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OPTIMAL WHEREABOUTS SEARCH FOR A MOVING TARGET

L.D. Stone and Joseph B. Kadane

Suppose we wish to find an object, the target, which is moving according to known probability laws. The search is assumed to last until time T where time is discrete (t = 0, ..., T) or continuous ($0 \leq t \leq T$).

Such a search may have a number of purposes. A common purpose is to maximize the probability of detection by time T. In this case the search is called a detection search. A second purpose is to localize the target to within one cell out of a finite number of cells defined by a grid system of the user's choice. In this case the searcher may succeed by detecting the target with the search up to time T, or failing that, he may guess one cell for the target's location at time T. If the guess is correct then the searcher also succeeds. This is called a where-abouts search. A third type is surveillance search in which one seeks to maximize the probability of correctly stating which cell contains the target at time T. The difference between a surveillance search and a whereabouts search is that a detection before time T in a surveillance search does not end the search. It merely helps to locate the target at time T.

This paper deals with the whereabouts search and shows that the optimal whereabouts search plan, i.e., an allocation of search effort and a choice of cell to guess, may be found by solving a finite number of optimal detection problems for a moving target, one for each cell in the grid. Having shown this we discuss how to use the optimal detection search algorithms of Brown [2], Stone et al [16], and the bounds given by Washburn [18] to compute optimal whereabouts search plans. In the case of a stationary target, as studied by Mela [6], Tognetti [17], and Kadane [5], optimal whereabouts search and optimal surveillance search coincide

-1-

because, once the target is detected, its location is known with certainty from then on and in particular at time T. In addition to the results mentioned above, optimal detection search for a moving target has been studied by Stone [13] and [14], Stone and Richardson [15], Persiheimo [7], Saretsalo [10], Hellman [4], Dobbie [3], Pollock [8], and Stewart [11].

In the case where false targets are present very few results have been obtained even when the target is stationary; see Chapter VI of Stone [12], Richardson [9], and Barker and Belkin [1]. In this paper, we do not consider the problem of false targets.

Whereabouts search might be used in a search and rescue situation in which it is known that those lost can survive no longer than T units of time in the environment. If the searcher finds the lost party in some search of a cell, the rescue can be effected immediately. If after time T the party has not been found, the rescue effort is assigned to some cell and will be successful if the lost party is in fact in that cell. Similarly it might be used in a military context in which the object is destroyed as soon as it is found, or a weapon is fired into the cell guessed after T if the target has not been previously found. By contrast, if it is desired to know where the object is at time T, without taking any action if it is found before T, a surveillance search problem results.

1. CHARACTERIZATION OF OPTIMAL WHEREABOUTS SEARCH IN TERMS OF OPTIMAL DETECTION SEARCH

Kadane [5] shows that solving the optimal whereabouts search for a stationary target may be reduced to solving J optimal detection problems for a stationary target where J is the number of cells in the grid of the target's probability distribution. In this section we generalize that result to moving targets.

Our results apply to targets moving in discrete or continuous space or time. Let $\{X_t, 0 \leq t \leq T\}$ be the stochastic process representing the target's motion. For discrete time, we shall understand $0 \leq t \leq T$ to mean $t = 0, 1, \ldots, T$ in order to state results simultaneously for continuous and discrete time. The search space may be continuous or discrete, but the searcher must specify a <u>whereabouts grid</u> of cells for time T. The size of these grid cells must correspond to the degree of localization required for the objective of the search. For example, the cells may be of the size of the lethal area of the weapon to be fired at time T. The whereabouts grid has no connection with the target motion which may be in continuous space or through a grid of cells entirely different from the whereabouts grid. Let the whereabouts grid have I cells numbered $i = 1, 2, \ldots, I$.

Let Ψ be the class of allowable detection search plans. A <u>whereabouts search plan</u> is a search plan $\psi \in \Psi$ with which to try to detect the target and a whereabouts cell i in the whereabouts grid which is guessed to contain the target if the detection search fails. Let $S_T[\psi, i]$ be the probability of success using the whereabouts search plan (ψ, i). We seek an <u>optimal whereabouts</u>

plan (ψ^*, i^*), i.e., a plan such that $\psi^* \in \Psi$, $1 \le i^* \le I$, and

$$S_T[\psi^*, i^*] = \max\{ S_T[\psi, i] : \psi \in \Psi \text{ and } 1 \le i \le I\} .$$

Let $C(i)$ denote the ith cell in the whereabouts grid and let $X^i = \{ X_t, \ 0 \le t \le T \ | \ X_T \notin C(i)\}$ denote the target process obtained by conditioning on the target not being in $C(i)$ at time T. Let $P_T^i[\psi]$ be the probability of detecting the target with plan ψ given that the target is not in cell i at time T. Then

$$S_T[\psi, i] = P_T^i[\psi]\left(1 - \Pr\{ X_T \in C(i)\}\right) + \Pr\{ X_T \in C(i)\} . \tag{1}$$

Given that we choose to guess cell i at time T, it is clear that we should choose the detection search plan ψ^i to use with i so that ψ^i maximizes P_T^i. Since I is finite it is also clear that we can find the optimal whereabouts plan by performing I optimizations to determine (ψ^{i^*}, i^*), such that

$$S_T[\psi^{i^*}, i^*] = \max_{1 \le i \le I} S_T[\psi^i, i].$$

Since finding ψ^i is simply finding an optimal detection plan for a target moving according to the stochastic process X^i, we have shown that the optimal whereabouts problem reduces to solving I moving target detection problems. We state this observation as a theorem.

