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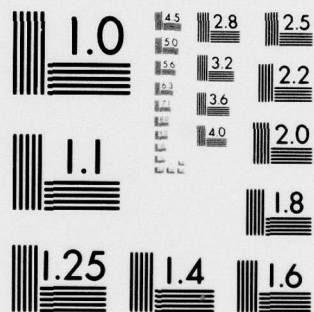
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OF A CONSTRAINED NONCONVEX PROGRAM

by

10 Garth P. McCormick

Serial T-409
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Program in Logistics

Abstract
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LOCATING AN ISOLATED GLOBAL MINIMIZER
OF A CONSTRAINED NONCONVEX PROGRAM

by

Garth P. McCormick

Abstract

Conditions are given which verify the existence and determine the location of an isolated local constrained minimizer of a constrained nonconvex program. An exact formula is given for the amount by which the value of the objective function at some point differs from the value at that isolated point. Further conditions are given which ensure that the local minimizer is a global minimizer.

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LOCATING AN ISOLATED GLOBAL MINIMIZER
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by

Garth P. McCormick

1. Introduction

In [2] conditions were given under which it could be proved that the problem

$$\inf_{x \in C \subseteq \mathbb{R}^n} f(x)$$

where C is a convex compact set, had only one local infimum which was taken on at a unique point x^* in C . The solution was shown to be unconstrained in the sense that the 1 by n derivative vanished there, i.e.,

$$f'(x^*) = 0.$$

Furthermore, the amount by which the function f evaluated at some point in C exceeded its global minimum value was given by an explicit formula. In this paper, similar results are obtained for the equality constrained and then the inequality constrained problem. In Section 2 some preliminary lemmas are developed for the problem of solving simultaneous nonlinear equations. In Sections 3 and 4 bounds are developed for an isolated local constrained minimizer in C , and on the minimum value associated with that minimizer. It is also shown that if slightly stronger conditions are placed on the problem (verifiable) then that local minimizer is a global one in C .

2. Solving Nonlinear Equations—Preliminary Lemmas

Let $a(z)$ be an $N \times 1$ vector function which is continuously differentiable in an open set containing a given compact convex set $C \subseteq \mathbb{R}^N$. Assume that $a'(z)$, the $N \times N$ derivative matrix of $a(z)$, has an inverse at every point in C . Let C^0 denote the interior of C . Define $U \equiv \{u | u = a(z) \text{ for some } z \in C\}$. Consider any point $\hat{z} \in C^0$. Put $\hat{u} = a(\hat{z})$. It follows from the inverse function theorem that there exists an open set $\hat{C} \subset C^0$ containing $\hat{z}$, an open set $\hat{U} \subset U$ containing $\hat{u}$, and a continuously differentiable function $g(\cdot)$ defined on $\hat{U}$ such that

$$g[a(z)] = z, \text{ for } z \text{ in } \hat{C}; \quad a[g(u)] = u, \text{ for } u \text{ in } \hat{U}.$$

Lemma 1. Suppose for some $z_0 \in C^0$,

$$N(z_0) \equiv \{z_0 - a'(y)^{-1}a(z_0) \mid y \in C\} \subseteq C. \quad (1)$$

Then:

$$a(z_0)t \in U, \text{ for all } t \text{ such that } 0 \leq t \leq 1, \quad (2)$$

and there exists a point z^* in $N(z_0)$ such that

$$a(z^*) = 0. \quad (3)$$

Proof: Let $\bar{t}$ be the smallest value greater than or equal to zero such that $g[a(z_0)t]$ is defined for $0 \leq \bar{t} < t \leq 1$. (Clearly $t < 1$, since $z_0 \in C^0$ and because of the inverse function theorem.) Assume t is such that $0 \leq \bar{t} < t < 1$. Now

$$dg[a(z)]/da = a'(z)^{-1}.$$

Thus,

$$\begin{aligned} g[a(z_0)t] &= g[a(z_0)] + \int_0^1 \{dg[a(z_0)(1-s) + (z_0)st]/da\} ds a(z_0)(t-1) \\ &= z_0 - \int_0^1 a'[g\{a(z_0)(1-s+st)\}]^{-1} ds a(z_0)(1-t). \end{aligned} \quad (4)$$

(Note because $0 < \bar{t} < 1-s+st < 1$, $g\{a(z_0)(1-s+st)\}$ is defined and in C .)

