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Gentlemen:

Enclosed please find five copies of the above-subject report. Transmittal of the enclosures completes the requirements of the referenced contract.

We thank you for the opportunity to pursue this unique analytical study and look forward to your continued interest in this work.

Sincerely yours,

DAVIDSON LABORATORY

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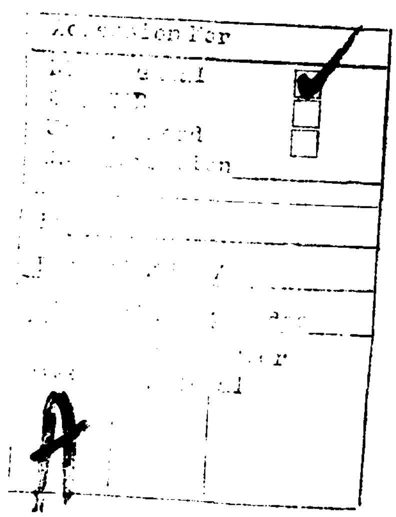
The series solution technique is further demonstrated by calculating the transient state density function with given initial condition for the same nonlinear, first order system. The results are discussed and elaborated in order to indicate the significance of this promising technique.



STEVENS INSTITUTE OF TECHNOLOGY
DAVIDSON LABORATORY
CASTLE POINT STATION
HOBOKEN, NEW JERSEY

Report SIT-DL-79-9-2033

September 1979

DEMONSTRATION OF A STOCHASTIC ANALYSIS TECHNIQUE FOR NONLINEAR DYNAMICAL SYSTEMS

by Charles J. Henry

Prepared for

Office of Naval Research
Statistics and Probability Program
Department of the Navy
Arlington, VA 22217

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In order to demonstrate a promising technique, the control gain for a nonlinear, first order system with limited control and with random input, required to obtain a limited range of state with given likelihood is evaluated. The proposed technique is based on a series approximation for the solution of the partial differential equation for the joint probability density function of the state variable, i.e., the Fokker-Planck Equation. For the simple illustrative example, the exact stationary density function is found to give control gains in good agreement with the proposed series approximation technique. The exact and approximate variances of state are also found in good agreement, but the fourth order moments begin to show some discrepancy and the detailed density functions do not show good agreement. However, for many control problems and for system performance analysis, accurate prediction of means and second order moments is sufficient, as was found for the illustrative example.

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KEYWORDS

Nonlinear Systems

Nonlinear Control

Nonlinear Filtering

TABLE OF CONTENTS

NOMENCLATURE

a	Control gain in system Equation (2.17)
a_i	i-th value of a in Newton-Raphson iteration
$A(i,j,k)$	Function defined in Appendix A, Eq.(A-13)
C	Constant of integration, Eq.(2.6)
$E\{\cdot\}$	Expectation operation or ensemble average
$erf(x)$	Error function as defined in Reference A1 (p.33)
$f[x(t),t]$	General vector system function
$f(x)$	Stationary system function
$f_a(x)$	Derivatives of $f(x)$ with respect to control gain a
f_i	i-th component of $f[x(t),t]$
f_ℓ	ℓ-th coefficient in series expansion of $f(x)$
f'_ℓ	ℓ-th coefficient in series expansion of $f_a(x)$
g	Constant driving noise gain
$G[x(t),t]$	General driving noise gain matrix
$(GQG^T)_{ij}$	i,j-th element of matrix product GQG^T
$h(a)$	Function defined in Eq.(2.22)
$h'(a)$	Derivative of $h(a)$
$h(x_k,t_k)$	Observation model
$He_n(x)$	Hermite polynomial of n-th order summarized in Appendix A
I_T	Target set
M	Truncation value of series approximation
M_k	k-th moment about mean
$p(\xi)$	Stationary state density function
$p_o(\xi)$	Density function for initial state

$p_x(\xi,t)$	Density function for state at time t
$p_x(\xi,t;x_o,t_o)$	Conditional density function for state at time t, given the density for the initial state x_o at time t_o
$p_x(\xi,t\|Y_k)$	Conditional density function for state at time t, given the observation set Y_k
$p_y(\eta\|x_k=\xi)$	Conditional density function for observation y_k given the state $x_k=\xi$
P_1	Desired likelihood level
p_n	n-th coefficient in series expansion of $p(\xi)$
$p_n(t;x_o,t_o)$	n-th coefficient in series expansion of $p_x(\xi,t;x_o,t_o)$
q	Variance of $w(t)$ in first order, stationary system
Q	Covariance matrix of $w(t)$
$r_x(\xi,t\|\alpha)$	Conditional density function for state at time t, given the initial state is α at time 0
$sat(x)$	Saturation function defined in Eq.(2.18)
$S_{xx}(\omega)$	Matrix of spectral densities for state vector $x(t)$
t	Time
t_o	Initial time
t_k	Time of k-th observation
T	Time to go
$U(x)$	Unit step function defined on page 9
v_k	Random error in k-th observation
$w(t)$	Driving noise process
$x(t)$	State vector
$\hat{x}(t)$	Conditional mean of $x(t)$, given Y_k
x_o	Initial state vector
x_1	Desired bound for first order state variable
x_k	Value of state vector at k-th observation time
y_k	k-th observation

Y_k	Set of observed values up to time t_k
α	Realization variable for initial state
β	Launching window width
$\delta_{i,j}$	Kronecker Delta function
$\mu_x(t)$	Mean value of state vector
μ_k	k-th moment about $\xi=0$
ξ	Realization variable for state
ξ_i	i-th component of ξ
$\sigma_x^2(t)$	Variance of state variable
σ_o^2	Initial state variance
τ	Lag
ω	Frequency

INTRODUCTION

A class of problems which frequently arises in control engineering is that of designing or analyzing a system in which some components are fixed before the control problem is attacked, either by the process it is desired to control or by restrictions on space, weight and cost. It is invariably found that some of the fixed attributes of the system are nonlinear since no element of a physical system can be linear over an infinite range of outputs. In the case of marine vehicle dynamics, some sources of nonlinear effects are: mechanical control limits, hydrodynamic pitch-yaw coupling due to vertical asymmetry, temporal changes in wetted or submerged geometry, hydrodynamic influences due to proximity of the free-surface. These attributes are inherently and significantly involved in the dynamics of marine vehicles so that ignoring them by treating a linearized system for control design will lead to vehicles which are ineffectual or unstable, or exhibit poor performance quality.

In practical applications, it is common to find that the input quantity to a control system is not known exactly, either because it is the result of random disturbances acting on the physical system or because the input signal results from an imperfect measurement of a physical quantity. Another source of uncertainty in control analysis and design problems is imperfect knowledge of system parameters in the final product. Recognition of these sources of uncertainty has led to the development of design techniques for linear systems with random inputs or uncertain parameters. However, for nonlinear systems with random inputs or uncertain parameters, the available design techniques are either cumbersome and financially impractical or they involve approximations which have not been found applicable to marine vehicle dynamics. For example, the Monte Carlo simulation technique requires the computation of an output sample from which the desired operational statistics can be estimated for a given design. Then control system parameters must be adjusted and another simulation carried out. This trial-and-error process must then be repeated until an adequate design is found. Consequently, for a complicated system such as a dynamic marine vehicle with six-degrees-of-freedom, the Monte Carlo technique is economically not attractive as a control design tool.

Equivalent linearization techniques yield results which are intuitively significant and reliably predict trends in performance, but are not sufficiently accurate for quantitative control system design. Stochastic approximation techniques rely on the assumption that the system response time is much larger or much smaller than the range of significant periods in the input process, neither of which is true for many marine vehicles.

Consequently, in the design or analysis of marine vehicle control systems, the designer is faced with a nonlinear system with random inputs but the available tools are limited to those for linear systems. This study presents a demonstration of one possible technique for stochastic analysis of nonlinear systems. The demonstration is carried out for the simplest of nonlinear systems since the purpose here is only to illustrate results and feasibility. Subsequent studies must deal with the practical problems of implementing the technique and defining its range of applicability. It should be noted that the rigorous mathematical foundation for this technique is not presently available. Therefore the mathematical conditions for which the proposed series solution converges and those for which a temporally stable approximation exists are not known. This of course is a disadvantage but it should be remembered that engineers have used mathematical tools previously without rigorous mathematical foundation. For example, Dirac Delta functions were used extensively before the theory of distributions or generalized functions was developed by the mathematician. Furthermore, the rigorous foundation for many of the approximate techniques which have been attempted is just as much undeveloped as that for the technique proposed here.

The proposed technique will be demonstrated for a continuous dynamical system with an n-component state vector $x(t)$ governed by a differential system which can be written formally as

$$\dot{x}(t) = f[x(t),t] + G[x(t),t]w(t) \quad ; \quad t \geq t_o$$

with initial condition

$$x(t_o) = x_o \quad .$$

The driving noise $w(t)$ must be a white, Gaussian, zero-mean vector with m-components. The initial condition x_o may be a random vector with density $p_o(\xi)$ not necessarily Gaussian and must be independent of $w(t)$ for $t \geq t_o$. The system function vector $f[x(t),t]$ and the driving noise gain matrix

$G[x(t),t]$ must satisfy the continuity and other conditions stated by Jazwinski[1] in his Theorem 4.5. Then, the conditional state density function given the density of x_o , satisfies

$$\frac{\partial}{\partial t} p_x(\xi,t;x_o,t_o) = -\sum_{i=1}^{n} \frac{\partial}{\partial \xi_i} (p_x f_i) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2}{\partial \xi_i \partial \xi_j}\left[p_x\left(GQG^T\right)_{ij}\right]$$

with initial condition

$$p_x(\xi,t_o;x_o,t_o) = p_o(\xi)$$

where ξ is an n-component vector, f_i denotes the i-th component of $f[x(t),t]$, Q is the covariance matrix of $w(t)$, and $(GQG^T)_{ij}$ denotes the i,j-th element of the matrix product. The resulting solution with $p_o(\xi)=\delta(\xi-x_o)$ is the fundamental statistical descriptor for the given system and the control analysis problem can be solved once this function is evaluated.

In this study, the state vector was limited to one component and the system treated is therefore a first order system. The system function is assumed time-invariant, and the driving noise gain is assumed constant. The system equation then reduces to

$$\dot{x}(t) = f[x(t)] + g\, w(t) \;\; ; \; t \geq t_o$$

with initial condition

$$x(t_o) = x_o$$

and the partial differential equation for the conditional state density reduces to

$$\frac{\partial}{\partial t} p_x(\xi,t;x_o,t_o) = - \frac{\partial}{\partial \xi} (p_x f) + \frac{1}{2} q g^2 \frac{\partial^2}{\partial \xi^2} p_x$$

with initial condition

$$p_x(\xi,t_o;x_o,t_o) = p_o(\xi)$$

and where the driving noise has also been assumed stationary. Under these conditions and if $f[x(t)]$ represents an asymptotically stable system, then a stationary density $p(\xi)$ is assumed to exist for t sufficiently greater than t_o. This stationary density function must satisfy

$$\frac{1}{2} q g^2 \frac{d^2}{d\xi^2} p(\xi) - \frac{d}{d\xi} (pf) = 0 \quad .$$

These equations for the conditional state density $p_x(\xi,t;x_o,t_o)$ and the stationary state density $p(\xi)$ are solved in this report by means of the proposed series approximation technique.

In the next section of this report, a very specific control problem is stated and solved using the stationary density. One of the reasons for treating the stationary first order system is that the above differential equation for $p(\xi)$ can be solved exactly, thus providing a basis for comparison with the approximate series solution. Results from the approximate series solution are illustrated and discussed. Then the series approximation technique for the transient conditional density is demonstrated. The results are elaborated and discussed.