THEOREM. <u>Let $\psi^i \in \Psi$ be an optimal detection plan for the target motion process</u> X^i, <u>i.e.</u>,

$$P_T^i[\psi^i] = \max\{ P_T^i[\psi] : \psi \in \Psi \} .$$

<u>Then an optimal whereabouts search plan is</u> $(\psi^{i*}, i*)$ <u>where</u>

$$S_T[\psi^{i*}, i*] = \max_{1 \leq i \leq I} S_T[\psi^i, i].$$

Observe that there are no restrictions on the class Ψ of detection plans from which we are allowed to choose. For example, the above result applies to whereabouts problems in which Ψ is a class of search paths or a collection of functions which specify the allocation of search effort in space and time.

Clearly the above result may be extended to whereabouts problems in which the searcher is allowed to choose $n \geq 1$ whereabouts cells for the target's location if the detection search fails.

2. ALGORITHMS FOR COMPUTING OPTIMAL WHEREABOUTS PLANS

Using the result of the theorem we now show how one may modify the algorithm of Brown [2] or the one of Stone, et. al. [16] (both of which are designed to compute optimal detection plans) to compute optimal whereabouts plans.

These algorithms are designed for discrete time searches and assume that there is a grid of J cells in the plane over which search is allocated. This is the search grid. A search plan ψ is a non-negative function of space and time such that

$$\psi(j,t) = \text{effort placed in cell j at time t} \quad \text{for } j = 1,\ldots,J, \ t = 1,\ldots,T.$$

We are restricted to the class $\Psi(m)$ of plans ψ such that

$$\sum_{j=1}^{J} \psi(j,t) = m(t) \quad \text{for } t = 0,\ldots,T.$$

Effort cannot be transformed from one time period to another.

The detection function is exponential with sweep width which may vary over space. Let W(j) and A(j) be the sweep width in an area of the jth cell for $j = 1,\ldots,J$. Then we assume

$$P_T[\psi] = E\left[1 - \exp\left(-\sum_{s=0}^{T} W(X_s)\,\psi(X_s,s)/A(X_s)\right)\right] \tag{2}$$

where E indicates expectation over the sample paths of the process and X_s is the cell that the process is in at time s.

When target motion is modeled by a mixture of discrete time and space Markov processes and the detection function is exponential, the algorithm of Brown [2] provides an extremely efficient method of computing optimal detection plans. If our target motion process X falls into this category, then we may modify

Brown's algorithm to find $\Pr\{X_T \in C(i)\}$, ψ^i, and $P_T^i[\psi^i]$ as follows.

Let $c(j)$ for $j = 1, \ldots, J$ indicate the cells in the search grid. The target moves among the cells in the search grid and each cell $C(i)$, $i = 1, \ldots, I$ in the whereabouts grid is composed of an integral number of cells from the search grid. Let ψ^0 be the search plan which assigns zero effort everywhere. Using Brown's notation we first choose a whereabouts cell $C(i)$ and compute

$$\Pr\{X_T \in C(i)\} = \sum_{\{j\,:\,c(j)\,\in\,C(i)\}} \text{reach}(c(j), T, \psi^0).$$

Here we are simply using the initial distribution at time 1 and iterating the transition matrix of the Markov process (or processes) to compute $\Pr\{X_T \in C(i)\}$.

We modify the initialization of the survive matrix by setting

$$\text{survive}(c(j), T, \psi) = \begin{cases} 0 & \text{if } c(j) \in C(i) \\[2mm] 1 & \text{if } c(j) \notin C(i) \end{cases} \quad \text{for } j = 1, \ldots, J,$$

for any search plan ψ. The result of modifying the survive matrix in the above manner is to remove the paths which end in whereabouts cell $C(i)$ at time T. Once can show that the resulting process X^i is still Markovian if X is Markovian or a mixture of Markov processes if X is a mixture of Markov processes. In either case Brown's [2] algorithm is applicable to X^i and is used to compute an optimal plan and probability of detection for that plan. The resulting plan will

be ψ^i and the resulting probability of detection will be $P_T^i[\psi^i]$ $(1 - Pr\{X_T \epsilon C(i)\})$.

By doing this for each whereabouts cell $C(i)$, $i = 1, \ldots, I$, and by choosing the

pair $(\psi^{i*}, i*)$ with the highest probability of success as calculated by equation (1),

we have found the optimal whereabouts plan.

