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THE EXACT AND ASYMPTOTIC FORMULAS
FOR THE STATE PROBABILITIES IN SIMPLE
EPIDEMICS WITH m KINDS OF SUSCEPTIBLES,

by

H. Lacayo[1] and Naftali A. Langberg[2]
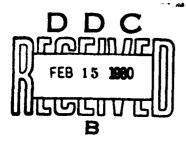
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The Florida State University
Department of Statistics
Tallahassee, Florida 32306

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The Exact And Asymptotic Formulas
For The State Probabilities In Simple
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ABSTRACT.

A population of susceptible individuals partitioned into m groups
and exposed to a contagious disease is considered. It is assumed that an
individual's susceptiblity at time t depends on the number of susceptible
individuals at time t in his group, and on the total number of infective
individuals at time t.

The progress of this simple epidemic is modeled by an m-dimensional
stochastic process. The components of this stochastic process represent the
number of infective individuals in the respective groups at time t. Exact
and approximate formulas for the joint and marginal state probabilities are
obtained. It is shown that the approximate formulas are simple functions of
time while, the derivations of the exact formulas involve tedious computations.

Key words: simple epidemics, different levels of susceptibility, con-
vergence in distribution, exponential, negative binomial, and multinomial
distributions.

1. Introduction and Summary.

We say that a population of susceptible individuals (susceptibles),
exposed to a contagious disease, (disease) undergoes a simple epidemic
if the following four assumptions hold [Bailey (1975)].

(1.1) At each point in time at most one susceptible contracts the disease.

(1.2) Once a susceptible contracts the disease he remains contagious for
the duration of the epidemic.

(1.3) Individuals neither join nor do they depart from the population. And

(1.4) All interactions between a susceptible and an infective individual
(infective) are "equally likely" to result in an infection.

Gart (1968) models the progress of a simple yaws epidemic based on a
data set obtained in New Guinea. He notices that the susceptiblity level
of an individual depends on his previous history of yaws. Gart divides the
susceptibles into two groups: those with a positive yaws history, and those
with a negative yaws history. He replaces, at least implicitly, Assumption
(1.4) by:

(1.5) All interactions between a susceptible that has a specified yaws
history and an infective are "equally likely" to result in an infection.

In this paper we follow Gart (1972) and consider a population of suscep-
tibles partitioned to m subpopulations, $m = 2, 3, \ldots$, and exposed to a disease.
We assume that the susceptibility level of an individual varies according to
his membership in the various subpopulations.

Let T_0 be the first time tnere is at least one infective in each of the
subpopulations, n_r be the number of susceptibles in the rth subpopulation

at T_o, b_r be the number of infectives in subpopulation r at T_o, r = 1, ..., m, and let $n = \sum_{r=1}^{m} n_r$. We describe the progress of the epidemic among suscepti- bles by an m-dimensional stochastic process: $\underline{X}_n(t) = [X_{n,1}(t),...,X_{n,m}(t)]$, $t\epsilon[0,\infty)$. The components of $\underline{X}_n(t)$ represent the number of infectives in the respective subpopulations at time t measured from T_o. Although the infection rates of the various subpopulations are not used explicitly we present them for reference purposes.

Definition 1.1. Let r = 1, ..., m, and $t\epsilon[0,\infty)$. Then the <u>infection rate of subpopulation r</u> at time t is given by

(1.6) $$\lim_{h\to o^+} h^{-1} P\{X_{n,r}(t+h)-X_{n,r}(t)=1|\underline{X}_n(t)\}.$$

We say that a population undergoes a <u>simple Gart epidemic</u> if Assumptions (1.1) through (1.3) and the following one are satisfied.

(1.7) The infection rate of subpopulation r at time t depends on the total number of infectives at time t, and on the number of susceptibles in subpop- ulation r at time t, r = 1, ..., m, $t\epsilon[0,\infty)$.

Note that Assumption (1.7) is the rigorous formulation of the statement: all interactions between a susceptible that belongs to a specified subpopulation and an infective are "equally likely" to result in an infection.

