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SHEWHART X-CHARTS (AVERAGE $\bar{X}$) WITH ESTIMATED PROCESS VARIANCE, (U)

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X-chart (average $\bar{x}$)

SHEWHART ($\bar{X}$ -CHART) WITH ESTIMATED PROCESS VARIANCE

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0. ABSTRACT

Properties of the Shewhart $\bar{X}$ -chart for controlling the mean of a process with a normal distribution are investigated for the situation where the process variance σ^2 must be estimated from initial sample data. The control limits of the $\bar{X}$ -chart depend on the estimate of σ^2 and thus, unlike the case when σ^2 is known, the $\bar{X}$ -chart is not equivalent to a sequence of independent tests. When σ^2 is estimated the distribution of the run length is not geometric and cannot be characterized simply in terms of the probability of a signal at a given point. The average run length (ARL) for the $\bar{X}$ -chart is expressed in terms of an integral involving the normal cdf, and it is shown that the chart signals with probability one, but the ARL may not be finite if the size of the sample used to estimate σ^2 is sufficiently small. In addition, certain bounds for the ARL are also derived. Numerical integration is used to show that the effect of using small sample sizes in estimating σ^2 is to increase the ARL and the variance of the run length distribution.

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KEY WORDS: $\bar{X}$ -Chart; Shewhart Chart; Process Control; Control Chart; Estimated Variance; Economic Model; Average Run Length.

1. INTRODUCTION

Control charts have long been used to detect changes in the distribution of observations taken from a process that operates continuously over time. Various criteria have been proposed to measure the performance of a control chart when it is used to control a process. These criteria aid in the design of charts to meet specific objectives and also serve as a basis for comparing several competing procedures. The purpose of this paper is to examine some of these criteria in terms of their appropriateness with special reference to the Shewhart $\bar{X}$ -chart for controlling the process mean when the process variance is unknown.

Consider the situation where an independent sequence $\bar{X}_1, \bar{X}_2, \dots$ of random samples is taken at regular intervals from the process, $\bar{X}_i = (X_{i1}, X_{i2}, \dots, X_{in})$ being the sample of n observations taken on the i th occasion. In most cases the distribution of the observations will depend on one or more parameters represented by θ , which may be vector-valued. The basic problem is to detect quickly any change in the value of θ by using the values of all observations taken up to the current time. In some situations the initial control value, say θ_0 , for θ will be known or specified in advance. In other situations one or more components of θ_0 may be unknown, and it is necessary to estimate these values using the observed data.

A process control procedure for detecting shifts away from θ_0 is a procedure that at time i decides on the basis of $\bar{X}_1, \bar{X}_2, \dots, \bar{X}_i$ whether to continue sampling or to stop and signal that a change has occurred. As long as the value of θ remains constant at θ_0 we want to continue sampling, but as soon as a significant shift occurs we want to stop. If N represents the random time at which the procedure stops and signals, then it is the distribution of N that determines the properties of the procedure. In process control terminology, N is called the *run length*.

As an example of a process control procedure, consider the standard Shewhart $\bar{X}$ -chart for detecting shifts in the mean μ of a process having a normal distribution. This procedure signals that a shift has occurred at the first value of i for which

$$\bar{X}_i \geq \mu_0 + k\sigma$$

where $\bar{X}_i$ is the mean of $X_{i1}, X_{i2}, \dots, X_{in}$, μ_0 is the target or control value for the process mean, k is a constant, and σ is the process standard deviation. In what follows it is convenient to express the process mean in terms of $\delta = (\mu - \mu_0)\sqrt{n}/\sigma$, the case $\mu = \mu_0$ then corresponding to $\delta = 0$. If both μ_0 and σ are known (or specified), then the distribution of the run length is geometric with parameter

$$p(\delta) = P(X_i \geq \mu_0 + k\sigma) = 1 - \Phi(\delta + k\sqrt{n})$$

since we are, in effect, performing a sequence of "independent tests" where the i th test involves only the i th sample. Here $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution. The parameter $p(\delta)$ completely determines the run-length distribution and thus the behavior of this procedure can be summarized conveniently and naturally in terms of this probability of a signal on a given occasion. In addition, if we let δ_1 be a value of δ that represents a shift of interest, then the quantities $p(0)$ and $1-p(\delta_1)$ can be thought of as the probabilities of errors of the first and second type, respectively, for testing the statistical hypothesis $H: \delta = 0$ against $A: \delta = \delta_1$. This formulation using the concepts and terminology of hypothesis testing is very natural to statisticians.

The drawback underlying $p(\delta)$ as a general parameter for process control procedures for detecting changes in δ is that it may not be meaningful for more complicated procedures such as cumulative sum (CUSUM) control charts, moving average charts, or Shewhart $\bar{X}$ -charts with estimated

variance. For example, the probability that a CUSUM chart signals on occasion i is not independent of i since, for this procedure, the decision at stage i depends on $\bar{X}_1, \bar{X}_2, \dots, \bar{X}_{i-1}$ in addition to $\bar{X}_i$. When the process variance is unknown and must be estimated from preliminary samples, the procedure no longer entails a sequence of independent tests and the run-length distribution cannot be characterized by the probability of a signal at a given point. Thus, for general process control procedures, some additional measures or parameters are needed to describe the run-length distribution. One commonly used parameter is the mean $E_\theta(N)$, called the *average run length* (ARL). Another useful parameter is $P_\theta(N \leq t)$, where t is a fixed value.

One objective of this paper is to determine which of the measures associated with the run-length distribution are most important in determining the characteristics of the procedure. This is done through the development of a simple economic model in Section 2. Another objective is to examine the properties of the $\bar{X}$ -chart when the variance of the process must be estimated. In Section 3, various methods of estimating the variance are discussed and the effect of estimated variance on the ARL is investigated. Properties of the $\bar{X}$ -chart are usually calculated under the assumption that σ is known, but our results indicate that the actual properties of the $\bar{X}$ -chart when σ is estimated can be quite different from the properties for the case when σ is known. This distinction is often ignored in the literature. Section 4 extends the results of Section 3 to the two-sided $\bar{X}$ -chart.