Rather than perform the two step procedure of calculating $Pr\{X_T \epsilon C(i)\}$ and

$P_T^i[\psi^i](1 - Pr\{X_T \epsilon C(i)\})$, it is more efficient to minimize failure to detect using

the defective process obtained from the above initialization of the survive matrix.

The resulting probability will be

$$F^i = (1 - P_T^i[\psi^i])(1 - Pr\{X_T \epsilon C(i)\})$$

and

$$1 - F^i = Pr\{X_T \epsilon C(i)\} + P_T^i[\psi^i](1 - Pr\{X_T \epsilon C(i)\})$$
$$= S_T[\psi^i, i].$$

Actually Brown's algorithm is an iterative one which converges to the optimal

plan in an infinite number of steps. However, after each iteration in the algorithm

one obtains a lower bound (the probability of detection for the plan obtained from that

iteration) and an upper bound by the method of Washburn [18] (see Section 4). These in turn

may be converted to upper $\bar{S}_T[\psi^i, i]$ and lower $\underline{S}_T[\psi^i, i]$ bounds for $S_T[\psi^i, i]$ by using equation (1).

With these bounds one can save computing time by using a branch and bound technique.

That is, one specifies a tolerance ϵ for how close to the success probability of

the optimal plan he wishes to come. Choose an initial whereabouts cell i' and

iterate until $\bar{S}_T[\psi^{i'}, i'] - \underline{S}[\psi^{i'}, i'] < \epsilon$. For each succeeding choice of a whereabouts cell

C(i) stop the iteration to determine ψ^i if either $\bar{S}_T[\psi^i, i] \leq \underline{S}_T[\psi^{i'}, i']$ or $\underline{S}_T[\psi^i, i] \geq \bar{S}_T[\psi^{i'}, i']$.

If the former occurs, retain $(\psi^{i'}, i')$ as the candidate optimal plan. If the latter occurs, then replace i' with i and continue iterating until the tolerance ϵ is met for the new candidate i'. If one cannot decide between two possible solutions on the basis given above (i.e., the tolerance interval for one plan overlaps the other), then one can choose between them in an arbitrary fashion. For example, one could choose the one with the higher lower bound. In this way one will be guaranteed to obtain a solution within the desired tolerance.

If the target motion is more general than a mixture of discrete space and time Markov processes, then one can modify the algorithm in Chapter IV of Stone, et. al. [16] as follows.

Using the notation of Stone, et. al., [16] we sort the sample paths by which cell in the whereabouts grid contains the target at time T. Choose a whereabouts cell C(i). To obtain $\Pr\{X_T \in C(i)\}$, we sum the probabilities of the paths ending in C(i). Removing these paths from the file of sample paths, we solve for ψ^i, the optimal detection search plan, using the remaining paths. The probability of detection resulting from this procedure will be $P_T^i[\psi^i](1 - \Pr\{X_T \in C(i)\})$. (Again it is more efficient to minimize the failure probability on the defective process obtained by removing the paths ending in cell C(i). One minus this probability is $S_T[\psi^i, i]$.) We then proceed using the above branch and bound technique to find the optimal whereabouts plan $(\psi^{i*}, i*)$ to the desired tolerance.

3. EXAMPLES OF OPTIMAL WHEREABOUTS PLAN

We give three examples of optimal whereabouts search plans. The first example shows that even when the detection function is exponential and the sweep width the same in all cells, it is not necessarily optimal to choose the cell with the highest prior probability at time T for the whereabouts cell. The second example shows a situation in which the optimal whereabouts cell is the one with the highest probability at time T when no search is applied at any time period. The third illustrates how the whereabouts search procedure balances detection capability against target location probability in choosing the whereabouts cell.

For the three examples presented in this section the whereabouts grid coincides with the search grid. The probability of detection using plan ψ is given by equation (2).

Example 1: For stationary whereabouts searches, Kadane [5] shows that when the cost and detection function are the same over all the cells in the whereabouts grid, then it is optimal to choose the cell with the highest prior probability for the whereabouts cell. For whereabouts searches involving moving targets, one might conjecture that this result would generalize by choosing as the

whereabouts cell the one with the highest probability at time T when no search is applied. We now show this conjecture to be false.

Let the search and whereabouts grid consist of two cells. Time is discrete and there are two time periods for search, $t = 0, 1$. There are three possible target paths. The paths and their associated probabilities are shown in Table 1. We use the notation $\omega_1 = (1, 1)$ to mean that path 1 is in cell 1 at time 0 and remains in cell 1 at time 1. Table 1 also shows the target distributions if no search is conducted. We see that cell 2 has the highest probability at time 1. There are two units of search effort available at each time period, and $W(j) = A(j) = 1$ for $j = 1, 2$.

TABLE 1

a. TARGET PATHS AND PROBABILITIES

Note: $\epsilon < 1/4$.

