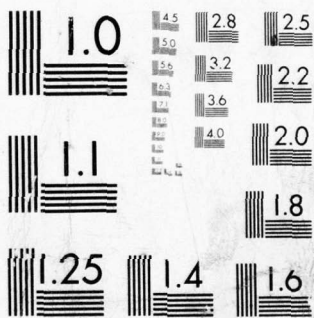


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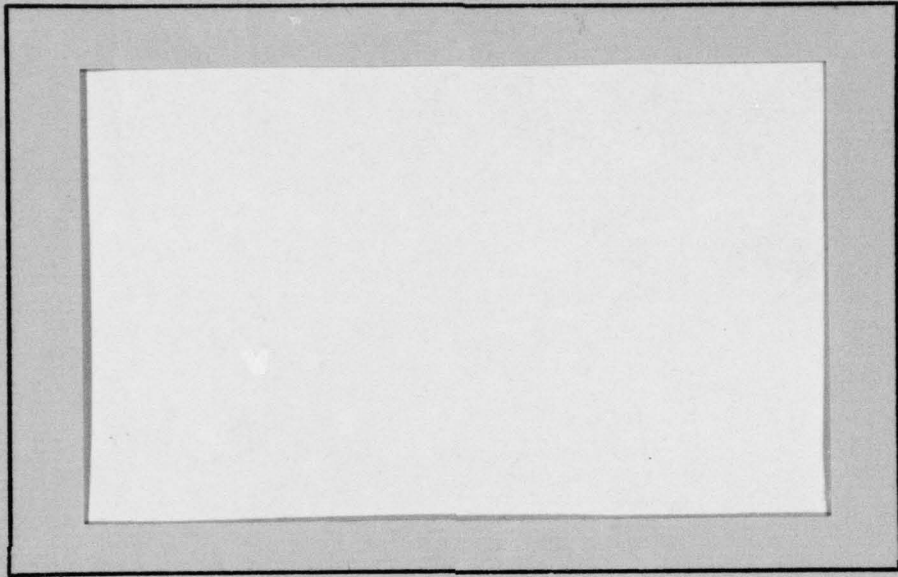
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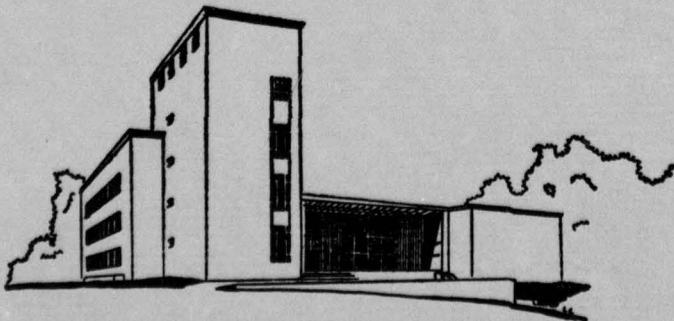
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9 Management Science Research Report No. 458

6 THE IMPLICIT COMPLEMENTARITY PROBLEM:  
PART I.

by

10 Jong-Shi/Pang

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THE IMPLICIT COMPLEMENTARITY PROBLEM: PART I

Jong-Shi Pang

ABSTRACT. Given an  $n$  by  $n$  matrix  $A$ ,  $n$ -vector  $b$  and a mapping  $m$  from  $\mathbb{R}^n$  into  $\mathbb{R}^n$ , the implicit complementarity problem is to find a vector  $x$  in  $\mathbb{R}^n$  so that

$$Ax + b \geq 0, \quad x \geq m(x) \quad \text{and} \quad (Ax + b)^T (x - m(x)) = 0.$$

This is the first of two papers in which we study this complementarity problem via an implicitly defined mapping  $F$  which depends on a given splitting of the matrix  $A$ . In the present paper, we derive sufficient conditions for the problem to have a unique solution for each vector  $b$  and study the problem in connection with a least-element theory.

Key Words. Implicit Complementarity      Homeomorphism  
Existence and Uniqueness                  Least-Element

## 1. INTRODUCTION

Given an  $n$  by  $n$  matrix  $A$ ,  $n$ -vector  $b$  and a mapping  $m$  from  $R^n$  into  $R^n$ , the implicit complementarity problem (ICP), denoted by the triple  $(A, b, m)$  is to find a vector  $x$  such that the conditions below are satisfied:

$$(1) \quad Ax + b \geq 0, \quad x \geq m(x) \quad \text{and} \quad (Ax + b)^T(x - m(x)) = 0.$$

If  $m$  is the constant mapping  $m(x) \equiv c$ , then the problem (1) becomes

$$Ax + b \geq 0, \quad x \geq c \quad \text{and} \quad (Ax + b)^T(x - c) = 0$$

which clearly is equivalent to the linear complementarity problem (LCP)

$$Ay + (b + Ac) \geq 0, \quad y \geq 0 \quad \text{and} \quad y^T[Ay + (b + Ac)] = 0$$

under the translation  $y = x - c$ . If the matrix  $A$  is nonsingular, then the ICP(1) is equivalent under the identification  $y = Ax + b$ , to the nonlinear complementarity problem (NLCP)

$$(2) \quad y \geq 0, \quad g(y) \geq 0 \quad \text{and} \quad y^T g(y) = 0$$

with  $g(y) = -A^{-1}b + A^{-1}y - m(-A^{-1}b + A^{-1}y)$ . Conversely, it is obvious that an arbitrary NLCP(2) can be written in the form (1) with  $A = I$ ,  $b = 0$  and  $m(x) = x - g(x)$ .

More generally, if the mapping  $m$  is piecewise linear, i.e. if

$$m(x) = \max_{1 \leq i \leq k} [A^i x - b^i]$$

where each  $A^i$  and each  $b^i$  are  $n$  by  $n$  matrix and  $n$ -vector respectively, then

the ICP(1) becomes the generalized linear complementarity problem (see Cottle and Dantzig [9] and Mangasarian [21])

$$\begin{aligned} Ax + b \geq 0, \quad (I - A^i) x + b^i \geq 0 \quad \text{for each } i = 1, \dots, k \\ (Ax + b)_j \prod_{i=1}^k [(I - A^i) x + b^i]_j = 0 \quad \text{for each } j = 1, \dots, n \end{aligned}$$

which obviously is equivalent to the problem of finding a solution to the system of piecewise linear equations (see Eaves [14] e.g.)

$$\min \{Ax + b, \min_{1 \leq i \leq k} [(I - A^i) x + b^i]\} = 0.$$

The ICP(1) has been studied in Dolcetta [13] as a discretized version of a continuous variational problem. There, a certain implicit (point-to-set) mapping was defined and used to analyze the problem.

