AD-A081 254 VIRGINIA POLYTECHNIC INST AND STATE UNIV BLACKSBURG --ETC F/G 20/4
EFFECT OF STREAMWISE VORTICES ON TOLLMIEN-SCHLICHTING WAVES.(U)
MAR 79 A H NAYFEH N00014-75-C-0381
UNCLASSIFIED VPI-E-79-12 NL

1 OF 1
AD
A081254

END
DATE
FILMED
3-80
DDC

College of Engineering
Virginia Polytechnic Institute
and State University
Blacksburg, Virginia 24061

VPI-E-79-12

EFFECT OF STREAMWISE VORTICES ON TOLLMIEN-SCHLICHTING WAVES

Ali H. Nayfeh

Department of Engineering Science and Mechanics
Virginia Polytechnic Institute and State University
Blacksburg, Virginia 20461

March 1979

A

ONR Contract: NR 061-201
N00014-75-C-0381
NASA Research Grant No.: NSG-1255

Letter on file

A 404 722

EFFECT OF STREAMWISE VORTICES ON TOLLMIEN-SCHLICHTING WAVES

Ali H. Nayfeh

Department of Engineering Science and Mechanics
Virginia Polytechnic Institute and State University
Blacksburg, Virginia 24061

The method of multiple scales is used to determine a first-order uniform expansion for the effect of counter-rotating steady streamwise vortices in growing boundary layers on Tollmien-Schlichting waves. The results show that such vortices have a strong tendency to amplify three-dimensional Tollmien-Schlichting waves having a spanwise wavelength that is twice the wavelength of the vortices. An analytical expression is derived for the growth rates of these waves. These growth rates increase linearly with increasing amplitudes of the vortices.

I. INTRODUCTION

We consider the effect of counter-rotating steady streamwise vortices on the instability of growing boundary layers. We describe a parametric instability mechanism by which such vorticies amplify selected three-dimensional Tollmien-Schlichting waves. To first order, the selected waves have a spanwise wavelength that is twice that of the vortices.

Weak and moderately strong steady streamwise vortices arise abundantly in boundary layers from many causes. In a series of wind-tunnel tests over flat plates with zero-pressure gradient, Klebanoff & Tidstrom (1959) observed steady quasi-periodic variations in the spanwise direction (streamwise vortices) evidently evoked by freestream conditions. Similar vortices were observed in a National Physical Laboratory tunnel specifically designed for the study of two-dimensional boundary layers. Bradshaw (1965) found that these variations may appear downstream of slightly nonuniform settling-chamber damping screens, depending on their solidity. Using the method of matched asymptotic expansions, Crow (1966) inferred the effect of a small, periodic incident transverse flow on the mean boundary layer over a flat plate.

Görtler (1941) found that a boundary layer over a concave surface is strongly unstable. The instability is manifested by the presence of counter-rotating vortices having their axis in the streamwise direction. Using the tellurium method, Wortman (1969) gave detailed flow visualization of these vortices on slightly curved walls. Bippes (1978) conducted experiments on walls with radii of curvature of 0.5 and 1m so that the generated Görtler vortices are fairly strong. He made the flow visible by using the hydrogen-bubble technique and photographed it with a photo-grogrammatic sterocamera. He analyzed photogrammetrically the photographs and obtained fairly accurate quantitative information of the flow field. Unlike the case of preexisting streamwise vortices, Görtler vortices generated by a concave surface are amplified with streamwise distance. Their amplification is exponential (Smith, 1955) when they are weak and it appears to be linear when they are strong (Bippes, 1978). Floryan and Saric (1979) gave a comprehensive review of the different analyses of these vortices and, using the model of Smith (1955), presented fairly accurate numerical results describing these vortices.

In their experiments on the nature of boundary-layer stability, Klebanoff, Tidstrom & Sargent (1962) established that streamwise vortices are associated with nonlinear three-dimensional wave motions. Benney & Lin (1966) modeled the generation of these vortices by the nonlinear interaction of a two-dimensional wave with a three-dimensional wave superimposed on the laminar profile. Using this model, Antar & Collins (1975) calculated these vortices in a boundary layer on a flat plate for different amplitudes of the waves.

A number of experimental studies investigated the influence of steady streamwise vortices on the transition from laminar to turbulent flow. Aihara (1962), Tani & Sakagami (1964), and Tani & Aihara (1969) studied the influence of steady streamwise vortices on two-dimensional Tollmien-Schlichting waves generated by a vibrating ribbon. They considered the case in which the vortices were generated naturally on a concave surface (Görtler vortices) as well as the case in which the vortices were generated artifically by a row of wings on a cancave surface. They measured the distributions of the mean velocity and wave intensity across the boundary layer for three spanwise positions at a number of streamwise stations. They concluded that the Görtler vortices indirectly affect the transition by inducing a spanwise variation in boundary-layer thickness, at least when the radii of curvature are not extremely small. However, they did not present any measurements of the growth rates of the Tollmien-Schlichting waves. Wortman (1969) investigated the development of natural transition downstream of Görtler vortices. Using the tellurium method, he determined the direction and relative magnitude of the unsteady velocities from the streaklines by confining his observations to the vicinity of the starting point of the streaklines. He observed a steady second-order instability that destroys the symmetry of the Görtler vortices. He suggested that this instability is caused by secondary vortices having spanwise wavelengths that are twice those of the Görtler vortices. Then he observed a third-order instability, consisting of regular three-dimensional oscillations.