In Section 2 we construct a variety of m-dimensional stochastic pro- cesses. These processes can be used to model the progress of simple Gart epidemics among susceptibles. Section 3 contains formulas for the joint and marginal state probabilities: $P\{X_{n,r}(t)-b_r=k_r,r=1,...,m\}$, and $P\{X_{n,r}(t)-b_r=q\}$, r = 1, ..., m, $t\epsilon[0,\infty)$, where k_1, ..., k_m, $q\epsilon\{0,1,...\}$. These formulas are calculated without the traditional use of the differential equations associated

with the state probabilities. This is done by utilizing a formula for the distribution function of a sum of independent exponential random variables (rva's) given by Billard, Lacayo, and Langberg (1979).

Let $\alpha_1, \ldots, \alpha_m \in (0,\infty)$, and $X_n(t) = \sum_{r=1}^{m} X_{n,r}(t)$. Gart (1968), following the classical approach, assumes that the infection rates at time t of the subpopulations are given by

(1.8) $\qquad \alpha_r(n_r+b_r-X_{n,r}(t))X_n(t)$, $r = 1, \ldots, m$, $t\in[0,\infty)$.

These rates have the property, shown in Section 2, that the duration time of the simple Gart epidemic tends to zero as $n_\ell \to \infty$ for $\ell = 1, \ldots, m$. To obtain approximations to the joint and marginal state probabilities when $n_1, \ldots, n_m$ are sufficiently large we adjust the above rates, and assume that the infection rates at time t of the subpopulations are given respectively by

(1.9) $\qquad n_r^{-1}\alpha_r(n_r+b_r-X_{n,r}(t))X_n(t)$, $r = 1, \ldots, m$, $t\in[0,\infty)$.

In Section 4, we obtain, under Assumption (1.9), approximations to the quantities: $P\{X_{n,r}(t)-b_r=k_r, r=1,\ldots,m\}$, $P\{X_{n,r}(t)-b_r=q\}$, $P\{X_n(t)-\sum_{\ell=1}^{m} b_\ell=q\}$, $EX_n(t)$, $EX_{n,r}(t)$, $Var\{X_n(t)\}$, and $Var\{X_{n,r}(t)\}$, when $n_1, \ldots, n_m$ are sufficiently large. These approximations are expressed as simple functions of t while, the derivations of the exact values of these quantities involve complicated computations, as we illustrate in Section 3.

2. Model Construction.

In this section we construct a variety of m-dimensional stochastic processes. These processes can be used to describe the progress of simple Gart epidemics among susceptibles.

First, we introduce some notation. Let $T_{n,k}$, the $k\underline{\text{th}}$ interinfection time, be defined as the time that elapses between the $b+k-1$ and the $b+k$ infection, $k=1,\ldots,n$. Let $S_{n,o}=0$, $S_{n,k}=\sum_{q=1}^{k}T_{n,q}$, $k=1,\ldots,n$, and $S_{n,n+1}=\infty$. Further, let $\xi_{n,o}=0$, and $\xi_{n,k}$, $k=1,\ldots,n$, be rva's assuming values in the set $\{1,\ldots,m\}$. The rva $\xi_{n,k}$ specifies the subpopulation membership of the $b+k$ infective, $k=1,\ldots,n$. Finally, let I be the indicator function, and $C_{n,k,r}=n_r-\sum_{q=0}^{k-1}I(\xi_{n,q}=r)$, $k=1,\ldots,n$, $r=1,\ldots,m$. Note that $C_{n,k,r}\geq C_{n,n,r}\geq 0$, $k=1,\ldots,n$, $r=1,\ldots,m$.

For $r=1,\ldots,m$, $k=0,\ldots,n$, and $t\epsilon(0,\infty)$, the following event equality holds.

$$(2.1)\qquad (X_{n,r}(t)-b_r=k)=\bigcup_{q=k}^{n}(S_{n,q}\leq t<S_{n,q+1},\ \sum_{j=1}^{q}I(\xi_{n,j}=r)=k).$$

Thus, to construct the stochastic process $\underline{X}_n(t)$ it suffices to determine the distribution function of the random vector (rve) $[T_{n,k},\xi_{n,k},k=1,\ldots,n]$.

