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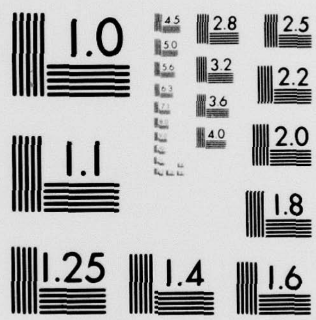

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ON THE PERFORMANCE OF LEAST SQUARES ESTIMATORS IN TYPE II CENSORED ACCELERATED LIFE TESTS

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(6) On the Performance of Least Squares Estimators in Type II Censored Accelerated Life Tests.

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ABSTRACT

In context of the analysis of accelerated life testing experiments where items are tested under more severe stress conditions than those arising in normal use, the least squares approach is noted for its simplicity. We consider a parametric family of life distributions with the pdf $\delta^{-1} f_\eta(z/\delta)$, $0 < z < \infty$, where the scale parameter is related to p stress variables as $\delta = \exp(\beta_0 + \beta_1 x_1 + \ldots + \beta_p x_p)$, and the shape parameter η is assumed to be independent of the x's. Expressions are derived for the asymptotic efficiency of the least squares estimators when the failure time data are type II censored. Applications of the general results are made to the Weibull and exponential models and the loss of efficiency is investigated in relation to the severity of censoring and the nature of spread of the design points.

KEY WORDS
Accelerated testing
Censored data
Asymptotic efficiency

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1. INTRODUCTION

Applications of the least squares method have been widely explored by Nelson and Hahn [10, 11], Mann, et al [7] and others in context of accelerated life testing experiments where items are tested under stress conditions more severe than those under normal use. Similar analyses are also useful in clinical trials where larger than normal doses of chemical agents are administered in order to expedite the response, see for instance, Prentice and Shillington [13] and Feigl and Zelen [5]. Aside from a rapidly growing literature on nonparametric approach, various parametric families including the exponential and Weibull distributions are used as the basic failure time distribution, and functional relationships are assumed between their parameter(s) and the concomitant (stress) variables. The object of this paper is to study the performance of the least squares estimators where the failure time data are type II censored, that is, a predetermined number of order statistics are recorded at each test condition.

We concentrate on a general model which assumes that the dependent variable Y (typically, logarithm of the failure time Z) has a specified probability density function (pdf) with a location parameter λ which depends on p stress variables $x_1, \ldots, x_p$ by a linear relation $\lambda = \beta_0 + \beta_1 x_1 + \ldots + \beta_p x_p$, and a scale parameter η which is independent of the x's. This includes some popular models in the engineering fields such as the Arrhenius, Eyring, generalized Eyring and the inverse power law where the scale parameter δ of Z is taken as $\delta = \exp(\alpha+\beta x)$, $x^{-1}\exp(\alpha+\beta x)$, $Ax_2^{-1}\exp(B/x_2)\exp(Cx_1+Dx_1/x_2)$ and $(\alpha x^\beta)^{-1}$, respectively, and $\log \delta = \lambda$.

We are concerned here with a balanced design and equal type II censoring. That is, the experiment consists of taking independent random samples of size k at each of the n stress settings; at the i-th setting $(x_{1i}, \ldots, x_{pi})$, k items are put on test and only the first $r(\leq k)$ ordered failure times $Z_{i(1)} \leq Z_{i(2)} \leq \cdots \leq Z_{i(r)}$ are recorded. Defining $Y_{i(j)} = \log Z_{i(j)}$, $V_{i(j)} = (Y_{i(j)} - \lambda_i)/\eta$, $\underset{\sim}{\beta}' = (\beta_0, \ldots, \beta_p)$, $\underset{\sim}{x}_i' = (1, x_{1i}, \ldots, x_{pi})$, and $\lambda_i = \underset{\sim}{x}_i'\underset{\sim}{\beta}$, we have the representation

$$Y_{i(j)} = \lambda_i + \eta V_{i(j)} = \underset{\sim}{x}_i'\underset{\sim}{\beta} + \eta V_{i(j)} ; \qquad j = 1, \ldots, r,\ i = 1, \ldots, n, \tag{1.1}$$

where $V_{i(j)}$, $j = 1, \ldots, r$ are the (unobserved) order statistics from the distribution of $V = (Y - \lambda)/\eta$. Since this distribution is completely specified, the moments $\mu_j \equiv EV_{i(j)}$ and $\sigma_{jj'} = \mathrm{cov}(V_{i(j)}, V_{i(j')})$ are known quantities.

