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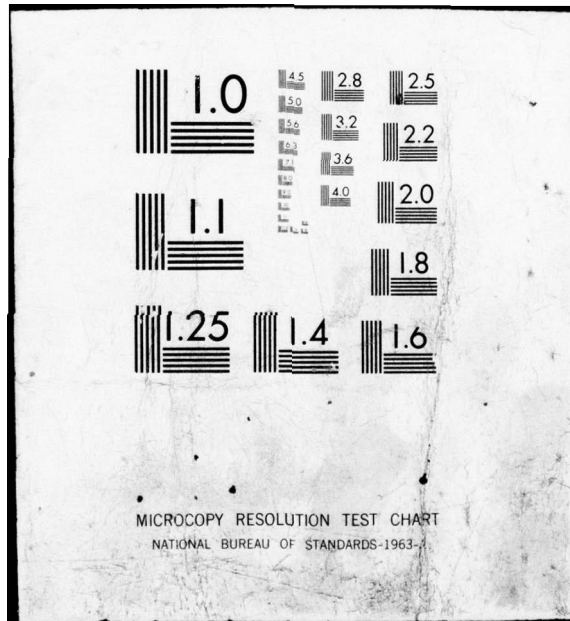
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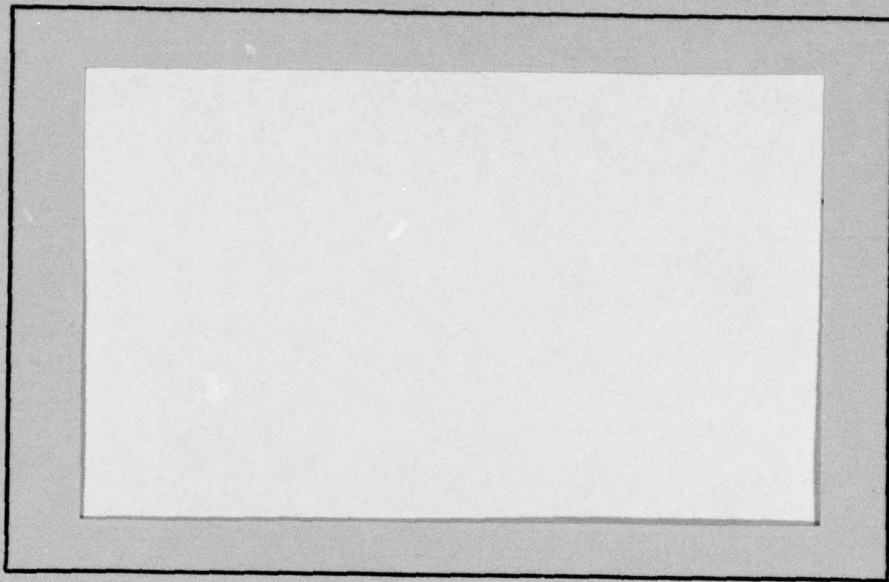
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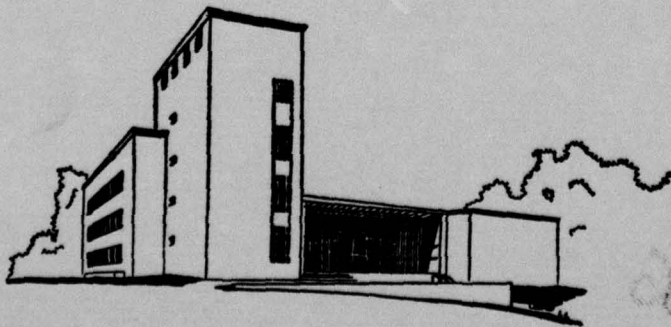
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by

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A UNIFICATION OF TWO CLASSES OF Q-MATRICES

Jong-Shi Pang

ABSTRACT. This note presents a class of Q-matrices which includes Saigal's class N of Q-matrices with negative principal minors and the class E of strictly semi-monotone Q-matrices.

Key Words: Class of matrices, Linear complementarity problem.

1. Introduction. Given a real square matrix  $M$  and a real vector  $q$  of the same size, the linear complementarity problem  $(q, M)$  is to find a vector  $x$  such that

$$q + Mx \geq 0, \quad x \geq 0 \quad \text{and} \quad x^T(q + Mx) = 0.$$

The matrix  $M$  is a Q-matrix if the problem  $(q, M)$  has a solution for all vectors  $q$ .

