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ONE-SIDED CONTINUOUS DEPENDENCE OF MAXIMAL SOLUTIONS.(U)

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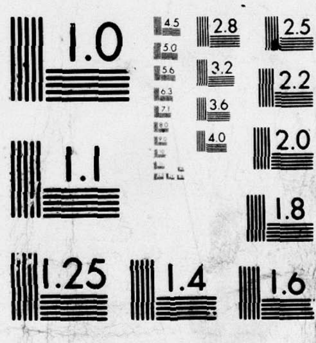

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6 ONE-SIDED CONTINUOUS DEPENDENCE OF MAXIMAL SOLUTIONS.

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ONE-SIDED CONTINUOUS DEPENDENCE OF MAXIMAL SOLUTIONS

Eric Schechter

Technical Summary Report #2016
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ABSTRACT

Existence of a maximal solution is proved for a differential equation satisfying a one-sided variant of Carathéodory's condition. The maximal solution is shown to dominate all solutions of a very general differential inequality. Also a best-possible condition is proved for the dependence of the maximal solution on the initial data and on the right-hand side of the equation.

AMS (MOS) Subject Classification: 34A10

Key Words: Maximal solution, Carathéodory condition, Continuous dependence, Differential inequality

Work Unit Number 1 (Applied Analysis)

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SIGNIFICANCE AND EXPLANATION

Many applied problems in fluid mechanics can be modelled by nonlinear evolution equations of the form

$$\text{(E)} \qquad u'(t) = A(t,u(t)) + B(t,u(t)) + g(t) .$$

Here $u(t) = u(t,x) = u(t,x_1,x_2,x_3)$ is a vector-valued function of time and space, which represents the state of the system at time t. The operators A and B are, typically, nonlinear partial differential operators in the spatial variables, and $g(t) = g(t,x)$ is a given forcing term.

Equation (E) usually is so complicated that there is no hope of finding explicit solutions in closed form. Thus it is important to obtain qualitative, and whenever possible, quantitative information about the solution u of (E). This often can be accomplished by showing that u is the limit of an approximating sequence of solutions u_n of simpler equations (E_n). For such an analysis (which will be given elsewhere) some crude estimates of $\|u(t)\|$ are needed, where $\| \ \|$ is a Banach space norm or some other measurement of how large $u(t) = u(t,x)$ is and how much $u(t,x)$ varies when x varies. Such estimates often can be obtained from a differential inequality of the form

$$\frac{d}{dt} \|u(t)\| \leq f(t,\|u(t)\|) .$$

That inequality implies $\|u(t)\| \leq z(t)$, where $z(t)$ is the maximal (i.e. largest) solution of the scalar ordinary differential equation

$$z'(t) = f(t,z(t)) \qquad (t \geq 0) ,$$

$$z(0) = \|u(0)\| .$$

In this paper we investigate the properties of maximal solutions $z(t)$, especially those properties relevant to the limiting behavior of (E). In particular, we determine in what sense the assumption $f_n \to f$ implies $z_n \to z$, under hypotheses on f which will make it possible to apply our results to (E).

ONE-SIDED CONTINUOUS DEPENDENCE OF MAXIMAL SOLUTIONS

Eric Schechter

1. Introduction

In this paper we consider the solutions of initial value problems of the form

$$(1.1)\qquad x'(t) = f(t,x(t)) \qquad (0 \le t < T)\ ,$$
$$x(0) = w\ ,$$

where T is some positive number or ∞. We assume that $w \ge 0$ and that

$$(1.2)\qquad \left\{\begin{array}{l} f : R_+ \times R_+ \to R_+ \text{ is a function such that } f(t,y) \text{ is locally} \\ \text{integrable in } t \text{ for each fixed } y, \text{ and increasing and} \\ \text{right-continuous in } y \text{ for almost every fixed } t\ . \end{array}\right.$$

These conditions do not determine $x(t)$ uniquely. (For instance, consider the equation $x'(t) = \sqrt{x(t)}$ with $x(0) = 0$.) In this paper we prove the existence of a maximal solution $x_{max}(t)$ of (1.1). We show that x_{max} not only dominates all solutions of (1.1), but also all generalized solutions (in a sense made precise in Section 2) of (1.1) and of the more general initial value problem

$$(1.3)\qquad \left\{\begin{array}{ll} v'(t) \le f(t,v(t)) & (0 \le t < T)\ , \\ v(t) \ge 0\ , & \\ v(0) = w\ . & \end{array}\right.$$

Also we obtain a best-possible result about the dependence of x_{max} on w and f.

The existence of maximal solutions of (1.1) can be shown by a variant of Filippov's methods; see [3]. However, we shall obtain the existence of maximal solutions as a byproduct of our proofs of other results.