Linear estimation techniques for the censored regression situation described above has been extensively developed in Nelson and Hahn [10,11] as computationally simpler competitors of the maximum likelihood method. One simple approach consists of two main steps: The first is to obtain the best linear unbiased estimators (BLUE) of λ_i and η based on the i-th set of ordered observations $Y_{i(1)}, \ldots, Y_{i(r)}$ and the linear model $EY_{i(j)} = \lambda_i + \eta\mu_j$, $j = 1, \ldots, r$. These estimators are given by

$$\tilde{\lambda}_i = \sum_{j=1}^{r} a_j Y_{i(j)} , \quad \tilde{\eta}_{io} = \sum_{j=1}^{r} b_j Y_{i(j)} \tag{1.2}$$

and their covariance matrix is of the form

$$\eta^2 \begin{pmatrix} d_1 & d_3 \\ d_3 & d_2 \end{pmatrix} . \tag{1.3}$$

The coefficients a_j, b_j as well as the quantities d_1, d_2 and d_3 depend on the known constants μ_j, $\sigma_{jj'}$, $j, j' = 1, \ldots, r$, and are tabulated for some distributions. For explicit formulas, see for instance, David [3]. The second step involves a consideration of the linear model $\underset{\sim}{\tilde{\lambda}} = \underset{\sim}{X}\underset{\sim}{\beta} + \underset{\sim}{e}$ where $\underset{\sim}{\tilde{\lambda}}' = (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n)$, $X' = (\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_n)$, and $\underset{\sim}{e}$ is a vector of iid random errors with means 0 and variances $\sigma^2 = d_1\eta^2$. From this structure, the BLUE of $\underset{\sim}{\beta}$ is then obtained as $\underset{\sim}{\tilde{\beta}}_n = \underset{\sim}{S}^{-1}\underset{\sim}{X}'\underset{\sim}{\tilde{\lambda}}$ where $\underset{\sim}{S} = \underset{\sim}{X}'\underset{\sim}{X}$, and the pooled estimator $\tilde{\eta}_n$ of η is taken as $\tilde{\eta}_n = n^{-1}\sum_{i=1}^{n} \tilde{\eta}_{io}$.

The most appealing feature of these estimators is that they are in simple closed forms as opposed to the maximum likelihood estimators which typically involve iterative solutions of complex equations where the computational difficulty rapidly increases with the number of concomitant variables in the model. It is therefore of considerable interest to examine the performance of these least squares estimators (LSE) which we do here by evaluating their asymptotic efficiency (AE). In Section 2, we derive expressions for the asymptotic efficiencies of the LSE for the general situation where the distribution of log-life constitutes a location-scale family with p concomitants. These results are employed in Section 3 to numerically evaluate the AE's for two important special cases, namely, the Weibull and the exponential models. We conclude with some comments in Section 4 on the scope and implications of the AE results vis-a-vis other variants of the estimation techniques.

2. ASYMPTOTIC DISTRIBUTION AND EFFICIENCY

We denote the parameter vector $\underset{\sim}{\theta}' = (\underset{\sim}{\beta}', \eta)$ and the two-stage least squares estimators described in Section 1 as $\underset{\sim}{\tilde{\theta}}_n' = (\underset{\sim}{\tilde{\beta}}_n', \tilde{\eta}_n)$. For the purposes of obtaining the limiting distribution and explicit expressions for the asymptotic efficiencies, the design matrix $\underset{\sim}{X}$ is assumed to be such that as $n \to \infty$ the limits $\lim n^{-1}\underset{\sim}{X}'\underset{\sim}{1} \equiv \underset{\sim}{c}$ and $\lim n^{-1}\underset{\sim}{S} = \underset{\sim}{B}$ exist and $\underset{\sim}{B}$ is non-singular. Let $\underset{\sim}{B}^{-1}$ be partitioned as

$$\underset{\sim}{B}^{-1} = \begin{pmatrix} b^{00} & \underset{\sim}{b}^{0p'} \\ \underset{\sim}{b}^{0p} & \underset{\sim}{B}^{pp} \end{pmatrix} \qquad (2.1)$$

where $\underset{\sim}{B}^{pp}$ is a symmetric $p \times p$ matrix. Since $(\tilde{\lambda}_i, \tilde{n}_{io}), i = 1,\ldots,n$ are independently distributed with the linear structure $\tilde{\underset{\sim}{\lambda}} = \underset{\sim}{X}\underset{\sim}{\beta}+\underset{\sim}{e}_1$, $\tilde{\underset{\sim}{n}}_o = n\underset{\sim}{1}+\underset{\sim}{e}_2$, the limiting joint normality of the LSE's $\tilde{\underset{\sim}{\beta}}_n$ and $\tilde{n}_n$ follows from an application of the multivariate central limit theorem along the lines of Eicker [4]. Specifically, $n^{1/2}(\tilde{\underset{\sim}{\theta}}_n - \underset{\sim}{\theta})$ has the limiting $(p+2)$-variate normal distribution $N_{p+2}(\underset{\sim}{0}, \underset{\sim}{\Sigma}_1)$ where

$$\underset{\sim}{\Sigma}_1 = n^2 \begin{pmatrix} d_1 b^{00} & d_1 \underset{\sim}{b}^{0p'} & d_3 \\ d_1 \underset{\sim}{b}^{0p} & d_1 \underset{\sim}{B}^{pp} & \underset{\sim}{0} \\ d_3 & \underset{\sim}{0}' & d_2 \end{pmatrix} . \qquad (2.2)$$