Next, we determine the distribution function of this rve. Let $a(n,k,j,r)$, $k=1,\ldots,n$, $j=1,\ldots,n_r$, $r=1,\ldots,m$, be positive real numbers. Assume:

$$(2.2)\qquad P\{\xi_{n,k}=r|\xi_{n,q},q=0,\ldots,k-1\}=$$

$$=a(n,k,C_{n,k,r},r)\{\sum_{\ell=1}^{m}a(n,k,C_{n,k,\ell},\ell)\}^{-1},\ k=0,\ldots,n,$$

$$r=1,\ldots,m,\ \text{and};$$

(2.3) The conditional rva's $\{T_{n,k}|\xi_{n,q}, q = 0,\ldots,k-1\}$, $k = 1, \ldots, n$, are
independent exponentially distributed with means equal respectively to
$\{\sum_{\ell=1}^{m} a(n,k,C_{n,k,\ell},\ell)\}^{-1}$.

Clearly, Assumptions (2.2) and (2.3) determine the distribution function of
the rve $[T_{n,k},\xi_{n,k},k=1,\ldots,n]$.

Note that by the memoryless property of exponential rva's [Barlow,
Proschan (1975), p. 56], Equation (2.1), and Assumptions (2.2), (2.3), the
infection rates at time t of the processes constructed in this section are
equal to $a(n,X_n(t)-b+1,n_r+b_r-X_{n,r}(t),r)$, $r = 1, \ldots, m$, $t\epsilon(0,\infty)$. Thus,
these processes can be used to describe the progress of simple Gart epi-
demics among susceptibles.

The classical model, considered by Gart (1968) and (1972), follows
by setting $a(n,k,j,r) = \alpha_r(b+k-1)j$. By Assumption (2.2) the duration time
of the classical simple Gart epidemic is equal to $E\{\sum_{q=1}^{n}[(b+q-1)\sum_{\ell=1}^{m}\alpha_r C_{n,q,r}]^{-1}\}$.
Thus, it is less than or equal to $(\min_{1\leq\ell\leq m}\alpha_\ell)^{-1}\sum_{q=1}^{n}[(b+q-1)(n-q+1)]^{-1}$. Conse-
quently, the duration time of classical simple Gart epidemic tends to zero
as $n_\ell \to \infty$ for $\ell = 1, \ldots, m$. In Section 4 we investigate the asymptotic
behaviour of simple Gart epidemic models determined by letting
$a(n,k,j,r) = n_r^{-1}\alpha_r(b+k-1)j$. Following Severo (1969) we can define
$a(n,k,j,r) = \alpha_r n_r^{\delta}(b+k-1)^{\theta}j^{\lambda}$, $\delta\epsilon(-\infty,\infty)$, θ, $\lambda\epsilon[0,\infty)$, and thus, construct
extensions to the simple epidemic models used and motivated by him. In
particular for $\delta = 0$, $\theta = \lambda = 1/2$, we get infection rates used by McNeil (1972)
to describe simple epidemic situations.

3. Formulas for the State Probabilities.

Let $q, k_1, \ldots, k_m \in \{0, \ldots, n\}$, $\underline{k} = [k_1, \ldots, k_m]$, and $k = \sum_{r=1}^{m} k_r$. Throughout we assume that $0 \le k \le n$. Further, let $U_1, U_2, \ldots,$ be a sequence of independent exponential rva's with means equal to 1, and $\mu_1, \mu_2, \ldots,$ be a sequence of positive real numbers.

This section contains formulas for the joint and marginal state probabilities: $P_{n,\underline{k}}(t) = P\{\underline{X}_n(t) - \underline{b} = \underline{k}\}$, and $P_{n,q,r}(t) = P\{X_{n,r}(t) - b_r = q\}$, $r = 1, \ldots, m, t \in (0, \infty)$. These formulas are calculated without the traditional use of the differential equations associated with the state probabilities. Rather, we utilize an available formula for the distribution function of a sum of independent exponential rva's. For the sake of completeness we present this formula.