2. AN ECONOMIC MODEL

To determine the characteristics of the run length distribution that are important determinants of the properties of a control chart, we may use

an economic model that evaluates the impact on income and costs associated with the use of a particular control chart (see Duncan (1956)). The economic models that have been developed in the literature usually express the long-term average net income per unit time for the process as a function of various process parameters and of various parameters associated with the control procedure. The usual objective in the design of control procedures is to maximize this net income.

A simplified economic model will now be developed to show how the characteristics of the run-length distribution enter into the model. This model is for a general process control procedure for controlling the general parameter θ in contrast to models in the literature which are specific to the procedure and parameter. Assume that the process is operating in control when the control procedure is first applied and, for simplicity, let the unit of time correspond to the time interval between samples. At some random time T_0 (counted from the start of the procedure), the process mean shifts away from the control value θ_0 . Assume for simplicity that there is only one possible out-of-control state represented by the value θ_1 . After the shift from θ_0 to θ_1 the process remains at θ_1 until the control procedure detects the out-of-control situation and rectifying action is taken to return θ to θ_0 . Another in-control-out-of-control production cycle then starts at the instant when θ returns to θ_0 . Thus, the long-term application of a control procedure involves an infinite sequence of these in-control-out-of-control cycles.

By considering income lost due to operating the process out of control as a cost, the objective of maximizing the long-term average net income per unit time can be reformulated as a problem of minimizing the long-term average cost per unit time. In each production cycle the primary costs are those due to false alarms, to operating out of control, and to sampling. The long-term average cost per unit time is obtained by finding the expected cost per cycle and then dividing the latter by the expected length of a cycle (see Johns and Miller (1963)).

Let T represent the time (counted from the start of the cycle) when the procedure detects the shift and θ is returned to θ_0 . Then T is just the random cycle length. Let W represent the number of false alarms during the part of the cycle when the process is in control. The cost per cycle can then be represented by

$$C = c_1(W) + c_2(T - T_0) + c_3(T),$$

where the functions c_1 , c_2 , and c_3 are given by

$c_1(w)$ = cost due to w false alarms during a cycle,

$c_2(t)$ = cost due to operating out of control for t time units,

$c_3(t)$ = cost of sampling during a cycle of length t .

The long run average cost per unit time is therefore

$$A = E(C)/E(T).$$

It seems reasonable to assume that the cost of false alarms, $c_1(W)$, is directly proportional to W so that $c_1(W) = a_1 W$ where a_1 is a constant. In this case $E(c_1(W))$ is approximately equal to $a_1 E(T_0)/E_{\theta_0}(N)$. This approximation can be justified as follows. Let $N_1, N_2, \dots, N_W$ be the sequence of run lengths between false signals during the in-control part of the cycle, and let N_{W+1} be the run length after N_W that would be required for a signal if the shift from θ_0 to θ_1 does not take place. Then N_{W+1} has the same distribution as N_1 . Thus

$$\sum_{i=1}^W N_i < T_0 \leq \sum_{i=1}^{W+1} N_i$$

and, since $E_{\theta_0} \left(\sum_{i=1}^W N_i \right) = E(W)E_{\theta_0}(N)$,

$$E(T_0)/E_{\theta_0}(N) - 1 \leq E(W) < E(T_0)/E_{\theta_0}(N).$$

In many cases it may be reasonable to assume also that $c_2(t)$ is of the form $a_2 t$ where a_2 is a constant. Neglecting the fact that a shift can occur between samples and assuming that the expected time to detect the shift does not depend significantly on how long the control procedure has been running in control, $E(c_2(T-T_0))$ is approximately equal to $a_2 E_{\theta_1}(N)$. The expected cycle length $E(T)$ is approximately $E(T_0) + E_{\theta_1}(N)$. The sampling cost $c_3(T)$ depends only on T and n . The long-term average cost per unit time then can be expressed as

$$A \approx \frac{a_1 E(T_0)/E_{\theta_0}(N) + a_2 E_{\theta_1}(N) + E(c_3(T))}{E(T_0) + E_{\theta_1}(N)}.$$

Thus A depends on the run-length distribution primarily through the ARL values $E_{\theta_0}(N)$ and $E_{\theta_1}(N)$. This suggests that, if the economic model is a reasonable representation of the actual cost and loss structure, the primary measure of the effectiveness of control procedures is the ARL.

There may be situations, however, where the function $c_2(t)$ is not linear in t . For example, a small amount of low quality material produced while operating out of control may not be particularly damaging, but if the out-of-control time exceeds a certain value then a complete production run may be jeopardized. In this case the cost function could take the form $c_2(t) = 0$ if $t \leq t_1$ and $c_2(t) = a'_2$ if $t > t_1$. In other cases a cost function of the form $c_2(t) = t^2$ might be appropriate. Thus, in addition to the ARL, parameters such as $P_{\theta_1}(N > t_1)$ or $E_{\theta_1}(N^2)$ may be of interest in particular applications.

3. ONE-SIDED $\bar{X}$ -CHARTS WITH UNKNOWN VARIANCE

In this section we consider the one-sided Shewhart $\bar{X}$ -chart for detecting shifts in the positive direction in the mean of a process having

a normal distribution. A one-sided $\bar{X}$ -chart for detecting shifts in the negative direction can be obtained by using a lower limit and the results obtained in this section can be applied by replacing δ with $-\delta$ where, as before, $\delta = (\mu - \mu_0)\sqrt{n}/\sigma$. The two-sided $\bar{X}$ -chart will be discussed in Section 4.

When the variance of the process is unknown, some form of estimate of the variance must be used in the construction of the control limits for the chart. The estimate of the variance can be obtained from the information in each sample, or from an initial sequence of samples, or from a combination of the two approaches where the initial variance estimate is updated as each new sample is taken. The variance estimates are traditionally based on either the sample variance or the sample range. Since the sample variance is more efficient than the sample range, we will restrict attention to situations where sample variances are used. We now look at the implications of the various ways of estimating σ^2 as they bear on the properties of the $\bar{X}$ -chart.