Now,

$$\hat{y} = z_0 - \int_0^1 a' [g\{a(z_0)(1-s+st)\}]^{-1} ds a(z_0) \quad (5)$$

is in C . To see this, assume the contrary. Because C is a convex set, there would exist a hyperplane $(\hat{\alpha}, \hat{\beta})$ which separates $\hat{y}$ from C with the properties that

$$\begin{aligned} \hat{\alpha}z - \hat{\beta} &\leq 0 \quad \text{for all } z \in C, \\ \text{if } \hat{\alpha}z - \hat{\beta} &> 0 \quad z \notin C, \end{aligned} \quad (6)$$

and

$$\hat{\alpha}\hat{y} - \hat{\beta} > 0.$$

Using (5) and the mean value theorem, there is an $\hat{s}$ where $0 \leq \hat{s} \leq 1$ such that

$$\hat{\alpha}[z_0 - a'[g\{a(z_0)(1-\hat{s}+\hat{s}t)\}]^{-1}a(z_0)] - \hat{\beta} > 0.$$

(Note because $\bar{t} < 1 - \hat{s} + \hat{s}t \leq 1$, $g\{a(z_0)(1-\hat{s}+\hat{s}t)\}$ is defined and in C .)

Using (6) this contradicts (1). Thus $\hat{y} \in C$. Now $z_0t + \hat{y}(1-t)$ gives

(4). Since C is a convex set and $0 < t < 1$, and $z_0 \in C^0$, (4) is also contained in C^0 . Since $a'(z)$ is continuous it follows that $g[a(z_0)\bar{t}] \in C$.

If $\bar{t} > 0$ the same reasoning as above implies $g[a(z_0)\bar{t}] \in C^0$, and the inverse function theorem implies the existence of values less than $\bar{t}$ and close to it for which $a(z_0)t \in U$. By contradiction then, $\bar{t} = 0$. This proves (2) and (3).

3. The Equality Constrained Problem

The equality constrained nonlinear programming problem can be written

$$\begin{aligned} &\text{minimize } f(x) \\ &\text{subject to } x \in R \equiv \{x | h_j(x) = 0, \text{ for } j=1, \dots, p\} \text{ and } x \in C_x \end{aligned} \quad (7)$$

where C_x is a compact convex subset of R^n . The functions f , $\{h_j\}$ are assumed to be twice continuously differentiable in an open set containing C_x . The vector $h(x)$ is the p by 1 vector whose j th component is $h_j(x)$. The following development will be general in that the particular numerical method for generating the null space of $h'(x)$ (the p by n derivative matrix of $h(x)$) will not be specified.

Suppose $S(x)$ denotes a matrix function which gives the null space of $h'(x)$. That is, a necessary and sufficient condition that $h'(x)z = 0$ is that $z = S(x)v$ for some v . Also associated with $h'(x)$ is a pseudo inverse $h'(x)^\#$, i.e., a matrix satisfying

$$h'(x) h'(x)^\# h'(x) = h'(x).$$

Note that when the rank of $h'(x)$ is p , $h'(x)h'(x)^\# = I$.

It is assumed only that $S(x)$ and $h'(x)^\#$ are related in that $[I - h'(x)^\# h'(x)] = S(x)W(x)$ for some matrix $W(x)$.

The Lagrangian function associated with (7) is

$$L(x, \alpha) = f(x) - \alpha^T h(x).$$

The vector z of Section 2 is to be identified with $z^T = (x^T, \alpha^T)$, and

$$a(z) = \begin{pmatrix} \nabla_x L(x, \alpha) \\ -h(x) \end{pmatrix}$$

is $(N=(n+p))$ by 1. The following result is easily obtained:

$$a'(z) = \begin{pmatrix} \nabla_{xx}^2 L(x, \alpha) & , & -h'(x)^T \\ -h'(x) & , & 0 \end{pmatrix}. \quad (8)$$

Let C_α be the convex hull of $\{f'(x)h'(x)^\#^T \mid x \in C_x\}$. Define $C = C_x \times C_\alpha$. Define $H(x, \alpha) = S(x)^T \nabla_{xx}^2 L(x, \alpha) S(x)$.

