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EXPECTED NUMBER OF VERTICES OF A RANDOM CONVEX POLYTOPE. I. INT--ETC(U)

JAN 80 D S KELLY, J W TOLLE

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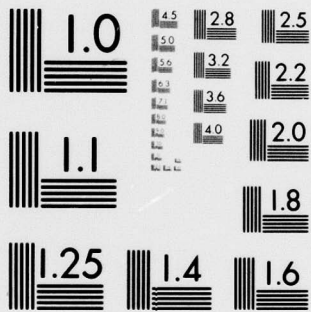
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Expected Number of Vertices of a Random Convex Polytope  
I. Integral Formula and Asymptotic Bounds

by

D.G. Kelly  
and  
J.W. Tolle

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Expected Number of Vertices of a Random Convex Polytope

I. Integral Formula and Asymptotic Bounds

by

D.G. Kelly\*

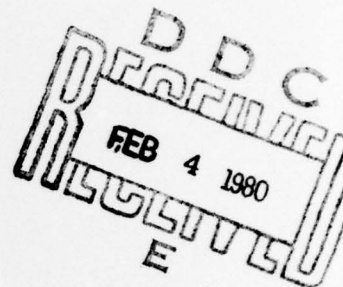
Department of Statistics  
Curriculum in Operations Research and Systems Analysis  
Department of Mathematics

and

J.W. Tolle\*\*

Curriculum in Operations Research and Systems Analysis  
Department of Mathematics

University of North Carolina at Chapel Hill



Abstract

Given  $m$  points on the unit sphere in  $n$ -space, the hyperplanes tangent to the sphere at the given points bound a convex polytope with  $m$  facets. If the points are chosen independently at random from the uniform distribution on the sphere, the number  $V_{mn}$  of the vertices of the polytope is a random variable. We obtain an integral expression for  $EV_{mn}$  and asymptotic bounds of the form

$$\alpha \frac{n(n-6)/2}{(m-n)} \leq EV_{mn} \leq \beta \frac{n(n-5)/2}{m} .$$

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## I. Introduction

The main results of this working paper are: the integral expression (3.3) for the expected number of vertices of a random convex polytope; the upper bound (4.1) and consequent asymptotic upper bound (4.2) on the expected number; and the asymptotic lower bounds (4.7) and (4.8).

Briefly and informally, the bounds are these: Let  $p_1, \dots, p_m$  be independent random points each having the uniform distribution on the unit sphere in  $\mathbb{R}^n$ , and let  $V_{mn}$  be the number of vertices of the (random) polytope whose facets are the hyperplanes tangent to the sphere at  $p_1, \dots, p_m$ . Then there exist constants  $\alpha$  and  $\beta$  independent of  $m$  and  $n$  such that asymptotically

$$\alpha^n n^{(n-6)/2} (m-n) \leq EV_{mn} \leq \beta^n n^{(n-5)/2} m.$$

The upper bound is valid for any  $n \geq 2$ , asymptotically as  $m \rightarrow \infty$ ; the lower bound is valid for any large  $n$ , asymptotically as  $m \rightarrow \infty$ .

In a later paper we will present the results of numerical approximations of the integral (3.3) for moderate values of  $m$  and  $n$ .

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## II. Formulation

A. Given an  $n$ -vector  $\underline{a}$  and a real number  $b$ , the inequality  $\underline{a}^T \underline{x} \leq b$  defines a half-space in  $\mathbb{R}^n$  with bounding hyperplane  $H = \{x: \underline{a}^T x = b\}$ . The non-empty intersection of a finite number, say  $k$ , of such half-spaces defines a polytope in  $\mathbb{R}^n$  with at most  $k$  facets. A *random polytope* is a polytope generated by choosing the components of the  $k$   $n$ -vectors,  $\underline{a}_1, \dots, \underline{a}_k$ , and the  $k$  real numbers,  $b_1, \dots, b_k$ , from some probability distribution on the real line. Obviously the structure of random polytopes will depend on the choice of the probability distribution from which the data are drawn. For examples of certain distribution schemes the reader is referred to [2,4].