The above shows that there are many theoretical and experimental studies relating to the generation of streamwise vortices and a number of experimental studies relating to their effect on transition, but to the author's knowledge, no theory yet exists on how these vortices affect the development of Tollmien-Schlichting waves. The purpose of the present paper is to present a parameteric instability mechanism by which the streamwise vortices increase the growth of selected Tollmien-Schlichting waves in growing boundary layers. To first order the selected waves have a spanwise wavelength that is twice that of the vortices, while to second order the selected waves have spanwise wavelengths that are equal and twice that of the vortices. To minimize the algebra, we consider flows with growing steady counter-rotating vortices over flat plates with or without pressure gradients.

II. Problem Formulation

We consider the stability of a basic flow that consists of the superposition of the Blasius or Falkner-Skan flow and a flow corresponding to growing steady quasi-periodic counter-rotating streamwise vortices. Thus we consider the stability of the flow described by

$$U = U_0(x_1,y) + \varepsilon_V U_1(x_1,z_1,y)\cos 2\beta z + \ldots, \tag{1}$$

$$V = \varepsilon V_0(x_1,y) + \varepsilon_V V_1(x_1,z_1,y)\cos 2\beta z + \ldots, \tag{2}$$

$$W = \varepsilon_V W_1(x_1,z_1,y)\sin 2\beta z + \ldots, \tag{3}$$

$$P = P_0(x_1) + \varepsilon_V P_1(x_1,z_1,y)\cos 2\beta z + \ldots, \tag{4}$$

where the subscript o refers to the Blasius of Falkner-Skan flow, the subscript 1 refers to the flow corresponding the the streamwise vortices, $x_1 = \varepsilon x$, $z_1 = \varepsilon z$, ε is a small dimensonless quantity that accounts for the quasi-periodicity of the vortices as well as for the streamwise growth of the vortices and the boundary layer, β is a real dimensionless spanwise wavenumber, and ε_V is a small dimensionless quantity that indicates the strength of the streamwise vortices. In Eqs. (1) - (4), velocities and lengths are made dimensionless by using a reference velocity U_r and a reference length δ_r. We superpose the small unsteady perturbation quantities $\varepsilon_T u(x,y,z,t)$, $\varepsilon_T v(x,y,z,t)$, $\varepsilon_T w(x,y,z,t)$, and $\varepsilon_T p(x,y,z,t)$ on those given in Eqs. (1) - (4) so that the total flow quantities become $U + \varepsilon_T u$, $V + \varepsilon_T v$, $W + \varepsilon_T w$, and $P + \varepsilon_T p$. Here, ε_T is a small dimensionless quantity that is the order of the amplitude of the Tollmien-Schlichting waves. In this paper, ε_T is assumed to be much smaller than ε_V and ε so that terms the order of ε_T^2 can be neglected compared with $\varepsilon_T\varepsilon_V$ and $\varepsilon_T\varepsilon$. Substituting these total flow quantities into the dimensionless Navier-Stokes equations, subtracting the basic-flow quantities, and keeping linear terms in the perturbation quantities, we obtain

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{5}$$

$$\frac{\partial u}{\partial t} + U \frac{\partial u}{\partial x} + u \frac{\partial U}{\partial x} + V \frac{\partial u}{\partial y} + v \frac{\partial U}{\partial y} + W \frac{\partial u}{\partial z} + w \frac{\partial U}{\partial z} = - \frac{\partial p}{\partial x} + \frac{1}{R} \nabla^2 u, \tag{6}$$

$$\frac{\partial v}{\partial t} + U \frac{\partial v}{\partial x} + u \frac{\partial V}{\partial x} + V \frac{\partial v}{\partial y} + v \frac{\partial V}{\partial y} + W \frac{\partial v}{\partial z} + w \frac{\partial V}{\partial z} = - \frac{\partial p}{\partial y} + \frac{1}{R} \nabla^2 v, \tag{7}$$

$$\frac{\partial w}{\partial t} + U \frac{\partial w}{\partial x} + u \frac{\partial W}{\partial x} + V \frac{\partial w}{\partial y} + v \frac{\partial W}{\partial y} + W \frac{\partial w}{\partial z} + w \frac{\partial W}{\partial z} = - \frac{\partial p}{\partial z} + \frac{1}{R} \nabla^2 w, \tag{8}$$

where t is made dimensionless by using δ_r/U_r, $R = U_r\delta_r/\nu$ is the Reynolds number, and ν is the kinematic viscosity of the fluid.

Substituting Eqs. (1)-(4) into Eqs. (6)-(8), we obtain

$$\begin{aligned}\frac{\partial u}{\partial t} + U_0 \frac{\partial u}{\partial x} + v \frac{\partial U_0}{\partial y} + \frac{\partial p}{\partial x} - \frac{1}{R} \nabla^2 u = -\varepsilon_v \Big\{ & [U_1 \frac{\partial u}{\partial x} + V_1 \frac{\partial u}{\partial y} + \frac{\partial U_1}{\partial y} v]\cos 2\beta z \\ & + [W_1 \frac{\partial u}{\partial z} - 2\beta U_1 w]\sin 2\beta z \Big\} - \varepsilon[\frac{\partial U_0}{\partial x_1} u + V_0 \frac{\partial u}{\partial y}] + O(\varepsilon\varepsilon_v)\end{aligned} \tag{9}$$