Path	Probability
$\omega_1 = (1, 1)$	ϵ
$\omega_2 = (2, 1)$	$\frac{1}{2} - 2\epsilon$
$\omega_3 = (2, 2)$	$\frac{1}{2} + \epsilon$

b. TARGET DISTRIBUTIONS

Target Distribution at Time 0		Target Distribution at Time 1	
Cell	Probability	Cell	Probability
1.	ϵ	1.	$\frac{1}{2} - \epsilon$
2.	$1 - \epsilon$	2.	$\frac{1}{2} + \epsilon$

Let F^j be the failure probability for the whereabouts plan which chooses cell j for the whereabouts cell and allocates its detection search optimally for the process X^j which has the sample paths ending in cell j removed. Then $F^j = 1 - S_T[\psi^j, j]$.

Suppose we choose 1 for the whereabouts cell. Since ω_3 is the only path not ending in cell 1, the plan ψ^1 puts 2 units of effort in cell 2 at time 0 and time 1. Thus,

$$F^1 = (\tfrac{1}{2} + \epsilon) \, e^{-4}.$$

If we choose 2 for the whereabouts cell, then ψ^2 places all the effort for time t = 1 in cell 1. By Brown [2] or Theorem 2 of Stone [14], we find the optimal allocation at time 0 by computing the posterior distribution at time 0 (using only ω_1 and ω_2), given failure to detect at time 1, and allocating the effort for time 0 to be optimal for this posterior distribution. If ϵ is small enough, then ψ^2 will place all effort in cell 2 at time 0 and

$$F^2 = (\tfrac{1}{2} - 2\epsilon) \, e^{-4} + \epsilon \, e^{-2}.$$

Since $F^2 - F^1 = \epsilon \, e^{-2}(1 - 3e^{-2}) > 0$, it follows that choosing cell 1 yields the lower failure probability (higher success probability) and that the optimal whereabouts plan chooses cell 1 for the whereabouts cell rather than the higher probability cell 2.

Observe that no matter how small ϵ is, cell 2 has the larger probability of containing the target at time 1 given no search. By making ϵ small enough the searcher is forced to place all his effort in cell 2 at time 0 regardless of which whereabouts cell he chooses. However, searching in cell 2 at time 0 causes cell 1 to be the high probability cell at time 1 given failure to detect at time 0. Thus cell 1 rather than 2 becomes the optimal whereabouts cell.

Example 2: Uniform Sweep Width. This example shows a situation in which

the cell having the highest probability of containing the target at time T, if no

search takes place, is the optimal one to choose for the whereabouts cell.

The target distribution at time 0 is bivariate normal with center at $30^{\circ} 10' N$

$30^{\circ} 10' W$. The major axis is oriented east-west, and the standard deviation along

the major and minor axes is 50 and 30 nautical miles, respectively. At time $t = 3$,

the target's distribution is circular normal with center at $28^{\circ} 10' N$, $30^{\circ} 10' W$

and standard deviation 10 nautical miles along any axis. The target paths are

obtained by making an independent draw for the target's position at time $t = 0$ and

$t = 3$ from the above distributions. The target then follows a constant course

and speed between these points. Four thousand sample paths were drawn to

represent the target motion process.

Figure 1 illustrates the target motion assumptions and shows some typical

target paths. Both the search and whereabouts grid consists of cells which

are 20' by 20' as indicated in the figure. The sweep width $W(j) = 1$ for all cells j,

and there are 1500 units of search effort available at each time period. In sub-

sequent figures we will show only the region of the grid in which the target location

probabilities are positive.

Figure 2 shows the probability distribution in the whereabouts grid at time 3 with

no search having taken place at any time period. The algorithm used to find the

optimal whereabouts plan is a modification of the one given in chapter IV of Stone et al

[16]. It proceeded by choosing the highest probability cell in this grid and computing

the optimal plan given this choice to within a tolerance of .001. This tolerance

was reached at 3 iterations. An iteration consists of one complete cycle through

FIGURE 1

TARGET MOTION ASSUMPTIONS FOR
EXAMPLES 2 AND 3

Target motion assumptions figure showing probability ellipses at t = 0 and t = 3, typical target paths, and a 20' by 20' grid cell on a latitude/longitude grid (27N–32N, 27W–34W).

