference [6], therefore, the characteristics of the problem are known from the outset. The characteristics are exactly what is to be found in the present problem, however, which gives it the nature of a free boundary problem. This nature is unavoidable in higher-dimensional inverse problems for elastic waves, and we meet in head-on, which causes some headaches. In particular, the necessary a priori estimates are more difficult to derive than are the corresponding estimates in [6].

The main tool is the progressing wave decomposition of the solution u of an initial value problem (1.1):

$$u(x,t) = \frac{1}{2}c^{1/2}(x)[\delta(t+T(x)) + \delta(t-T(x))] + K(x,t)$$

where

$$T(x) = \int_0^x c^{-1}$$

is the <u>travel-time function</u>, and K is smooth inside the light cone with apex (0,0) (Section 2).

This expansion is established in §2 under a condition on c which guarantees that distribution solutions of singular initial value problems for the operator $\Box_c = \frac{\partial^2}{\partial t^2} - c^2 \frac{\partial}{\partial x^2}$ are unique. This condition is not necessary. It is proved in ref. [12] that a sharp condition is $c \in W^{3,1}_{loc}$. Also, a converse to <u>Theorem A</u> is given in [12], which amounts to a complete solution of the inverse problem (necessary and sufficient conditions on F).

In §3, it is shown that the pair (K,c) solves the GL system of Volterra equations (<u>Theorem</u> B):

$$H(T(x),t) = \frac{1}{2} c^{-1/2}(x)K(x,t) + \int_0^x dy c^{-2}(y)K(y,T(x))K(y,t)$$

$$\text{for } t \geq T(x) \tag{1.2a}$$

$$K(T(x),x) = \frac{1}{4} c^{1/2}(x)\int_0^x c^{1/2}(c^{1/2})'' \tag{1.2b}$$

The initials GL stand for Gel'fand and Levitan, for they introduced integral equations of this sort into inverse scattering theory in their seminal paper [13]. Our work is in direct line of descent from theirs; the "nonlinear integral equation" of [13] is derived by the present techniques in [6]. GL also stands for <u>group law</u>; indeed, the equation (1.2a) expresses in compact form the group property of the solution operators for the Cauchy problem for $\Box_c$, which follows from the time-independence of its coefficient.

A number of crucial <u>a priori</u> estimates are determined in Section 4 for the solution {K,c} of (1.2). These involve sup norms of F and the number ϵ^{-1}. This latter number, although in principle present in the data F, is in practice difficult to extract. On the other hand, in practical problems one often has known bounds on the index of refraction. It is shown in Section 6 that a priori bounds on c and its first two derivatives determine a lower bound for ϵ, hence can be employed in place of ϵ in the a priori and stability estimates. We use the results of Section 4 in Section 5 to show that an iteration scheme converges to a global solution of the GL system (1.2). The solution defines a continuous map

$$\mathcal{F}_T \to \mathcal{F} = \{(X,c,K) : X > 0\},$$
$$c \in C^3[0,X],\ K \in C^2(\mathcal{C}(T,c))\}$$

where

$$\mathcal{C}(T,c) = \{(x,t) : 0 \leq x \leq X,\ T(x) \leq t \leq 2T - T(x)\}.$$

Finally, we establish in Section 7 that the solution (K,c) of the GL system actually solves the Chudov boundary value problem

$$(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2})K = 0 \quad \text{in} \quad \mathcal{C}(T,c) \tag{1.3a}$$

$$K(0,t) = F(t), \quad (\frac{\partial}{\partial t} - \frac{\partial}{\partial x})K(0,t) \equiv 0 \tag{1.3b}$$

$$K(x,T(x)) = \frac{1}{4} c^{1/2}(x)\int_0^x c^{1/2}(c^{1/2})''. \tag{1.3c}$$

Since it was established in Section 2 that the (necessarily unique) solution of (1.3) is the smooth part of the solution of the singular initial value problem (1.1), it follows that c is the solution of the inverse problem sought in Theorem A, which completes its proof.

The result of Section 7 also establishes that the GL system (1.2) and the Chudov problem (1.3) are completely equivalent. In particular, the a priori estimates of Section 4 also hold for the solution of (1.3).

In the context of the inverse problem of [6], a similar observation was used in [7] to prove stability of an optimally efficient numerical scheme based on a Chudov problem. In fact, the Chudov problem of [6], [7] was suggested as an approach to the inverse spectral problem for the Schrödinger operator by Chudov in the mid 50's (see [14], appendix), hence the name. The present results will likewise provide the base for a stability result and consequent a priori error estimates for efficient numerical solution of the present problem. This matter will be discussed elsewhere.

To end this introduction, we note that the present results can be combined with well-known techniques from exploration seismology to solve the inverse problem for piecewise C^3 index of refraction with jump discontinuities. Finally, we mention the work of O. Hald [8] on inverse Sturm-Liouville problems and P. Deift and E. Trubowitz [9] on inverse Schrödinger Scattering. These authors also prove stability results, and their techniques are actually closely related to those presented here.

§2. The progressing wave expansion

The goal of this section is to express the solution u of the singular initial value problem

$$\Box_c \bar{u} = \left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}\right)u \equiv 0 \tag{2.1a}$$

$$\bar{u}(x,0) = \delta(x - x_0) \tag{2.1b}$$

$$\frac{\partial \bar{u}}{\partial t}(x,0) \equiv 0$$

as a superposition of progressing waves

$$u_s(x,t) = \frac{1}{2} c^{1/2}(x) c^{-3/2}(x_0)[\delta(t + T(x,x_0)) + \delta(t - T(x,x_0))] \tag{2.2}$$

where

$$T(x,x_0) = \int_{x_0}^{x} c^{-1}$$

plus a remainder $K(x,t;x_0)$ which is smooth inside the light cone with apex $(x_0,0)$ and has a jump discontinuity on the boundary of the light cone given by

$$K(x,T(x,x_0):x_0) = \frac{1}{4}\, c(x_0)^{-3/2}\, c^{1/2}(s) \int_{x_0}^{x} c^{1/2}(c^{1/2})'' \quad x \geq x_0$$

$$K(x,-T(x,x_0):x_0) = \frac{1}{4}\, c(x_0)^{-3/2}\, c^{1/2}(x) \int_{x_0}^{x} c^{1/2}(c^{1/2})'' \quad x \leq x_0 \quad (2.3)$$

The first step is the determination of conditions on a compound distribution (see [10], Ch. 7)

$$v(x,t) = g(x,t)\delta(\varphi(x,t))$$

which ensures that $\Box_c v$ has singularities no worse than those of v, i.e., in this case that no derivatives of the delta distribution appear in $\Box_c v$ along the surface $\varphi = 0$. Here g and φ are understood to be smooth functions, with φ playing a role analogous to that of the phase function in geometric optics.

A short calculation, using the chain rule for compound distributions ([10], Ch. 7) yields

1) the eikonal equation

$$\left(\frac{\partial^2\varphi}{\partial t^2}\right)^2 - c^2\left(\frac{\partial^2\varphi}{\partial x^2}\right) = 0$$

2) the transport equation

$$2\,\frac{\partial g}{\partial t}\,\frac{\partial \varphi}{\partial t} - 2c^2\,\frac{\partial g}{\partial x}\,\frac{\partial \varphi}{\partial x} + g(\varphi\Box_c\varphi) = 0.$$

Two convenient independent solutions to the eikonal equation are

$$\varphi^{\pm}(x,t) = t \pm T(x,x_0).$$

The curves $\{\varphi^{\pm} = 0\}$ are characteristics of $\Box_c$ passing through $(x_0,0)$.

The corresponding transport coefficients must solve

$$0 = 2\frac{\partial g^{\pm}}{\partial t} \mp 2c\frac{\partial g^{\pm}}{\partial x} \pm c' g^{\pm}, \qquad (2.4)$$

along $\{\varphi^{\pm} = 0\}$ to first order. Now demand that for suitable choice of $g^{\pm}$, the superposition

$$u_s = g^{+}\delta(\varphi^{+}) + g^{-}\delta(\varphi^{-})$$

should satisfy the initial conditions (2.1b). We obtain

$$\begin{aligned} &g^{+}(x,0) + g^{-}(x,0) = c^{-1}(x_0) \\ &\frac{\partial g^{+}}{\partial t}(x,0) + \frac{\partial g^{-}}{\partial t}(x,0) = 0 \\ &g^{+}(x,0) - g^{-}(x,0) = 0 \qquad x = x_0 \end{aligned} \qquad (2.5)$$

The last is in fact required to hold to first order at $x = x_0$. One easily checks that, if

$$g^{\pm}(x_0,0) = \frac{1}{2}c^{-1}(x_0)$$

then the equations (2.5) are satisfied, including the last one to first order, provided that the transport equations (2.4) are taken into account.

To solve the transport equations, notice that

$$\tilde{g}^{\pm}(x,t) = c^{-1/2}(x)g^{\pm}(x,t)$$

must solve

$$\frac{\partial \tilde{g}^{\pm}}{\partial t} \mp c\frac{\partial \tilde{g}^{\pm}}{\partial x} = 0, \qquad (2.6)$$

along $\{\varphi^{\pm} = 0\}$, with initial condition

$$\tilde{g}^{\pm}(x_0,0) = \frac{1}{2}c^{-3/2}(x_0).$$

The initial condition is satisfied if we take

$$\tilde{g}^{\pm}(x,0) \equiv \frac{1}{2}c^{-3/2}(x_0), \text{ all } x$$

and the equation (2.6) is certainly satisfied along $\{\varphi^{\pm} = 0\}$ to first order if it is satisfied identically in x and t, in which case the solution is

$$\tilde{g}^{\pm}(x,t) \equiv \frac{1}{2} c^{-3/2}(x_0)$$

It follows that suitable solutions to the transport equations (2.4) are

$$\tilde{g}^{\pm}(x,t) = \frac{1}{2} c^{1/2}(x) c^{-3/2}(x_0)$$

and we have established the form (2.2) for the singular progressing wave.

