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MULTI-BODY DYNAMICS
INCLUDING THE EFFECTS OF
FLEXIBILITY AND COMPLIANCE.

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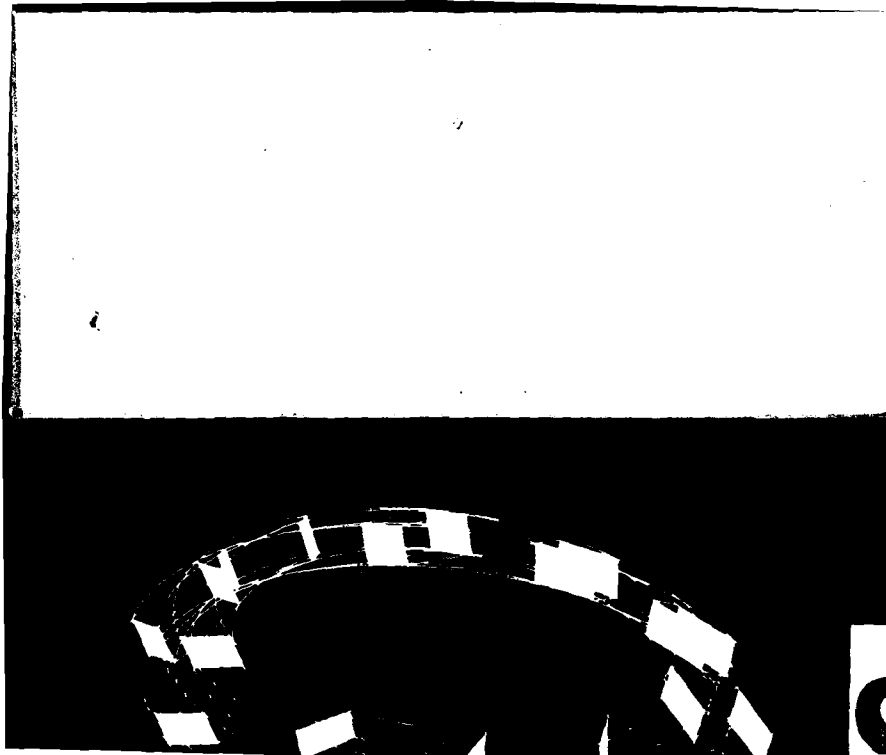
ABSTRACT

New and recently developed concepts and ideas useful
in obtaining efficient computer algorithms for solving the
equations of motion of multi-body mechanical systems with
flexible links are presented and discussed. These ideas
include the use of Euler parameters, Lagrange's form of
d'Alembert's principle, generalized speeds, quasi-coordinates,
relative coordinates, structural analysis techniques and body
connection arrays. The mechanical systems considered are
linked bodies forming a tree structure, but with no "closed
loops" permitted. An explicit formulation of the equations
of motion is presented.

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INTRODUCTION

This report discusses the development of new methods for including the effects of flexibility and link and joint compliance in the governing dynamical equations of multi-body systems. Specifically, new, computationally-oriented techniques with potential for efficient, automated, and comprehensive analyses of multi-body system dynamics are presented and discussed.

The development of equations of motion of multi-body mechanical systems has received considerable attention of analysts in recent years. There are several reasons for this: Foremost, is the fact that many mechanical systems and devices can be effectively modelled by systems of linked bodies. But, another reason is the fact that it has just recently been possible with the aid of high-speed digital computers, to obtain efficient numerical solutions of the governing dynamical equations. Hence, the emphasis of researchers and analysts working with multi-body systems has been the formulation of equations of motion which can easily be developed into numerical algorithms for a computer code.

Most of the recent efforts in obtaining these dynamic formulations and their corresponding computational algorithms has been with systems of linked rigid bodies. Recently, however, a few researchers have attempted to include the effects of flexibility, compliance, and relative translation of the links by using a variety of approaches such as quasi-static methods, finite-element methods, modal analysis, and the strategic positioning of the flexible bodies,(for example, to the extremities of the system). Many of these efforts and the corresponding methodologies have been stimulated and motivated by specific application areas such as mechanism vibration and flexible satellite oscillations. In this report, these ideas are used and extended in the outline of new procedures for efficiently modelling the dynamics of multi-body systems with flexible links and joints.

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If a mechanical system consists of connected bodies such that no closed loops or circuits are formed, the system is called a "general-chain", "open-chain", or "open-tree" system. Figure 1. depicts such a system. References [1-81]*provide summary of approaches taken to obtain efficient, computer-oriented formulation of equations of motion for such systems and related systems. If the mechanical system model of Figure 1. is generalized to include translation and compliance at the joints, it might appear as shown in Figure 2. References 2-143 provide a summary of approaches taken to include the effects of flexibility, translation, and link and joint compliance of these systems.

In one of these approaches it is shown [28, 29, 37, 38, 39, 51] that it is possible to obtain expressions for the governing equations in a form where the coefficients are directly related through computer algorithms. This approach uses Lagrange's form of d'Alembert's principle, as exposited by Kane and others [30, 52, 144, 145, 146] together with body connection arrays [37, 38, 39] and relative orientation coordinates [31, 32, 37, 50] to obtain the governing equations. Lagrange's form of d'Alembert's principle - a virtual work type approach combines the computational advantages of both Newton's laws and Lagrange's equations. That is, it has the advantage of automatic elimination of non-working internal constraint forces but without the introduction of tedious differentiation or other similar calculations.

Recently, it has been suggested by Huston, et. al. [32, 33, 37] that further efficiencies could be obtained through the use of Euler parameters as described by Whittaker [146] and Kane and Likins [147], together with the quasi-coordinates suggested by Kane and Wang [148]. Specifically, it is claimed [32, 33, 37] that using Euler parameters together with relative angular velocity components as generalized coordinate derivatives allows for the avoidance of geometrical singu-

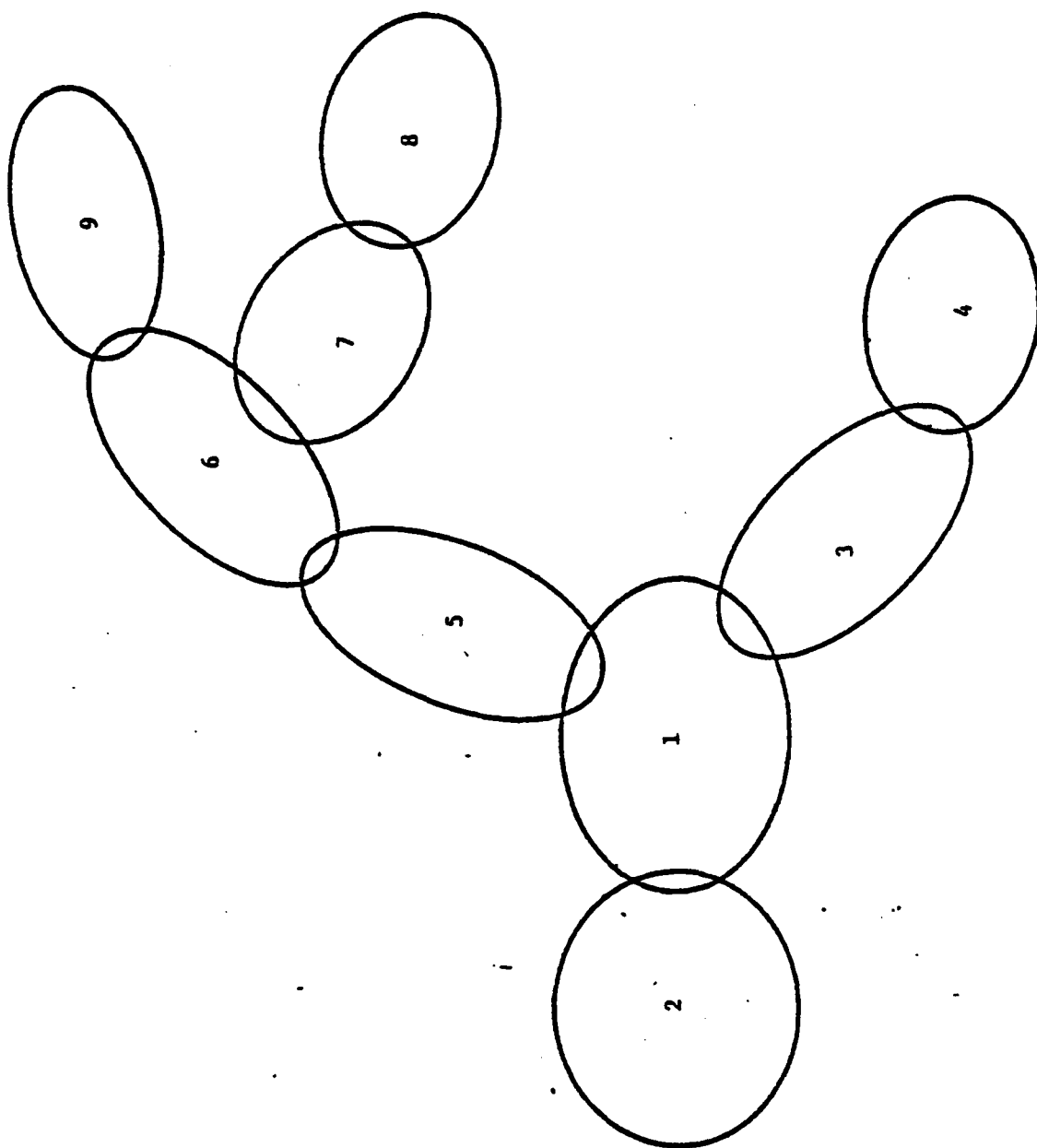* Numbers in brackets refer to References at the end of the report.