FIGURE 2

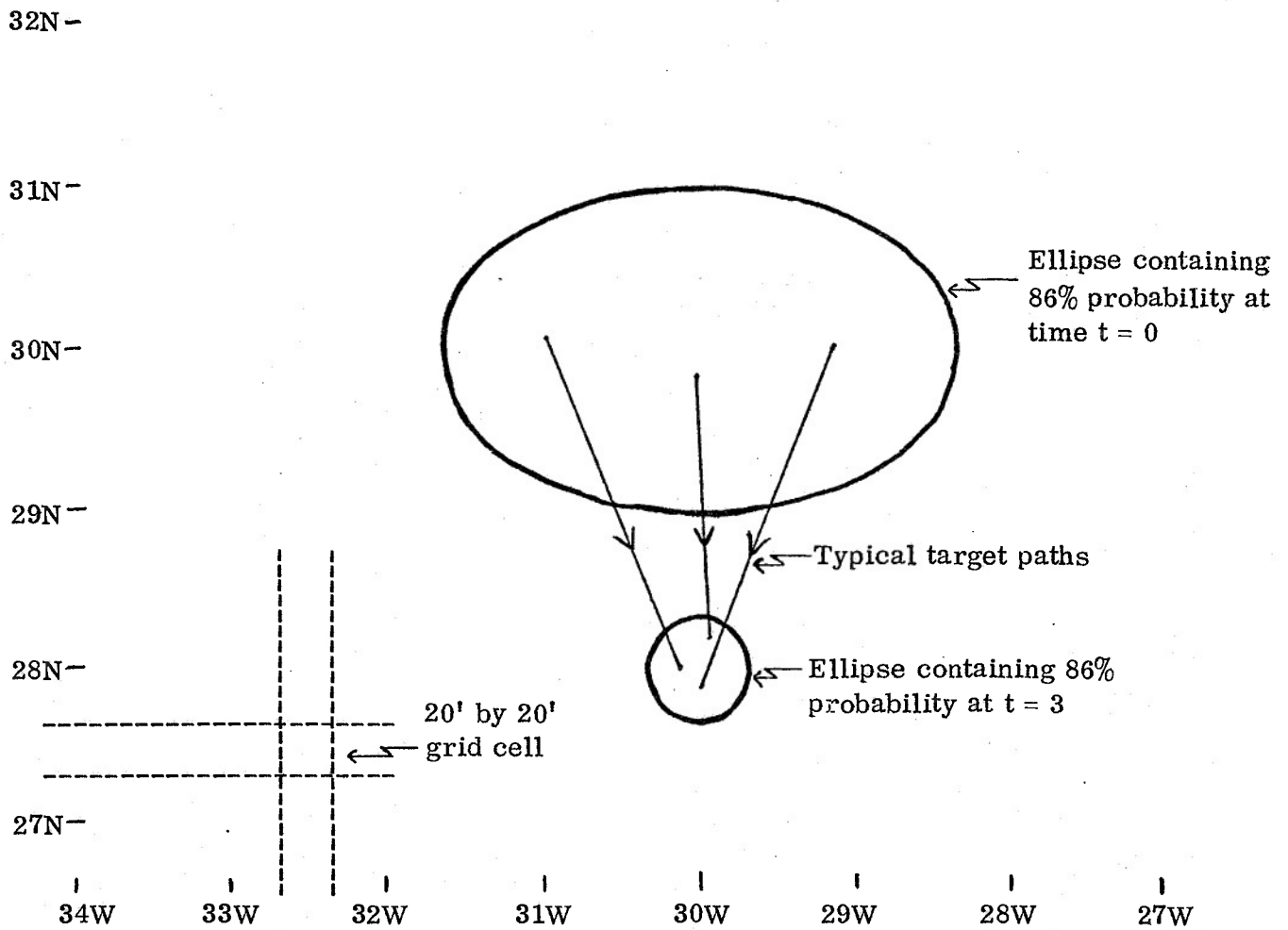TARGET DISTRIBUTION IN WHEREABOUTS GRID AT TIME 3 WHEN NO
SEARCH IS APPLIED DURING ANY TIME PERIOD

NOTE: ENTRIES HAVE BEEN MULTIPLIED BY 10

	0	0	1	0	0
	0	27	96	31	0
28°N	2	133	426	126	3
	0	30	91	31	0
	0	0	0	0	0

30°W

the time periods. For convenience in referring to the cells in the whereabouts grid in Figure 2, we number them starting with the bottom row and numbering from left to right. The highest probability cell is cell 13. Table 2 shows the convergence of the algorithm, which chooses cells in the order of highest to lowest probability in Figure 2. Since the high probability cell in Figure 2 is the optimal whereabouts cell for this problem, the initial guess is optimal and the remaining guesses are eliminated very quickly after one or two iterstions (see Table 2). The optimization algorithm required 702 CPU seconds on a Prime 400 minicomputer. This time does not include the time necessary to generate the sample paths for the target motion process.

Figure 3 shows the target distribution before search and the optimal allocation of search at times $t = 0, 1, 2, 3$ for this example. The rectangle in the target distribution outlines the region in which search is placed at that time period. Observe that no effort is placed in the high probability cell at time 0 and time 3 whereas substantial amounts of effort are placed in the high probability cell at times 1 and 2. Since the high probability cell at $t = 3$ is also the whereabouts cell, it is clear that no search should be placed in this cell at time 3. Also, points that start in the high probability cell at time 0 are likely to end up in the high probability cell at time 3 by the way we have chosen our target paths. This explains why no effort is placed in the high probability cell at time 0. However, at times 1 and 2 the high probability cell contains points from paths that start and end on the edges of the distributions at times $t = 0$ and $t = 3$ but cross over the center at times 1 or 2. Thus it is profitable to search these cells. Figure 4 shows the target distribution given

TABLE 2

CONVERGENCE OF ALGORITHM TO OPTIMAL WHEREABOUTS PLAN

Note: Solution tolerance .001

Whereabouts Cell Chosen	Success Probability		Number of Iterations
	Lower Bound	Upper Bound	
13	.8619	.8623	3
12	.8370	.8447	2
14	.8358	.8432	2
18	.8303	.8382	2
8	.8292	.8340	2
9	.7996	.8470	1
19	.8010	.8470	1
7	.7979	.8447	1
17	.7980	.8436	1
15	.7843	.8303	1
11	.7849	.8308	1
23	.7837	.8300	1

failure of the detection search.

