

AD-A083 807

WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER

F/S 12/1

EQUILIBRIA OF THE CURVATURE FUNCTIONAL AND MANIFOLDS OF NONLINE--ETC(U)

NOV 79 M GOLOMB, J JEROME

DAA829-75-C-0024

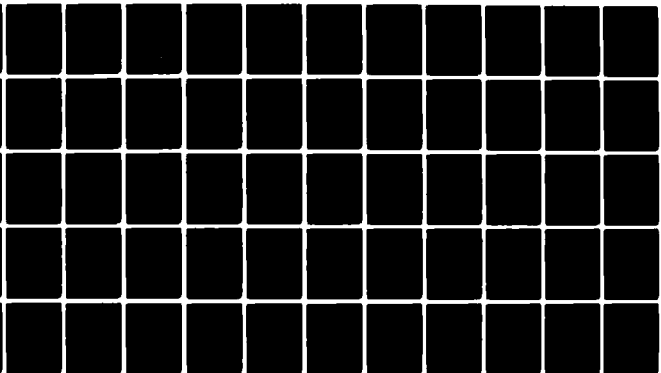
NL

UNCLASSIFIED

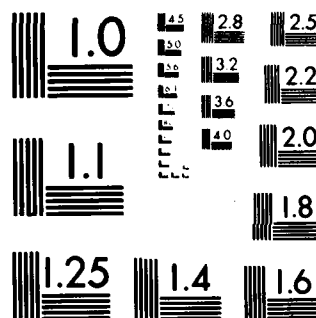
MRC-TSR-2024

[1 of 1]

Microfilm



END
DATE
FILMED
6-80
DTIC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

EVER

2 B.S.

ADA 083807

9 NRC Technical Summary Report, 2024

6 EQUILIBRIA OF THE CURVATURE FUNCTIONAL AND MANIFOLDS OF NONLINEAR INTERPOLATING SPLINE CURVES.

14 MRC-TSR-2024

10 Michael Golomb and Joseph Jerome

Mathematics Research Center
University of Wisconsin-Madison
610 Walnut Street
Madison, Wisconsin 53706

41 Nov 79

12 70

DTIC ELECTE
MAY 6 1980
D
C

(Received September 4, 1979)

15 DAAG29-75-C-0024,
- NSF-MPS 74-02292

Approved for public release
Distribution unlimited

FILE COPY

Sponsored by
U. S. Army Research Office and National Science Foundation
P. O. Box 12211 Washington, D. C. 20550
Research Triangle Park
North Carolina 27709

221200 80 4 9 113

UNIVERSITY OF WISCONSIN - MADISON
MATHEMATICS RESEARCH CENTER

EQUILIBRIA OF THE CURVATURE FUNCTIONAL AND MANIFOLDS
OF NONLINEAR INTERPOLATING SPLINE CURVES

Michael Golomb¹ and Joseph Jerome²

Technical Summary Report # 2024

November 1979

ABSTRACT

A detailed global and local analysis of smooth solutions of the variational problem

$$(li) \quad \delta \int_0^s \kappa^2(s) ds = 0 ,$$

subject to position function constraints

$$(lii) \quad x(s_i) = p_i, \quad 0 \leq s_0 < s_1 < \dots < s_m \leq \bar{s} ,$$

is carried out. Here $\{p_i\}_0^m \subset \mathbb{R}^2$ is prescribed, x is a vector-valued function with curvature $\kappa(s)$ at arc length s and the interpolation nodes s_i are free. Problem (1) may be viewed as the mathematical formulation of the draftsman's technique of curve fitting by mechanical splines.

Although most of the basic equations satisfied by these nonlinear spline curves have been known for a very long time, calculation via elliptic integral functions has been hampered by a lack of understanding concerning what precise information must be specified for the stable determination of a smooth, unique interpolant modelling the thin elastic beam. In this report, sharp characterizations are derived for the extremal interpolants as well as structure theorems in terms of inflection point modes which guarantee uniqueness and well-posedness.

A certain type of stability is introduced and studied and shown to be related to (linearization) concepts associated with piecewise cubic spline

¹Division of Mathematical Sciences, Purdue University, W. Lafayette, Indiana 47907.

²Department of Mathematics, Northwestern University, Evanston, Illinois 60201.

ABSTRACT (continued)

functions, which have been studied for decades as a simplification of the nonlinear spline curves. Many examples are introduced and studied.

AMS (MOS) Subject Classifications: 41A05, 49B15, 49B50, 49F22, 65D10

Key Words: Nonlinear spline curves, elastica, manifolds of extremals, perturbation stability, mode, ray configuration, rectangular configuration.

Work Unit Number 6 - Spline Functions and Approximation Theory

SIGNIFICANCE AND EXPLANATION

The mathematical formulation of curve fitting by mechanical splines, i.e. thin, flexible, elastic beams passing through freely rotating sleeves anchored at fixed locations, is studied in this report. These are called elastica or nonlinear spline curves.

As contrasted with the mathematically idealized splines, which have proven to be of considerable utility and concerning which much information is available, the nonlinear splines are relatively poorly understood. The writers are attempting to understand and systematically construct these curves. Computer graphics obtained by other workers suggest remarkable efficiency of the elastica for curve fitting. This is perhaps not too surprising since the nonlinear spline represents an equilibrium position of a thin beam.

