ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF HYPERBOLIC BALANCE LAWS

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C. M. Dafermos

Lefschetz Center for Dynamical Systems
Division of Applied Mathematics
Brown University
Providence, Rhode Island   02912   USA

We survey the asymptotic behavior of solutions to hyperbolic conservation laws and balance laws, as well as systems of such laws with the property that each characteristic field is either genuinely nonlinear or linearly degenerate.

1.   INTRODUCTION

We discuss the asymptotic behavior, as $t \to \infty$, of solutions to the   problem for systems of strictly hyperbolic balance laws

$$u_t + f(u)_x + g(u,x,t) = 0 \qquad (1.1)$$

in one space variable. The vector field $u(x,t)$ takes values in R^n and the functions $f: R^n \to R^n$ and $g: R^n \times R \times R^+ \to R^n$ are assumed smooth. Strict hyperbolicity means that for each $u \in R^n$ the $n \times n$ matrix $\nabla f(u)$ has n real distinct eigenvalues (characteristic speeds) $\lambda_1(u),\ldots,\lambda_n(u)$.

When the supply term g vanishes, (1.1) reduces to the system of hyperbolic conservation laws

$$u_t + f(u)_x = 0. \qquad (1.2)$$

Solutions of (1.2) vanishing at $x = \pm\infty$ conserve the quantity

$$\int_{-\infty}^{\infty} u(x,t)dx.$$

Even so, L^p norms of $u(x,t)$, $p > 1$, may decay, as $t \to \infty$, provided that the solution is "spreading out". When $f(u)$ is linear, characteristic speeds are constant so that each mode travels with its fixed speed and does not decay. When $f(u)$ is nonlinear, the dependence of characteristic speeds upon u causes the convergence of characteristics so that, even if the initial data $u(x,0)$ are very smooth, smooth solutions generally break down in a finite time and shock waves develop. Shocks and simple waves interact and cancel each other out thus inducing decay. Another indication of decay is provided by "entropy inequalities" [1]

$$\eta(u)_t + q(u)_x \leq 0, \tag{1.3}$$

satisfied by admissible weak solutions of certain systems (1.2), as abstract forms of the second law of thermodynamics. The entropy decay mechanism is particularly effective in the presence of strong shocks.

Since in the linear case there is no dissipation, one should expect that solutions of (1.2) will decay faster when $f(u)$ is "very nonlinear". It turns out that the relevant condition on $f(u)$ is <u>genuine nonlinearity</u> [2],

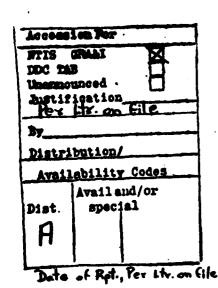$$\nabla \lambda_k(u) \cdot r_k(u) \neq 0, \qquad u \in R^n, \tag{1.4}$$

where $r_k(u)$ is the right eigenvector of $\nabla f(u)$ associated with $\lambda_k(u)$. Condition (1.4) is not always satisfied by all characteristic fields in systems of conservation laws arising in continuum physics. As a matter of fact, there are classical examples (e.g. the system of conservation laws of gas dynamics) where certain characteristic fields are <u>linearly degenerate</u>:

$$\nabla \lambda_j(u) \cdot r_j(u) \equiv 0, \qquad u \in R^n. \tag{1.5}$$

One should not expect decay of components of solutions corresponding to such fields.

Returning to general balance laws (1.1) we remark that whenever the supply term g is dissipative it collaborates with the dissipative mechanism of (1.2) to speed up decay. On the other hand, when g is not dissipative, there is competition with the dissipative mechanism of (1.2) the outcome of which depends on relative strength.

In the case of a single genuinely nonlinear conservation law very precise information is available which is described in Section 2. In Sections 3 and 4 we discuss asymptotic behavior

for the single genuinely nonlinear balance law and the general single conservation law. Known results for systems are surveyed in Sections 5 and 6.

2. SINGLE CONSERVATION LAW. THE CONVEX CASE

In this section we discuss the asymptotic behavior of solutions to the single conservation law

$$u_t + f(u)_x = 0, \quad f(u) \text{ strictly convex.} \qquad (2.1)$$

In the genuinely nonlinear case, $f''(u) > 0$, Lax [2] discovered an explicit solution for the initial value problem of (2.1) which yields precise information on the asymptotic behavior of solutions. A different approach, which also applies when $f''(u)$ may vanish at isolated points, employs the concept of generalized characteristics [3].

The following proposition [3] exhibits the influence on decay of (a) the "strength of convexity" of $f(u)$ and (b) the deployment of initial data.