Example 3: Variable Sweep Width. The assumptions are the same as in example 2, except the sweep width is 1 to the east of 30° 20' W and 0.1 to the west. If no search takes place the distribution at $t=3$ remains as given in Figure 2. However, the optimal whereabouts plan for this search does not choose the high probability cell in Figure 2 for the whereabouts cell. Instead the cell directly to the west is chosen. The reason for this is that the optimization algorithm takes into account the fact that it is very difficult to search west of 30° 20' W (i.e., $W = 0.1$) and that it is more efficient to use search effort where the detection capability is high (i.e., in the high probability cell) and make the wereabouts guess in a cell where detection capability is poor.

Figure 5 shows the target distribution and optimal search allocation for the optimal whereabouts search plan. The rectangle in the target distribution outlines the area in which search effort is applied. Figure 6 shows the target distribution given the detection search has failed to find the target. Note, that the whereabouts cell is now the high probability cell. Observe that all search takes place in the area where detection capability is good. However, no effort is placed in the high probability area at time 0 even though it is in the good detection region!

The detection probability at the end of time 3 is 0.74 which is higher than the detection probability of 0.66 at the end of time 3 in example 2 even though the sweep width is uniformly as large or larger in example 2 than in example 3. The reason for this is that the detection search in example 3 searches the high probability cell at time 3 in contrast to example 2 which puts no search in that cell because it is the whereabouts cell. However the probability of success in example 2 (0.86) is higher than in example 3 (0.82).

A glance at Figure 6 explains why no search effort is placed in the poor detection region. Virtually all the target paths that terminate in this region are located in the whereabouts cell so that no search is required.

FIGURE 3

UNIFORM SWEEP WIDTH EXAMPLE

TARGET DISTRIBUTION BEFORE SEARCH AT TIME 0

NOTE: ENTRIES HAVE BEEN MULTIPLIED BY 10^4

```
       0    0    0    0    0    0    0    2    0    2    0    0    0    0    0    0    0    0   0
       0    0    0    2    5    2    5    5    2    7   17   15    0    0    0    2    2    0   0
31°N─0    0    5    5   10   17   35   35   45   52   57   45   32   12   17    2    2    0   2
       0    0    5   17   30  │57   75  145  155  147  177   90   95   55   37│  22    0    5   7
       2   10    7   52   67  │122  187  205  277  270  245  232  180  107   65│  27   22    7   0
30°N─2   12    7   35   77  │117  227  290  332  370  362  302  215  132   62│  37   22    7   5
       0    7    7   47   65  │120  192  245  257  275  290  232  172   90   97│  30   25    5   7
       0    5   15   22   35   55  135  105  122  145  135  107   77   67   47   17   10    2   0
29°N─0    0    2    5   17   20   40   35   45   47   40   32   30   17   10    2    2    0   0
       0    0    0    0    0    2    0   10   17   10   22   15    2    5    2    0    2    0   0
       0    0    0    0    0    0    0    0    0    5    2    0    0    0    0    0    0    0   0
      33°W          32°W           31°W           30°W           29°W           28°W
```

OPTIMAL ALLOCATION OF EFFORT FOR TIME 0

```
31°N──  0    0    0    0    0   18    0    0    0    0
         0  144  119   78   75    0   24  114   31    0
30°N── 21   66   68   79    0   62   66   21  147    0
        54   23   96   20   45   12   65    0    0   48
             31°W          30°W          29°W
```

PROBABILITY OF DETECTION BY THE END OF TIME 0 = 0.09

FIGURE 3 (continued)

TARGET DISTRIBUTION BEFORE SEARCH AT TIME 1

NOTE: ENTRIES HAVE BEEN MULTIPLIED BY 10^4

```
           0    0    2    5    7    7   10   22    5    0    2    0   0
        ─  0    5   15   45   75  115  132  116   70   37   15    2   5
30°N  ─  5   15   82 │140  222  389  427  416  262  159│  40   22   5
          10   15   90 │223  388  551  661  622  392  206│  72   47   7
           2   22   62 │159  326  356  433  446  218  175│  76   22   5
29°N  ─  0    2    5   52   77  100  150  102   65   45    7    2   0
           0    0    0    2    5   10   12   22    5    0    2    0   0
           0    0    0    0    0    0    2    0    0    0    0    0   0
```