Accession For	
NTIS	GENERAL
DDC	TAB
Unannounced	
Justification	
By	
Distribution/	
Availability Codes	
Dist	Avail and/or special
A	

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

EQUILIBRIA OF THE CURVATURE FUNCTIONAL AND MANIFOLDS
OF NONLINEAR INTERPOLATING SPLINE CURVES

Michael Golomb¹ and Joseph Jerome²

§1. Introduction

Let $P = \{p_0, p_1, \dots, p_m\}$ be an ordered set of points in the Euclidean plane (the p_i need not be distinct) and let it be required to pass a smooth curve through these points in the prescribed order. It is an old technique of draftsmen to use a mechanical spline to accomplish this. If the spline is considered as a thin elastic beam of uniform cross section with a central fiber that is inextensible, then the strain energy of the bent spline of length $\bar{s}$ is given by

$$A \int_0^{\bar{s}} \kappa^2(s) ds + B$$

where $\kappa(s)$ is the curvature of the fiber at arc length s and A, B are constants. An equilibrium position of the spline makes the energy functional stationary, hence satisfies

$$(1.1i) \quad \delta \int_0^{\bar{s}} \kappa^2(s) ds = 0.$$

This equation together with the interpolation conditions

$$(1.1ii) \quad x(s_i) = p_i, \quad 0 \leq s_0 < s_1 < \dots < s_m \leq \bar{s}$$

for the position function x , constitute the mathematical formulation of the draftsman's technique. The present article deals with analytical (not graphical nor computational) problems arising from system (1.1). In elasticity theory the solutions of (1.1i) or of the more general equation

¹Division of Mathematical Sciences, Purdue University, W. Lafayette, Indiana 47907.

²Department of Mathematics, Northwestern University, Evanston, Illinois 60201.

$$(1.2) \quad \delta \left(\int_0^{\bar{s}} [\lambda + \kappa^2(s)] ds \right) = 0 ,$$

where λ is a constant, are known as elastica. Their study dates back to the Bernoulli brothers, Euler and others (see Love [8, Ch. XIX] for classical results). The boundary conditions in traditional elasticity theory have little in common with the interpolation conditions (1.1ii). To materialize the latter ones in the beam model one may think of freely rotating small sleeves, anchored at the points $p_0, \dots, p_m$, through which the spline can slide without friction. We refer to the solutions $x = \bar{x}$ of the variational problem (1.1i, ii) as extremal P-interpolants. In some parts of the present paper we deal with extremal length-prescribed P-interpolants, in which $\bar{s} = s_m - s_0$ is given in addition to P . To materialize this condition one replaces the sleeves at p_0, p_m by pins which allow no sliding. In other parts we consider extremal angle-prescribed P-interpolants, in which the angles that the spline makes at p_0 and p_m with a reference line are given. This situation prevails if the sleeves at p_0 and p_m are not allowed to rotate.

Stable equilibrium positions in mechanics are sought as positions that minimize the potential energy functional. In one of the earliest papers discussing nonlinear interpolating splines [2] it was pointed out that the infimum of $\int_0^{\bar{s}} \kappa^2(s) ds$ is 0 for any configuration P , hence can be attained only in the trivial case where P is interpolated by a straight segment. Lee and Forsythe [7], who make a substantial study of the variational problem (1.1i, ii), call the solutions (when existence is hypothesized) local minima. However, it will be proved in §6 below that, in the simple case where P consists of 2 points, there are countably many nontrivial extremal P-interpolants; none of them constitutes a local minimum of the energy. This makes it evident that an extremal P-interpolant is, in general, not a local minimum (for a detailed discussion of the stability problem see [5]).

The existence questions for interpolating elastica are much more subtle. For length-constrained or length-prescribed interpolants one can prove existence of extremals (actually global minima) by the direct methods of the calculus of variations, because one has compactness in a suitably chosen function space (this was done in [3] and [6]; see also the

Appendix of this paper). This is not the case for interpolants with no length restriction, and the existence of such extremals interpolating n points in general position, and whether they are local minima or not remains an open question (some progress along these lines has been achieved by M. Golomb [4], [5]). Computational work on extremal interpolants is more advanced (cf. M. Malcolm [10]), although decisive progress in this area is also hampered by the lack of general existence and uniqueness theorems.

We now give a brief account of the content of this paper. In §2 we define the function classes in which the extremal interpolants are sought. We also characterize them by Euler equations (for the Cartesian coordinates), boundary and regularity conditions. In §3 we do the same for the "normal representation" of the extremals, by which we mean the function $s \mapsto \theta(s)$, which is the angle that the extremal makes at arc length s with a reference line.

The normal representation θ of a length-prescribed extremal P -interpolant appears as the solution of a free multi-point boundary value problem for $s_1, \dots, s_{m-1}, \theta$ with $s_m - s_0$ prescribed:

$$\begin{aligned}
 & \text{(i)} \quad \ddot{\theta}(s) - \mu_1^1 \sin \theta(s) + \mu_1^2 \cos \theta(s) = 0, \quad s_{i-1} < s < s_i, \\
 & \text{(1.3)} \quad \text{(ii)} \quad \dot{\theta}(s_0) = \dot{\theta}(s_m) = 0, \\
 & \text{(iii)} \quad (\mu_{i+1}^1 - \mu_i^1) \cos \theta(s_i) + (\mu_{i+1}^2 - \mu_i^2) \sin \theta(s_i) = 0, \quad i = 1, \dots, m-1.
 \end{aligned}$$

For the general extremal P -interpolant, (1.3iii) holds also for $i = m$ and s_0 and s_m are free as well with $\mu_{m+1}^1 = \mu_{m+1}^2 = 0$. The function θ and the knot abscissas s_i are the unknowns; the multipliers μ_i^1, μ_i^2 are determined from the interpolation conditions. In §4 it is shown how certain families $E_{(k_1, \dots, k_m)}$ of extremal interpolants with prescribed numbers $(k_1, \dots, k_m)$ of inflection points between knots ("mode"), can be realized as smooth $2m$ -dimensional manifolds $M_{(k_1, \dots, k_m)}$. The inverse mapping from $M_{(k_1, \dots, k_m)}$ into the position function space of the extremals is continuous when the latter is topologized by a suitable metric. Thus, the mode $(k_1, \dots, k_m)$ of an extremal suitably delineates uniqueness and well-posedness. The union of the $E_{(k_1, \dots, k_m)}$ consists of only those extremals which have a

genuine knot and no inflection point at each interior interpolation node. Points in the intersection of the boundaries of the manifolds $M_{(k_1, \dots, k_m)}$ correspond to singular points in position function space. For these boundary elements some interior knot is spurious (γ is not discontinuous) or is an inflection point. We give two examples to demonstrate this.