The next step in the progressing wave expansion is to add a correction u_c, again in the form of a progressing wave

$$u_c(x,t) = g_1^+(x,t) H(t + T(x,x_0)) + g_1^-(x,t) H(t - T(x,x_0))$$

whose singularity

$$H(\varphi) = \begin{cases} 1 & \varphi \geq 0 \\ 0 & \varphi \leq 0 \end{cases}$$

is the <u>primitive</u> of the preceding (delta) singularity.

One requires that $\Box_c(u_s + u_c)$ have singularities no worse than those of u_c, hence that $\Box_c u_c$ must <u>cancel</u> the singularity of $\Box_c u_s$. This requirement leads to the <u>second transport equation</u>

$$2 \frac{\partial g_1^{\pm}}{\partial t} \mp 2c \frac{\partial g_1^{\pm}}{\partial x} \pm c' g_1^{\pm} = \frac{1}{2} c^2 (c^{1/2})'' c^{-3/2}(x_0) \tag{2.7}$$

The initial values $g_1^{\pm}(x,0)$ must be specified in such a way that $u_c(x,0) \equiv 0$. By computations very similar to the preceding, one arrives at the solutions

$$g_1^+(x,t) = \begin{cases} \frac{1}{4} c^{1/2}(x) c^{-3/2}(x_0) \int_{x_0}^{x} c^{1/2} (c^{1/2})'' & x \leq x_0 \\ 0 & x \geq x_0 \end{cases}$$

$$g_1^-(x,t) = \begin{cases} \frac{1}{4} c^{1/2}(x) c^{-3/2}(x_0) \int_{x_0}^{x} c^{1/2} (c^{1/2})'' & x \geq x_0 \\ 0 & x \leq x_0 \end{cases} \tag{2.8}$$

Finally, one shows that the problem

$$\Box_c w = -\Box_c(u_s + u_c)$$

$$w \equiv \frac{\partial w}{\partial t} \equiv 0 \quad \text{for} \quad t = 0$$

has a unique continuous solution w supported in the light cone $C(x_0)$. This is spelled out in ([11], Ch. VI, §4) and ([10], Ch. 7) under stringent differentiability assumptions on c, and in ([12], §2) under weaker restrictions more suited to the discussion of the inverse problem. In any case, we can now represent the solution u of (2.1) as

$$u = u_s + u_c + w.$$

Since w vanishes on $\partial C(x_0)$ and is continuous, the summand $K = u_c + w$ has the asserted jump along $\partial C(x_0)$, by virtue of (2.8).

§3. The GL equation

Since any function may be written as a superposition of delta functions, we can write the general solution $u(x,t)$ of the partial differential equation

$$\Box_c u = 0$$

satisfying the initial condition

$$\frac{\partial u}{\partial t}(x,0) \equiv 0$$

as

$$u(x,t) = \int_{-\infty}^{\infty} dx_0 \; S(x,t;x_0,0)u(x_0,0)$$

where

$$S(x,t;x_0,0) = \bar{u}(x,t)$$

is the solution examined in the previous section: that is

$$\square_c(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}) S(x,t;x_0,0) \equiv 0$$

$$S(x,0;x_0,0) = \delta(x - x_0)$$

$$D_2 S(x,0;x_0,0) \equiv 0.$$

S is related to the Riemann function $\mathcal{R}$ of $\square_c$, which solves the initial value problem

$$\square_c(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}) R(x,t;x_0,0) \equiv 0$$

$$R(x,0;x_0,0) \equiv 0$$

$$D_2 R(x,0;x_0,0) = \delta(x - x_0)$$

(see [11], Ch. V, §5) by

$$R(x,t;x_0,0) = \int_0^t ds\ S(x,s;x_0,0).$$

Define

$$R(x,t;x_0,t_0) = R(x,t - t_0;x_0,0)$$

$$S(x,t;x_0,t_0) = S(x,t - t_0;x_0,0) \qquad (3.1)$$

Then the general solution of $\square_c u = 0$ with arbitrary initial data is given by

$$u(x,t) = \int_{-\infty}^{\infty} dx_0 \{S(x,t;x_0,t_0) u(x_0,t_0) + R(x,t;x_0,t_0) \frac{\partial}{\partial t_0} u(x_0,t_0)\}$$

This works because the coefficient of $\square_c$ is independent of t. For the same reason, the initial value vector $(u(\cdot,t_0); \frac{\partial u}{\partial t}(\cdot,t_0))^{\dagger}$ is propagated in t by a group of operators (bounded, in fact, in a suitable function space). The distribution kernel of this group of solution operators is the matrix

$$\mathcal{R} = \begin{pmatrix} S & R \\ D_2 S & D_2 R \end{pmatrix}$$

The group law is expressed in terms of $\mathcal{R}$ by

$$\mathcal{R}(x,t;x_0,t_0) = \int_{-\infty}^{\infty} dy\mathcal{R}(x,t;y,t')\mathcal{R}(y,t',x_0,t_0) \tag{3.2}$$

This relation is fundamental for what follows. Since the components of $\mathcal{R}$ may all be expressed in terms of the scalar kernel S, it is plausible that (2.2) may be expressed in terms of a property of S. This is indeed the case.

To derive this relation, we require some further symmetries of S, which are as follows:

1) S is even in $t-t_0$, whereas R is odd in $t-t_0$

2) The kernel

$$S^*(x,t;x_0,t_0) = S(x_0,t_0;x,t)$$

is a solution of the adjoint equation

$$\Box_c^* S^* = 0$$

(see [11], Ch. V, §5.3).

Now

$$\Box_c^* = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} c^2$$

So (we suppress for the moment the dependence on x_0,t_0)

$$0 = (\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} c^2(x))S^*(x,t) = \frac{\partial^2}{\partial t^2} S^*(x,t) - \frac{\partial^2}{\partial x^2}(c^2(x)S^*(x,t))$$

$$= (c^{-2}(x) \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2})(c^2(x)S^*(x,t)) = c^{-2}(x)\Box_c(c^2(x)S^*(x,t)).$$

Since $c > 0$, it follows that $c^2(x)S^*(x,t)$ solves

$$\Box_c c^2(x)S^*(x,t) = 0.$$

Now

$$c^2(x)S^*(x,t_0) = c^2(x)\delta(x-x_0) = c^2(x_0)\delta(x-x_0)$$

and

$$\frac{\partial}{\partial t} c^2(x)S^*(x,t)|_{t=t_0} = 0.$$

We assume, without further explanation, that the coefficient c is sufficiently well-behaved that distribution solutions of the initial value problem for the wave equation are unique (conditions on c under which this is so are discussed in [12]; §2, certainly sufficient is $c \in C^3$). Therefore

$$c^2(x)S^*(x,t) = c^2(x_0)S(x,t)$$

i.e.,

$$S(x_0,t_0;x,t) = c^{-2}(x_0)c^2(x)S(x,t;x_0,t_0)$$

which is the desired symmetry relation.

Now examine the (1,1) component of the group law equation (3.3) which reads

$$S(x,t;x_0,t_0) = \int_{-\infty}^{\infty} dy\ \{S(x,t;y,t')S(y,t'x_0,t_0) + R(x,t;y,t')D_2S(y,t';x_0,t_0)\}.$$

We set $x = x_0$, $t_0 = 0$ and replace t by $t+s$, t' by s to obtain

$$S(x_0,t+s;x_0,0) = \int_{-\infty}^{\infty} dy\ \{S(x_0,t+s;y,s)S(y,s,x_0,0) + R(x_0,t+s;y,s)D_2S(y,S;x_0,0)\}.$$

Using the time-translation symmetry (2.1) rewrite the r.h.s. as

$$= \int_{-\infty}^{\infty} dy\{S(x_0,t;y,0)S(y,s;x_0,0) + R(x_0,t;y,0)D_2S(y,s;x_0,0)\}.$$

Now note that, since $S(y,s;x_0,0)$ is <u>even</u> in s, its s-derivative $D_2S(y,s;x_0,0)$ is odd. The second term in the integrand is therefore odd in s, whereas the first is even. Replace s by $-s$, add, and divide by two to obtain

$$\frac{1}{2}[S(x_0,t+s;x_0,0)+S(x_0,t-s;x_0,0)] = \int_{-\infty}^{\infty} dy\ S(x_0,t;y,0)S(y,s;x_0,0)$$

Now use the adjoint symmetry (2) above to interchange the arguments in the first factor in the integrand:

$$= c^2(x_0)\int_{-\infty}^{\infty} dy\ c^{-2}(y)S(y,0;x_0,t)S(y,s;x_0,0)$$

$$= c^2(x_0)\int_{-\infty}^{\infty} dy\ c^{-2}(y)S(y,-t;x_0,0)S(y,s;x_0,0)$$

$$= c^2(x_0)\int_{-\infty}^{\infty} dy\ c^{-2}(y)S(y,t;x_0,0)S(y,s;x_0,0). \qquad (3.3)$$

Conversely, one can show that (3.3) entails the group law equation for the full Riemann matrix $\mathcal{R}$.