        32°W   31°W   30°W   29°W

OPTIMAL ALLOCATION OF EFFORT FOR TIME 1

```
30°N  ─   0   32  127  147  182   39    0
         18    6  163  119  188    8    5
29°N  ─   0  169   97   59  140    0    2
         31°W            30°W            29°W
```

PROBABILITY OF DETECTION BY THE END OF TIME 1 = 0.25

FIGURE 3 (continued)

TARGET DISTRIBUTION BEFORE SEARCH AT TIME 2

NOTE: ENTRIES HAVE BEEN MULTIPLIED BY 10^3

```
              0    0    1    2    1    0    0
    29°N —    0   13   37   54   38    9    2
              4   35   96  138  107   37    5
              1   15   37   62   40   13    1
              0    0    1    3    0    0    0
                   |              |
                  31°W          30°W
```

OPTIMAL ALLOCATION OF EFFORT FOR TIME 2

```
    29°N —      0  125   62    0
              282  388  342   13
                0  232   55    0
                        |
                      30°W
```

PROBABILITY OF DETECTION BY THE END OF TIME 2 = 0.52

FIGURE 3 (continued)

TARGET DISTRIBUTION BEFORE SEARCH AT TIME 3

NOTE: ENTRIES HAVE BEEN MULTIPLIED BY 10^3

```
        0    0   1   0    0
        0   14  48  17    0
28°N __ 2   63 202  60    1
        0   14  45  15    0
        0    0   0   0     0
                  |
                30°W
```

OPTIMAL ALLOCATION OF EFFORT FOR TIME 3

```
           0  337   0
28°N __  433    0 415
           0  315   0
               |
             30°W
```

PROBABILITY OF DETECTION BY THE END OF TIME 3 = 0.66

FIGURE 4

TARGET DISTRIBUTION FOLLOWING UNSUCCESSFUL
SEARCH AT TIME 3 IN UNIFORM SWEEP WIDTH EXAMPLE

Notes: 1. Entries are multiplied by 10.

2. The cell enclosed by the dashed lines
is the optimal whereabouts cell. If the
target is not detected by the end of time 3,
this cell is guessed to contain the target.

```
29°N-0    0    |    0    0
      0   28   67   33    0
28°N-5   54  [594]  54    3
      |   42   55   44    |
      0    0    0    0    0
      |              |
    31°W           30°W
```

PROBABILITY OF SUCCESS IN WHEREABOUTS SEARCH = 0.86

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FIGURE 5

VARIABLE SWEEP WIDTH EXAMPLE

TARGET DISTRIBUTION BEFORE SEARCH AT TIME 0

NOTE: ENTRIES HAVE BEEN MULTIPLIED BY 10^4

```
           0   0   0   0   0   0   0   2   0   2   0   0   0   0   0   0  0  0  0
           0   0   0   2   5   2   5   5   2   7  17  15   0   0   0   2  2  0  0
31°N —     0   0   5   5  10  17  35  35  45  52  57  45  32  12  17   2  2  0  2
           0   0   5  17  30  57  75 145 155 |147 177  90  95  55  37| 22  0  5  7
           2  10   7  52  67 122 187 205 277 |270 245 232 180 107  65| 27 22  7  0
           2  12   7  35  77 117 227 290 332 |370 362 302 215 132  62| 37 22  7  5
30°N —     0   7   7  47  65 120 192 245 257 |275 290 232 172  90  97| 30 25  5  7
           0   5  15  22  35  55 135 105 122  145 135 107  77  67  47  17 10  2  0
           0   0   2   5  17  20  40  35  45   47  40  32  30  17  10   2  2  0  0
           0   0   0   0   0   2   0  10  17   10  22  15   2   5   2   0  2  0  0
           0   0   0   0   0   0   0   0   0    5   2   0   0   0   0   0  0  0  0

          23°N        32°W        31°W        30°W        29°W        28°W
```

OPTIMAL ALLOCATION OF EFFORT FOR TIME 0

```
                  0 101    0    0    0    0
                 98  18   86  207   65    0
30°N —            0  95  127   86  176    0
                 95  28  111  104    0  105

                    30°W          29°W
```

PROBABILITY OF DETECTION BY THE END OF TIME 0 = 0.08

FIGURE 5 (continued)

TARGET DISTRIBUTION BEFORE SEARCH AT TIME 1

NOTE: ENTRIES HAVE BEEN MULTIPLIED BY 10^4

```
            0    0    2    5    7    7   10   22    5    0    2    0    0
30°N ——     0    5   15   45   75  115  131  110   70   37   15    2    5
            5   15   82  142  277  454  420  369  232  152   40   22    5
           10   15   90  232  470  679  659  552  323  195   72   47    7
29°N ——     2   22   62  167  350  393  414  422  188  168   74   22    5
            0    2    5   52   77  100  150  102   65   45    7    2    0
            0    0    0    2    5   10   12   22    5    0    2    0    0
            0    0    0    0    0    0    2    0    0    0    0    0    0
               32°W          31°W          30°W          29°W
```

OPTIMAL ALLOCATION OF EFFORT FOR TIME 1

```
30°N  ——
            186  215   88    6
            188  235   72   66
29°N  ——    125  229   14   75
                30°W         29°W
```