$$\begin{aligned}\frac{\partial v}{\partial t} + U_0 \frac{\partial v}{\partial x} + \frac{\partial p}{\partial y} - \frac{1}{R} \nabla^2 v = - \varepsilon_v \Big\{ & [U_1 \frac{\partial v}{\partial x} + V_1 \frac{\partial v}{\partial y} + \frac{\partial V_1}{\partial y} v]\cos 2\beta z \\ & + [W_1 \frac{\partial v}{\partial z} - 2\beta V_1 w]\sin 2\beta z \Big\} - \varepsilon[V_0 \frac{\partial v}{\partial y} + \frac{\partial V}{\partial y} v] + O(\varepsilon^2, \varepsilon\varepsilon_v)\end{aligned} \tag{10}$$

$$\begin{aligned}\frac{\partial w}{\partial t} + U_0 \frac{\partial w}{\partial x} + \frac{\partial p}{\partial z} - \frac{1}{R} \nabla^2 w = - \varepsilon_v \Big\{ & [U_1 \frac{\partial w}{\partial x} + V_1 \frac{\partial w}{\partial y} + 2\beta W_1 w]\cos 2\beta z \\ & + [\frac{\partial W_1}{\partial y} v + W_1 \frac{\partial w}{\partial z}]\sin 2\beta z \Big\} - \varepsilon V_0 \frac{\partial w}{\partial y} + O(\varepsilon\varepsilon_v)\end{aligned} \tag{11}$$

Equations (5) and (9)-(11) need to be supplemented by initial and boundary conditions. The initial conditions are specified later, while the boundary conditions for an impermeable flat surface are

$$u = v = w = 0 \text{ at } y = 0, \tag{12}$$

$$u, v, w \to 0 \quad \text{at } y \to \infty. \tag{13}$$

III. Solution

We use the method of multiple scales (e.g., Nayfeh, 1973) to determine a first-order uniform expansion for Eqs. (5) and (9)-(13). To accomplish this, we let $\varepsilon_v = O(\varepsilon)$ and write $\varepsilon_v = \chi\varepsilon$, where $\chi = O(1)$. If $\varepsilon \ll \varepsilon_v$, the effect of the growth of the boundary layer is small compared with the effect of the vortices. If $\varepsilon_v \ll \varepsilon$, the effect of the vortices is small compared with that due to the growth of the boundary layer, and the solution accounts for the nonparallel effects only. Thus, the above ordering yields an expansion that accounts for the effects of the streamwise vortices and the growth of the boundary layer, and it includes the cases $\varepsilon \ll \varepsilon_v$ and $\varepsilon_v \ll \varepsilon$ as special cases.

We seek a uniform expansion for Eqs. (5) and (9)-(13) in the form

$$u = \sum_{n=0}^{1} \varepsilon^n u_n(x_0,x_1,y,z_0,z_1,t_0,t_1) + O(\varepsilon^2), \tag{14}$$

$$v = \sum_{n=0}^{1} \varepsilon^n v_n(x_0,x_1,y,z_0,z_1,t_0,t_1) + O(\varepsilon^2), \tag{15}$$

$$w = \sum_{n=0}^{1} \varepsilon^n w_n(x_0,x_1,y,z_0,z_1,t_0,t_1) + O(\varepsilon^2), \tag{16}$$

$$p = \sum_{n=0}^{1} \varepsilon^n p_n(x_0,x_1,y,z_0,z_1,t_0,t_1) + O(\varepsilon^2), \tag{17}$$

where

$$x_n = \varepsilon^n x, \qquad z_n = \varepsilon^n z, \qquad t_n = \varepsilon^n t \; . \tag{18}$$

Substituting Eqs. (14)-(18) into Eqs. (5) and (9)-(13) and equating coefficients of like powers ε, we obtain

Order ε^0

$$\mathcal{L}_1(u_0,v_0,w_0) = \frac{\partial u_0}{\partial x_0} + \frac{\partial v_0}{\partial y} + \frac{\partial w_0}{\partial z_0} = 0, \tag{19}$$

$$\mathcal{L}_2(u_0,v_0,p_0) = \frac{\partial u_0}{\partial t_0} + U_0 \frac{\partial u_0}{\partial x_0} + v_0 \frac{\partial U_0}{\partial y} + \frac{\partial p_0}{\partial x_0} - \frac{1}{R} \nabla_0^2 u_0 = 0, \tag{20}$$

$$\mathcal{L}_3(v_0,p_0) = \frac{\partial v_0}{\partial t_0} + U_0 \frac{\partial v_0}{\partial x_0} + \frac{\partial p_0}{\partial y} - \frac{1}{R} \nabla_0^2 v_0 = 0, \tag{21}$$

$$\mathcal{L}_4(w_0,p_0) = \frac{\partial w_0}{\partial t_0} + U_0 \frac{\partial w_0}{\partial x_0} + \frac{\partial p_0}{\partial z_0} - \frac{1}{R} \nabla_0^2 w_0 = 0, \tag{22}$$

$$u_0 = v_0 = w_0 = 0 \quad \text{at } y = 0, \tag{23}$$

$$u_0, v_0, w_0 \to 0 \quad \text{as } y \to \infty, \tag{24}$$

Order ε

$$\mathcal{L}_1(u_1,v_1,w_1) = -\frac{\partial u_0}{\partial x_1} - \frac{\partial w_0}{\partial z_1}, \tag{25}$$