If the sample variance

$$s_i^2 = \frac{1}{n} \sum_{j=1}^n (X_{ij} - \bar{X}_i)^2 / (n-1)$$

for stage i is used as the variance estimator at stage i , then the procedure that signals at the first i for which

$$\bar{X}_i \geq \mu_0 + ks_i$$

is a sequence of independent tests. The parameter

$$p^*(\delta) = P(\bar{X}_i \geq \mu_0 + ks_i)$$

can be calculated using the noncentral t-distribution for any value of δ and the ARL of the procedure is $1/p^*(\delta)$. This procedure is not used widely in practice because n is usually too small to provide a reasonable estimate of σ^2 when each sample is used independently. In addition, the procedure must be carried out in its standardized version

$$(\bar{X}_1 - \mu_0)/S_1 \geq k$$

in order to have the same upper limit at each stage, and this may inhibit a visual interpretation of the plots since the chart is not showing the actual sample means.

A second approach to the estimation of σ^2 involves the use of the same initial estimate for each stage. In most cases this initial estimate is based on a sequence of r initial samples, each of size m . In addition to providing a variance estimate, these r samples can be examined further to determine whether the process was in control with respect to variance at the time that these samples were taken. If the r sample variances are pooled to obtain the estimator

$$S^2 = \frac{r}{\sum_{i=1}^r} S_i^2 / r$$

then the procedure signals at the first i for which

$$\bar{X}_1 \geq \mu_0 + kS.$$

This approach has the advantage that the chart shows the actual sample means. Since the same S is used at each point, the procedure no longer entails a sequence of independent tests, and the run-length distribution will not be geometric. In this case the parameter

$$p^+(\delta) = P(\bar{X}_1 \geq \mu_0 + kS)$$

will not suffice to characterize the run-length distribution.

As an alternative to the pooled estimate of σ^2 from a sequence of initial samples, one large initial sample can be taken to estimate σ^2 . A disadvantage of this approach is that it is not possible to test for control of the variance at the time the sample was taken.

A third approach to estimate σ^2 uses some kind of initial estimate and then updates the estimate as each new sample is taken. Although this method appears reasonable as long as the process variance is fixed, additional computational effort would be required and the control limits would change after each sample. In addition, the properties of such a procedure are quite difficult to determine.

A common practice in setting up a control chart is to take $k = 3/\sqrt{n}$ regardless of whether σ^2 is known or estimated from a small sample (for example, see Duncan (1974)). Various authors (cf. King (1954), Proschan and Savage (1960), Hillier (1964), and Yang and Hillier (1970)) have considered the problem of choosing k when σ^2 is estimated. Yang and Hillier (1970) proposed that k be chosen to give a specified value of $p^+(0)$. If the estimator S^2 has $v = r(m-1)$ degrees of freedom, then the choice

$$k = t_{1-\alpha}(v)/\sqrt{n}$$

gives $p^+(0) = \alpha$, where $t_{1-\alpha}(v)$ is the $1-\alpha$ fractile of the t -distribution having v degrees of freedom. By choosing $t_{1-\alpha}(v)$ corresponding to $\alpha = .00135$, the value of $p^+(0)$ will be the same as the value of $p(0)$ for $k = 3/\sqrt{n}$. Of course, $t_{1-\alpha}(v)$ can be chosen to give any other desired value of $p^+(0)$. The problem with this approach is that the run-length distribution and ARL are still unknown. It will be shown later in this paper that the actual ARL for the procedure using S as an estimator for σ can differ substantially from the ARL calculated under the assumption that $p^+(0) = p(\delta)$.

If the estimator S^2 for σ^2 is a sample variance or a pooled sample variance based on ν degrees of freedom, then $\nu S^2/\sigma^2$ has a chi-squared distribution with ν degrees of freedom when the process distribution is normal. Consequently the run length N satisfies

$$\begin{aligned} P(N > t) &= P(\bar{X}_i < \mu_0 + kS, i = 1, 2, \dots, t) \\ &= \int_0^\infty \phi^t \left(-\delta + \frac{ks}{\sigma} \sqrt{n} \right) f_S(s) ds \\ &= \frac{1}{\Gamma(\frac{1}{2}\nu)} \int_0^\infty \phi^t (-\delta + c\sqrt{u}) u^{\frac{1}{2}(\nu-2)} e^{-u} du \end{aligned} \quad (3.1)$$

where $f_S(s)$ is the density of S , and $c = k\sqrt{2n/\nu}$. Unfortunately the integral in (3.1) cannot be expressed in a simpler closed form. Nevertheless, certain results for the run-length distribution can be established using (3.1). These results are given in the following theorems, and the proofs are given in the Appendix.

THEOREM 1. For all values of k and δ , $P(N < \infty) = 1$ and $\lim_{\nu \rightarrow \infty} P(N > t) = \phi^t(-\delta + k\sqrt{n})$.

This theorem implies that the $\bar{X}$ -chart using an initial estimate of σ eventually will signal, whatever the values of μ and σ may be. Moreover, as the number of degrees of freedom for the initial estimate of σ^2 grows larger, the distribution of the run length converges to the geometric distribution for the case where σ^2 is known. The results given in this theorem hold without the assumption of normality.

The ARL of the $\bar{X}$ -chart can be expressed as

$$E(N) = \sum_{t=0}^{\infty} P(N > t).$$

Using this expression the following theorem can be proved.

THEOREM 2.

- (a) If $c < \sqrt{2}$ then $E(N) < \infty$ for all δ .
- (b) If $c > \sqrt{2}$ then $E(N) = \infty$ for all δ .
- (c) If $c = \sqrt{2}$ then $E(N) < \infty$ for $\delta > 0$ and $E(N) = \infty$ for $\delta \leq 0$.
- (d) If $E(N) < \infty$ then

$$E(N) = \frac{1}{\Gamma(\frac{1}{2}v)} \int_0^{\infty} \frac{u^{\frac{1}{2}(v-2)} e^{-u}}{1 - \Phi(-\delta + c\sqrt{u})} du. \quad (3.2)$$

- (e) $\lim_{v \rightarrow \infty} E(N) = [1 - \Phi(-\delta + k\sqrt{n})]^{-1}$.