Now set

$$F(t) = S(x_0,t;x_0,0), \quad t \neq 0.$$

One can show ([12], §2) that, under suitable smoothness hypotheses on c, one can also set

$$F(0) = 0$$

to obtain a continuous, even function of all t. Define also

$$G(s,t) = \frac{1}{2}[F(t+s)+F(t-s)]$$

Now

$$S(x_0,t;x_0,0) = \delta(t)+F(t)$$

so the l.h.s. of (2.3) is

$$\frac{1}{2}[S(x_0,t+s;x_0,0)+S(x_0,t-s;x_0,0)] = \frac{1}{2}[\delta(s+t)+\delta(s-t)]+G(s,t).$$

Recall also from §2 the expansion

$$S(y,t;x_0,0) = \frac{1}{2}\, c^{1/2}(x)c^{-3/2}(x_0)[\delta(t+T(x,x_0))+\delta(t-T(x,x_0))] + K(x,t;x_0).$$

Since x_0 will remain fixed for the rest of this discussion, we shall henceforth set $x_0 = 0$. The number $c(x_0) = c(0)$ is now surely a positive constant which can be set equal to 1 by <u>scaling t</u>, which we assume has been done. We write

$$T(x) = T(x,0) = \int_0^x c^{-1}.$$

We now have in place of (3.3)

$$\frac{1}{2}[\delta(s+t)+\delta(s-t)] + G(s,t) = \int_{-\infty}^{\infty} dy\, c^{-2}(y)S(y,t;0,0)S(y,s;0,0) \quad (3.4)$$

with

$$S(y,t;0,0) = \frac{1}{2}\, c^{1/2}(x)[\delta(t+T(x))+\delta(T-T(x))] + K(x,t) \quad (3.5)$$

One now substitutes (3.5) into (3.4), and after some computation eliminates the singular terms to obtain

$$G(s,t) = \frac{1}{2}[c^{-1/2}(X^-(s))K(X^-(s),t) + c^{-1/2}(X^+(s))K(X^+(s),t)] + \int_{X^-(s)}^{X^+(s)} dy\, c^{-2}(y)K(y,t)K(y,s) \quad (3.6)$$

Here $s \mapsto X^{\pm}(s) = x$ is the inverse function to $x \mapsto \pm T(x) = s$, and is thus the solution of

$$\frac{d}{ds} X^{\pm} = \pm c(X^{\pm}), \quad X^{\pm}(0) = 0.$$

The light cone through (0,0) is thus described by $\{(x,t) : X^-(t) \leq x \leq X^+(t)\}$, and $t \mapsto (X^{\pm}(t),t)$, $x \mapsto (x, \pm T(x))$ are equivalent descriptions of the characteristic curves emanating from (0,0). Recall that (Section 2, (2.3))

$$K(x,T(x)) = \frac{1}{4}\, c^{1/2}(x)\int_0^x c^{1/2}(c^{1/2})'' \quad x \geq 0$$

$$K(x,-T(x)) = \frac{1}{4}\, c^{1/2}(x)\int_0^x c^{1/2}(c^{1/2})'' \quad x \leq 0 \quad (3.7)$$

In view of the formulation of the inverse problem (Section 1), we now assume that

$$c(x) \equiv 1, \quad x \leq 0.$$

Lemma A.

$$K(y,t) = F(t+y), \quad y \bar{<} 0.$$

Proof: K, being the smooth part of a solution of the wave equation, must solve it in the interior of the light cone. For $x \le 0$, $T(x) = x$, and (3.7) shows that

$$K(x,-x) = 0, \quad x \le 0 \tag{3.8}$$

Finally

$$K(0,t) = F(t). \tag{3.9}$$

A solution to the problem

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)K = 0$$

together with the conditions (3.8) and (3.9) in the region $\{-t \le x \le 0, t \ge 0\}$ is

$$K(x,t) = F(x+t).$$

However, since one of the boundaries of this region is characteristic, the solution is unique (see [11], Ch. V). q.e.d.

Now $X^-(s) = -s$, and set $X(s) = X^+(s)$. Then (2.6) reads for $0 \le s \le t$

$$\begin{aligned} G(s,t) &= \tfrac{1}{2}\, c^{-1/2}(X(s))K(X(s),t) + \tfrac{1}{2}\, K(-s,t) \\ &\quad + \int_{-s}^{X(s)} dy\, c^{-2}(y)K(y,s)K(y,t) \\ &= \tfrac{1}{2}\, F(t-s) + \int_{-s}^{0} dy\, F(y+t)F(y+s) + \tfrac{1}{2}\, c^{-1/2}(X(s))K(X(s),t) \\ &\quad + \int_{0}^{X(s)} dy\, c^{-2}(y)K(y,s)K(y,t). \end{aligned}$$

Now set

$$H(s,t) = \frac{1}{2} F(t+s) - \int_0^s d\tau\, F(t-\tau)F(s-\tau) \qquad (3.10)$$

We have proved

Theorem B. The smooth part K of the solution u of the singular initial value problem (2.1) satisfies for $0 \le s \le t$

$$H(s,t) = \frac{1}{2} c^{-1/2}(X(s))K(X(s),t) + \int_0^{X(s)} dy\, c^{-2}(y)K(y,s)K(y,t) \qquad (3.10a)$$

and is related to the coefficient c by the transport equation

$$K(x,T(x)) = \frac{1}{4} c^{1/2}(x)\int_0^x c^{1/2}(c^{1/2})'' \qquad (3.10b)$$

We shall refer to the system of integral equation (3.10) as the G L system, partly because it expresses the group law for the Cauchy problem for $\Box_c$, and partly because it is related to an equation discovered by Gel'fand and Levitan and derived in a completely different way in their fundamental paper [13]. For application of the methods developed here to the problem considered in [6], and a discussion of various aspects of inverse problems, see [6],[7].

The converse of Theorem B will be proved in Section 7.

§4. A Priori Estimates

In this section we derive some necessary conditions for scattering data. We shall make use of the substitution $x \mapsto s = T(x)$ throughout. We point out that this is not the same as reduction to the case of the Schrödinger equation. Indeed, in the first part of this section, leading up to the estimate (4.11), the kernel $\tilde{K}$ can be replaced by the kernel J, defined in the next section. The operator $\mathbb{K}$ can be replaced by an operator $\mathbb{J} : L^2(dt) \to L^2(c^{-1}(x)dx)$, and all mention of the coordinate transformation $x \mapsto s$ can be eliminated. In higher-dimensional problems, the volume element of the Riemannian metric associated with the relevant hyperbolic p.d.e. will play the role of $c^{-1}(x)dx$, so our methods conform to the rubric laid down in Section 1. Also, the last part of the section, leading up to the bounds (4.16), depends only on $x \mapsto s$ as arc-length parameterization of the geodesics of the above-mentioned metric, so this is again an admissible trick.

Note that the kernel H is symmetric. It follows that equation (3.10a) also holds with s and t interchanged for $0 \le t \le s$.

Now define

$$\tilde{K}(s,t) = 2c^{-1/2}(X(s))K(X(s),t) \quad \text{for } 0 \le s \le t \tag{4.1}$$

$$\tilde{K}(s,t) = 0 \quad \text{for } s > t \ge 0$$

Since $c^{-1}(y)dy = ds$ with $s = T(y)$, i.e., $y = X(s)$, we can re-write (3.10) as

$$4H(s,t) = \tilde{K}(s,t) + \int_0^s d\tau\, \tilde{K}(\tau,s)\tilde{K}(\tau,t) \tag{4.2}$$

For $T > 0$ denote by $\mathbb{K}$ the Volterra operator on $L^2[0,T]$ defined by

$$\mathbb{K}\varphi(\tau) = \int_\tau^T dt\, \tilde{K}(\tau,t)\varphi(t)$$

for $\varphi \in L^2[0,T]$. Denote by $\mathbb{H}$ the symmetric operator with kernel H. The hypotheses on c are sufficient to ensure that $\mathbb{H}$ is a Hilbert-Schmidt operator on $L^2[0,T]$. Denote finally by $\mathbb{I}$ the identity operator on $L^2[0,T]$ (with kernel $\delta(\tau - t)$). Then (4.2) can be written

$$\mathbb{I} + 4\,\mathbb{H} = (\mathbb{I} + \mathbb{K})^\dagger(\mathbb{I} + \mathbb{K})$$

(see [6], eqn. 4.10, also [14], eqn. 8.1,2). This shows that $\mathbb{I} + 4\,\mathbb{H}$ must be positive definite (since $\mathbb{I} + \mathbb{K}$ is invertible). According to the Fredholm character of $\mathbb{I} + 4\,\mathbb{H}$, we must in fact have

$$\mathbb{I} + 4\,\mathbb{H} \geq \epsilon(T) > 0. \tag{4.3}$$

Obviously $\epsilon(T)$ is a monotone nonincreasing function of T. We note for later use the identity

$$\epsilon(T)^{-1} = \sup\{\|(\mathbb{I} + \mathbb{K})^{-1}(\varphi)\|_{L^2[0,T]} : \|\varphi\|_{L^2[0,T]} = 1\} \tag{4.4}$$

which follows immediately from (4.3).

Next, we extract from (4.3) some a priori estimates which will be crucial for the next section. Denote by H_t, $\tilde{K}_t$ the functions

$$H_t(s) = H(s,t), \qquad \tilde{K}_t(s) = \tilde{K}(s,t)$$

and note that (4.2) may be written

$$4H_t(s) = ((\mathbb{I} + \mathbb{K})^{+}\tilde{K}_t)(s), \quad 0 \leq s \leq t.$$

It follows that

$$16\|H_t\|^2_{L^2[0,T]} \geq 16\|H_t\|^2_{L^2[0,t]} = \langle \tilde{K}_t, (\mathbb{I} + \mathbb{K})(\mathbb{I} + \mathbb{K}^{\dagger})\tilde{K}_t \rangle_{L^2[0,t]} \tag{4.5}$$

Now the invertible self-adjoint operator $(\mathbb{I} + \mathbb{K})(\mathbb{I} + \mathbb{K})^{\dagger}$ has the same spectrum as the operator $(\mathbb{I} + \mathbb{K})^{\dagger}(\mathbb{I} + \mathbb{K}) = \mathbb{I} + 4\,\mathbb{H}$, hence in particular the same lower bound. Therefore you may combine (4.3) and (4.5) to obtain

$$16\|H_t\|^2_{L^2[0,T]} \geq \epsilon(T)\|\tilde{K}_t\|^2_{L^2[0,T]} \tag{4.6}$$

Now $\|H_t\|$ may be estimated in the following truly crude fashion

$$\|H_t\|^2_{L^2[0,T]} \leq 2(\|F\|^2_{L^2[0,2T]} + T^2\|F\|^4_{L^2[0,T]}). \tag{4.7}$$

Also

$$\left|\int_0^s d\tau\, \tilde{K}(\tau,s)\tilde{K}(\tau,t)\right| = \left|\langle \tilde{K}_s,\tilde{K}_t\rangle\right|_{L^2[0,t]} \le \|\tilde{K}_s\|_{L^2[0,s]}\|\tilde{K}_t\|_{L^2[0,t]}$$

Hence, recalling (4.6) and using (4.7)

$$\left|\int_0^{X(s)} dy\, c^{-2}(y)K(y,s)K(y,t)\right|$$
$$\le 8\epsilon(T)^{-1}\left(\|F\|^2_{L^2[0,2T]} + T^2\|F\|^4_{L^2[0,T]}\right) \tag{4.8}$$

Combined with (3.10a), this yields the estimate

$$\left|\tfrac{1}{2}\, c^{-1/2}(X(s))K(X(s),t)\right| \le \|F\|_{L^\infty[0,2T]} + \|F\|^2_{L^2[0,T]}$$
$$+ 8\epsilon(T)^{-1}\left[\|F\|^2_{L^2[0,2T]} + T^2\|F\|^4_{L^2[0,T]}\right] \tag{4.9}$$

valid in the range $0 \le s \le t \le T$.