THEORETICAL ANALYSIS

The Stationary Density

The system considered here has scalar state variable $x(t)$ which satisfies a system equation of the form

$$\dot{x}(t) = f[x(t)] + gw(t) \tag{2.1}$$

where $f(x)$ is the system function, g is the driving noise gain assumed constant, and $w(t)$ is a stationary white noise process with normal probability density with zero mean and unit variance. To insure the existence of the probability density for the state variable $x(t)$ and to insure that it approaches a stationary form when initial transients are ignorable, it is assumed that $f(x)$ and g are time-invariant as indicated, that $w(t)$ is stationary and that there exist equilibrium states which are asymptotically stable. These mathematical concepts are discussed by Leipholz[2] and Jazwinski[1] and will not be developed further here.

The stationary probability density function $p(\xi)$ for the state variable $x(t)$ must satisfy the stationary form of the Fokker-Planck Equation[1] given by

$$\frac{1}{2} g^2 \frac{d^2 p(\xi)}{d\xi^2} - \frac{d}{d\xi} [f(\xi)p(\xi)] = 0 \tag{2.2}$$

as well as the conditions

$$\lim_{|\xi|\to\infty} p(\xi) = 0 \tag{2.3}$$

and

$$\int_{-\infty}^{\infty} p(\xi)d\xi = 1 \quad . \tag{2.4}$$

In this section, the exact solution of this boundary value problem will be derived, then a series solution technique will be derived. These solutions will be illustrated by application to a specific system.

Equation (2.2) can be integrated once to obtain

$$\frac{1}{2} g^2 \frac{dp}{d\xi} - f(\xi)p = C_1$$

where C_1 is an arbitrary constant of integration. This equation is a first order ordinary differential equation with general solution given by

$$p(\xi) = \exp\left\{\frac{2}{g^2}\int f(\xi)\,d\xi\right\}\left[\frac{2C_1}{g^2}\int \exp\left\{-\frac{2}{g^2}\int f(\xi)\,d\xi\right\}d\xi + C_2\right] . \quad (2.5)$$

No systems have yet been found by this author which admit asymptotically stable equilibrium states and for which the C_1-term in Eq.(2.5) satisfies Eq.(2.3). Thus, it is presumed here that $p(\xi)$ must be of the form

$$p(\xi) = C \exp\left\{\frac{2}{g^2}\int f(\xi)\,d\xi\right\} \quad (2.6)$$

where the arbitrary constant C is evaluated from Eq.(2.4) as

$$C^{-1} = \int_{-\infty}^{\infty} \exp\left\{\frac{2}{g^2}\int f(\xi)\,d\xi\right\}d\xi \quad . \quad (2.7)$$

Equations (2.6) and (2.7) define the exact solution for the stationary probability density function of the state of the system given by Eq.(2.1). This general result was obtained easily for the first order system, but for higher order systems the exact solution of the corresponding Fokker-Planck Equation is not generally available. Consequently, an approximate solution technique is attempted here in the hope that it may be more generally applicable to higher order systems.

It is assumed that the stationary probability density $p(\xi)$ can be expanded in a series of Hermite polynomials $He_n(\xi)$ of order n, in the form

$$p(\xi) = e^{-\xi^2/2}\sum_{n=0}^{\infty} p_n He_n(\xi) \quad (2.8)$$

where p_n; $n=0,1,2,\ldots$ are unknown constants. Properties of the complete orthogonal set of functions $He_n(\xi)$ are briefly described in Appendix A. Furthermore, it is shown in Appendix B that

$$p_o = 1/\sqrt{2\pi} \quad (2.9)$$

since $p(\xi)$ must satisfy Eq.(2.4). The exponential term in Eq.(2.8) insures that $p(\xi)$ will satisfy Eq.(2.3) and it remains to determine the constants p_n such that $p(\xi)$ satisfies Eq.(2.2).

Substituting Eq.(2.8) into Eq.(2.2), multiplying by $He_m(\xi)d\xi$ and integrating over all ξ yields

$$\sum_{n=0}^{\infty} p_n \left\{ \frac{1}{2} g^2 \int_{-\infty}^{\infty} He_m(\xi) \frac{d^2}{d\xi^2} \left[e^{-\xi^2/2} He_n(\xi) \right] d\xi \right.$$

$$\left. - \int_{-\infty}^{\infty} He_m(\xi) \frac{d}{d\xi} \left[e^{-\xi^2/2} He_n(\xi) f(\xi) \right] d\xi \right\} = 0 \quad . \quad (2.10)$$

The first integral can be integrated by parts twice, making use of Eq.(A-9) and (A-1), to obtain

$$\int_{-\infty}^{\infty} He_m(\xi) \frac{d^2}{d\xi^2} \left[e^{-\xi^2/2} He_n(\xi) \right] d\xi = 0 \quad ; \quad m = 0,1$$

$$= \sqrt{2\pi}\, m!\, \delta_{n,m-2} ; \; m \geq 2 \quad . \quad (2.11)$$

The second integral in Eq.(2.10) can be integrated by parts in the same fashion to obtain

$$\int_{-\infty}^{\infty} He_m(\xi) \frac{d}{d\xi} \left[e^{-\xi^2/2} He_n(\xi) f(\xi) \right] d\xi = 0 \quad ; \quad m = 0$$

$$= -m \int_{-\infty}^{\infty} He_{m-1}(\xi) e^{-\xi^2/2} He_n(\xi) f(\xi)\, d\xi ; \; m \geq 1 \; .$$

It is now assumed that the system function can be expressed as a Hermite series in the form (A-5)

$$f(\xi) = \sum_{\ell=0}^{\infty} f_\ell He_\ell(\xi) \quad (2.12)$$

where

$$f_\ell = \frac{1}{\sqrt{2\pi}\, \ell!} \int_{-\infty}^{\infty} f(\xi) e^{-\xi^2/2} He_\ell(\xi)\, d\xi \quad . \quad (2.13)$$

Equation (2.12) is then substituted in the above integral and Eq.(A-13) is used to obtain

$$\int_{-\infty}^{\infty} He_m(\xi) \frac{d}{d\xi} \left[e^{-\xi^2/2} He_n(\xi) f(\xi) \right] d\xi = 0 \quad ; \quad m=0$$

$$= -m \sqrt{2\pi} \sum_{\ell=0}^{\infty} f_\ell A(\ell, m-1, n) \; ; \; m \geq 1 \; . \quad (2.14)$$

With Eqs.(2.9), (2.11) and (2.14), Eq.(2.10) now gives

<u>m=1</u> :

$$\sum_{n=1}^{\infty} p_n \left[\sum_{\ell=0}^{\infty} f_\ell A(\ell,0,n) \right] = - \frac{1}{\sqrt{2\pi}} \sum_{\ell=0}^{\infty} f_\ell A(\ell,0,0)$$

m ≥ 2:

$$\sum_{n=1}^{\infty} p_n \left[\frac{1}{2} g^2 m! \delta_{n,m-2} + m \sum_{\ell=0}^{\infty} f_\ell A(\ell,m-1,n)\right]$$

$$= -\frac{1}{\sqrt{2\pi}} \left[\frac{1}{2} g^2 m! \delta_{0,m-2} + m \sum_{\ell=0}^{\infty} f_\ell A(\ell,m-1,0)\right] .$$

But,

$$A(\ell,0,n) = n!\delta_{\ell,n} \;;\; A(\ell,0,0) = \delta_{\ell,0} \;;\; A(\ell,m-1,0) = (m-1)!\delta_{\ell,m-1}$$

so that these results become

m = 1: $$\sum_{n=1}^{\infty} p_n n! f_n = -\frac{1}{\sqrt{2\pi}} f_o \tag{2.15}$$

and

m ≥ 2: $$\sum_{n=1}^{\infty} p_n \left[\frac{1}{2} g^2 \delta_{n,m-2} + \sum_{\ell=0}^{\infty} f_\ell \frac{A(\ell,m-1,n)}{(m-1)!}\right]$$

$$= -\frac{1}{\sqrt{2\pi}} \left[\frac{1}{2} g^2 \delta_{m,2} + f_{m-1}\right] . \tag{2.16}$$

If the series expansion for $p(\xi)$ converges, then it can be truncated at some order M and Eqs.(2.15) and (2.16) then give a system of M linear non-homogeneous equations in M unknown constants p_n ; $n=1,2, \ldots, M$, so that the series expansion coefficients can be determined.

The exact and series solution techniques will be demonstrated by application to a nonlinear system with limited control. In particular, consider the system defined by

$$\dot{x}(t) = x - x^3 + a\,\mathrm{sat}(x) + \sqrt{2}\, w(t) \tag{2.17}$$

where

$$\mathrm{sat}(x) = \begin{cases} -1 \;;\; x < -1 , \\ x \;;\; |x| \le 1 , \\ 1 \;;\; x > 1 . \end{cases} \tag{2.18}$$

This system with no control (a=0) has two asymptotically stable equilibrium states at $x = \pm 1$, while $x=0$ is an unstable equilibrium state. The control function has gain a for $|x| \le 1$ and is limited to $\pm a$ for $|x| > 1$. The objective of the control in this example is to obtain a stationary probability p_1 that the state $x(t)$ at any time t is between $-x_1$ and $+x_1$ for a given $x_1 > 0$, i.e., find a such that

$$\int_{-x_1}^{x_1} p(\xi)d\xi = p_1 \quad . \tag{2.19}$$

No attempt will be made to argue that this problem is related to any specific practical situation although it is hoped that the interested reader will see the many practical aspects embodied in this simplified example. The purpose of this application is simply to demonstrate the series solution technique in comparison with an exact solution of the same problem, as well as to indicate the usefulness of the probabilistic design approach.

The system function for this example is seen to be

$$f(x) = x - x^3 + a \text{ sat}(x) \quad , \text{ with } g = \sqrt{2} \quad . \tag{2.20}$$

Substituting this function in the general solution for $p(\xi)$ given by Eq.(2.6) leads to

$$p(\xi) = C \exp\left\{ \frac{\xi^2}{2} - \frac{\xi^4}{4} + \frac{a\xi^2}{2} U(1-\xi^2) + a(|\xi| - \tfrac{1}{2})U(\xi^2-1) \right\}$$

with (2.21)

$$C^{-1} = 2 \int_0^\infty \exp\left\{ \frac{\xi^2}{2} - \frac{\xi^4}{4} + \frac{a\xi^2}{2} U(1-\xi^2) + a(|\xi| - \tfrac{1}{2})U(\xi^2-1) \right\} d\xi$$

where $U(x)$ is the unit step function

$$U(x) = \begin{cases} 1 & ; x > 0 \\ 0 & ; x < 0 \end{cases} \quad .$$

In the expression for C^{-1} , use was made of the symmetry of the integral of an antisymmetric system function.