$$\begin{aligned}\mathcal{L}_2(u_1,v_1,p_1) = {}& -\frac{\partial u_0}{\partial t_1} - U_0 \frac{\partial u_0}{\partial x_1} - \frac{\partial p_0}{\partial x_1} + \frac{2}{R}\left[\frac{\partial^2 u_0}{\partial x_0 \partial x_1} + \frac{\partial^2 u_0}{\partial z_0 \partial z_1}\right] \\ & - \chi\Big\{\Big[U_1 \frac{\partial u_0}{\partial x_0} + V_1 \frac{\partial u_0}{\partial y} + \frac{\partial U_1}{\partial y} v_0\Big]\cos 2\beta z_0 + \Big[W_1 \frac{\partial u_0}{\partial z_0} \\ & - 2\beta U_1 w_0\Big]\sin 2\beta z_0\Big\} - \frac{\partial U_0}{\partial x_1} u_0 - V_0 \frac{\partial u_0}{\partial y},\end{aligned} \tag{26}$$

$$\begin{aligned}\mathcal{L}_3(v_1,p_1) = {}& -\frac{\partial v_0}{\partial t_1} - U_0 \frac{\partial v_0}{\partial x_1} + \frac{2}{R}\left[\frac{\partial^2 v_0}{\partial x_0 \partial x_1} + \frac{\partial^2 v_0}{\partial z_0 \partial z_1}\right] - \chi\Big\{\Big[U_1 \frac{\partial v_0}{\partial x_0} \\ & + V_1 \frac{\partial v_0}{\partial y} + \frac{\partial V_1}{\partial y} v_0\Big]\cos 2\beta z_0 + \Big[W_1 \frac{\partial v_0}{\partial z_0} - 2\beta V_1 w_0\Big]\sin 2\beta z_0\Big\} \\ & - V_0 \frac{\partial v_0}{\partial y} - \frac{\partial V_0}{\partial y} v_0,\end{aligned} \tag{27}$$

$$\begin{aligned}\mathcal{L}_4(w_1,p_1) = {}& -\frac{\partial w_0}{\partial t_1} - U_0 \frac{\partial w_0}{\partial x_1} - \frac{\partial p_0}{\partial z_1} + \frac{2}{R}\left[\frac{\partial^2 w_0}{\partial x_0 \partial x_1} + \frac{\partial^2 w_0}{\partial z_0 \partial z_1}\right] \\ & - \chi\Big\{\Big[U_1 \frac{\partial w_0}{\partial x_0} + V_1 \frac{\partial w_0}{\partial y} + 2\beta W_1 w_0\Big]\cos 2\beta z_0 + \Big[\frac{\partial W_1}{\partial y} v_0 \\ & + W_1 \frac{\partial w_0}{\partial z_0}\Big]\sin 2\beta z_0\Big\} - V_0 \frac{\partial w_0}{\partial y},\end{aligned} \tag{28}$$

$$u_1 = v_1 = w_1 = 0 \qquad \text{at } y = 0, \tag{29}$$

$$u_1, v_1, w_1 \to 0 \qquad \text{as } y \to \infty, \tag{30}$$

where

$$\nabla_0^2 = \frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z_0^2} \; .$$

The initial conditions are taken such that the solution of the zeroth-order problem, Eqs. (19)-(24), consists of two wave packets centered around the frequency ω, the streamwise wavenumber α, and the spanwise wavenumbers β_1 and $-\beta_1$; that is,

$$u_0 = A_1(x_1,z_1,t_1)\zeta_{11}(y,x_1)\exp(i\theta_1) + A_2(x_1,z_1,t_1)\zeta_{21}(y,x_1)\exp(i\theta_2), \tag{31}$$

$$v_0 = A_1\zeta_{12}(y,x_1)\exp(i\theta_1) + A_2\zeta_{22}(y,x_1)\exp(i\theta_2), \tag{32}$$

$$\omega_0 = A_1\zeta_{13}(y,x_1)\exp(i\theta_1) + A_2\zeta_{23}(y,x_1)\exp(i\theta_2), \tag{33}$$

$$p_0 = A_1\zeta_{14}(y,x_1)\exp(i\theta_1) + A_2\zeta_{24}(y,x_1)\exp(i\theta_2), \tag{34}$$

where

$$\theta_n = \int \alpha(x_1)dx - \beta_n z_0 - \omega t_0, \quad \beta_2 = -\beta_1 \; , \tag{35}$$

and the functions A_1 and A_2 are undetermined at this level of approximation; they are determined by imposing the solvability conditions at the next level of approximation. Substituting Eqs. (31)-(35) into Eqs. (19)-(24) yields the following eigenvalue problems:

$$i\alpha\zeta_{n_1} + D\zeta_{n_2} - i\beta_n\zeta_{n_3} = 0 \ , \tag{36}$$

$$i(U_0\alpha - \omega)\zeta_{n_1} + \zeta_{n_2}DU_0 + i\alpha\zeta_{n_4} - \frac{1}{R}(D^2 - \alpha^2 - \beta_1^2)\zeta_{n_1} = 0, \tag{37}$$

$$i(U_0\alpha - \omega)\zeta_{n_2} + D\zeta_{n_4} - \frac{1}{R}(D^2 - \alpha^2 - \beta_1^2)\zeta_{n_2} = 0, \tag{38}$$

$$i(U_0\alpha - \omega)\zeta_{n_3} - i\beta_n\zeta_{n_4} - \frac{1}{R}(D^2 - \alpha^2 - \beta_1^2)\zeta_{n_3} = 0, \tag{39}$$

$$\zeta_{n_1} = \zeta_{n_2} = \zeta_{n_3} = 0 \qquad \text{at } y = 0, \tag{40}$$

$$\zeta_{n_1}, \zeta_{n_2}, \zeta_{n_3} \to 0 \qquad \text{as } y \to \infty. \tag{41}$$

where $D\zeta = \partial\zeta/\partial y$. For a given ω, β_1, and R, one can solve Eqs. (36)-(41) numerically to determine the complex eigenvalue α and the eigenfunctions ζ_{nm}.