In §5 we study the existence of elastica spline interpolation in the small. Does the set of configurations P for which extremal interpolants exist have nonempty interior in $\mathbb{R}^{2m}$? More specifically, which P in $\mathbb{R}^{2m}$ are interior points of this set? We show, by use of the implicit function theorem that near a given configuration $\bar{P}$ with extremal interpolant $\bar{E}$ there is a local diffeomorphism between configurations and extremal interpolants if $(\bar{E}, \bar{P})$ satisfies a certain hypothesis (A). It requires that a homogeneous linear differential equation with variable coefficients depending on $\bar{E}$ and with homogeneous linear side conditions has no nontrivial solution. Another formulation of this condition is that a computable function (involving many quadratures) be $\neq 0$ at the end point of $\bar{E}$. It is easily verified that the ray configuration P_0 , with the trivial extremal interpolant E_0 , satisfies (A), so that the existence of extremal P -interpolants for all configurations P in some Euclidean neighborhood of any ray configuration is thereby demonstrated. The differential equation problem of hypothesis (A) reduces to the natural cubic spline interpolation problem in the case (P_0, E_0) . This demonstrates that cubic spline interpolation can be interpreted as the result of linearization of extremal interpolation (in the sense of making $\int \kappa^2 ds$ stationary) near the trivial interpolant for the ray configuration. This proof makes precise the old idea that cubic splines are in some sense the "smoothest" interpolants. Of course, it has long been known that cubic spline functions arise from minimizing the quadratic functional $\int (D^2 f)^2$ among the interpolating functions f . Since the linear operator D^2 supposedly approximates the nonlinear curvature operator, the cubic splines recommend themselves as near optimally smooth interpolants. The "hairpin" configuration $\tilde{P}$ with a loop interpolant $\tilde{E}$ is given as an example where hypothesis (A) is not satisfied. There are configurations close to $\tilde{P}$ for which there exists no extremal interpolant near $\tilde{E}$ and there are other configurations close to $\tilde{P}$ for which there do exist extremal interpolants

near $\bar{E}$. This seems to be the first known example demonstrating singular behavior in non-linear spline interpolation.

§6 contains an exhaustive study of extremal P-interpolants for the case where P consists of two points. It is shown that there exist, besides the trivial extremal, countably many non-trivial ones of distinct integral mode, that all of them are obtained by simple transformations from a basic one, all have the same length and (cf. [5]) none makes the potential energy a local minimum. Composition of these 2-point extremals yields countably many extremal P-interpolants for various special configurations P. §6 also exhibits countably many angle-prescribed and countably many length-prescribed 2-point extremals.

In §7 some special cases of closed extremal P-interpolants are considered. It is shown that the only closed length-prescribed extremal without knots are the repeatedly traversed circle and figure eight configurations. Formally, the Euler equation is the limiting case of the Euler equation for an elastic circular ring under hydrostatic pressure p as $p \rightarrow 0$ (cf. [1] and [12]). The ring, however, is not an elastica since its deformations satisfy stress-strain relationships. We also consider closed extremals which are not length-prescribed. Here the extremals in §6 are used to construct infinitely many closed extremals for several special P-configurations, for example where P is the set of vertices of a regular polygon. In particular, if a regular m-gon, $m \geq 2$, is inscribed in the unit circle, then a circumscribed extremal exists with length

$$s_m = \frac{2m \sin \frac{\pi}{m} F(\frac{1}{2} \sqrt{2}; \beta_m)}{\sqrt{2} [2E(\frac{1}{2} \sqrt{2}; \beta_m) - F(\frac{1}{2} \sqrt{2}; \beta_m)]}$$

where $\cos \beta_m = \sqrt{\cos \pi/m}$. There are similar formulas for the energy U_m and the arc length $s_m(\theta)$; of interest is the result that $s_m(\theta)/\theta \rightarrow 1$ as $m \rightarrow \infty$, so that the circumscribed extremals have the unit circle as a limiting configuration. These extremals are stable, i.e. they make the potential energy a local minimum, as proved in [5].

§2. Regularity and Characterizations of Open Extremals

For two points $p = (p^1, p^2)$ and $q = (q^1, q^2)$ in real Euclidean space $\mathbb{R}^2$ we employ the inner product $pq = p^1q^1 + p^2q^2$, the distance $|p-q| = [(p-q)(p-q)]^{1/2}$, and the exterior product $[p, q] = p^1q^2 - p^2q^1$ of such points. We consider mappings $x = (x^1, x^2)$ of the unit interval $I = [0, 1]$ to $\mathbb{R}^2$. We denote by $H_2(I)$ the real Hilbert space of those mappings x such that the derivative $\dot{x}$ is absolutely continuous and $\ddot{x} \in L_2(I)$, equipped with the inner product

$$(2.1) \quad (x, y)_{H_2} = \int_I (xy + \dot{x}\dot{y} + \ddot{x}\ddot{y}) \, dt.$$

We say x is a regular element of H_2 if $|\dot{x}(t)| > 0$ for all $t \in I$. We observe that the regular elements of H_2 form an open subset H_2^{reg} of H_2 .