In order to obtain expressions for the asymptotic efficiencies of the LSE's we next proceed to derive the limiting covariance matrix of the maximum likelihood estimators (MLE). The log-likelihood function of the censored samples $\underset{\sim}{Y}' = (\underset{\sim}{Y}_1', \ldots, \underset{\sim}{Y}_n')$ with $\underset{\sim}{Y}_i' = (Y_{i(1)}, \ldots, Y_{i(r)})$ is given by $\ell_n = \sum_{i=1}^{n} \log g^*(\underset{\sim}{Y}_i; \underset{\sim}{\theta})$ where

$$g^*(\underset{\sim}{Y}_i;\underset{\sim}{\theta}) = \frac{k!}{(k-r)!}\left[\prod_{j=1}^{r} \frac{1}{n} g\left(\frac{Y_{i(j)} - \underset{\sim}{x}_i'\underset{\sim}{\beta}}{n}\right)\right]\left[1 - G\left(\frac{Y_{i(r)} - \underset{\sim}{x}_i'\underset{\sim}{\beta}}{n}\right)\right]^{k-r}, \qquad (2.3)$$

and $g(\cdot)$ and $G(\cdot)$ respectively denote the pdf and cdf of $V = (Y-\lambda)/n$ so the components of $\underset{\sim}{Y}_i$ are the order statistics from the pdf $n^{-1}g[(y-\underset{\sim}{x}_i'\underset{\sim}{\beta})/n]$. We now introduce a few notations:

$$V_{i(j)}(\underset{\sim}{\theta}) = (Y_{i(j)} - \underset{\sim}{x}_i'\underset{\sim}{\beta})/n, \quad u(v) = -d \log g(v)/dv,$$
$$h(v) = -d \log[1-G(v)]/dv \quad u(v) = du(v)/dv, \; h'(v) = dh(v)/dv$$

$$Q_1(\underset{\sim}{V}_i(\underset{\sim}{\theta})) = \sum_{j=1}^{r} u(V_{i(j)}(\underset{\sim}{\theta})) + (k-r)h(V_{i(r)}(\underset{\sim}{\theta}))$$
$$Q_2(\underset{\sim}{V}_i(\underset{\sim}{\theta})) = \sum_{j=1}^{r} V_{i(j)}(\underset{\sim}{\theta})u(V_{i(j)}(\underset{\sim}{\theta})) + (k-r)V_{i(r)}(\underset{\sim}{\theta})h(V_{i(r)}(\underset{\sim}{\theta}))$$
$$Q_{11}(\underset{\sim}{V}_i(\underset{\sim}{\theta})) = \sum_{j=1}^{r} u'(V_{i(j)}(\underset{\sim}{\theta})) + (k-r)h'(V_{i(r)}(\underset{\sim}{\theta})) \qquad (2.4)$$
$$Q_{12}(\underset{\sim}{V}_i(\underset{\sim}{\theta})) = \sum_{j=1}^{r} V_{i(j)}(\underset{\sim}{\theta})u'(V_{i(j)}(\underset{\sim}{\theta})) + (k-r)V_{i(r)}(\underset{\sim}{\theta})h'(V_{i(r)}(\underset{\sim}{\theta}))$$
$$Q_{22}(\underset{\sim}{V}_i(\underset{\sim}{\theta})) = \sum_{j=1}^{r} V_{i(j)}^2(\underset{\sim}{\theta})u'(V_{i(j)}(\underset{\sim}{\theta})) + (k-r)V_{i(r)}^2(\underset{\sim}{\theta})h'(V_{i(r)}(\underset{\sim}{\theta})) .$$

Then the likelihood equations are

$$\frac{\partial \ell_n}{\partial \underset{\sim}{\beta}} = \frac{1}{n}\sum_{j=1}^{n} \underset{\sim}{x}_i Q_1(\underset{\sim}{V}_i(\underset{\sim}{\theta})) = \underset{\sim}{0}$$
$$\frac{\partial \ell_n}{\partial n} = \frac{1}{n}\sum_{i=1}^{n} [-r + Q_2(\underset{\sim}{V}_i(\underset{\sim}{\theta}))] = 0 , \qquad (2.5)$$

and the second partial derivatives of ℓ_n are given by

$$\left(\frac{\partial^2 \ell_n}{\partial \beta_\ell \partial \beta_{\ell'}}\right) = -\frac{1}{n^2}\sum_{i=1}^{n} \underset{\sim}{x}_i \underset{\sim}{x}_i' Q_{11}(\underset{\sim}{V}_i(\underset{\sim}{\theta}))$$
$$\frac{\partial^2 \ell_n}{\partial \underset{\sim}{\beta} \partial n} = -\frac{1}{n^2}\sum_{i=1}^{n} \underset{\sim}{x}_i [Q_1(\underset{\sim}{V}_i(\underset{\sim}{\theta})) + Q_{12}(\underset{\sim}{V}_i(\underset{\sim}{\theta}))] \qquad (2.6)$$
$$\frac{\partial^2 \ell_n}{\partial n^2} = -\frac{1}{n^2}\sum_{i=1}^{n} [-r + Q_{22}(\underset{\sim}{V}_i(\underset{\sim}{\theta})) + 2Q_2(\underset{\sim}{V}_i(\underset{\sim}{\theta}))] .$$

