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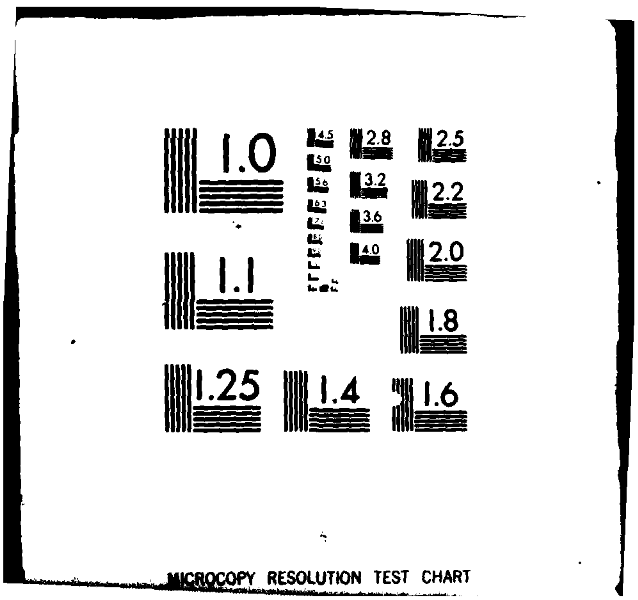
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RESULTS ON CUMULANT CONTROL AND ESTIMATING SYSTEMS: A SUMMARY

By

Stanley R. Liberty
Principal Investigator

and

Clifford R. Parten

Final Report
For the period May 16, 1978 - December 31, 1979

Prepared for the
Office of Naval Research
800 North Quincy Street
Arlington, Virginia

Under
Contract No. N00014-78-C-0443

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A description of a previously unpublished linear stochastic control problem class called "cumulant control" is presented. A summary of an "accessible state" formulation of problems in this class is given. This result is key to the applicability of this generalized Linear-Quadratic-Gaussian theory.

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I. INTRODUCTION

This report is a summary of the major research results obtained under Contract N00014-78-C-0443. Work under this contract was direced at a general class of Linear-Quadratic-Gaussian control problems herein called "cumulant control" problems.

Over the past few years, Liberty and colleagues [1],[2] and [3] have investigated probabilistic questions surrounding the random performance measure normally associated with linear stochastic control systems. Early work was aimed at finding an analytical means of evaluating the performance of such systems [3]. Results of this effort consisted of a solution to the "performance analysis" problem. These results allow the determination of complete statistical descriptions of system performance without requiring stochastic simulation.

Work, immediately preceeding that reported here, was directed at finding statistical descriptions of performance for design purposes [4]. This development was a natural analytical extension of the performance analysis results. It led to a promising analytical structure for design as well as a demonstration of the existence of control laws that achieve less variance of performance than the classical Linear-Quadratic-Gaussian controller.

The original statement of a "cumulant control problem" was contained in [4] but not actually solved. Despite the promise of the new formulation, it was not clear how to actually solve a cumulant control problem involving statistical parameters other than the mean. The primary objective of the research summarized here was to resolve the principal difficulty blocking solution of problems in the cumulant class.

Loosely speaking, the difficulty involves enforcement of an "admissibility" constraint on the set of control actions over which optimization of cumulant indices of system performance is carried out. In this class of stochastic optimal control problems, performance indices consist of linear combinations of mean-conditional-cumulants of a quadratic performance measure attached to a linear stochastic system. The investigation was carried out for discrete time systems over a finite time interval. The resulting finite dimensional setting removed difficulty encountered in the continuous time case in seeing the essence of the admissibility enforcement question.

Mean-conditional-cumulants appear in this work as $E\{\kappa_{\alpha|N}\}$ where $\kappa_{\alpha|N}$ is the α-order cumulant of performance, conditioned on system output observations from time 0 to time N. Linear combinations of these objects are to be minimized over all control actions in an "admissible" class. Essentially, to be admissible, a control action must be determined by a physically realizable operation on system output observations. The key result reported here is the discovery of a set of random variables $\{\kappa_\alpha^*\}$ with the property that

$$E\{\kappa_\alpha^*\} = E\{\kappa_{\alpha|N}\}. \tag{1}$$

It can be shown that constrained (with respect to admissibility) optimization over linear combinations of the latter objects is equivalent to constrained optimization over mean-conditional-cumulants.

II. SYSTEM DESCRIPTION

Let $\mathcal{P}$ denote the set of integers $\{0,1,\ldots,N\}$ and let R^q denote the q-fold Cartesian product of the real line. In this work, several vector-valued random sequences will be defined on the discrete time interval $\mathcal{P}$. It will be understood that all expressions containing these random sequences hold on the interval $\mathcal{P}$ unless otherwise specified.