In order to find the value of a such that $p(\xi)$ in Eq.(2.21) satisfies Eq.(2.19), a numerical iteration procedure, the Newton-Raphson technique, was used which requires that $p(\xi)$ be considered a function of the parameter a , so that for the i-th estimate a_i of the desired value, Eq.(2.19) becomes

$$h(a_i) = \int_{-x_1}^{x_1} p(\xi, a_i)d\xi - p_1 \quad . \tag{2.22}$$

Then the next estimate a_{i+1} is obtained from the Taylor series expansion

$$h(a_{i+1}) = h(a_i) + (a_{i+1} - a_i)h'(a_i) = 0$$

or

$$a_{i+1} = a_i - h(a_i)/h'(a_i) \tag{2.23}$$

where $h'(a)$ is the derivative of $h(a)$ with respect to a . From Eq.(2.21) and Eq.(2.22), $h'(a)$ is found to be

$$h'(a) = \int_{-x_1}^{x_1} p(\xi)\left\{\left[\frac{\xi^2}{2} U(1-\xi^2) + (|\xi| - \tfrac{1}{2})U(\xi^2-1)\right] + \frac{1}{C}\frac{\partial C}{\partial a}\right\} d\xi$$

where (2.24)

$$\frac{1}{C}\frac{\partial C}{\partial a} = -2C\int_0^\infty \exp\left\{\frac{\xi^2}{2} - \frac{\xi^4}{4} + \frac{a\xi^2}{2} U(1-\xi^2) + a(|\xi| - \tfrac{1}{2})U(\xi^2-1)\right\}$$

$$\cdot\left[\frac{\xi^2}{2} U(1-\xi^2) + (|\xi| - \tfrac{1}{2})U(\xi^2-1)\right] d\xi \quad .$$

The solution to this same problem is obtained via the series solution technique by substituting the system function Eq.(2.20) into Eq.(2.13) and using the resulting coefficients in Eq.(2.15) and Eq.(2.16) to find the unknown coefficients p_n for the expansion of the density function shown in Eq.(2.8). The coefficients for the expansion of the system function are obtained from Tables A-1 and A-3 as

$$f_1 = a\ \mathrm{erf}\left(\frac{1}{\sqrt{2}}\right) - 2$$

$$f_3 = -a\frac{e^{-1/2}}{3\sqrt{2\pi}} He_1(1) - 1$$

$$f_{2\ell+1} = -a\frac{2e^{-1/2}}{\sqrt{2\pi}}\frac{He_{2\ell-1}(1)}{(2\ell+1)!} \quad ; \quad \ell = 2,3, \ldots$$

$$f_{2\ell} = 0 \ ; \ \ell = 0,1,2, \ldots$$

Due to the rapid increase in numerical difficulties with increasing number of terms in series expansions, it is important to take account of any simplifications that can be obtained from further analytical considerations before going to Eq.(2.15) and Eq.(2.16), such as, accounting for the symmetries in the particular system function being considered. In this application, it is seen that the system function in Eq.(2.20) is antisymmetric about x=0, i.e.,

$$f(-x) = -f(x)$$

leading to zero values for the even order coefficients shown above. But if $f(\xi)$ in Eq.(2.2) is antisymmetric then $p(\xi)$ must be symmetric about $\xi=0$ so that all odd order coefficients p_{2n+1} must vanish. These symmetries can be accounted for by relabeling coefficients in Eqs.(2.8) and (2.12) and (2.13) in the form

$$p(\xi) = e^{-\xi^2/2} \sum_{n=0}^{\infty} p_n He_{2n}(\xi) \;\; ; \;\; p_n = \frac{1}{\sqrt{2\pi}\,(2n)!} \int_{-\infty}^{\infty} p(\xi) He_{2n}(\xi)\, d\xi \tag{2.25}$$

and

$$f(\xi) = \sum_{\ell=0}^{\infty} f_\ell He_{2\ell+1}(\xi) \, ; \;\; f_\ell = \frac{1}{\sqrt{2\pi}\,(2\ell+1)!} \int_{-\infty}^{\infty} f(\xi)\, e^{-\xi^2/2} He_{2\ell+1}(\xi)\, d\xi \tag{2.26}$$

As before, Eq.(2.4) still requires $p_o = 1/\sqrt{2\pi}$. Equations (2.15) and (2.16) become

$$\sum_{n=1}^{\infty} p_n \left[\frac{1}{2} g^2 \delta_{n,m-1} + \sum_\ell f_\ell \frac{A(2\ell+1,\ 2m-1,\ 2n)}{(2m-1)!}\right]$$

$$= - \frac{1}{\sqrt{2\pi}} \left[\frac{1}{2} g^2 \delta_{m,1} + \frac{f_{m-1}}{(2m-1)!}\right] \; ; \; m = 1,2,\ \ldots \tag{2.27}$$

which again gives a linear system of non-homogeneous equations for the unknown coefficients p_n , once the n and ℓ summations are truncated.

In order to apply Eq.(2.23) to the given problem by means of the series solution technique, a representation for $\partial p/\partial a$ must be developed so that $h'(a_i)$ can be calculated from

$$h'(a) = \int_{-x_1}^{x_1} \frac{\partial}{\partial a} p(\xi)\, d\xi \tag{2.28}$$

where as before, $p(\xi)$ is considered a function of the parameter a . But then Eq.(2.2) can be differentiated with respect to a to give

$$\frac{1}{2} g^2 \frac{\partial^2}{\partial \xi^2} \frac{\partial p}{\partial a} - \frac{\partial}{\partial \xi} \left[f(\xi) \frac{\partial p}{\partial a}\right] = \frac{\partial}{\partial \xi} \left[f_a(\xi) p(\xi)\right] \tag{2.29}$$

where

$$f_a(\xi) = \frac{\partial}{\partial a} f(\xi) \quad .$$

In Eq.(2.29), the series expansion of $f(\xi)$ is given by Eq.(2.26) and that for $p(\xi)$ is given by Eq.(2.25) with coefficients obtained from Eq.(2.27) for a given value of a. Furthermore, the coefficients f_ℓ are seen to be linear in the parameter a so that the corresponding coefficients in the expansion of $f_a(\xi)$ are easily found to be

$$f_a(\xi) = \sum_{\ell=0}^{\infty} f'_\ell \, He_{2\ell+1}(\xi) \; ; \; f'_\ell = \begin{cases} \mathrm{erf}(1/\sqrt{2}) \; ; \; \ell=0 \\ -He_1(1)/3\sqrt{2\pi e} \; ; \; \ell=1 \\ -2He_{2\ell-1}(1)/\sqrt{2\pi e}\,(2\ell+1)! \, ; \, \ell \geq 2 \end{cases} \tag{2.30}$$

With $f(\xi)$, $f_a(\xi)$ and $p(\xi)$ all known in series form, $\partial p/\partial a$ can also be expressed in series form as

$$\frac{\partial}{\partial a} p(\xi) = e^{-\xi^2/2} \sum_{n=1}^{\infty} p'_n He_{2n}(\xi) \; ; \; p'_n = \frac{1}{\sqrt{2\pi}\,(2n)!} \int_{-\infty}^{\infty} \frac{\partial}{\partial a} p(\xi) e^{-\xi^2/2} He_{2n}(\xi)\, d\xi \tag{2.31}$$

and the unknown coefficients p'_n can be found from the differential Eq.(2.29). The equations for p'_n become

$$\sum_n p'_n \left[\frac{1}{2} g^2 \delta_{n,m-1} + \sum_\ell f_\ell \frac{A(2\ell+1,\, 2m-1, 2n)}{(2m-1)!}\right]$$

$$= - \sum_n p_n \sum_\ell f'_\ell \frac{A(2\ell+1,\, 2m-1,\, 2n)}{(2m-1)!} \tag{2.32}$$

which can be solved numerically once the n and ℓ summations are truncated. Thus, the Newton-Raphson iteration technique in Eq.(2.23) can be applied when $P(\xi)$ is expressed in series form, to determine the desired value of the control gain a.

The Transient Density

The transient density function $p_x(\xi,t;x_o,t_o)$ can also be evaluated by means of the series approximation by allowing the coefficients to be time dependent, i.e.,

$$p_x(\xi,t;x_o,t_o) = e^{-\xi^2/2}\sum_{n=0}^{\infty} p_n(t;x_o,t_o)He_n(\xi) \quad ,$$

where (3.1)

$$p_n(t;x_o,t_o) = \frac{1}{\sqrt{2\pi}\; n!}\int_{-\infty}^{\infty} p_x(\xi,t;x_o,t_o)He_n(\xi)d\xi .$$

The conditional arguments x_o,t_o will be dropped for clarity. Equation(2.4) still requires

$$p_o(t) = 1/\sqrt{2\pi} \quad . \tag{3.2}$$

Substituting Eq.(3.1) into the Fokker-Planck Equation given by

$$\frac{\partial}{\partial t}p_x(\xi,t) = \frac{1}{2}qg^2\frac{\partial^2}{\partial\xi^2}p_x - \frac{\partial}{\partial\xi}(p_x f) \tag{3.3}$$

then multiplying by $He_m(\xi)d\xi$ and integrating over all ξ leads to

$$\sum_{n=0}^{\infty}\dot{p}_n(t)\int_{-\infty}^{\infty}e^{-\xi^2/2}He_m(\xi)He_n(\xi)d\xi$$

$$= \sum_{n=0}^{\infty}p_n(t)\left\{\frac{1}{2}qg^2\int_{-\infty}^{\infty}He_m(\xi)\frac{\partial^2}{\partial\xi^2}\left[e^{-\xi^2/2}He_n(\xi)\right]d\xi - \int_{-\infty}^{\infty}He_m(\xi)\frac{\partial}{\partial\xi}\left[e^{-\xi^2/2}He_n(\xi)f(\xi)\right]d\xi\right\} .$$

Using Eqs.(3.2) and (2.12) and the results in Appendix A gives

$$\dot{p}_o(t) = 0$$

$$\dot{p}_1(t) = \sum_{n=1}^{\infty}p_n(t)n!\sum_{\ell=0}^{\infty}f_\ell\delta_{2\ell+1,n}$$

$$\dot{p}_m(t) \quad \sum_{n=1}^{\infty}\left[\frac{1}{2}qg^2\delta_{n,m-2} + \sum_{\ell=0}^{\infty}f_\ell\frac{A(2\ell+1,m-1,n)}{(m-1)!}\right] \; ; \; m=2,3,\ldots$$

which is a linear system of first order differential equations. The time history of the unknown coefficients can be evaluated by numerical integration, once the n and ℓ summations are truncated and once an initial condition is specified. In general, the initial condition is obtained in the form

$$p_o(0) = 1/\sqrt{2\pi} \quad , \tag{3.5a}$$

$$p_n(0) = \frac{1}{\sqrt{2\pi}\, n!} \int_{-\infty}^{\infty} p_o(\xi) He_n(\xi) d\xi \quad ; n=1,2,\ldots \tag{3.5b}$$

An illustrative calculation of the transient density function is shown in Figure 5 where the initial condition is

$$p_o(\xi) = \frac{e^{-\xi^2/2\sigma_o^2}}{\sqrt{2\pi\sigma_o^2}} \quad . \tag{3.6}$$

The practical value of this initial condition will be demonstrated in the following section, Discussion of Results. In Figure 5, it is seen that the transient density evolves from a fairly peaked initial condition into the flat stationary density function shown in Figure 3a for the same value of control gain.

DISCUSSION OF RESULTS

The Stationary Density

The control gain a required to obtain a limited range of state variable $-x_1 < x(t) < x_1$ with 0.90 likelihood was calculated for the first order system defined by Eq.(2.17) using the exact stationary density function, Eq.(2.21) as well as the approximate series expansion Eq.(2.25) with coefficients given by the solution of Eq.(2.27). The results obtained are compared in Figures 1, 2, 3 and 4.

The exact system function given by Eq.(2.20) is exhibited in Figure 1 for the case $a=-3$, together with truncated series approximations with 5 odd terms (up to 9th order Hermite polynomial) and 13 odd terms (25th order polynomial). The coefficients used in the series expansion are shown in Eq.(2.26). It is seen in the Figure that the exact and approximate system function agree very well for the range of x shown except near the discontinuity in $f(x)$ at $x=1$. At this point, the polynomial approximation is continuous and therefore tends to smooth out the discontinuity. In subsequent calculations, a nine odd term series was used (17th order polynomial), which lies between the two approximations shown in Figure 1.