Theorem 1. Suppose for every $z \in C$,

- (i) $h'(x)$ has full rank (equal to p)
- (ii) $S(x)$ is a continuously differentiable n by $(n-p)$ matrix function generating the null space of $h'(x)$, and that
- (iii) $H(x, \alpha)$ is a positive definite matrix.

Then $a'(z)^{-1}$ exists for all $z \in C$ and is given by

$$a'(z)^{-1} = \begin{bmatrix} S(x)H(x, \alpha)^{-1}S(x)^T, -[I - S(x)H(x, \alpha)^{-1}S(x)^T \nabla_{xx}^2 L(x, \alpha)]h'(x)^\# \\ -h'(x)^\#^T [I - \nabla_{xx}^2 L(x, \alpha)S(x)H(x, \alpha)^{-1}S(x)^T], F \end{bmatrix} \quad (9)$$

where

$$F = -h'(x)^\#^T [\nabla_{xx}^2 L(x, \alpha) - \nabla_{xx}^2 L(x, \alpha)S(x)H(x, \alpha)^{-1}S(x)^T \nabla_{xx}^2 L(x, \alpha)]h'(x)^\#.$$

Let $g(\cdot)$ denote the inverse function as defined in Section 2. The notation $S(t)$ will mean $S[g\{a(z_0)t\}]$ and similar notation used for the other quantities. Suppose $x_0 \in C_x$ is given. Define $\alpha_0 = [f'(x_0)h'(x_0)^\#]^T$, and set $z_0^T = (x_0^T, \alpha_0^T)$. Define $N(z_0) = \{z_0 - a'(y)^{-1}a(z_0) \mid y \in C\}$. Assume further (iv) that $N(z_0) \subseteq C$ and that $z_0 \in C^0$.

Then there exists a $z^* \in N(z_0)$ such that $\nabla_x L(x^*, \alpha^*) = 0$, $h(x^*) = 0$, x^* is an isolated local minimizer to Problem (7), and

$$f(x^*) - f(x_0) = \int_1^0 \{ t \nabla_x L(x_0, \alpha_0)^T S(t) H(t)^{-1} S(t)^T \nabla_x L(x_0, \alpha_0) + f'(t) [I - S(t) H(t)^{-1} S(t)^T \nabla_{xx}^2 L(t)] h'(t)^{\#} h(x_0) \} dt . \quad (10)$$

Proof: Because $H(x, \alpha)$ is positive definite, its inverse exists. The remainder of the first assertion above can be verified by multiplying the two matrices together.

The existence of z^* is guaranteed by Lemma 1. Since $\nabla_x L(x^*, \alpha^*) = 0$, (x^*, α^*) satisfies the first order necessary conditions for an equality constrained extremum. Assumption (iii) means then that the second order sufficiency conditions for an isolated local minimizer are satisfied at x^* .

From Lemma 1 it also follows that $g\{a(z_0)t\}$ is defined and in C for all $0 \leq t \leq 1$. Define the composite function

$$F(t) = f[g\{a(z_0)t\}] .$$

Then

$$f(x^*) = F(0) = F(1) + \int_1^0 F'(t) dt, \text{ where } F(1) = f(x_0) .$$

Using the chain rule of differentiation,

$$\begin{aligned} F'(t) &= df[g\{a(z_0)t\}]/dt = df[g\{a(z_0)t\}]/dz \cdot dg\{a(z_0)t\}/da \cdot d[a(z_0)t]/dt \\ &= f'(t) S(t) H(t)^{-1} S(t)^T \nabla_x L(x_0, \alpha_0) + f'(t) [I - S(t) H(t)^{-1} S(t)^T \nabla_{xx}^2 L(t)] \cdot \\ &\quad h'(t)^{\#} h(x_0) . \end{aligned}$$

Now $h'(t) S(t) = 0$, so

$$f'(t) S(t) = [f'(t) - \alpha(t) h'(t)] S(t) = \nabla_x L(t)^T S(t) = t \nabla_x L(x_0, \alpha_0)^T S(t) . \quad (11)$$

Using this fact completes the proof of the theorem.

4. The Inequality Constrained Problem

The notation of the previous section will be used and the preceding results extended to the inequality constrained problem

$$\begin{aligned} & \text{minimize } f(x) \\ & \text{subject to } x \in R \equiv \{x \mid h_j(x) \geq 0, \text{ for } j=1, \dots, m\}, \end{aligned} \quad (12)$$

and $x \in C_x$ where C_x is a compact convex subset of R^n .