Theorem III in this paper is analogous to, and may be motivated by, the following simpler result: Let T be a positive number. Let A be a directed set. Let f_∞ and $\{f_a : a \in A\}$ be measurable functions from $[0,T] \times \mathbf{R}^n$ into $\mathbf{R}^n$ satisfying

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$$
(1.4) \quad \begin{cases} |f_\infty(t,x) - f_\infty(t,y)| \le K(t)|x-y| \ , \\ |f_a(t,x) - f_a(t,y)| \le K(t)|x-y| \ , \\ |f_\infty(t,x)| \le M(t), \quad |f_a(t,x)| \le M(t) \ , \end{cases}
$$

for some fixed $K, M \in L^1[0,T]$. Then

$$
\lim \int_0^t f_a(s,y)ds = \int_0^t f_\infty(s,y)ds
$$

for every t in $[0,T]$ and y in $\mathbb{R}^n$ if and only if for every p in $[0,T]$ and w in $\mathbb{R}^n$, the unique solution of

$$
x'(t) = f_a(t,x(t)) \qquad (p \le t \le T) \ ,
$$
$$
x(p) = w
$$

converges (in the sense of nets, as a increases in A) to the unique solution of

$$
x'(t) = f_\infty(t,x(t)) \qquad (p \le t \le T) \ ,
$$
$$
x(p) = w \ .
$$

A variant of the above theorem was proved in [1]; however, the methods in [1] rely on the Lipschitz condition in (1.4) and do not generalize to cover the case described in (1.2). The above theorem and an assortment of generalizations can also be proved using the methods in [2].

One reason for interest in problem (1.1), (1.2) is the following: Let $(U, \| \ \|)$ be some Banach space of functions, and let $A(t)$ be some time-dependent nonlinear partial differential operator. Consider the initial value problem

$$
u'(t) = A(t)u(t) \qquad (t \ge 0) \ ,
$$
$$
u(0) = u_0 \ .
$$

For many purposes it is important to have estimates on $\|u(t)\|$. In many applications $u(t)$ satisfies some condition such as

$$
\frac{d}{dt} \|u(t)\| \le f(t, \|u(t)\|)
$$

where f satisfies (1.2). Let $w = \|u(0)\|$; then $\|u(t)\| \le x_{max}(t)$. Theorems II and III of this paper will be used in [2] for estimates of this sort.

2. Statement of results

Notation.

Let $R_+ = [0,+\infty)$. A function g is increasing if $y \geq z$ implies $g(y) \geq g(z)$, and right-continuous if $y_n \downarrow y$ implies $g(y_n) \to g(y)$. A function $x : [0,T) \to R_+$ is nonextendable (or T is final for x) if either $T = \infty$ or $x(t) \to \infty$ as t increases to T.

A function $x : [0,T) \to R_+$ is a solution of (1.1) if x is absolutely continuous on compact subsets of $[0,T)$, satisfies the differential equation almost everywhere in $[0,T)$, and satisfies the initial condition. Note that since w and f are nonnegative, $x(t)$ must be nonnegative and increasing.

Theorem I.

Assume f satisfies (1.2) and w is some nonnegative number.

Then (1.1) has at least one solution for some $T > 0$. Every solution $x(t)$ of (1.1) can be continued to a nonextendable solution $x : [0,T_x) \to R_+$.

Among the nonextendable solutions of (1.1) there exists a maximal solution $x_m : [0,T_m) \to R_+$. That is: x_m is a nonextendable solution; and if x is any other nonextendable solution, then $0 < T_m \leq T_x$ and $x(t) \leq x_m(t)$ for all t in $[0,T_m)$.

Clearly, the maximal solution is unique, since any two maximal solutions must dominate each other.

Theorem II, below, concerns solutions of (1.3) in a generalized sense. The differential inequality $v'(r) \leq f(r,v(r))$ is not suitable for some purposes, e.g. when v is not differentiable. So we shall replace that inequality with the following more general condition:

$$
(2.1)\quad \begin{cases} \text{either (i)} \int_r^t f(s,v(s))ds > 0 \text{ as } t \downarrow r, \text{ and} \\ \qquad \limsup_{t\downarrow r}[v(t)-v(r)]/\int_r^t f(s,v(s))ds \le 1 , \\ \text{or (ii)} \int_r^{r+\varepsilon} f(s,v(s))ds = 0 \text{ for some } \varepsilon > 0 , \\ \qquad \text{and } v(\cdot) \le v(r) \text{ on } [r,r+\varepsilon) . \end{cases}
$$

The following notations will be used in Theorem II: f is a function satisfying (1.2), and $v : [0,T) \to \mathbf{R}_+$ is some function. Hence (by Theorem I) for each p in $[0,T)$ there exists a unique maximal solution $x_p : [p,T_p) \to \mathbf{R}_+$ of

$$
(2.2)\quad \begin{cases} x_p'(t) = f(t,x_p(t)) \qquad (p \le t < T_p) , \\ x_p(p) = v(p) , \end{cases}
$$

with final time T_p.

For motivation note that if v is a nonextendable solution of (1.1) then v satisfies condition (2.3) of Theorem II, and hence also the other conditions.

<u>Theorem II.</u>

Let f be a function satisfying (1.2). Let $v : [0,T) \to \mathbf{R}_+$ be a measurable function which is bounded on compact subsets of $[0,T)$. Assume $T > 0$, and either $T = \infty$ or $\limsup_{t\uparrow T} v(t) = \infty$. Define maximal solutions x_p and final times T_p as in (2.2).