This theorem gives the rather surprising result that under certain conditions the ARL of the $\bar{X}$ -chart with estimated variance is not finite. The condition $c > \sqrt{2}$ is equivalent to the condition $v < k^2 n$. Thus if v , the number of degrees of freedom in the estimate of σ^2 , is sufficiently small, the ARL is not finite. In particular, if k is taken as the $3/\sqrt{n}$ (the standard 3-sigma limits) then $c > \sqrt{2}$ if $v < 9$. If we take $k = t_{1-\alpha}^2(v)/\sqrt{n}$, following the suggestion in Yang and Hillier (1970), then $c > \sqrt{2}$ whenever $v < t_{1-\alpha}^2(v)$ (e.g. when $\alpha = 0.001$ the ARL is not finite if $v \leq 14$). However, if v is large, say 100, then $c < \sqrt{2}$ if $k\sqrt{n} < 9.95$ and the ARL will be finite for values of k and n that are likely to be used in applications.

The following general bound for the run-length distribution can be obtained.

THEOREM 3. For all values of δ , k , and v ,

$$P(N \leq t) \leq 1 - (1 - p^+(\delta))^t.$$

Thus the run-length distribution is stochastically larger than a geometric distribution with parameter $p^+(\delta)$. This theorem holds regardless of the form of

the underlying distribution of the observations, although computation of $p^+(\delta)$ may be difficult for non-normal distributions. If the underlying distribution is normal then

$$p^+(\delta) = P(t(v, \delta) \geq k\sqrt{n})$$

where $t(v, \delta)$ has a noncentral t-distribution with v degrees of freedom and noncentrality parameter δ . Suppose k is chosen as $k = t_{1-\alpha}(v)/\sqrt{n}$, following the suggestion of Yang and Hillier (1960). Then, when the process is in control, the geometric distribution with parameter α serves as a stochastic lower bound for the run-length distribution. The theorem also provides a lower bound for the ARL since

$$E(N) \geq 1/p^+(\mu) .$$

Sharper bounds for the ARL under the assumption of normality are given in the following theorems.

THEOREM 4. If $0 < c < \sqrt{2}$ and $\delta = 0$, then

$$b_1 b_2 \leq E(N) \leq b_1 b_3 \quad (3.3)$$

where

$$\begin{aligned} b_1 &= \frac{(2\pi)^{\frac{1}{2}} \Gamma(\frac{1}{2}(v+1))}{\Gamma(\frac{1}{2}v)} c(1-\frac{1}{2}c^2)^{-\frac{1}{2}(v+1)} , \\ b_2 &= \frac{1}{2} + \max \left\{ \frac{1}{2}, \frac{(2-c^2)^{\frac{1}{2}} \Gamma(\frac{1}{2}v)}{c\pi^{\frac{1}{2}} \Gamma(\frac{1}{2}(v+1))} \right\} , \\ b_3 &= 1 + \min \left\{ \frac{2-c^2}{(v-1)c^2}, \frac{(2-c^2)^{\frac{1}{2}} \Gamma(\frac{1}{2}v)}{c2^{\frac{1}{2}} \Gamma(\frac{1}{2}(v+1))} \right\} . \end{aligned}$$

THEOREM 5. If $0 < c < \sqrt{2}$ and $\delta > 0$, then

$$\begin{aligned} & \{\phi(\delta)\}^{-1} P(\chi^2(v) \leq 2\delta^2/c^2) + D \max [A - \delta B, \frac{1}{2}A + ((2/\pi)^{\frac{1}{2}} - \frac{1}{2}\delta)B] \\ & \leq E(N) \leq 2P(\chi^2(v) \leq 2\delta^2/c^2) + D\{A + (1-\delta)B\}, \end{aligned} \quad (3.4)$$

where

$$\begin{aligned} D &= (\frac{1}{2}\pi)^{\frac{1}{2}} \Gamma(\frac{1}{2}(v+1)) (1 - \frac{1}{2}c^2)^{-\frac{1}{2}(v+1)} \exp\{\delta^2/(2-c^2)\} . \\ A &= vc \sum_{j=0}^v a_j P(\chi^2(v+1-j) > d), \quad B = 2(1 - \frac{1}{2}c^2)^{\frac{1}{2}} \sum_{j=0}^{v-1} b_j P(\chi^2(v-j) > d), \\ d &= 2\delta^2 \{c^2(1 - \frac{1}{2}c^2)\}^{-1}, \\ a_j &= (-1)^j \{\delta c(1 - \frac{1}{2}c^2)^{-\frac{1}{2}}\}^j \{j! \Gamma(\frac{1}{2}(v+2-j))\}^{-1}, \\ b_j &= (-1)^j \{\delta c(1 - \frac{1}{2}c^2)^{-\frac{1}{2}}\}^j \{j! \Gamma(\frac{1}{2}(v+1-j))\}^{-1} \end{aligned}$$

and where $\chi^2(v)$ represents a chi-square random variable having v degrees of freedom.

THEOREM 6. If $c = \sqrt{2}$ and $\delta > 0$, then

$$\begin{aligned} & A_1/\phi(\delta) + K \max[vA_3 - \delta^2 A_2, \frac{1}{2}vA_3 + ((\frac{1}{2}\pi)^{\frac{1}{2}} - \frac{1}{2}\delta)A_2] \\ & \leq E(N) \leq 2A_1 + K[vA_3 + \delta(1-\delta)A_2], \end{aligned} \quad (3.5)$$

where

$$\begin{aligned} K &= 2^{\frac{1}{2}(v+1)} \Gamma(\frac{1}{2}(v+1)) e^{\frac{1}{2}\delta^2} / \delta^{v+1}, \quad A_1 = P(\chi^2(v) \leq \delta^2), \\ A_2 &= P(\chi^2(2v) < 2\delta^2), \quad A_3 = P(\chi^2(2v+2) > 2\delta^2). \end{aligned}$$

Thus far we have examined the run-length distribution under the assumption that δ is constant for each sample. In many situations the shift away from the control value may take place gradually as a drift over a period of time. Suppose that δ_1 is the value of δ when the sample $\bar{X}_1$ is taken so that $\underline{\delta} = (\delta_1, \delta_2, \dots)$ represents the set of values of δ corresponding to the sampling occasions. We now investigate the distribution of N as a function of $\underline{\delta}$.

THEOREM 7. If $\underline{\delta}$ and $\underline{\delta}'$ are two sequences such that $\delta_i \leq \delta'_i$ for all i , then $P_{\underline{\delta}}(N \leq t) \leq P_{\underline{\delta}'}(N \leq t)$.