The problem of constructively identifying a Q-matrix has yet to be solved. Over the past years, numerous classes of matrices have been shown to belong to this large yet very much unknown class of Q-matrices. Two such classes are discovered by Eaves [2] and Saigal [7]. Eaves' class is  $L_1 \cap L_2 \cap S$  where  $L_1$  consists of the semi-monotone matrices  $M$  which are square matrices such that for each vector  $0 \neq x \geq 0$ , there is an index  $k$  such that  $x_k > 0$  and  $(Mx)_k \geq 0$ ;  $L_2$  consists of the square matrices  $M$  such that if  $x$  is a nonzero solution of the problem  $(0, M)$ , then there exist nonnegative diagonal matrices  $D_1$  and  $D_2$  with  $D_2 x \neq 0$  and  $(D_1 M + M^T D_2) x = 0$ ; and  $S$  consists of matrices  $M$  for which there is a vector  $x > 0$  such that  $Mx > 0$ . Saigal's class is  $N \cap S$  where  $N$  consists of square matrices  $M$  with negative principal minors. These two classes, namely,  $L \cap S$  and  $N \cap S$  where  $L = L_1 \cap L_2$ , are in fact distinct because

$$\begin{bmatrix} -1 & 2 \\ 4 & -1 \end{bmatrix} \in (N \cap S) \setminus L_1 \quad \text{and} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in (L \cap S) \setminus N.$$

Included among Eaves' class of Q-matrices is the subclass  $E$  of strictly semi-monotone matrices which are square matrices  $M$  such that for each  $0 \neq x \geq 0$ , there is an index  $k$  with  $x_k > 0$  and  $(Mx)_k > 0$ . The two matrices just given illustrate that the two classes  $E$  and  $N \cap S$  are not contained in one another.

Our purpose in the note is to present a class of Q-matrices which properly contains the two classes E and  $N \cap S$ . The construction of this unifying class is very much motivated by the proof used in [7] to establish that  $N \cap S \subseteq Q$ .

2. Main Results. Let A be an n by n matrix and  $\alpha$  an index subset of  $\{1, \dots, n\}$ . Suppose that the principal submatrix  $A_{\alpha\alpha}$  is nonsingular. Let P be a permutation matrix such that  $P^T A P$  has  $A_{\alpha\alpha}$  as a leading principal submatrix. The  $\alpha$ -principal pivot transform of A is defined as the matrix  $PA^*P^T$  where

$$(1) \quad A^* = \begin{pmatrix} A_{\alpha\alpha}^{-1} & & -A_{\alpha\alpha}^{-1} A_{\alpha\beta} \\ A_{\beta\alpha} A_{\alpha\alpha}^{-1} & A_{\beta\beta} - A_{\beta\alpha} A_{\alpha\alpha}^{-1} A_{\alpha\beta} \end{pmatrix} .$$

Following the notations of Garcia [4], we let  $E^*(d)$  for  $d > 0$  or  $d = 0$ , to denote the class of square matrices M for which the problem  $(d, M)$  has zero as the unique solution.

We note that A is a Q-matrix if and only if each of its principal pivot transforms is so, i.e., Q-matrices are invariant under principal pivot transforms. Notice however, that the two classes E and  $N \cap S$  are not invariant under such transforms.

We say that an n by n matrix A is an  $\tilde{N}$ -matrix if there is a vector  $f > 0$  and a subset  $\alpha$  of  $\{1, \dots, n\}$  satisfying the conditions below

- (i)  $Af > 0$
- (ii)  $A_{\alpha\alpha}$  is nonsingular
- (iii) For any  $(n - 1)$  by  $(n - 1)$  principal submatrix  $\tilde{A}$  of the  $\alpha$ -principal pivot transform of A, it holds that  $\tilde{A} \in E^*(\tilde{f}) \cap E^*(0)$  where  $\tilde{f}$  is the  $(n - 1)$ -

subvector of the (positive) vector  $\bar{f}$  which in partitioned form (according to  $A^*$  given in (1)) is defined as

$$\bar{f} = \begin{pmatrix} f_\alpha \\ A_{\beta\alpha} f_\alpha + A_{\beta\beta} f_\beta \end{pmatrix}.$$

The components of  $\bar{f}$  are in correspondence with the rows of  $\tilde{A}$

In the above definition of an  $\tilde{N}$ -matrix, we allow  $\alpha = \beta$ . Of course, the  $\phi$ -principal pivot transform of  $A$  is  $A$  itself. Note that condition (i) implies that an  $\tilde{N}$ -matrix is necessarily an  $S$ -matrix. According to [4], the principal submatrix  $\tilde{A}$  is a  $Q$ -matrix. The result below shows that an  $\tilde{N}$ -matrix is in fact a  $Q$ -matrix.

Theorem 1. An  $\tilde{N}$ -matrix is a  $Q$ -matrix.