The asymptotic theory of MLE under general situations of independent and non-identically distributed observations has been treated in Hoadley [6], Philippou and Roussas [12], among others. However, the linear regression and constant scale structure of our model permits relaxation of some of their moment conditions and a simplification of the derivations. Bhattacharyya and Soejoeti [2] contains the relevant developments for the uncensored case, and only with minor modifications the proofs extend to the type II censoring scheme considered here. Some mild regularity conditions entail that for a consistent sequence of roots $\{\hat{\theta}_n\}$ of the likelihood equations, the limiting distribution of $n^{1/2}(\hat{\theta}_n-\theta)$ is $N_{p+2}(0,\Sigma)$ where

$$\Sigma = n^2\begin{pmatrix} i_{11}B & i_{12}c \\ i_{12}c' & i_{22}\end{pmatrix}^{-1}$$

$$i_{11} = EQ_{11}(V) \ , \quad i_{12} = E[Q_1(V) + Q_{12}(V)] \qquad (2.7)$$

$$i_{22} = -r + E[Q_{22}(V) + 2Q_2(V)] \ ,$$

and the r-component vector V is distributed as the first r order statistics of a random sample of size k from the pdf $g(\cdot)$. Using the partitioned form (2.1) of B^{-1} and a result on the inverse of a partitioned matrix (c.f. Rao [16], pp 33), we obtain the following explicit form

$$\Sigma = n^2\begin{pmatrix} \gamma_{11} & \Gamma_{12}' & \gamma_{13} \\ \Gamma_{12} & \Gamma_{22} & 0 \\ \gamma_{13} & 0' & \gamma_{33}\end{pmatrix} \qquad (2.8)$$

where

$$\gamma_{11} = i_{11}^{-1}(b^{00} + i_{12}^2\Delta^{-1}) \ , \quad \Delta = i_{11}i_{22} - i_{12}^2$$
$$\Gamma_{12} = b^{0p}i_{11}^{-1} \ , \quad \gamma_{13} = -i_{12}\Delta^{-1} \qquad (2.9)$$
$$\Gamma_{22} = i_{11}^{-1}B^{pp} \ , \quad \gamma_{33} = i_{11}\Delta^{-1} \ .$$

From (2.2), (2.8) and (2.9), the asymptotic efficiencies of the individual LSE's are obtained as

$$AE(\tilde{\beta}_0) = (d_1 i_{11} b^{00})^{-1}(b^{00} + i_{12}^2\Delta^{-1})$$
$$AE(\tilde{\beta}_\ell) = (d_1 i_{11})^{-1} \ , \quad \ell = 1,\dots,p \qquad (2.10)$$
$$AE(\tilde{\eta}) = i_{11}(d_2\Delta)^{-1} \ .$$

As for the vector LSE $\tilde{\theta}_n$, we consider the asymptotic efficiency as the inverse ratio of the asymptotic generalized variances, that is,

$$AE(\tilde{\theta}) = |\Sigma|/|\Sigma_1| = [|\Sigma_1|\cdot|\Sigma^{-1}|]^{-1}$$
$$= [(d_1 i_{11})^p\Delta(d_1d_2-d_3^2)]^{-1} \ . \qquad (2.11)$$

We note that $AE(\beta_0)$ depends on the design matrix X through the element b^{00} of B^{-1}, the limit of nS^{-1}, whereas the others are independent of X.

3. APPLICATIONS

Here we present some applications of the general results of the previous section to the Hiebull and exponential distributions which are the two most widely used models for failure times. Although exponential is a special case of the Weibull, it is separately treated to bring out an interesting feature that an initial reduction of data by sufficiency leads to an improvement over the previously discussed LSE. Although our general expressions permit any number of concomitant variables in the model, for simplicity of numerical illustrations, we limit discussions to the case of a single variable x whose values are denoted by $x_1,\dots,x_n$. The limits of $\Sigma x_i/n$ and $\Sigma(x_i-\bar{x})^2/n$ as $n\to\infty$ are denoted by m_1 and m_2, respectively.

3.1 WEIBULL DISTRIBUTION

If the failure time Z has a Weibull distribution with $\text{pdf} = z^{1/\eta-1}\exp[-(z/\delta)^{1/\eta}]$, then the distribution of $Y = \log Z$ constitutes a location-scale family. Also, with $\delta = \exp(\beta_0+\beta_1 x)$, the distribution of $V = (Y-\beta_0-\beta_1 x)/\eta$ has the pdf

$$g(v) = \exp(v)\exp[-\exp(v)] \tag{3.1}$$

which is a standard extreme value distribution. For this distribution, tables of the coefficients a_j, b_j, and d_i, $i = 1,2,3$ defined in (1.2) and (1.3) are available in Nelson and Hahn [9] for some sample sizes. From (2.4) and (3.1) we have $u(v) = -1 + \exp(v)$, $h(v) = \exp(v)$, and hence

$$\begin{aligned} i_{11} &= E[\sum_{j=1}^{r} \exp(V_{(j)}) + (k-r)\exp(V_{(r)})] \\ i_{12} &= E[\sum_{j=1}^{r} \exp(V_{(j)}) + (k-r)\exp(V_{(r)}) \\ &\quad + \sum_{j=1}^{r} V_{(j)}\exp(V_{(j)}) + (k-r)V_{(r)}\exp(V_{(r)})] \\ i_{22} &= E[-r-2\sum_{j=1}^{r} V_{(j)} + 2(\sum_{j=1}^{r} V_{(j)}\exp(V_{(j)}) + (k-r)V_{(r)}\exp(V_{(r)})) \\ &\quad + \sum_{j=1}^{r} V_{(j)}^2\exp(V_{(j)}) + (k-r)V_{(r)}^2\exp(V_{(r)})] . \end{aligned} \tag{3.2}$$