In this paper it is desired to consider only random polytopes in  $\mathbb{R}^n$  with a fixed number,  $m$ , of facets. The general scheme given above for generating polytopes will lead to redundant constraints especially in the case where  $k$  is large relative to  $n$ . This difficulty can be avoided if  $b_1, \dots, b_k$  are chosen as appropriate functions of the  $\underline{a}_1, \dots, \underline{a}_k$ . In particular, if for  $j = 1, \dots, m$ , each  $b_j$  satisfies

$$(2.1) \quad b_j = |\underline{a}_j| = \left( \sum_{i=1}^n a_{ji}^2 \right)^{1/2},$$

then the random polytope generated by the vectors  $\underline{a}_1, \dots, \underline{a}_m$  is circumscribed about the unit sphere,  $S_{n-1}$ , and has exactly  $m$  facets. Each facet lies in one of the hyperplanes  $H_j = \{x: \underline{a}_j^T x = |\underline{a}_j|\}$  and is tangent to  $S_{n-1}$  at  $\underline{p}_j = \underline{a}_j / |\underline{a}_j|$ . The polytope is thus completely determined by the unit vectors  $\underline{p}_1, \dots, \underline{p}_m$ . Consequently, an alternative generating scheme for this type of random polytope is to choose the vectors  $\underline{p}_1, \dots, \underline{p}_m$  from some distribution on  $S_{n-1}$ .

Of special interest here will be the case where  $\underline{p}_1, \dots, \underline{p}_m$  are independently and uniformly distributed on  $S_{n-1}$ . As is well-known ([3]) this choice of the

tangent points is equivalent to choosing the components of the vectors  $\underline{a}_j$ ,  $j = 1, \dots, m$ , independently from the standard normal distribution on the real line and the  $b_j$  according to (2.1).

It should be pointed out that requiring the polytopes to be circumscribed on the unit sphere as described above certainly limits their possible structure. Polytopes of certain combinatorial types cannot be realized with all of their facets tangent to a sphere. Therefore it may be argued that the more pathological polytopes are excluded by the method of generation.

B. Let  $p_1, \dots, p_m$  be independent and identically distributed points on  $S_{n-1}$ . As discussed above, these points determine a unique polytope in  $\mathbb{R}^n$  containing  $S_{n-1}$  and having exactly  $m$  facets. Let  $V_{mn}(p_1, \dots, p_m)$  denote the number of vertices of this polytope. The function  $V_{mn}$  is a random variable on the space of  $m$ -tuples of  $n$ -vectors, with values in the nonnegative integers. The purpose of this paper is to investigate the expected value of  $V_{mn}$ , specifically when the  $p_j$  have the uniform distribution on  $S_{n-1}$ .

### III. Integral Expression for $EV_{mn}$ .

A. Fix integers  $m$  and  $n$ ,  $2 < n < m$ , and let  $p_1, \dots, p_m$  be independent random points chosen from some distribution with density  $g(p)$  on  $S_{n-1}$ . Let  $H_i$  be the hyperplane tangent to  $S_{n-1}$  at  $a_i$ ,  $i = 1, \dots, n$ , and let  $P(p_1, \dots, p_n)$  be the polytope bounded by  $H_1, \dots, H_n$ . Then with probability 1 any  $n$  of  $H_1, \dots, H_n$  intersect in a point of  $\mathbb{R}^n$  and there are  $\binom{m}{n}$  such points of intersection, among which are the vertices of  $P(p_1, \dots, p_n)$ .

Denote by  $A_{mn}$  the family of all  $n$ -subsets of  $1, 2, \dots, m$ . For any  $A = \{i_1, \dots, i_n\} \in A_{mn}$  let  $V_A$  be the event that the point of intersection of  $H_{i_1}, \dots, H_{i_n}$  is a vertex of  $P(p_1, \dots, p_n)$ . Then  $V$  is the sum of the indicator functions of the events  $V_A$ , and so

$$EV_{mn} = \sum_{A \in A_{mn}} \Pr(V_A) .$$

Because the  $p_i$  are identically distributed, we have by symmetry

$$EV_{mn} = \binom{m}{n} \Pr(V_n) ,$$

where  $V_n$  denotes  $V_{\{1, \dots, n\}}$ .