The value of control gain a required to limit the state $x(t)$ to the range $-x_1 < x(t) < x_1$ with 0.90 likelihood is plotted in Figure 2 as calculated by means of the exact density and by means of the approximate series expansion of the density. It is seen that the two solutions for the stated control problem agree well. The estimated control gains based on the approximate density show some scatter or irregularity which could be reduced by improving the iteration. Consequently, the proposed series approximation technique for nonlinear systems analysis is considered worthy of further development and study.

It should be noted here that the level of effort required to obtain the results shown in Figure 1 by means of the Monte Carlo simulation technique (the best known technique when the exact solution is not obtainable) is orders of magnitude greater than that utilized here in applying the series approximation technique to derive the density function. The required steps would be as follows. A simulator or model of the system defined by Eq.(2.17) would be developed and subjected to a sample input process $w(t)$, for a given control gain a. The output sample would be analyzed statistically in order to estimate the probability of the event $\{-x_1 < x(t) < x_1\}$ as well as the confidence of this estimate. This process would then be repeated for a new value of a until the desired likelihood was obtained, with appropriate confidence. The only way to increase confidence in the estimated likelihood is to increase the sample length of the simulation or model test. Finally, all the above steps must be repeated for each x_1 shown in Figure 1. On the other hand, the series approximation technique requires one solution of a linear algebraic system of equations for each iteration step and the results shown in Figure 2 were obtained with 2 to 3 iterations.

Some calculated density functions are shown in Figure 3a, b and c, using the exact and approximate solutions, for various values of x_1. When the desired range x_1 is large (x_1=1.5 and 1.35,Fig.3a), the $\exp(-\xi^4)$ behavior for ξ large in the exact solution Eq.(2.21) produces a rapid fall-off of $p(\xi)$ which the series solution attempts to follow. However, the approximate solution behaves like $\exp(-\xi^2)$ times a polynomial so that in attempting to produce the more rapid fall-off, poor agreement for small ξ results. On the other hand, for small values of x_1 (0.60 and 0.75 in Fig.3c), the large peak at $\xi=0$ leads to poor agreement between exact and approximate solutions. In between these extremes, some good approximations for the exact density function are shown. In particular, see the result for x_1=0.90 in Figure 3c. It should be kept in mind here that good estimates of the desired control gain a were obtained for all cases shown in Figure 3 so that a solution technique which duplicates the exact density function in every detail is not always required. Methods for improving the convergence of the series solution technique and for handling the numerical difficulties of large order systems must be developed during the subsequent evolution of the proposed technique.

A further comparison between the exact and approximate solution is illustrated in Table 1 which shows corresponding values of the second order moment $M_2=\sigma_x^2$, the variance, and the ratio M_4/M_2^2, the kurtosis, which is a measure of the flatness or peakedness of the density function. From Eq.(B-3) it is seen that $M_4/M_2^2 = 3$ for a normal density. The comparison between exact and approximate variance is seen to be very good for all cases except x_1=0.60

where the exact solution becomes very peaked at $\xi=0$. The discrepancy in density function in most cases does not effect the accuracy of the series solution for second order statistics just as for estimating the control gain a . The comparison of kurtosis is seen to be very good except for the last two values (x_1 = 0.75, 0.60). Thus, the accuracy of the series solution diminishes for predicting higher order moments as should be expected. For performance analysis and for control system design, estimates of the mean and variance are the most important statistics and these are reliably predicted here.

The series for $p_x(\xi)$ was truncated at $M=8$ in all the results shown in Figures 2 and 3 and in Table 1. The effect of decreasing the number of terms in the series expansion is shown in Figure 4. It is seen that the agreement between exact and approximate solutions diminishes as the series is truncated at lower order. Solutions attempted with larger numbers of terms showed signs of numerical problems which were beyond the scope of this initial study.

The Transient Density

Utilization of the series approximation technique to solve the Fokker-Planck Equation has been demonstrated in Figure 5 for a nonlinear first order system with limited control. Suppose that this technique could yield a means of approximating the density function $r_x(\xi,t|\alpha)$ defined as the transient density for the given system with initial condition

$$r_x(\xi,0|\alpha) = \delta(\xi-\alpha)$$

where α is any given initial state and where a time invariant system has been assumed so that we can take $t_o=0$ for simplicity. It should be noted that

$$\lim_{\sigma_o \to 0} \frac{e^{-\xi^2/2\sigma_o^2}}{\sqrt{2\pi\sigma_o^2}} = \delta(\xi)$$

so that the results shown in Figure 5 are an approximation for $r_x(\xi,t|0)$. In any event, $r_x(\xi,t|\alpha)$ is the conditional transition density function for the system from which any first or second order statistic can be calculated.

The joint state density, for example, is given by

$$p_x(\xi,t+\tau;\alpha,t) = r_x(\xi,\tau|\alpha)\; p_x(\alpha,t) \quad , \quad \tau \geq 0$$

from which the power spectral density for stationary operating condition $S_{xx}(\omega)$ can be calculated by taking $p_x(\alpha,\tau)$ to be the stationary density $p_x(\alpha)$ and using

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} d\tau \; e^{-i\omega\tau} \iint_{-\infty}^{\infty} d\xi \; d\alpha \; r_x(\xi,\tau|\alpha) \; p_x(\alpha) \quad .$$

The first order state density is given by

$$p_x(\xi,t+\tau) = \int_{-\infty}^{\infty} p_x(\xi,t+\tau;\alpha,\tau)\,d\alpha$$

$$= \int_{-\infty}^{\infty} r_x(\xi,\tau|\alpha) \; p_x(\alpha,t)\,d\alpha$$

which is the Chapman-Kolmogrov Equation. From the first order state density, the mean $\mu_x(t)$ and variance $\sigma_x^2(t)$ can be calculated by means of

$$\mu_x(t) = \int_{-\infty}^{\infty} \xi \; p_x(\xi,t)\,d\xi$$

$$\sigma_x^2(t) = \int_{-\infty}^{\infty} (\xi-\mu_x)^2 \; p_x(\xi,t)\,d\xi \quad .$$

As an illustration of how these results can be utilized for performance analysis, suppose it is desired that the state of the system after time T be in an interval I_T . The probability of obtaining the desired state, when the initial state is a random variable x_o with density $p_o(\alpha)$, is given by

$$\text{Prob}\{x(T)\in I_T\} = \int_{I_T} p_x(\xi,T)\,d\xi$$

$$= \int_{I_T} d\xi \int_{-\infty}^{\infty} r_x(\xi,T|\alpha)\,p_0(\alpha)\,d\alpha \quad .$$

The series approximation technique illustrated herein shows promise of providing a relation for the fundamental statistical descriptor of the system $r_x(\xi,T|\alpha)$ in terms of system parameters such as the control gain a , and in terms of time T. Supposing further that the initial density is of the form

$$p_o(\alpha) = \begin{cases} 1/2\beta & ; \; |\alpha| < \beta \\ 0 & ; \; |\alpha| > \beta \end{cases}$$

then the designer could find the combinations of launching window width β, time to go T and control gain a , which would yield an acceptable risk of missing the target set I_T given by $1\text{-Prob}\{x(T)\in I_T\}$.

Finally, the statistical estimation of state of a nonlinear system driven by noise, from observations of state corrupted by noise, can be carried out with the series approximation technique outlined here. The problem of parameter identification falls in this category. In particular, suppose that the state $x(t)$ of a system is modeled by an equation of the form

$$\dot{x}(t) = f[x(t),t] + G[x(t),t]w(t)$$

$$t \geq t_o \quad ; \quad x(t_o) = x_o$$

where $w(t)$ is a white, normal, zero mean process, where x_o may be a random vector which is independent of $w(t)$, and that the observation model is given by

$$y_k = h(x_k,t_k) + v_k \quad ; \; k=1,2, \; \ldots$$

where v_k are independent, normal, zero mean random vectors which are independent of x_o and $w(t)$. The statistical problem is to derive an estimate $\hat{x}(t)$ for the state based on measurements

$$Y_k = \{y_1,y_2, \; \ldots, \; y_k\} \; ; \; t_k < t \leq t_{k+1} \quad .$$

One good estimator is the conditional mean defined as

$$\hat{x}(t) = E\{x(t)|Y_k\} = \int_{-\infty}^{\infty}\xi \; p_x(\xi,t|Y_1)d\xi \quad .$$

It has been shown[1] that between observations the conditional density of function $p_x(\xi,t|Y_k)$ must satisfy the Fokker-Planck Equation

$$\frac{\partial}{\partial t} p_x = -\sum_{i=1}^{n} \frac{\partial}{\partial \xi_i} (f_i p_x) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2}{\partial \xi_i \partial \xi_j} \left[(GQG^T)_{ij} p_x\right]$$

for $t_k < t < t_{k+1}$ with initial condition

$$p_x(\xi,t_k|Y_k) \quad .$$

At an observation the conditional density function is updated by means of Bayes's Theorem in the form

$$p_x(\xi, t_k | Y_k) = \frac{p_y\left(y_k | x_k = \xi\right) \, p_x\left(\xi, t_k | Y_{k-1}\right)}{\int_{-\infty}^{\infty} p_y\left(y_k | x_k = \eta\right) \, p_x\left(\eta, t_k | Y_{k-1}\right) d\eta}$$

which relates the new initial condition to the density at the end of the previous interval between measurements. Thus, the required conditional density, and thereby $\hat{x}(t)$, can be evaluated for all t by alternately using the Fokker-Planck Equation and Bayes's Theorem. The series approximation provides the appropriate means for solving the Fokker-Planck Equation.

CONCLUSIONS

For effective design of nonlinear systems subjected to random inputs, a stochastic analysis technique is needed in order to estimate statistics of output or performance for given input statistics. A technique for providing this relationship is illustrated herein, based on a series approximation of the solution to the governing equation for the fundamental statistical quantity, the transient conditional state density function.

Using the governing equation, the Fokker-Planck Equation, a stationary control problem for a first order nonlinear system with limited control, was stated and solved exactly and by means of the series approximation technique. The exact and approximate control gain were found to be in good agreement as were the variances of response. The fourth order response moment began to show some discrepancy between the exact and approximate solutions and the detailed density functions were in poor agreement, but the stated problem did not require detailed agreement as is the case for many design and analysis problems.

Also, based on the governing equation, the series solution technique for evaluating the transient conditional density function was demonstrated for the same first order nonlinear system and control. Then, in discussing these results, it is briefly shown how the transient conditional density function can be used to analyze problems in systems analysis, control design and state estimation.

REFERENCES

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2. Leipholz, H., Stability Theory, Academic Press, 1970.