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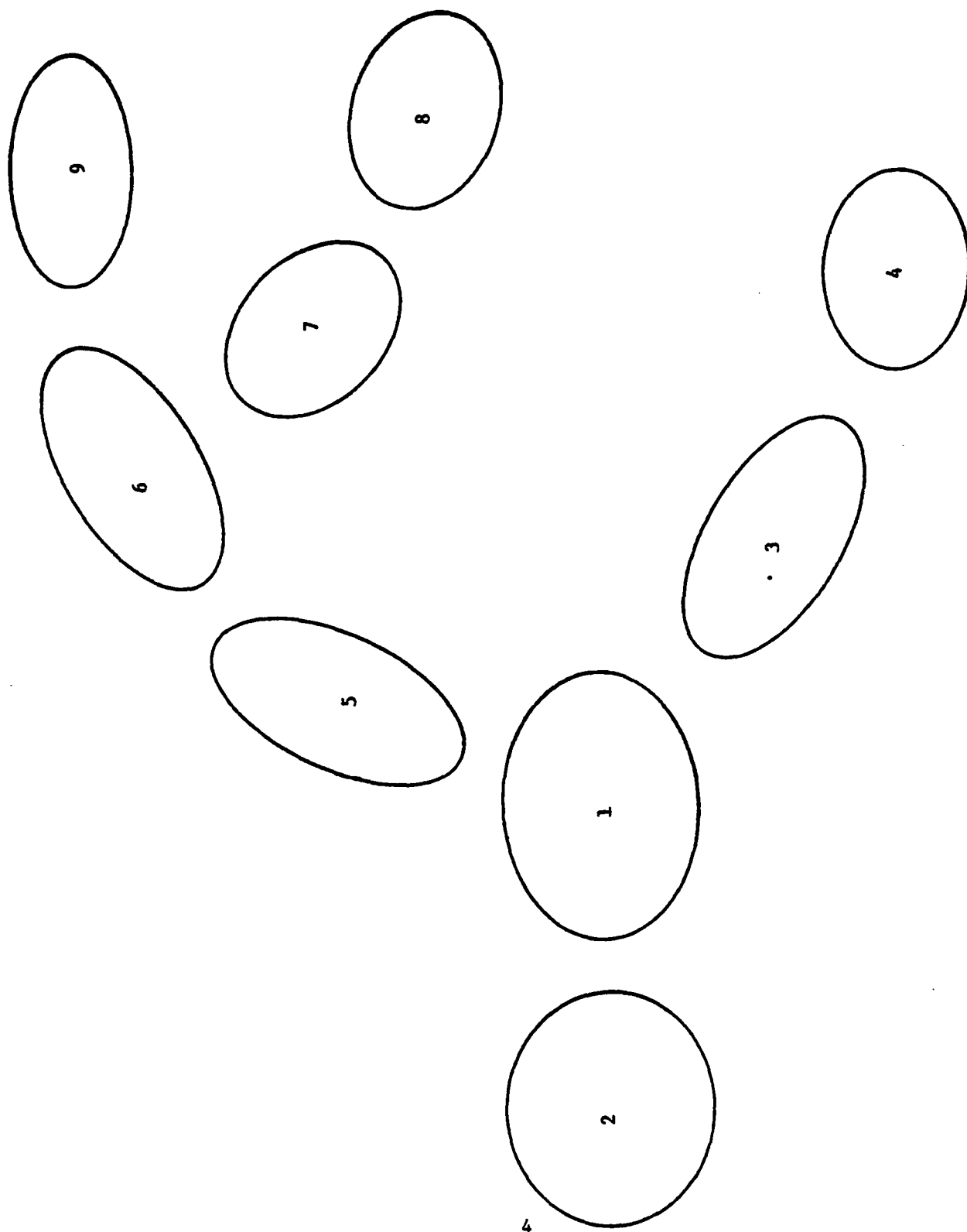Figure 1. A General Chain System

Figure 2. A General Chain System with Translation Between the Bodies

larities encountered with using Euler angles or dextral orientation angles to
define the relative orientation of the bodies.

The use of Lagrange's form of d'Alembert's principle, body connection arrays,
relative orientation coordinates, quasi-coordinates, and Euler parameters also
promises to provide an effective and efficient approach in the modelling, gover-
ning equation formulation, and analysis of multi-body systems with flexible links
and joints. The exposition of these ideas is a primary objective of this report.

The balance of the report is divided into five parts with the first two
parts containing the geometrical and kinematical development. The governing
dynamical equations are developed in the third part. This is followed by an
analysis of the flexibility and compliance effects in the fourth part. The
final part contains a discussion of the developed procedure together with con-
cluding remarks.

PRELIMINARY GEOMETRICAL CONSIDERATIONS

Body Connection Array

Consider a mechanical system such as depicted in Figure 1. To develop an accounting routine for the system's geometry arbitrarily select one of the bodies as a reference body and call it B_1. Next, number or label the other bodies of the system in ascending progression away from B_1 as shown in Figure 1. Now, although this numbering procedure does not lead to a unique labeling of the bodies, it can nevertheless be used to describe the chain structure or topology through the "body connection array" as follows: Let $L(k)$, $k=1,\ldots,N$ be an array of the adjoining lower numbered body of body B_k. For example, for the system shown in Figure 1., $L(k)$ is:

$$L(k) = (0,1,1,3,1,5,6,7,6) \tag{1}$$

where

$$(k) = (1,2,3,4,5,6,7,8,9) \tag{2}$$

and where 0 refers to an inertial reference frame R. It is not difficult to see that, given $L(k)$, one could readily describe the topology of the system. That is, Figure 1. could be drawn by simply knowing $L(k)$. It is shown in the sequel that $L(k)$ is useful in the development of expressions of kinematical quantities needed for analysis of the system's dynamics.

Transformation Matrices

Next, consider a typical pair of adjoining bodies such as B_j and B_k as shown in Figure 3. The general orientation of B_k relative to B_j may be defined in terms of the relative orientation of the dextral orthogonal unit vector sets n_{ji} and n_{ki} ($i=1,2,3$) fixed in B_j and B_k as shown in Figure 2. Specifically n_{ji} and n_{ki} are related to each other as

$$n_{ji} = SJK_{im} n_{km} \tag{3}$$

where SJK is a 3 x 3 orthogonal transformation matrix defined as [47]:

$$SJK_{im} = n_{ji} \cdot n_{km} \tag{4}$$

(Regarding notation, the J and K in SJK and the first subscripts on the unit vectors refer to bodies B_j and B_k, and repeated indices, such as the m, in Equation (3) signify a sum over the range (eg. 1,...,3) of that index. Thus, with a computer SJK_{im} would be the array SJK(I,M).)

From Equation (3), it is easily seen that with three bodies B_j, B_k, B_ℓ, the transformation matrix obeys the following chain and identity rules:

$$SJL = SJK \ SKL \tag{5}$$

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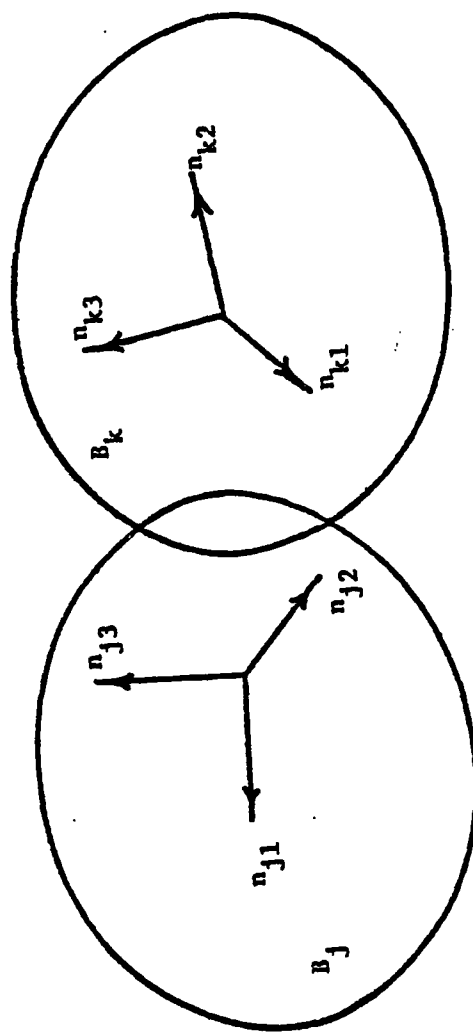
Figure 3. Two Typical Adjoining Bodies

and

$$SJJ = I = SJK \ SKJ = SJK \ SJK^{-1} \tag{6}$$

where I is the identity matrix.

These expressions allow for the transformation of components of vectors referred to one body of the system into components referred to any other body of the system and, in particular, to the inertial reference frame, R. For example, if a typical vector, $\underset{\sim}{V}$, is expressed as

$$\underset{\sim}{V} = V_i^{(k)} \ \underset{\sim}{n}_{ki} = V_i^{(0)} \underset{\sim}{n}_{0i} \tag{7}$$

then

$$V_i^{(0)} = SOK_{ij} \ V_j^{(k)} \tag{8}$$

where 0 refers to the inertial frame, R.

Since these transformation matrices play a central role throughout the analysis, it is helpful to also have an algorithm for their derivative, especially the derivative of SOK. Using Equation (3), and noting that $\underset{\sim}{n}_{0i}$ are fixed in R, the following is obtained:

$$d(SOK_{ij})/dt = \underset{\sim}{n}_{0i} \cdot {}^R d \ \underset{\sim}{n}_{kj}/dt \tag{9}$$

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where the R in $^R d\, n_{kj}/dt$ indicates that the derivative is computed in R. However, since the n_{kj} are fixed in B_k, their derivatives may be written as $\omega_k \times n_{kj}$ where ω_k is the angular velocity of B_k in R. Equation (9) may then be written as:

$$d(SOK_{ij})/dt = -e_{imn}\omega_{kn}\, n_{0m} \cdot n_{kj} \tag{10}$$

or as

$$d(SOK)/dt = WOK\ SOK \tag{11}$$

where WOK is a matrix defined as

$$WOK_{im} = -e_{imn}\omega_{kn} \tag{12}$$

and where ω_{kn} are the components of ω_k referred to n_{on} and e_{imn} is the standard permutation symbol [150]. (WOK is simply the matrix whose dual vector [150] is ω_k.) Equation (11) thus shows that the transformation matrix derivative may be computed by a simple matrix multiplication.