The point of the theorem is that, whatever the form of the deviation from the control value, the larger the shift the smaller the detection time. The computation of the distribution of N for arbitrary $\underline{\delta}$ is difficult. However, if $\underline{\delta}$ is bounded as $\underline{\delta} \leq \delta_i \leq \bar{\delta}$, for all i , then letting $\underline{\delta} = (\underline{\delta}, \underline{\delta}, \dots)$ and $\bar{\delta} = (\bar{\delta}, \bar{\delta}, \dots)$ we have that

$$P_{\underline{\delta}}(N \leq t) \leq P_{\underline{\delta}}(N \leq t) \leq P_{\bar{\delta}}(N \leq t).$$

Thus the run-length distribution under the assumption of a gradual drift can be bounded by run-length distributions for constant process mean. The conclusion of this theorem can be shown to hold under the general condition that the observations are from a location parameter family with location parameter δ .

In order to determine the effect of estimated variance on the properties of the $\bar{X}$ -chart, $P(N > t)$ and the ARL were calculated using numerical integration. For the case where $n = 4$ and $k = 3/\sqrt{n}$ (the standard three-sigma limits), Table 1 gives the ARL and Table 2 gives $P(N > t)$ for several values of v and $\delta/\sqrt{n} = (\mu - \mu_0)/\sigma$. From Table 1 we see that the ARL is larger for small values of v than it is for the case where σ is known ($v = \infty$). From Table 2 we see that $P(N > t)$ is smaller than expected when $\delta = 0$ and $t = 100$ but larger than expected for $\delta > 0$. For example, if $v = 29$ the probability of stopping before 100 observations when in control is 0.2038 compared to 0.1264 when $v = \infty$. But if $\delta/\sqrt{n} = 1$, the probability of taking more than 20 samples to detect the out-of-control situation is 0.0796 instead of 0.0316 when $v = \infty$. One effect of using an estimated variance is to increase the variance of N . This explains the fact that, when $\delta = 0$, $E(N)$ increases as v decreases while $P(N > 100)$ decreases as v decreases.

The ARL values are larger for small values of v , and the effect of using limits based on the t -distribution in place of the three-sigma limits is to increase the ARL values even further. This suggests that if the desired ARL values are those calculated for three-sigma limits when σ is known, then the limits for the case where σ is estimated should be reduced rather than increased. (Note that this is contrary to the recommendation of Yang and Hillier (1970)).

An additional complication arises in the estimation of σ when r initial samples of size m are tested for control with respect to variance before being used to estimate σ . The usual practice is to set up a control limit and discard any sample from among the r samples that exceeds the control limit. If we let S^* be the resulting estimate of σ then S^* and S have different distributions. The run length N^* for the $\bar{X}$ -chart in this case satisfies

$$P(N^* > t) = \int_0^{\infty} P^t(\bar{X}_1 \leq \mu_0 + ks | S^* = s) f_{S^*}(s) ds.$$

Now S based on the original sample is stochastically larger than S^* and (assuming $k > 0$) $P(\bar{X}_1 \leq \mu_0 + ks | S^* = s)$ is increasing in s so that $P(N^* > t) \leq P(N > t)$. This implies that $E(N^*) \leq E(N)$. Thus the effect of testing for control when the initial samples are taken is to reduce the ARL.

4. RESULTS FOR TWO-SIDED SHEWHART CHARTS

In some situations it is desirable to detect shifts in either direction from the control value μ_0 . For this problem it is not necessary that the control limits be symmetric about μ_0 , but symmetric limits usually are used in practice and we consider this case for simplicity. If an initial estimate S based on ν degrees of freedom is used to estimate σ , then the two-sided $\bar{X}$ -chart signals at the first i for which

$$|\bar{X}_i - \mu_0| \geq kS.$$

Let N , N' , and N'' denote the run lengths of the one-sided chart for positive deviations, the two-sided chart, and the one-sided chart for negative deviations, respectively. If the three procedures are applied simultaneously using the same value of k then it follows that

$$N' = \min\{N, N''\}.$$

Using the results and approach of Theorem 1 for the one-sided case we have that $P(N' < \infty) = 1$ and $\lim_{\nu \rightarrow \infty} P(N < t) = [\Phi(-\delta + k\sqrt{n}) - \Phi(-\delta - k\sqrt{n})]^t$.

Corresponding to THEOREM 2 for the one-sided case we have the following result for the two-sided case.

THEOREM 8.

- (a) If $c < \sqrt{2}$ then $E(N') < \infty$.
- (b) If $c > \sqrt{2}$ then $E(N') = \infty$.
- (c) If $c = \sqrt{2}$ then $E(N') < \infty$ for $\delta \neq 0$ and $E(N') = \infty$ for $\delta = 0$.
- (d) If $E(N') < \infty$ then

$$E(N') = \frac{1}{\Gamma(\frac{1}{2}\nu)} \int_0^{\infty} \frac{u^{\frac{1}{2}(\nu-2)} e^{-u}}{1 - \Phi(-\delta + c\sqrt{u}) + \Phi(-\delta - c\sqrt{u})} du$$

- (e) $\lim_{\nu \rightarrow \infty} E(N') = [1 - \Phi(-\delta + k\sqrt{n}) + \Phi(-\delta - k\sqrt{n})]^{-1}$

The stochastic bound for the run-length distribution based on the

geometric distribution also holds for the two-sided case when the parameter $p^+(\delta)$ is replaced by

$$p'(\delta) = P(|\bar{X}_1 - \mu_0|/S \geq k).$$

When the underlying distribution is normal,

$$p'(\delta) = P(|t(v, \delta)| \geq k\sqrt{n}).$$

When $\delta = 0$, the expression for $E(N')$ given in Theorem 8(d) reduces to $E(N)/2$ where $E(N)$ is given by Theorem 2(d). Thus bounds for $E(N')$ for the case $0 < c < \sqrt{2}$ and $\delta = 0$ can be obtained from the bounds for $E(N)$ given in Theorem 4.