The present paper is the first of a two-part research on the ICP. Here, we study the (global) existence and uniqueness of solution to the ICP and investigate the connection of the ICP with a least-element theory. In the second part of this research (which is presented in an accompanying paper [28]), we establish a convergence theory for a certain iterative algorithm to solve the ICP and show how this theory unifies and extends many existing convergence results for the successive overrelaxation methods to solve a LCP. The fundamental device employed throughout is a certain implicitly defined mapping  $F$  which is constructed from a splitting of the matrix  $A$ . The construction of  $F$  is largely based on ideas extending those used in Dolcetta's work [13] and Aganagic's recent doctoral dissertation [1].

The organization of the remainder of the paper is as follows. Section 2 contains two parts. In the first part, we shall establish some preliminary

results about the ICP. In particular we shall derive a basic relationship between the ICP and a certain variational problem. In the second part, we shall introduce the mapping  $F$ , lay down a fundamental relationship between  $F$  and solution to the ICP and review some basic facts which are useful in the subsequent study. Section 3 is concerned with the existence and uniqueness of solution to ICP. Our main goal is to establish some sufficient conditions (on the matrix  $A$  and the mapping  $m$ ) under which the  $ICP(A,b,m)$  will have a unique solution for all vectors  $b$ . The conditions derived are closely related to those obtained in Megiddo and Kojima [22] for the NLCP. The approach taken is based on ideas used in Aganagic [1], Eaves [14] and Megiddo and Kojima [22]. In Section 4, we shall study the ICP in connection with a least-element theory. In particular, we present some conditions under which the ICP will have a solution characterized as the least-vector of the feasible set, and establish the monotone convergence of a certain iterative scheme (defined by the mapping  $F$ ) to solve the ICP. Some results of this last section have previously appeared in the author's dissertation [25].

## 2. PRELIMINARY DISCUSSION

2.1. Some Basic Facts. It is well-known that there is a fundamental connection between the NLCP(2) and the variational inequality problem. The latter is to find, given a set  $K$  in  $R^n$  and a mapping  $g$  from  $R^n$  into itself, a vector  $u \in K$  such that

$$(x - u)^T g(u) \geq 0 \quad \text{for all } x \in K.$$

See Cottle [7] and Karamardian [16]. In the proposition below, we establish an extension of this connection for the ICP.

Proposition 2.1. A vector  $u$  solves the ICP(1) if and only if it satisfies the conditions

$$(3) \quad u \in K(u) \quad \text{and} \quad (x - u)^T g(u) \geq 0 \quad \text{for all } x \in K(u)$$

where  $g(x)$  is the affine mapping

$$g(x) = Ax + b$$

and  $K(u)$  is the set

$$K(u) = \{x \in R^n : x \geq m(u)\}.$$

The problem of finding a vector  $u$  satisfying (3) is known as a quasi-variational inequality problem (QVI). This terminology was due to A. Bensoussan and J. L. Lions. In fact in a series of papers [2, 3, 4, 5] these authors have defined both the QVI and the ICP in an infinite-dimensional-space setting and showed how various impulsive control and optimal stopping problems can be formulated as (infinite-

dimensional) QVI's and ICP's. Here we shall not go into further discussion of this matter, but simply refer the interested readers to the cited references for more details. We would like to point out however, that these impulsional control and optimal stopping problems in their discretized versions, have provided another source of application of the ICP defined in (1). See Dolcetta [13].

Proof of Proposition 2.1. Suppose that  $u$  satisfies (3). Then  $u \geq m(u)$  and  $(m(u) - u)^T g(u) \geq 0$  as  $m(u) \in K(u)$ . Moreover,  $2u - m(u) \in K(u)$  so that

$$0 \leq (2u - m(u) - u)^T g(u) = (u - m(u))^T g(u).$$

Hence  $(u - m(u))^T g(u) = 0$ . Finally, if  $x \in \mathbb{R}_+^n$ , then  $x + u \in K(u)$ . Thus,

$$0 \leq (x + u - u)^T g(u) = x^T g(u).$$

Consequently,  $g(u) \geq 0$ . This shows that  $u$  solves the ICP(1).

Conversely, suppose that  $u$  solves the ICP(1). Clearly  $u \in K(u)$ . If  $x \in K(u)$ , then  $x \geq m(u)$ . Hence, we have

$$\begin{aligned} (x - u)^T g(u) &= (x - m(u))^T g(u) + (m(u) - u)^T g(u) \\ &= (x - m(u))^T g(u) \geq 0. \end{aligned}$$

In other words,  $u$  satisfies (3).

Q.E.D.

An immediate consequence of the proposition is that one may formulate the ICP as a fixed point problem. More specifically, we have Corollary 2.2. A vector  $u$  solves the ICP(1) if and only if  $u \in \bar{K}(u)$  where

$$\bar{K}(u) = \{v \in K(u) : v^T (Au + b) = \min_{x \in K(u)} x^T (Au + b)\}.$$

It is a known fact that the NLCP(2) has at most one solution if  $g$  is a P-function, i.e., if for every vectors  $y^1 \neq y^2$ , it holds that

$$\max_{1 \leq i \leq n} (y^1 - y^2)_i (g(y^1) - g(y^2))_i > 0.$$

See Moré [23] e.g. Specializing this to the ICP(1), we obtain

Proposition 2.3. Suppose that the mapping  $m$  and the matrix  $A$  satisfy the condition: for every  $x^1 \neq x^2$ ,

$$(4) \quad \max_{1 \leq i \leq n} [(x^1 - m(x^1))_i - (x^2 - m(x^2))_i] (A(x^1 - x^2))_i > 0.$$

Then the ICP(1) has at most one solution for every vector  $b$ .

Proof. In fact, the given condition (4) implies that the matrix  $A$  is nonsingular. Hence, as pointed out in the Introduction, the ICP(1) is then equivalent to the NLCP(2) where the mapping  $g$  is given by

$$g(y) = A^{-1}(y - b) - m(A^{-1}(y - b)).$$

It is easy to deduce that condition (4) is equivalent to the fact that  $g$  is a P-function. Thus the desired conclusion follows readily. Q.E.D.

Note that the condition (4) is not sufficient to imply the existence of solution to the ICP.

2.2. Definition of F. A splitting of the matrix  $A$  is a representation

$$(5) \quad A = B - C.$$

A P-splitting (Z-splitting) is one where  $B$  is a P-matrix (Z-matrix).

Recall that a P-matrix (Z-matrix) is a real square matrix whose

principal minors are all positive (whose off-diagonal entries are all non-positive). A K-splitting is one where B is a K-matrix (i.e. both a P- and Z-matrix). The trivial splitting is where B is the identity matrix. Note that the trivial splitting is a K-splitting. We shall denote the splitting (5) by the pair (B,C).