2.1. <u>Theorem</u>. Assume that

$$c|u|^p \le f''(0) \le C|u|^p, \qquad p \ge 0, \quad 0 < c < C, \qquad (2.2)$$

for u in some neighborhood of 0. Let $u(x,t)$ be an admissible solution of (2.1) on $(-\infty,\infty) \times [0,\infty)$ with

$$\int_x^{x+L} u(y,0)dy = O(L^s), \quad \text{as} \quad L \to \infty, \qquad (2.3)$$

uniformly in $x \in (-\infty,\infty)$, for some $s \in [0,1)$. Then

$$u(x,t) = O(t^{\frac{s-1}{p(1-s)+2-s}}), \quad \text{as} \quad t \to \infty, \qquad (2.4)$$

uniformly in $x \in (-\infty,\infty)$.

2.2. <u>Corollary</u>. Assume that $f(u)$ satisfies (2.2) in some neighborhood of 0 and let $u(x,t)$ be an admissible solution of (2.1) on $(-\infty,\infty) \times [0,\infty)$ with $u(x,0) \in L^q(-\infty,\infty)$, $1 \le q < \infty$. Then

$$u(x,t) = O(t^{-\frac{1}{p+q+1}}), \quad \text{as} \quad t \to \infty, \qquad (2.5)$$

uniformly in $x \in (-\infty,\infty)$.

The variation of a certain function of the solution also decays, namely,

2.3. Theorem. Let $u(x,t)$ be an admissible solution of (2.1) on $(-\infty,\infty) \times [0,\infty)$ with initial data $u(x,0) \in L^1(-\infty,\infty)$. Then, for any $t \in (0,\infty)$,

$$\operatorname*{Var}_{x}\{f(u(x,t)) - u(x,t)f'(u(x,t))\} \leq \frac{2}{t} \int_{-\infty}^{\infty} |u(x,0)|dx. \qquad (2.6)$$

In connection to the above theorem, see the discussion following Theorem 4.3, in Section 4 below.

When the initial data are periodic, one has information on the asymptotic shape of the solution:

2.4. Theorem. Let $u(x,t)$ be an admissible solution of (2.1) on $(-\infty,\infty) \times [0,\infty)$ with initial data $u(x,0)$ periodic of mean $\bar{u}$. Assume $f''(\bar{u}) \neq 0$. Then, as $t \to \infty$, $u(x,t)$ is asymptotic to order $o(t^{-1})$ to a periodic sawtooth function. The number of "teeth" per period equals the number of points in a period interval in which the function

$$\int_0^x [u(y,0) - \bar{u}]dx$$

attains its minimum.

Interesting asymptotic shapes also emerge when initial data have compact support. In that case solutions behave asymptotically as <u>N-waves</u>.

2.5. Theorem. Assume $f''(0) > 0$. Let $u(x,t)$ be an admissible solution of (2.1) on $(-\infty,\infty) \times [0,\infty)$ with initial data of compact support. Then

$$u(x,t) = \begin{cases} 0 & x \leq \eta_-(t) \\ \dfrac{1}{f''(0)} [\dfrac{x}{t} - f'(0)] + 0(\dfrac{1}{t}) & \eta_-(t) < x \leq \eta_+(t) \\ 0 & \eta_+(t) < x \end{cases} \qquad (2.7)$$

where

$$\eta_{\mp}(t) = tf'(0) \mp [2I_{\mp}tf''(0)]^{1/2} + 0(1) \qquad (2.8)$$

with I_-, I_+ constants (invariants of the solution) depending upon the initial data.

When $f''(0)$ vanishes, the N-wave is deformed, as the following example indicates.

2.6. Theorem. Let $u(x,t)$ be an admissible solution on $(-\infty,\infty) \times [0,\infty)$ of the equation

$$u_t + (u^{2k})_x = 0, \qquad k = 1,2,\ldots, \tag{2.9}$$

with initial data of compact support. Then

$$u(x,t) = \begin{cases} 0 & x \le \eta_-(t) \\ (\frac{1}{2k} \frac{x}{t})^{\frac{1}{2k-1}} + O(t^{-\frac{1}{k}}) & \eta_-(t) < x \le \eta_+(t) \\ 0 & \eta_+(t) < x \end{cases} \tag{2.10}$$

where

$$\eta_{\substack{-\\+}}(t) = \mp 2k(\frac{1}{2k-1} I_{\substack{-\\+}})^{\frac{2k-1}{2k}} t^{\frac{1}{2k}} + O(1) \tag{2.11}$$

with I_-, I_+ the same invariants appearing in (2.8).