Theorem 3.1. [Billard, Lacayo and Langberg (1979), Theorem 1]. Let M be a positive integer, and $\ell_M(j) = (-1)^{M-1} \sum_{j_1 + \ldots + j_M = j} \prod_{q=1}^{M} (\mu_q)^{j_q + 1}$, $j = M, M+1, \ldots$. Then for $t \in (0, \infty)$.

$$(3.1) \qquad P\{\sum_{q=1}^{M} \mu_q^{-1} U_q \le t\} = \sum_{j=M}^{\infty} \ell_M(j)(-t)^j / (j!).$$

To aid in computing the joint and marginal state probabilities we introduce the following notation. Let $\underline{\ell}_k = [\ell_0, \ldots, \ell_k]$, $\ell_0 = 0$, $\ell_1, \ldots, \ell_k \in \{1, \ldots, m\}$, $B_{\underline{k}} = \{\underline{\ell}_k : \sum_{q=0}^{k} I(\ell_q = r) = k_r, r = 1, \ldots, m\}$, and $A_{r,j,q} = \{\underline{\ell}_j : \sum_{e=0}^{j} I(\ell_e = r) = q\}$, $j = q, q+1, \ldots$.

By Equation (2.1) we obtain that for $t \in (0, \infty)$

$$(3.2) \quad P_{n,\underline{k}}(t) =$$

$$= \sum_{\underline{\ell}_k} P\{S_{n,k} \le t < S_{n,k+1} | \xi_{n,q} = \ell_q, q = 0, \ldots, k\} P\{\xi_{n,q} = \ell_q, q = 0, \ldots, k\} I(\underline{\ell}_k \in B_{\underline{k}}),$$

and that

(3.3) $P_{n,q,r}(t) =$

$$= \sum_{j=q}^{n} \sum_{\underline{\ell}_j} P\{S_{n,j} \le t < S_{n,j+1} | \xi_{n,e} = \ell_e, e = 0, \ldots, j\} P\{\xi_{n,e} = \ell_e, e = 0, \ldots, j\} I(\underline{\ell}_j \epsilon A_{r,j,q}).$$

Thus, to compute $P_{n,\underline{k}}(t)$ and $P_{n,q,r}(t)$ it suffices to evaluate
$P\{\xi_{n,q} = \ell_q, q = 0, \ldots, k\}$ and $P\{S_{n,k} \le t < S_{n,k+1} | \xi_{n,q} = \ell_q, q = 0, \ldots, k\}$.

Now, we present formulas for these probabilities. Let

$$D_{n,k,r}(\underline{\ell}_k, q) = n_r - \sum_{j=0}^{q-1} I(\ell_j = r), \quad q = 1, \ldots, k+1, \text{ and}$$

$$\eta(n, k, \underline{\ell}_k, q) = \sum_{r=1}^{m} a(n, q, D_{n,k,r}(\underline{\ell}_k, q), r), \quad q = 1, \ldots, k+1.$$

First, by Assumption (2.2)

(3.4) $P\{\xi_{n,q} = \ell_q, q = 0, \ldots, k\} =$

$$= \prod_{q=1}^{k} [\{\eta(n, k, \underline{\ell}_k, q)\}^{-1} \prod_{r=1}^{m} \{a(n, q, D_{n,k,r}(\underline{\ell}_k, q), r)\}^{I(\ell_q = r)}].$$

Next, let $f_k(\mu_1, \ldots, \mu_k, t)$ be the density function of the
rva $\sum_{q=1}^{k} \mu_q^{-1} U_q$. Then for $k = 1, 2, \ldots,$ and $t \epsilon (0, \infty)$

(3.5) $P\{\sum_{q=1}^{k} \mu_q^{-1} U_q \le t < \sum_{q=1}^{k+1} \mu_q^{-1} U_q\} =$

$$= \int_0^t e^{-\mu_{k+1}(t-u)} f_k(\mu_1, \ldots, \mu_k, u) du = \mu_{k+1}^{-1} f_{k+1}(\mu_1 \ldots, \mu_{k+1}, t).$$