The expected values $EV_{(j)} = \mu_j$ for the extreme value distribution (3.1) are tabulated in White [17]. Since $\exp(V)$ has the exponential distribution, $\sum_{j=1}^{r} \exp(V_{(j)}) + (k-r)\exp(V_{(r)})$ is distributed as gamma and hence $i_{11} = r$. The expressions for i_{12} and i_{22} involve the quantities

$$B_j \equiv E[V_{(j)}\exp(V_{(j)})] , \quad D_j = E[V_{(j)}^2\exp(V_{(j)})] . \tag{3.3}$$

In order to calculate these we note that the pdf of $W_j \equiv \exp(V_{(j)})$ is

$$f_j(w) = \frac{k!}{(j-1)!(k-j)!} \sum_{\ell=0}^{j-1} (-1)^{\ell}\binom{j-1}{\ell} e^{-(k-j+\ell+1)w} ,$$

and $B_j = \int_0^{\infty} w(\log w) f_j(w)dw$, $D_j = \int_0^{\infty} w(\log w)^2 f_j(w)dw$. Relating these to the derivatives of the gamma integral, the following expressions are obtained

$$\begin{aligned} B_j &= \frac{k!}{(j-1)!(k-j)!} \sum_{\ell=0}^{j-1} (-1)^{\ell}\binom{j-1}{\ell} \frac{[\Gamma'(2)-\log(k-j+\ell+1)]}{(k-j+\ell+1)^2} \\ D_j &= \frac{k!}{(j-1)!(k-j)!} \sum_{\ell=0}^{j-1} (-1)^{\ell}\binom{j-1}{\ell} \\ &\quad \times \frac{[\Gamma''(2)-2\Gamma'(2)\log(k-j+\ell+1)+\log^2(k-j+\ell+1)]}{(k-j+\ell+1)^2} . \end{aligned} \tag{3.4}$$

Table 1 provides the numerical values of B_j and D_j for sample sizes $k \leq 10$, and these are used to compute i_{12} and i_{22}. Finally, expressions (2.10) and (2.11) along with $b^{00} = 1+m_1^2/m_2$ are used to compute the asymptotic efficiencies $AE(\tilde{\theta})$, $AE(\tilde{\beta}_1)$ and $AE(\tilde{\eta})$. These are presented in Table 2 for replication sizes $k \leq 10$. The numerical values indicate that serious losses of efficiency could result when either the replication size k is too small or r is too small compared to k, that is, when the censoring is severe. For these situations, the simplicity of the LSE ceases to be an overriding consideration. A close examination of Table 2 reveals an interesting feature as to the manner in which the AE changes with r for a fixed k. For $k \geq 8$, $AE(\tilde{\theta})$ attains its maximum value at an $r < k$. Likewise, $AE(\tilde{\beta}_1)$ has its maximum at an $r < k$ when $k \geq 6$, whereas $AE(\tilde{\eta})$ increases with r for all $k \leq 10$.

Aside from depending on (r,k), $AE(\tilde{\beta}_0)$ also depends on the location and spread of the design points through the ratio $\rho = m_1^2/m_2$ which is scale invariant but depends on the origin. For a fixed (r,k), viewing $AE(\tilde{\beta}_0)$ as a function of ρ, it is easy to see that $\frac{d}{d\rho} AE(\tilde{\beta}_0) < 0$. So, the minimum and maximum of $AE(\tilde{\beta}_0)$ are associated with $\rho \to \infty$ and $\rho \to 0$, respectively. In order to further relate its behavior to the manner of spread of the design points, we consider an illustrative situation where $x_1, x_2, \ldots$ are generated from a beta distribution with pdf $\propto x^{\omega_1 - 1}(1-x)^{\omega_2 - 1}$, $0 < x < 1$ in which case $\rho = \omega_1(\omega_1+\omega_2+1)/\omega_2$. For different choices of (ω_1, ω_2) a wide variety of shapes of the design scatter can be realized. For instance, if ω_1 is fixed and $\omega_2 \to 0$, we have $\rho \to \infty$ and $AE(\tilde{\beta}_0) \to (i_{11}d_1)^{-1}$. This corresponds to $m_1 \to 1$, $m_2 \to 0$, so the x's are concentrated at 1. When $\rho \to 0$, we have $AE(\tilde{\beta}_0) \to i_{22}/(d_1\Delta)$. This happens, for instance, when ω_2 is fixed and $\omega_1 \to 0$, so $m_1 \to 0$, $m_2 \to 0$ and the x's are concentrated at 0.