Substituting Eqs. (31)-(35) into Eqs. (25)-(28) yields

$$\begin{aligned}\mathcal{L}_1(u_1,v_1,w_1) = &- \sum_{n=1}^{2} \left(\frac{\partial A_n}{\partial x_1}\zeta_{n_1} + \frac{\partial A_n}{\partial z_1}\zeta_{n_3}\right)\exp(i\theta_n) \\ &- \sum_{n=1}^{2} A_n \frac{\partial \zeta_{n_1}}{\partial x_1}\exp(i\theta_n),\end{aligned} \tag{42}$$

$$\begin{aligned}\mathcal{L}_2(u_1,v_1,p_1) = &- \sum_{n=1}^{2} \left[\frac{\partial A_n}{\partial t_1}\zeta_{n_1} + \left(U_0\zeta_{n_1} + \zeta_{n_4} - \frac{2i\alpha}{R}\zeta_{n_1}\right)\frac{\partial A_n}{\partial x_1}\right. \\ &\left. + 2i\beta_n\zeta_{n_1}\frac{\partial A_n}{\partial z_1}\right]\exp(i\theta_n) - \sum_{n=1}^{2}\left[\left(U_0 - \frac{2i\alpha}{R}\right)\frac{\partial\zeta_{n_1}}{\partial x_1} - \frac{i}{R}\zeta_{n_1}\frac{d\alpha}{dx_1}\right. \\ &\left. + \frac{\partial\zeta_{n_4}}{\partial x_1} + \frac{\partial U_0}{\partial x_1}\zeta_{n_1} + V_0\frac{\partial\zeta_{n_1}}{\partial y}\right]A_n\exp(i\theta_n) - \frac{1}{2}\chi\left[iU_1\alpha\zeta_{11}\right. \\ &\left. + V_1\frac{\partial\zeta_{11}}{\partial y} + \frac{\partial U_1}{\partial y}\zeta_{12} - W_1\beta_1\zeta_{11} + 2i\beta U_1\zeta_{13}\right]A_1\exp[i\theta_2 \\ &+ 2i(\beta - \beta_1)z_0] - \frac{1}{2}\chi\left[iU_1\alpha\zeta_{21} + V_1\frac{\partial\zeta_{21}}{\partial y} + \frac{\partial U_1}{\partial y}\zeta_{22}\right. \\ &\left. - W_1\beta_1\zeta_{21} - 2i\beta U_1\zeta_{23}\right]A_2\exp[i\theta_1 - 2i(\beta - \beta_1)z_0] + \text{NST},\end{aligned} \tag{43}$$

$$\mathcal{L}_3(v_1,p_1) = -\sum_{n=1}^{2}\Big[\frac{\partial A_n}{\partial t_1}\zeta_{n_2} + \Big(U_0 - \frac{2i\alpha}{R}\Big)\zeta_{n_2}\frac{\partial A_n}{\partial x_1} + \frac{2i\beta_n}{R}\zeta_{n_2}\frac{\partial A_n}{\partial z_1}\Big] \times$$

$$\exp(i\theta_n) - \sum_{n=1}^{2}\Big[\Big(U_0 - \frac{2i\alpha}{R}\Big)\frac{\partial\zeta_{n_2}}{\partial x_1} - \frac{i}{R}\zeta_{n_2}\frac{d\alpha}{dx_1}$$

$$+ \frac{\partial}{\partial y}(V_0\zeta_{n_2})\Big]\exp(i\theta_n) - \frac{1}{2}\chi[i\alpha U_1\zeta_{12} + \frac{\partial}{\partial y}(V_1\zeta_{12}) - \beta_1 W_1\zeta_{12}$$

$$+ 2i\beta V_1\zeta_{13}]A_1\exp[i\theta_2 + 2i(\beta - \beta_1)z_0] - \frac{1}{2}\chi[i\alpha U_1\zeta_{22}$$

$$+ \frac{\partial}{\partial y}(V_1\zeta_{22}) - \beta_1 W_1\zeta_{22} - 2i\beta V_1\zeta_{23}]A_2\exp[i\theta_1 - 2i(\beta - \beta_1)z_0]$$

$$+ \text{NST}, \tag{44}$$

$$\mathcal{L}_4(w_1,p_1) = -\sum_{n=1}^{2}\Big[\frac{\partial A_n}{\partial t_1}\zeta_{n_3} + \Big(U_0 - \frac{2i\alpha}{R}\Big)\zeta_{n_3}\frac{\partial A_n}{\partial x_1} + \Big(\zeta_{n_4} + \frac{2i\beta_n}{R}\zeta_{n_3}\Big) \times$$

$$\frac{\partial A_n}{\partial z_1}\Big]\exp(i\theta_n) - \sum_{n=1}^{2}\Big[\Big(U_0 - \frac{2i\alpha}{R}\Big)\frac{\partial\zeta_{n_3}}{\partial x_1} - \frac{i}{R}\zeta_{n_3}\frac{d\alpha}{dx_1}$$

$$+ V_0\frac{\partial\zeta_{n_3}}{\partial y}\Big]A_n\exp(i\theta_n) - \frac{1}{2}\chi[iU_1\alpha\zeta_{13} + (2\beta - \beta_1)W_1\zeta_{13}$$

$$+ V_1\frac{\partial\zeta_{13}}{\partial y} - i\frac{\partial W_1}{\partial y}\zeta_{12}]A_1\exp[i\theta_2 + 2i(\beta - \beta_1)z_0]$$

$$- \frac{1}{2}\chi[iU_1\alpha\zeta_{23} + (2\beta - \beta_1)W_1\zeta_{23} + V_1\frac{\partial\zeta_{23}}{\partial y}$$

$$+ i\frac{\partial W_1}{\partial y}\zeta_{22}]A_2\exp[i\theta_1 - 2i(\beta - \beta_1)z_0], \tag{45}$$

where NST stands for terms that are proportional to $\exp[\pm i(\beta + \beta_1)z_0]$, which do not produce secular terms in u_1, v_1, w_1, and p_1.