Then conditions (2.3), (2.4), and (2.5) are equivalent:

$$
(2.3)\quad v(t) - v(r) \le \int_r^t f(s,v(s))ds \quad \text{whenever} \quad 0 \le r \le t < T .
$$

$$
(2.4)\quad \begin{cases} v(t) \le \liminf_{r\uparrow t} v(r) \quad \text{for every } t \text{ in } (0,T), \\ \text{and (2.1) holds for every } r \text{ in } [0,T) . \end{cases}
$$

$$
(2.5)\quad \begin{cases} \text{For every } p \text{ in } [0,T),\ T_p \le T \text{ and } v(t) \le x_p(t) \\ \text{for all } t \text{ in } [p,T_p) . \end{cases}
$$

Moreover, if (2.3), (2.4), (2.5) hold, then also

(2.6) $\quad v$ has bounded variation on $[0,t]$ for every t in $[0,T)$, and

(2.7) $\quad v$ is nonextendable, i.e. either $T = \infty$ or $\liminf_{t\uparrow T} v(t) = \infty$.

Theorem III is stated in the terminology of nets. The reader may read "sequence" for "net" and let $A = \{\text{positive integers}\}$ if he so chooses. The additional generality of nets is needed for an application in [2].

The following notations will be used in Theorem III: f_∞ and $\{f_a : a \in A\}$ are functions satisfying (1.2); w_∞ and $\{w_a : a \in A\}$ are nonnegative numbers; p is a nonnegative number. Hence we can define the maximal solutions x_∞ and x_a of the initial value problems

$$(2.8) \quad \begin{cases} x'_\infty(t) = f_\infty(t, x_\infty(t)) & (p \le t < T_\infty), \\ x_\infty(p) = w_\infty, & \\ \text{and} & \\ x'_a(t) = f_a(t, x_a(t)) & (p \le t < T_a), \\ x_a(p) = w_a, & \\ \text{with final times } T_\infty \text{ and } T_a. & \end{cases}$$

The lim inf's and lim sup's in the theorem are with respect to the ordering of A.

<u>Theorem III</u>.

Let A be a directed set. Let f_∞ and $\{f_a : a \in A\}$ be functions satisfying (1.2). Define maximal solutions x_∞, x_a and final times T_∞, T_a as in (2.8). Then the following conditions (2.9), (2.10), and (2.11) are equivalent:

(2.9) $\quad \limsup \int_r^t f_a(s,y)\,ds \le \int_r^t f_\infty(s,y)\,ds$ for every choice of $y \ge 0$ and $t \ge r \ge 0$

$$(2.10)\quad \begin{cases} \text{for every choice of nonnegative numbers } p \text{ and } w_\infty, \text{ if} \\ w_a = w_\infty \text{ for all } a, \text{ then } \liminf T_a \geq T_\infty \text{ and} \\ \limsup x_a(t) \leq x_\infty(t) \text{ for all } t \text{ in } [p,T_\infty) \end{cases}$$

$$(2.11)\quad \begin{cases} \text{for every choice of nonnegative numbers } p,\ w_\infty, \text{ and} \\ \{w_a : a \in A\}, \text{ if } \limsup w_a \leq w_\infty, \text{ then } \liminf T_a \geq T_\infty \text{ and} \\ \limsup x_a(t) \leq x_\infty(t) \text{ for all } t \text{ in } [p,T_\infty)\ . \end{cases}$$

In particular, the implication (2.9) ==> (2.11) tells us that maximal solutions are increasing and right-continuous, in the sense that if

$$\int_r^t f_a(s,y)ds \downarrow \int_r^t f_\infty(s,y)ds \quad \text{for all } y \geq 0 \text{ and } t \geq r \geq 0\ ,$$

and $w_a \downarrow w_\infty$, then $T_a \uparrow T_\infty$ and $x_a(t) \downarrow x_\infty(t)$ for every t in $[p,T_\infty)$.

The reader may feel uneasy about condition (2.9), and may prefer the simpler and more familiar condition

$$(2.12)\qquad \limsup f_a(s,y) \leq f_\infty(s,y) \quad \text{for all } s \geq 0,\ y \geq 0\ .$$

In fact, (2.12) implies (2.9), at least for L^1-dominated sequences, by Fatou's Lemma. The reverse implication does not hold. For instance, take A = {positive integers}, and $f_a(s,y) = 1 + \sin(as)$, $f_\infty(s,y) = 1$. Then (2.9) holds but (2.12) does not. One purpose of Theorem III is to show that condition (2.9) is in some sense natural, and best possible.

3. Auxiliary constructions

The proofs of the theorems will be based partly on some technical lemmas given below.

Lemma 1.

Let $v : [0,T) \to \mathbf{R}_+$ be a measurable function which is bounded on compact subsets of $[0,T)$. Assume that $T > 0$, and that either $T = \infty$ or $\limsup_{t\uparrow T} v(t) = \infty$. Let $e : \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}_+$ be a function satisfying (1.2). Assume that

$$v(t) - v(r) \leq \int_r^t e(s,v(s))ds \quad \text{whenever} \quad 0 \leq r \leq t < T . \tag{3.1}$$

Then:

(i) $v(t) \leq \liminf_{r\uparrow t} v(r)$ for every t in $(0,T)$.