Another interesting situation arises when $\omega_1 = \omega_2 = \omega$, that is, the beta distribution is symmetric with $m_1 = .5$, $m_2 = 1/4(2\omega+1)$ and

$\rho = 2\omega+1$. For $\omega = 1$ we have Uniform (0,1) and $\rho = 3$. For $\omega \to 0$ the x's are equally divided at 0 and 1, and $\rho \to 1$. As for $\omega \to \infty$ the x's are concentrated at .5, $\rho \to \infty$, and so $AE(\tilde{\beta}_0) \to (i_{11}d_1)^{-1}$. Table 3 provides (limiting) values of $AE(\tilde{\beta}_0)$ for the special cases I. $\omega_1 = \omega_2 = 1$, II. $\omega_1 = \omega_2 \to 0$, and III. $\omega_1 \to 0$, ω_2 fixed. As in Table 2, here the AE is very low if r/k is small, it increases as r/k increases. For Cases I and II the AE reaches its maximum value at an $r < k$, if $k \geq 6$ for Case I and $k \geq 9$ for Case II. As for Case III, the AE increases with r for all $k \leq 10$. Finally, for complete samples $(r = k)$, the AE increases as k increases in all three cases.

TABLE 1 — Values of B_j and D_j for replication sizes $k \leq 10$.

k	j	B_j	D_j
2	1	-.13518142	.35901503
	2	.98075009	1.28834615
3	1	-.22527598	.36722581
	2	.04500770	.34259349
	3	1.44862129	1.76122249
4	1	-.24087751	.39332139
	2	-.17847142	.28893905
	3	.26848682	.39624792
	4	1.84199945	2.21621401
5	1	-.23733072	.41061614
	2	-.25506467	.32414239
	3	-.06358154	.23613405
	4	.48986573	.50299050
	5	2.18003288	2.64451989
6	1	-.22816252	.41983782
	2	-.28317168	.36450777
	3	-.19885065	.24341162
	4	.07168757	.22885648
	5	.69895481	.64005751
	6	2.47525849	3.04541236
7	1	-.21758940	.42354946
	2	-.29160125	.39756796
	3	-.26209777	.28185724
	4	-.11452115	.19215090
	5	.21134412	.25638557
	6	.89399908	.79352634
	7	2.73995673	3.42072668
8	1	-.20708215	.42368089
	2	-.29114016	.42262950
	3	-.29298451	.32238335
	4	-.21061986	.21435574
	5	-.01842246	.16998801
	6	.34920406	.30822407
	7	1.07559744	.95529377
	8	2.97772234	3.77293139
9	1	-.19716003	.42150802
	2	-.28645914	.44106383
	3	-.30752371	.35810934
	4	-.26390609	.25093137
	5	-.14401208	.16854170
	6	.09204923	.17114514
	7	.49278147	.37676354
	8	1.24497341	1.12058812
	9	3.19431596	4.10447430
10	1	-.18798008	.41785849
	2	-.27977959	.45435380
	3	-.31317736	.38790392
	4	-.29433188	.28858864
	5	-.21826741	.19444547
	6	-.06975675	.14263792
	7	.18325322	.19014996
	8	.61115072	.45674078
	9	1.40342908	1.28654996
	10	3.39330339	4.41757700

TABLE 2 — Asymptotic efficiencies for the Weibull model.

k	r	$AE(\tilde{\theta})$	$AE(\tilde{\beta}_1)$	$AE(\tilde{\eta})$
2	2	.248	.758	.427
3	2	.169	.546	.402
	3	.404	.827	.588
4	2	.113	.375	.390
	3	.368	.770	.570
	4	.450	.852	.675
5	2	.083	.279	.383
	3	.296	.630	.560
	4	.498	.857	.664
	5	.562	.864	.730
6	2	.065	.223	.379
	3	.237	.511	.552
	4	.444	.772	.656
	5	.581	.894	.724
	6	.607	.872	.768
7	2	.054	.186	.376
	3	.195	.423	.547
	4	.383	.671	.650
	5	.549	.850	.718
	6	.637	.912	.766
	7	.660	.877	.796
8	2	.047	.161	.373
	3	.165	.360	.544
	4	.329	.580	.645
	5	.497	.775	.713
	6	.622	.893	.761
	7	.698	.921	.796
	8	.684	.880	.818
9	2	.041	.142	.372
	3	.152	.313	.541
	4	.286	.506	.642
	5	.444	.696	.709
	6	.577	.842	.748
	7	.670	.918	.786
	8	.705	.926	.818
	9	.689	.883	.835
10	2	.037	.128	.370
	3	.126	.277	.539
	4	.251	.447	.639
	5	.396	.623	.706
	6	.537	.777	.753
	7	.647	.884	.789
	8	.711	.932	.817
	9	.729	.928	.837
	10	.708	.885	.849

TABLE 3 — (Limiting) values of $AE(\tilde{\beta}_0)$ for the Weibull model with x's generated from beta (ω_1,ω_2).