There are also results [3,4] on the asymptotic behavior of solutions of (2.1) when the initial data approach different constants u_- and u_+ as $x \to -\infty$ and $x \to \infty$. It turns out that the solution approaches the wave fan that solves the Riemann Problem [2] corresponding to (u_-,u_+) .

3. SINGLE BALANCE LAW. THE CONVEX CASE

We now consider the single balance law

$$u_t + f(u)_x + g(u,x,t) = 0, \quad f(u) \text{ strictly convex.} \tag{3.1}$$

Although no analog of Lax's explicit solution of (2.1) is known for (3.1), the method of generalized characteristics [5] applies and yields precise information.

As a first example we consider the case where g is dissipative and we generalize Corollary 2.2.

3.1. Theorem. Assume that $f(u)$ satisfies (2.2) in some neighborhood of 0 and that

$$ug(u,x,t) \geq 0, \quad u \in (-\infty,\infty), \quad x \in (-\infty,\infty), \quad t \in [0,\infty). \tag{3.2}$$

Let $u(x,t)$ be an admissible solution of (3.1) on $(-\infty,\infty) \times [0,\infty)$ with $u(x,0) \in L^q(-\infty,\infty)$, $1 \leq q < \infty$. Then

$$u(x,t) = O(t^{-\frac{1}{p+q+1}}), \quad \text{as} \quad t \to \infty, \tag{3.3}$$

uniformly in $x \in (-\infty,\infty)$.

In the next example g is not necessarily dissipative but it decays at infinity so that the dissipative mechanism of the conservation law prevails.

3.2. Theorem. Assume that

$$|g(u,x,t)| \leq a(x)b(t), \quad u \in (-\infty,\infty), \quad x \in (-\infty,\infty), \quad t \in [0,\infty), \tag{3.4}$$

where $a(x)$ is a bounded function such that $a(x) \to 0$ as $|x| \to \infty$ while $b(t) \in L^1(0,\infty)$. Let $u(x,t)$ be an admissible solution of (3.1) on $(-\infty,\infty) \in [0,\infty)$ with $u(x,0) \in L^1_{loc}(-\infty,\infty)$ such that $u(x,0) \to 0$ as $|x| \to \infty$. Then

$$u(x,t) = o(1), \quad \text{as} \quad t \to \infty, \tag{3.5}$$

uniformly in $x \in (-\infty,\infty)$.

It is also possible to establish decay of the variation of solutions. Here is a typical result for the periodic case.

3.3. Theorem. Consider the balance law

$$u_t + f(u)_x + g(u) = 0 \tag{3.6}$$

with $f''(u) \geq a > 0$, $g(0) = 0$, and $g'(u) \geq 0$. Let $u(x,t)$ be an admissible solution of (3.6) on $(-\infty,\infty) \times [0,\infty)$ with initial data periodic of period T. Then

$$\underset{[0,T]}{\text{Var}} \, u(x,t) \leq \frac{2T}{at}, \quad t \in (0,\infty). \tag{3.7}$$

When the initial data have compact support, the solution attains an asymptotic profile analogous to the N-wave of the conservation law (Theorem 2.5). We present the result in the context of a simple concrete example.

3.4. Theorem. Let $u(x,t)$ be an admissible solution on $(-\infty,\infty) \times [0,\infty)$ of the balance law

$$u_t + (\frac{u^2}{2})_x + u^3 = 0 \qquad (3.8)$$

with initial data of compact support. Then

$$u(x,t) = \begin{cases} 0 & x \leq \eta_-(t) \\ \dfrac{2x}{x^2+2t} + O(\frac{1}{t}) & \eta_-(t) < x \leq \eta_+(t) \\ 0 & \eta_+(t) < x. \end{cases} \qquad (3.9)$$

The fronts $\eta_-(t), \eta_+(t)$ of the wave are determined asymptotically as solutions of an equation which is too complicated to be illuminating.

4. SINGLE CONSERVATION LAW. THE GENERAL CASE

Next we consider the single conservation law

$$u_t + f(u)_x = 0 \qquad (4.1)$$

without any convexity restrictions on $f(u)$. No explicit representation of solutions is known. The method of generalized characteristics works but not as effectively as in the convex case.

In general, decay is to be expected when $f''(u)$ vanishes only at isolated points. The rate of decay will depend on the "flattness" of $f(u)$ at points of inflexion. The following result, established [6,7] by methods of topological dynamics, starts from very weak assumptions on $f(u)$ but, in return, yields no information whatsoever on the rate of decay.