For $x \in H_2$ we define the arc length map $s_x : I \rightarrow \mathbb{R}_+$ by

$$(2.2) \quad s_x(t) = \int_0^t |\dot{x}|, \quad t \in I.$$

If $x \in H_2^{\text{reg}}$ then s_x has an inverse $s_x^{-1} : [0, \bar{s}] \rightarrow [0, 1]$, where $\bar{s} = s_x(1)$, and in this case the function $x \circ s_x^{-1} : [0, \bar{s}] \rightarrow \mathbb{R}^2$ has an absolutely continuous derivative, and square-integrable second derivative. We identify x with the oriented curve C in the x^1x^2 -plane which has parametric representation $x = x(t)$. Writing $\bar{x} = x \circ s_x^{-1}$, we say that $\bar{x}$ is the arc length parametrization of the curve C . Clearly $\bar{x} \in H_2(0, \bar{s})$ and we have:

$$\dot{\bar{x}} = \dot{x} \circ s_x^{-1} / |\dot{x} \circ s_x^{-1}|, \quad \ddot{\bar{x}} = \frac{\ddot{x} \circ s_x^{-1} |\dot{x} \circ s_x^{-1}| - \dot{x} \circ s_x^{-1} (\dot{x} \ddot{x} \circ s_x^{-1})}{|\dot{x} \circ s_x^{-1}|^4}.$$

If $x \in H_2^{\text{reg}}$, then its curvature $\kappa_x : I \rightarrow \mathbb{R}$ is defined by

$$(2.3) \quad \kappa_x(t) = [\dot{\bar{x}}, \ddot{\bar{x}}] \circ s_x(t) = [\dot{x}, \ddot{x}] |\dot{x}|^{-3}(t), \quad t \in I.$$

Suppose $x \in H_2^{\text{reg}}$ and $s_x(1) = \bar{s}$. Then we define the curvature functional,

$$(2.4) \quad U(x) = \int_0^{\bar{s}} \kappa_x^2 = \int_0^{\bar{s}} [\dot{x}, \ddot{x}]^2 = \int_0^{\bar{s}} \ddot{x}^2.$$

Note that (2.4) defines U as a mapping of H_2^{reg} into $\mathbb{R}_+$. The equivalent expression,

$$(2.5) \quad U(x) = \int_I [\dot{x}, \ddot{x}]^2 |\dot{x}|^{-5},$$

is independent of the parametrization of x in the following sense. If u is a C^∞ -map of I onto itself with $\dot{u} > 0$ and $x = y \circ u$ then

$$U(x) = \int_I [\dot{y}, \ddot{y}]^2 |\dot{y}|^{-5} = U(y).$$

If $x \in H_2^{\text{reg}}$ then U is Fréchet-differentiable at x and, for any increment $y \in H_2$,

$$(2.6) \quad U'(x)[y] = \int_I \{2[\dot{x}, \ddot{x}]([\dot{x}, \ddot{y}] + [\dot{y}, \ddot{x}])|\dot{x}|^{-5} - 5[\dot{x}, \ddot{x}]^2 \dot{x} \dot{y} |\dot{x}|^{-7}\}.$$

If the variable of integration is chosen to be s_x , (2.6) simplifies to

$$(2.7) \quad U'(x)[y] = \int_0^{\bar{s}} (2\ddot{x}\ddot{y} - 3\kappa_x^2 \dot{x} \dot{y}) ds_x; \quad \bar{x} = x \circ s_x, \quad \bar{y} = y \circ s_x.$$

Let $0 = p_0, p_1, \dots, p_m$ be fixed points in $\mathbb{R}^2$, not necessarily distinct, but $p_{i-1} \neq p_i$, $i = 1, \dots, m$, and let P denote the ordered set $\{p_0, p_1, \dots, p_m\}$. We refer to P as a configuration in $\mathbb{R}^2$. If $x \in H_2^{\text{reg}}$ is such that $x(t_i) = p_i$ ($i = 0, 1, \dots, m$) for some $0 \leq t_0 < t_1 < \dots < t_m \leq 1$ we say the curve x is an admissible P-interpolant, with knots p_i ($i = 0, \dots, m$). The terminals $x(0), x(1)$ may or may not be coincident with the terminal knots p_0, p_m . The P-interpolants defined here are to be considered as open even if $x(0) = x(1)$. In the physical interpretation $p_i = p_j$ for some $i \neq j$ means that the beam is constrained to pass through two sleeves which are fixed at the same point p_i but can rotate independently of each other.

Suppose x is a fixed admissible P-interpolant and $\bar{x} = x \circ s_x^{-1}$ is its arc length parametrization, $s_x(1) = \bar{s}$ its length, $\bar{x}(\bar{s}_i) = p_i$ ($i = 0, 1, \dots, m$) its knots. Given any $z \in H_2$ let $\bar{z} = z \circ s_x^{-1}$, be the parametrization of z which uses the arc length of x as the parameter, and assume $\bar{z}(\bar{s}_i) = 0$ ($i = 0, 1, \dots, m$). For $|\epsilon|$ sufficiently small, $x + \epsilon z$ is an admissible P-interpolant and

$$(2.8) \quad U(x + \epsilon z) - U(x) = \epsilon U'(x)[z] + o(\epsilon) \text{ as } \epsilon \rightarrow 0.$$

This justifies the following

Definition 2.1. The admissible P-interpolant x , with arc length parametrization $\bar{x} = x \circ s_x^{-1}$, knots $p_i = \bar{x}(\bar{s}_i)$ ($i = 0, 1, \dots, m$), length $\bar{s} = s_x(1)$, is an extremal P-interpolant if

$$(2.9) \quad U'(x)[z] = 0,$$

i.e.,

$$\int_0^{\bar{s}} (2\bar{x}\bar{z} - 3\kappa_{\bar{x}}^2 \bar{x}\bar{z}) = 0, \quad \bar{z} = z \circ s_x^{-1},$$

for every $z \in H_2$ satisfying $\bar{z}(\bar{s}_i) = 0$ ($i = 0, 1, \dots, m$).