Lemma 2. If x^* is an isolated local minimizer for the Problem

$$\begin{aligned} & \text{minimize } f(x) \\ & \text{subject to } x \in \{x \mid h_j(x) \geq 0, \text{ for } j=1, \dots, p \leq m\} \cap C_x, \end{aligned} \quad (13)$$

and if $x^* \in R$, then x^* is an isolated local minimizer for Problem (12). (Note: the proof is obvious and therefore omitted.)

In what follows, conditions will be given which guarantee the existence of an isolated local constrained minimizer in C for the Problem (13). Without loss of generality it will be assumed that the vector $h(x)$ will consist of the first p functions.

Theorem 2. For the inequality constrained Problem (13) assume as in Theorem 1. Then there exists a $z^* \in N(z_0)$ such that $\nabla_x L(x^*, \alpha^*) = 0$, and $h(x^*) = 0$.

Proof: The proof of this theorem parallels that for Theorem 1 and will not be given. Note that it is not possible to conclude that x^* is a local minimizer for the inequality constrained problem. Several more conditions must be placed on the problem quantities.

Corollary 1. Assume as in Theorem 2. If at x^* , $\alpha^* > 0$ and $x^* \in R$, then x^* is an isolated local minimizer for the Problem (12).

Proof: The proof follows because under the assumptions, x^* satisfies the second order sufficiency conditions for a constrained minimizer to (13). Using Lemma 2, the corollary is proved.

5. Conditions Under Which the Isolated Local Minimizer Is a Global Minimizer in the Convex Compact Set

Some notation needs to be established. Let x be any point in $C_x \cap R$. Define for $0 \leq t \leq 1$, $y(t) = x^*(1-t) + xt$, and $\hat{y}(s, t) = x^*(1-s) + [x^*(1-t) + xt]s$, for $0 \leq s \leq 1$.

Theorem 3. Assume as in Theorem 1. In addition the following assumptions are made.

If

- (i) $\int_0^1 h'[y(t)]dt$ has rank p ,
- (ii) $v^T \int_0^1 \int_0^1 \nabla_{xx}^2 L[\hat{y}(s,t), \alpha^*] ds dt v > 0$ for all $v \neq 0$ where $\int_0^1 h'[y(t)]dt v = 0$,

then

$$\begin{pmatrix} \int_0^1 \int_0^1 \nabla_{xx}^2 L[\hat{y}(s,t), \alpha^*] ds dt, & - \int_0^1 h'[y(t)]^T dt \\ - \int_0^1 h'[y(t)] dt, & 0 \end{pmatrix} \quad (14)$$

has an inverse. Denote this by $\begin{pmatrix} A, B^T \\ B, D \end{pmatrix}$ where A is n by n , B^T is n by p .

If (iii), $-\int_0^1 f'[y(t)] dt B^T > 0$, (component by component) then $f(x) > f(x^*)$. If (i), (ii), and (iii) hold for all $x \in C_x$, then x^* is the unique global minimizer for f in $R \cap C_x$.

Proof: Consider the following (recall $\nabla_x L(x^*, \alpha^*) = 0$, $h(x^*) = 0$):

$$\begin{pmatrix} \int_0^1 \nabla_x L[y(t), \alpha^*] \\ - h'(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \int_0^1 \int_0^1 \nabla_{xx}^2 L[\hat{y}(s,t), \alpha^*] ds dt, & - \int_0^1 h'[y(t)]^T dt \\ - \int_0^1 h'[y(t)] dt, & 0 \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta \alpha \end{pmatrix},$$

where $\Delta x = (x - x^*)$, $\Delta \alpha = (\alpha^* - \alpha^*) = 0$.

The existence of the above inverse is guaranteed by the theorem hypotheses. It can be written explicitly after the following notational considerations are given. Let S be any n by $(n-p)$ matrix generating the null space of $N = \int_0^1 h'[y(t)]dt$. Let $N^\#$ be a psueudo inverse of N related to S in that there exists a matrix W such that $[I - N^\#N] = SW$.