Given a P-splitting (B,C) of the matrix A, we define a mapping F from  $R^n$  into  $R^n$  as follows. For each vector  $u \in R^n$ , F(u) is the unique solution to the LCP

$$Bx + (b - Cu) \geq 0, \quad x \geq m(u) \quad \text{and} \quad (x - m(u))^T (Bx + (b - Cu)) = 0.$$

Since B is a P-matrix, F(u) is well-defined (see Samelson et al [30]). In general, the mapping F depends on the splitting and possesses no explicit representation (except in certain special cases). However, each F(u) can be computed by a number of efficient algorithms, see Cottle [8]. If (B,C) is the trivial splitting, then

$$F(u) = \min(m(u), (I - A)u - b).$$

We point out that if m is the zero mapping, then the latter F is precisely the mapping used in Aganagic [1] to study the LCP. If (B,C) is a K-splitting, then F(u) has the characterization below. Moreover, it can be computed by a special-purpose algorithm due to Chandrasekaran [6].

Theorem 2.4. (Cottle and Veinott [12]). Let (B,C) be a K-splitting. Then F(u) is characterized as the least-element of the polyhedral set

$$X(u) = \{x \in R^n : Bx + (b - Cu) \geq 0, \quad x \geq m(u)\}.$$

Moreover, it is the unique solution to the linear program

$$\min e^T x \quad \text{subject to} \quad x \in X(u)$$

where e is any positive vector.

Recall that the least-element  $\bar{x}$  of a set  $S$  in  $R^n$  is one such that  $\bar{x} \leq x$  for all  $x \in S$ . Another P-splitting that is of particular interest is the one where  $B$  is a lower or upper triangular matrix with positive diagonal entries. In fact, it is easy to verify that in this case, the components of  $F(u)$  are given inductively by : for each  $i$ ,

$$F(u)_i = \max \{m(u)_i, - [(b - Cu)_i - \sum_{j < i} B_{ij} F(u)_j] / B_{ii}\}$$

or

$$F(u)_i = \max \{m(u)_i, - [(b - Cu)_i - \sum_{j > i} B_{ij} F(u)_j] / B_{ii}\}$$

depending on whether  $F$  is lower or upper triangular.

If the mapping  $m$  is piecewise linear (as in the case of the generalized linear complementarity problem), then so is the mapping  $F$ . The proposition below asserts a fundamental relationship between solution(s) to the ICP(1) and the mapping  $F$ . Its proof is easy and is thus omitted.

Proposition 2.5. A vector  $u$  is a solution to the ICP(1) if and only if it is a fixed point of the mapping  $F$  for any P-splitting of the matrix  $A$ . In particular, if some  $F$  has a unique fixed point, then the ICP(1) has a unique solution.

We state an important property for a  $K$ -matrix in the theorem below. A proof can be found in the reference cited.

Theorem 2.6. (Fiedler and Pták [15]). If  $B$  is a  $K$ -matrix, then so is each of its principal submatrices. The inverse  $B^{-1}$  exists and is non-negative.

We close this section by noting that the (point-to-set) mapping used in Dolcetta [13] to study the ICP corresponds to the mapping  $F$  defined by the splitting  $(B,0)$  with  $B(=A)$  being a  $Z$ -matrix. More about this will be said in Section 4.

### 3. EXISTENCE AND UNIQUENESS

Associated with the mapping  $F$  defined by a  $P$ -splitting of the matrix  $A$ , we define the mapping

$$G(x) = x - F(x).$$

In view of Proposition 2.5, we have

Theorem 3.1. A sufficient condition for the ICP(1) to have a unique solution is that the mapping  $G$  is a bijection for some  $P$ -splitting of the matrix  $A$ .

Our goal in this section is to derive some sufficient conditions (on the function  $m$  and matrix  $A$ ) so that for each vector  $b$ , the mapping  $G$  is a global homeomorphism and is thus a bijection. This then implies that the ICP(1) will have a unique solution for each vector  $b$ .

This approach of using a global homeomorphic mapping to study the (global) existence and uniqueness of solution to a complementarity problem has previously been taken by Megiddo and Kojima [22] in the case of the NLCP(2). It is not clear to the author, however, whether the specific mapping  $g(y^+) + y^-$  where  $y^+$  and  $y^-$  are the positive and negative part of the vector  $y$  used in the reference can be obtained by our approach.

The rest of this section is divided into two subsections. In the next subsection, we shall first derive conditions under which the mapping  $G$  is a local homeomorphism. In Subsection 3.2 we shall study the global property of  $G$ .

3.1. Local Properties. Throughout the rest of this section, we let the mappings  $F$  and  $G$  be defined with respect to a fixed but arbitrary  $K$ -splitting  $(B,C)$  of the matrix  $A$ .

Proposition 3.2. If the mapping  $m$  is convex, then so is  $F$ .

Proof. For any two vectors  $u^1$  and  $u^2$  and scalar  $\lambda \in [0,1]$ , the vector  $\lambda F(u^1) + (1-\lambda)F(u^2)$  belongs to the set  $X(\bar{u})$  where  $\bar{u} = \lambda u^1 + (1-\lambda)u^2$ .

As  $F(\bar{u})$  is the least element of this latter set (Theorem 2.4), it follows that  $F(\bar{u}) \leq \lambda F(u^1) + (1-\lambda)F(u^2)$ , proving the convexity of

$F$ .

Q.E.D.

Recall that if  $f$  is a convex mapping from  $R^n$  into  $R^n$ , then an  $n$  by  $n$  matrix  $E$  is a subgradient of  $f$  at the point  $x$  if

$$f(y) \geq f(x) + E(y - x) \quad \text{for all } y \in R^n.$$

In particular, each row  $E_i$  of  $E$  is a subgradient of the  $i$ -th component function  $f_i$  at the point  $x$ . We denote by  $\partial f(x)$  the set of subgradients of  $f$  at  $x$ . If  $\alpha$  is a subset of  $\{1, \dots, n\}$ , then by  $\partial f_\alpha(x)$  we denote those rows of  $\partial f(x)$  indexed by  $\alpha$ . In particular, each  $\partial f_i(x)$  is a set of row vectors.

Returning to our mapping  $F$ , we define for each vector  $u$ , three partitioning subsets of  $\{1, \dots, n\}$ :

$$\begin{aligned} \alpha_1(u) &= \{i_1 : F(u)_{i_1} = m(u)_{i_1} \quad \text{and} \quad (BF(u) + (b - Cu))_{i_1} > 0\} \\ \alpha_2(u) &= \{i_2 : F(u)_{i_2} > m(u)_{i_2} \quad \text{and} \quad (BF(u) + (b - Cu))_{i_2} = 0\} \\ \alpha_3(u) &= \{i_3 : F(u)_{i_3} = m(u)_{i_3} \quad \text{and} \quad (BF(u) + (b - Cu))_{i_3} = 0\}. \end{aligned}$$

For each subset  $\gamma(u)$  of  $\alpha_3(u)$ , we let  $\beta(u)$  be its complement in  $\alpha_3(u)$

and let  $\bar{\gamma}(u) = \alpha_2(u) \cup \gamma(u)$ . If the point of reference  $u$  is obvious in the context, we shall omit it from these index sets. For each subset  $\gamma$  of  $\{1, \dots, n\}$ , we shall denote  $(B_{\gamma\gamma})^{-1}$  by  $\bar{B}_{\gamma\gamma}$ . By Theorem 2.6, each  $\bar{B}_{\gamma\gamma}$  is nonnegative.