(ii) $v(r) \geq \limsup_{t\downarrow r} v(t)$ for every r in $[0,T)$.

(iii) If $a \in [0,T)$ and $b \in [a,\infty)$ and $h > 0$ satisfy $\int_a^b e(s,v(a) + h)ds < h$, then $b < T$ and $v(t) \leq v(a) + \int_a^t e(s,v(a) + h)ds < v(a) + h$ for all t in $[a,b]$.

(iv) v is nonextendable, i.e. $T = \infty$ or $\liminf_{t\uparrow T} v(t) = \infty$.

Note that in particular any nonextendable solution v of $v'(t) = e(t,v(t))$ satisfies the hypotheses of Lemma 1.

Proof of Lemma 1.

Since $v(s)$ is bounded on any compact subset of $[0,T)$, $e(s,v(s))$ is integrable there. So (i) and (ii) follow immediately from (3.1).

Let a,b,h satisfy the hypotheses of (iii). By (ii), since $v(a) < v(a) + h$, there is some c in $(a,T]$ such that $v(\cdot) < v(a) + h$ on $[a,c)$. Choose the largest such c. If $c < T$, then (again by (ii)) $v(c) \geq v(a) + h$. If $c = T$, then $\limsup_{t\uparrow T} v(t) \leq v(a) + h < \infty$, so we must have $T = \infty$.

For any t in $[a,b] \cap [a,c] \cap [a,T)$, compute

$$v(t) \le v(a) + \int_a^t e(s,v(s))ds \le v(a) + \int_a^t e(s,v(a) + h)ds$$

$$\le v(a) + \int_a^b e(s,v(a) + h)ds < v(a) + h .$$

In particular, if $c < T$ and $c \le b$, then $v(a) + h \le v(c) < v(a) + h$, a contradiction. So either $c = T = \infty$ (in which case $b < \infty = c = T$), or $c < T$ (in which case $b < c$). In either case we obtain $b < c \le T$. Hence

$$t \in [a,b] \implies t \in [a,b] \cap [a,c] \cap [a,T) \implies v(t) \le v(a) + \int_a^t e(s,v(a) + h)ds < v(a) + h.$$

This proves (iii).

To prove (iv), suppose T and $M \equiv \liminf_{t \uparrow T} v(t)$ are both finite. Choose some $\varepsilon > 0$ small enough so that $\int_{T-\varepsilon}^T e(s,M + 2)ds < 1$. Then choose some a in $(T - \varepsilon,T)$ such that $v(a) < M + 1$. Then $\int_a^T e(s,v(a) + 1)ds \le \int_{T-\varepsilon}^T e(s,M + 2)ds < 1$. Apply (iii) with $h = 1$ and $b = T$. This proves $T < T$, a contradiction. Hence (iv) holds. This completes the proof of the lemma.

Let $[[y]]$ be the greatest integer less than or equal to y. For each positive integer n, let $h_n(y) = 2^{-n}([[2^n y]] + 1)$. Then $h_n(y)$ is the first multiple of 2^{-n} after y. Hence $y < h_{n+1}(y) \le h_n(y) \le y + 2^{-n}$, and $h_n(y) \downarrow y$ as $n \to \infty$.

<u>Lemma 2.</u>

Let f satisfy (1.2), and let w be a nonnegative number. Fix some positive integer n. Then there exists a unique nonextendable solution $x_n : [0,T_n) \to R_+$ of the initial value problem

$$(3.2)\qquad \begin{aligned} x'_n(t) &= f(t,h_n(x_n(t))) + 2^{-n} \qquad (0 \le t < T_n)\ ,\\ x_n(0) &= h_n(w) + 2^{-n+1} = 2^{-n}([[2^n w]] + 3)\ . \end{aligned}$$

This solution has the following further properties:

Let $i = i(n) = [[2^n w]] + 3$. Then for every integer $j \geq i$ there exists a unique number t_j (depending on n) such that $x_n(t_j) = 2^{-n}j$. These numbers satisfy

$$0 = t_i < t_{i+1} < t_{i+2} < \dots < T_n\ ,$$

and $t_j \to T_n$ as $j \to \infty$.

Furthermore:

Suppose $v : [0,T) \to R_+$ and $e : R_+ \times R_+ \to R_+$ satisfy the hypotheses of Lemma 1. Suppose that $v(0) \leq 2^{-n}([[2^n w]] + 2)$, and

$$(3.3)\qquad \int_{t_j}^{t_{j+1}} e(t,2^{-n}j)dt < \int_{t_j}^{t_{j+1}} [f(t,2^{-n}(j+1)) + 2^{-n}]dt$$

for $j = i,i+1,i+2,\dots,k-1$, where k is some integer greater than i.

Then $T > t_k$; $v(t_j) \leq 2^{-n}(j-1)$ for $j = i,i+1,\dots,k$; and $v(t) < x_n(t)$ for all t in $[0,t_k]$.