Euler Parameters

Finally, consider describing the relative orientation of B_j and B_k by using the so-called Euler parameters as discussed by Whittaker[147] and Kane and Likins [148] It is well known [147] that B_k may be brought into any

general orientation relative to B_j by means of a single rotation about an appropriate axis. If λ_k is a unit vector along this axis and if θ_k is the rotation angle, the four Euler parameters describing the orientation of B_k relative to B_j may be defined as:

$$\epsilon_{k1} = \lambda_{k1} \quad \sin(\theta_k/2)$$

$$\epsilon_{k2} = \lambda_{k2} \quad \sin(\theta_k/2)$$

$$\epsilon_{k3} = \lambda_{k3} \quad \sin(\theta_k/2) \tag{13}$$

$$\epsilon_{k4} = \cos(\theta_k/2)$$

where the λ_{ki} (i=1,2,3) are the components of λ_k referred to n_{ji}, the unit vectors fixed in B_j. Clearly, the ϵ_{ki} (i=1,2,3,4) are not independent since:

$$\epsilon_{k1}^2 + \epsilon_{k2}^2 + \epsilon_{k3}^2 + \epsilon_{k4}^2 = 1 \tag{14}$$

These parameters may be related to angular velocity components by using the transformation matrices as follows: It is shown in [147, 148] that SJK may be expressed in terms of these parameters as:

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$$SJK = \begin{bmatrix} \epsilon_{k1}^2 - \epsilon_{k2}^2 - \epsilon_{k3}^2 + \epsilon_{k4}^2 & 2(\epsilon_{k1}\epsilon_{k2} - \epsilon_{k3}\epsilon_{k4}) & 2(\epsilon_{k1}\epsilon_{k3} + \epsilon_{k2}\epsilon_{k4}) \\ 2(\epsilon_{k1}\epsilon_{k2} + \epsilon_{k3}\epsilon_{k4}) & -\epsilon_{k1}^2 + \epsilon_{k2}^2 - \epsilon_{k3}^2 + \epsilon_{k4}^2 & 2(\epsilon_{k2}\epsilon_{k3} - \epsilon_{k1}\epsilon_{k4}) \\ 2(\epsilon_{k1}\epsilon_{k3} - \epsilon_{k2}\epsilon_{k4}) & 2(\epsilon_{k2}\epsilon_{k3} + \epsilon_{k1}\epsilon_{k4}) & -\epsilon_{k1}^2 - \epsilon_{k2}^2 + \epsilon_{k3}^2 + \epsilon_{k4}^2 \end{bmatrix} \tag{15}$$

Now, by solving Equations (11) and (12) for the angular velocity components, one obtains:

$$\omega_{k1} = SOK_{21}\, S\dot{O}K_{31} + SOK_{22}\, S\dot{O}K_{32} + SOK_{23}\, S\dot{O}K_{33}$$

$$\omega_{k2} = SOK_{31}\, S\dot{O}K_{11} + SOK_{32}\, S\dot{O}K_{12} + SOK_{33}\, S\dot{O}K_{13} \tag{16}$$

$$\omega_{k3} = SOK_{11}\, S\dot{O}K_{21} + SOK_{12}\, S\dot{O}K_{22} + SOK_{13}\, S\dot{O}K_{23}$$

where the dot designates time differentiation. By using Equation (15), these expressions may be used to express the n_{ji} components of the angular velocity of B_k relative to B_j in terms of the Euler parameters as:

$$\hat{\omega}_{k1} = 2(\epsilon_{k4}\dot{\epsilon}_{k1} - \epsilon_{k3}\dot{\epsilon}_{k2} + \epsilon_{k2}\dot{\epsilon}_{k3} - \epsilon_{k1}\dot{\epsilon}_{k4})$$

$$\hat{\omega}_{k2} = 2(\epsilon_{k3}\dot{\epsilon}_{k1} + \epsilon_{k4}\dot{\epsilon}_{k2} - \epsilon_{k1}\dot{\epsilon}_{k3} - \epsilon_{k2}\dot{\epsilon}_{k4}) \tag{17}$$

$$\hat{\omega}_{k3} = 2(-\epsilon_{k2}\dot{\epsilon}_{k1} + \epsilon_{k1}\dot{\epsilon}_{k2} + \epsilon_{k4}\dot{\epsilon}_{k3} - \epsilon_{k3}\dot{\epsilon}_{k4})$$

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(Regarding notation, in the sequel "hats" refer to relative angular velocity vectors or their components. That is the $\underset{\sim}{\omega}_k$ represent the angular velocity of B_k in R and $\hat{\underset{\sim}{\omega}}_k$ represent the angular velocity of B_k relative to B_j, its adjoining lower numbered body.) Equation (17) may now be solved for the $\hat{\varepsilon}_{ki}$ (i=1,...,4) in terms of the $\hat{\omega}_{ki}$, leading to the expressions:

$$\dot{\varepsilon}_{k1} = \tfrac{1}{2}(\varepsilon_{k4}\,\hat{\omega}_{k1} + \varepsilon_{k3}\,\hat{\omega}_{k2} - \varepsilon_{k2}\,\hat{\omega}_{k3})$$

$$\dot{\varepsilon}_{k2} = \tfrac{1}{2}(-\varepsilon_{k3}\,\hat{\omega}_{k1} + \varepsilon_{k4}\,\hat{\omega}_{k2} + \varepsilon_{k1}\,\hat{\omega}_{k3})$$

$$\dot{\varepsilon}_{k3} = \tfrac{1}{2}(\varepsilon_{k2}\,\hat{\omega}_{k1} - \varepsilon_{k1}\,\hat{\omega}_{k2} + \varepsilon_{k4}\,\hat{\omega}_{k3})$$ (18)

$$\dot{\varepsilon}_{k4} = \tfrac{1}{2}(-\varepsilon_{k1}\,\hat{\omega}_{k1} - \varepsilon_{k2}\,\hat{\omega}_{k2} - \varepsilon_{k3}\,\hat{\omega}_{k3})$$

This solution is quickly obtained by observing that if Equation (14) is differentiated and placed with Equation (17), the resulting set of equations could be written in the matrix form:

$$
\begin{bmatrix} \hat{\omega}_{k1} \\ \hat{\omega}_{k2} \\ \hat{\omega}_{k3} \\ \hat{\omega}_{k4} \end{bmatrix}
= 2
\begin{bmatrix}
\varepsilon_{k4} & -\varepsilon_{k3} & \varepsilon_{k2} & -\varepsilon_{k1} \\
\varepsilon_{k3} & \varepsilon_{k4} & -\varepsilon_{k1} & -\varepsilon_{k2} \\
-\varepsilon_{k2} & \varepsilon_{k1} & \varepsilon_{k4} & -\varepsilon_{k3} \\
\varepsilon_{k1} & \varepsilon_{k2} & \varepsilon_{k3} & \varepsilon_{k4}
\end{bmatrix}
\begin{bmatrix} \dot{\varepsilon}_{k1} \\ \dot{\varepsilon}_{k2} \\ \dot{\varepsilon}_{k3} \\ \dot{\varepsilon}_{k4} \end{bmatrix}
$$ (19)

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where ω_{k4} is equal to the derivative of Equation (14) and has the value zero. The square matrix in Equation (19) is seen to be orthogonal (ie. the inverse is the transpose) and hence, Equations (18) follow immediately from (19) upon letting ω_{k4} be zero.

Coordinates

A multibody system of N bodies, with translation permitted between the bodies will, in general, have 6N degrees of freedom. Let these be described by 6N generalized coordinates x_ℓ ($\ell=1,\ldots,6N$) and let the first 3N of these be divided into N triplets describing the relative orientation of the successive bodies of the system. Let the remaining 3N x_ℓ also be divided into N triplets representing the relative displacement of the successive bodies of the system. As before, let B_k be a typical body of the system and let B_j be its adjacent lower numbered body, as in Figure 3. The angular velocity of B_k relative to B_j (that is, the relative rate of change of *orientation*) may then be written as:

$$\hat{\underset{\sim}{\omega}}_k = \hat{\omega}_{k1}\, \underset{\sim}{n}_{j1} + \hat{\omega}_{k2}\, \underset{\sim}{n}_{j2} + \hat{\omega}_{k3}\, \underset{\sim}{n}_{j3} \tag{20}$$

where $\underset{\sim}{n}_{ji}$ ($j=1,\ldots,N$, $i=1,2,3$) are mutually perpendicular dextral unit vectors fixed in B_j. Next, let these bodies be displaced relative to each other with the displacement measured by the vector $\underset{\sim}{\xi}_k$ as shown in Figure 4., where 0_j and 0_k are arbitrarily selected reference points of B_j and B_k. Q_k, which is fixed in B_j, is the connection point or "origin" of B_k. Then $\underset{\sim}{\xi}_k$ may be written in the form:
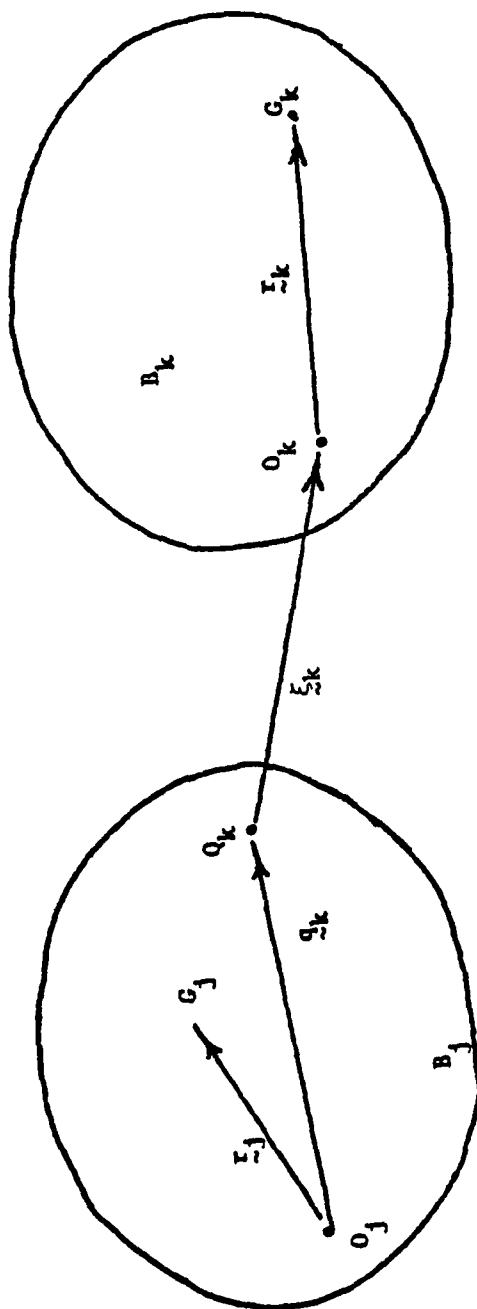
Figure 4. Reference Points and Position Vectors of Two Typical Adjoining Bodies