Proposition 3.3. Let  $m$  be convex and let  $u \in \mathbb{R}^n$ . Then

$$(6a) \quad \partial F_{\alpha_1}(u) = \partial m_{\alpha_1}(u)$$

$$(6b) \quad \partial F_{\alpha_2}(u) \supseteq \left\{ - \sum_{\gamma \subseteq \alpha_3(u)} \Lambda_{\gamma} \bar{B}_{\alpha_2 \bar{\gamma}} [B_{\bar{\gamma}\alpha_1} \cup \beta \partial m_{\alpha_1} \cup \beta(u) - C_{\bar{\gamma}}] : \right. \\ \left. \Lambda_{\gamma} \in \mathbb{R}^{|\alpha_2| \times |\alpha_2|}, \Lambda_{\gamma} \geq 0 \text{ and } \sum_{\gamma \subseteq \alpha_2(u)} \Lambda_{\gamma} = I \right\}$$

(6c) For each  $i_3 \in \alpha_3(u)$ ,

$$\partial F_{i_3}(u) \supseteq \left\{ - \sum_{i_3 \in \gamma \subseteq \alpha_3(u)} \lambda_{\gamma} \bar{B}_{i_3 \bar{\gamma}} [B_{\bar{\gamma}\alpha_1} \cup \beta \partial m_{\alpha_1} \cup \beta(u) - C_{\bar{\gamma}}] \right. \\ \left. + (1 - \lambda_{\emptyset}) \partial m_{i_3}(u) : \lambda_{\gamma}, \lambda_{\emptyset} \in \mathbb{R}; \lambda_{\gamma}, \lambda_{\emptyset} \geq 0 \text{ and } \right. \\ \left. \sum_{i \in \gamma \subseteq \alpha_3(u)} \lambda_{\gamma} + \lambda_{\emptyset} = 1 \right\}$$

where in (6b) and (6c),  $C_{\bar{\gamma}}$  denotes the rows of the matrix  $C$  indexed by  $\bar{\gamma}$ .

To prove the proposition, we need the following lemma.

Lemma 3.4. The vector  $h$  is a subgradient of a convex function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  at the point  $x$  if and only if there is a neighborhood  $N_x$  of  $x$  such that for all  $y \in N_x$

$$(7) \quad f(y) \geq f(x) + h^T(y - x).$$

The lemma says that the vector  $h$  is a subgradient if and only if the subgradient inequality (7) is satisfied locally.

Proof of Lemma 3.4. Only the sufficiency part needs to be proved. Let  $z$  be an arbitrary point in  $R^n$ . Then there is a  $\lambda \in (0,1)$  such that the vector  $y = \lambda z + (1 - \lambda)x$  belongs to the neighborhood  $N_x$ . By the monotonicity property of a convex function (see Stoer and Witzgall [31, p. 135]) we have

$$f(z) - f(x) \geq (f(y) - f(x))/\lambda \geq h^T(y - x)/\lambda = h^T(z - x)$$

where the second inequality follows from the assumption. Q.E.D.

Proof of Proposition 3.3. Since both  $m$  and  $F$  are convex, they are continuous (see Stoer and Witzgall [31, p. 135]). Hence, there exists a neighborhood  $N_u$  of  $u$  such that for all  $v \in N_u$ , we have

$$[BF(v) + (b - Cv)]_{\alpha_1(u)} > 0 \quad \text{and} \quad (F(v) > m(v))_{\alpha_2(u)}.$$

By complementarity, it follows that for all  $v \in N_u$ ,

$$(F(v) = m(v))_{\alpha_1(u)}.$$

Consequently, (6a) follows from the last lemma. To prove (6b), observe that for each  $\gamma \subseteq \alpha_3(u)$ , we have

$$(F(u))_{\gamma}^- = -(B_{\gamma\gamma})^{-1} [B_{\gamma\alpha_1}^- \cup \beta \ F(u)_{\alpha_1} \cup \beta + (b - Cu)_{\gamma}^-]$$

and for all  $v$

$$[BF(v) + (b - Cv)]_{\gamma}^- \geq 0.$$

Thus, it follows that,

$$\begin{aligned} F(v)_{\alpha_2} &\geq -\bar{B}_{\alpha_2\gamma} [B_{\gamma\alpha_1}^- \cup \beta \ F(v)_{\alpha_1} \cup \beta + (b - Cv)_{\gamma}^-] \\ &\geq -\bar{B}_{\alpha_2\gamma} [B_{\gamma\alpha_1}^- \cup \beta \ m(v)_{\alpha_1} \cup \beta + (b - Cv)_{\gamma}^-] \\ &\geq -\bar{B}_{\alpha_2\gamma} [B_{\gamma\alpha_1}^- \cup \beta \ (m(u) + E(v - u))_{\alpha_1} \cup \beta + (b - Cv)_{\gamma}^-] \end{aligned}$$

where  $E_{\alpha_1} \cup \beta \in \partial^m_{\alpha_1} \cup \beta(u)$ . Note that we have used the nonnegativity of the matrix  $-\bar{B}_{\alpha_2 \bar{\gamma}} \bar{B}_{\bar{\gamma} \alpha_1} \cup \beta$  to get the last two inequalities. Hence, for each vector  $v$

$$F(v)_{\alpha_2} \geq F(u)_{\alpha_2} - \bar{B}_{\alpha_2 \bar{\gamma}} \bar{B}_{\bar{\gamma} \alpha_1} \cup \beta E_{\alpha_1} \cup \beta - C_{\bar{\gamma}}] (v - u).$$

Assertion (6b) now follows from the convexity of the subgradient set  $\partial F_{\alpha_2}(u)$ . The last assertion (6c) can be proved in a similar fashion.

Q.E.D.

Corollary 3.4. Let  $m$  be convex and  $u \in R^n$ . Then

$$(8a) \quad \partial G_{\alpha_1}(u) = I_{\alpha_1} - \partial^m_{\alpha_1}(u)$$

$$(8b) \quad \partial G_{\alpha_2}(u) \supseteq \{I_{\alpha_2} + \sum_{\gamma \subseteq \alpha_3(u)} \Lambda_{\gamma} \bar{B}_{\alpha_2 \bar{\gamma}} \bar{B}_{\bar{\gamma} \alpha_1} \cup \beta \partial^m_{\alpha_1} \cup \beta(u) - C_{\bar{\gamma}} :$$

$$\Lambda_{\gamma} \in R^{|\alpha_2| \times |\alpha_2|}, \Lambda_{\gamma} \geq 0 \quad \text{and} \quad \sum_{\gamma \subseteq \alpha_3(u)} \Lambda_{\gamma} = I\}$$

$$(8c) \quad \text{For each } i_3 \in \alpha_3(u)$$

$$\partial G_{i_3}(u) \supseteq \{I_{i_3} + \sum_{i \in \gamma \subseteq \alpha_3(u)} \lambda_{\gamma} \bar{B}_{i_3 \bar{\gamma}} \bar{B}_{\bar{\gamma} \alpha_1} \cup \beta \partial^m_{\alpha_1} \cup \beta(u) - C_{\bar{\gamma}}]$$

$$- (1 - \lambda_{\phi}) \partial^m_{i_3}(u) : \lambda_{\gamma}, \lambda_{\phi} \in R; \lambda_{\gamma}, \lambda_{\phi} \geq 0$$

$$\text{and } i \in \gamma \subseteq \alpha_3(u) \lambda_{\gamma} + \lambda_{\phi} = 1\}.$$

and

Here  $I_{\gamma}$  denotes the rows of the identity matrix indexed by  $\gamma$ .