B. Now with probability 1,  $p_1, \dots, p_n$  lie on a unique small hypercircle on  $S_{n-1}$ , which divides  $S_{n-1}$  into two caps. Let  $C(p_1, \dots, p_n)$  be the smaller of these two caps. Then  $V_n$  occurs if and only if none of  $p_{n+1}, \dots, p_m$  is in  $C(p_1, \dots, p_n)$ . Given  $p_1, \dots, p_n$ , therefore, the conditional probability of  $V_n$  is

$$\Pr(V_n | p_1, \dots, p_n) = (1 - \int_{C(p_1, \dots, p_n)} g(p) dp)^{m-n} .$$

So  $EV_{mn}$  is  $\binom{m}{n}$  times the expected value of this function of  $p_1, \dots, p_n$ ; that is,

$$EV_{mn} = \binom{m}{n} E \Pr(V_n | p_1, \dots, p_n) = \int_{S_{n-1}} \dots \int_{S_{n-1}} (1 - \int_{C(p_1, \dots, p_n)} g(p) dp)^{m-n} \cdot dp_1, \dots, dp_n . \quad (3.1)$$

C. Now we assume that the common distribution of  $p_1, \dots, p_n$  is the uniform distribution on  $S_{n-1}$ . In this case

$$\int_{C(p_1, \dots, p_n)} g(p) dp = \frac{\text{area of } C(p_1, \dots, p_n)}{\text{area of } S_{n-1}} = \frac{I_{n-2}(r)}{2I_{n-2}(\pi/2)},$$

where  $r = r(p_1, \dots, p_n)$  is the angular radius of  $C(p_1, \dots, p_n)$  ( $0 < r < \pi/2$ ) and where

$$I_k(r) = \int_0^r \sin^k x \, dx \quad (0 < r < \pi/2, k = 0, 1, 2, \dots)$$

is the area of a cap of angular radius  $r$  on  $S_{k+1}$ . Thus (3.1) can be rewritten

$$EV_{mn} = \binom{m}{n} E \left( 1 - \frac{I_{n-2}(r)}{2I_{n-2}(\pi/2)} \right)^{m-n}$$

where the random variable  $r$  is the radius of  $C(p_1, \dots, p_n)$ .

Now it follows from results of R.A. Miles ([5], theorem 4, p. 368) that  $t = \sin^2 r$  has the beta  $((n-1)^2/2, 1/2)$  distribution, whose density is proportional to  $t^{((n-1)^2/2)-1} (1-t)^{-1/2}$  on  $(0, 1)$ . From this it follows that the density of  $r$  is  $f_{n(n-2)}(r)$ , where

$$f_k(r) = C_k \sin^k r, \quad 0 < r < \pi/2$$

and

$$C_k = \frac{2\Gamma(\frac{k+2}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{k+1}{2})}, \quad k = 0, 1, 2, \dots \quad (3.2)$$

Moreover, the distribution function of this density is

$$F_k(r) = \frac{I_k(r)}{I_k(\pi/2)}, \quad 0 < r < \pi/2.$$

Using the above notation we obtain the expression

$$EV_{mn} = \binom{m}{n} \int_0^{\pi/2} (1 - \frac{1}{2} F_{n-2}(r))^{m-n} f_{n(n-2)}(r) dr . \quad (3.3)$$

Although we will not use it, the following alternative expression may be of interest. If  $B_k$  and  $b_k$  denote the distribution function and density of the beta( $k/2$ ,  $1/2$ ) distribution, then the substitution  $t = \sin^2 r$  in (3.3) gives

$$EV_{mn} = \binom{m}{n} \int_0^1 (1 - \frac{1}{2} B_{n-1}(t))^{m-n} b_{(n-1)2}(t) dt .$$

A. The evaluation of (3.3) is not difficult in case  $n$  is 2 or 3:

$$EV_{m2} = \binom{m}{2} \int_0^{\pi/2} \left(1 - \frac{r}{\pi}\right)^{m-2} \cdot \frac{2}{\pi} dr = m \left(1 - \left(\frac{1}{2}\right)^{m-1}\right),$$

and

$$\begin{aligned} EV_{m3} &= \binom{m}{3} \int_0^{\pi/2} \left(1 - \frac{1}{2}(1 - \cos r)\right)^{m-3} \cdot \frac{3}{2} \sin^3 r dr \\ &= 2m-4 - \left(\frac{1}{2}\right)^{m-1} (m+1)(m-2). \end{aligned}$$

(We note in passing that  $m$  and  $m-1$  are the only possible values of  $V_{m2}$ , the latter occurring only when  $p_1, \dots, p_m$  are all in one semicircle. The value of  $E_{m2}V$  given above gives the well-known value  $m\left(\frac{1}{2}\right)^{m-1}$  for the probability of this event.)