The following proposition follows from (2.8) by the usual arguments of the calculus of variations. It helps to explain the interest in extremal P-interpolants.

Proposition 2.1. Suppose the admissible P-interpolant x minimizes the curvature functional U locally, i.e.

$$U(x) \leq U(y)$$

for every admissible P-interpolant y in a neighborhood of x in H_2 . Then x is an extremal P-interpolant.

The three major propositions of this section follow. We use the notation y_J for the restriction of a map y to the interval J .

Proposition 2.2. The admissible P-interpolant $\bar{x}$ with arc length parametrization $\bar{x}$, knots $p_i = \bar{x}(\bar{s}_i)$ ($i = 0, \dots, m$), and length $\bar{s}$, is extremal if and only if the conditions

$$(2.10) \quad \begin{aligned} (i) \quad & \bar{x} \in C^2[0, \bar{s}], \quad \ddot{\bar{x}}(s) = 0 \quad \text{for } 0 \leq s \leq \bar{s}_0 \quad \text{and} \quad \bar{s}_m \leq s \leq \bar{s} \\ (ii) \quad & (2\ddot{\bar{x}} + 3\kappa_{\bar{x}}^2)(s) = c_i \in \mathbb{R}^2 \quad \text{for } s \in (\bar{s}_{i-1}, \bar{s}_i), \quad i = 1, \dots, m \end{aligned}$$

hold with $\bar{x}|_{(\bar{s}_{i-1}, \bar{s}_i)} \in C^\infty(\bar{s}_{i-1}, \bar{s}_i)$ ($i = 1, \dots, m$).

Remark 2.1. Throughout the paper we use the same symbol to denote regularity classes for both scalar and vector functions.

Proof: The implication (2.10) $\Rightarrow$ (2.9) is routine and follows upon decomposing $[0, \bar{s}]$ into subintervals determined by the $\bar{s}_i$, dot-multiplying (2.10ii) by $\dot{\bar{z}}$, integrating by parts and summing; the continuity of $\ddot{\bar{x}}\dot{\bar{z}}$, the equations $\ddot{\bar{z}}(\bar{s}_i) = 0$ ($i = 0, \dots, m$) and the equations of (2.10) easily yield $U'(\bar{x})[\bar{z}] = 0$.

Conversely, if (2.9) holds then, selecting $\bar{z} \in C^\infty[0, \bar{s}]$ with support in $(\bar{s}_i, \bar{s}_{i+1})$, i fixed, we have

$$\int_{\bar{s}_i}^{\bar{s}_{i+1}} (2\ddot{\bar{x}} + F)\ddot{\bar{z}} = 0,$$

where $F = 3\kappa_{\bar{x}}^2$. By elementary distribution theory, $(2\ddot{\bar{x}} + F)|_{(\bar{s}_i, \bar{s}_{i+1})}$ is in $C^\infty(\bar{s}_i, \bar{s}_{i+1})$ and

$$D^2(2\ddot{\bar{x}} + F)|_{(\bar{s}_i, \bar{s}_{i+1})} = 0.$$

It follows that

$$(2\ddot{\bar{x}} + 3\kappa_{\bar{x}}^2)|_{(\bar{s}_i, \bar{s}_{i+1})} = c_i$$

and, recursively, $\bar{x}|_{(\bar{s}_i, \bar{s}_{i+1})} \in C^\infty(\bar{s}_i, \bar{s}_{i+1})$. To prove the continuity of $\ddot{\bar{x}}$ at an interior knot $\bar{s}_i$, select u in $C^\infty[0, \bar{s}]$ with support in $(\bar{s}_{i-1}, \bar{s}_{i+1})$ satisfying $u(\bar{s}_i) = 0$.

$u'(\bar{s}_i) = 1$ and put $z = (u, 0)$. Then, from (2.9) and integration by parts,

$$0 = \int_{\bar{s}_{i-1}}^{\bar{s}_{i+1}} \{ (2\ddot{x} + 3\kappa_x^2 \dot{x}) \dot{z} \} + 2(1, 0) (\ddot{x}(\bar{s}_i + 0) - \ddot{x}(\bar{s}_i - 0)) .$$

Since the first term equals

$$c_i(z(\bar{s}_i) - z(\bar{s}_{i-1})) + c_{i+1}(z(\bar{s}_{i+1}) - z(\bar{s}_i)) ,$$

which is clearly zero, we conclude that $(\bar{x}^1)_{(\bar{s}_0, \bar{s}_m)}$ is in $C^2(\bar{s}_0, \bar{s}_m)$. A similar argument works for $\bar{x}^2$.