I: $\omega_1 = \omega_2 = 1$, II: $\omega_1 = \omega_2 \to 0$, III: ω_2 fixed, $\omega_1 \to 0$

k	r	I	II	III
2	2	.760	.762	.767
3	2	.546	.547	.557
	3	.850	.872	.917
4	2	.384	.394	.413
	3	.770	.770	.770
	4	.875	.898	.944
5	2	.298	.317	.355
	3	.635	.641	.653
	4	.858	.860	.863
	5	.888	.911	.958
6	2	.249	.275	.327
	3	.528	.545	.580
	4	.774	.776	.781
	5	.898	.901	.908
	6	.895	.919	.966
7	2	.212	.249	.312
	3	.451	.480	.536
	4	.682	.694	.717
	5	.850	.851	.852
	6	.918	.923	.933
	7	.901	.924	.972
8	2	.196	.232	.302
	3	.397	.434	.509
	4	.603	.626	.672
	5	.782	.788	.801
	6	.893	.893	.893
	7	.921	.921	.922
	8	.904	.928	.976
9	2	.181	.219	.297
	3	.357	.402	.491
	4	.550	.594	.682
	5	.714	.732	.769
	6	.845	.848	.854
	7	.918	.919	.920
	8	.934	.942	.959
	9	.907	.932	.980
10	2	.169	.210	.293
	3	.327	.378	.479
	4	.490	.533	.618
	5	.650	.677	.731
	6	.788	.799	.821
	7	.886	.887	.890
	8	.934	.935	.937
	9	.937	.947	.966
	10	.909	.933	.981

3.2 EXPONENTIAL DISTRIBUTION

This is a special case of the Weibull model where the shape parameter $\eta = 1$ so the two-stage least squares estimation involves one fewer regression parameter at each stage as compared to the Weibull situation. Specifically, in terms of $V_{i(j)} = \log Z_{i(j)}$, we have the linear model $Y^*_{i(j)} = (Y_{i(j)} - \mu_j) = \lambda_i + e_j$, $j = 1,\ldots,r$ where μ_j is as defined in Subsection 3.1. Having estimated λ_i by linear function of order statistics at the i-th design point, $i = 1,\ldots,n$, the second stage proceeds with the model $\tilde{\lambda}_i = \beta_0 + \beta_1 x_i + e_i'$, $i = 1,\ldots,n$. The details being analogous to the Weibull case are omitted and we only give the expressions for the AE's of $\tilde{\theta}' = (\tilde{\beta}_0, \tilde{\beta}_1)$: $AE(\tilde{\theta}) = (r\tau^2)^{-2}$, $AE(\tilde{\beta}_0) = AE(\tilde{\beta}_1) = (r\tau^2)^{-1}$ where

$$\tau^2 = \Big(\sum_{j=1}^{r}\sum_{j'=1}^{r}\sigma^{jj'}\Big)^{-1},\quad (\sigma^{jj'}) = (\sigma_{jj'})^{-1} \text{ and } \sigma_{jj'} = \mathrm{cov}(V_{(j)}, V_{(j')}).$$

Table 4 gives the numerical values of $AE(\tilde{\beta}_0)$ for $r \le k = 6$ which have been computed by using Table 12C.2 in Sarhan and Greenberg [14].

TABLE 4 — $AE(\tilde{\beta}_0)$ for exponential model, $r \le k = 6$.

r	1	2	3	4	5	6
$AE(\tilde{\beta}_0)$	.608	.775	.842	.877	.897	.907

The reason we make a special mention of the exponential case is that the two-stage least squares approach outlined in Section 2 and applied to the Weibull model can be readily improved upon by an initial reduction of the data to sufficient statistics, as is considered in Singpurwalla, et al [15]. Denoting, as before, the observed ordered failure times $Z_{i(j)}$, $j = 1,\ldots,r$

at the i-th design point x_i, $i = 1,\ldots,n$, and defining $Z_i^* = \sum_{j=1}^{r} Z_{i(j)} + (k-r)Z_{i(r)}$, note that $\{Z_i^*, i = 1,\ldots,n\}$ are sufficient statistics for our model, and are independently distributed as gamma $(1,r)$. Further, $E \log Z_i^* = \beta_0+\beta_1 x_i+\psi(r)$ and $\mathrm{Var}(\log Z_i^*) = \psi'(r)$ where ψ and ψ' are respectively the digamma and trigamma functions which are tabulated (c.f. Abramowitz and Stegun [1]). The BLUE can now be obtained from the simple linear model

$$\log Z_i^* = [\beta_0+\psi(r)] + \beta_1 x_i + e_i, \quad i = 1,\ldots,n .$$

The AE's of the resulting estimators $\tilde{\theta}^* = (\tilde{\beta}_0^*, \tilde{\beta}_1^*)$ are given by

$$AE(\tilde{\theta}^*) = [r\psi'(r)]^{-2}, \; AE(\tilde{\beta}_0^*) = AE(\tilde{\beta}_1^*) = [r\psi'(r)]^{-1} .$$

Table 5 gives the values of $AE(\tilde{\beta}_0^*)$ for $r \leq 12$. Note that this method provides estimators which have higher asymptotic efficiencies than those obtained by the previous method. Furthermore, to use the previous method, one has first to invert the matrix $(\sigma_{jj'})_{r\times r}$, which may be computationally involved, especially when r is not too small. On the other hand, the present method needs only the tabulated values of the digamma and trigamma functions.