Proof. Let  $A$  be a  $\tilde{N}$ -matrix and let  $A^*$  be the matrix given in (1). Let  $q$  be a given vector and let

$$q^* = \begin{pmatrix} -A_{\alpha\alpha}^{-1} q_\alpha \\ q_\beta - A_{\beta\alpha} A_{\alpha\alpha}^{-1} q_\alpha \end{pmatrix}.$$

It suffices to show that the linear complementarity problem  $(q^*, A^*)$  has a solution. We may write this latter problem as

$$x_\alpha = -A_{\alpha\alpha}^{-1} q_\alpha + A_{\alpha\alpha}^{-1} u_\alpha - A_{\alpha\alpha}^{-1} A_{\alpha\beta} x_\beta \geq 0, \quad u_\alpha \geq 0$$

$$u_\beta = q_\beta - A_{\beta\alpha} A_{\alpha\alpha}^{-1} q_\alpha + A_{\beta\alpha} A_{\alpha\alpha}^{-1} u_\alpha + (A_{\beta\beta} - A_{\beta\alpha} A_{\alpha\alpha}^{-1} A_{\alpha\beta}) x_\beta \geq 0, \quad x_\beta \geq 0$$

$$(x_\alpha)^T u_\alpha = (x_\beta)^T u_\beta = 0.$$

Consider the solution of the problem by Lemke's almost complementary

pivoting algorithm [6] using  $\bar{f}$  defined in condition (iii) above as the artificial vector. If at some point in the solution process, both  $x_\alpha$  and  $u_\beta$  become nonbasic, then the system below has a solution:

$$0 = -A_{\alpha\alpha}^{-1} q_\alpha + \lambda f_\alpha + A_{\alpha\alpha}^{-1} u_\alpha - A_{\alpha\alpha}^{-1} A_{\alpha\beta} x_\beta, \quad u_\alpha \geq 0, \quad \lambda \geq 0, \quad x_\beta \geq 0$$

$$0 = q_\beta - A_{\beta\alpha} A_{\alpha\alpha}^{-1} q_\alpha + \lambda(A_{\beta\alpha} f_\alpha + A_{\beta\beta} f_\beta) + A_{\beta\alpha} A_{\alpha\alpha}^{-1} u_\alpha + (A_{\beta\beta} - A_{\beta\alpha} A_{\alpha\alpha}^{-1} A_{\alpha\beta}) x_\beta.$$

This latter system is clearly equivalent to the one

$$0 = q_\beta + A_{\beta\beta}(x_\beta + \lambda f_\beta), \quad x_\beta \geq 0, \quad \lambda \geq 0$$

$$\bar{u}_\alpha = q_\alpha - \lambda(A_{\alpha\alpha} f_\alpha + A_{\alpha\beta} f_\beta) + A_{\alpha\beta}(x_\beta + \lambda f_\beta) \geq 0.$$

The consistency of the last system implies that the one below is solvable

$$0 = q_\beta + A_{\beta\beta} \bar{x}_\beta, \quad \bar{x}_\beta \geq 0$$

$$\bar{u}_\alpha = q_\alpha + A_{\alpha\beta} \bar{x}_\beta \geq 0.$$

In fact, we have  $\bar{x}_\beta = x_\beta + \lambda f_\beta$  and  $\bar{u}_\alpha = u_\alpha + \lambda(A_{\alpha\alpha} f_\alpha + A_{\alpha\beta} f_\beta)$ . Therefore in this case, the problem  $(q, A)$  has a solution. So suppose that throughout the solution of the problem  $(q^*, A^*)$  by Lemke's algorithm (with the above choice of artificial vector  $\bar{f}$ ), at least one variable in  $\begin{pmatrix} x_\alpha \\ u_\beta \end{pmatrix}$  is basic. If the algorithm terminates in a ray, then the problem  $(\bar{\lambda}\bar{f}, A^*)$  for some nonnegative  $\bar{\lambda}$ , has a nonzero solution. This implies by the fact that at least one variable in  $\begin{pmatrix} u_\alpha \\ x_\beta \end{pmatrix}$  must be zero, that a certain principal subproblem  $(\tilde{f}, \tilde{A})$  (in the case  $\bar{\lambda} > 0$ ) or  $(0, \tilde{A})$  (in the case  $\bar{\lambda} = 0$ ) where  $\tilde{A}$  is an  $(n - 1)$  principal submatrix of  $A^*$  and  $\tilde{f}$  the corresponding  $(n - 1)$ -subvector of  $\bar{f}$ , would have a nonzero solution. But this contradicts

condition (iii). Consequently, Lemke's algorithm must compute a solution of  $(q^*, A^*)$ . This completes the proof of the theorem.