If $\bar{s}_i$ is either $\bar{s}_0 = 0$ or $\bar{s}_m = \bar{s}$, jumps are replaced by one-sided limits and one concludes $\ddot{x}(\bar{s}_0 + 0) = \ddot{x}(\bar{s}_m - 0) = 0$. Assume now $\bar{s}_m < \bar{s}$. One argues as above that

$$(2\ddot{x} + 3\kappa_x^2 \dot{x})_{(\bar{s}_m, \bar{s})} = c_m ,$$

and that $\ddot{x}$ is continuous at $\bar{s}_m$. We show that $c_m = 0$. Indeed, select $u \in C^\infty[0, \bar{s}]$, with $u \equiv 0$ for $0 \leq s \leq \bar{s}_m$, satisfying $u(\bar{s}) = 1$, $\dot{u}(\bar{s}) = 0$, and put $z = (u, 0)$. Then, from (2.9) and integration by parts over $[\bar{s}_m, \bar{s}]$,

$$0 = c_m(z(\bar{s}) - z(\bar{s}_m)) = c_m^1$$

and a similar result holds for c_m^2 . Thus $c_m = 0$. By (2.15i) of Proposition 2.4 to follow, we conclude that $\kappa_x^2(s) = 0$ for $\bar{s}_m < s < \bar{s}$. In particular, $\ddot{x}(s) = 0$ for $\bar{s}_m < s < \bar{s}$, and, by continuity, for $\bar{s}_m \leq s \leq \bar{s}$. A similar proof holds if $0 < \bar{s}_0$. This completes the proof of the proposition.

We introduce the following notation for the jump of the third derivative of the extremal $\bar{x}$ at the knot $\bar{s}_i$

$$(2.11) \quad \Delta_i \ddot{\bar{x}} = \ddot{\bar{x}}(\bar{s}_i + 0) - \ddot{\bar{x}}(\bar{s}_i - 0), \quad i = 0, 1, \dots, m .$$

If $\ddot{\bar{x}}(\bar{s}_0 - 0)$ and/or $\ddot{\bar{x}}(\bar{s}_m + 0)$ are not defined, they are to be replaced by 0.

The following proposition gives expressions for the extremal value $U(x)$ which do not involve quadratures.

Corollary 2.1. If $\bar{x}$ is the extremal P-interpolant of Proposition 2.2, then

$$(2.12i) \quad U(\bar{x}) = \sum_{i=0}^{m-1} c_i (p_{i+1} - p_i) ,$$

and

$$(2.12ii) \quad U(\bar{x}) = -2 \sum_{i=0}^m p_i \Delta_i \bar{x}$$

Proof. If we dot-multiply (2.10ii) by $\dot{\bar{x}}$ and integrate over $[0, \bar{s}]$, using integration by parts, we obtain

$$-2U(\bar{x}) + 3U(\bar{x}) = \sum_{i=0}^{m-1} c_i (p_{i+1} - p_i) ,$$

which is (2.12i). Now use (2.10ii) at $\bar{s}_i + 0$ and $\bar{s}_i - 0$ and subtract to obtain

$2\Delta_i \bar{x} = c_i - c_{i-1}$, which holds for $i = 0, 1, \dots, m$ if we define $c_{-1} = c_m = 0$. By (2.12i) we have

$$U(\bar{x}) = \sum_{i=0}^m c_i (p_{i+1} - p_i) = - \sum_{i=0}^m p_i (c_i - c_{i-1}) = -2 \sum_{i=0}^m p_i \Delta_i \bar{x} ,$$

so (2.12ii) is also proved.

Remark 2.2. Since $U(x) = 0$ only if x is linear, it follows from (2.12ii) that an extremal P-interplant $\bar{x}$ that is not linear must have a discontinuity of $\bar{x}$ at some of the knots (or else, $\bar{x}(p_0 + 0) \neq 0$ or $\bar{x}(p_m - 0) \neq 0$).

From an extremal x , as characterized in Proposition 2.2, one can obtain infinitely many other extremals by shifting the terminals $x(0)$ and $x(1)$ along the rays that are tangent to x at p_0 and p_m . The value of $U(x)$ is not changed by these variations. We wish to ignore these trivial portions of an extremal and will for this reason adopt the

following convention. If we speak of an extremal P-interpolant x with arc parametrization $\bar{x}$, knots $p_i = \bar{x}(\bar{s}_i)$ ($i = 0, \dots, m$) and length $\bar{s}$ then, unless stated otherwise, $\bar{s}_0 = 0$, $\bar{s}_m = \bar{s}$, and the terminals are $p_0 = \bar{x}(0) = 0$, $p_m = \bar{x}(\bar{s}_m) = \bar{x}(\bar{s})$.

For some applications one wishes to constrain the p-interpolants further by prescribing the length $\bar{s}$ of the arc between the terminals. Let the class of these P-interpolants be called length-prescribed (they differ from the "length-constrained" interpolants of [3]). The next definition deals with the extremals for U in this class.

Definition 2.2. The admissible P-interpolant x , with arc length parametrization $\bar{x} = x \circ s_x^{-1}$, knots $p_i = \bar{x}(\bar{s}_i)$ ($i = 0, \dots, m$), length $\bar{s} = s_x(1)$, is a length-prescribed extremal P-interpolant if

$$(2.13i) \quad U'(x)[z] + \lambda S'(x)[z] = 0$$

for any $z \in H_2(I)$ for which $z \circ s_x^{-1}(\bar{s}_i) = 0$ ($i = 0, \dots, m$) and $\lambda \in \mathbb{R}$ determined so that

$$(2.13ii) \quad S(x) := \int_I |\dot{x}| = \bar{s}.$$

For these extremals we have a characterization similar to that of Proposition 2.2; note that $S'(x)$ is given by

$$S'(x)[y] = \int_0^{\bar{s}} \frac{\bar{x} \cdot \dot{\bar{y}}}{\bar{x} \cdot \dot{\bar{x}}}, \text{ for all } y \in H_2(I); \bar{x} = x \circ s_x^{-1}, \bar{y} = y \circ s_x^{-1}.$$