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$$\underset{\sim}{\xi}_k = \xi_{k1} \, \underset{\sim}{n}_{j1} + \xi_{k2} \, \underset{\sim}{n}_{j2} + \xi_{k3} \, \underset{\sim}{n}_{j3} \tag{21}$$

Following Kane and Wang[149] introduce 6N parameters y_ℓ ($\ell = 1, \ldots, 6N$) defined as:

$$y_\ell = \dot{x}_\ell \quad \ell = 1, \ldots, 6N \tag{22}$$

where the first 3N of these are called "generalized speeds", are

$$y_{3k-2} = \hat{\omega}_{k1}$$

$$y_{3k-1} = \hat{\omega}_{k2} \tag{23}$$

$$y_{3k} = \hat{\omega}_{k3}$$

and the remaining 3N are:

$$y_{3(N+k)-2} = \dot{\xi}_{k1}$$

$$y_{3(N+k)-1} = \dot{\xi}_{k2} \tag{24}$$

$$y_{3(N+k)} = \dot{\xi}_{k3}$$

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In general, Equations (23) are non-integrable. That is, they cannot
be integrated to obtain generalized orientation coordinates x_{3k-2} ,
x_{3k-1} , x_{3k} . Thus, explicit parameters x_{3k-2}, x_{3k-1}, and x_{3k} do not
in general exist—hence, the name "quasi-coordinates". However, since
parameters are needed to relate the relative orientation of the bodies
to the respective relative angular velocities, let the Euler parameters
introduced in the foregoing section be used for this purpose. Hence, if the
orientation of a typical body B_k relative to B_j is described by the four
parameters ε_{ki} (i=1,...,4),the geometry and kinematics of the entire system
may be expressed in terms of the 4N Euler parameters ε_{ki} (k=1,...,N; i=1,...,4),
the 3N relative angular velocity components $\hat{\omega}_{ki}$ (k=1,...,N; i=1,2,3), and the
3N displacement components ξ_{ki} (k=1,...,N; i=1,2,3).

Angular Velocity

The angular velocity of a typical body B_k in the inertial frame R is
readily obtained by the addition formula as [145]

$$\underline{\omega}_k = \hat{\underline{\omega}}_1 + \ldots + \hat{\underline{\omega}}_k \tag{25}$$

where the relative angular velocities on the right side of this expression
are each with respect to the respective adjacent lower numbered bodies and
where the sum is taken over the bodies of the chain from B_1 outward through
the branch containing B_k. The L(k) array introduced in the foregoing section
can be useful in computing this sum: Consider for example, the system shown
in Figure 1. The angular velocity of B_9 is:

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$$\omega_9 = \hat{\omega}_1 + \hat{\omega}_5 + \hat{\omega}_6 + \hat{\omega}_9 \qquad (26)$$

The subscript indices (ie. 9,6,5,1) may be obtained from L(k) as follows: Consider L(k) as a function mapping the (k) array (See Equation (2)) into the L(k) array. Then, using the notation that $L^0(k) = (k)$, $L^1(k) = L(L^0(k))$, $L^2(k) = L(L^1(k))$, ..., $L^j(k) = L(L^{j-1}(k))$, it is seen (see Equation (1)) that:

$$L^0(9) = 9, \ L^1(9) = 6, \ L^2(9) = 5, \ L^3(9) = 1 \qquad (27)$$

Therefore, ω_9 may be written as:

$$\omega_9 = \sum_{p=0}^{3} \hat{\omega}_q \ , \quad q = L^p(9) \qquad (28)$$

Hence, in general, the angular velocity of B_k may be written as:

$$\omega_k = \sum_{p=0}^{r} \hat{\omega}_q, \quad q = L^p(k) \qquad (29)$$

where r is the index such that $L^r(k) = 1$ and it is obtained by comparing $L^p(k)$ to 1. The index r represents the number of bodies from B_1 to B_k in that branch of the chain system B_k. For example, for the system of Figure 1., if k=9, r=3. Equation (29) is thus an algorithm for determining ω_k once $\hat{\omega}_k$ and L(k) are known.

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By examining Equations (20, (23), and (25) it is seen that ω_k may be written in the form

$$\omega_k = \omega_{k\ell m} \, y_\ell \, n_{om} \tag{30}$$

where there is a sum over the repeated indices and where $\omega_{k\ell m}$ $(k=1,\ldots,N;$ $\ell=1,\ldots,3N; m=1,2,3)$ form a block array of coefficients needed to express ω_k in terms of n_{om}. In view of Equations (3), (16), (20), and (23), it is seen that the elements of the $\omega_{k\ell m}$ array may be obtained from the SOK transformation matrices. Moreover, it can be shown that the matching between the elements of the $\omega_{k\ell m}$ and SOK arrays is solely dependent upon the body connection array $L(k)$.

To see this, consider for example the angular velocity of B_4 of the system of Figure 1: From Equation (25), ω_4 is

$$\omega_4 = \hat{\omega}_1 + \hat{\omega}_3 + \hat{\omega}_4 \tag{31}$$

where from Equations (3), (20), and (23) $\hat{\omega}_1$, $\hat{\omega}_3$, and $\hat{\omega}_4$ may be written as:

$$\hat{\omega}_1 = y_1 n_{01} + y_2 \, n_{02} + y_3 \, n_{03} = y_j \, \delta_{mj} \, n_{0m} \tag{32}$$

$$\hat{\omega}_3 = y_7 \, n_{11} + y_8 \, n_{12} + y_9 \, n_{13} = y_{6+j} \, SO1_{mh} \, n_{0m} \tag{33}$$

$$\hat{\omega}_4 = y_{10} \, n_{31} + y_{11} \, n_{32} + y_{12} \, n_{33} = y_{9+j} \, SO3_{mj} \, n_{0m} \tag{34}$$

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Hence, the $\omega_{4\ell m}$ are:

$$\omega_{4\ell m} = \begin{cases} \delta_{m\ell} & \ell = 1,2,3 \\ 0 & \ell = 4,5,6 \\ SO1_{m\ell-9} & \ell = 7,8,9 \qquad m = 1,2,3 \\ SO3_{m\ell-9} & \ell = 10,11,12 \\ 0 & \ell > 12 \end{cases} \qquad (35)$$

where δ_{ij} are the identity matrix components [150].

Next, consider that the results such as Equation (35) may be obtained for the entire system of Figure 1. or Figure 2. from a table such as Table 1., where the "m" entries of the $\omega_{k\ell m}$ array are the column of the transformation matrices. Finally, note that the non-zero entries in a typical row, say the kth row of Table 1. are obtained as follows: Let P = L(k). Then SOP is placed in the kth column of triplets of $\dot{x}_\ell$. Next, let Q=L(P). The SOQ is placed in the Pth column to triplets of $\dot{x}_\ell$, etc. That is, SOM is placed in column $L^{j-1}(k)$ where $M = L^j(k)$, j=1,...,r+1 with r determined from $L^r(k) = 1$.

Finally, it is interesting to note that the elements of the $\omega_{k\ell m}$ array (and hence, the transformation matrix columns of Table 1.) are components of the "partial angular velocity vectors" as originally defined by Kane [144].

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$$\omega_{k\ell m}$$

x_ℓ / k	1,2,3	4,5,6	7,8,9	10,11,12	13,14,15	16,17,18	19,20,21	22,23,24	25,26,27
1	I	0	0	0	0	0	0	0	0
2	I	S01	0	0	0	0	0	0	0
3	I	0	S01	0	0	0	0	0	0
4	I	0	S01	S03	0	0	0	0	0
5	I	0	0	0	S01	0	0	0	0
6	I	0	0	0	S01	S05	0	0	0
7	I	0	0	0	S01	S05	S06	0	0
8	I	0	0	0	S01	S05	S06	S07	0
9	I	0	0	0	S01	S05	0	0	S06

Table 1. $\omega_{k\ell m}$ for the System of Figure 1.

Angular Acceleration

The angular acceleration of B_k in R may be obtained by differentiating Equation (30). Noting that the $\underset{\sim}{n}_{om}$ are constant, this leads to:

$$\underset{\sim}{\alpha}_k = (\dot{\omega}_{k\ell m}\ y_\ell + \omega_{k\ell m}\ \dot{y}_\ell)\ \underset{\sim}{n}_{om} \qquad (36)$$

A table containing the $\dot{\omega}_{k\ell m}$ can be constructed directly form the corresponding table for the $\omega_{k\ell m}$. For example, for the system of Figure 1., such a table is shown in Table 2.