The above calculations suggest that for fixed  $n$ ,  $EV_{mn}$  is asymptotically of the form  $K_n m$ . This section is devoted to finding upper and lower bounds on  $EV_{mn}$  which establish this asymptotic growth rate and bound the growth of  $K_n$  for large  $n$ . (Unfortunately, the tempting conjecture that  $K_n = n-1$  is false:  $K_n$  is approximately of order  $(\beta n)^{n/2}$  as we have noted.)

B. First we will establish an upper bound by showing that for  $2 \leq n < m$ ,

$$EV_{mn} \leq A_n m (1 - B_{mn}) \tag{4.1}$$

where

$$A_n = C_{n(n-2)} \left(\frac{2}{C_{n-2}}\right)^{n-1} \frac{(n-1)^{n-3}}{n},$$

$$B_{mn} = \left(\frac{1}{2}\right)^{m-1} \sum_{j=0}^{n-2} \binom{m-1}{j} \left(\frac{C_{n-2}}{n-1}\right)^j.$$

Before proceeding, we remark that  $B_{mn}$  is negligible:

$$B_{mn} < \left(\frac{1}{2}\right)^{m-1} \left(1 + \frac{C_{n-2}}{n-1}\right)^{m-1},$$

and since  $\frac{C_{n-2}}{n-1} < \frac{3}{4}$  when  $n \geq 3$ ,  $B_{mn} \leq \left(\frac{7}{8}\right)^{m-1}$ .

We observe also that Wallis's formula ([1], p. 258), viz.

$$\frac{\Gamma(\alpha + \frac{1}{2})}{\Gamma(\alpha + 1)} \sim \frac{1}{\sqrt{\alpha}} \left(1 - \frac{1}{8\alpha} + O\left(\frac{1}{\alpha^2}\right)\right) \text{ as } \alpha \rightarrow \infty,$$

implies

$$C_{n(n-2)} \sim \sqrt{2/\pi} n$$

and

$$\frac{2}{C_{n-2}} \sim \sqrt{2\pi/(n-2)} \left(1 - \frac{1}{4(n-2)}\right).$$

Thus

$$A_n \sim \frac{(2\pi)^{n/2}}{\pi e^{1/4}} n^{(n-5)/2}$$

and so if  $m \geq n$ , then asymptotically as  $n \rightarrow \infty$

$$EV_{mn} \leq \frac{(2\pi)^{n/2}}{\pi e^{1/4}} n^{(n-5)/2} m. \quad (4.2)$$

The proof of (4.1) begins with (3.3), which says that

$$EV_{mn} = \binom{m}{n} C_{n(n-2)} S_{n-2, m-n, n(n-2)} \quad (4.3)$$

where

$$S_{jkl} = \int_0^{\pi/2} \left(1 - \frac{1}{2} F_j(r)\right)^k \sin^\ell r \, dr. \quad (4.4)$$

If  $\ell \geq j+1$ , an integration by parts using

$$u = -\frac{2}{C_j} \sin^{\ell-j} r, \quad dv = \left(1 - \frac{1}{2} F_j(r)\right)^k \left(-\frac{1}{2} C_j \sin^j r\right) dr$$

yields

$$S_{jkl} = \frac{-2}{C_j} \frac{1}{k+1} \left(\frac{1}{2}\right)^{k+1} + \frac{2}{C_j} \frac{\ell-j}{k+1} \int_0^{\pi/2} \left(1 - \frac{1}{2} F_j(r)\right)^{k+1} \sin^{\ell-j-1} r \cos r \, dr$$

and hence

$$S_{jkl} \leq -\frac{2}{C_j} \frac{1}{k+1} \left(\frac{1}{2}\right)^{k+1} + \frac{2}{C_j} \frac{\ell-j}{k+1} S_{j,k+1,\ell-j-1} \quad \text{for } \ell \geq j+1. \quad (4.5)$$