Since $N' = \min(N, N'')$ it follows that an upper bound for $E(N')$ is given by $\min(E(N), E(N''))$. When $0 < c < \sqrt{2}$ and $\delta > 0$ an upper bound for $E(N)$ is given by Theorem 5. When $0 < c < \sqrt{2}$ and $\delta < 0$ an upper bound for $E(N'')$ can be obtained from Theorem 5 by using $-\delta$ in the bound for $E(N)$.

Corresponding to Theorem 7 for the one-sided case, we have the following theorem concerning the behavior of N' as a function of δ .

THEOREM 9. If δ and δ' are two sequences such that $|\delta_i| \leq |\delta'_i|$ for all i , then $P_{\delta}(N \leq t) \leq P_{\delta'}(N \leq t)$.

This result holds for any underlying distribution which is symmetric and unimodal.

APPENDIX

PROOF OF THEOREM 1. For any positive integer t

$$P(N > t) = \int_0^{\infty} \phi^t(-\delta + \frac{ks}{\sigma} \sqrt{n}) dF_S(s). \quad (A1)$$

Since $\phi^t(-\delta + \frac{ks}{\sigma} \sqrt{n}) \rightarrow 0$ as $t \rightarrow \infty$ and $\phi(-\delta + \frac{ks}{\sigma} \sqrt{n}) \leq 1$ for every s , it follows from the dominated convergence theorem that

$$\lim_{t \rightarrow \infty} P(N > t) = \int_0^{\infty} \lim_{t \rightarrow \infty} \phi^t(-\delta + \frac{ks}{\sigma} \sqrt{n}) dF_S(s) = 0.$$

Hence $P(N < \infty) = 1$. Since $S \rightarrow \sigma$ a.s. as $v \rightarrow \infty$ and $\phi(-\delta + \frac{ks}{\sigma} \sqrt{n})$ is continuous and bounded, it follows that

$$\lim_{v \rightarrow \infty} P(N > t) = \lim_{v \rightarrow \infty} \int_0^{\infty} \phi^t(\delta + \frac{ks}{\sigma} \sqrt{n}) dF_S(s) = \phi^t(\delta + k\sqrt{n}).$$

The conclusions of the theorem hold under the more general condition that the distribution of the observations be continuous with finite second moments and support $(-\infty, \infty)$.

PROOF OF THEOREM 2. Denote the right-hand side of (A1) by a_t for $t = 0, 1, 2, \dots$, and let

$$g(\zeta) = \sum_{t=0}^{\infty} a_t \zeta^t \quad \text{for } |\zeta| < 1. \quad (A2)$$

Then $a_t \geq 0$ for all t and $E(N) = \sum_{t=0}^{\infty} a_t$.

For part (a), if $c \leq 0$ then $\phi(c\sqrt{u}-\delta) \leq \phi(-\delta)$ for all $u \geq 0$, and $a_t \leq \phi^t(-\delta)$ for all t . Since $0 < \phi(-\delta) < 1$ for all δ , $\sum a_t$ is absolutely convergent and $E(N)$ is given by (3.2). If $0 < c < \sqrt{2}$ then by Tauber's second theorem,

$$\lim_{\zeta \rightarrow 1^-} g(\zeta) = M < \infty \implies \sum_{t=0}^{\infty} a_t = M.$$

Now, for sufficiently small $|\zeta|$, we find

$$g(\zeta) = \frac{1}{\Gamma(\frac{1}{2}\nu)} \int_0^{\infty} \frac{u^{\frac{1}{2}(\nu-2)} e^{-u}}{1 - \zeta \phi(-\delta + c\sqrt{u})} du \quad (A3)$$

and we shall show that $g(\zeta)$ is convergent for all $|\zeta| \leq 1$. Using (see Birnbaum, 1942)

$$1 - \phi(y) > \frac{2\phi(y)}{y + (y^2 + 4)^{\frac{1}{2}}} \text{ for } y \geq 0, \quad (A4)$$

where $\phi(y)$ is the standard normal density, we get

$$1 - \phi(c\sqrt{u} - \delta) > \frac{2\phi(c\sqrt{u} - \delta)}{c\sqrt{u} - \delta + \{(c\sqrt{u} - \delta)^2 + 4\}^{\frac{1}{2}}} \text{ if } \sqrt{u} > \delta/c.$$

Hence, if $\delta \leq 0$ and $0 \leq \zeta \leq 1$,

$$[e^{\frac{1}{2}\delta^2} \sqrt{2\pi}/2\zeta\Gamma(\frac{1}{2}\nu)] \int_0^{\infty} [c\sqrt{u} - \delta + \{(c\sqrt{u} - \delta)^2 + 4\}^{\frac{1}{2}}] u^{\frac{1}{2}(\nu-2)} e^{-u(1-\frac{1}{2}c^2) - \delta c\sqrt{u}} du \leq H_1 < \infty.$$

Thus, $g(\zeta)$ exists for all $|\zeta| \leq 1$, and (3.2) follows by letting $\zeta \rightarrow 1^-$. If $\delta > 0$, the proof is similar. One now splits the range of integration in (A3) into $[0, \delta^2/c^2]$ and $[\delta^2/c^2, \infty]$, and applies the technique above to the second integral.

For part (b), by Abel's theorem,

$$\sum_{t=0}^{\infty} a_t < \infty \implies \lim_{\zeta \rightarrow 1^-} g(\zeta) = \sum_{t=0}^{\infty} a_t.$$

Using (see Pollak, 1957)

$$1 - \phi(y) < \frac{2\phi(y)}{y + (y^2 + 8/\pi)^{\frac{1}{2}}} \text{ for } y > 0 \quad (A5)$$

we get

$$1 - \phi(c\sqrt{u} - \delta) < \frac{2\phi(c\sqrt{u} - \delta)}{c\sqrt{u} - \delta + \{(c\sqrt{u} - \delta)^2 + 8/\pi\}^{\frac{1}{2}}} \text{ if } \sqrt{u} > \delta/c.$$

Hence, if $\delta \leq 0$,

$$g(\zeta) > [e^{\frac{1}{2}\delta^2} \sqrt{2\pi}/2\Gamma(\frac{1}{2}\nu)] \int_0^\infty [c\sqrt{u}-\delta + \{(c\sqrt{u}-\delta)^2 + 8/\pi\}^{\frac{1}{2}}] e^{-u(1-\frac{1}{2}c^2)-\delta c\sqrt{u}} du$$

The right-hand side diverges for $\zeta = 1$ and, since $g(\zeta)$ is continuous at $\zeta = 1$, $\lim_{\zeta \rightarrow 1^-} g(\zeta) = \infty$. Consequently, $E(N) = \sum a_t = \infty$ for all $\delta \leq 0$. The proof is similar when $\delta > 0$.