A remark is in order. If  $m$  is convex, then the mapping  $G$  is concave. A subgradient of a concave function is a vector whose negative is a subgradient of the negative of the function (which is convex).

It has been shown in Eaves [14] and Aganagic [1] that if the mapping  $m$  is piecewise linear, then the mapping  $G$  defined with respect

to the trivial splitting of A is locally univalent at u provided that each subgradient matrix in  $\partial G(u)$  is nonsingular. The proposition below extends this result.

**Proposition 3.5.** Let m be convex and  $u \in \mathbb{R}^n$ . Suppose that

(9) For each  $E \in \partial m(u)$ , and each  $(\Lambda_Y, \Lambda'_Y) \in \mathbb{R}^{|\alpha_2| \times |\alpha_2|} \times \mathbb{R}^{|\alpha_3| \times |\alpha_3|}$  satisfying

$$(10) \quad 0 \leq \Lambda_Y, \sum_{Y \subseteq \alpha_3} \Lambda_Y = I \quad \text{and} \quad 0 \leq \Lambda'_Y, \sum_{Y \subseteq \alpha_3(u)} \Lambda'_Y \leq I,$$

the matrix below is nonsingular

$$(11) \quad \begin{pmatrix} I_{\alpha_1} - E_{\alpha_1} \\ I_{\alpha_2} + \sum_{Y \subseteq \alpha_3(u)} \Lambda_Y \bar{B}_{\alpha_2 Y} [B_{Y\alpha_1} \cup B E_{\alpha_1} \cup B - C_Y] \\ I_{\alpha_3} + \sum_{Y \subseteq \alpha_3(u)} \Lambda'_Y \bar{B}_{\alpha_3 Y} [B_{Y\alpha_1} \cup B E_{\alpha_1} \cup B - C_Y] - (I - \sum_{Y \subseteq \alpha_3(u)} \Lambda'_Y) E_{\alpha_3} \end{pmatrix}$$

Then the mapping G is locally univalent at u.

**Proof.** By [Thm. 24.7, 29], each  $\partial m_i(u)$  is a compact set. Thus so is  $\partial m(u) = \partial m_1(u) \times \dots \times \partial m_n(u)$ . Hence, there exists an  $\epsilon > 0$  such that if N is the unit ball in the space of n by n matrices (considered as the product space of n copies of  $\mathbb{R}^n$ ), then the matrix  $\tilde{E}$  is nonsingular for every  $\tilde{E} \in H + \epsilon N$  for some matrix H of the form (11).

Suppose that G is not locally univalent at u. Then there exist sequences  $\{x^i\}$  and  $\{y^i\}$  both converging to u such that for each i,  $G(x^i) = G(y^i)$  and  $x^i \neq y^i$ . With no loss of generality, we may assume that the two sequences are chosen so that for each i,

$$[BF(x^i) + (b - Cx^i)]_{\alpha_1(u)} > 0, [BF(y^i) + (b - Cy^i)]_{\alpha_1(u)} > 0$$

$$(F(x^i) > m(x^i))_{\alpha_2(u)}, \quad (F(y^i) > m(y^i))_{\alpha_2(u)}.$$

Let  $\delta > 0$ . By [Thm. 24.5, 29], there is an index  $j$  such that  $\partial m(x^j) \cup \partial m(y^j) \subseteq \partial m(u) + \delta N$ . Let  $E^1 \in \partial m(x^j)$  and  $E^2 \in \partial m(y^j)$ . Define four mutually disjoint subsets of  $\alpha_3(u)$  as follows:

$$\beta_1(u) = \{i_1 \in \alpha_3(u) : F_{i_1}(x^j) = m_{i_1}(x^j) \quad \text{and} \quad F_{i_1}(y^j) = m_{i_1}(y^j)\}$$

$$\beta_2(u) = \{i_2 \in \alpha_3(u) : F_{i_2}(x^j) = m_{i_2}(x^j) \quad \text{and} \quad F_{i_2}(y^j) > m_{i_2}(y^j)\}$$

$$\beta_3(u) = \{i_3 \in \alpha_3(u) : F_{i_3}(x^j) > m_{i_3}(x^j) \quad \text{and} \quad F_{i_3}(y^j) = m_{i_3}(y^j)\}$$

$$\beta_4(u) = \alpha_3(u) \setminus \bigcup_{i=1}^3 \beta_i(u).$$

Now for each  $i \in \beta_1(u) \cup \alpha_1(u)$ , we have

$$x_i^j - m_i(x^j) = G_i(x^j) = G_i(y^j) = y_i^j - m_i(y^j)$$

By the fact that  $(E_i^1, E_i^2) \in \partial m_i(x^j) \times \partial m_i(y^j)$ , it follows that

$$(I_i - E_i^1)(y^j - x^j) \geq 0 \quad \text{and} \quad (I_i - E_i^2)(x^j - y^j) \geq 0$$

where  $I_i$  is the  $i$ -th row of the identity matrix and where  $E_i^1$  and  $E_i^2$  are the  $i$ -th rows of the matrices  $E^1$  and  $E^2$  respectively. Hence, there exists a  $\lambda_i \in [0, 1]$  such that for  $\tilde{E}_i = (I_i - E_i) = \lambda_i(I_i - E_i^1) + (1 - \lambda_i)(I_i - E_i^2)$ , we have

$$(12) \quad \tilde{E}_i(x^j - y^j) = 0.$$

Note that  $E_i \in \partial m_i(u) + \delta N$ . Moreover, for  $i \in \beta_2(u)$ , we have

$$x_i^j - m_i(x^j) = G_i(x^j) = G_i(y^j) \leq y_i^j - m_i(y^j).$$

Thus,

$$(13) \quad (I_i - E_i^1) (y^j - x^j) \geq 0.$$

On the other hand, we have by complementarity,

$$\begin{aligned} [BF(y^j) + (b - Cy^j)]_i &= [B(y^j - G(y^j)) + (b - Cy^j)]_i = 0 \\ &\leq [BF(x^j) + (b - Cy^j)]_i = [B(x^j - G(x^j)) + (b - Cx^j)]_i \end{aligned}$$