Proposition 2.3. The admissible P-interpolant x with arc length parametrization $\bar{x}$, knots $p_i = \bar{x}(\bar{s}_i)$ ($i = 0, 1, \dots, m$) and length $\bar{s} = \bar{s}_m$ is a length-prescribed extremal if and only if

$$(2.14) \quad \begin{aligned} (i) & \quad \bar{x} \in C^2[0, \bar{s}], \quad \ddot{\bar{x}}(0) = 0, \quad \bar{x}(\bar{s}) = 0, \\ (ii) & \quad (2\ddot{\bar{x}} + 3\kappa_{\bar{x}}^2 \dot{\bar{x}} - \lambda \dot{\bar{x}})(\bar{s}_{i-1}, \bar{s}_i) = c_i \in \mathbb{R}^2, \quad i = 1, \dots, m, \\ (iii) & \quad \lambda \bar{s} = U(\bar{x}) - \sum_{i=1}^m c_i (p_i - p_{i-1}) = U(\bar{x}) + 2 \sum_{i=0}^m p_i \Delta_i \ddot{\bar{x}}, \end{aligned}$$

hold with $\bar{x}_{(\bar{s}_{i-1}, \bar{s}_i)} \in C^{\infty}(\bar{s}_{i-1}, \bar{s}_i)$ ($i = 1, \dots, m$).

Proof. To prove the implication (2.14i, ii) $\Rightarrow$ (2.13i) one proceeds as in the first part of the proof of Proposition 2.2. If, next, (2.14ii) is dot-multiplied by $\frac{1}{x}$ and integrated over $[0, \bar{s}]$ one obtains, using integration by parts,

$$-2U(\bar{x}) + 3U(\bar{x}) - \lambda \bar{s} = \sum_{i=1}^m c_i (p_i - p_{i-1}) = -2 \sum_{i=0}^m p_i \Delta_i \bar{x}$$

which give (2.14iii), which is seen to be equivalent to (2.13ii).

Conversely, if (2.13i) holds then selecting $z \in C^{\infty}[0, \bar{s}]$ with support in $[\bar{s}_i, \bar{s}_{i+1}]$, i fixed, we have

$$\int_{\bar{s}_i}^{\bar{s}_{i+1}} (2\bar{x} + F - \lambda \bar{x}) \bar{z} = 0$$

where F is as in the proof of Proposition 2.2. It follows that

$$(2\bar{x} + 3\kappa_x^2 \bar{x} - \lambda \bar{x})_{(\bar{s}_i, \bar{s}_{i+1})} = c_i.$$

The regularity properties of x are proved as before. The argument in the first part of the proof shows, that given (2.14i, ii), then (2.14iii) and (2.13ii) are equivalent.

Remark 2.3. For any configuration $P = \{p_0, p_1, \dots, p_m\}$, there exists an extremal P -interpolant satisfying (2.14) with $\lambda \in \mathbb{R}$. Indeed, if

$$L_0 = \sum_{i=0}^{m-1} |p_{i+1} - p_i|,$$

then the length-prescribed extremal x , which minimizes $\int_0^L \kappa_x^2$ among all admissible P -interpolants with length equal to $L > L_0$ is guaranteed to exist [6] and satisfies (2.13i) [cf. Appendix].

Remark 2.4. Curves $\bar{x} : \mathbb{R} \rightarrow \mathbb{R}^2$, satisfying the equation

$$2\ddot{\bar{x}} + 3\kappa_x^2 \dot{\bar{x}} - \lambda \ddot{\bar{x}} = c$$

are called elastica (cf. [8]), more specifically inflexional elastica if the curve has inflection points (which is the case if and only if $\lambda^2 \leq c^2$). Curves for which $\lambda = 0$, the case of primary interest in this paper, will be referred to as simple elastica. Geometrically, simple elastica are characterized by the property that the angular variation between consecutive inflection points is exactly π (for all inflexional elastica the angular variation is $\geq \pi$). A smooth oriented curve in $\mathbb{R}^2$ with continuous curvature which consists of finitely many subarcs of the simple elastica and has (possibly) discontinuities of the curvature derivative at the interpolation points $p_1, \dots, p_{m-1}$ only is called an interpolating elastica (cf. [8]).

The next proposition deals with implications and equivalences of (2.14ii). It should be observed that these results apply to Equation (2.10ii) as well, since the latter is the special case of (2.14ii) with $\lambda = 0$.

Proposition 2.4. Condition (2.14ii) implies each of the following four conditions on

$(\bar{s}_{i-1}, \bar{s}_i)$ ($i = 1, \dots, m$):

- (2.15)
- (i) $\kappa_x^2 = c_1 \dot{\bar{x}} + \lambda$,
 - (ii) $\dot{\kappa}_x = \frac{1}{2}[\bar{x}, c_1]$,
 - (iii) $\kappa_x = \frac{1}{2}[\bar{x}, c_1] + \gamma_i, \gamma_i \in \mathbb{R}$,
 - (iv) $\ddot{\kappa}_x + \frac{1}{2}\kappa_x^3 - (\lambda/2)\kappa_x = 0$.

Moreover, (2.15i) also implies (2.14ii).

Proof: If (2.14ii) is dot-multiplied by $\dot{\bar{x}}$, one obtains, since $|\dot{\bar{x}}| = 1, |\ddot{\bar{x}}|^2 = \kappa_x^2, \ddot{\bar{x}}\ddot{\bar{x}} = 0, \dot{\bar{x}}\ddot{\bar{x}} + |\ddot{\bar{x}}|^2 = 0$:

$$3\kappa_x^2 + 2\ddot{\bar{x}}\ddot{\bar{x}} = \kappa_x^2 + 2|\ddot{\bar{x}}|^2 + 2\dot{\bar{x}}\ddot{\bar{x}} = \kappa_x^2 = c_1 \dot{\bar{x}} + \lambda,$$