Finally we give a priori bounds for c, in terms of T and $K^* \equiv \sup_{0\le t\le T} |2c^{-1/2}(x(t))K(x(t),t)|$. Note that the latter quantity has just been estimated in terms of F.

According to (3.10b), for $0 \le x \le X(T)$,

$$4c^{-1/2}(x)K(x,T(x)) = \int_0^x c^{1/2}\,(c^{1/2})''$$
$$= \tfrac{1}{2}\,c'(x) - \tfrac{1}{4}\int_0^x c^{-1}(c')^2 \tag{4.10}$$

Set

$$g(x) = 8\, c^{-1/2}(x)\, K(x,T(x))$$

Then

$$c(x)^{-1}\, g(x) = (\log c)'(x) - \tfrac{1}{2}\, c^{-1}(x)\int_0^x c^{-1}(c')^2$$

Since $c(0) = 1$, obtain

$$\log c(x) \ge \int_0^x c^{-1}\, g \ge -4K^*\int_0^x c^{-1} \ge -4K^*T$$

i.e. $$c \ge \exp(-4K^*T) \tag{4.11}$$

which is the required lower bound for c.

To obtain upper bounds for c, write

$$\frac{1}{4}\, g(X(s)) = \bar{K}(s) = \tilde{K}(s,s), \quad \varphi = c^{-1/2}, \; \cdot = \frac{d}{ds}$$

Then (4.10) may be rewritten

$$2\bar{K}(t) = (\log \varphi)^{\cdot}(t) - \int_0^t ((\log \varphi)^{\cdot})^2 .$$

It follows that φ obeys the equation

$$\ddot{\varphi} = Q\varphi$$

where $Q = 2\dot{\bar{K}}$. (This relation is also part of the Liouville reduction which figures in other treatments of this inverse *problem* -- see e.g. [2]). Note that

$$\varphi(0) = 1, \quad \dot{\varphi}(0) = 0, \quad \bar{K}(0) = 0.$$

Hence φ *is the solution of*

$$\varphi(t) = 1 + \int_0^t ds\,(t-s)\varphi(s)Q(s) = 1 + 2\int_0^t ds\; \varphi(s)\bar{K}(s) - 2\int_0^t ds\,(t-s)\dot{\varphi}(s)\bar{K}(s) \qquad (4.12a)$$

Also $\dot{\varphi}$ *is the solution of*

$$\dot{\varphi}(t) = 2\varphi(t)\bar{K}(t) - 2\int_0^t ds\; \dot{\varphi}(s)\bar{K}(s) \qquad (4.12b)$$

Define $\psi = \dot{\varphi} - 2\bar{K}\varphi$. Then the Volterra system (4.12) *may be rewritten as*

$$\varphi(t) = 1 + \int_0^t ds\; 2\bar{K}(s)(1 + 2(s-t)K(s))\varphi(s) + \int_0^t ds\; 2\bar{K}(s)(s-t)\psi(s) \qquad (4.13a)$$

$$\psi(t) = -4\int_0^t ds\,(\bar{K}(s))^2\varphi(s) - 2\int_0^t ds\,\bar{K}(s)\psi(s) \tag{4.13b}$$

Set

$$K^{**} = \max\{4K^* + 8T(K^*)^2, 8(K^*)^2, 4TK^*, 4K^*\}.$$

Then the following estimate is easily derived for the solution of (4.13):

$$\|\varphi\|_{L^\infty[0,T]} \leq \exp TK^{**} \tag{4.14a}$$

Also

$$\|\psi\|_{L^\infty[0,T]} \leq \exp TK^{**} \tag{4.14b}$$

In view of the definition of ψ, this entails

$$\|\dot{\varphi}\|_{L^\infty[0,T]} \leq (1 + 2K^*)\exp TK^{**} \tag{4.15}$$

Now $\dot{\varphi} = \frac{d}{ds}(c^{-1/2}) = c\frac{d}{dx}(c^{1/2}) = \frac{1}{2}c^{1/2}c'$. Thus, combining (4.15), (4.14a) and (4.11), we get the estimates, valid for $0 \leq x \leq X(T)$,

$$\exp(-4K^*T) \leq |c(x)| \leq \exp(2TK^{**}) \tag{4.16a}$$

and

$$|c'(x)| \leq 2(1 + 2K^*)\exp T(K^{**} + 4K^*) \tag{4.16b}$$

§5. <u>Solution of the GL system</u>

With this section we begin the solution of the inverse problem as stated in Section 1. The first step is to show that the GL system as presented in <u>Theorem B</u>:

$$H(s,t) = \frac{1}{2}\, c^{-1/2}(X(s))K(X(s),t) + \int_0^{X(s)} dy\; c^{-2}(y)K(y,s)K(y,t)$$

$$0 \leq s \leq t \leq T \tag{5.1a}$$

$$K(x,T(x)) = \frac{1}{4}\, c^{1/2}(x)\int_0^x c^{1/2}(c^{1/2})'' \tag{5.1b}$$

has a unique solution $\{K,c\}$, where c is defined on $0 \leq x \leq X(T)$, and K is defined in $C_T = \{(x,t) : 0 \leq x \leq X(T),\ T(x) \leq t \leq 2T - T(x)\}$. Since the domains on which the solutions are defined are themselves defined by part of the solution (namely c), the problem has something of the nature of a free boundary problem.

We note that continuous solutions are trivially unique, in view of the Volterra character of (5.1).

The system (5.1) will only have a solution as described when H has the positivity property

$$\mathbb{I} + 4\,\mathbb{H} \geq \epsilon(T) > 0 \tag{5.2}$$

in the notation of the last section, for the reasons explained there. Our goal is to show that this necessary condition is also sufficient.

First introduce the function

$$J(x,t) = 2c^{-1/2}(x)K(x,t)$$

and rewrite (5.1) as

$$4H(s,t) = J(X(s),t) + \int_0^{X(s)} dy\; c^{-1}(y)J(y,s)J(y,t) \tag{5.3a}$$

$$J(x,T(x)) = \frac{1}{2}\int_0^x c^{1/2}(c^{1/2})'' \tag{5.3b}$$

We have from (5.3b), (4.10)

$$J(x,T(x)) = \frac{1}{4} c'(x) - \frac{1}{8} \int_0^x c^{-1}(c')^2$$

so

$$c(x) = 1 + 4\int_0^x dy\ J(y,T(y)) + \frac{1}{2} \int_0^x dy \int_0^y c^{-1}(c')^2 \qquad (5.4)$$

We shall suppose that (5.3) has been solved for $0 \leq x \leq x_0$. Set

$$\tilde{H}(x,t) = 4H(s,t) - \int_0^{x_0} dy\ c^{-1}(y)J(y,s)J(y,t).$$

Then for $x \geq x_0$ (5.3) may be rewritten

$$\tilde{H}(s,t) = J(X(s),t) + \int_{x_0}^{X(s)} dy\ c^{-1}(y)J(y,s)J(y,t) \qquad (5.5a)$$

$$c(x) = k(x,x_0) + 4\int_{x_0}^x dy\ J(y,T(y)) + \frac{1}{2} \int_{x_0}^x dy \int_{x_0}^y c^{-1}(c')^2 \qquad (5.5b)$$

where

$$k(x,x_0) = c(x_0) + \frac{1}{2} \int_0^{x_0} dy \int_0^y c^{-1}(c')^2 + \frac{1}{2}(x-x_0)\int_0^{x_0} c^{-1}(c')^2 \qquad (5.5c)$$

and (5.5a) is to be construed for $T(x_0) \leq s \leq t$.