Thus, by Assumption (2.3) and Theorem 3.1

$$(3.6) \qquad P\{S_{n,k} \le t < S_{n,k+1} | \xi_{n,q} = \ell_q, q=0,\ldots,k\} =$$

$$= \begin{cases} e^{-\eta(n,k,\underline{\ell}_k,1)t} & k = 0 \\[2ex] \sum_{j=n}^{\infty} \frac{(-1)^{n+j-1} t^j}{j!} \sum_{j_1+\ldots+j_n=j} \prod_{q=1}^{n} \{\eta(n,k,\underline{\ell}_k,q)\}^{j_q+1} & k = n \\[3ex] \{\eta(n,k,\underline{\ell}_k,k+1)\}^{-1} \sum_{j=k+1}^{\infty} \frac{(-1)^{k+j} t^{j-1}}{(j-1)!} \sum_{j_1+\ldots+j_{k+1}=j} \\[1ex] \prod_{q=1}^{k+1} \{\eta(n,k,\underline{\ell}_k,q)\}^{j_q+1} & 1 \le k < n. \end{cases}$$

Consequently, the formulas for the joint and marginal state probabilities can be obtained from (3.2) and (3.3) by subtitution.

4. The Asymptotic Behaviour.

This section contains our main asymptotic results. All limits are calulated as $n_\ell \to \infty$ for $\ell = 1, \ldots, m$.

For the sake of completeness we present two definitions.

Definition 4.1. Let $\ell \in \{1,2,\ldots\}$, and $p \in (0,1)$. We say that the rva Y has a negative binomial distribution with parameters ℓ, and p, and write $Y \sim NB(\ell,p)$ if

$$(4.1) \qquad P\{Y=q\} = \binom{\ell+q-1}{q} p^\ell (1-p)^q, \quad q = 0, 1, \ldots .$$

Definition 4.2. Let e, $\ell \in \{1,2,\ldots\}$, and $\gamma_1, \ldots, \gamma_e \in (0,1)$ such that $\sum_{j=1}^{e} \gamma_j = 1$. We say that the rve $\underline{W}_e = \{W_1,\ldots,W_e\}$ has a multinomial distribution with parameters ℓ, $\gamma_1,\ldots, \gamma_e$, and write $\underline{W}_e \sim MN(\ell,\gamma_1,\ldots,\gamma_e)$ if

$$(4.2) \qquad P\{W_j=\ell_j, j=1,\ldots,e\} = (\ell! \prod_{j=1}^{e} \gamma_j^{\ell_j})/(\prod_{j=1}^{e} \ell_j!),$$

$$\ell_1, \ldots, \ell_e \in \{0,\ldots,\ell\}, \quad \sum_{j=1}^{e} \ell_j = \ell.$$

Let $\alpha = \sum_{r=1}^{m} \alpha_r$, $\beta_r(t) = e^{-\alpha t}[1-(1-e^{-\alpha t})(1-\alpha_r \alpha^{-1})]^{-1}$, $Y(t) \sim NB(b,e^{-\alpha t})$, $Y_r(t) \sim NB(b,\beta_r(t))$, and $\underline{W}_m(\kappa) \sim MN(k,\alpha_1\alpha^{-1},\ldots,\alpha_m\alpha^{-1})$, $r = 1, \ldots, m$, $k = 1, 2, \ldots, t \in (0,\infty)$.

Assume throughout that $a(n,k,j,r) = n_r^{-1} \alpha_r(b+k-1)j$, $j = 1, \ldots, n_r$, $r = 1, \ldots, m$, $k = 1, \ldots, n$, $n = 1, 2, \ldots$. First, we show that for $t \in (0,\infty)$

$$(4.3) \qquad \lim P_{n,\underline{k}}(t) = P\{Y(t)=\sum_{r=1}^{m}k_r\}P\{\underline{W}_m(\sum_{r=1}^{m}k_r)=\underline{k}\},$$

$$(4.4) \qquad \lim P\{X_n(t)-b=q\} = P\{Y(t)=q\}, \text{ and}$$

$$(4.5) \qquad \lim P_{n,q,r}(t) = P\{Y_r(t)=q\}, \quad r = 1, \ldots, m.$$

Next, we show that for t, $\beta \in (0, \infty)$

(4.6) $\qquad \lim E\{X_n(t)\}^\beta = E\{Y(t)\}^\beta$, and that

(4.7) $\qquad \lim E\{X_{n,r}(t)\}^\beta = E\{Y_r(t)\}^\beta$, $r = 1, \ldots, m$.