Thus, it follows that

$$(C_i - B_i) (x^j - y^j) \leq 0.$$

Similarly, we can show that for each  $i \in \beta_3(u)$ ,

$$(I_i - E_i^2) (x^j - y^j) \geq 0 \quad \text{and} \quad (C_i - B_i) (y^j - x^j) \leq 0.$$

Moreover, for each  $i \in \beta_4(u) \cup \alpha_2(u)$ , we have

$$(C_i - B_i) (x^j - y^j) = 0.$$

Now, for an index  $i \in \beta_2(u)$ , let  $\gamma_i = \beta_2(u) \cup \beta_4(u)$ . Then

$$\begin{aligned} & \{I_i + \bar{B}_{i\gamma_i} [B_{\gamma_i\alpha_1} \cup \beta_1^E \alpha_1 \cup \beta_1 + B_{\gamma_i\beta_3} E_{\beta_3}^2 - C_{\gamma_i}]\} (x^j - y^j) \\ & \geq \{I_i + \bar{B}_{i\gamma_i} [B_{\gamma_i\alpha_1} \cup \beta_1^I \alpha_1 \cup \beta_1 + B_{\gamma_i\beta_3} I_{\beta_3} - B_{\gamma_i}]\} (x^j - y^j) \\ & = [I_i + \bar{B}_{i\gamma_i} (-B_{\gamma_i\gamma_i} I_{\gamma_i})] (x^j - y^j) = 0. \end{aligned}$$

Since  $(I_i - E_i^1) (x^j - y^j) \leq 0$  by (13), there exists a  $\lambda_i \in [0,1]$  such that

$$\text{for } \tilde{E}_i = I_i + \lambda_i \bar{B}_{i\gamma_i} [B_{\gamma_i\alpha_1} \cup \beta_1^E \alpha_1 \cup \beta_1 + B_{\gamma_i\beta_3} E_{\beta_3}^2 - C_{\gamma_i}] - (1 - \lambda_i) E_i^1,$$

(12) holds.

Similarly for  $i \in \beta_3(u)$  let  $\gamma_i = \beta_3(u) \cup \beta_4(u)$ . Then there exists a  $\lambda_i \in [0,1]$  such that for  $\tilde{E}_i = I_i + \lambda_i \bar{B}_i \bar{\gamma}_i [B_{\gamma_i}^- \alpha_1 \cup \beta_1 E_{\alpha_1} \cup \beta_1 + B_{\gamma_i}^- \beta_2 E_{\beta_2}^1 - C_{\gamma_i}^-]$   
 $- (1 - \lambda_i) E_i^2$ , (12) holds as well. Finally, for  $i \in \beta_4(u) \cup \alpha_2(u)$ ,  
 let  $\gamma_i = \beta_4(u) \cup \beta_2(u)$  and  $\delta_i = \beta_4(u) \cup \beta_3(u)$ . Then there exists a  $\lambda_i \in [0,1]$  such that for  $\tilde{E}_i = I_i + \lambda_i \bar{B}_i \bar{\gamma}_i [B_{\gamma_i}^- \alpha_1 \cup \beta_1 E_{\alpha_1} \cup \beta_1 + B_{\gamma_i}^- \beta_3 E_{\beta_3}^2 - C_{\gamma_i}^-]$   
 $+ (1 - \lambda_i) \bar{B}_i \bar{\delta}_i [B_{\delta_i}^- \alpha_1 \cup \beta_1 E_{\alpha_1} \cup \beta_1 + B_{\delta_i}^- \beta_2 E_{\beta_2}^1 - C_{\delta_i}^-]$ , (12) holds.

Let  $\tilde{E}$  be the matrix whose rows  $\tilde{E}_i$  are as defined above. By choosing  $\delta > 0$  small enough, we have  $\tilde{E} \in H + \epsilon N$  where  $H$  is a matrix of the form (11). Since  $\tilde{E}(x^j - y^j) = 0$ , we obtain a contradiction to the nonsingularity of  $\tilde{E}$ .

Q.E.D.

Remark. By Corollary 3.4, each matrix given in (11) is a subgradient in  $\partial G(u)$ .

A mapping  $f$  from  $R^n$  into itself is said to be inverse locally Lipschitz continuous at the point  $x$  if there is a neighborhood  $N_x$  of  $x$  and a constant scalar  $k$  such that

$$\|x - y\| \leq k \|f(x) - f(y)\| \quad \text{for all } y \in N_x.$$

It was shown in Aganagic [1] that in the case of the LCP, the mapping  $G$  defined with respect to the trivial splitting of the matrix  $A$  is always inverse locally Lipschitz continuous provided that certain

regularity conditions are satisfied. In what follows, we extend this result. We define for each subset  $\gamma$  of  $\{1, \dots, n\}$ , a mapping  $\mathcal{L}^\gamma$  from  $\mathbb{R}^n$  into itself by

$$(\mathcal{L}^\gamma(x))_\gamma = A_\gamma x \quad \text{and} \quad (\mathcal{L}^\gamma(x))_\beta = x_\beta - m_\beta(x)$$

with  $\beta$  the complement of  $\gamma$ .

**Proposition 3.6.** Suppose that for each subset  $\gamma$  of  $\{1, \dots, n\}$ , the mapping  $\mathcal{L}^\gamma$  defined above is inverse locally Lipschitz continuous at the point  $u$ . Then so is  $G$ .

**Proof.** Let  $\bar{N}_u$  be a neighborhood of the point  $u$  so that for each  $y \in \bar{N}_u$ , the following holds

$$[BF(y) + (b - Cy)]_{\alpha_1(u)} > 0, \quad (F(y) > m(y))_{\alpha_2(u)}$$

and for all subsets  $\gamma$  containing  $\alpha_2(u)$ ,

$$\|u - y\| \leq k \|\mathcal{L}^\gamma(u) - \mathcal{L}^\gamma(y)\|$$

for some scalar  $k$ . Let  $y$  be an arbitrary point in  $\bar{N}_u$  and define two index sets  $\beta'$  and  $\gamma'$  by

$$\beta' = \{i \in \alpha_3(u) : F(y)_i = m(y)_i\}, \quad \gamma' = \{i \in \alpha_3(u) : F(y)_i > m(y)_i\}$$

By letting  $\gamma = \alpha_2(u) \cup \gamma'$  and  $\beta = \alpha_1(u) \cup \beta'$ , we have

$$[G(u) - G(y)]_\beta = (u - m(u))_\beta - (y - m(y))_\beta$$

$$[B(u - G(u)) + (b - Cu)]_{\gamma'} - [B(y - G(y)) + (b - Cy)]_{\gamma'} = 0$$

or equivalently,

$$\mathcal{L}^\gamma(u) - \mathcal{L}^\gamma(y) = \begin{pmatrix} I_\beta \\ B_{\gamma'} \end{pmatrix} (G(u) - G(y)).$$

Consequently, for each  $y \in \bar{N}_u$ ,

$$\|u - y\| \leq K \|G(u) - G(y)\|$$

where

$$K = k \cdot \max_{\beta \supseteq \alpha_1(u), \gamma \supseteq \alpha_2(u)} \left\| \begin{pmatrix} I & \beta \\ B & \gamma \end{pmatrix} \right\|.$$

Therefore, the mapping  $G$  is inverse locally Lipschitz continuous at the point  $u$ . Q.E.D.