Now define a sequence of approximate solutions by iterating the right-hand sides of (5.5): select $\Delta x > 0$ and define

$$c_0(x) \equiv c(x_0) \quad \text{for} \quad x_0 \leq x \leq x_0 + \Delta x$$

$$J_0(x,t) \equiv 0 \quad \text{for} \quad x_0 \leq x \leq x_0 + \Delta x$$

$$t \geq 0.$$

For $n \geq 1$, define

$$c_n(x) = k(x,x_0) + 4\int_{x_0}^x dy\ J_{n-1}(y,T_{n-1}(y))$$

$$+ \frac{1}{2} \int_{x_0}^x dy \int_{x_0}^y c_{n-1}^{-1}(c'_{n-1})^2$$

$$\text{for} \quad x_0 \leq x \leq x_0 + \Delta x$$

Here $T_n(x) = \int_0^{x_0} c^{-1} + \int_{x_0}^{x} c_n^{-1}$ for $x_0 \leq x \leq x_0 + \Delta x$. Similarly,

$$J_n(x,t) = \tilde{H}(T_{n-1}(x),t) - \int_{x_0}^{x} dy\, c_{n-1}^{-1}(y) J_{n-1}(y, T_{n-1}(x)) J_{n-1}(y,t)$$

$$\text{for } x_0 \leq x \leq x_0 + \Delta x$$
$$t \geq 0.$$

Now suppose $\delta > 0$, and select c_* with $0 < c_* \leq c(x_0) - \delta$. We claim that, for Δx small enough, we have $c_n(x) \geq c_*$, $x_0 \leq x \leq x + \Delta x$. In fact, suppose that this is so for c_k, $k = 0,1,\ldots,n-1$ (it is obviously true for $n = 0$). As the first part of the induction, we estimate J_{n-1}:

$$|J_{n-1}(x,t)| \leq |\tilde{H}(T_{n-2}(x),t)| + \int_{x_0}^{x} dy\, c_*^{-1} |J_{n-2}(y, T_{n-2}(x))| |J_{n-2}(y,t)| \tag{5.6}$$

Now

$$\tilde{H}(s,t) = 4[F(s+t) + \int_0^g d\tau\, F(s-\tau)F(t-\tau)] - \int_0^{x_0} dy\, c^{-1}(y) J(y,s) J(y,t)$$

So long as $T_{n-2}(x_0 + \Delta x) \leq T$, which we assume for the moment, we have

$$|H(s,t)| \leq \|F\|_{L^\infty[0,2T]} + \|F\|^2_{L^2[0,T]}.$$

The second summand on the r.h.s. is estimated by (4.10):

$$\left|\int_0^{x_0} dy\, c^{-1}(y) J(y,s) J(y,t)\right| \leq 32\epsilon(T)^{-1} [\|F\|^2_{L^2[0,2T]} + T^2 \|F\|^4_{L^2[0,T]}]$$

So

$$|\tilde{H}(s,t)| \leq 4\|F\|_{L^\infty[0,2T]} + (4 + 32\epsilon(t)^{-1}) \|F\|_{L^2[0,2T]} + 32\epsilon(T)^{-1}\, T^2 \|F\|^4_{L^2[0,T]}$$

$$\equiv N(F,T,\epsilon) \tag{5.7}$$

Combined with (5.6) this yields

$$\|J_{n-1}\|_\infty \leq N(F,T,\epsilon) + (x - x_0)c_*^{-1}\|J_{n-2}\|_\infty^2$$

where $\|J\|_\infty$ means, for the moment, $\sup\{|J(x,t)| : x_0 \leq x \leq x_0 + \Delta x,\ \ t \geq 0\}$.

Now suppose that

$$\|J_{n-2}\|_\infty \leq (1 + \delta')N(F,T,\epsilon).$$

Then

$$\|J_{n-1}\|_\infty \leq [1 + (x - x_0)c_*^{-1}(1 + \delta')N(F,T,\epsilon)]N(F,T,\epsilon)$$

Suppose that Δx is so small that

$$\Delta x\, c_*^{-1}(1 + \delta')N(F,T,\epsilon) \leq \delta'. \tag{5.8}$$

Then we have once again that

$$\|J_{n-1}\|_\infty \leq (1 + \delta')N(F,T,\epsilon) \tag{5.9}$$

To complete the induction, notice from (5.5.c) that

$$k(x,x_0) \geq c(x_0) \geq c_* + \delta$$

so that (from the definition of c_n)

$$c_n(x) \geq c_* + \delta - 4\left|\int_{x_0}^{x} dy\, J_{n-1}(y,T_{n-1}(y))\right| \geq c_* + \delta - 4(1 + \delta')N(F,T,\epsilon)\Delta x$$

so we have proved

<u>Lemma B</u>. <u>Suppose</u> $\delta, \delta' > 0$, <u>and</u>

$$\Delta x \leq \min\{c_*[(1 + \delta')N(F,T,\epsilon)]^{-1}\delta', [4(1 + \delta')N(F,T,\epsilon)]^{-1}\delta\}.$$

<u>Then</u>

$$c_n(x) \geq c_*, \quad n = 0,1,2,\ldots$$
$$x_0 \leq x \leq x_0 + \Delta x.$$

Note that $\varepsilon' > 0$ is arbitrary here, whereas δ must be chosen so that $0 < c(x_0) - \varepsilon$. The next step begins with the assumption of a Lipschitz bound on F:

$$|F(t) - F(s)| \le L|s - t| \tag{5.10}$$

It follows that

$$|H(\tau,t) - H(\tau,s)| \le |F(\tau + t) - F(\tau + s)| + \left|\int_0^{\tau} d\sigma\, F(\tau - \sigma)(F(t-\sigma) - F(s-\sigma))\right|$$
$$\le L|s - t|(1 + T\|F\|_{L^\infty[0,T]}) \equiv L_1|s - t| \tag{5.11}$$

Set $J_{s,t}(y) = J(y,s) - J(y,t)$. It follows from (5.5a) that

$$J_{s,t}(y) = 4[H(T(y),s) - H(T(y),t)] + \int_0^y d\tau\, c^{-1}(\tau)J(\tau,y)J_{s,t}(\tau)$$

This is a linear Volterra equation, whence follows the estimate (for $0 \le y \le \min(X(s), X(t))$

$$|J(y,s) - J(y,t)| \le \|J_{s,t}\|_{L_\infty[0,x_0]} \le L_1(1 + T\|F\|_{L^\infty[0,T]})$$
$$\times \exp[x_0 c_*^{-1}\|J\|_\infty](|s - t|)$$
$$= L_2|s - t| \tag{5.12}$$

Here $\|J\|_\infty = \sup\{|J(x,t)| : 0 \le x \le x_0,\ t \ge 0\}$ is estimated by (4.11) in terms of F, so L_2 is estimated in terms of F, c_*, T, x_0.

Next we observe that

$$T_n(x) - T_{n-1}(x) = \int_{x_0}^{x} c_n^{-1} - c_{n-1}^{-1} = \int_{x_0}^{x} c_n^{-1}c_{n-1}^{-1}(c_{n-1} - c_n)$$

So

$$|T_n(x) - T_{n-1}(x)| \le (x - x_0)c_*^{-2} \sup_{x_0 \le y \le x} |c_{n-1}(y) - c_n(y)|$$
$$\le \Delta x c_*^{-2}\|c_{n-1} - c_n\|_\infty \tag{5.13}$$

where for the movement

$$\|c_{n-1} - c_n\|_\infty = \sup_{x_0 \le y \le x_0 + \Delta x} |c_{n-1}(y) - c_n(y)|$$

and so on.

According to the definition of $\tilde{H}$,

$$\tilde{H}(s,t) - \tilde{H}(t,\tau) = 4[H(s,\tau) - H(t,\tau)] - \int_0^{x_0} dy\, c^{-1}(y)[J(y,s) - J(y,t)]J(y,\cdot)$$

Hence

$$|\tilde{H}(s,\tau) - \tilde{H}(t,\tau)| \le 4L_1|s-t| + x_0 c_*^{-1}\|J\|_\infty L_2|s-t|$$

$$= L_3|s-t| \qquad (5.14)$$

where again L_3 is estimated in terms of F, c_*, T, and x_0. Next estimate for $x_0 \le y \le x$

$$|J_n(y,t) - J_n(y,s)| \le |\tilde{H}(T_{n-1}(y),t) - \tilde{H}(T_{n-1}(y),s)$$

$$+ |\int_{x_0}^{y} dz\, c_{n-1}^{-1}(z) J_{n-1}(z, T_{n-1}(y))|$$

$$\times \{J_{n-1}(z,t) - J_{n-1}(z,s)\}|$$

$$\le L_3|t-s|$$

$$+ (y - x_0) c_*^{-1}\|J_{n-1}\|_\infty \sup_{x_0 \le z \le y} [J_{n-1}(z,t) - J_{n-1}(z,s)] \qquad (5.15)$$

According to (5.8), (5.9),

$$(y - x_0)c_*^{-1}\|J_{n-1}\|_\infty \le \delta'.$$

Select L_4 so that

$$L_4 \le (1-\delta')^{-1} L_3$$

(which introduces the new restriction $\delta' < 1$). Then (5.15) and obvious induction guarantees that

$$\sup_{x_0 \le y \le x_0+\Delta x} |J_n(y,s) - J_n(y,t)| \le L_4 |s-t|, \quad n = 0,1,2,\ldots \tag{5.16}$$

Note that c_n' satisfies

$$c_n'(x) = \frac{\partial K}{\partial x}(x,x_0) + 4J_{n-1}(x,T_{n-1}(x)) + \frac{1}{2}\int_{x_0}^{x} c_{n-1}^{-1}(c_{n-1}')^2$$

Hence,

$$|c_n'(x)| \le \frac{1}{2}\int_0^{x_0} c^{-1}(c')^2 + 4|J_{n-1}(x,T_{n-1}(x))| + \frac{1}{2}\int_{x_0}^{x} c_{n-1}^{-1}(c_{n-1}')^2$$

We suppose that $d^* > 0$ such that

$$\|c'\|_{L^\infty[0,x_0]} \le d^*$$

and $|c_k(x)| \le d^{**}$, $k = 0,1,\ldots,n-1$, $x_0 \le x \le x_0 + \Delta x$, where

$$d^{**} = \frac{1}{2} c_*^{-1}(d^*)^2 x_0 + 4(1+\delta)N(F,T,\epsilon) + \delta.$$