Consider the problem of controlling the noisy system described on $\mathcal{P}$ by

$$x(k+1) = A(k)x(k) + B(k)u(k) + \xi(k) \tag{2}$$

and

$$z(k) = C(k)x(k) + \theta(k) \tag{3}$$

where the state $x(k)\epsilon R^n$, the control action $u(k)\epsilon R^m$, and the observation $z(k)\epsilon R^r$. The initial condition for (2), $x(0)$, is assumed Gaussian with mean

$$x_o = E\{x(0)\} \tag{4}$$

and covariance

$$\Sigma_o = E\{[x(0) - x_o][x(0) - x_o]^T\}, \tag{5}$$

where $E\{\cdot\}$ symbolizes the expectation operation and $(^T)$ denotes matrix transposition.

Let the observation noise, Θ, and the state process noise, ξ, be zero-mean Gaussian white processes with

$$E\{\xi(k)\theta^T(\ell)\} = 0, \tag{6}$$

$$E\{[x(0) - x_o]\xi^T(\ell)\} = 0, \tag{7}$$

$$E\{[x(0) - x_o]\theta^T(\ell)\} = 0, \tag{8}$$

$$E\{\xi(k)\xi^T(\ell)\} = \Xi(k)\delta_{k\ell}, \tag{9}$$

$$E\{\theta(k)\theta^T(\ell)\} = \Theta(k)\delta_{k\ell}, \tag{10}$$

where $\Theta(k)$ and $\Xi(k)$ are positive definite and positive semi-definite on P. Here, $\delta_{k\ell}$ is the Kroneker delta.

Attach to (2) a performance measure, J, defined by

$$J \triangleq \sum_{k=1}^{N} [x^T(k)Q(k)x(k) + u^T(k-1)R(k-1)u(k-1)], \qquad (11)$$

where the weighting matrix, Q(k), is positive semi-definite on P and the weighting matrix, R(k), is positive definite on P. Both Q(k) and R(k) are symmetric matrices.

The control objective is to find a control sequence, u(·), in a particular class (called admissible) such that a certain statistical performance index (formed from mean-conditional-cumulants of J) is minimized. Mean-conditional-cumulants are introduced in Section III. The admissible class of control actions is described as follows.

Let

$$u(k) = f(k;z) \qquad (12)$$

where

$$f: P \times Z[P] \rightarrow R^m$$

is an operator mapping time k and an output sequence z(·) to a control sequence value u(k). The set of output sequences on P is denoted by $Z[P]$.

The admissible class of feedback control laws is determined by f operators that are

1. causal

and

2. satisfy the growth condition

$$||f(k;g) - f(k;h)|| < \varepsilon ||g-h|| \qquad (13)$$

for all $k \in P$, for all $g,h \in Z[P]$, and for some $\varepsilon > o$. An operator, f, is causal if whenever $g(\sigma) = h(\sigma)$ for $\sigma \leq k \in P$, then $f(k;g) = f(k;h)$.

III. CONDITIONAL CUMULANTS OF J

Since feedback control laws are being sought, it is desirable to formulate statistical indices of J such that optimization over these indices will directly couple control action to past observations. The secret to such a formulation lies in the use of the conditional expectation operator, $E\{\cdot|F_k\}$ where F_k is the minimal σ-algebra induced by the set of observations $\{z(\tau):\tau\varepsilon[0,k]\}$. By definition, there are no observations when $k = -1$. When $k = N$, F_N will be written without a subscript. An interpretation of F_k for $k>0$ is that it represents the minimal information necessary to describe uncertainty about $x(0)$, $\{\xi(\tau):\tau\varepsilon[0,k-1]\}$, and $\{\theta(\tau):\tau\varepsilon[0,k]\}$ after $\{z(\tau):\tau\varepsilon[0,k]\}$ is known.

The particular statistical indices presented in this section are conditional cumulants. These objects are selected bacause of their inherent quadratic structure. This property of cumulants of quadratic performance measures was first observed by Liberty in [1] and exploited in [2], [3] and [4]. Optimization over quadratic performance indices will lead to linear control structures.

It is necessary at this point to introduce some notation. It is well known that $E\{x(k)|F\}$ is the smoothed least squares estimate of $x(k)$ which will be denoted by

$$\hat{x}(k|N) \triangleq E\{x(k)|F\}. \tag{14}$$

Let the smoothed estimate error covariance kernel, $\Gamma_s(k,\ell)$ be defined as

$$\begin{aligned}\Gamma_s(k,\ell) &\triangleq E\{[x(k) - \hat{x}(k|N)][x(\ell) - \hat{x}(\ell|N]^T|F\},\\ &= E\{\tilde{x}(k|N)\tilde{x}^T(\ell|N)|F\},\end{aligned} \tag{15}$$

where

$$\tilde{x}(k|N) \triangleq x(k) - \hat{x}(k|N), \tag{16}$$

is the error in the smoothed estimate of the state vector. Then, if the linear Gaussian assumptions of equations (2) through (10) are satisfied, and if the control action $u(\cdot)$ is admissible.

$$E\{\Gamma_s(k,\ell)\} = \Gamma_s(k,\ell). \tag{17}$$

That is, $\Gamma_s(k,\ell)$ is not random.