Let $H = \int_0^1 \int_0^1 \nabla_{xx} L[\hat{y}(s,t), \alpha^*] ds dt$. The explicit inverse (which is invariant to the choice of the particular null space matrix S and pseudo inverse $N^\#$) is then

$$\begin{pmatrix} H, & -N^T \\ -N, & 0 \end{pmatrix}^{-1} = \begin{pmatrix} S(S^T H S)^{-1} S^T & , & -[I - S(S^T H S)^{-1} S^T] N^\# \\ -N^\#^T [I - H S (S^T H S)^{-1} S^T] & , & -N^\#^T [H - H S (S^T H S)^{-1} S^T] N^\# \end{pmatrix}.$$

Now

$$\begin{aligned} f(x) - f(x^*) &= \int_0^1 f'[y(t)] dt \Delta x \\ &= \int_0^1 f'[y(t)] dt S(S^T H S)^{-1} S^T \int_0^1 \nabla_x L[y(t), \alpha^*] dt \\ &\quad - \int_0^1 f'[y(t)] dt B^T h(x) \\ &= \int_0^1 \nabla_x L[y(t), \alpha^*]^T dt S(S^T H S)^{-1} S^T \int_0^1 \nabla_x L[y(t), \alpha^*] dt \\ &\quad - \int_0^1 f'[y(t)] dt B^T h(x). \end{aligned}$$

Since $x \in R$, $h(x) \geq 0$. Thus both terms on the right above are nonnegative. If $\int_0^1 \nabla_x L[y(t), \alpha^*]^T dt S \neq 0$, the first term above is > 0 . The second term is > 0 if $h_j(x) > 0$ for one of $j = 1, \dots, p$. The only way both terms can be equal to zero is if $\int_0^1 \nabla_x L[y(t), \alpha^*]^T dt S = 0$ and $h(x) = 0$.

If both these conditions hold it is easy to show that $\Delta x = x - x^* = 0$. Thus $f(x) > f(x^*)$ when $x \neq x^*$.

Several interesting results come out of this analysis. There are two estimates of the generalized Lagrange multipliers associated with a local minimizer. The first one is

$$\alpha_0 = f'(x_0) h'(x_0)^\# . \quad (15)$$

This is used in the computation of the Lagrangian Hessian matrix. The second estimate is

$$f'(x_0)[I - S(x_0)H(x_0, \alpha_0)^{-1}S(x_0)^T \nabla_{xx}^2 L(x_0, \alpha_0)]h'(x_0)^{\#}. \quad (16)$$

This is the one which determines whether or not a constraint is to be considered binding at the local solution. The latter estimate is really the estimate of the multipliers (15) at the minimizer of f in the sub-space $h(x) = h(x_0)$. In algorithm development, then, the choice of whether or not to include a constraint in the set of those considered active should be made on the basis of (16) rather than (15).

6. Discussion and Example

The importance of the second order estimate of the multipliers was first pointed out by Gill and Murray in their seminal paper [1]. The estimate in (16) above corresponds exactly to their "refinement" given in (7.4) of [1]. Their vector p there is the approximation required to find the minimizer of the function in the equality constrained sub-space. In the current paper this is the transpose of the vector

$$-f'(x_0)S(x_0)H(x_0, \alpha_0)^{-1}S(x_0)^T$$

in (16) above.

The quantity (10) which gives the exact difference between the value of the objective function at the minimizer in the space where $h(x) = 0$ and its value at some point x_0 may be considered (for intuitive explanatory purposes) to consist of two parts. Let $\bar{x}$ denote the minimizer for the problem: minimize $f(x)$ subject to $h(x) = h(x_0)$, $x \in C_x$. The first term under the integral in (10) is an approximation to $f(\bar{x}) - f(x_0)$. It is well known that the difference in the optimal objective function value resulting from a perturbation in the right-hand side of the constraints is approximated by the sum of the Lagrange multipliers times the perturbations. The vector premultiplying $h(x_0)$ is the

integral of a vector of second order multiplier estimates at unconstrained minimizers of the form (16). This can be regarded as the appropriate vector of Lagrange multipliers needed to approximate $f(x^*) - f(\bar{x})$. Thus (10) says (in effect) that $f(x^*) - f(x_0) = [f(\bar{x}) - f(x_0)] + [f(x^*) - f(\bar{x})]$.