Then

$$|c_n'(x)| \le \frac{1}{2} c_*^{-1}(d^*)^2 x_0 + 4(1+\delta)N(F,T,\epsilon) + \frac{1}{2}\Delta x\, c_*^{-1}(d^{**})^2$$

where we have used (5.9) and Lemma A. So we have proved

Lemma C. *Provided* Δx *satisfies the bounds of Lemma B and additionally*

$$\Delta x \le 2c_*(d^{**})^{-2}\delta$$

we have the estimates for all $n \ge 0$:

$$|c_n'(x)| \le d^{**}, \quad x_0 \le x \le x_0 + \Delta x \tag{5.17}$$

We are now ready for the main estimates. First,

$$\begin{aligned} |J_n(x,t) - J_{n-1}(x,t)| \le\ & |\tilde{H}(T_{n-1}(x),t) - \tilde{H}(T_{n-2}(x),t)| \\ & + \Big|\int_{x_0}^{x} dy\{c_{n-1}^{-1}(y)J_{n-1}(y,T_{n-1}(x))J_{n-1}(y,t) \\ & - c_{n-2}^{-1}(y)J_{n-2}(y,T_{n-2}(x))J_{n-2}(y,t)\}\Big| \end{aligned}$$

$$\leq \Delta x\, L_3\, c_*^{-2} \|c_{n-1} - c_{n-2}\|_\infty$$

$$+ \int_{x_0}^{x} dy\, |c_{n-1}^{-1}(y) - c_{n-2}^{-1}(y)|\, |J_{n-1}(y, T_{n-1}(x)) J_{n-1}(y,t)|$$

$$+ \int_{x_0}^{x} dy\, c_{n-2}^{-1}(y)\, |J_{n-1}(y, T_{n-1}(x) - J_{n-1}(y, T_{n-2}(x))|\, |J_{n-1}(y,t)|$$

$$+ \int_{x_0}^{x} dy\, c_{n-2}^{-1}(y)\, |J_{n-1}(y, T_{n-2}(x)) - J_{n-2}(y, T_{n-2}(x))|\, |J_{n-1}(y,t)|$$

$$+ \int_{x_0}^{x} dy\, c_{n-2}^{-1}(y)\, |J_{n-2}(y, T_{n-2}(x))|\, |J_{n-1}(y,t) - J_{n-2}(y,t)|$$

$$\leq \Delta x\, c_*^{-2} [L_3 + \|J_{n-1}\|_\infty^2] \|c_{n-1} - c_{n-2}\|_\infty$$

$$+ \Delta x\, c_*^{-3} \|J_{n-1}\|_\infty\, L_4 \|c_{n-1} - c_{n-2}\|_\infty$$

$$+ \Delta x\, c_*^{-1} (\|J_{n-1}\|_\infty + \|J_{n-2}\|_\infty) \|J_{n-1} - J_{n-2}\|_\infty$$

$$\leq \Delta x\, A \|c_{n-1} - c_{n-2}\|_\infty + \Delta x\, B \|J_{n-1} - J_{n-2}\|_\infty \tag{5.18}$$

where

$$A = c_*^{-2} [L_3 + (1 + \delta')N(F,T,\epsilon)[(1 + \delta')N(F,T,\epsilon) + c_*^{-1}\, L_4]]$$

$$B = 2c_*^{-1} (1 + \delta')N(F,T,\epsilon).$$

Next estimate

$$|c_n(x) - c_{n-1}(x)| \leq 4\int_{x_0}^{x} dy\, |J_{n-1}(y, T_{n-1}(y)) - J_{n-2}(y, T_{n-2}(y))|$$

$$+ \frac{1}{2} \int_{x_0}^{x} dy \int_{x_0}^{y} |c_{n-1}^{-1}(c_{n-1}')^2 - c_{n-2}^{-1}(c_{n-2}')^2|$$

$$\leq 4\int_{x_0}^{x} dy\, |J_{n-1}(y, T_{n-1}(y)) - J_{n-1}(y, T_{n-2}(y))|$$

$$+ 4\int_{x_0}^{x} dy\, |J_{n-1}(y, T_{n-2}(y) - J_{n-2}(y, T_{n-2}(y))|$$

$$+ \frac{1}{2} \int_{x_0}^{x} dy \int_{x_0}^{y} c_{n-1}^{-1} c_{n-2}^{-1} |c_{n-1} - c_{n-2}|\, (c_{n-1}')^2$$

$$\sup_{x_0 \le y \le x_0 + \Delta x} |J_n(y,s) - J_n(y,t)| \le L_4 |s-t|, \; n = 0,1,2,\ldots \qquad (5.16)$$

Note that c_n' satisfies

$$c_n'(x) = \frac{\partial K}{\partial x}(x,x_0) + 4J_{n-1}(x,T_{n-1}(x)) + \frac{1}{2}\int_{x_0}^{x} c_{n-1}^{-1}(c_{n-1}')^2$$

Hence,

$$|c_n'(x)| \le \frac{1}{2}\int_0^{x_0} c^{-1}(c')^2 + 4|J_{n-1}(x,T_{n-1}(x))| + \frac{1}{2}\int_{x_0}^{x} c_{n-1}^{-1}(c_{n-1}')^2$$

We suppose that $d^* > 0$ such that

$$\|c'\|_{L^\infty[0,x_0]} \le d^*$$

and $|c_k(x)| \le d^{**}$, $k = 0,1,\ldots,n-1$, $x_0 \le x \le x_0 + \Delta x$, where

$$d^{**} = \frac{1}{2} c_*^{-1}(d^*)^2 x_0 + 4(1+\delta)N(F,T,\epsilon) + \delta.$$

Then

$$|c_n'(x)| \le \frac{1}{2} c_*^{-1}(d^*)^2 x_0 + 4(1+\delta)N(F,T,\epsilon) + \frac{1}{2}\Delta x \, c_*^{-1}(d^{**})^2$$

where we have used (5.9) and Lemma A. So we have proved

Lemma C. *Provided* Δx *satisfies the bounds of Lemma B and additionally*

$$\Delta x \le 2c_*(d^{**})^{-2}\delta$$

we have the estimates for all $n \ge 0$:

$$|c_n'(x)| \le d^{**}, \quad x_0 \le x \le x_0 + \Delta x \qquad (5.17)$$

We are now ready for the main estimates. First,

$$\begin{aligned} |J_n(x,t) - J_{n-1}(x,t)| \le\ & |\tilde{H}(T_{n-1}(x),t) - \tilde{H}(T_{n-2}(x),t)| \\ & + \Big|\int_{x_0}^{x} dy\{c_{n-1}^{-1}(y)J_{n-1}(y,T_{n-1}(x))J_{n-1}(y,t) \\ & - c_{n-2}^{-1}(y)J_{n-2}(y,T_{n-2}(x))J_{n-2}(y,t)\}\Big| \end{aligned}$$

$$+ \frac{1}{2} \int_{x_0}^{x} dy \int_{x_0}^{y} c_{n-2}^{-1} |c'_{n-1} + c'_{n-2}| \, |c'_{n-1} - c'_{n-2}|$$

$$\leq 4(\Delta x)^2 c_*^{-2} L_4 \|c_{n-1} - c_{n-2}\|_\infty + 4\Delta x \|J_{n-1} - J_{n-2}\|_\infty$$

$$+ \frac{1}{4}(\Delta x)^2 c_*^{-2} (d^{**})^2 \|c_{n-1} - c_{n-2}\|_\infty$$

$$+ \frac{1}{2}(\Delta x)^2 c_*^{-1} d^{**} \|c'_{n-1} - c'_{n-2}\|_\infty \qquad (5.19)$$

Finally,

$$|c'_n(x) - c'_{n-1}(x)| \leq 4|J_{n-1}(x, T_{n-1}(x)) - J_{n-2}(x, T_{n-2}(x))|$$

$$+ \frac{1}{2} \int_{x_0}^{x} |c_{n-1}^{-1} (c'_{n-1})^2 - c_{n-2}^{-1} (c'_{n-2})^2|$$

$$\leq 4|J_{n-1}(x, T_{n-1}(x)) - J_{n-1}(x, T_{n-2}(x))|$$

$$+ 4|J_{n-1}(x, T_{n-2}(x)) - J_{n-2}(x, T_{n-2}(x))|$$

$$+ \frac{1}{2} \int_{x_0}^{x} c_{n-1}^{-1} c_{n-2}^{-1} |c_{n-1} - c_{n-2}| (c'_{n-1})^2$$

$$+ \frac{1}{2} \int_{x_0}^{x} c_{n-2}^{-1} |c'_{n-1} + c'_{n-2}| \, |c'_{n-1} - c'_{n-2}|$$

$$\leq 4\Delta x \, c_*^{-2} \|c_{n-1} - c_{n-2}\|_\infty + 4\|J_{n-1} - J_{n-2}\|_\infty$$

$$+ \frac{1}{2} \Delta x \, c_*^{-2} (d^{**})^2 \|c_{n-1} - c_{n-2}\|_\infty$$

$$+ \Delta x \, c_*^{-1} d^{**} \|c'_{n-1} - c'_{n-2}\|_\infty \qquad (5.20)$$

The estimates (5.18), (5.19) and (5.20), taken together, show that, provided that $\Delta x > 0$ satisfies the bounds in Lemmas B and C, and is possibly smaller yet, the sequences $\{J_n\}$, $\{c_n\}$, and $\{c'_n\}$ converge uniformly on $x_0 \leq x \leq x_0 + \Delta x$ to solutions of (5.5). The numbers which determine how small Δx must be are c_*, δ', $N(F,T,\epsilon)$, δ, L (in (5.10)), $\|F\|_\infty$, T, $\|J\|_\infty$, d^*, and x_0. Of these, $N(F,T,\epsilon)$, L, $\|F\|_\infty$ and T are determined by the data, c_* is estimated from below by (4.16a), d^* is

$$\leq \Delta x\ L_3\ c_*^{-2} \|c_{n-1} - c_{n-2}\|_\infty$$

$$+ \int_{x_0}^{x} dy |c_{n-1}^{-1}(y) - c_{n-2}^{-1}(y)| |J_{n-1}(y, T_{n-1}(x)) J_{n-1}(y,t)|$$

$$+ \int_{x_0}^{x} dy\ c_{n-2}^{-1}(y) |J_{n-1}(y, T_{n-1}(x) - J_{n-1}(y, T_{n-2}(x))| |J_{n-1}(y,t)|$$

$$+ \int_{x_0}^{x} dy\ c_{n-2}^{-1}(y) |J_{n-1}(y, T_{n-2}(x)) - J_{n-2}(y, T_{n-2}(x))| |J_{n-1}(y,t)|$$