For part (c), if $\delta \leq 0$, we find $g(1) = \infty$ by the method of part (b), and $E(N) = \infty$. If $\delta > 0$, we find

$$g(\zeta) = \frac{1}{\Gamma(\frac{1}{2}\nu)} \left[\int_0^{\frac{1}{2}\delta^2} \frac{u^{\frac{1}{2}(\nu-2)} e^{-u}}{1-\zeta\phi(-\delta+c\sqrt{u})} du + \int_{\frac{1}{2}\delta^2}^\infty \frac{u^{\frac{1}{2}(\nu-2)} e^{-u}}{1-\zeta\phi(-\delta+c\sqrt{u})} du \right] \leq \frac{1}{(1-\frac{1}{2}\zeta)\Gamma(\frac{1}{2}\nu)}$$

$$\int_0^{\frac{1}{2}\delta^2} u^{\frac{1}{2}(\nu-2)} e^{-u} du + \frac{e^{\frac{1}{2}\delta^2} \sqrt{2\pi}}{2\zeta\Gamma(\frac{1}{2}\nu)} \int_0^{\frac{1}{2}\delta^2} [\sqrt{2u}-\delta + \{(\sqrt{2u}-\delta)^2 + 4\}^{\frac{1}{2}}] u^{\frac{1}{2}(\nu-2)} e^{-\delta\sqrt{2u}} du$$

$$\leq H_2 < \infty$$

and (3.2) follows.

To prove the last part, we note $S \rightarrow \sigma$ in probability, and $h(S) = [1 - \phi(-\delta + \frac{k\sqrt{n}}{\sigma} S)]^{-1}$ is a continuous function. Consequently, $h(S) \rightarrow h(\sigma)$ in probability. It suffices now to show that $\sup_\nu E(h^2(S)) < \infty$ (see Loeve, 1963, Corollary 2 on p. 164). Using the facts that $1 - \phi(x) \geq 1 - \phi(x_0)$ for $x < x_0$ and $1 - \phi(x) \geq \phi(x)(x^{-1} - x^{-3})$ for $x > x_0$, where $x_0 > 0$ is fixed, it can be shown after some algebra that

$$[1 - \phi(-\delta + \frac{k\sqrt{n}}{\sigma} S)]^{-2} \leq [1 - \phi(2+|\delta|)]^{-2} \text{ if } S < (2+|\delta|+\delta)\sigma/k\sqrt{n}$$

$$\leq 2\pi e^{2\delta^2 + S^2} \text{ if } S \geq (2+|\delta|+\delta)\sigma/k\sqrt{n}$$

where

$$\lambda = \frac{2k^2 n}{\sigma^2} \left(1 + \frac{2|\delta|}{2+|\delta|+\delta}\right).$$

Since

$$E(e^{\lambda S^2}) = (1 - 2\lambda\sigma^2/\nu)^{-\nu/2} < (2\lambda\sigma^2 + 1)^{\lambda\sigma^2 + 1/2}$$

for $\nu > 2\lambda\sigma^2 + 1$, it follows that $\sup_{\nu} E(h^2(S)) \leq M$, where

$$M = [1 - \phi(2 + |\delta|)]^{-2} + 2e^{2\delta^2} (1 + 2\lambda\sigma^2)^{\lambda\sigma^2 + 1/2}.$$

Consequently, $E(N) \rightarrow [1 - \phi(-\delta + k\sqrt{n})]^{-1}$ as $\nu \rightarrow \infty$.

PROOF OF THEOREM 3. From (3.1) we have

$$P(N > t) = E(\phi^t(-\delta + \frac{kS}{\sigma} \sqrt{n})).$$

Since ϕ^t is a convex function of ϕ , Jensen's inequality gives

$$\begin{aligned} P(N > t) &\geq E(\phi(-\delta + \frac{kS}{\sigma} \sqrt{n}))^t \\ &= P((\bar{X}_t - \mu_0)/S \leq k)^t \end{aligned}$$

The theorem follows easily from this result. Note that the argument would apply to any distribution function so that the conclusion of the theorem is also valid for non-normal distributions with finite variance.

PROOF OF THEOREM 4. We use the basic expression (3.2) where $\delta = 0$. Using (A5) for $1 - \phi(c\sqrt{u})$ we get

$$E(N) \geq \frac{(2\pi)^{1/2}}{2\Gamma(1/2\nu)} \int_0^\infty [c\sqrt{u} + (c^2u + 8\pi)^{1/2}] u^{1/2(\nu-2)} e^{-u(1-1/2c^2)} du.$$

Since $c^2u + 8/\pi \geq \max\{c^2u, 8/\pi\}$ for all $u \geq 0$, the right hand side above yields, upon integration, the first inequality in (3.3). Similarly, using (A4) and

$$c\sqrt{u} + \{c^2u + 4\}^{1/2} \leq \min[2(c\sqrt{u} + (1/c\sqrt{u})), 2(c\sqrt{u} + 1)], \quad u \geq 0,$$

in (3.2) we get the second inequality in (3.3).

PROOF OF THEOREM 5. We shall use the basic expression (3.2) where $\delta > 0$.