TABLE 1

COMPARISON OF MOMENTS

x_1	VARIANCE (M_2)			KURTOSIS (M_4/M_2^2)		
	Exact	Approx.	Error	Exact	Approx.	Error
1.50	0.921	0.926	-0.5%	1.99	1.97	1.0%
1.35	0.705	0.725	-2.8	2.22	2.18	1.8
1.20	0.541	0.522	3.6	2.45	2.50	-2.0
1.05	0.411	0.402	2.2	2.65	2.67	-0.7
0.90	0.302	0.300	0.7	2.72	2.84	-4.2
0.75	0.209	0.207	1.0	2.92	3.16	-7.6
0.60	0.133	0.120	10.8	2.95	6.32	-53.3

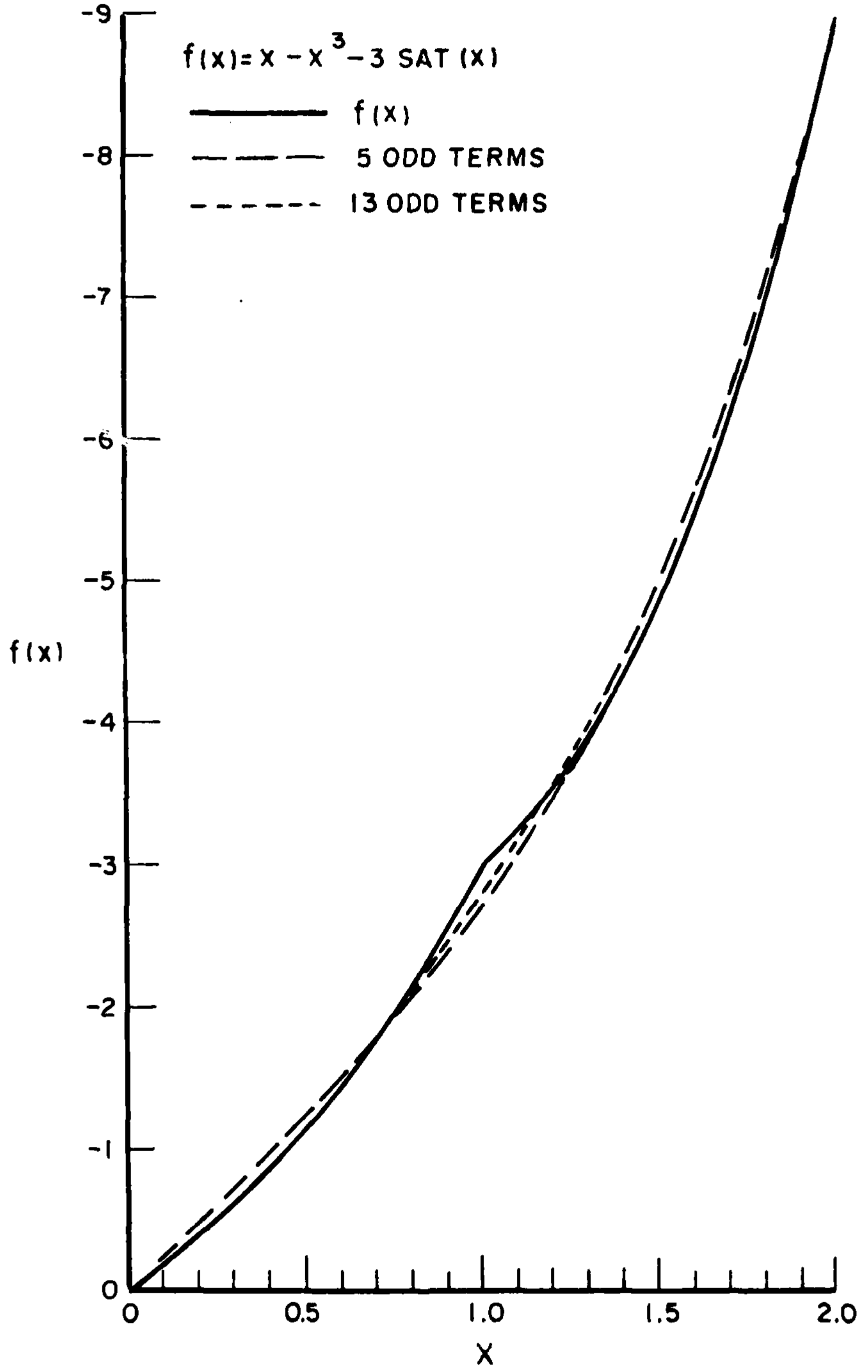

FIG. 1. SERIES APPROXIMATION FOR SYSTEM FUNCTION WITH DISCONTINUOUS SLOPE

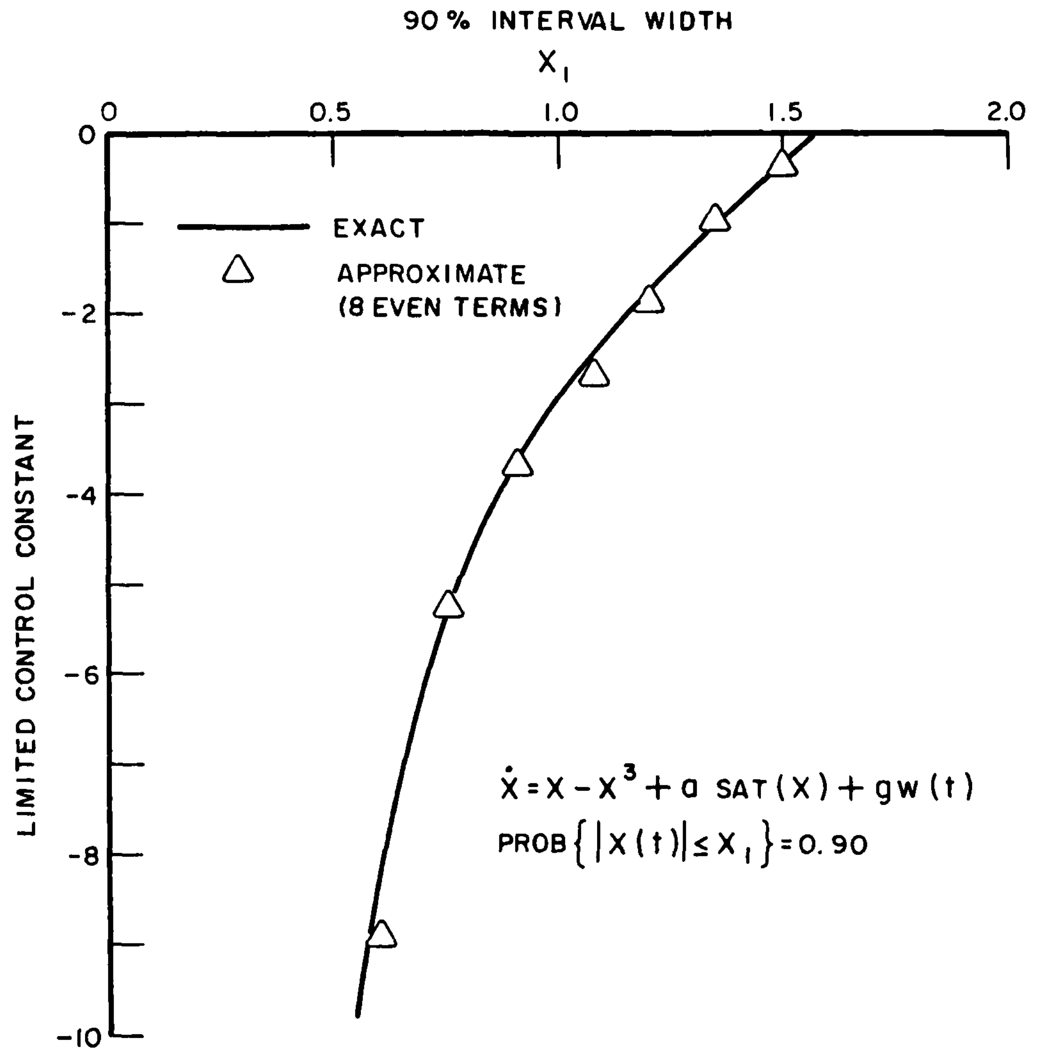

FIG. 2. LIMITED CONTROL CONSTANT REQUIRED TO OBTAIN 90 PER CENT LIKLIHOOD THAT STATE IS BOUNDED BY X_1

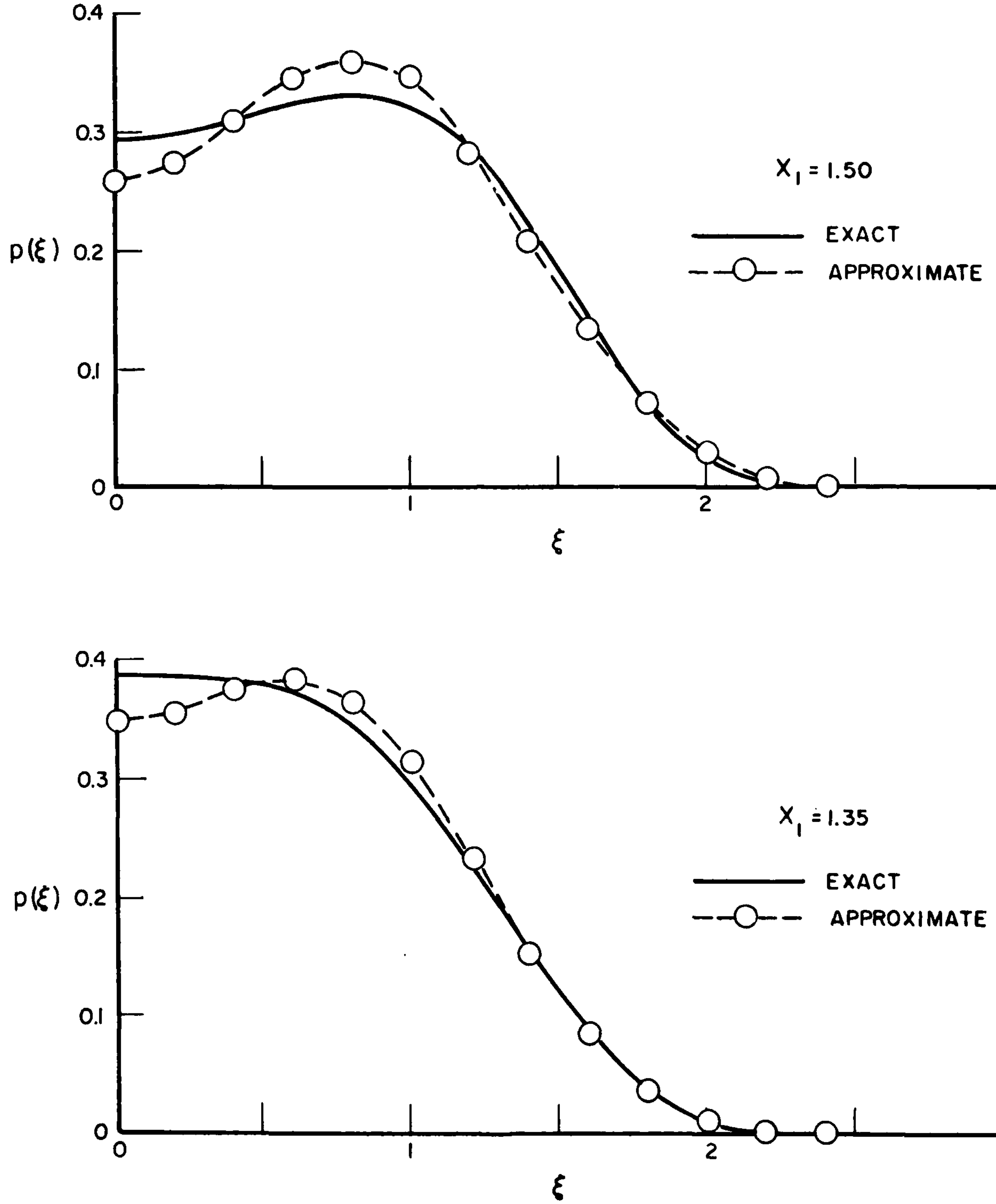

FIG. 3a. COMPARISON OF EXACT AND APPROXIMATE STATE DENSITIES

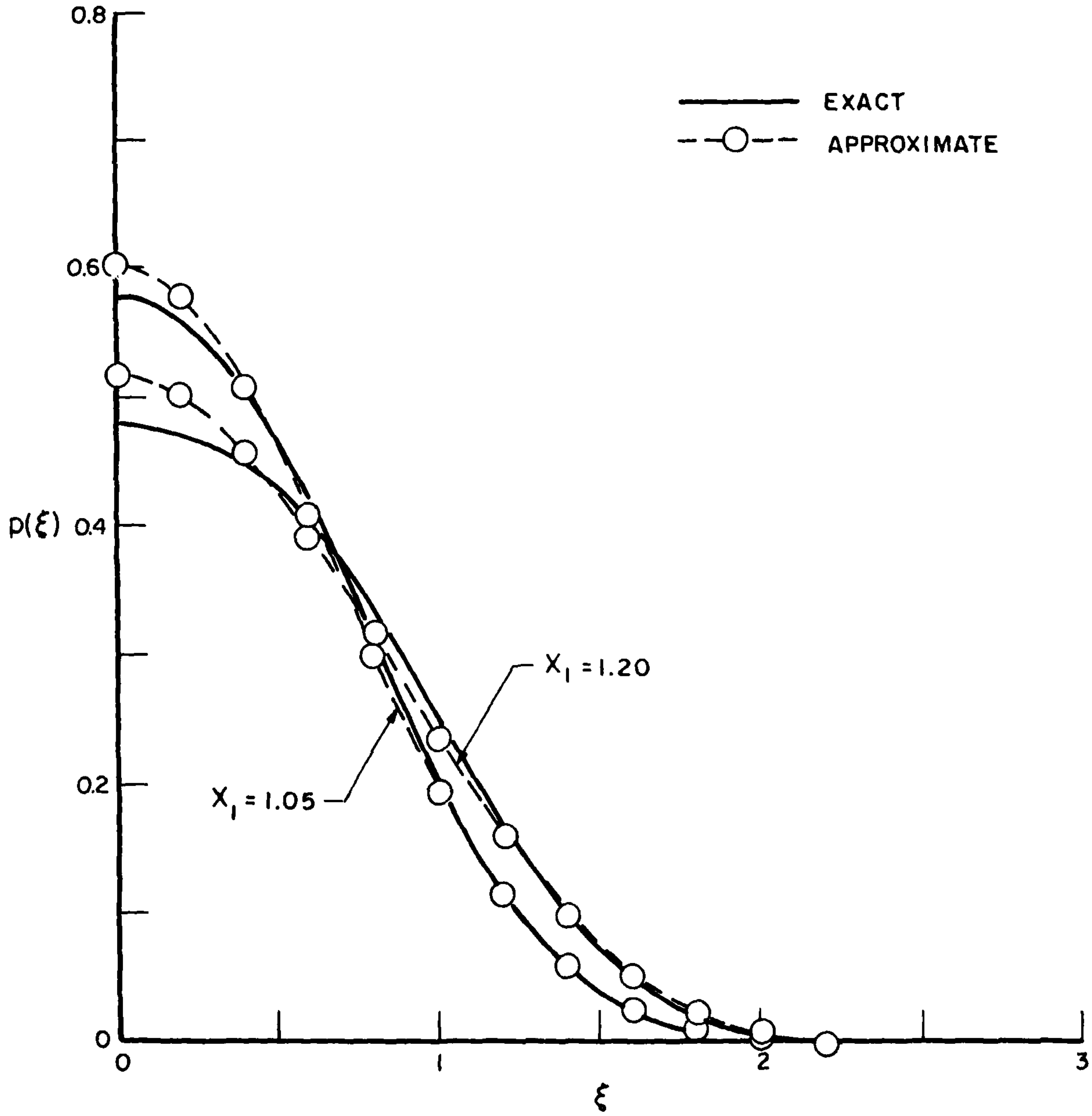

FIG. 3b. COMPARISON OF EXACT AND APPROXIMATE STATE DENSITIES

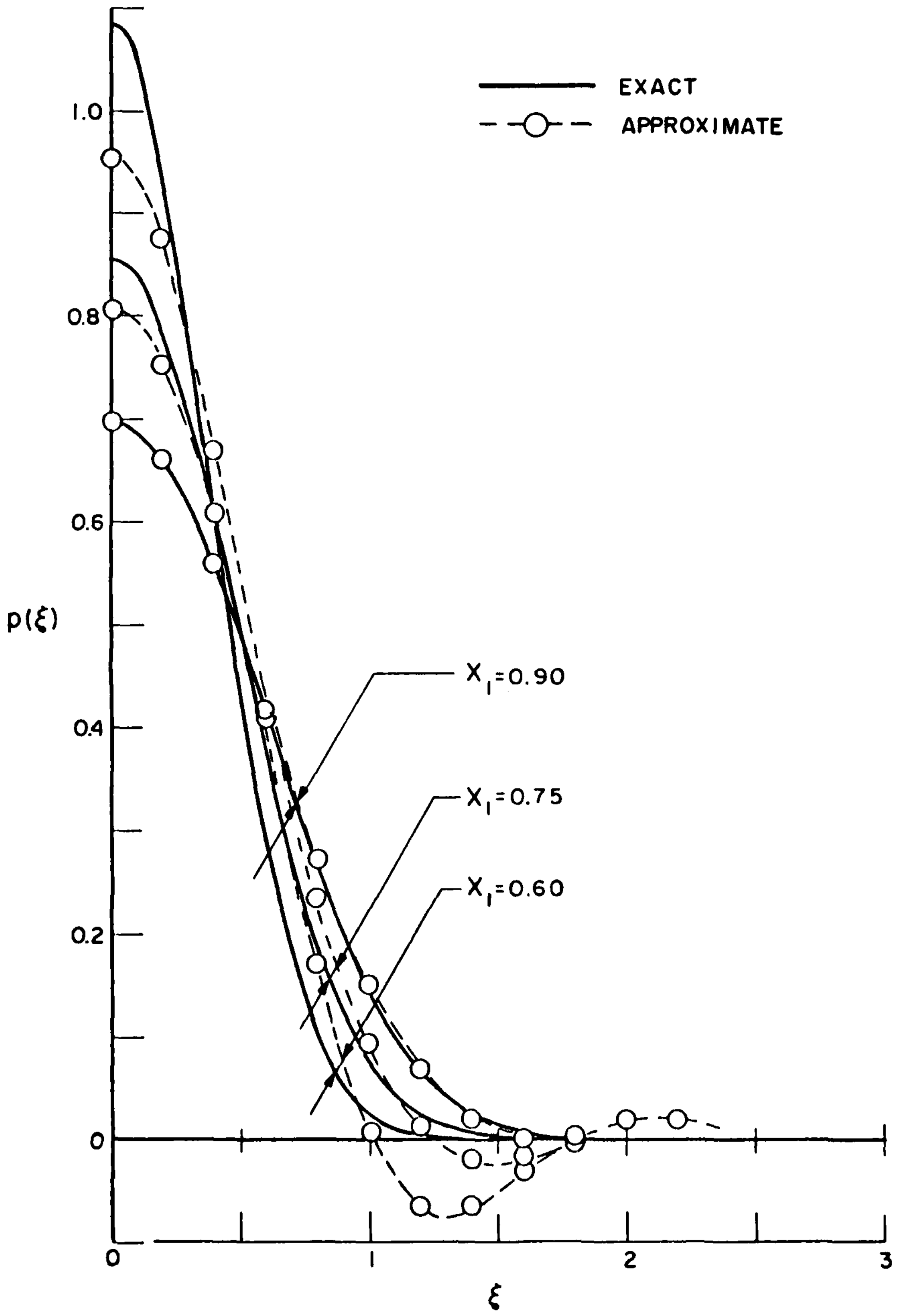

FIG. 3c. COMPARISON OF EXACT AND APPROXIMATE STATE DENSITIES

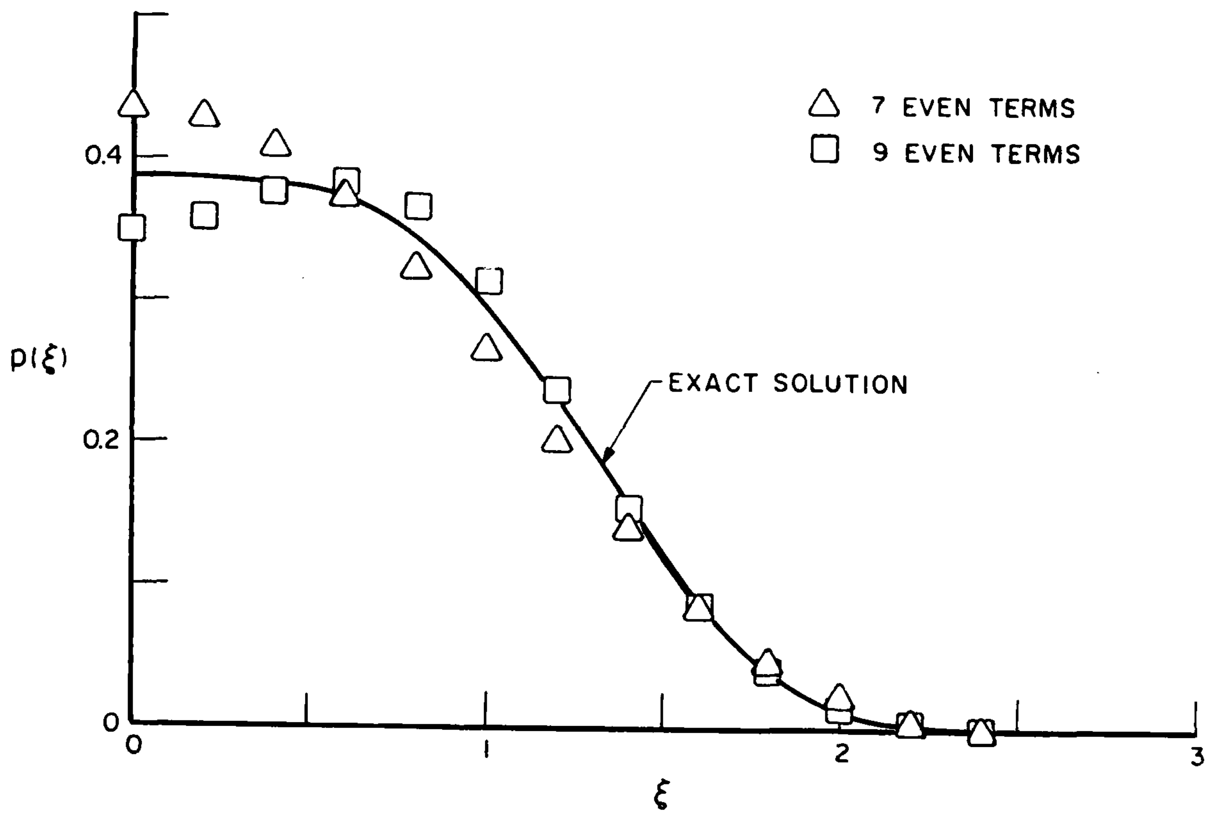

FIG. 4. EFFECT OF SERIES TRUNCATION ON APPROXIMATE DENSITY FOR $X_1 = 1.35$

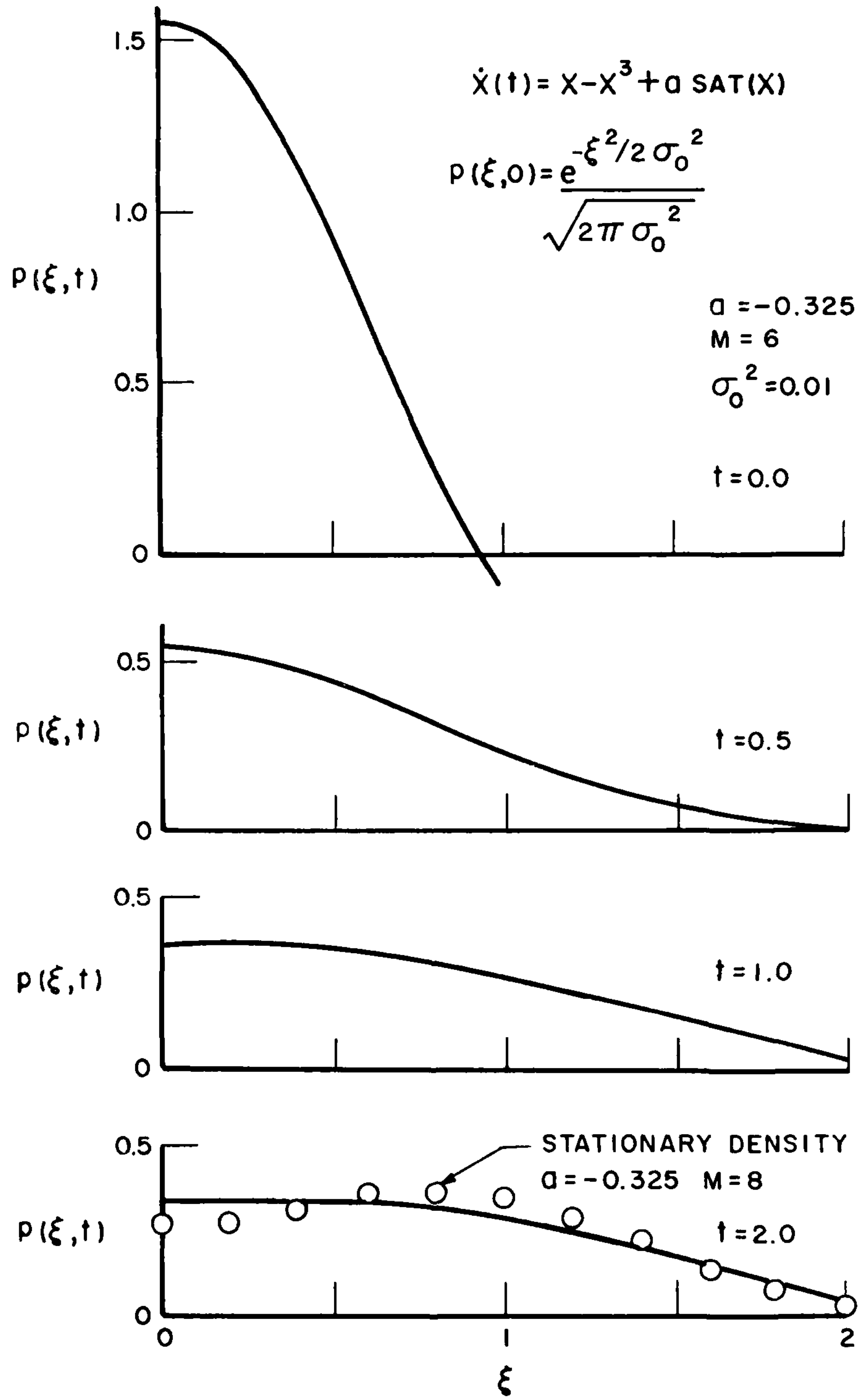

FIG. 5. TRANSIENT DENSITY WITH SYMMETRIC INITIAL CONDITION

APPENDIX A

HERMITE POLYNOMIAL — SUMMARY

The following summary of properties of the Hermite polynomial $He_n(x)$ of degree n and argument x, follows the more detailed and broader development of Abramowitz and Stegun.[A1] This set of functions satisfies the orthogonality condition

$$\int_{-\infty}^{\infty} e^{-x^2/2} He_n(x) He_m(x)\, dx = \sqrt{2\pi}\; n!\; \delta_{m,n} \tag{A-1}$$

where the order is a non-negative integer, the argument is a real variable and $\delta_{m,n}$ is the Kronecker delta function defined as

$$\delta_{m,n} = \begin{cases} 0 \; ; \; m \neq n \\ 1 \; ; \; m = n \end{cases} \quad . \tag{A-2}$$

The Hermite polynomial is further defined by the finite series expansion

$$He_n(x) = n! \sum_{m=0}^{[n/2]} \frac{(-1)^m \, x^{n-2m}}{m!\; 2^m (n-2m)!} \tag{A-3}$$

where the notation $[n/2]$ denotes the largest integer not greater than $n/2$. A brief list of coefficients for the Hermite polynomial is shown in Table A-1.

Using the orthogonality condition, Eq.(A-1), a function $f(x)$ for which the expansion exists, can be expressed as an infinite series in two ways, namely, either

$$f(x) = e^{-x^2/2} \sum_{n=0}^{\infty} f_n He_n(x) \quad ; \quad f_n = \frac{1}{\sqrt{2\pi}\; n!} \int_{-\infty}^{\infty} f(x) He_n(x)\, dx \tag{A-4}$$