$$+ \int_{x_0}^{x} dy\ c_{n-2}^{-1}(y) |J_{n-2}(y, T_{n-2}(x))| |J_{n-1}(y,t) - J_{n-2}(y,t)|$$

$$\leq \Delta x\ c_*^{-2}[L_3 + \|J_{n-1}\|_\infty^2] \|c_{n-1} - c_{n-2}\|_\infty$$

$$+ \Delta x\ c_*^{-3} \|J_{n-1}\|_\infty\ L_4 \|c_{n-1} - c_{n-2}\|_\infty$$

$$+ \Delta x\ c_*^{-1} (\|J_{n-1}\|_\infty + \|J_{n-2}\|_\infty) \|J_{n-1} - J_{n-2}\|_\infty$$

$$\leq \Delta x\ A \|c_{n-1} - c_{n-2}\|_\infty + \Delta x\ B \|J_{n-1} - J_{n-2}\|_\infty \tag{5.18}$$

where

$$A = c_*^{-2}[L_3 + (1 + \delta')N(F,T,\epsilon)[(1 + \delta')N(F,T,\epsilon) + c_*^{-1}\ L_4]]$$

$$B = 2c_*^{-1}(1 + \delta')N(F,T,\epsilon).$$

Next estimate

$$|c_n(x) - c_{n-1}(x)| \leq 4\int_{x_0}^{x} dy |J_{n-1}(y, T_{n-1}(y)) - J_{n-2}(y, T_{n-2}(y))|$$

$$+ \frac{1}{2} \int_{x_0}^{x} dy \int_{x_0}^{y} |c_{n-1}^{-1}(c'_{n-1})^2 - c_{n-2}^{-1}(c'_{n-2})^2|$$

$$\leq 4\int_{x_0}^{x} dy |J_{n-1}(y, T_{n-1}(y)) - J_{n-1}(y, T_{n-2}(y))|$$

$$+ 4\int_{x_0}^{x} dy |J_{n-1}(y, T_{n-2}(y) - J_{n-2}(y, T_{n-2}(y))|$$

$$+ \frac{1}{2} \int_{x_0}^{x} dy \int_{x_0}^{y} c_{n-1}^{-1} c_{n-2}^{-1} |c_{n-1} - c_{n-2}| (c'_{n-1})^2$$

$$+ \frac{1}{2} \int_{x_0}^{x} dy \int_{x_0}^{y} c_{n-2}^{-1} |c'_{n-1} + c'_{n-2}| \, |c'_{n-1} - c'_{n-2}|$$

$$\leq 4(\Delta x)^2 \, c_*^{-2} \, L_4 \|c_{n-1} - c_{n-2}\|_\infty + 4\Delta x \|J_{n-1} - J_{n-2}\|_\infty$$

$$+ \frac{1}{4}(\Delta x)^2 \, c_*^{-2} (d^{**})^2 \|c_{n-1} - c_{n-2}\|_\infty$$

$$+ \frac{1}{2}(\Delta x)^2 \, c_*^{-1} \, d^{**} \|c'_{n-1} - c'_{n-2}\|_\infty \qquad (5.19)$$

Finally,

$$|c'_n(x) - c'_{n-1}(x)| \leq 4 |J_{n-1}(x, T_{n-1}(x)) - J_{n-2}(x, T_{n-2}(x))|$$

$$+ \frac{1}{2} \int_{x_0}^{x} |c_{n-1}^{-1} (c'_{n-1})^2 - c_{n-2}^{-1} (c'_{n-2})^2|$$

$$\leq 4 |J_{n-1}(x, T_{n-1}(x)) - J_{n-1}(x, T_{n-2}(x))|$$

$$+ 4 |J_{n-1}(x, T_{n-2}(x)) - J_{n-2}(x, T_{n-2}(x))|$$

$$+ \frac{1}{2} \int_{x_0}^{x} c_{n-1}^{-1} c_{n-2}^{-1} |c_{n-1} - c_{n-2}| (c'_{n-1})^2$$

$$+ \frac{1}{2} \int_{x_0}^{x} c_{n-2}^{-1} |c'_{n-1} + c'_{n-2}| \, |c'_{n-1} - c'_{n-2}|$$

$$\leq 4\Delta x \, c_*^{-2} \|c_{n-1} - c_{n-2}\|_\infty + 4 \|J_{n-1} - J_{n-2}\|_\infty$$

$$+ \frac{1}{2} \Delta x \, c_*^{-2} (d^{**})^2 \|c_{n-1} - c_{n-2}\|_\infty$$

$$+ \Delta x \, c_*^{-1} \, d^{**} \|c'_{n-1} - c'_{n-2}\|_\infty \qquad (5.20)$$

The estimates (5.18), (5.19) and (5.20), taken together, show that, provided that $\Delta x > 0$ satisfies the bounds in Lemmas B and C, and is possibly smaller yet, the sequences $\{J_n\}$, $\{c_n\}$, and $\{c'_n\}$ converge uniformly on $x_0 \leq x \leq x_0 + \Delta x$ to solutions of (5.5). The numbers which determine how small Δx must be are c_*, δ', $N(F, T, \epsilon)$, δ, L (in (5.10)), $\|F\|_\infty$, T, $\|J\|_\infty$, d^*, and x_0. Of these, $N(F, T, \epsilon)$, L, $\|F\|_\infty$ and T are determined by the data, c_* is estimated from below by (4.16a), d^* is estimated by (4.16b), $\|J\|_\infty$ is governed by the main a-priori estimate (4.11), and

$$x_0 \leq c^* T$$

where c^* is an upper bound for c, given by (4.16a). It follows that, for given T and F satisfying the positivity condition (5.2), Δx may be chosen independently of x_0 so long as $x_0 \leq X(T)$. Thus finitely many repetitions of the iteration scheme outlined above suffice to determine c on the interval $[0,X(T)]$, and J on the corresponding domain. The system (5.5), (and with it (5.1)) has therefore been solved, as promised.

By differentiating the G-L system (5.1), one obtains systems of Volterra equations for the derivatives of c and the partial derivatives of K. These are (essentially) linear systems. Without carrying out the details, we state that these systems for the derivatives possess continuous solutions. As in the case of the GL equation itself, c winds up with one more derivative than K, and K has as many derivatives as F. The solution of the GL system therefore defines a map $F \mapsto \{K,c\}$. It is easily verified that the positivity condition is stable under perturbation. It follows that the map $F \mapsto \{K,c\}$ is continuous in the obvious sense of C^m norms, as outlined in the introduction.

§6. The lower bound on ϵ

For reasons explained in the introduction, we now estimate $\epsilon(T)$ (see (4.3)) in terms of the bounds c_*, c^*, d^* and e^* which we suppose given:

$$c_* \leq c(x) \leq c^*,$$

$$|c'(x)| \leq d^*,$$

$$|c''(x)| \leq e^*, \quad 0 \leq x \leq X(t).$$

Recall that K solves the boundary value problem

$$\left(\frac{\partial^2}{\partial t^2} - c^2(x)\,\frac{\partial^2}{\partial x^2}\right)K(x,t) = 0, \quad t \geq T(x)$$

$$K(x,T(x)) = \frac{1}{4}\,c^{1/2} \int_0^x c^{1/2}(c^{1/2})''$$

$$= \frac{1}{8}\,c^{1/2}(x)c'(x) - \frac{1}{16}\,c^{1/2}(x) \int_0^x c^{-1}(c')^2$$

$$(x \geq 0) \tag{6.1}$$

Also, according to Lemma C (Sec. 3) (recall $c \equiv 1$ for $x < 0$):

$$K(x,-x) = 0, \; x < 0 .$$

It follows, as in [11], *Ch. V, that K is the solution of an integral* equation of Volterra type. In fact, if you denote by $C(x,t)$ the intersection of the backward light cone with vertex (x,t) with the forward light cone with vertex $(0,0)$, and by $\Gamma(x,t)$ the x-coordinate of the intersection of the characteristic curve of negative shape through (x,t) with the characteristic curve of positive slope through $(0,0)$, you eventually obtain

$$J(x,t) = \frac{1}{2} \int_{C(x,t)} \{\tfrac{1}{2}\,c'' + \tfrac{1}{4}\,c^{-1}(c')^2\}J + \frac{1}{8}[c'(\Gamma(x,t) - \frac{1}{2} \int_0^{\Gamma(x,t)} c^{-1}(c')^2]$$

(recalling that $J(x,t) = 2c^{1/2}(x)K(x,t))$. Now

$$\text{vol } C(x,t) \leq \frac{1}{2}\,c^* t^2.$$

It follows easily that

$$|J(x,t)| \leq Pf(\sqrt{c^* Q}\;t) \tag{6.2}$$

where f is the entire function with power series

$$f(z) = \frac{1}{2}\{1 + z^2 + \frac{z^4}{3} + \frac{z^6}{5 \cdot 3} + \frac{z^8}{7 \cdot 5 \cdot 3} + \ldots\}$$

and

$$P = \sup_{0 \leq x \leq X(T)} \frac{1}{8}[c'(x) - \frac{1}{2}\int_0^x c^{-1}(c')^2]$$
$$\leq \frac{1}{8}[d^* + \frac{1}{2} c^* c_*^{-1}(d^*)^2 T]$$

$$Q = \sup_{0 \leq x \leq X(T)} [\frac{1}{2} c''(x) + \frac{1}{4} c^{-1}(x)(c'(x))^2]$$
$$\leq \frac{1}{2} e^* + \frac{1}{4} c_*^{-1}(d^*)^2.$$

Recalling the definition (4.1) of $\tilde{K}$ and of the operator $\mathbb{K}$, one sees that $\tilde{K}$ obeys the same sup norm bound as J, namely (6.2) above. The kernel of the inverse operator $(\mathbb{I} + \mathbb{K})^{-1}$, say $\hat{K}$, is then easily bounded:

$$\|\hat{K}\|_\infty \leq \|\tilde{K}\|_\infty \exp(\|\tilde{K}\|_\infty T) \tag{6.3}$$

It follows immediately that

$$\|(\mathbb{I} + \mathbb{K})^{-1}\varphi\|^2 \leq (1 + \|\hat{K}\|_\infty T)^2 \|\varphi\|^2$$

So (see (4.4))

$$\epsilon(T)^{-1} \leq (1 + \|\tilde{K}\|_\infty T \exp(\|\tilde{K}\|_\infty T))^2$$

which together with the bound (6.2) estimates $\epsilon(T)$ in terms of T and the a priori information on c, as desired.