Write

$$E(N) = I_1 + I_2,$$

where

$$I_1 = \{\Gamma(\frac{1}{2}\nu)\}^{-1} \int_0^{\delta^2/c^2} \frac{u^{\frac{1}{2}(\nu-2)} e^{-u} du}{\phi(\delta - c\sqrt{u})}, \quad I_2 = \{\Gamma(\frac{1}{2}\nu)\}^{-1} \int_{\delta^2/c^2}^{\infty} \frac{u^{\frac{1}{2}(\nu-2)} e^{-u} du}{1 - \phi(c\sqrt{u} - \delta)}.$$

Since $\phi(0) \leq \phi(\delta - c\sqrt{u}) \leq \phi(\delta)$ for all $0 \leq u \leq \delta^2/c^2$, we find

$$\{\phi(\delta)\}^{-1} P(\chi^2(\nu) \leq 2\delta^2/c^2) \leq I_1 \leq 2P(\chi^2(\nu) \leq 2\delta^2/c^2). \quad (A8)$$

Consider now I_2 . Using (A4) and the fact $\{(c\sqrt{u} - \delta)^2 + 4\}^{\frac{1}{2}} \leq c\sqrt{u} - \delta + 2$ for $u \geq \delta^2/c^2$ one gets

$$I_2 \leq \frac{\sqrt{2\pi} e^{\frac{1}{2}\delta^2}}{\Gamma(\frac{1}{2}\nu)} \int_{\delta^2/c^2}^{\infty} \{c\sqrt{u} + 1 - \delta\} u^{\frac{1}{2}(\nu-2)} e^{-u(1 - \frac{1}{2}c^2) - \delta c\sqrt{u}} du. \quad (A9)$$

Using (A5) and the fact $\{(c\sqrt{u} - \delta)^2 + 8/\pi\}^{\frac{1}{2}} \geq \max[c\sqrt{u} - \delta, \sqrt{8/\pi}]$ for $u \geq \delta^2/c^2$ one gets

$$I_2 \geq \frac{\sqrt{2\pi} e^{\frac{1}{2}\delta^2}}{\Gamma(\frac{1}{2}\nu)} \int_{\delta^2/c^2}^{\infty} M(u) u^{\frac{1}{2}(\nu-2)} e^{-u(1 - \frac{1}{2}c^2) - c\sqrt{u}} du, \quad (A10)$$

where $M(u) = 2(c\sqrt{u} - \delta)$ or $c\sqrt{u} - \delta + \sqrt{8/\pi}$.

Inequalities (A8) through (A10) lead to (3.4).

PROOF OF THEOREM 6. The result follows by writing $c = \sqrt{2}$ in (A8) through (A10) and using the duplication formula for gamma functions.

PROOF OF THEOREM 7.

$$\begin{aligned} P_{\delta}(N > t) &= P_{\delta}(\bar{X}_1 - \mu_0 \leq kS, i=1, 2, \dots, t) \\ &= \int_0^{\infty} \prod_{i=1}^t \phi(-\delta_1 + \frac{ks}{\sigma} \sqrt{n}) f_S(s) ds. \end{aligned}$$

Since $\Phi(-\delta_1 + \frac{ks}{\sigma} \sqrt{n})$ is decreasing in δ_1 the result follows. Note that the theorem holds under the more general condition that the observations are from a location parameter family with location parameter δ .

PROOF OF THEOREM 8. If either $E(N) < \infty$ or $E(N'') < \infty$ then $E(N') < \infty$ since $N' = \min\{N, N''\}$. Part (a) holds since $E(N) < \infty$ for $c < \sqrt{2}$. Now

$$E(N') = \frac{1}{\Gamma(\frac{1}{2}v)} \sum_{t=0}^{\infty} \int_0^{\infty} [\Phi(-\delta + c\sqrt{u}) - \Phi(-\delta - c\sqrt{u})] t u^{\frac{1}{2}(v-2)} e^{-u} du \quad (A11)$$

which is symmetric about $\delta = 0$. Moreover, $E(N')$ decreased in $|\delta|$ so that $E(N')$ is maximized when $\delta = 0$. Using methods similar to the proof for Theorem 2(b), part (b) can be proved by showing $E(N') = \infty$ when $c > \sqrt{2}$ and $\delta = 0$. When $\delta \neq 0$ part (c) follows from the fact one of $E(N)$ and $E(N'')$ is finite. If $\delta = 0$ we must again use methods similar to the proof for Theorem 2. If $E(N') < \infty$ then part (d) holds since the summation and integration can be interchanged in (A11). The proof for part (e) uses the same methods as the proof of Theorem 7(e).

PROOF OF THEOREM 9. Following the proof of Theorem 7,

$$P_{\delta_1}(N' > t) = \int_0^{\infty} \prod_{i=1}^t P_{\delta_1}(|\bar{X}_i - \mu_0| \leq ks | S=s) f_S(s) ds$$

Since $P_{\delta_1}(|\bar{X}_i - \mu_0| \leq ks | S=s)$ is decreasing in $|\delta_1|$, the result follows. Note that the theorem holds for any underlying distribution which is symmetric and unimodal since this condition is sufficient to guarantee that the distribution of $\bar{X}_i$ is symmetric and unimodal and this, in turn, implies that $P_{\delta_1}(|\bar{X}_i - \mu_0| \leq ks | S=s)$ is decreasing in $|\delta_1|$.

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$\frac{\mu-\mu_0}{\sigma}$	$\nu=29$	$\nu=69$	$\nu=\infty$
0	2100.0	1088.6	740.8
0.5	75.9	53.8	44.0
1	8.0	6.9	6.3
2	1.2	1.2	1.2
3	1.0	1.0	1.0

Table 1. ARL values when σ is estimated with ν degrees of freedom.

$\frac{\mu-\mu_0}{\sigma}$	t	$\nu=9$	$\nu=29$	$\nu=69$	$\nu=\infty$
0	100	.6900	.7962	.8384	.8736
0.5	100	.2329	.1774	.1427	.1001
1	20	.1375	.0796	.0543	.0316
2	2	.0639	.0382	.0307	.0252
3	1	.0064	.0025	.0018	.0013

Table 2. Value of $P(N>t)$ when σ is estimated with ν degrees of freedom.

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20.

to a sequence of independent tests. When σ^2 is estimated the distribution of the run length is not geometric and cannot be characterized simply in terms of the probability of a signal at a given point. The average run length (ARL) for the $\bar{X}$ -chart, ^(average $\bar{x}$) is expressed in terms of an integral involving the normal cdf, and it is shown that the chart signals with probability one, but the ARL may not be finite if the size of the sample used to estimate σ^2 is sufficiently small. In addition, certain bounds for the ARL are also derived. Numerical integration is used to show that the effect of using small sample sizes in estimating σ^2 is to increase the ARL and the variance of the run-length distribution. ^{sigma-squared}