PROBABILITY OF DETECTION BY THE END OF TIME 1 = 0.22

TARGET DISTRIBUTION BEFORE SEARCH AT TIME 2

NOTE: ENTRIES HAVE BEEN MULTIPLIED BY 10^3

```
         0   0    1 | 2    1    0 | 0
         0   13   46|57   33    9 | 2
29°N —   4   37  134|143  81   33 | 5
         1   16   47|61   35   12 | 1
         0   0    1 | 3    0    0 | 0
             |          |
           31°W       30°W
```

OPTIMAL ALLOCATION OF EFFORT FOR TIME 2

```
29°N —   221   71    0
         483  314   57
         284   70    0
              |
            30°W
```

PROBABILITY OF DETECTION BY THE END OF TIME 2 = 0.45

FIGURE 5 (continued)

TARGET DISTRIBUTION BEFORE SEARCH AT TIME 3

NOTE: ENTRIES HAVE BEEN MULTIPLIED BY 10^3

```
            0    0    1    0    0
            0   16   57   17    0
    28°N—   2   79  231   61    1
            0   18   50   15    0
            0    0    0    0    0
                      |
                    30°W
```

OPTIMAL ALLOCATION OF EFFORT FOR TIME 3

```
    257    0
    751  282
    210    0
```

PROBABILITY OF DETECTION BY THE END OF TIME 3 = 0.74

FIGURE 6

TARGET DISTRIBUTION FOLLOWING UNSUCCESSFUL
SEARCH AT TIME 3 IN VARYING SWEEP WIDTH EXAMPLE

Notes: 1. Entries have been multiplied by 10.

2. The cell enclosed by the dashed lines is the optimal
   whereabouts cell. If the target is not detected by the
   end of time 3, this cell is guessed to contain the
   target.

```
0    0    2    0    0
0    33   84   33   0
7   [306] 106  106   3
2    71   106  57   1
0    0    0    0    0
```

PROBABILITY OF SUCCESS IN WHEREABOUTS SEARCH = 0.82.

4. WASHBURN'S UPPER BOUND

This section contains a brief description of the upper bound given by Washburn [18].

Let P_T be defined as in equation (2) and let ψ and $\tilde{\psi} \in \Psi$. Then

$$P'_T[\psi, \tilde{\psi}-\psi] \equiv \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(P_T[\psi+\epsilon(\tilde{\psi}-\psi)] - P_T[\psi] \right)$$

is the Gateaux differential of P_T at ψ in the direction $\tilde{\psi}-\psi$. Let E_{jt} indicate expectation conditioned on $X_t = j$, and define

$$D_T[\psi, j, t] = E_{jt}\left[\frac{W(j)}{A(j)} \exp\left(-\sum_{t=0}^{T} W(X_t)\, \psi(X_t, t)/A(X_t)\right] p_t(j) \text{ for } \psi \in \Psi, \; 1 \le j \le J, \; 1 \le t \le T,$$

where $p_t(j) = \Pr\{X_t = j\}$. Then Stone [14] shows that

$$P'_T[\psi, \tilde{\psi}-\psi] = \sum_{t=0}^{T} \sum_{j=1}^{J} D_T[\psi, j, t]\, [\tilde{\psi}(j, t) - \psi(j, t)].$$

Since P_T is a concave function, one can show that (see the proof of Theorem 1 in Stone [14])

$$P_T[\tilde{\psi}] - P_T[\psi] \le P'_T[\psi, \tilde{\psi}-\psi]$$

$$= \sum_{t=0}^{T} \sum_{j=1}^{J} D_T[\psi, j, t]\, \tilde{\psi}(j, t) - D_T[\psi, j, t]\, \psi(j, t).$$

For $t = 0, \ldots, T$, let

$$\bar{\lambda}(t) = \max\{D_T[\psi, j, t] : 1 \le j \le J\}$$

$$\underline{\lambda}(t) = \min\{D_T[\psi, j, t] : 1 \le j \le J \text{ and } \psi(j, t) > 0\}.$$

Then

$$P_T[\tilde{\psi}] - P_T[\psi] \leq \sum_{t=1}^{T} (\bar{\lambda}(t) - \underline{\lambda}(t)) \, m(t) \quad \text{for } \psi, \tilde{\psi} \in \Psi(m).$$

This is Washburn's upper bound. Note that the right-hand side depends only on ψ and the bound holds for any $\tilde{\psi} \in \Psi(m)$, in particular for the plan ψ^* which maximizes detection probability at time T within $\Psi(m)$. The necessary conditions of Stone [14] guarantee that $\bar{\lambda}(t) = \underline{\lambda}(t)$ when $\psi = \psi^*$.

5. ACKNOWLEDGEMENTS

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This paper shows that solving the optimal whereabouts search problem for a moving target is equivalent to solving a finite number of optimal detection problems for moving targets. This generalizes the result of Kadane (5) for stationary targets.

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