4.1. Theorem. Assume that the set of points on which $f''(u)$ vanishes has no (finite) accumulation point on the real axis. Let $u(x,t)$ be an admissible solution of (4.1) on $(-\infty,\infty) \times [0,\infty)$ with initial data that are L^1-almost periodic on $(-\infty,\infty)$ of mean 0. Then $u(x,t)$ decays to 0, as $t \to \infty$, in $L^1_{loc}(-\infty,\infty)$.

In the test case where $f(u)$ has only one inflexion point, one may derive the asymptotic behavior of solutions from the convex case (Corollary 2.2) by employing the ordering property of

admissible solutions of (4.1), i.e., that $u(x,0) \leq \bar{u}(x,0)$, $x \in (-\infty,\infty)$, implies $u(x,t) \leq \bar{u}(x,t)$, $x \in (-\infty,\infty)$, $t \in [0,\infty)$.

4.2. Theorem. Assume that $f''(u) > 0$ for $u > 0$, $f''(u) < 0$ for $u < 0$ and

$$c|u|^p \leq |f''(u)| \leq C|u|^p, \quad p > 0, \ 0 < c < C, \qquad (4.2)$$

for u in some neighborhood of 0. Let $u(x,t)$ be an admissible solution of (4.1) on $(-\infty,\infty) \times [0,\infty)$ with $u(x,0) \in L^q(-\infty,\infty)$, $1 \leq q < \infty$. Then

$$u(x,t) = O(t^{-\frac{1}{p+q+1}}), \quad \text{as } t \to \infty, \qquad (4.3)$$

uniformly in $x \in (-\infty,\infty)$.

Using a clever scaling argument, Benilan and Crandall [8] obtain a decay estimate for orbits of Lipschitz continuous semigroups generated by homogeneous generators. Since $-f(u)_x$ generates a contraction semigroup on $L^1(-\infty,\infty)$, the above result applies to (4.1), when $f(u)$ is homogeneous, and gives

4.3. Theorem. Assume that for some $\alpha \geq 2$

$$f(\lambda u) = \lambda^\alpha f(u), \quad u \in (-\infty,\infty), \ \lambda \in [0,\infty). \qquad (4.4)$$

Let $u(x,t)$ be an admissible solution of (4.1) on $(-\infty,\infty) \times [0,\infty)$ with $u(x,0) \in L^1(-\infty,\infty)$. Then

$$\operatorname*{Var}_x f(u(x,t)) \leq \frac{2}{(\alpha-1)t} \int_{-\infty}^{\infty} |u(x,0)|\,dx, \quad t \in (0,\infty). \qquad (4.5)$$

In particular, (4.5) implies $|u(x,t)|^\alpha = O(t^{-1})$ which is in agreement with (2.5), (4.3). Note that (4.5) also yields information on the decay of $u_t(x,t)$. In the convex case, $f(u) = |u|^\alpha$, (4.5) reduces to (2.6). It is not known whether (2.6), or any analog of it, holds for general conservation laws.

The decay of solutions with periodic initial data was invesitgated by Greenberg and Tong [9] who obtain the following

4.4. Theorem. Assume that $f(u)$ is as in Theorem 4.2 and let $u(x,t)$ be an admissible solution of (4.1) on $(-\infty,\infty) \times [0,\infty)$ with initial data periodic of mean 0. Then there is a sequence

$\{t_n\}$, with $t_n \to \infty$, such that

$$u(x,t_n) = O(t_n^{-\frac{1}{p+1}}), \quad n \to \infty, \tag{4.6}$$

uniformly in $x \in (-\infty,\infty)$.

The asymptotic shape of periodic solutions, under the assumptions of Theorem 4.4, has also been studied [9,10].

Our understanding of the asymptotic behavior of solutions to the general conservation law (4.1) is still imperfect. Additional research will be required for the completion of the program.

5. SYSTEMS OF CONSERVATION LAWS

All available information on the asymptotic behavior of solutions to systems of conservation laws has been obtained through explicit studies of wave interactions, within the frame-work of the construction scheme of Glimm [11], and is thus restricted to solutions of small oscillation or variation.