In particular it follows from Statements (4.6) and (4.7) that for $t \in (0, \infty)$

(4.8) $\qquad \lim EX_n(t) = b(e^{\alpha t} - 1)$,

(4.9) $\qquad \lim \text{Var}\{X_n(t)\} = b(e^{2\alpha t} - e^{\alpha t})$,

(4.10) $\qquad \lim EX_{n,r}(t) = b((\beta_r(t))^{-1} - 1)$, $r = 1, \ldots, m$, and

(4.11) $\qquad \lim \text{Var}\{X_{n,r}(t)\} = b((\beta_r(t))^{-2} - (\beta_r(t))^{-1})$, $r = 1, \ldots, m$.

Note that the approximate formulas given in Statements (4.3) through (4.5),

and in Statement (4.8) through (4.11) are indeed simple functions of time.

The following three lemmas are needed to prove Statements (4.3) through (4.5).

Lemma 4.1. Let $k \in \{1, 2, \ldots\}$, and $\ell_1, \ldots, \ell_k \in \{1, \ldots, m\}$. Then

(4.12) $\qquad \lim P\{\xi_{n,q} = \ell_q, q = 1, \ldots, k\} = \prod_{r=1}^{m} (\alpha_r \alpha^{-1})^{\sum_{q=1}^{k} I(\ell_q = r)}$.

Proof. Let $\ell_0 = 0$. Then $P\{\xi_{n,q} = \ell_q, q = 1, \ldots, k\} =$

$= \prod_{q=1}^{k} P\{\xi_{n,q} = \ell_q | \xi_{n,j} = \ell_j, j = 0, \ldots, q-1\}$. Consequently the result of the lemma

follows by Assumption (2.2). ||

Lemma 4.2. Let $k \in \{1, 2, \ldots\}$, $\ell_1, \ldots, \ell_k \in \{1, \ldots, m\}$, and $\ell_0 = 0$. Then

the sequence of conditional rva's: $\{S_{n,k} | \xi_{n,q} = \ell_q, q = 0, \ldots, k-1\}$, $n = 1, 2, \ldots$,

converges in distribution to the rva $\sum_{q=1}^{k} \alpha^{-1} (b+q-1)^{-1} U_q$

<u>Proof</u>. To prove the result of the lemma it suffices, by the Cramer-Wold device [Billingsley (1968), p. 49], to show that the sequence of conditional rve's: $[\{T_{n,1},\ldots,T_{n,k}|\xi_{n,q}=\ell_q,q=0,\ldots,k-1\}]$, n = 1, 2, $\ldots$, converges in distribution to the rve $\{\alpha^{-1}b^{-1}U_1,\ldots,\alpha^{-1}(b+k-1)^{-1}U_k\}$.

The preceeding statement follows from Assumption (2.3). ||

<u>Lemma 4.3</u>. Let d, k, be two positive integers. Further, let $\{U_{1:d+k-1},\ldots,U_{d+k-1:d+k-1}\}$ be the order statistic of a sample of size d + k - 1 taken from the population U_1. Then the rva's $\sum_{q=1}^{k}(d+q-1)^{-1}U_q$ and $U_{k:d+k-1}$ are equal in distribution.

<u>Proof</u>. Let $U_{0,d+k-1} = 0$. Then the spacings: $U_{k-q+1:d+k-1}-U_{k-q:d+k-1}$, q = 1, $\ldots$, k, are independent exponentially distributed rva's with means respectively equal to $(d+q-1)^{-1}$ [Barlow Proschan (1975) p. 59]. Thus, the rva's $\sum_{q=1}^{k}(d+q-1)^{-1}U_q$ and $\sum_{q=1}^{k}\{U_{k-q+1:d+k-1}-U_{k-q:d+k-1}\}$ are equal in distribution. To complete the proof of the lemma we note that

$$\sum_{q=1}^{k}\{U_{k-q+1:d+k-1}-U_{k-q:d+k-1}\} = U_{k:d+k-1}.\ ||$$

We are ready now to prove Statements (4.3) through (4.5). First, we prove Statement (4.3).