Next, define the iterated kernel, $\Gamma_s^{\alpha}(k,\ell)$, as

$$\Gamma_s^{\alpha}(k,\ell) \triangleq \sum_{\alpha=1}^{N} \Gamma_s(k,\sigma)Q(\sigma)\Gamma_s^{\alpha-1}(\sigma,\ell), \qquad \alpha>1, \tag{18}$$

where

$$\Gamma_s^{1}(k,\ell) \triangleq \Gamma_s(k,\ell). \tag{19}$$

Also define

$$\eta_{\alpha-1}(k|N) \triangleq \sum_{\ell=1}^{k} \Gamma_s^{\alpha-1}(k,\ell)Q(\ell)\hat{x}(\ell|N)$$

$$- \frac{1}{2}\Gamma_s^{\alpha-1}(k,k)Q(k)\hat{x}(k|N), \ \alpha>1. \tag{20}$$

It can be shown that the conditional cumulants of J can be expressed as

$$\kappa_{1|N} = \sum_{k=1}^{N} TR[\Gamma_s(k,k)Q(k)] +$$

$$+ \sum_{k=1}^{N} [\hat{x}^T(k|N)Q(k)\hat{x}(k|N) + u^T(k-1)R(k-1)u(k-1)] \tag{21}$$

and

$$\kappa_{\alpha|N} = (\alpha-1)!2^{\alpha-1} \sum_{k=1}^{N} TR[\Gamma_s^{\alpha}(k,k)Q(k)]$$

$$+ \alpha!2^{\alpha} \sum_{k=1}^{N} \hat{x}^T(k|N)Q(k)\eta_{\alpha-1}(k|N), \ \alpha>1, \tag{22}$$

where $TR[\cdot]$ denotes the matrix trace operator. Equations (21) and (22) are single summation expressions of the conditional cumulants in terms of dynamical (but physically unrealizable) variables $x(\cdot|N)$ and $\eta_{\alpha-1}(\cdot|N)$. Note that both (21) and (22) contain uncontrollable terms.

IV. THE CUMULANT CONTROL PROBLEM CLASS

The conditional cumulant expressions given by (21) and (22) can be placed under expectation to yield interesting statistical parameters of J. They are interesting because they have a quadratic structure in the dynamical variables $\hat{x}(\cdot|N)$ and $\eta_\alpha(\cdot|N)$, and because these variables can be expressed explicitly in terms of the observation sequence $z(\cdot)$. The precise statistical interpretation of mean-conditional-cumulants is given in [4].

It is now possible to define a class of optimal control problems, called "cumulant control" problems, in terms of mean-conditional-cumulants.

> Let A denote the class of admissible control actions for the system described by (2)-(10) with performance measure J as given in (11). Let $\kappa_{\alpha|N}$, $\alpha\geq 1$, be conditional cumulants of J, and let μ_α, be non-negative scalar constants for $\alpha\geq 2$. Then, for each set of K scalar constants a cumulant control problem is specified by the statement
>
> $$\underset{u\epsilon A}{\text{Minimize}}\ E\{\kappa_{1|N} + \sum_{\alpha=2}^{K} \mu_\alpha \kappa_{\alpha|N}\}$$
>
> subject to the dynamical constraints on $\dot{x}(\cdot|N)$ and $\eta_\alpha(\cdot|N)$.

Note that when $\mu_\alpha = 0$, $2\leq\alpha\leq K$, the cumulant control problem specified is precisely the minimum mean Linear-Quadratic-Gaussian problem since

$$E\{\kappa_{1|N}\} = E\{J\}.$$

Although the class of cumulant control problems is clearly specified, it is not so clear how one should carry out the indicated minimization. The specific point of concern is enforcement of the admissibility requirement. This point is particularly bothersome when one notes (as at the end of Section III) that the conditional cumulants have been explicitly expressed in terms of noncausal dynamical variables. The results summarized in this section resolve this difficulty.

If the conditional cumulants were expressed in terms of causal dynamical variables, which in turn were driven by the observation sequence, then standard optimization techniques, applied under expectation to all control-observation sequence pairs, would couple the control action to these variables and an admissible, optimal-control action would be realized. Clearly, this is not the case for the conditional cumulants. However, it can be shown that portions of the conditional cumulant formulations are zero under expectation. The portions that remain contain only causal variables and optimization over these new objects yields physically realizable control action that solves the original cumulant control problem.