We discuss first the case where the system is genuinely nonlinear, i.e., (1.4) is satisfied for $k = 1,\ldots,n$. The crucial observation is that weak waves propagate and interact as waves of a single conservation law, modulo an "error" which is at most of second order in the strength of interacting waves. The general strategy is to show that the cumulative effect of these errors is insignificant and thus decay of waves in systems is governed by precisely the same laws that rule decay of waves in a single conservation law. Systems of two equations are special in that, in Riemann invariants coordinates, interaction errors are actually of third order and this makes the study of decay easier. The earliest result is due to Glimm and Lax [12]:

5.1. Theorem. When the initial data $u(x,0)$ have small oscillation, there is a solution $u(x,t)$ on $(-\infty,\infty) \times [0,\infty)$ to the strictly hyperbolic, genuinely nonlinear system (1.2) of two equations. Furthermore, if $u(x,0)$ is periodic of mean 0,

$$u(x,t) = O(\frac{1}{t}), \quad \text{as} \quad t \to \infty, \tag{5.1}$$

while if $u(x,0)$ has compact support,

$$\operatorname*{Var}_{x} u(x,t) = O(t^{-\frac{1}{2}}), \quad \text{as} \quad t \to \infty. \tag{5.2}$$

Starting from the above result, DiPerna [13] was able to show that in systems of two equations the two characteristic fields decouple at a rate $O(t^{-3/2})$ and each one approaches, at a rate $O(t^{-1/6})$, an individual N-wave of the type exhibited in Theorem 2.5. After a careful analysis, Liu [14] observed that in a genuinely nonlinear system of n equations interactions between waves of different families, which produce errors of second order, are rapidly completed, since these fields propagate with distinct characteristic speeds. On the other hand, interactions between waves of the same family, which persist longer, produce errors of the third order. He thus establishes the following

5.2. Theorem. When the initial data $u(x,0)$ have compact support and small total variation, there is a solution $u(x,t)$ on $(-\infty,\infty) \times [0,\infty)$ to the strictly hyperbolic, genuinely nonlinear system (1.2) of n equations and

$$\text{Var } u(x,t) = O(t^{-\frac{1}{2}}), \quad \text{as} \quad t \to \infty. \tag{5.3}$$
$$x$$

Furthermore, as $t \to \infty$, $u(x,t)$ approaches in L^1 at a rate $O(t^{-1/6})$, a system of n N-waves propagating with speeds $\lambda_1(0),\ldots,\lambda_n(0)$.

It is not known whether the rate $O(t^{-1/6})$ in the above theorem is optimal.

Next, we consider systems with the property that each characteristic field is either genuinely nonlinear or linearly degenerate. DiPerna [15] studied the asymptotic behavior of solutions to systems in the above class that are endowed with an entropy. Subsequently, Liu [16] established more precise results based exclusively upon considerations on wave interactions.

5.3. Theorem. Consider a strictly hyperbolic system of n equations each characteristic field of which is either genuinely nonlinear or linearly degenerate. When the initial data have compact support and small total variation, a global solution to the Cauchy problem exists and, as $t \to \infty$, each genuinely nonlinear characteristic field approaches in L^1, at a rate $O(t^{-1/6})$, an N-wave, while every linearly degenerate characteristic field approaches in L^1, at a rate $O(t^{-1/2})$, a travelling wave (generalized contact).

Liu also considers [16,17,18] initial data which approach different constant states u_- and u_+, as $x \to -\infty$ and $x \to \infty$, and shows that the resulting solution approaches the wave fan

that solves the Riemann problem associated with (u_-, u_+). Thus, the investigation of asymptotic behavior within the framework of small solutions and Glimm's method is virtually complete. In contrast, very little is known about the asymptotic behavior of solutions with large initial data or of solutions to systems that are not genuinely nonlinear.

6. SYSTEMS OF BALANCE LAWS

The investigation of the asymptotic behavior of solutions to systems of balance laws (1.1) has only begun recently. Using an adaptation of the method of Glimm, Liu [19] studies systems of balance laws

$$u_t + f(u)_x + g(u,x) = 0 \qquad \qquad (6.1)$$

under the assumption that g decays rapidly as $|x| \to \infty$ and no characteristic speed is 0. Liu's scheme is based upon an approximate resolution of discontinuities by (6.1), analogous to the Riemann problem solution for (1.2) which is the building block of Glimm's scheme. A typical result is that when the initial data $u(x,0)$ have small total variation and approach constant states u_- and u_+ as $x \to -\infty$ and $x \to +\infty$, there is a solution $u(x,t)$ of (6.1) on $(-\infty,\infty) \times [0,\infty)$ which approaches, as $t \to \infty$, a fan of waves, which do not produce interactions and cancellations and essentially consist of simple and shock waves of the associated conservation law, together with a steady state solution

$$\frac{df(u)}{dx} + g(u,x) = 0 \qquad \qquad (6.2)$$

of (6.1) which occupies the center.

It is not known what happens in the "resonance" situation where one of the characteristic speeds is 0. Clearly, a lot of work is still needed in order to understand completely asymptotic behavior in systems of balance laws.

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