<u>Theorem 4.4</u>. Let k_1, $\ldots$, $k_m\epsilon\{0,1,\ldots\}$, $\underline{k} = \{k_1,\ldots,k_m\}$, $k = \sum_{r=1}^{m}k_r$, and $t\epsilon(0,\infty)$. Then $\ell im\ P_{n,\underline{k}}(t) = P\{Y(t)=k\}P\{\underline{W}_m(k)=\underline{k}\}$.

<u>Proof</u>. Let B_k be defined as in Section 3, and let $\underline{\ell}_k \epsilon B_k$. Then by Lemma 4.1 $\ell im\ P\{\xi_{n,q}=\ell_q,q=1,\ldots,k\} = \prod_{r=1}^{m}(\alpha_r\alpha^{-1})^{k_r}$. Further, by Lemmas 4.2, 4.3, and equation (3.5) $\ell im\ P\{S_{n,k}\leq t<S_{n,k+1}|\xi_{n,q}=\ell_q,\ q = 0,\ldots,k\} =$

$$P\{\sum_{q=1}^{k}(b+q-1)^{-1}U_q\leq t<\sum_{q=1}^{k+1}(b+q-1)^{-1}U_q\} = (b+k)^{-1}\begin{pmatrix}b+k\\k+1\end{pmatrix}(k+1)e^{-\alpha bt}(1-e^{-\alpha t})^k.$$

Finally, we note that B_k contains $k!/(\prod_{r=1}^{m} k_r!)$ elements. Consequently, the

result of the theorem follows from Equation (3.2). ||

Let $t\epsilon(0,\infty)$, and $\underline{Z}_m(t) = \{Z_1(t),\ldots,Z_m(t)\}$ be a rve such that

$\{\underline{Z}_m|\sum_{r=1}^{m} Z_r(t)\} \sim MN(\sum_{r=1}^{m} Z_r(t),\alpha_1\alpha^{-1},\ldots,\alpha_m\alpha^{-1})$, and $\sum_{r=1}^{m} Z_r(t) \sim NB(b,e^{-\alpha t})$.

By Theorem 4.4 and a well known result [Billinsgley (1968), p. 16] we conclude,

Corollary 4.5. Let $t\epsilon(0,\infty)$. Then the sequence of rve's $\underline{X}_n(t) - \underline{b}$,

$n = 1, 2, \ldots$, converges in distribution to the rve $\underline{Z}_m(t)$.

From Corollary 4.5 and the Cramer-Wold device we obtain,

Corollary 4.6. Let $r = 1, \ldots, m$, and $t\epsilon(0,\infty)$. Then the two sequences

of rva's $X_n(t)-b$, and $X_{n,r}(t)-b_r$, $n = 1, 2, \ldots$, converge in distribution to

$\sum_{\ell=1}^{m} Z_\ell(t)$ and $Z_r(t)$ respectively.

Now, we prove Statements (4.4) and (4.5).

Theorem 4.7. Let $q\epsilon\{0,1,\ldots\}$, $r = 1, \ldots, m$, and $t\epsilon(0,\infty)$. Then

(a) $\lim P\{X_n(t)-b=q\} = P\{Y(t)=q\}$, and (b) $\lim P_{n,q,r}(t) = P\{Y_r(t)=q\}$.

Proof. Part (a) follows clearly from Corollary 4.6. To prove part (b)

it suffices by Corollary 4.6 to evaluate $P\{Z_r(t)=q\}$.

It is well known that for $k = 1, 2, \ldots$, $P\{Z_r(t)=q|\sum_{\ell=1}^{m} Z_\ell(t)=k\} = $

$\binom{k}{q}(\alpha_r\alpha^{-1})^q(1-\alpha_r\alpha^{-1})^{k-q}$, $q = 0, \ldots, k$. Thus, $P\{Z_r(t)=q\} = $

$= \sum_{k=q}^{\infty}\binom{b+k-1}{k}\binom{k}{q}e^{-\alpha bt}(1-e^{-\alpha t})^k(\alpha_r\alpha^{-1})^q(1-\alpha_r\alpha^{-1})^{k-q} = $

$= [q!(b-1)!]^{-1}e^{-\alpha bt}[(1-e^{-\alpha t})\alpha_r\alpha^{-1}]^q\sum_{k=q}^{\infty}(b+k-1)![(k-q)!]^{-1}[(1-e^{-\alpha t})(1-\alpha_r\alpha^{-1})]^{k-q}$