In terms of the inverse function approach $\bar{x}$ consists of the first n components of

$$g \left\{ \begin{matrix} 0 \\ h(x_0) \end{matrix} \right\}.$$

The exact difference $f(x_0) - f(\bar{x})$ can be computed by integrating from $(x_0^T, \alpha_0^T)^T$ to $(\bar{x}^T, \bar{\alpha}^T)^T$ along $g[(t \nabla_x L(x_0, \alpha_0)^T, h(x_0)^T)^T]$. The exact second term, $f(x^*) - f(\bar{x})$, can then be obtained by integrating from $(\bar{x}^T, \bar{\alpha}^T)^T$ to $(x^{*T}, \alpha^{*T})^T$ along the curve $g[(0^T, h(x_0)^T)^T]$. The reason that the current approach is taken is that the existence of the inverse function along the "diagonal path" can be checked by (1). The piecewise two-path segment described above may not exist in the region $C_x \times C_\alpha$ without stronger assumptions.

The practical application of this theory is that from (1), it is possible to place upper and lower bounds on each component of the minimizer and multipliers, and from (10) it is possible to place an upper and lower bound on the optimal objective function value. It is in general not possible to compute the exact set $N(z_0)$ of (1) or the right-hand side of (10). Using numerical techniques, one can compute upper and lower bounds on the desired quantities. For the purely unconstrained case, the computational requirements were discussed in [3]. The constrained case is more complicated and a future paper will discuss the linear algebra required to do this.

Essentially what has to be done is to replace each scalar function required in the formulas by two numbers, an upper and a lower bound on

the range that that scalar function can take when x is restricted to C_x , and α to C_α . Techniques required to do this have, to a certain extent, been developed into a general theory of "interval arithmetic" [4]. Every element of a vector and matrix can be systematically replaced by two numbers. The problem becomes more difficult when an interval inverse matrix is required, i.e., a matrix of interval elements which contains the inverse of all possible matrices implied by the interval matrix. In order to get a set bounding (1), an interval inverse matrix is required for $a'(y)^{-1}$, where the form taken for the constrained problem is given by (8). In the following example, *ad hoc* procedures are used to get the interval bounds. A few of the details are given, but the bulk of the computations must be taken on faith.

The rules for computing intervals for the range of functions (or interval extensions as they are sometimes called) are very simple. The notation $[a,b]$ is used to indicate that an upper bound on the range of a scalar function is b and a lower bound is a . If two scalar valued functions have interval extensions $[a,b]$ and $[c,d]$, respectively, then the sum has the bounds $[a+b, c+d]$. The range on the product is $[\min(ac, ad, bc, bd), \max(ac, ad, bc, bd)]$. Subtraction is similar to addition. Division is similar to multiplication as long as the divisor does not contain zero. The other major requirement is that interval extensions be computable for functions of a single variable operating on an interval. In one sense this can be viewed as an optimization problem. For most functions of a single variable, the min and max of that function over an interval are easily discernible. Suppose a scalar t is known to have a range of $[a,b]$. Then the scalar function t^2 has a range of $[\min(a^2, b^2), \max(a^2, b^2)]$ if $0 \notin [a,b]$, and $[0, \max(a^2, b^2)]$ if $a \leq 0 \leq b$. A scalar function which has the range $[a,a]$ will in the following be indicated by a for simplicity. Thus, the notation $a[c,d]$ will indicate the range of the product of two functions, one of which has range $[a,a]$, the other $[c,d]$. The resulting range is $[\min(ac, ad), \max(ac, ad)]$. The major research area needing to be investigated is obtaining an interval extension of a matrix inverse.

The problem is

$$\begin{aligned} &\text{minimize } -x + y \\ &\text{subject to } h = -x^2 - y^2 + 1 \geq 0, \\ &C_1 = \{(x,y) | .2 \leq x \leq 2, -2 \leq y \leq -.2\}. \end{aligned}$$

The quantities required are

$$\begin{aligned} h' &= (-2x, -2y), \quad (h')^\# = \begin{pmatrix} -x \\ -y \end{pmatrix} [2(x^2+y^2)]^{-1} \\ s &= \begin{pmatrix} -y \\ x \end{pmatrix} (x^2+y^2)^{-1}, \quad \nabla_{xx}^2 L = \begin{pmatrix} 2\alpha & \\ & 2\alpha \end{pmatrix}. \end{aligned}$$