or

$$f(x) = \sum_{n=0}^{\infty} f_n He_n(x) \quad ; \quad f_n = \frac{1}{\sqrt{2\pi}\; n!} \int_{-\infty}^{\infty} e^{-x^2/2} f(x) He_n(x)\, dx \quad . \tag{A-5}$$

The choice of which form to use depends mainly on the behavior of $f(x)$ as $|x| \to \infty$. If the integral defining f_n in Eq.(A-4) exists, then either form may be utilized but if this integral does not exist, then Eq.(A-5) must be used. The coefficients for the expansion of several functions are shown in

Tables A-2 and A-3, using Eqs.(A-4) and (A-5), respectively.

From Eq.(A-3), it is seen that

$$He_n(-x) = (-1)^n He_n(x) \tag{A-6}$$

that is, even order Hermite polynomials are symmetric functions of x while odd orders are antisymmetric. The Hermite polynomial of n^{th} order satisfies the ordinary differential equation

$$\frac{d^2}{dx^2} He_n(x) - x \frac{d}{dx} He_n(x) + nHe_n(x) = 0 \quad , \tag{A-7}$$

it satisfies the recurrence relation with respect to n given by

$$He_{n+1}(x) = xHe_n(x) - nHe_{n-1}(x) \quad , \tag{A-8}$$

its derivative with respect to x is given by

$$\frac{d}{dx} He_n(x) = nHe_{n-1}(x) \tag{A-9}$$

and its generating function is $e^{xz-z^2/2}$ so that

$$e^{xz-z^2/2} = \sum_{n=0}^{\infty} \frac{z^n}{n!} He_n(x) \tag{A-10}$$

and

$$He_n(x) = \frac{\partial^n}{\partial z^n} e^{xz-z^2/2}\Big|_{z=0} \quad . \tag{A-11}$$

Rodriques' Formula for the Hermite polynomial of n^{th} order becomes

$$He_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2} \quad . \tag{A-12}$$

Several definite integrals involving Hermite polynomials are shown in Table A-4. The notation in Table A-4 follows that of Abramowitz and Stegun.[A1]

Using the generating function in the form of Eq.(A-11), it can be shown that

$$\int_{-\infty}^{\infty} \frac{e^{-t^2/2}}{\sqrt{2\pi}} He_i(t) He_j(t) He_k(t)\,dt \equiv A(i,j,k)$$

$$= \frac{i!\ j!\ k!}{\left(\frac{i+j-k}{2}\right)!\ \left(\frac{i-j+k}{2}\right)!\ \left(\frac{-i+j+k}{2}\right)!}\ ;\ \text{if}\ \begin{cases} i+j+k \text{ is even and} \\ i \le j+k \text{ and} \\ j \le k+i \text{ and} \\ k \le i+j\ , \end{cases}$$

$$= 0\ ;\ \text{otherwise.} \qquad \text{(A-13)}$$

References

A1. Abramowitz, M. and Stegun, I., Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables, National Bureau of Standards, Applied Mathematics Series, 55, June 1964.