Since the homogeneous parts of Eqs. (42)-(45), (29), and (30) are the same as Eqs. (19)-(24) and since the latter have a nontrivial solution, the inhomogeneous equations (42)-(45), (29), and (30) have a solution if, and only if, the inhomogeneous parts are orthogonal to every solution of the adjoint homogeneous problem. These solvability conditions depend on whether $\beta \simeq \beta_1$ or not. If β is away from β_1, the solvability conditions yield two uncoupled equations describing the effect of nonparallelism on A_1 and A_2. If $\beta \simeq \beta_1$, we introduce a detuning parameter σ defined by

$$\beta = \beta_1 + \epsilon\sigma \tag{46}$$

where $\sigma = O(1)$ and express $(\beta - 2\beta_1)z_0$ as σz_1. Then, imposing the solvability condition that the inhomogeneities be orthogonal to every solution of the adjoint homogeneous problem, we obtain

$$g_{11}\frac{\partial A_1}{\partial t_1} + g_{12}\frac{\partial A_1}{\partial x_1} + g_{13}\frac{\partial A_1}{\partial z_1} = h_1 A_1 + \chi h_{12} A_2 \exp(-i\sigma z_1) \tag{47}$$

$$g_{21}\frac{\partial A_2}{\partial t_1} + g_{22}\frac{\partial A_2}{\partial x_1} + g_{23}\frac{\partial A_2}{\partial z_1} = h_2 A_2 + \chi h_{21} A_1 \exp(i\sigma z_1) \tag{48}$$

where the g's and h's are given in Appendix A together with the adjoint problems. Differentiating Eqs. (19) - (24), respectively, with respect to α and β_n and imposing the solvability conditions, one can show that

$$\frac{g_{n_2}}{g_{n_1}} = \omega_\alpha \; , \qquad \frac{g_{n_3}}{g_{n_1}} = \omega_{\beta_n} \tag{49}$$

where ω_α and ω_β are the complex group velocities in the x and z directions.

Since the solutions of Eqs. (47) and (48) for general initial conditions are not available yet, we consider next the special case of periodic streamwise vortices and a single-frequency disturbance that is perfectly tuned in the spanwise wavenumber. The single-frequency assumption corresponds to the case of a disturbance generated by a vibrating ribbon. The second assumption demands that $\beta = \beta_1$ and that the waves are modulated in the streamwise position only. Thus, we consider the case in which $\partial A_n/\partial t_1 = \partial A_n/\partial z_1 = 0$ and $\sigma = 0$. Then, Eqs. (47) and (48) can be rewritten as

$$\frac{dA_1}{dx} = \varepsilon\hat{h}_1 A_1 + \varepsilon_v \hat{h}_{12} A_2 \tag{50}$$

$$\frac{dA_2}{dx} = \varepsilon\hat{h}_2 A_2 + \varepsilon_v \hat{h}_{21} A_1 \tag{51}$$

where

$$\hat{h}_n = h_n/g_{n_2}, \quad \hat{h}_{12} = h_{12}/g_{12}, \quad \hat{h}_{21} = h_{21}/g_{22}$$

It follows from Eqs. (36) - (41) that $\zeta_{13} = -\zeta_{23}$ and $\zeta_{1_n} = \zeta_{2_n}$ for $n \neq 3$, while it follows from Eqs. (A.7) - (A.12) that the adjoint solutions are related by $\zeta^*_{14} = -\zeta^*_{24}$ and $\zeta^*_{1_n} = \zeta^*_{2_n}$ for $n \neq 4$. Hence, it follows from Eqs. (A.2), (A.4), (A.5), and (A.6) that

$$g_{12} = g_{22}, \quad h_1 = h_2, \quad h_{12} = h_{21}.$$

Thus,

$$\hat{h}_1 = \hat{h}_2 \quad \text{and} \quad \hat{h}_{12} = \hat{h}_{21}.$$

Therefore, adding Eqs. (50) and (51) yields

$$\frac{d}{dx}(A_2 + A_1) = (\varepsilon\hat{h}_1 + \varepsilon_v\hat{h}_{12})(A_2 + A_1) \tag{52}$$

Subtracting Eq. (50) from Eq. (51) yields

$$\frac{d}{dx}(A_2 - A_1) = (\varepsilon\hat{h}_1 - \varepsilon_v\hat{h}_{12})(A_2 - A_1) \tag{53}$$

The solutions of Eqs. (52) and (53) are

$$A_2 + A_1 = 2c_1\exp[\int(\varepsilon\hat{h}_1 + \varepsilon_v\hat{h}_{12})dx] \tag{54}$$

$$A_2 - A_1 = 2c_2\exp[\int(\varepsilon\hat{h}_1 - \varepsilon_v\hat{h}_{12})dx] \tag{55}$$

where c_1 and c_2 are arbitrary constants that can be determined from the initial conditions. Solving Eqs. (54) and (55) gives

$$A_2 = c_1\exp[\int(\varepsilon\hat{h}_1 + \varepsilon_v\hat{h}_{12})dx] + c_2\exp[\int(\varepsilon\hat{h}_1 - \varepsilon_v\hat{h}_{12})dx]$$

$$A_1 = c_1\exp[\int(\varepsilon\hat{h}_1 + \varepsilon_v\hat{h}_{12})dx] - c_2\exp[\int(\varepsilon\hat{h}_1 - \varepsilon_v\hat{h}_{12})dx] \tag{57}$$