If (3.3) holds for all integers $j \geq i$ (in particular, if $e(t,y) < f(t,y) + 2^{-n}$ for all $t \geq 0$ and $y \geq 0$), then $T \geq T_n$, and $v(t) < x_n(t)$ for all t in $[0,T_n)$.

Proof.

First suppose that (3.2) does have at least one nonextendable solution $x_n : [0,T_n) \to R_+$. Since $x'_n(t) \geq 2^{-n}$, the numbers t_j satisfying $x_n(t_j) = 2^{-n}j$ are uniquely determined by this solution x_n; and t_j increases to T_n as $j \to \infty$. For $t_{j-1} \leq t < t_j$ we have $h_n(x_n(t)) = 2^{-n}j$, hence $x'_n(t) = f(t,2^{-n}j) + 2^{-n}$. Therefore

$$(3.4)\qquad x_n(t) = 2^{-n}(j-1) + \int_{t_{j-1}}^{t} [f(s,2^{-n}j) + 2^{-n}]ds \qquad (t_{j-1} \leq t \leq t_j)\ .$$

Since $x_n(t_j) - x_n(t_{j-1}) = 2^{-n}j - 2^{-n}(j-1) = 2^{-n}$, we must have

$$2^{-n} = \int_{t_{j-1}}^{t_j} [f(s,2^{-n}j) + 2^{-n}]ds \qquad (j = i, i+1, i+2, \ldots) \tag{3.5}$$

if a nonextendable solution x_n exists.

Formula (3.5) recursively determines t_j uniquely from t_{j-1}. Then (3.4) uniquely determines $x_n(t)$. We easily verify that the function $x_n(t)$ constructed in this fashion is a nonextendable solution of (3.2). This completes the proof of the first part of the lemma.

Now suppose v, e, and k satisfy the hypotheses stated in the lemma. As an induction hypothesis, assume that $t_j < T_v$ and $v(t_j) \leq 2^{-n}(j - 1)$, for some $j < k$. (This is clear for $j = i$, since $t_i = 0$.) Then

$$\int_{t_j}^{t_{j+1}} e(s,v(t_j) + 2^{-n})ds \leq \int_{t_j}^{t_{j+1}} e(s,2^{-n}j)ds$$
$$< \int_{t_j}^{t_{j+1}} [f(s,2^{-n}(j+1)) + 2^{-n}]ds = 2^{-n} ,$$

by (3.3) and (3.5). Hence, by part (iii) of Lemma 1, $T > t_{j+1}$, and

$$v(t) < v(t_j) + 2^{-n} \leq 2^{-n}j = x_n(t_j) \leq x_n(t)$$

for all t in $[t_j,t_{j+1}]$. This completes the induction, and the proof of Lemma 2.

<u>Lemma 3.</u>

Let f satisfy (1.2), and assume $w \geq 0$. Then there exists a maximal solution $x_\infty : [0,T_\infty) \to \mathbf{R}_+$ of (1.1). Moreover:

For each positive integer n, define $x_n : [0,T_n) \to \mathbf{R}_+$ as in Lemma 2. Then the numbers T_n increase to T_∞, and $x_n(t) \downarrow x_\infty(t)$ for each t in $[0,T_\infty)$, uniformly on compact subsets of $[0,T_\infty)$.

<u>Proof.</u>

We easily verify that the hypotheses of Lemma 2 are satisfied by $e(t,y) = f(t,h_{n+1}(y)) + 2^{-n-1}$ and $v(t) = x_{n+1}(t)$. By Lemma 2, then, $T_{n+1} \geq T_n$ and $0 \leq x_{n+1}(t) \leq x_n(t)$ for all t in $[0,T_n)$. Hence the numbers T_n increase to some limit T_∞ (possibly ∞), and $x_n(t)$ decreases to a limit $x_\infty(t)$ for every t in $[0,T_\infty)$, uniformly on compact subsets of $[0,T_\infty)$.

Since h_n is an increasing function and $y < h_{n+1}(y) \leq h_n(y) \leq y + 2^{-n}$, it follows easily that $h_n(x_n(t))$ decreases to $x_\infty(t)$. For almost every s in $[0,T_\infty)$, $f(s,\cdot)$ is increasing and right-continuous, hence $f(s,h_n(x_n(s)))$ decreases to $f(s,x_\infty(s))$. Take limits in the equation

$$x_n(t) - x_n(r) = \int_r^t [f(s,h_n(x_n(s))) + 2^{-n}]ds .$$

By the Lebesgue Monotone Convergence Theorem, we obtain

$$x_\infty(t) - x_\infty(r) = \int_r^t f(s,x_\infty(s))ds .$$

Therefore $x_\infty : [0,T_\infty) \to \mathbf{R}_+$ is a solution of (1.1).