Mass Center Velocities

The velocity and acceleration of the mass center G_k of a typical body B_k (k=1,...,N) may be obtained as follows: Let $\underset{\sim}{r}_k$ locate G_k relative 0_k as shown in Figure 4. Since 0_k is located relative to Q_k by $\underset{\sim}{\xi}_k$ and if Q_k is located relative to 0_j by the vector $\underset{\sim}{q}_k$ (See Figure 4.), then by continuing this procedure, G_k may ultimately be located relative to a fixed point 0 in R, the inertial reference frame. For example, for Body B_8 of Figure 2., the position vector $\underset{\sim}{P}_8$ of G_8 relative to 0 is:

$$\underset{\sim}{P}_8 = \underset{\sim}{\xi}_1 + \underset{\sim}{q}_5 + \underset{\sim}{\xi}_5 + \underset{\sim}{q}_6 + \underset{\sim}{\xi}_6 + \underset{\sim}{q}_7 + \underset{\sim}{\xi}_7 + \underset{\sim}{q}_8 + \underset{\sim}{\xi}_8 + \underset{\sim}{r}_8 \qquad (37)$$

In general, for Body B_k, the position vector $\underset{\sim}{P}_k$ of B_k relative to 0 is:

$$\dot{\omega}_{k\ell m}$$

B_k \ x_ℓ k	1,2,3	4,5,6	7,8,9	10,11,12	13,14,15	16,17,18	19,20,21	22,23,24	25,26,27
1	0	0	0	0	0	0	0	0	0
2	0	$s\dot\delta_1$	0	0	0	0	0	0	0
3	0	0	$s\dot\delta_1$	0	0	0	0	0	0
4	0	0	$s\dot\delta_1$	$s\dot\delta_3$	0	0	0	0	0
5	0	0	0	0	$s\dot\delta_1$	0	0	0	0
6	0	0	0	0	$s\dot\delta_1$	$s\dot\delta_5$	0	0	0
7	0	0	0	0	$s\dot\delta_1$	$s\dot\delta_5$	$s\dot\delta_6$	0	0
8	0	0	0	0	$s\dot\delta_1$	$s\dot\delta_5$	$s\dot\delta_6$	$s\dot\delta_7$	0
9	0	0	0	0	$s\dot\delta_1$	$s\dot\delta_5$	0	0	$s\dot\delta_6$

Table 2. $\dot{\omega}_{k\ell m}$ for the System of Figure 1.

$$\underset{\sim}{P}_k = [SOK_{1h} \; r_{kh} + \sum_{q=0}^{u} \; SOS_{1h} \; (q_{sh} + \xi_{sh})] \underset{\sim}{n}_{o1} \qquad (38)$$

where $s = L^q(k)$, $S = L^{q+1}(k)$, and u is the index such that $L^u(k) = 1$,

and where $\underset{\sim}{q}_1$ is 0. By differentiating, the velocity of G_k in R is obtained

as:

$$\underset{\sim}{v}_k = \{S\dot{O}K_{1h} \; r_{ks} + \sum_{q=0}^{u} \; [S\dot{O}S_{1h} \; (q_{sh} + \xi_{sh})$$

$$+ SOS_{1h} \; \dot{\xi}_{sh}] \; \underset{\sim}{n}_{o1} \qquad (39)$$

By using Equations (11), (12), and (30), $\underset{\sim}{v}_k$ may be written in the form:

$$\underset{\sim}{v}_k = v_{k\ell m} \; y_\ell \; \underset{\sim}{n}_{om} \qquad (40)$$

where $v_{k\ell m}$ (k=1,...,N; ℓ=1,...,6N; m=1,2,3) form a block array of coefficients

needed to express $\underset{\sim}{v}_k$ in terms of $\underset{\sim}{n}_{om}$. In view of Equation (39), the

non-zero $v_{k\ell m}$ are:

$$v_{k\ell m} = WK_{mh\ell} \; r_{kj} + \sum_{q=0}^{u} \; WS_{mh\ell} (\xi_{sh} + q_{sh})$$

$$(k=1,...,N; \; \ell=1,...,3N; \; m=1,2,3) \qquad (41)$$

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where WK_{mh} is defined as:

$$WK_{mh\ell} = \frac{\partial WOK_{mp}}{\partial 2y_\ell} \, SOK_{ph} = -e_{mpi} \, \omega_{k\ell i} \, SOK_{ph} \tag{42}$$

and

$$v_k(3N+\ell)m = \omega_{k\ell m} \quad (k=1,\dots,N; \; \ell=1,\dots,3N; \; m=1,2,3) . \tag{43}$$

The elements of the $v_{k\ell m}$ array are components of the "partial velocity vectors" as originally defined by Kane [144].

Mass Center Accelerations

Similarly, by differentiation of Equations (40), the acceleration of G_k in R is

$$\underline{a}_k = (\dot{v}_{k\ell m} \, y_\ell + v_{k\ell m} \, \dot{y}_\ell)\underline{n}_{om} \tag{44}$$

where the non-zero $\dot{v}_{k\ell m}$ are, by Equations (41) to (43),

$$\dot{v}_{k\ell m} = W\dot{K}_{mh\ell} \, r_{kh} + \sum_{q=0}^{u-1} \, [W\dot{S}_{mh\ell} \, (\xi_{sh} + q_{sh}) + WS_{mh\ell}\xi_{sh}] \quad (k=1,\dots,N;$$

$$\ell=1,\dots,3N, \; m=1, 2, 3) \tag{45}$$

where $W\dot{K}_{mh\ell}$ is:

$$W\dot{K}_{mh\ell} = -e_{mpi} \, (\dot{\omega}_{k\ell i} \, SOK_{ph} + \omega_{k\ell i} \, S\dot{O}K_{ph}) \tag{46}$$

and

$$\dot{v}_k (3N+\ell)_m = \dot{\omega}_{k\ell m} \quad (k=1,\ldots,N; \; \ell=1,\ldots,3N, \; m=1,2,3) \tag{47}$$

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EQUATIONS OF MOTION

Consider again a general chain system such as shown in Figure 2., and imagine the system to be subjected to an externally applied force field. Let the force field on a typical body B_k, be replaced by an equivalent force field consisting of a single force $\underset{\sim}{F}_k$, passing through G_k together with a couple with torque $\underset{\sim}{M}_k$. Then Lagrange's form of d'Alembert's principle leads to governing dynamical equations of motion of the form [38]:

$$F_\ell + F_\ell{}^* = 0 \qquad \ell = 1,\ldots,6N \tag{48}$$

F_ℓ ($\ell=1,\ldots,6N$) is called the generalized active force and is given by:

$$F_\ell = v_{k\ell m} F_{km} + \omega_{k\ell m} M_{km} \tag{49}$$

where there is a sum from 1 to N on k and from 1 to 3 on m, and where F_{mk} and M_{km} are the components of $\underset{\sim}{F}_k$ and $\underset{\sim}{M}_k$ with respect to $\underset{\sim}{n}_{om}$. $F_\ell{}^*$ ($\ell=1,\ldots,6N$) is called the generalized inertia force and is given by:

$$F_\ell{}^* = v_{k\ell m} F_{km}{}^* + \omega_{k\ell m} M_{km}^* \tag{50}$$

where the indices follow the same rules as in Equation (48), and where F_{km}^* and M_{km}^* are $\underset{\sim}{n}_{om}$ components of inertia forces, $\underset{\sim}{F}_k^*$, and inertia torques, $\underset{\sim}{M}_k^*$, given by [145].

$$\underset{\sim}{F}_k^* = -m_k \, \underset{\sim}{a}_k \qquad \text{(no sum)} \tag{51}$$

and

$$\underset{\sim}{M}_k^* = -\underset{\sim}{I}_k \cdot \underset{\sim}{\alpha}_k - \underset{\sim}{\omega}_k \times (\underset{\sim}{I}_k \cdot \underset{\sim}{\omega}_k) \qquad \text{(no sum)} \tag{52}$$

where m_k is the mass of B_k and $\underset{\sim}{I}_k$ is the inertia dyadic of B_k relative to G_k ($k=1,\ldots,N$). ($\underset{\sim}{F}_k^*$, with line of action passing through G_k together with $\underset{\sim}{M}_k^*$ are equivalent to the inertia forces on B_k [145]. Through use of the shifter transformation matrices, $\underset{\sim}{I}_k$ may be written in the form:

$$\underset{\sim}{I}_k = I_{kmn} \, \underset{\sim}{n}_{om} \, \underset{\sim}{n}_{on} \tag{53}$$

By substituting Equations (36) and (44) into Equations (51) and (52) and ultimately into Equation (47), the equations of motion may be written in the form:

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$$a_{\ell p} \, \dot{y}_p = f_\ell \quad (\ell = 1, \ldots, 6N) \tag{54}$$

where there is a sum from 1 to 6N on p and where $a_{\ell p}$ and f_ℓ are
given by:

$$a_{\ell p} = m_k \, v_{kpm} \, v_{k\ell m} + I_{kmn} \, \omega_{kpm} \, \omega_{k\ell n} \tag{55}$$

and

$$f_\ell = - (F_\ell + m_k \, v_{k\ell m} \, v_{kqm} \, y_q + I_{kmn} \, \omega_{k\ell m} \, \dot{\omega}_{kqn} \, y_q$$

$$+ \, e_{nmh} \, I_{kmr} \, \omega_{kqn} \, \omega_{ksr} \, \omega_{k\ell n} \, y_q \, y_s) \tag{56}$$

where there is a sum from 1 to N on k, from 1 to 6N on q and s, and from
1 to 3 on the other repeated indices.

Recall that the first 3N y_p are relative angular velocity components.
These may be related to the Euler parameters by N sets of first order
equations of the form of Equations (18).

Equations (54), (20), and the 4N equations of the form of Equations
(18) form a set of 13N simultaneous first-order differential equations for
the 6N y_p, the 3N ξ_{ki}, and the 4N Euler parameters ϵ_{ki} (h=1,...,N;
i=1,...,4). Since the coefficients $a_{\ell p}$ and f_ℓ in Equations (54) are
algebraic functions of the physical parameters and the four block arrays
$\omega_{k\ell m}$, $\dot{\omega}_{k\ell m}$, $v_{k\ell m}$ and $\dot{v}_{k\ell m}$, computer algorithms can be written for the
numerical development of these governing equations. Moreover, once these

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arrays are developed, the system of equations consisting of Equations (54),
(20), and 4N equations of the form of Equations (18), may also be solved
numerically by using one of the standard numerical integration routines
and a linear equation solver.