Substituting for A_1 and A_2 in Eq. (31), using Eq. (35), substituting the results into Eq. (14), and recalling that $\beta_1 = \beta$, we obtain

$$u = \zeta_{11}\exp[\int(i\alpha + \varepsilon\hat{h}_1)dx - i\beta z - i\omega t] \times$$
$$[c_1\exp(\varepsilon_v\int\hat{h}_{12}dx) - c_2\exp(-\varepsilon_v\int\hat{h}_{12}dx)]$$
$$+ \zeta_{21}\exp[\int(i\alpha + \varepsilon\hat{h}_1)dx + i\beta z - i\omega t] \times$$
$$[c_1\exp(\varepsilon_v\int\hat{h}_{12}dx) + c_2\exp(-\varepsilon_v\int\hat{h}_{12}dx)] + \ldots \qquad (58)$$

Equations (A.2) and (A.5) show that, in general, $\hat{h}_{12}$ is a complex number. Hence, one of the terms multipling c_1 and c_2 decays while the other grows exponentially with distance. Thus, the growth rate σ based on u for either the wave with the positive or negative β is

$$\sigma = -\alpha_i + \varepsilon[\mathrm{Real}(\hat{h}_1) + \frac{1}{\zeta_{11}}\frac{\partial\zeta_{11}}{\partial x_1}] + \varepsilon_v|\,\mathrm{Real}(\hat{h}_{12})| \qquad (59)$$

because $\zeta_{11} = \zeta_{21}$.

Equation (59) shows that the growth rate is the sum of three quantities: $-\alpha_i$, the quasi-parallel growth rate; the term proportional to ε, the effect of nonparallelism; and the term proportional to ε_v, the effect of the streamwise vortices. Thus, in a given physical situation, the relative influence of the vortices and nonparallelism depends on the relative magnitudes of ε and ε_v. For maximum amplified waves, $\varepsilon = O(10^{-3})$, while for flows over concave surfaces, ε_v can be (0.10), depending on the radius of curvature. In such situations, the effects of the vortices dominate the effects of nonparallelism, and the presence of the vortices is a very powerful instability mechanism. Hence, the presence of this mechanism may not be difficult to check experimentally. This requires an experiment in which the amplitudes and spanwise variations of Görtler vortices and Tollmien-Schlichting waves on a concave wall are measured. The spanwise variations of the Görtler vortices and the generated Tollmien-Schlichting waves can be checked to see whether the wavelengths of the latter are twice those of the vortices. The modification of the mean flow due to the presence of the vortices can be determined and the interaction coefficient $\hat{h}_{12}$ can be calculated from Eqs. (A.2) and (A.5) by quadrature. Then, the variation of u with x and y can be calculated from Eq. (58) and compared with the experimental results.

It should be noted that the present analysis is valid only when the amplitude of the Tollmien-Schlichting waves ε_T is small compared with the amplitude of the vortices ε_V. As the Tollmien-Schlichting waves grow, one needs to account for their influence on the vortices. In fact, they will generate streamwise vortices having an amplitude $O(\varepsilon_T^2)$ (Klebanoff, Tidstrom & Sargent, 1962; Benney & Lin, 1960), which may strengthen or weeken the primary vortices, depending on their phasings. This effect has not been taken into account in this paper.

ACKNOWLEDGMENT

The comments and discussions of Drs. Mark Morkovin and William S. Saric are greatly appreciated. This work was supported by the Fluid Dynamics Program of the United States Office of Naval Research and the National Aeronautics and Space Administration, Langley Research Center under Grant NSG 1255.

APPENDIX A

$$g_{n_1} = \int_0^\infty (\zeta_{n_1}\zeta^*_{n_2} + \zeta_{n_2}\zeta^*_{n_3} + \zeta_{n_3}\zeta^*_{n_4})dy \tag{A.1}$$

$$g_{n_2} = \int_0^\infty [\zeta_{n_1}\zeta^*_{n_1} + \zeta_{n_4}\zeta^*_{n_2} + (U_0 - \frac{2i\alpha}{R})(\zeta_{n_1}\zeta^*_{n_2} + \zeta_{n_2}\zeta^*_{n_3} + \zeta_{n_3}\zeta^*_{n_4})]dy \tag{A.2}$$

$$g_{n_3} = \int_0^\infty [\zeta_{n_3}\zeta^*_{n_1} + \zeta_{n_4}\zeta^*_{n_4} + \frac{2i\beta_n}{R}(\zeta_{n_1}\zeta^*_{n_2} + \zeta_{n_2}\zeta^*_{n_3} + \zeta_{n_3}\zeta^*_{n_4})]dy \tag{A.3}$$

$$-h_n = \int_0^\infty \Big\{\frac{\partial\zeta_{n_1}}{\partial x_1}\zeta^*_{n_1} + [(U_0 - \frac{2i\alpha}{R})\frac{\partial\zeta_{n_1}}{\partial x_1} - \frac{i}{R}\zeta_{n_1}\frac{d\alpha}{dx_1} + \frac{\partial\zeta_{n_4}}{\partial x_1} + \frac{\partial U_0}{\partial x_1}\zeta_{n_1}$$