TABLE 5 — $AE(\tilde{\beta}_0^*)$ for exponential model

r	1	2	3	4	5	6
$AE(\tilde{\beta}_0^*)$	.608	.775	.844	.881	.904	.919
r	7	8	9	10	11	12
$AE(\tilde{\beta}_0^*)$	.930	.939	.946	.951	.955	.959

4. CONCLUDING REMARKS

Generally speaking, asymptotic theory is a useful indicator of the results one would expect in practical situations involving large or at least moderately large samples. Some caution is needed in interpreting the asymptotic efficiency comparisons of the present work in so far as our censored regression model includes three sampling 'parameters': the number of stress settings (n), the number of items (k) tested at each setting, and the number of order statistics (r) observed out of k. Since our asymptotic theory is founded on $n \to \infty$ with (r,k) fixed, the results are relevant for an experimental situation where small sets of censored data are recorded from relatively larger number of independent samples corresponding to the stress settings which need not be all distinct. A different type of asymptotics comes into the picture when one deals with a small number of independent samples in each of which the numbers (r,k) are large. For the latter case, results on the asymptotic normality of linear functions of order statistics can be invoked to justify a normal approximation for the first stage estimators $(\tilde{\lambda}_i, \tilde{n}_{io})$ defined in (1.2). As an alternative method of estimation, one may opt for the maximum likelihood estimates $(\hat{\lambda}_i, \hat{n}_{io})$ in the first stage followed by its normal approximation and use of least squares in the second stage. Such mixtures of MLE and LSE have been considered in Nelson [8] and Mann, et al [7] in context of complete as well as censored samples. Our results pertain to the complementary situation where a normal approximation of the initial estimates is not appropriate for reason of small sample sizes.

REFERENCES

[1] ABRAMOWITZ, M. and STEGUN, I. A. (1965). Handbook of Mathematical Functions, National Bureau of Standards Appl. Math. Series 55.

[2] BHATTACHARYYA, G. K. and SOEJOETI, Z. (1979). Asymptotic Normality and Efficiency of Modified Least Squares Estimators in Some Accelerated Life Test Models, to appear in Sankhyā, Series B.

[3] DAVID, H. A. (1970). Order Statistics, John Wiley, New York.

[4] EICKER, F. (1963). Asymptotic Normality and Consistency of the Least Squares Estimators for Families of Linear Regressions. Ann. Math. Statist., 34, 447-456.

[5] FEIGL, P. and ZELEN, M. (1965). Estimation of Exponential Survival Probabilities With Concomitant Information, Biometrics, 21, 943-970.

[6] HOADLEY, B. (1971). Asymptotic Properties of Maximum Likelihood Estimators for the Independent Non Identically Distributed Case. Ann. Math. Statist., 42, 1977-1991.

[7] MANN, N. R., SCHAEFER, R. E. and SINGPURWALLA, N. D. (1974). Methods for Statistical Analysis of Reliability and Life Data, John Wiley, New York.

[8] NELSON, W. B. (1970). Statistical Methods for Accelerated Life Test Data - The Inverse Power Law Model, General Electric Corporate Research and Development TIS Report 71-C-120.

[9] NELSON, W. B. and HAHN, G. J. (1971). Regression Analysis of Censored Data - Linear Estimation Using Order Observations, General Electric Corporate Research and Development TIS Report 71-C-122.

[10] NELSON, W. and HAHN, G. (1972). Linear Estimation of a Regression Relationship from Censored Data Part I — Simple Methods and Their Application, Technometrics, 14, 247-276.

[11] NELSON, W. and HAHN, G. (1973). Linear Estimation of a Regression Relationship from Censored Data Part II — Best Linear Unbiased Estimation and Theory, Technometrics, 15, 133-150.

[12] PHILIPPOU, A. N. and ROUSSAS, G. G. (1975). Asymptotic Normality of the Maximum Likelihood Estimate in the Independent Not Identically Districuted Case, Ann. Inst. Math. Statist., 27, 45-55.

[13] PRENTICE, R. L. and SHILLINGTON, R. (1975). Regression Analysis of Weibull Data and the Analysis of Clinical Trials, Utilitas Mathematica, 8, 257-276.

[14] SARHAN, A. E. and GREENBERG, B. G. Eds. (1962). Contribution to Order Statistics, John Wiley, New York.

[15] SINGPURWALLA, N. D., CASTELLINO, V. C., GOLDSCHEN, D. Y. (1975). Inference from Accelerated Life Tests Using Eyring Type Re-parameterizations, Naval Research Logistic Quarterly, 22, 289-296.

[16] RAO, C. R. (1973). Linear Statistical Inference and Its Applications, 2nd Ed., John Wiley, New York.

[17] WHITE, J. S. (1969). The Moments of the Log-Weibull Order Statistics, Technometrics, 11, 373-386.

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20. In context of the analysis of accelerated life testing experiments where items are tested under more severe stress conditions than those arising in normal use, the least squares approach is noted for its simplicity. We consider a parametric family of life distributions with the pdf $\delta^{-1} f_\eta(z/\delta)$, $0 < z < \infty$, where the scale parameter is related to ρ stress variables as $\delta = \exp(\beta_0 + \beta_1 x_1 + \ldots + \beta_\rho x_\rho)$, and the shape parameter η is assumed to be independent of the x's. Expressions are derived for the asymptotic efficiency of the least squares estimators when the failure time data are type II censored. Applications of the general results are made to the Weibull and exponential models and the loss of efficiency is investigated in relation to the severity of censoring and the nature of spread of the design points.