Iterating (4.5) gives, for  $\ell \geq v(j+1)$ ,

$$S_{jkl} \leq -\frac{2}{C_j} \frac{1}{k+1} \left(\frac{1}{2}\right)^{k+1} - \sum_{i=2}^v \left(\frac{2}{C_j}\right)^i \frac{(\ell-j)(\ell-2j-1)\dots(\ell-(i-1)j-(i-2))}{(k+1)(k+2)\dots(k+i)} \left(\frac{1}{2}\right)^{k+i} \\ + \left(\frac{2}{C_j}\right)^v \frac{(\ell-j)(\ell-2j-1)\dots(\ell-vj-(v-1))}{(k+1)(k+2)\dots(k+v)} S_{j,k+v,\ell-v(j+1)}. \quad (4.6)$$

In particular this is valid for  $j = n-2$ ,  $k = m-n$ ,  $\ell = n(n-2)$ , and  $v = n-2$ .

In this case for  $i = 2, 3, \dots, n-2$  we have

$$(\ell-j)(\ell-2j-1)\dots(\ell-(i-1)j-(i-2)) \\ = [(n-1)(n-2)][(n-1)(n-3)]\dots[(n-1)(n-i)] = (n-1)^{i-1} \frac{(n-2)!}{(n-i-1)!}$$

and also by a simple integration

$$S_{j,k+v,\ell-v(j+1)} = S_{n-2,m-2,n-2} = \frac{2}{C_{n-2}} \frac{1}{m-1} \left(1 - \left(\frac{1}{2}\right)^{m-1}\right).$$

Putting the two expressions **above** into (4.6) yields

$$S_{n-2,m-n,n(n-2)} \leq \left(\frac{2}{C_{n-2}}\right)^{n-1} \frac{(n-2)!(m-n)!}{(m-1)!} (n-1)^{n-2} \\ - \sum_{i=1}^{n-1} \left(\frac{2}{C_{n-2}}\right)^i \frac{(n-2)!(m-n)!}{(n-i-1)!(m-n+i)!} (n-1)^{i-1} \left(\frac{1}{2}\right)^{m-n+i}.$$

Multiplying by  $C_{n(n-2)} \binom{m}{n}$  gives an upper bound on  $EV_{mn}$ ; simplifying and letting the sum run over  $j = n-1-i$  then gives (4.1).

C. Next we show that for any  $n \geq 2$  and any  $\epsilon \in (0,1)$ ,

$$EV_{mn} \geq \frac{d_{n(n-2)}^2}{d_{n-1}^{n-1}} \frac{2^{n-1} \epsilon^n}{n! e^{n-1}} (n-1)^{n-1} (m-n), \quad \text{asymptotically as } m \rightarrow \infty. \quad (4.7)$$

where

$$d_j = \frac{\Gamma(\frac{j+1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{j+2}{2})} .$$

Before proceeding we notice that Wallis's formula implies

$$d_j \sim \sqrt{2/\pi j} \text{ as } j \rightarrow \infty .$$

This and Stirling's formula provide that

$$\frac{d_{(n-1)^2}}{d_{n-1}^{n-1}} \frac{2^{n-1} \epsilon^n}{n! e^{n-1}} (n-1)^{n-1} \sim \frac{(2\pi)^{\frac{n-1}{2}}}{\pi e^{1/2}} \epsilon^n n^{\frac{n-6}{2}} \text{ as } n \rightarrow \infty .$$

Therefore, for fixed large  $n$ , (4.7) implies that for any  $\epsilon \in (0,1)$ , we have asymptotically as  $m \rightarrow \infty$

$$E_{mn} V \geq \frac{(2\pi)^{\frac{n-1}{2}}}{\pi e^{1/2}} \epsilon^n n^{\frac{n-6}{2}} (m-n) . \quad (4.8)$$

To prove (4.7) we begin again with (3.3). For any  $\alpha \in (0, \frac{\pi}{2})$

$$\begin{aligned} EV_{mn} &\geq \binom{m}{n} \int_0^\alpha (1 - \frac{1}{2} F_{n-2}(\alpha))^{m-n} f_{n(n-2)}(r) dr \\ &= \binom{m}{n} (1 - \frac{1}{2} F_{n-2}(\alpha))^{m-n} F_{n(n-2)}(\alpha) . \end{aligned} \quad (4.9)$$

It can be checked that for  $0 \leq \alpha < \frac{\pi}{2}$

$$F_k(\alpha) = \cos \alpha (d_{k+1} \sin^{k+1} \alpha + d_{k+3} \sin^{k+3} \alpha + \dots)$$

by differentiating and using elementary properties of the gamma function.