$$+ V_0\frac{\partial\zeta_{n_1}}{\partial y}]\zeta^*_{n_2} + [(U_0 - \frac{2i\alpha}{R})\frac{\partial\zeta_{n_2}}{\partial x_1} - \frac{i}{R}\zeta_{n_2}\frac{d\alpha}{dx_1} + \frac{\partial}{\partial y}(V_0\zeta_{n_2})]\zeta^*_{n_3}$$

$$+ [(U_0 - \frac{2i\alpha}{R})\frac{\partial\zeta_{n_3}}{\partial x_1} - \frac{i}{R}\zeta_{n_3}\frac{d\alpha}{dx_1} + V_0\frac{\partial\zeta_{n_3}}{\partial y_1}]\zeta^*_{n_4}\Big\}dy \tag{A.4}$$

$$-2h_{12} = \int_0^\infty \Big\{[i\alpha U_1\zeta_{21} + V_1\frac{\partial\zeta_{21}}{\partial y} + \frac{\partial U_1}{\partial y}\zeta_{22} - \beta_1 W_1\zeta_{21} - 2i\beta U_1\zeta_{23}]\zeta^*_{12}$$

$$+ [i\alpha U_1\zeta_{22} + \frac{\partial}{\partial y}(V_1\zeta_{22}) - \beta_1 W_1\zeta_{22} - 2i\beta V_1\zeta_{23}]\zeta^*_{13}$$

$$+ [i\alpha U_1\zeta_{23} + (2\beta - \beta_1)W_1\zeta_{23} + V_1\frac{\partial\zeta_{23}}{\partial y} + i\frac{\partial W_1}{\partial y}\zeta_{22}]\zeta^*_{14}\Big\}dy \tag{A.5}$$

$$-2h_{21} = \int_0^\infty \Big\{[i\alpha U_1\zeta_{11} + V_1\frac{\partial\zeta_{11}}{\partial y} + \frac{\partial U_1}{\partial y}\zeta_{12} - \beta_1 W_1\zeta_{11} + 2i\beta U_1\zeta_{13}]\zeta^*_{22}$$

$$+ [i\alpha U_1\zeta_{12} + \frac{\partial}{\partial y}(V_1\zeta_{12}) - \beta_1 W_1\zeta_{12} + 2i\beta V_1\zeta_{13}]\zeta^*_{23} + [i\alpha U_1\zeta_{13}$$

$$+ (2\beta - \beta_1)W_1\zeta_{13} + V_1\frac{\partial\zeta_{13}}{\partial y} - i\frac{\partial W_1}{\partial y}\zeta_{12}]\zeta^*_{24}\Big\}dy \tag{A.6}$$

where the ζ^*_{nm} are solutions of the following adjoint problems:

$$i\alpha\zeta^*_{n_1} + i(U_0\alpha - \omega)\zeta^*_{n_2} - \frac{1}{R}(D^2 - \alpha^2 - \beta_n^2)\zeta^*_{n_2} = 0 \qquad \text{(A.7)}$$

$$- D\zeta^*_{n_1} + DU_0\zeta^*_{n_2} + i(U_0\alpha - \omega)\zeta^*_{n_3} - \frac{1}{R}(D^2 - \alpha^2 - \beta_n^2)\zeta^*_{n_3} = 0 \qquad \text{(A.8)}$$

$$- i\beta_n\zeta^*_{n_1} + i(U_0\alpha - \omega)\zeta^*_{n_4} - \frac{1}{R}(D^2 - \alpha^2 - \beta_n^2)\zeta^*_{n_4} = 0 \qquad \text{(A.9)}$$

$$i\alpha\zeta^*_{n_2} - D\zeta^*_{n_3} - i\beta_n\zeta^*_{n_4} = \qquad \text{(A.10)}$$

$$\zeta^*_{n_2} = \zeta^*_{n_3} = \zeta^*_{n_4} = 0 \quad \text{at} \quad y = 0 \qquad \text{(A.11)}$$

$$\zeta^*_{nm} \to 0 \quad \text{as} \quad y \to \infty \qquad \text{(A.12)}$$

REFERENCES

AIHARA, Y. 1962 Bull. Aero. Res. Inst., Tokoyo Univ., 3, 195.

ANTAR, B. N. & COLLINS, F. G. 1975 Phys. Fluids, 18, 289.

BENNEY, D. J. & LIN, C. C. 1960 Phys. Fluids, 3, 656.

BIPPES, H. 1978 NASA-TM-75243.

BRADSHAW, P. 1965 J. Fluid Mech., 22, 679.

CROW, S. C. 1966 J Fluid Mech., 24, 153.

FLORYAN, J. M. & SARIC, W. S. 1979 AIAA Paper No. 79-1497.

GÖRTLER, H. 1941 ZAMM, 21, 250.

KLEBANOFF, P. S. & TIDSTROM, K. D. 1959 NASA TN D 195.

KLEBANOFF, P. S., TIDSTROM, K. D. & SARGENT, L. M. 1962 J. Fluid Mech., 12, 1.

NAYFEH, A. H. 1973 Perturbation Methods, Wiley.

SMITH, A. M. O. 1955 Quart. Appl. Math., 13, 233.

TANI, I. & AIHARA, Y. 1969 ZAMP, 20/39, 609.

TANI, I. & SAKAGAMI, J. 1962 Proc. Int. Counc. Aero. Sci., Stockholm, Spartan.

WORTMAN, F. X. 1969 J. Fluid Mech., 38, 473.