§7. Equivalence of GL and Chudov systems

We show by direct computation that the solution of the GL system constructed in Section 5 solves the Chudov boundary value problem

$$\left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial}{\partial x^2}\right)K \equiv 0 \quad \text{in} \quad \{t \geq 0,\ 0 \leq x \leq X(t)\} \tag{7.1a}$$

$$K(0,t) = F(t)$$

$$(\frac{\partial}{\partial t} - \frac{\partial}{\partial x})K(0,t) \equiv 0 \tag{7.1b}$$

$$K(x,T(x)) = \frac{1}{4} c^{1/2}(x)\int_0^x c^{1/2}(c^{1/2})'' \tag{7.1c}$$

Now it was shown in §2 that the smooth part of the solution of the initial value problem (2.1) solves (7.1). Since the solution of (7.1) can easily be shown to be unique, it follows that the solution of the GL system is in fact the smooth part of the solution of (2.1) with corresponding coefficient c, hence that we have solved the inverse problem.

First note from (3.10c) that

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial s^2}\right)H(s,t) = F(s)F'(t) - F'(s)F(t) \tag{7.2}$$

Now denote by $\{K,c\}$ the solution of the GL system as constructed in Section 5. We assume of course that F satisfies the positivity condition (5.2) and is of class C^2 on its interval of definition so that K and c are of classes C^2 and C^3 respectively on their domains of definition.

Setting $s = 0$ in (5.10) and recalling the definition (3.10c) of H, one obtains

$$F(t) = K(0,t).$$

Using the definition (3.10c) again, and the requirement $F(0) = 0$, one sees that

$$(\frac{\partial}{\partial t} - \frac{\partial}{\partial s})H(0,t) \equiv 0.$$

On the other hand, from (5.1) one obtains (recalling $c(0) = 1$, $c'(0) = 0$ and $K(0,0) = F(0) = 0$)

$$0 = (\frac{\partial}{\partial t} - \frac{\partial}{\partial s})H(0,t) = \frac{1}{2}(D_2K(0,t) - D_1K(0,t)).$$

Thus K obeys the boundary conditions on the Chudov system. It remains only to verify that K solves the wave equation in the interior of the light cone. To do this first compute

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial s^2}\right)H(s,t) = \frac{1}{2}\, c^{-1/2}(X(s))D_2^2K(X(s),t)$$
$$+ \left[\int_0^{X(s)} dy\; c^{-2}(y)K(y,s)D_2^2K(y,t)\right]$$
$$- \frac{1}{8}\, c^{-1/2}(X(s))(c'(X(s)))^2K(X(s),t)$$
$$+ \frac{1}{4}\, c^{1/2}(X(s))c''(X(s))K(X(s),t)$$
$$- \frac{1}{2}\, c^{3/2}(X(s))D_1^2K(X(s),t)$$
$$- \frac{d}{ds}[c^{-1}(X(s))K(X(s),s)K(X(s),t)]$$
$$- c^{-1}(X(s))D_2K(X(s),s)K(X(s),t)$$
$$- \int_0^{X(s)} dy\; c^{-2}(y)D_2^2K(y,s)K(y,t)$$

Using (7.1c) one sees that

$$c^{-1/2}(X(s))\,\frac{d}{ds}[2c^{-1/2}(X(s))K(X(s),s)]$$
$$= 2c^{-1}(X(s))\,\frac{d}{ds}\,K(X(s),s) - c^{-1}(X(s))c'(X(s))K(X(s),s)$$
$$= \frac{1}{4}\, c^{1/2}(X(s))c''(X(s)) - \frac{1}{8}\, c^{-1/2}(X(s))(c'(X(s)))^2$$

Also

$$\frac{d}{ds}[c^{-1}(X(s))K(X(s),s)K(X(s),t)] + c^{-1}(X(s))D_2K(X(s),s)K(X(s),t)$$
$$= -c^{-1}(X(s))c'(X(s))K(X(s),s)K(X(s),t)$$
$$+ 2c^{-1}(X(s))\,\frac{d}{ds}\,K(X(s),s)K(X(s),t) - D_1K(X(s),s(K(X(s),t)$$
$$+ K(X(s),s)D_1K(X(s),t)$$
$$= \left[\frac{1}{4}\, c^{1/2}(X(s))c''(X(s)) - \frac{1}{8}\, c^{-1/2}(X(s))(c'(X(s)))^2\right]K(X(s),t)$$
$$- D_1K(X(s),s)K(X(s),t) + K(X(s),s)D_1K(X(s),t)$$

Hence

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial s^2}\right)H(s,t) = \frac{1}{2}\, c^{-1/2}(X(s))[D_2^2K(X(s),t) - c^2(X(s))D_1^2K(X(s),t)]$$
$$+ \int_0^{X(s)} dy\; c^{-2}(y)[K(y,s)D_2^2K(y,t) - D_2^2K(y,s)K(y,t)]$$
$$+ D_1K(X(s),s)K(X(s),t) - K(X(s),s)D_1K(X(s),t) \quad (7.3)$$

Now add to (7.3) the identity

$$0 = -D_1K(X(s),s)K(X(s),t) + K(X(s),s)D_1K(X(s),t) + D_1K(0,s)K(0,t)$$
$$- K(0,s)D_1K(0,t)$$
$$- \int_0^{X(s)} dy\; c^{-2}(y)[K(y,s)(c_2(y)D_1^2K(y,t))$$
$$- (c^2(y)D_1^2K(y,s))K(y,t)]$$

obtained by integration by parts. After a little manipulation, one gets

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial s^2}\right)H(s,t) = D_1K(0,s)K(0,t) + K(0,s)D_1K(0,t)$$
$$= \frac{1}{2}\, c^{-1/2}(X(s))[D_2^2K(X(s),t) - c^2(X(s))D_1^2K(X(s),t)]$$
$$+ \int_0^{X(s)} dy\; c^{-2}(y)[K(y,s)\{D_2^2K(y,t) - c^2(y)D_1^2K(y,t)\}$$
$$- \{D_2^2K(y,s) - c^2(y)D_1^2K(y,s)\}K(y,t)] \quad (7.4)$$

Comparing with (7.2) and using (7.1b), one sees that the l.h.s. of (7.4) in fact vanishes identically. Eq. (7.4) is therefore a linear integral equation of Volterra type, since K is continuous and $c > 0$, for $D_2^2K - c^2D_1^2K$. It follows that this expression vanishes in the domain covered by the limits of integration, that is, in the light cone. This result shows that the solution K of the GL equation solves the Chudov system also. Since we showed in Sections 2 and 3 that the solution of the Chudov system solves the GL system, it follows that these two systems are completely equivalent. It also follows that c, as part of the solution of the GL system for prescribed F, solves the inverse problem stated in Section 1. Together with the stability statement of Section 5, this constitutes a proof of Theorem A.

References

1. J. Berryman, Bell Laboratories, Preprint (1979).

2. J. Ware and K. Aki, J. Acoust. Soc. Am., 45, 911-921 (1969).

3. D. Cohen and N. Bleistein, SIAM J. Appl. Math., 32, 784-799 (1977).

4. K. Chad , and P. Sabatier, Inverse Problems in Quantum Scattering Theory, Springer, New York, 1977.

5. P. Gopilland, Geophys., 26, 754-760 (1961).

6. W. Symes, MRC TSR #1846, UW/Madison, Wisconsin (1978), to appear J. Math. Anal. Appl., 7 (1), 378-402 (1979).

7. W. Symes, MRC TSR #1990, UW/Madison, Wisconsin (1979), submitted to SIAM J. Num. Anal.

8. O. Hald, Acta Math. (Sweden) 141, 264-291 (1979).

9. P. Deift and E. Trubowitz, Comm. Pure Appl. Math., 32, 121-251 (1979).

10. P. Lax, Notes on Hyperbolic P.D.E., Stanford (1963).

11. R. Courant and D. Hilbert, Methods of Mathematical Physics, Vol. II, Wiley-Interscience, New York (1962).

12. W. Symes, (in preparation).

13. I.M. Gel'fand and B.M. Levitan, Isv. Akad. Nauk. SSSR Ser. Matem., 15 (2), 309-360 (1951); English translation: AMS Tr. Ser. 2, 1, 253-304 (1955).

14. L.D. Fadeev, J. Math. Phys., 4 (1), 72-104 (1963).

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER 2007	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) THE INVERSE REFLECTION PROBLEM FOR A SMOOTHLY STRATIFIED ELASTIC MEDIUM		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) W. Symes		8. CONTRACT OR GRANT NUMBER(s) MCS78-09525 DAAG29-75-C-0024
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 1 - Applied Analysis
11. CONTROLLING OFFICE NAME AND ADDRESS See Item 18 below		12. REPORT DATE October 1979 13. NUMBER OF PAGES 39
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

U.S. Army Research Office
P.O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, D.C. 20550

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Inverse scattering, Nonlinear Volterra systems, Stratified elastic medium

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The subject of this paper is a version of the inverse reflection problem for a smoothly stratified elastic medium. The same mathematics describes the inverse problem of the vibrating string. This problem is solved in a constructive way. Also, *a priori* estimates are derived which exhibit the continuous dependence of the solution (index of refraction, relative sound speed) on the data (scattering or reflection measurements).