Moreover,  $d_0 \geq d_1 \geq d_2 \geq \dots$ . Therefore

$$1 - \frac{1}{2} F_{n-2}(\alpha) \geq 1 - \frac{\cos \alpha}{2} (d_{n-1} (\sin^{n-1} \alpha + \sin^{n+1} \alpha + \dots))$$

$$= 1 - \frac{d_{n-1} \sin^{n-1} \alpha}{2 \cos \alpha}$$

and

$$F_{n(n-2)}(\alpha) \geq d_{(n-1)^2} \cos \alpha \sin^{(n-1)^2} \alpha.$$

So, writing  $\tau$  for  $\sin \alpha$ , we have

$$E_{mn} V \geq \binom{m}{n} d_{(n-1)^2} \tau^{(n-1)^2} \sqrt{1-\tau^2} \left( 1 - \frac{d_{n-1}}{2} \frac{\tau^{n-1}}{\sqrt{1-\tau^2}} \right)^{m-n} \text{ for any } \tau \in (0,1).$$

Let  $\varepsilon$  be an arbitrary number in  $(0,1)$ ; then we have

$$E_{mn} V \geq \binom{m}{n} d_{(n-1)^2} \tau^{(n-1)^2} \varepsilon \left( 1 - \frac{d_{n-1}}{2\varepsilon} \tau^{n-1} \right)^{m-n} \text{ if } 0 < \varepsilon < 1$$

and  $0 < \tau < \sqrt{1-\varepsilon^2}$ .

If we now write  $\sigma$  for  $\frac{d_{n-1}}{2\varepsilon} \tau^{n-1}$ , then this inequality becomes

$$E_{mn} V \geq \binom{m}{n} d_{(n-1)^2} \left( \frac{2\varepsilon\sigma}{d_{n-1}} \right)^{n-1} \varepsilon (1-\sigma)^{m-n} \text{ if } 0 < \varepsilon < 1$$

and  $0 < \sigma < \frac{d_{n-1}}{2\varepsilon} (1-\varepsilon^2)^{\frac{n-1}{2}}$ .

Notice that the value of  $\sigma$  maximizing the right side of the above inequality is  $\frac{n-1}{m-1}$ . So if  $\frac{n-1}{m-1} < \frac{d_{n-1}}{2\varepsilon} (1-\varepsilon^2)^{(n-1)/2}$ , then

$$E_{mn} V \geq \binom{m}{n} \frac{d_{(n-1)^2}}{d_{n-1}^{n-1}} 2^{n-1} \varepsilon^n \left( \frac{n-1}{m-1} \right)^{n-1} \left( 1 - \frac{n-1}{m-1} \right)^{m-n}.$$

Since  $\binom{m}{n} \geq \frac{(m-n)^n}{n!}$ ,

$$E_{mn} V \geq (m-n) \frac{d_{(n-1)^2}}{d_{n-1}^{n-1}} \frac{2^{n-1} \varepsilon^n}{n!} \left( 1 - \frac{n-1}{m-1} \right)^{m-n} (n-1)^{n-1},$$

and (4.7) follows from this.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) Given $m$ points on the unit sphere in $n$ -space, the hyperplanes tangent to the sphere at the given points bound a convex polytope with $m$ facets. If the points are chosen independently at random from the uniform distribution on the sphere, the number $V_{m,n}$ of the vertices of the polytope is a random variable. We obtain an integral expression for $EV_{m,n}$ and asymptotic bounds of the form $\alpha n^{(n-6)/2} (m-n) \leq EV_{m,n} \leq \beta n^{(n-5)/2} m$ alpha to the $n$ power, $n$ to the $(n-6)/2$ power, $\beta$ to the $n$ power, $n$ to the $(n-5)/2$ power		

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