The proof of the theorem is based on an extension of the argument used in Saigal [7] for the special case  $N \cap S$ . As a matter of fact, the proof also suggests a constructive method for actually computing a solution to  $(q, A)$  with  $A$  an  $\tilde{N}$ -matrix, provided that the index set  $\alpha$  and vector  $f$  are available readily. Indeed, one can apply Lemke's algorithm to the problem  $(q^*, A^*)$  using  $\bar{f}$  as the artificial vector. As soon as the artificial variable  $\lambda$  reaches zero or all the  $u_\beta$  and  $x_\alpha$  variables become nonbasic, a solution to the given problem  $(q, A)$  can be obtained easily.

If a square matrix  $A$  is such that some  $\alpha$ -principal pivot transform is strictly semi-monotone, then  $A$  is an  $\tilde{N}$ -matrix. This follows from the fact that principal submatrices of strictly semi-monotone matrices are themselves strictly semi-monotone and that a strictly semi-monotone matrix must be in  $E^*(d) \cap E^*(0)$  for any positive  $d$  (see [2] e.g.). Hence in particular, if  $A$  is itself strictly semi-monotone, then  $A$  is an  $\tilde{N}$ -matrix.

On the other hand, if a square  $S$ -matrix  $A$  is such that some  $\alpha$ -principal pivot transform  $\bar{A}$  has all proper principal minors positive, then  $A$  is an  $\tilde{N}$ -matrix. This is because any  $(n-1)$  by  $(n-1)$  principal submatrix of  $\bar{A}$  must then be a  $P$ -matrix, i.e. has all principal minors positive, and thus belong to  $E^*(d) \cap E^*(0)$  for any positive  $d$  (see [8]). Hence, in particular, if  $A$  is in  $N \cap S$  then  $A$  is an  $\tilde{N}$ -matrix. This follows from the fact that  $A^{-1}$  which is the  $\{1, \dots, n\}$ -principal pivot transform of  $A$ , has all proper principal minors positive (see [7]).

That the class of  $\tilde{N}$ -matrices properly contains the union  $E \cup (N \cap S)$  can be seen from the example

$$\begin{bmatrix} -1 & 2 \\ -4 & 7 \end{bmatrix}$$

which is an  $\tilde{N}$ -matrix (its  $\{2\}$ -principal pivot transform is a positive matrix) but certainly not strictly semi-monotone or has negative principal minors.

In [5], it is shown that the linear complementarity problem  $(q, A)$  with  $A \in N$  has 0, 1, 2 or 3 solutions. The theorem below extends this result. Recall that a square matrix is nondegenerate if all its principal minors are nonzero.

Theorem 2. Let  $A$  be a nondegenerate matrix such that some  $\alpha$ -principal pivot transform  $\bar{A}$  has all proper principal minors positive. Then for every vector  $q$ , the linear complementarity problem  $(q, A)$  has 0, 1, 2, or 3 solutions.

Proof. If  $\bar{A}$  has positive determinant, then  $\bar{A}$  and thus  $A$  is a P-matrix. Hence the problem  $(q, A)$  has a unique solution for all vectors  $q$ . On the other hand, if  $\bar{A}$  does not have positive determinant, then it must have negative determinant. This is because  $\bar{A}$  must be nonsingular. In fact, its inverse is a principal rearrangement of the  $\beta$ -principal pivot transform of  $A$  with  $\beta$  the complement of  $\alpha$ . This latter principal pivot transform is well-defined by the nondegeneracy of  $A$ . Consequently, it follows that the inverse of  $\bar{A}$  is in class  $N$ . Hence, by the result established in [5], the linear complementarity problem  $(\bar{q}, \bar{A}^{-1})$  has 0, 1, 2 or 3 solutions. As  $\bar{A}^{-1}$  is also a principal pivot transform of  $A$ , the same conclusion is true for each  $(q, A)$ . This completes the proof of the theorem.

We gave an example earlier to show that there are matrices in  $N \cap S$  which are not in the class E. The following result establishes that the inverse of a matrix in  $N \cap S$  in fact belongs to E.

Theorem 3. Let A be in  $N \cap S$ . Then the inverse of A is in E.

Proof. In fact, if A is in  $N \cap S$ , then each proper principal submatrix of  $A^{-1}$  is a P-matrix. In particular, each  $(n - 1)$  by  $(n - 1)$  principal submatrix of  $A^{-1}$  is strictly semi-monotone (see [3]). As  $A^{-1}$  is also an S-matrix, the desired conclusion now follows from a result established in [1].

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