$= [q!(b-1)!]^{-1}e^{-\alpha bt}[(1-e^{-\alpha t})\alpha_r\alpha^{-1}]^q\int_0^{\infty}e^{-u}u^{b+q-1}\sum_{k=q}^{\infty}[(k-q)!]^{-1}[u(1-e^{-\alpha t})(1-\alpha_r\alpha^{-1})]^{k-q}du$

$= \binom{b+q-1}{q}(\beta_r(t))^b(1-\beta_r(t))^q$. ||

Now we determine the convergence of the moments.

Theorem 4.8. Let t, $\beta \in (0, \infty)$. Then $\lim E\{X_n(t)\}^\beta = E\{Y(t)\}^\beta$.

Proof. First, we note that

$$(4.13) \qquad E\{X_n(t)\}^\beta = \beta \int_0^\infty y^{\beta-1} P\{X_n(t) > y\} dy =$$

$$= \beta \int_0^b y^{\beta-1} dy + \beta \int_0^\infty (b+z)^{\beta-1} P\{X_n(t) - b > z\} dz =$$

$$= b^\beta + \sum_{q=1}^\infty [(b+q)^\beta - (b+q-1)^\beta] P\{X_n(t) - b \geq q\}.$$

Next, for $k = 1, 2, \ldots, \sum_{r=1}^m n_r^{-1} \alpha_r C_{n,k,r} \leq \alpha$, and for $q = 1, 2, \ldots,$ $b + q - 1 \leq 2bq$. Thus, by Lemma 4.3

$$(4.14) \qquad P\{X_n(t) - b \geq k\} = P\{S_{n,k} \leq t\} \leq P\{\sum_{q=1}^k q^{-1} U_q \leq 2\alpha bt\} =$$

$$= (1 - e^{-2\alpha bt})^k, \quad k = 1, 2, \ldots .$$

Finally, we note that

$$(4.15) \quad \sum_{q=1}^\infty [(b+q)^\beta - (b+q-1)^\beta](1 - e^{-2\alpha bt})^q \leq (2b)^\beta \sum_{q=1}^\infty q^\beta (1 - e^{-2\alpha bt})^q < \infty.$$

Consequently, the result of the theorem follows by Corollary 4.6 and the dominated convergence theorem [Loève (1963), p. 125]. ||

Finally, we prove Statement (4.7).

Theorem 4.9. Let $r = 1, \ldots, m$, and t, $\beta \in (0, \infty)$. Then $\lim E\{X_{n,r}(t)\}^\beta = E\{Y_r(t)\}^\beta$.

Proof. Note that $E\{X_{n,r}(t)\}^\beta =$
$= b_r + \sum_{q=1}^\infty [(b_r+q)^\beta - (b_r+q-1)^\beta] P\{X_{n,r}(t) - b_r \geq q\}$, that by Inequality (4.14) $P\{X_{n,r}(t) - b_r \geq q\} \leq P\{X_n(t) - b \geq q\} \leq (1 - e^{-2\alpha bt})^q$, $q = 1, 2, \ldots,$ and that $\sum_{q=1}^\infty [(b_r+q)^\beta - (b_r+q-1)^\beta](1 - e^{-2\alpha bt})^q \leq (2b_r)^\beta \sum_{q=1}^\infty q^\beta (1 - e^{-2\alpha bt})^q < \infty$.

Consequently, the result of the theorem follows by Corollary 4.6 and the dominated convergence theorem. ||

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20. ABSTRACT

A population of susceptible individuals partitioned into m groups and exposed to a contagious disease is considered. It is assumed that an individual's susceptibility at time t depends on the number of susceptible individuals at time t in his group, and on the total number of infective individuals at time t.

The progress of this simple epidemic is modeled by an m-dimensional stochastic process. The components of this stochastic process represent the number of infective individuals in the respective groups at time t. Exact and approximate formulas for the joint and marginal state probabilities are obtained. It is shown that the approximate formulas are simple functions of time while, the derivations of the exact formulas involve tedious computations.