It is easy to see that if the mapping  $m$  is a constant, a sufficient condition for each mapping  $\mathcal{L}^Y$  to be inverse locally Lipschitz continuous (everywhere) is that the matrix  $A$  is nondegenerate, i.e. each principal submatrix of  $A$  is nonsingular. This nondegeneracy of  $A$  is precisely the regularity property used in Aganagic [1] to establish Proposition 3.6 for the LCP in the case of the trivial splitting. We remark that no convexity assumption on the mapping  $m$  is needed in order for this last proposition to hold.

The next theorem gives sufficient conditions for  $G$  to be a local homeomorphism.

Theorem 3.7. Let  $m$  be convex and  $u \in R^n$ . Suppose that condition (9) is satisfied at  $u$ . Then the mapping  $G$  is a local homeomorphism mapping a neighborhood of  $u$  onto a neighborhood of  $G(u)$ .

Proof. Proposition 3.5 implies that there is a neighborhood  $N_u$  of  $u$  where  $G$  is injective. Since  $G$  is continuous, by [Theorem A.1, 22],  $G$  maps  $N_u$  homeomorphically onto the neighborhood  $G(N_u)$ . Q.E.D.

Remark. The inverse continuity of  $G$  is a consequence rather than an assumption of Theorem 3.7.

3.2. Global Homeomorphism. To continue the discussion, we say that a mapping  $f : R^n \rightarrow R^n$  is norm-coercive if for every sequence  $\{x^k\}$  with  $\|x^k\| \rightarrow \infty$ , we have  $\|f(x^k)\| \rightarrow \infty$ .

Proposition 3.8. Suppose that for each subset  $\gamma$  of  $\{1, \dots, n\}$ , the mapping  $\mathcal{L}^\gamma$  is norm-coercive. Then so is  $G$ .

Proof. Let  $\{x^k\}$  be a sequence with  $\|x^k\| \rightarrow \infty$ . It suffices to show that for any two partitioning subsets  $\alpha_1$  and  $\alpha_2$  of  $\{1, \dots, n\}$  and for any infinite subsequence  $\{x^{k_i}\}$  with

$$[F(x^{k_i}) = m(x^{k_i})]_{\alpha_1} \quad \text{and} \quad (BF(x^{k_i}) + (b - Cx^{k_i}))_{\alpha_2} = 0$$

for each  $k_i$ , the subsequence  $\{\|G(x^{k_i})\|\}$  diverges to  $\infty$ . For any such subsets and subsequences we have

$$[G(x^{k_i}) = \mathcal{L}^{\alpha_2}(x^{k_i})]_{\alpha_1} \quad \text{and} \quad G(x^{k_i})_{\alpha_2} = -(B_{\alpha_2\alpha_2})^{-1} [B_{\alpha_2\alpha_1} G(x^{k_i})_{\alpha_1} - (b + Ax^{k_i})_{\alpha_2}]$$

or equivalently,

$$\begin{pmatrix} G(x^{k_i})_{\alpha_1} \\ G(x^{k_i})_{\alpha_2} \end{pmatrix} = \begin{pmatrix} 0 \\ (B_{\alpha_2\alpha_2})^{-1} b_{\alpha_2} \end{pmatrix} + \begin{pmatrix} I & 0 \\ B_{\alpha_2\alpha_1} & B_{\alpha_2\alpha_2} \end{pmatrix}^{-1} \begin{pmatrix} \mathcal{L}^{\alpha_2}(x^{k_i})_{\alpha_1} \\ \mathcal{L}^{\alpha_2}(x^{k_i})_{\alpha_2} \end{pmatrix}$$

As  $\|\mathcal{L}^{\alpha_2}(x^{k_i})\| \rightarrow \infty$ , therefore so does  $\|G(x^{k_i})\|$ . Hence the mapping  $G$  is norm-coercive. Q.E.D.

Combining Theorem 3.7 and Proposition 3.8, we obtain the following result concerning the (global) existence and uniqueness of solution to the ICP(1).

Theorem 3.9. Let  $m$  be convex. Suppose that

(14a) For each subgradient matrix  $E$  of  $m$ , the matrix given by (11) is nonsingular for arbitrary partitioning subsets  $\alpha_1$ ,  $\alpha_2$ , and  $\alpha_3$  of  $\{1, \dots, n\}$ ;

(14b) For each subset  $\gamma$  of  $\{1, \dots, n\}$ , the mapping  $\mathcal{L}^\gamma$  is norm-coercive.

Then the mapping  $G$  is a global homeomorphism mapping  $\mathbb{R}^n$  onto  $\mathbb{R}^n$ . In particular, the ICP(1) has a unique solution for all vectors  $b$  in  $\mathbb{R}^n$ .

Proof. Conditions (14a) and (14b) and Theorem 3.7 imply that the mapping  $G$  is a local homeomorphism at every point  $u \in \mathbb{R}^n$ . The first conclusion now follows from the norm-coerciveness theorem in Ortega and Rheinbolt [24, p. 136-137]. For the second conclusion, it suffices to combine Theorem 3.1 with the observation that the conditions (14a) and (14b) are independent of the vector  $b$ . Q.E.D.

In [22], Megiddo and Kojima have derived some fairly general necessary and sufficient conditions as well as several more specific sufficient conditions for the (global) existence and uniqueness of solution to the NLCP(2). As pointed out in the Introduction, if the matrix  $A$  is nonsingular, the ICP(1) is equivalent to an NLCP. This observation implies that if  $A$  is nonsingular, then the results of Megiddo and Kojima would provide conditions for the ICP(1) to have a unique solution for all vectors  $b$ . This nonsingularity assumption of the matrix  $A$ , although not explicitly made in Theorem 3.9, is nevertheless a consequence of the norm-coerciveness of the mapping  $\mathcal{L}^N(x)$  where  $N = \{1, \dots, n\}$ . In fact, the norm-coerciveness of the mappings  $\mathcal{L}^\gamma(x)$  is equivalent to that of the principal subfunctions of the mapping  $g$  appearing in the corresponding NLCP(2). The latter is one of the conditions needed in Theorem 3.1 of [22].