V. CONDITIONAL CUMULANTS UNDER EXPECTATION

Through Section IV only one state estimate has been encountered, namely, the so-called "fixed interval" smoothed estimate, $\hat{x}(k|N)$. Its error covariance kernel has been given the symbol $\Gamma_s(k,\ell)$. In this section, several other well-known least-squares state estimates will arise. These are: the "filtered state estimate,"

$$\hat{x}(k|k) \triangleq E\{x(k)|F_k\}, \tag{23}$$

the "one-step predicted state estimate,"

$$\hat{x}(k|k-1) \triangleq E\{x(k)|F_{k-1}\}, \tag{24}$$

and the "fixed-point smoothed state estimate,"

$$\hat{x}(\tau|k) \triangleq E\{x(\tau)|F_k\},\ \tau\leq k. \tag{25}$$

The error covariance kernels associated with (23) and (24) are useful. They are denoted by $\Gamma_F(k,\ell)$ and $\Gamma_p(k,\ell)$, respectively. Defining a new matrix

$$W_{\alpha-1}(k,\sigma) \triangleq \sum_{\ell=1}^{k} \Gamma_s^{\alpha-1}(k,\ell)Q(\ell)\Gamma_p(\ell,\sigma)-\tfrac{1}{2}\Gamma_s^{\alpha-1}(k,k)Q(k)\Gamma_p(k,\sigma), \tag{26}$$

(20) can be rewritten as

$$\eta_{\alpha-1}(k|N) = \hat{\eta}_{\alpha-1}(k|k) + \sum_{\sigma=k+1}^{N} W_{\alpha-1}(k,\sigma)M(\sigma)\nu(\sigma|\sigma-1), \tag{27}$$

where

$$\hat{\eta}_{\alpha-1}(k|k) \triangleq \sum_{\ell=1}^{k} \Gamma_s^{\alpha-1}(k,\ell)Q(\ell)\hat{x}(\ell|k)-\tfrac{1}{2}\Gamma_s^{\alpha-1}(k,k)Q(k)\hat{x}(k|k). \tag{28}$$

In addition, the fixed-interval smoothed estimate can be written as

$$\hat{x}(\ell|N) = \hat{x}(\ell|k) + \sum_{\sigma=k+1}^{N} \Gamma_p(\ell,\sigma)M(\sigma)\nu(\sigma|\sigma-1), \tag{29}$$

where $\nu(\sigma|\sigma-1)$ is the "innovations process". This separation of the dynamical variables (present in the conditional cumulants) into causal and noncausal parts is the key to resolution of the admissibility difficulty.

It can be shown that the mean-conditional-cumulants can be written entirely in terms of causal variables. These expressions are

$$E\{\kappa_{1|N}\} = E\{\sum_{k=1}^{N} TR[\Gamma_F(k,k)Q(k)]$$

$$+ \sum_{k=1}^{N} [\hat{x}^T(k|k)Q(k)\hat{x}(k|k)+u^T(k-1)R(k-1)u(k-1)]\}$$

$$\triangleq E\{\kappa_1^*\}, \tag{30}$$

and

$$E\{\kappa_{\alpha|N}\} = E\{(\alpha-1)!2^{\alpha-1} \sum_{k=1}^{N} TR[\{\Gamma_s^{\alpha}(k,k)+2\alpha \sum_{\tau=1}^{k} [\Gamma_F(k,\tau)$$

$$- \Gamma_s(k,\tau)]Q(\tau)\Gamma_s^{\alpha-1}(\tau,k)-\alpha[\Gamma_F(k,k)-\Gamma_s(k,k)]Q(k)\Gamma_s^{\alpha-1}(k,k)\}Q(k)]$$

$$+ \alpha!2^{\alpha} \sum_{k=1}^{N} \hat{x}^T(k|k)Q(k)\hat{\eta}_{\alpha-1}(k|k)\}$$

$$\triangleq E\{\kappa_\alpha^*\},\ \alpha>1. \tag{31}$$

Optimization, in the cumulant control problem, over these objects will couple control action to the fixed point smooth estimate of the state. Thus, causal feedback control can be realized.

VI. CONCLUSION

The results summarized in this report open the door to several avenues of investigation and should enhance the applicability of linear stochastic control theory. The details of this work are currently being prepared for journal publication. In addition to the results presented here, an analogous investigation has been conducted for continuous time systems and minor investigations of the cumulant structure as applied to estimation problems have also been carried out. The latter studies have led to interesting observations, but inconclusive results. More research is required here. It would be desirable to relax the Gaussian assumption in this class of problems, but to date no mechanism for doing so has been found for cumulants of order two or more.

Dynamical feedback control laws for certain cumulant control problems have been derived in recursive form and will be published along with the details of the cumulant class derivation. Properties of cumulant controllers are totally unknown. Several suboptimal structures are appealing and should be investigated. Some of these lead to interesting classes of Riccati-type equations about which little is known [5].

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