The development of these computer algorithms and the numerical development
of Equations (54) might proceed as follows: First, let the body connection
array $L(k)$ (See Equation (1)) together with the geometrical and physical
parameters r_k, ξ_k, I_k, and m_k (See Equations (38), (51), and (52).) and
the applied forces and moments F_k and M_k (See Equation (48).) be read into
the computer. (Let r_k, ξ_k, I_k and, if desired, F_k and M_k be expressed in
terms of n_{ki}.) Next, from assumed initial values of ϵ_{ki} form the
transformation matrix arrays SOK using Equations (15) and (5). Use these
arrays to express r_k, ξ_k, I_k and possibly F_k and M_k in terms of n_{ok}. Next,
using $L(k)$ and SOK write an algorithm, with Tables 1. and 2. as a guide, to
form $\omega_{k\ell m}$ and $\dot{\omega}_{k\ell m}$. For example, to obtain the non-zero $\omega_{k\ell m}$, observe that
if $L(k) = p$, then $\omega_{k\ell m} = SOP_{m\ell}$ ($m=1,2,3$; $\ell=3p+1$, $3p+2$, $3p+3$). Then, if
$L(p) = q$, $L^2(k) = q$ and $\omega_{k\ell m} = SOQ_m$ ($m=1,2,3$; $\ell=3q+1$, $3q+2$, $3q+3$).
This assignment procedure is continued until unity is reached or r times
where r is given by $L^r(k) = 1$ (See the remark following Equation (29).).
$v_{k\ell m}$ and $\dot{v}_{k\ell m}$ may then be obtained using Equations (40) to (47). Finally,
numerical values of the coefficients $a_{\ell p}$ and f_ℓ of the governing differential
equations (54) may then be obtained from Equations (55) and (56). These
equations may then be integrated numerically to obtain incremental values to the
initial values of the parameters y_p, ϵ_{ki}, and x_q ($p=1,\ldots,3N+3$; $k=1,\ldots,N$; $i=1,2,3,4$
and $q=1,2,3$), at the end of a time interval, say t_1. New values of the

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transformation matrix arrays SOK may then be obtained and the entire process repeated until a history of the configuration and motion of the system is determined.

The application of these expressions and ideas in an analysis of the flexibility and compliance effects is developed in the following part of the report.

EFFECTS OF COMPLIANCE AND FLEXIBILITY

Compliance

Let the term "compliance" refer to the yielding or deformation of the system due to the externally applied forces and due to the inertia forces. If the assumption is made that the compliance of a link or joint is "small" compared with the general dimensions of the system, then the effects of the compliance can be determined directly from the integration of Equations (54).

To see this, consider a typical integration step as described at the end of the foregoing section. If the $v_{k\ell m}$, $\dot{v}_{k\ell m}$, $\omega_{k\ell m}$, $\dot{\omega}_{k\ell m}$, ε_{ki}, y_ℓ, and $\dot{y}_\ell$ are known, all the kinematics is known. That is, by using Equations (30) and (36) the angular velocity and angular acceleration of each body is determined. Similarly, by using Equations (40) and (44), the velocities and accelerations of the mass centers of each of the bodies is determined. Then, by using Equations (51) and (52) the equivalent inertia force system on each body is determined. Hence, since the externally applied force field on each body is also known, the entire force system on each body is determined. Therefore, by taking successive free body diagrams of the bodies of the system, starting with the Nth body and working backward through the chain, the force system transmitted across each connection joint may be determined. Finally, by knowing the complete force system acting on each body, including the forces transmitted across the connection joints, the physical force–deformation relations may be used to determine the compliance. Then by addition and superposition the compliance of the entire system is determined.

To illustrate this procedure in more detail, assume, for example, that the bodies of the system are long slender members which can be modelled as beams with uniform cross section. Hence, the system of Figures 1. and 2. might appear as

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shown in Figure 5., and a typical member of this system might be depicted as in Figure 6., where a rectangular shape is assumed and where an axes system is introduced. As before O_j is the connection point of the adjacent lower numbered body B_i and Q_k is the connection point with the adjacent higher numbered body B_k.

Let the forces exerted on B_j by B_i and B_k at the connection joints O_j and Q_k be represented by single forces $\hat{\underset{\sim}{f}}_{i/j}$ and $\hat{\underset{\sim}{f}}_{k/j}$ passing through O_j and Q_k together with couples with torques $\hat{\underset{\sim}{m}}_{i/j}$ and $\hat{\underset{\sim}{m}}_{k/j}$. Similarly, let the externally applied force system on B_j together with the inertia force system of B_j be represented by equivalent force systems at the ends O_j and Q_k of B_j. Hence, let the <u>resultant</u> force system exerted on B_j at Q_k be represented by the single force $\underset{\sim}{f}_{k/j}$ passing through Q_k, together with a couple with torque $\underset{\sim}{m}_{k/j}$ and let $\underset{\sim}{f}_{k/j}$ and $\underset{\sim}{m}_{k/j}$ be expressed in the forms:

$$\underset{\sim}{f}_{k/j} = f_{j1} \underset{\sim}{n}_{j1} + f_{j2} \underset{\sim}{n}_{j2} + f_{j3} \underset{\sim}{n}_{j3} \tag{57}$$

and

$$\underset{\sim}{m}_{k/j} = m_{j1} \underset{\sim}{n}_{j1} + m_{j2} \underset{\sim}{n}_{j2} + m_{j3} \underset{\sim}{n}_{j3}$$

(Note, that from equilibrium considerations, the force system exerted on B_j at O_j is equivalent to a single force $-\underset{\sim}{f}_{k/j}$ passing through O_j together with a couple with torque $-\underset{\sim}{m}_{k/j}$.)

Let the displacement of Q_k relative to O_j due to the beam compliance, or deformation, be represented by $\underset{\sim}{u}_j$. Let $\underset{\sim}{u}_j$ be written in the form:

$$\underset{\sim}{u}_j = u_{j1} \underset{\sim}{n}_{j1} + u_{j2} \underset{\sim}{n}_{j2} + u_{j3} \underset{\sim}{n}_{j3} \tag{59}$$

Similarly, let the rotation of the beam cross section at Q_k relative to the cross section at O_j, due to the beam compliance, be represented by $\underset{\sim}{\phi}_j$. Let $\underset{\sim}{\phi}_j$ be

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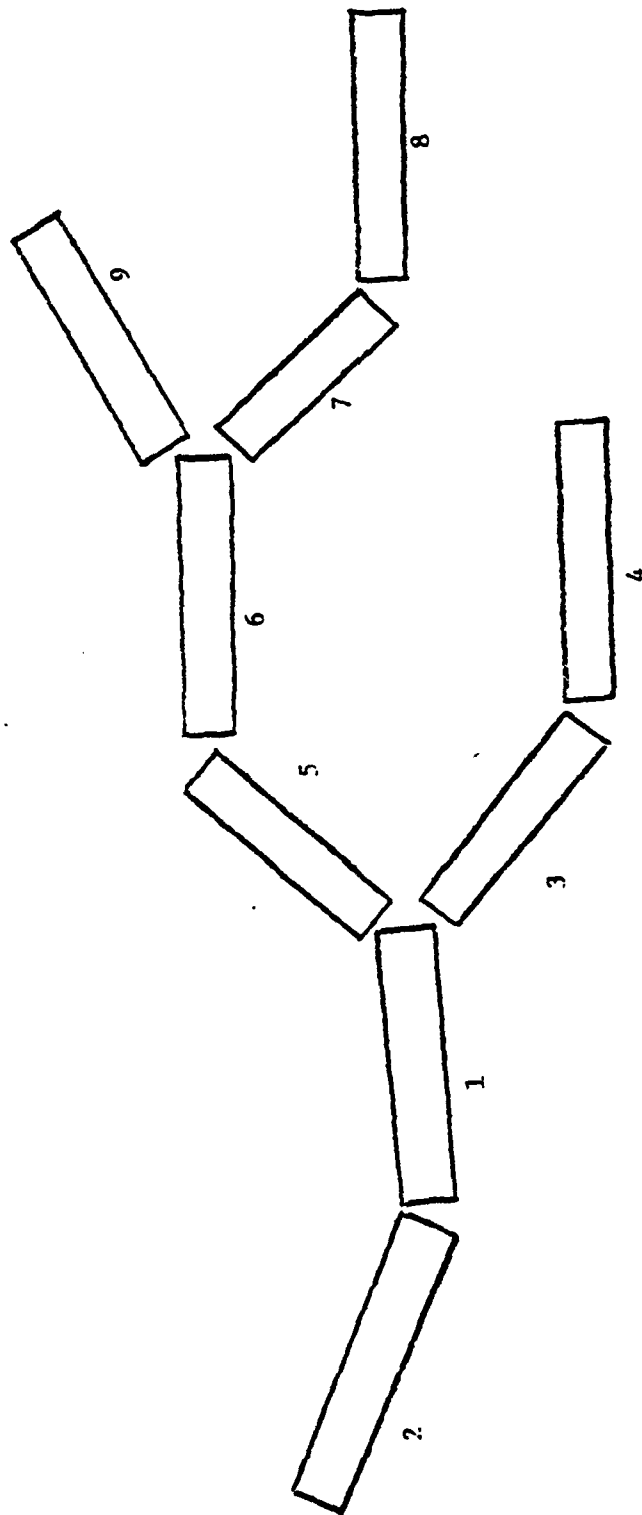
Figure 5. A General Chain System of Slender Members
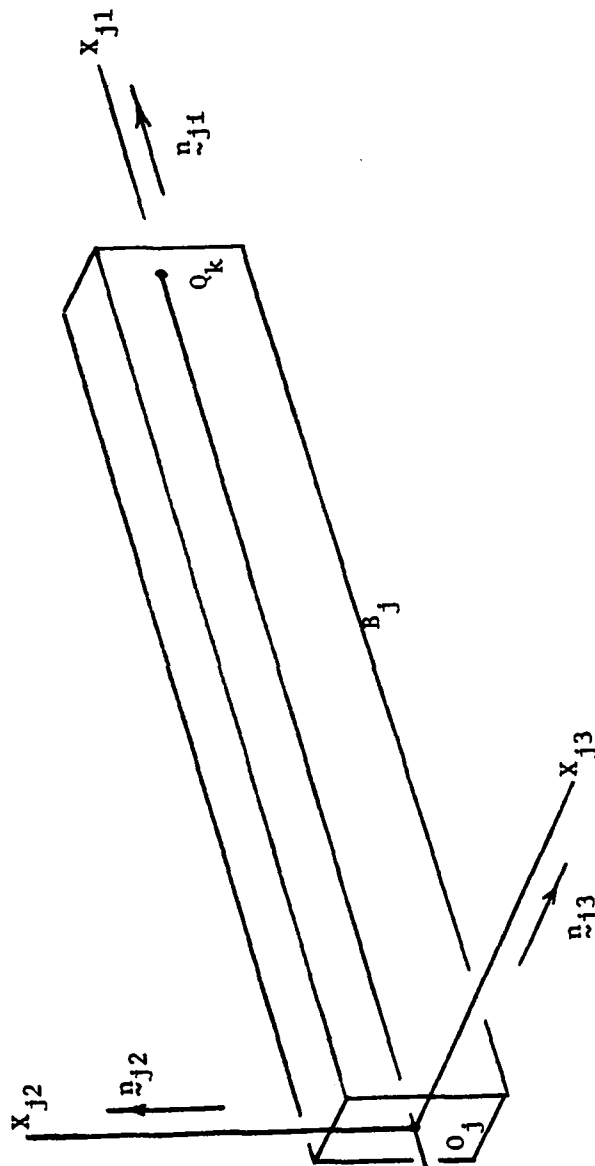
Figure 6. A Typical Member of a Beam Chain System