Suppose T_∞ is not final for x_∞. Then $T_\infty < \infty$, and $x_\infty(t)$ increases to some finite limit M when $t \uparrow T_\infty$. Fix some $\varepsilon > 0$ small enough so that

$$\int_{T_\infty-\varepsilon}^{T_\infty} f(s,M + 2)ds < 1 .$$

Since $f(s,h_n(M + 2)) + 2^{-n}$ decreases to $f(s,M + 2)$, for all n sufficiently large we have

$$\int_{T_\infty-\varepsilon}^{T_\infty} [f(s,h_n(M + 2)) + 2^{-n}]ds < 1 .$$

Since $T_n \uparrow T_\infty$ and $x_n \downarrow x_\infty$, for all n sufficiently large we have $T_\infty - \varepsilon < T_n \le T_\infty$ and $x_n(T_\infty - \varepsilon) < x_\infty(T_\infty - \varepsilon) + 1 \le M + 1$. Then

$$\int_{T_\infty-\varepsilon}^{T_\infty} \{f(s,h_n[x_n(T_\infty - \varepsilon) + 1]) + 2^{-n}\}ds < 1 .$$

Apply part (iii) of Lemma 1 with $v(s) = x_n(s)$ and $e(s,y) = f(s,h_n(y)) + 2^{-n}$. We obtain $T_\infty < T_n$, a contradiction. So T_∞ is final for x_∞, i.e. x_∞ is nonextendable.

To show the solution x_∞ is maximal, let $v : [0,T) \to R_+$ be any other nonextendable solution of (1.1). It follows easily from Lemma 2 that $T \ge T_n$ and $v(t) \le x_n(t)$ for all t in $[0,T_n)$. Taking limits, we find that $T \ge T_\infty$ and $v(t) \le x_\infty(t)$ for all t in $[0,T_\infty)$. This completes the proof of Lemma 3.

4. Proofs of theorems

Proof of Theorem I.

Most of Theorem I was proved in Lemma 3. It suffices to show every solution $x : [0,T) \to \mathbf{R}_+$ of (1.1) can be continued to a nonextendable solution. Suppose T is not final for x. Then T is finite, and $x(t)$ increases to some finite limit L when $t \uparrow T$. By Lemma 3, there exists a maximal, hence nonextendable, solution of

$$x'(t) = f(t,x(t)) \qquad (T \le t < T_x) ,$$
$$x(T) = L .$$

This completes the proof of Theorem I.

Proof of Theorem II.

(2.3) implies (2.4):

Trivial.

(2.4) implies (2.3):

Fix some q in $[0,T)$ and some $\varepsilon > 0$. It suffices to show that

$$v(t) - v(q) \le (1 + \varepsilon) \int_q^t f(s,v(s))ds \tag{4.1}$$

for all t in $[q,T)$ (for then let $\varepsilon \downarrow 0$). Let $S = \{t \in [q,T): \text{(4.1) holds}\}$. Then $q \in S$.

Fix any $r \in S$. Then $v(t) - v(r) \le (1 + \varepsilon) \int_r^t f(s,v(s))ds$ for all t greater than r and sufficiently close to r, by (2.1). For any such t,

$$\begin{aligned} v(t) - v(q) &= [v(t) - v(r)] + [v(r) - v(q)] \\ &\le (1+\varepsilon) \int_r^t f(s,v(s))ds + (1+\varepsilon) \int_q^r f(s,v(s))ds = (1+\varepsilon) \int_q^t f(s,v(s))ds \end{aligned}$$

and so $t \in S$. Thus S is open on the right in $[q,T)$. On the other hand, since $(1+\varepsilon) \int_q^t f(s,v(s))ds$ is a continuous function of t and $v(t) \le \liminf_{r \uparrow t} v(r)$, S is closed on the right in $[q,T)$. Therefore $S = [q,T)$. This completes the proof of (2.3).

(2.3) implies (2.7):

Immediate from part (iv) of Lemma 1.

(2.3) implies (2.5):

By a translation of p, we may assume without loss of generality that $p = 0$. The function $x_0 : [0,T_0) \to \mathbf{R}_+$ of Theorem II is the same as the function $x_\infty : [0,T_\infty) \to \mathbf{R}_+$ of Lemma 3, with $w = v(0)$. By Lemma 2, $T_n \leq T$ and $x_n \geq v$. Hence, by Lemma 3, $T_\infty \leq T$ and $x_\infty \geq v$.

(2.5) implies (2.6):

Fix t in $[0,T)$. By hypothesis, $M \equiv \sup\{v(s) : 0 \leq s \leq t\}$ is finite. Since $f(\cdot,M + 1)$ is integrable on $[0,t]$, there is some $\mu > 0$ such that

$$\int_a^b f(s,M + 1)ds < 1 \quad \text{if} \quad 0 \leq a \leq b \leq t \quad \text{and} \quad b - a \leq \mu .$$

It follows by Lemma 1, part (iii), that

$$(4.2) \qquad \begin{cases} \text{if } 0 \leq a \leq b \leq t,\ b - a \leq \mu, \text{ then} \\ T_a > b \text{ and } x_a(r) \leq v(a) + \displaystyle\int_a^r f(s,v(a) + 1)ds \\ \text{for all } r \text{ in } [a,b], \text{ hence in particular} \\ v(b) - v(a) \leq x_a(b) - v(a) \leq \displaystyle\int_a^b f(s,M + 1)ds . \end{cases}$$