TABLE A-1

LIST OF COEFFICIENTS FOR $He_n(x)$ and x^n

$$He_n(x) = \sum_{m=0}^{[n/2]} (-1)^m h_m^{(n)} x^{n-2m}, \quad x^n = \sum_{m=0}^{[n/2]} h_m^{(n)} He_{n-2m}(x)$$

Table of $h_m^{(n)}$

n \ m=	0	1	2	3	4	5	6	7
0	1							
1	1							
2	1	1						
3	1	3						
4	1	6	3					
5	1	10	15					
6	1	15	45	15				
7	1	21	105	105				
8	1	28	210	420	105			
9	1	36	378	1260	945			
10	1	45	630	3150	4725	945		
11	1	55	990	6930	17325	10395		
12	1	66	1485	13860	51975	62370	10395	
13	1	78	2145	25740	135135	270270	135135	
14	1	91	3003	45045	315315	756756	945945	135135
15	1	105	4095	75075	675675	1891890	4729725	2027025
16	1	120	5460	120120	1351350	5045040	18918900	16216200
17	1	136	7140	185640	2552550	12252240	64324260	91891800
18	1	153	9180	278460	4594590	27567540	192972780	413513100
19	1	171	11628	406980	7936110	58198140	523783260	1571349780
20	1	190	14535	581400	13226850	116396280	1309458150	5237832600

n \ m=	8	9	10
16	2027025		
17	34459425		
18	310134825	34459425	
19	1473140419	654729075	
20	7365702095	6547290750	654729075

$$h_m^{(n)} = \frac{n!}{m!\,2^m (n-2m)!}$$

TABLE A-2

HERMITE SERIES EXPANSIONS

$$f(x) = e^{-x^2/2} \sum_{n=0}^{\infty} f_n He_n(x) \quad ; \quad \sqrt{2\pi}\, n!f_n = \int_{-\infty}^{\infty} f(x) He_n(x)\, dx$$

$f(x)$	f_n
$1 \; ; \; \lvert x \rvert < 1$ $0 \; ; \; \lvert x \rvert > 1$	$0 \quad ; \; n$ odd $\frac{2}{n+1} He_{n+1}(1) \quad ; \; n$ even
$\frac{e^{-(x-\mu)^2/2\sigma^2}}{\sqrt{2\pi}\,\sigma}$	$n!\mu^n \sum_{m=0}^{[n/2]} \frac{(-1)^m}{m!2^m \mu^{2m}} \sum_{k=0}^{[\frac{n-2m}{2}]} \frac{M_{2k}(\sigma/\mu)^{2k}}{(2k)!(n-2m-2k)!}$

NOTES:

$$\text{sat}(x) = \begin{cases} -1; & x < -1 \\ x; & \lvert x \rvert < 1 \\ +1; & x > 1 \end{cases} \quad , \quad M_{2k} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \xi^{2k} e^{-\xi^2/2} d\xi = 1.3 \ldots (2k+1)$$

TABLE A-3

HERMITE SERIES EXPANSIONS

$$f(x) = \sum_{n=0}^{\infty} f_n He_n(x) \quad ; \quad \sqrt{2\pi}\, n!\, f_n = \int_{-\infty}^{\infty} e^{-x^2/2} f(x) He_n(x)\, dx$$

$f(x)$	f_n	
$\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right)$	0	; n even
	$\frac{2}{\sqrt{3\pi}} \frac{(-1/3)^{(n-1)/2}}{n\left(\frac{n-1}{2}\right)!}$	; n odd
$\mathrm{sat}\left(\frac{x}{b}\right)$	0	; n even
	$\frac{1}{b}\, \mathrm{erf}\left(\frac{b}{\sqrt{2}}\right)$	; n=1
	$\frac{-2e^{-b^2/2} He_{n-2}(b)}{\sqrt{2\pi}\, b\, n!}$	; n=3,5, ...