written in the form:

$$\underset{\sim}{\mathfrak{L}}_j = \phi_{j1}\, \underset{\sim}{n}_{j1} + \phi_{j2}\, \underset{\sim}{n}_{j2} + \phi_{j3}\, \underset{\sim}{n}_{j3} \tag{60}$$

By following the precedures of matrix structural analysis [15] u_{ji} and ϕ_{ji} may be expressed in terms of f_{ji} and m_{ji} ($i = 1, 2, 3$) as:

$$u_{j1} = (\ell_j / A_j E_j)\, f_{j1} \tag{61}$$

$$u_{j2} = (\ell_j^3 / 3 E_j J_{j2})\, f_{j2} + (\ell_j^2 / 2 E_j J_{j2})\, m_{j3} \tag{62}$$

$$u_{j3} = (\ell_j^3 / 3 E_j J_{j3})\, f_{j3} - (\ell_j^2 / 2 E_j J_{j3})\, m_{j2} \tag{63}$$

$$\phi_{j1} = (\ell_j / J_{j1} G_j)\, m_{j1} \tag{64}$$

$$\phi_{j2} = (\ell_j^2 / 2 E_j J_{j2})\, f_{j2} + (\ell_j / E_j J_{j2})\, m_{j3} \tag{65}$$

$$\phi_{j3} = (\ell_j^2 / 2 E_j J_{j3})\, f_{j3} - (\ell_j / E_j J_{j3})\, m_{j2} \tag{66}$$

where ℓ_j is the axial length of B_j, A_j is the cross-sectional area, J_{ji} ($i = 1,2, 3$) are the centroidal second moments of area of the cross section relative to the X_{ji} axes, E_j is the elastic modulus, and G_j is the shear modulus.

$\underset{\sim}{u}_j$ and $\underset{\sim}{\phi}_j$ thus represent the compliance as yielding of member B_j due to the holding and system motion. In an automated analysis as outlined in the foregoing section, $\underset{\sim}{u}_j$ and $\underset{\sim}{\phi}_j$ would be calculated and then used to adjust the geometrical parameters at each integration step. Specifically, $\underset{\sim}{q}_k$ (See Figure 4.) and SOK are adjusted as:

$$\underset{\sim}{q}_k \rightarrow \underset{\sim}{q}_k + \underset{\sim}{u}_j \tag{67}$$

and

$$\text{SOK} \rightarrow \text{SOK CJ} \tag{68}$$

where CK is

$$CJ = \begin{bmatrix} 1 & -\phi_{j3} & \phi_{j2} \\ \phi_{j3} & 1 & -\phi_{j1} \\ \phi_{j2} & \phi_{j1} & 1 \end{bmatrix} \tag{69}$$

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(The development of Equation (69) follows from the successive multiplications of matrices of the form of Equation (15) for θ being ϕ_{jk}, ϕ_{j2}, and ϕ_{j3} (small angles) about the X_{j1}, X_{j2}, and X_{j3} axes respectively.) The integration would then proceed with the adjusted values of $\underset{\sim}{q}_k$ and SOK.

Vibration and Impact Response

The above compliance analysis is a quasi static analysis and as such it does not directly account for oscillatory or vibration phenomena due to the flexibility of the system and the externally applied (for example, impact) and inertia forces. If, as before, it is assumed that the vibrations have relatively small amplitude, then a modelling and description of the vibration phenomena may be obtained through torsion and translation springs introduced at the connection points, or joints, of the system.

To illustrate this, consider again a system consisting of long slender members which can be modelled as beams as in Figure 5. Consider two typical adjoining members of such a system as shown in Figure 7. The contribution to the systems oscillation due to the flexibility of B_j can be modelled by 1) three torsion springs connecting the surfaces of Q_k and O_k with spring constants $G_j J_{j1}/\ell_j$, and $E_j J_{j3}/\ell_j$ and governing the relative rotation of B_j and B_k about axes parallel to $\underset{\sim}{n}_{j1}$, $\underset{\sim}{n}_{j2}$ and $\underset{\sim}{n}_{j3}$ respectively; and by 2) three translation springs connecting Q_k and O_k with spring constants $A_j E_j/\ell_j$, $3E_j J_{j3}/\ell_j^3$ and $3E_j J_{j2}/\ell_j^3$ and governing the relative translation of B_j and B_k along axes parallel to $\underset{\sim}{n}_{j1}$, $\underset{\sim}{n}_{j2}$, $\underset{\sim}{n}_{j3}$. (these constants are determined from elementary structural analysis as in Equations (61) to (66).)

Discussion

Both of the above analyses involve the effects of forces and moments transmitted across connection joints. The dynamics analysis of the preceding part is
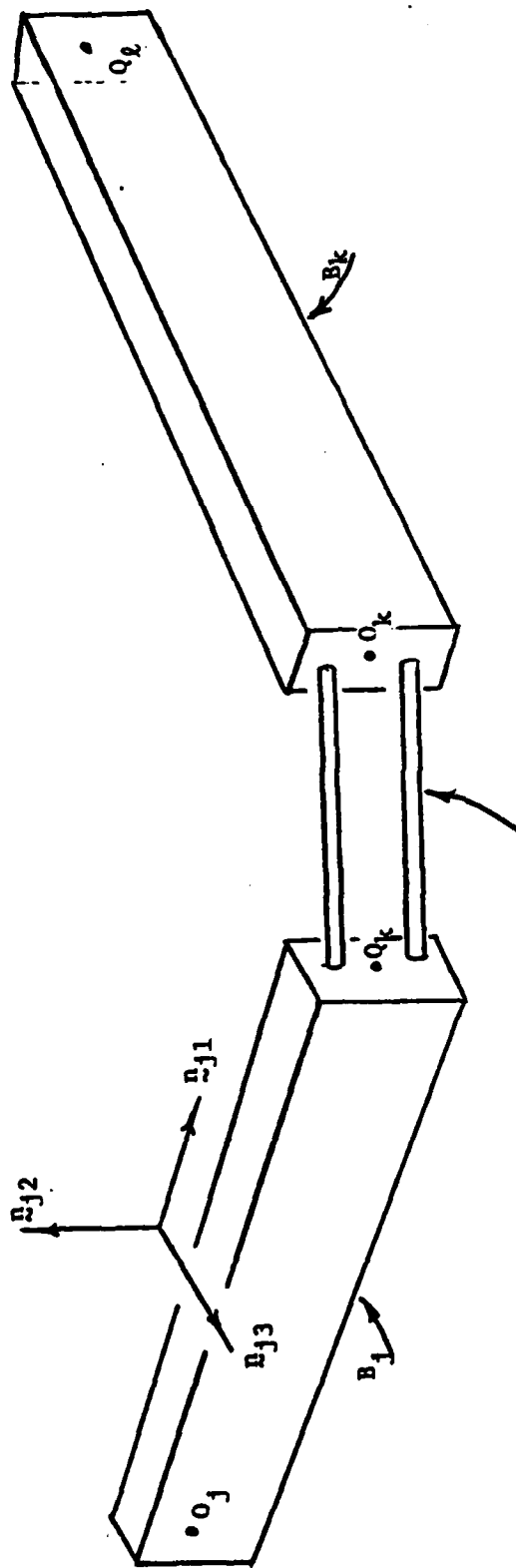
Torsion and translation springs

Figure 7. Two Typical Adjoining Slender Members

particularly well suited for accommodating the introduction of these forces and moments and for obtaining their contributions to the generalized forces. That is, although the compliance procedure above suggests the use of successive free body diagrams to obtain the force and moment components, and although this procedure could be automated, these components as well as the spring and moment components of the above oscillation analysis, can be readily obtained and directly incorporated into the governing equations by using the partial velocity and partial angular velocity vectors of the preceding dynamics analysis.