4. MONOTONE ITERATES

In this section, we study the ICP(A,b,m) under the assumptions that m is monotonically nondecreasing and A has a Z-splitting (B,C) with C nonnegative. Note that the matrix A possessing such a splitting must necessarily be a Z-matrix. The ICP(A,b,m) with A and m having these properties arises for instance, from discretization of the impulsive control and optimal stopping problems mentioned earlier and from a certain generalized LCP (or equivalently, the problem of solving a system of piecewise linear equations) solvable as a single linear program. We refer to Dolcetta [13] for more details on the former application and briefly discuss the latter. Consider the generalized LCP

$$(15) \quad M^i x + q^i \geq 0 \quad \text{for } i = 1, \dots, k \quad \text{and} \quad \prod_{i=1}^k (M^i x + q^i)_j = 0 \quad \text{for } j = 1, \dots, n$$

where each  $M^i$  is  $n$  by  $n$ . It was shown in Mangasarian [21] that if there exist Z-matrices  $Y^1, \dots, Y^k$  and a nonsingular matrix  $Q$  such that the two conditions below are satisfied

$$(16a) \quad M^i Q = Y^i$$

$$(16b) \quad \sum_{i=1}^k (s^i)^T Y^i > 0 \quad \text{for some } (s^1, \dots, s^k) \geq 0 \quad \text{with} \quad \sum_{i=1}^k s^i > 0,$$

then a solution to the generalized LCP(15), if it exists, can be obtained by solving the linear program

$$\min p^T x \quad \text{subject to} \quad M^i x + q^i \geq 0 \quad \text{for } i = 1, \dots, k$$

where  $p = \sum_{i=1}^k (M^i)^T s^i$ . We remark that the result stated in the reference

is somewhat more general than the one presented here. (For instance  $Q$  needs not be nonsingular and the two conditions (16a) and (16b) can be relaxed.) We have simplified the general result in order to explain our point more easily. (See Cottle and Pang [11] for more discussion.)

Since  $Q$  is nonsingular, it is obvious that we can convert the generalized LCP to the following equivalent one

$$Y^i z + q^i \geq 0 \text{ for } i = 1, \dots, k \text{ and } \prod_{i=1}^k (Y^i z + q^i)_j = 0 \text{ for } j = 1, \dots, n$$

where  $z = Q^{-1}x$ . As each  $Y^i$  is a Z-matrix, there exists positive diagonal matrix  $D^i$  such that  $D^i \geq Y^i$ . Hence, with  $A = Y^1$  and

$$m(z) = \max_{2 \leq i \leq k} [(I - (D^i)^{-1} Y^i)z - (D^i)^{-1} q^i], \text{ the latter generalized LCP is equivalent to the ICP } (A, q^1, m) \text{ where } A \text{ is a Z-matrix and } m \text{ is non-}$$

decreasing. The condition (16b) implies that the feasible set of the ICP  $(A, q^1, m)$  is bounded below. See Cottle and Pang [10].

Formally, the feasible set of the ICP  $(A, b, m)$  is

$$X(A, b, m) = \{x \in R^n : Ax + b \geq 0 \text{ and } x \geq m(x)\}.$$

It is bounded below if there is a vector  $s \in R^n$  such that  $x \geq s$  for all  $x$  in  $X(A, b, m)$ . The following result shows that under certain conditions the ICP  $(A, b, m)$  has a least-element solution. Its proof can be found in Pang [25].

Theorem 5.1. Let  $m$  be a continuous and nondecreasing mapping from  $R^n$  into itself. Let  $A$  be a Z-matrix. Suppose that the feasible set  $X(A, b, m)$  is nonempty and bounded below. Then  $X(A, b, m)$  has a least-element  $x^*$  which is the unique solution of the program

$$\min e^T x \text{ subject to } x \in \bar{X}(A,b,m)$$

where  $e$  is any positive vector. Moreover,  $x^*$  solves the ICP  $(A,b,m)$ .

A mapping  $f$  from  $R^n$  into  $R^n$  is inverse-isotone if  $f(x) \geq f(y)$  implies  $x \geq y$  for each  $x$  and  $y$ . The next result gives a sufficient condition for the feasible set  $X(A,b,m)$  to be bounded below. Its proof is easy.

Proposition 5.2. Suppose that for some subset  $\gamma$  of  $\{1, \dots, n\}$ , the mapping  $\mathcal{L}^\gamma$  defined in Section 3 is surjective and inverse isotone. Then the feasible set  $X(A,b,m)$  is bounded below.

Theorem 5.1 has provided a least-element interpretation to Mangasarian's result of solving the generalized LCP as a linear program. For more discussion on related results, see the series of papers [10, 11, 17, 18, 19, 20, 26, 27].

If  $(B,C)$  is a Z-splitting of the matrix  $A$ , then the mapping  $F$  is not always well-defined for two reasons. First,  $F(u)$  may not exist. Second,  $F(u)$  may not be unique. However, if the set  $X(u)$  defined in Theorem 2.4 is nonempty, it has a least element by Theorem 5.1 (see also Tamir [32]). We shall let  $F(u)$  be this element, so that it is well-defined provided that it exists.

Under appropriate conditions, Theorem 5.1 shows that a solution to the ICP  $(A,b,m)$  can be obtained by solving a certain nonlinear program. The next theorem describes an alternate way for computing a (possible different) solution of the problem under the same set of conditions.

Theorem 5.3. Let  $m$  be a continuous and nondecreasing mapping from  $R^n$  into itself. Let  $(B,C)$  be a  $Z$ -splitting of the matrix  $A$  with  $C$  nonnegative. Suppose that the feasible set  $X(A,b,m)$  is nonempty and bounded below. Let  $u^0$  be an arbitrary vector in  $X(A,b,m)$ . Then with  $u^0$  as the initial vector, the sequence of iterates  $\{u^k\}$  defined by :  $u^{k+1} = F(u^k)$  is well-defined, nonincreasing and converges to a solution of the ICP  $(A,b,m)$ .

Proof. By induction, suppose that  $\{u^0, u^1, \dots, u^k\}$  is well-defined and nonincreasing. Moreover, we may assume that each  $u^i$  for  $i \leq k$  is feasible to the ICP  $(A,b,m)$ . We now consider  $u^{k+1}$ . First of all,  $u^k \in X(u^k)$ , so that  $u^{k+1} = F(u^k)$  is well-defined. Since  $u^{k+1}$  is the least element of  $X(u^k)$ , thus  $u^k \geq u^{k+1}$ . Moreover, we have

$$u^{k+1} \geq m(u^k) \geq m(u^{k+1})$$

and

$$Bu^{k+1} + (b - Cu^{k+1}) \geq Bu^{k+1} + (b - Cu^k) \geq 0.$$

Hence  $u^{k+1}$  is feasible to the ICP as well. Consequently, the entire sequence  $\{u^k\}$  is well-defined, nonincreasing as well as feasible. Since the feasible set  $X(A,b,m)$  is bounded below, the sequence  $\{u^k\}$  converges to a vector  $u^*$ . For each  $k \geq 0$ , we have

$$u^{k+1} \geq m(u^k), Bu^{k+1} + (b - Cu^k) \geq 0 \quad \text{and} \quad (u^{k+1} - m(u^k))^T (Bu^{k+1} + b - Cu^k) = 0.$$

Hence by passing the limit  $k \rightarrow \infty$ , we conclude that the vector  $u^*$  is a solution to the ICP  $(A,b,m)$ . Q.E.D.

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