TABLE A-4

DEFINITE INTEGRALS

$$\int_0^x He_n(t)\,dt = \frac{1}{(n+1)}\left[He_{n+1}(x) - He_{n+1}(0)\right]$$

$$\int_0^x e^{-t^2/2}He_n(t)\,dt = He_{n-1}(0) - e^{-x^2/2}He_{n-1}(x)$$

$$\int_{-\infty}^{\infty} e^{-t^2/2}He_{2n}(xt)\,dt = \sqrt{2\pi}\,\frac{(2n)!}{n!}\left(\frac{x^2-1}{2}\right)^n$$

$$\int_{-\infty}^{\infty} e^{-t^2/2}tHe_{2n+1}(xt)\,dt = \sqrt{2\pi}\,\frac{(2n+1)!}{n!}\,x\left(\frac{x^2-1}{2}\right)^n$$

$$\int_{-\infty}^{\infty} e^{-t^2/2}t^nHe_n(xt)\,dt = \sqrt{2\pi}\;n!\;P_n(x)$$

$$\int_{-\infty}^{\infty} e^{-t^2/2}[He_n(t)]^2\cos(xt)\,dt = \sqrt{\frac{\pi}{2}}\;n!\;e^{-x^2/2}\;L_n(x^2)$$

APPENDIX B

THE NORMAL PROBABILITY DENSITY FUNCTION

One of the most important probability density functions is the normal or Gaussian density, with mean μ and variance σ^2, which is defined as

$$p(\xi) = \frac{e^{-(\xi-\mu)^2/2\sigma^2}}{\sqrt{2\pi}\,\sigma} \tag{B-1}$$

The moments of the normal density are given by

$$\mu = \int_{-\infty}^{\infty} \xi \, p(\xi)\, d\xi$$

$$\sigma^2 = \int_{-\infty}^{\infty} (\xi-\mu)^2 p(\xi)\, d\xi$$

$$M_k = \int_{-\infty}^{\infty} (\xi-\mu)^k p(\xi)\, d\xi = \begin{cases} 0 & ; \text{ k odd} \\ \sigma^k \dfrac{k!}{\sqrt{2}^k \left(\frac{k}{2}\right)!} & ; \text{ k even} \end{cases} \tag{B-2}$$

$$\mu_k = \int_{-\infty}^{\infty} \xi^k \, p(\xi)\, d\xi$$

where it is seen that

$$M_o = \mu_o = 1 \;,\; \mu_1 = \mu \;,\; M_2 = \sigma^2 \;,\; M_{k+2} = (k+1)\sigma^2 M_k \quad . \tag{B-3}$$

Furthermore, by introducing the binomial expansion

$$(\xi-\mu)^k = \sum_{\ell=0}^{k} \binom{k}{\ell} \xi^{\ell} (-\mu)^{k-\ell} \tag{B-4}$$

the moments M_k and μ_k are related by

$$M_k = \sum_{\ell=0}^{k} \binom{k}{\ell} \mu_{\ell} (-\mu)^{k-\ell} \tag{B-5}$$

and by inversion

$$\mu_k = \sum_{\ell=0}^{k} \binom{k}{\ell} M_{\ell} \mu^{k-\ell} \quad . \tag{B-6}$$

Finally, the standard normal density is obtained by setting $\mu=0$ and $\sigma=1$ in Eq.(B-1), leading to

$$p(\xi) = \frac{e^{-\xi^2/2}}{\sqrt{2\pi}} \tag{B-7}$$

which is seen to be proportional to the weighting function for the Hermite polynomial shown in Eq.(A-1).

The Hermite series expansion of the normal density is given by

$$p(\xi) = e^{-\xi^2/2} \sum_{n=0}^{\infty} f_n He_n(\xi) \quad . \tag{B-8}$$

Multiplying by $He_m(\xi)$ and integrating leads to

$$\int_{-\infty}^{\infty} p(\xi) He_m(\xi)\, d\xi = \sum_n f_n \int_{-\infty}^{\infty} e^{-\xi^2/2} He_n(\xi) He_m(\xi)\, d\xi$$

so that the orthogonality condition in Eq.(A-1) gives

$$f_m = \frac{1}{\sqrt{2\pi}\, m!} \int_{-\infty}^{\infty} p(\xi) He_m(\xi)\, d\xi \quad .$$

Introducing Eqs.(B-1) and (A-3) gives

$$f_m = \frac{1}{2\pi\sigma} \sum_{n=0}^{[m/2]} \frac{(-1)^n}{n!2^n(m-2n)!} \int_{-\infty}^{\infty} \xi^{m-2n}\, e^{-(\xi-\mu)^2/2\sigma^2} d\xi$$

where the integral is seen to be the $m-2n^{th}$ moment. Thus,

$$f_m = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{[m/2]} \frac{(-1)^n \mu_{m-2n}}{n!2^n(m-2n)!} \tag{B-9}$$

which defines the Hermite series coefficients in terms of the moments of the normal density function.

Multiplying both sides of Eq.(B-8) by $He_o(\xi)=1$ and integrating gives

$$\int_{-\infty}^{\infty} p(\xi)\, d\xi = \sum_{n=0}^{\infty} f_n \int_{-\infty}^{\infty} e^{-\xi^2/2} He_n(\xi) He_o(\xi)\, d\xi \quad .$$

But the area under any density function is unity and the orthogonality condition brings this result to the form

$$f_o = 1/\sqrt{2\pi} \tag{B-10}$$

which agrees with Eq.(B-9) and demonstrates that the only term in the Hermite series expansion which contributes to area is the zeroth-order term.

Finally, combining Eqs.(B-2), (B-8) and (B-9) leads to the final relationships between the moments and the expansion coefficients:

$$\mu = \sqrt{2\pi}\, f_1$$

$$\mu_m = \sqrt{2\pi}\, m! \sum_{\ell=0}^{[m/2]} f_{m-2\ell}/2^{\ell}\ell! \tag{B-11}$$

The central moments can then be formed from Eq.(B-5) or from the recursion relationship in Eqs.(B-3). A number of explicit expressions for expansion coefficients and moments are given in Table B1.

After several pages of algebraic manipulation, it can be shown that the Hermite expansion coefficients can be written in terms of the mean μ and variance σ^2, in the form

$$f_{2n} = \frac{1}{\sqrt{2\pi}} \sum_{\ell=0}^{n} \frac{(2n)!}{(2\ell)!(n-\ell)!} \mu^{2\ell} \left(\frac{\sigma^2-1}{2}\right)^{n-\ell} \quad ; \; n=0,1,\ldots$$

$$f_{2n+1} = \frac{\mu}{\sqrt{2\pi}} \sum_{\ell=0}^{n} \frac{(2n)!}{(2\ell+1)!(n-\ell)!} \mu^{2\ell} \left(\frac{\sigma^2-1}{2}\right)^{n-\ell} \quad ; \; n=0,1,\ldots \tag{B-12}$$

With these results it is then found that for the special case $\mu=0$, the expansion of the normal density becomes

$$p(\xi) = \frac{e^{-\xi^2/2\sigma^2}}{\sqrt{2\pi}\,\sigma} = \frac{e^{-\xi^2/2}}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\sigma^2-1}{2}\right)^n He_{2n}(\xi) \tag{B-13}$$

which is seen to be symmetric in ξ as required. Furthermore, for the special case $\sigma=1$, Eq.(B-12) gives

$$p(\xi) = \frac{e^{-(\xi-\mu)^2/2}}{\sqrt{2\pi}} = \frac{e^{-\xi^2/2}}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{\mu^n}{n!} He_n(\xi) \tag{B-14}$$

while for the case when $\mu=0$ and $\sigma=1$, the expansion reduces to

$$p(\xi) = \frac{e^{-\xi^2/2}}{\sqrt{2\pi}} = \frac{e^{-\xi^2/2}}{\sqrt{2\pi}} He_o(\xi) \quad . \qquad \text{(B-15)}$$

Equations (B-13) and (B-14) clearly demonstrate that the number of terms required in the expansion to obtain a desired accuracy increases as σ^2 deviates further from unity or as μ deviates further from 0 .

The general relationships between the probability density and distribution functions are

$$p(\xi) = \frac{d}{d\xi} P(\xi) \quad , \qquad P(\xi) = \int_{-\infty}^{\xi} p(\xi')d\xi' \quad . \qquad \text{(B-16)}$$

Thus, the normal distribution function for a random variable with mean μ and variance σ^2 is given by

$$P(\xi) = \int_{-\infty}^{\xi} \frac{e^{-(\xi'-\mu)^2/2\sigma^2}}{\sqrt{2\pi}\,\sigma} d\xi'$$

$$= \frac{1}{2}\left[1 + \text{erf}\left(\frac{\xi-\mu}{\sqrt{2\pi}\,\sigma}\right)\right] \qquad \text{(B-17)}$$

where

$$\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_{o}^{z} e^{-t^2} dt$$

following Reference A1. Integrating the expansion of the density function given in Eq.(B-8), term by term yields

$$P(\xi) = \frac{1}{2\sqrt{2\pi}}\left[1 + \text{erf}\left(\frac{\xi}{\sqrt{2\pi}}\right)\right] - \frac{e^{-\xi^2/2}}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \frac{f_n}{n!} He_{n-1}(\xi) \quad . \qquad \text{(B-18)}$$

TABLE B1

SUMMARY OF EXPLICIT EQUATIONS FOR COEFFICIENTS AND MOMENT FOR THE NORMAL PROBABILITY DENSITY FUNCTION

μ = Mean σ^2 = Variance

$$f_0 = 1/\sqrt{2\pi}$$

$$f_1 = \mu/\sqrt{2\pi} \qquad p(\xi) = e^{-\xi^2/2} \sum_{n=0}^{\infty} f_n He_n(\xi)$$

$$f_2 = (\mu^2 + \sigma^2 - 1)/2\sqrt{2\pi}$$

$$f_3 = \mu[\mu^2 + 3(\sigma^2 - 1)]/6\sqrt{2\pi}$$

$$f_4 = [\mu^4 + 6\mu^2(\sigma^2 - 1) + 3(\sigma^2 - 1)^2]/24\sqrt{2\pi}$$

$$f_5 = \mu[\mu^4 + 10\mu^2(\sigma^2 - 1) + 15(\sigma^2 - 1)^2]/120\sqrt{2\pi}$$

$$\mu_0 = \sqrt{2\pi}\, f_0$$

$$\mu_1 = \sqrt{2\pi}\, f_1$$

$$\mu_2 = \sqrt{2\pi}\,(2f_2 + f_0) \qquad \mu_m = \frac{m!}{\sqrt{2\pi}} \sum_{\ell=0}^{[m/2]} f_{m-2\ell}/2^{\ell}\ell!$$

$$\mu_3 = \sqrt{2\pi}\,(6f_3 + 3f_1)$$

$$\mu_4 = \sqrt{2\pi}\,(24f_4 + 12f_2 + 3f_0)$$

$$\mu_5 = \sqrt{2\pi}\,(120f_5 + 60f_3 + 15f_1)$$

$$M_0 = 1$$

$$M_1 = 0$$

$$M_2 = \sqrt{2\pi}\,(2f_2 - f_1 + f_0)$$

$$M_3 = \sqrt{2\pi}\,(6f_3 - 6f_1f_2 + 2f_1^3)$$

$$M_4 = \sqrt{2\pi}\left[24f_4 - 24f_1f_3 + 12f_2(f_1^2 + f_0) - 3f_1^4 - 6f_1^2 + 3f_0\right]$$

$$M_5 = \sqrt{2\pi}\left[120f_5 - 120f_1f_4 + 60f_3(f_1^2 + f_0) - 20f_2f_1(f_1^2 + 3f_0) + 20f_1^3\right]$$