The multiplier estimate is given by $f'(h')^\# = (x-y)/[2(x^2+y^2)]$. The top component of $(h')^\#$ is $-x/[2(x^2+y^2)]$. One can show that the minimum of this function in the rectangle is taken on at $x = .2$, $y = -.2$ and the maximum at $x = 2$, $y = -2$. Thus the range of the first component is $[-1.25, -.125]$. Similar considerations yield the range of the second component to be $[.125, 1.25]$. The range on α is therefore $(-1)[-1.25, -.125] + (1)[.125, 1.25] = [.25, 2.5]$. This then is the set C_α .

The matrix whose inverse is desired [from (8)] is

$$\begin{pmatrix} 2\alpha & 0 & 2x \\ 0 & 2\alpha & 2y \\ 2x & 2y & 0 \end{pmatrix}.$$

The inverse matrix has the form

$$\begin{pmatrix} -y^2 & xy & -\alpha x \\ xy & -x^2 & -\alpha y \\ -\alpha x & -\alpha y & \alpha^2 \end{pmatrix} [-2\alpha(x^2+y^2)]^{-1}.$$

The interval extension of this inverse when x, y , and α are restricted as indicated above is

$$\bar{B} = \begin{pmatrix} [.001980, 1.98020] & [.019802, 1.0] & [.024752, 1.25] \\ [.019802, 1.0] & [.001980, 1.98020] & [-1.25, -.024752] \\ [.024752, 1.25] & [-1.25, -.024752] & [-15.625, -.015625] \end{pmatrix} \quad (17)$$

Let $(x_0, y_0) = [.7, -.71]$. Thus $\alpha_0 = .709184$, $h_0 = .0059$, and

$\nabla_x L_0 = [-.007142, -.007042]^T$. Then $N(z_0)$ of Theorem 2 is contained in

$$\begin{aligned} \begin{pmatrix} x_0 \\ y_0 \\ \alpha_0 \end{pmatrix} - \bar{B} \begin{pmatrix} \nabla_x L_0 \\ h_0 \end{pmatrix} &= \begin{pmatrix} .7 \\ -.71 \\ .709184 \end{pmatrix} - \bar{B} \begin{pmatrix} -.007142 \\ -.007042 \\ .0059 \end{pmatrix} \\ &= \begin{pmatrix} [.700300, .728560] \\ [-.717220, -.689060] \\ [.608371, .717845] \end{pmatrix} \subset \begin{pmatrix} [0.2, 2] \\ [-2, -0.2] \\ [0.25, 2.5] \end{pmatrix}. \end{aligned}$$

which is clearly in the original rectangle of bounds.

The bound on the optimal objective function value is calculated using (10). The first term under the integral involves $S(t)H(t)^{-1}S(t)$. Note from (9) that this occurs, for any (x, α) , in the upper left-hand corner of the inverse of a' . This is thus available from (17) above. An interval extension (for every t) of $\nabla_x L(x_0, \alpha_0)^T S(t)H(t)^{-1}S(t)^T \nabla_x L(x_0, \alpha_0)$ is

$$\begin{aligned} &(-.007142, -.007042) \begin{pmatrix} [.001980, 1.98020], & [.019802, 1.0] \\ [0.19802, 1.0] & , [.001980, 1.98020] \end{pmatrix} \begin{pmatrix} -.007142 \\ -.007042 \end{pmatrix} \\ &= [.000002191, .00029976]. \end{aligned}$$

A term of one-half comes from the integration of a constant times t for t between 0 and 1. Thus the contribution of the first term is $[\text{.0000010955, .00014988}]$.

The second term under the integral in (10) involves the negative of the matrix which occurs in the upper right-hand corner of (9). This needs to be premultiplied by the objective function derivative $(-1, 1)$ and postmultiplied by $h(x_0)$ to yield

$$(-1,1) \begin{pmatrix} -[.024752, 1.25] \\ -[-1.25, -.024752] \end{pmatrix} (.0059) = [.00029208, .01475] .$$

Adding the two contributions yields the bounds

$$[.0002931755, .01489988] .$$

Since $f_0 = -1.41$, the value of f^* must lie in the interval

$$[-1.42490, -1.41029] .$$

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