To see this, consider again two typical adjoining bodies such as B_j and B_k of Figures 4. and 7. As in Figure 4., let ξ_k measure the displacement of O_k relative to Q_k. Then by differentiation of ξ_k [145] the velocity of O_k may be expressed as:

$$\underset{\sim}{v}^{O_k} = \underset{\sim}{v}^{Q_k} + \underset{\sim}{\omega}_j \times \underset{\sim}{\xi}_k + \dot{\xi}_{ki} \, n_{ji} \tag{70}$$

As before, let the orientation and rotation of B_k relative to B_j be defined in terms of Euler parameters and relative angular velocity components. Then, from Equation (25) the angular velocity of B_k may be expressed as:

$$\underset{\sim}{\omega}_k = \underset{\sim}{\omega}_j + \underset{\sim}{\hat{\omega}}_k = \underset{\sim}{\omega}_j + \hat{\omega}_{ki} n_{ji} \tag{71}$$

Let the force system which B_k exerts on B_j be equivalent to a single force $\underset{\sim}{f}_{k/j}$ passing through Q_k together with a couple with torque $\underset{\sim}{m}_{k/j}$, as in Equations (57) and (58). Then by the law of action-reaction, the force system which B_j exerts on B_k is equivalent to a single force $-\underset{\sim}{f}_{k/j}$ passing through Q_k together with a couple with torque $-\underset{\sim}{m}_{k/j}$.

Let y_ℓ ($\ell=1, \ldots, 6N$) be the generalized coordinate derivatives as defined by Equations (23) and (24). Let the contribution to the generalized active force F_ℓ by these forces transmitted across the connecting joint be $\hat{F}_\ell$. Then $\hat{F}_\ell$ is given by the expression [145]:

$$\hat{F}_\ell = (\partial \underset{\sim}{V}^{Q_k}/\partial y_\ell) \cdot (\underset{\sim}{f}_{k/j}) + (\partial \underset{\sim}{\omega}_j/\partial y_\ell) \cdot (\underset{\sim}{m}_{k/j}) \tag{72}$$

$$+ (\partial \underset{\sim}{V}^{O_k}/\partial y_\ell) \cdot (-\underset{\sim}{f}_{k/j}) + (\partial \underset{\sim}{\omega}_k/\partial y_\ell) \cdot (-\underset{\sim}{m}_{k/j})$$

Consider the following cases:

Case 1: y_ℓ is not equal to either ξ_{ki} or $\hat{\omega}_{ki}$. In this case the partial velocities and partial angular velocities of Q_k, O_k, B_j, and B_k may be expressed by using Equations (70) and (71) as:

$$\partial \underset{\sim}{V}^{Q_k}/\partial y_\ell = \partial \underset{\sim}{V}^{O_k}/\partial y_\ell \tag{73}$$

and

$$\partial \underset{\sim}{\omega}_k/\partial y_\ell = \partial \underset{\sim}{\omega}_j/\partial y_\ell \tag{74}$$

Then, by Equation (72), $\hat{F}_\ell$ becomes:

$$\hat{F}_\ell = 0 \tag{75}$$

Case 2: y_ℓ is equal to one of the $\dot{\xi}_{ki}$ ($i=1, 2, 3$). In this case, the partial velocities and partial angular velocities of Q_k, O_k, B_j, and B_k become:

$$\partial \underset{\sim}{V}^{Q_k}/\partial y_\ell = \partial \underset{\sim}{V}^{Q_k}/\partial \xi_{ki} = 0 \tag{76}$$

$$\partial \underset{\sim}{V}^{O_k}/\partial y_\ell = \partial \underset{\sim}{V}^{O_k}/\partial \xi_{ki} = \underset{\sim}{n}_{ji} \tag{77}$$

$$\partial \underset{\sim}{\omega}_j/\partial y_\ell = \partial \underset{\sim}{\omega}_j/\partial \xi_{ki} = 0 \tag{78}$$

and

$$\partial \underset{\sim}{\omega}_k/\partial y_\ell = \partial \underset{\sim}{\omega}_k/\partial \xi_{ki} = 0 \tag{79}$$

Hence, by Equation (72) and (57), $\hat{F}_\ell$ becomes:

$$\hat{F}_\ell = \underset{\sim}{n}_{ji} \cdot (-\underset{\sim}{f}_{k/j}) = -f_{ji} \tag{80}$$

Case 3: y_ℓ is equal to one of the $\hat{\omega}_{ki}$ ($i=1, 2, 3$). In this case, the partial velocities and partial angular velocities of Q_k, O_k, B_j, and B_k become:

$$\partial \underset{\sim}{y}^{Q_k}/\partial y_\ell = \partial \underset{\sim}{y}^{Q_k}/\partial \hat{\omega}_{ki} = 0 \qquad (81)$$

$$\partial \underset{\sim}{y}^{O_k}/\partial y_\ell = \partial \underset{\sim}{y}^{O_k}/\partial \hat{\omega}_{ki} = 0 \qquad (82)$$

$$\partial \underset{\sim}{\omega}_j/\partial y_\ell = \partial \underset{\sim}{\omega}_j/\partial \hat{\omega}_{ki} = 0 \qquad (83)$$

$$\partial \underset{\sim}{\omega}_k/\partial y_\ell = \partial \underset{\sim}{\omega}_k/\partial \hat{\omega}_{ki} = \underset{\sim}{n}_{ji} \qquad (84)$$

Hence, by Equation (72) and (58), $\hat{F}_\ell$ becomes:

$$\hat{F}_\ell = \underset{\sim}{n}_{ji} \cdot (-\underset{\sim}{m}_{k/j}) = -m_{ji} \qquad (85)$$

The above three cases include the contribution to the generalized active forces for each of the y_ℓ ($\ell = 1,\ldots, 6N$). Moreover, each of the non-zero contributions (from Equations (80) and (85) occurs individually; that is, each contribution occurs separately in one of the governing equations. Hence, if in a particular configuration or motion of the system, y_ℓ is specified (for example, y_ℓ is zero) then the ℓ th governing equation becomes an uncoupled linear espression for the unknown restraining force or moment, thus determining the compliance. Conversely, if y_ℓ is an unknown variable (representing a degree of freedom) the contribution to F_ℓ due to the flexibility as modelled by the translation and torsion springs is determined directly by Equations (61) to (66), (80), and (85).

CONCLUDING REMARKS

The results of numerically solving the governing differential equations (54) where the coefficients are given by Equations (55) and (56) are reported and discussed in References [1, 32, 34, 35, 36, 77, 152-155] for a number of physical systems and configurations (e.g. human-body models, head-neck models, and flexible cables).

The application of Equations (54) with these systems, however, is based on the use of relative orientation angles between the respective bodies of the system as the generalized coordinates (x_ℓ) as opposed to the use of Euler parameters, quasi-coordinates, and generalized speeds as outlined herein. A problem which arises in the numerical solution of Equations (54) where orientation angles are used is that there always exists values of the angles and hence, configurations of the system, for which the determinant of $a_{k\ell}$ is zero. A numerical solution will, of course, fail to converge at these singular configurations of the system, and convergence is very slow for configurations in the vicinity of a singularity. This problem is avoided by using Euler parameters to relate the orientation geometry to the angular velocity.

The advantages of using Lagrange's form of d'Alembert's principle to obtain the governing equations of motion for multi-body mechanical systems has been exposited in detail in References [28-30]. Basically, this principle has the advantages of Lagrange's equations or of virtual work in that non-working internal constraint forces, between the bodies of the system, are automatically eliminated from the analysis, and may therefore be ignored in the formulation of the governing equations. The principle, however, has the additional advantage of avoiding the

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differentiation of scalar energy functions. Indeed, the differentiation required to obtain velocities and accelerations are performed by vector cross products and multiplication algorithms -- procedures which are ideally suited for numerical computation. As with Lagrange's equations, Lagrange's form of d'Alembert's principle requires the use f generalized coordinates to define the system geometry. The use of Euler parameters to avoid problems with singularities, as discussed above, leads naturally to the use of generalized speeds - that is, relative angular velocity components as the generalized coordinate derivatives. This, in turn, leads to additional computational advantages as observed by Kane and Wang [149] and Likins [122]. Specifically, by using generalized speeds (relative angular velocity components) as the principle parameters of the analysis, the coefficient matrices in the governing equations can be obtained directly from the body connection array $L(k)$ (see Tables 1. and 2.).

The use of "relative" coordinates, that is, angular velocity components of the bodies with respect to their adjoining bodies, as opposed to "absolute" coordinates, (for example, angular velocity components in inertial space) also contributes to the computational advantage. In applications with specific geometrical configurations [1, 31, 32, 34-36, 50, 77, 152-155], it is seen that the geometry is more easily described in terms of relative coordinates.

The generalization to allow translation between the bodies of the system makes the analysis applicable to a much broader class of problems than was possible with those previous analyses which are restricted to linked multi-body systems. For example, with the head-neck systems of References [152, 154, 155] the use of translation variables between the vertebrae is necessary to obtain satisfactory models of such systems. But, and of perhaps greater significance, the generalization to include translation between the bodies of the system is necessary for an efficient

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analysis of the flexibility and compliance effects as discussed earlier.

In this regard, the compliance can be modelled with a quasi-static approach whereas the oscillations and impact response require a dynamic analysis with the introduction of additional degrees of freedom. In both of these cases the analysis outlined herein (using Lagrange's form of d'Alembert's principle together with the use of generalized speeds) accomodates the effects of flexibility in an extremely efficient manner. That is, the forces and moments transmitted across the system joints are directly determined and incorporated into the governing equations. Moreover, the modelling may be made as detailed as necessary by introducing non-linearities through the elastic springs and dampers and by increasing the number of joints and bodies of the system.

Finally, the entire analysis outlined herein is developed with the intent of obtaining efficiencies in a computer oriented development of the governing dynamical equations. As such, its most productive application will be with large multi-body systems such as finite-segment models of the human body, chains, cables, robots, manipulators, and teleoperators.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

New and recently developed concepts and ideas useful in obtaining efficient computer algorithms for solving the equations of motion of multi-body mechanical systems with flexible links are presented and discussed. These ideas include the use of Euler parameters, Lagrange's form of d'Alembert's principle, generalized speeds, quasi-coordinates, relative coordinates, structural analysis techniques and body connection arrays. The mechanical systems considered are linked bodies forming a tree structure, but with no "closed loops" permitted. An explicit formulation of the equations of motion is presented.

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