Let any partition

$$(4.3) \qquad 0 = t_0 < t_1 < t_2 < \dots < t_m = t$$

of $[0,t]$ be given. Choose a refinement

$$0 = s_0 < s_1 < s_2 < \dots < s_n = t$$

such that $\max_i (s_i - s_{i-1}) < \mu$. We have

$$\begin{aligned} v(s_n) - v(s_0) &= \sum_{i=1}^n [v(s_i) - v(s_{i-1})] \\ &= \sum_{i=1}^n [v(s_i) - v(s_{i-1})]^+ - \sum_{i=1}^n [v(s_i) - v(s_{i-1})]^- , \end{aligned}$$

where $[w]^+ = \max\{w,0\}$, $[w]^- = \max\{-w,0\}$. Hence

$$\sum_{j=1}^{m} |v(t_j) - v(t_{j-1})| \le \sum_{i=1}^{n} |v(s_i) - v(s_{i-1})|$$

$$= \sum_{i=1}^{n} [v(s_i) - v(s_{i-1})]^+ + \sum_{i=1}^{n} [v(s_i) - v(s_{i-1})]^-$$

$$= 2 \sum_{i=1}^{n} [v(s_i) - v(s_{i-1})]^+ + v(s_0) - v(s_n)$$

$$\le 2 \sum_{i=1}^{n} \int_{s_{i-1}}^{s_i} f(s,M+1)ds + M + 0 = 2 \int_0^t f(s,M+1)ds + M .$$

The right side is independent of the choice of partition (4.3). This proves (2.6).

<u>(2.5) and (2.6) together imply (2.3):</u>

Fix t and r, $0 \le r < t < T$. By hypothesis, $M = \sup\{v(s) : 0 \le s \le t\}$ is finite. Choose $\mu > 0$ to satisfy (4.2).

By (2.6), v has at most countably many discontinuities in $[0,t]$, each discontinuity is a jump, and the magnitudes of the jumps are summable. Hence for each positive integer n, the set

$$A_n = \{s \in (r,t) : |v(s+) - v(s)| > 2^{-n} \text{ or } |v(s-) - v(s)| > 2^{-n}\}$$

is finite. The sets A_n form an increasing sequence, and v is continuous at every $s \in (r,t)\setminus\cup_{n=1}^{\infty} A_n$.

The sets

$$B_n = \{r + 2^{-n}k(t - r): \ k = 0,1,2,\ldots,2^n\}$$

also form an increasing sequence of finite sets; hence so do the sets $C_n = A_n \cup B_n$.

Temporarily fix any integer $n > \log_2((t - r)/\mu)$. Suppose C_n consists of the points

(4.4) $$C_n : r = s_0 < s_1 < s_2 < \ldots < s_m = t .$$

Then $s_i - s_{i-1} \le 2^{-n}(t-r) < \mu$, so

$$T_{s_{i-1}} > s_i$$

by (4.2). Therefore we can define a function $w_n : [r,t] \to \mathbf{R}_+$ by taking

$$w_n(s_i) = v(s_i) \quad \text{for } 0 \le i \le m ,$$

$$w_n(s) = x_{s_{i-1}}(s) \quad \text{for } s_{i-1} \le s < s_i, \quad 1 \le i \le m .$$

Hence

$$v(s) \leq w_n(s) \leq M + 1 \quad \text{for all } s \text{ in } [r,t], \text{ and}$$

$$(4.5) \quad \begin{cases} w_n(s) = v(s_{i-1}) + \int_{s_{i-1}}^{s} f(q, x_{s_{i-1}}(q))dq \leq v(s_{i-1}) + \int_{s_{i-1}}^{s} f(q, M+1)dq \\ \text{for all } s \text{ in } [s_{i-1}, s_i), \text{ and} \end{cases}$$

$$(4.6) \quad \begin{cases} v(t) - v(r) = \sum_{i=1}^{m} [v(s_i) - v(s_{i-1})] \leq \sum_{i=1}^{m} [x_{s_{i-1}}(s_i) - v(s_{i-1})] \\ = \sum_{i=1}^{m} \int_{s_{i-1}}^{s_i} f(s, x_{s_{i-1}}(s))ds = \int_r^t f(s, w_n(s))ds . \end{cases}$$

Now suppose $n > \log_2((t - r)/\mu) + 1$. For any s_{i-1} and s_i in C_n, both w_n and w_{n-1} are defined on $[s_{i-1}, s_i)$ as maximal solutions of $w'(s) = f(s,w(s))$, with initial values $w_n(s_{i-1})$ and $w_{n-1}(s_{i-1})$, respectively. But $w_n(s_{i-1}) = v(s_{i-1}) \leq w_{n-1}(s_{i-1})$ since $s_{i-1} \in C_n$. Hence $w_n \leq w_{n-1}$ on $[s_{i-1}, s_i)$. This holds for $1 \leq i \leq m$; so $w_n \leq w_{n-1}$ on $[r,t]$. Thus

$$(4.7) \quad w_n \geq w_{n+1} \geq w_{n+2} \geq \cdots \geq v \quad \text{on } [r,t] ,$$

and $w_n(s) = v(s)$ for all s in C_n.

We wish to show $w_n(s)$ decreases to $v(s)$ for every s in $[r,t]$. This is clear for every s in $\cup_{n=1}^{\infty} C_n$. Fix any s in $[r,t]\backslash\cup_{n=1}^{\infty} C_n$. Then $v(.)$ is continuous at s. Temporarily fix some large n, and let C_n be as in (4.4). Choose i so that $s_{i-1} < s < s_i$. We shall apply inequality (4.5). As $n \to \infty$, $s_{i-1} \to s$, hence $v(s_{i-1}) \to v(s)$ and $\int_{s_{i-1}}^{s} f(q, M+1)dq \to 0$. Taking limits in (4.5), we obtain $\limsup_{n\to\infty} w_n(s) \leq v(s)$. In view of (4.7), then, $w_n(s)$ decreases to $v(s)$.

For almost every s in $[r,t]$, $f(s,.)$ is increasing and right-continuous, so $f(s,w_n(s))$ decreases to $f(s,v(s))$. Also $f(s,w_n(s)) \leq f(s,M+1)$, which is an integrable function of s. By Lebesgue's Dominated Convergence Theorem, from (4.6) we obtain

$$v(t) - v(r) \leq \int_r^t f(s,v(s))ds \; .$$

This proves (2.3), and completes the proof of Theorem II.

Proof of Theorem III.

(2.11) implies (2.10):

Trivial.

(2.9) implies (2.11):

Without loss of generality let $p = 0$. Let $f = f_\infty$, $w = w_\infty$. Fix any positive integer n, and define x_n, i, and $\{t_j\}$ as in Lemma 2. Fix any integer $k > i$. Then for all $a \in A$ sufficiently large, $w_a \leq 2^{-n}([[2^n w_\infty]] + 2)$ (since $\limsup w_a \leq w_\infty$) and

$$\int_{t_j}^{t_{j+1}} f_a(t,2^{-n}j)dt < \int_{t_j}^{t_{j+1}} [f_\infty(t,2^{-n}(j + 1)) + 2^{-n}]dt$$

$(j = i, i + 1, i + 2, \ldots, k - 1)$, by (2.9). Hence, by Lemma 2, $T_a > t_k$, and $x_a \leq x_n$ on $[0,t_k]$. Therefore $\liminf T_a \geq t_k$, and $\limsup x_a(t) \leq x_n(t)$ for all t in $[0,t_k]$. Let $k \to \infty$ and then let $n \to \infty$; this proves $\liminf T_a \geq T_\infty$ and $\limsup x_a(t) \leq x_\infty(t)$ for all t in $[0,T_\infty)$.

(2.10) implies (2.9):

Suppose $\limsup \int_r^t f_a(s,y)ds > \int_r^t f_\infty(s,y)ds$ for some $y \geq 0$ and $t > r \geq 0$. Since $\int_r^t f_\infty(s,.)ds$ is right-continuous, we have in fact

$$\limsup \int_r^t f_a(s,y)ds > \int_r^t f_\infty(s,y + \varepsilon)ds$$

for some $\varepsilon > 0$. Partition the interval $[r,t]$ into n pieces $[r',t']$ of length $(t - r)/n$. For n large enough, all of the pieces must satisfy

$$\int_{r'}^{t'} f_\infty(s,y + \varepsilon)ds < \varepsilon \; . \tag{4.8}$$

On the other hand, at least one of the pieces must satisfy

$$\text{(4.9)} \qquad \limsup \int_{r'}^{t'} f_a(s,y)ds > \int_{r'}^{t'} f_\infty(s,y+\varepsilon)ds \ .$$

Fix this choice of r' and t'. Let $p = r'$ and $w_a = w_\infty = y$, and define maximal solutions x_a and x_∞ and final times T_a and T_∞ as in (2.8). By hypothesis (2.10) we have $\liminf T_a \geq T_\infty$ and

$$\text{(4.10)} \qquad \limsup x_a(s) \leq x_\infty(s) \quad \text{for every } s \text{ in } [r',T_\infty) \ .$$

Since $x_\infty(r') = y$, we can use (4.8) and part (iii) of Lemma 1 to show that $T_\infty > t'$ and that

$$\text{(4.11)} \qquad x_\infty(t') \leq y + \int_{r'}^{t'} f_\infty(s,y+\varepsilon)ds \ .$$

Then for all $a \in A$ sufficiently large we have $T_a > t'$ and

$$\text{(4.12)} \qquad \begin{cases} x_a(t') = x_a(r') + \int_{r'}^{t'} f_a(s,x_a(s))ds \\ \geq x_a(r') + \int_{r'}^{t'} f_a(s,x_a(r'))ds = y + \int_{r'}^{t'} f_a(s,y)ds \ . \end{cases}$$

Combine (4.10) (with $s = t'$) and (4.9), (4.11), (4.12); this gives us a contradiction. So (2.10) implies (2.9). This completes the proof of Theorem III.

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Existence of a maximal solution is proved for a differential equation satisfying a one-sided variant of Carathéodory's condition. The maximal solution is shown to dominate all solutions of a very general differential inequality. Also a best-possible condition is proved for the dependence of the maximal solution on the initial data and on the right-hand side of the equation.

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