hence (2.15i). Differentiating (2.15i) we obtain

$$\kappa_x \dot{\kappa}_x = \frac{1}{2} c_1 \ddot{x}.$$

If $\kappa_x(s) \neq 0$ for some s , then $|\kappa_x(s)| = |\ddot{x}(s)| > 0$ and $\ddot{x}(s)/\kappa_x(s)$ is the unit vector $(-\dot{x}^2(s), \dot{x}^1(s))$. Therefore, $\dot{\kappa}_x(s) = \frac{1}{2}[\dot{x}(s), c_1]$ and (2.15ii) holds in this case. If $\kappa_x(s) = 0$ and s is a limit of s_n such that $\kappa_x(s_n) \neq 0$, then continuity of $\dot{x}$ and $\dot{\kappa}_x$ together with the previous argument give the same equation. On a fixed interval $(\bar{s}_{i-1}, \bar{s}_i)$, let $\Gamma_i = \{s : \kappa_x(s) \neq 0\}$ and let (α, β) be any subinterval in the decomposition of $(\bar{s}_{i-1}, \bar{s}_i) \setminus \bar{\Gamma}_i$. We must show that (2.15ii) holds on (α, β) . Now κ_x and $\dot{\kappa}_x$ are zero on (α, β) . By the continuity of $\dot{\kappa}_x$ on $(\bar{s}_{i-1}, \bar{s}_i)$ it follows that $\dot{\kappa}_x(\beta) = 0 = [\dot{x}(\beta), c_1]$. Moreover, since $\kappa_x = 0$ on (α, β) ,

$$(2.16) \quad \ddot{x}(s) = as + b, \quad s \in (\alpha, \beta), \quad a, b \in \mathbb{R}^2.$$

Thus, from (2.16),

$$[\ddot{x}(s), c_1] = [a, c_1], \quad s \in (\alpha, \beta).$$

In particular, $[a, c_1] = 0$ and thus (2.15ii) holds on (α, β) . Next, (2.15iii) follows from (2.15ii) by integration. To show that (2.15iv) holds, let $\kappa_x(s) \neq 0$; then $\ddot{x}(s) = \kappa_x(s)(-\dot{x}^2(s), \dot{x}^1(s))$ and one obtains upon differentiating (2.15ii),

$$\ddot{\kappa}_x(s) = \frac{1}{2}[\ddot{x}(s), c_1] = -\frac{1}{2}\kappa_x(s)c_1\dot{x}(s) = \frac{1}{2}(\lambda\kappa_x(s) - \kappa_x^3(s)).$$

Thus, (2.15iv) holds at every point s for which $\kappa_x(s) \neq 0$. The case when s is a limit point of s_n for which $\kappa_x(s_n) \neq 0$ then follows immediately. The case when s is not such a limit point is of course trivial.

To show that (2.15i) implies (2.14ii), assume $\ddot{x}(s) \neq 0$ for some $s \in (\bar{s}_i, \bar{s}_{i+1})$. Since $\dot{x}(s)$ and $\ddot{x}(s)/|\ddot{x}(s)|$ are orthogonal unit vectors in $\mathbb{R}^2$, we have, using (2.15i) and the fact that $\dot{x}(s)\ddot{x}(s) + |\ddot{x}(s)|^2 = 0$,

The authors thank S. D. Fisher for a helpful suggestion here.

$$\begin{aligned}
c_1 &= (c_1 \ddot{x}(s)) \dot{x}(s) + (c_1 \ddot{x}(s)) \ddot{x}(s) / |\ddot{x}(s)|^2 \\
&= (\kappa_x^2(s) - \lambda) \dot{x}(s) + \frac{d(\kappa_x^2(s))}{ds} \ddot{x}(s) / |\ddot{x}(s)|^2 \\
&= (3\kappa_x^2(s) - \lambda) \dot{x}(s) + (2\ddot{x}(s) \dot{x}(s)) \dot{x}(s) + (2\ddot{x}(s) \ddot{x}(s)) \ddot{x}(s) / |\ddot{x}(s)|^2 \\
&= (3\kappa_x^2(s) - \lambda) \dot{x}(s) + 2\ddot{x}(s),
\end{aligned}$$

which is just (2.14ii). If s is a limit point of s_n for which $\ddot{x}(s_n) \neq 0$, then the same equation holds by continuity. If s is not such a limit point, then (2.14ii) reduces to $c_1 = -\lambda \dot{x}(s)$, which is clearly obtained by dot-multiplication of (2.15i) by $\dot{x}(s)$. This concludes the proof of Proposition 2.4.

At the end of this section we mention still another constraining condition for extremal interpolants. It consists in fixing the angle that the interpolant x makes with a fixed line at the terminal knot $p_0 = \bar{x}(\bar{s}_0)$ and/or $p_m = \bar{x}(\bar{s}_m)$. Thus, the condition is

$$(2.17) \quad \dot{\bar{x}}(\bar{s}_0) = e_0 \text{ and/or } \dot{\bar{x}}(\bar{s}_m) = e_m$$

where e_0, e_m are unit vectors in $\mathbb{R}^2$. We refer to these extremals as angle-constrained.

If x is an admissible P-interpolant with knots $p_i = \bar{x}(\bar{s}_i)$ which satisfies the constraint (2.17) then any other P-interpolant in a sufficiently small H_2 -neighborhood of x , satisfying the same condition, is of the form $x + \varepsilon z$ where $z \in H_2$, $z \circ s_x^{-1}(\bar{s}_i) = 0$ ($i = 0, 1, \dots, m$), $\dot{z} \circ s_x^{-1}(\bar{s}_0) = 0$ and/or $\dot{z} \circ s_x^{-1}(\bar{s}_m) = 0$. It is easily seen that if $\bar{x}$ is an extremal P-interpolant with the added constraint $\dot{\bar{x}}(\bar{s}_0) = e_0$ then the "free boundary" condition $\ddot{\bar{x}}(\bar{s}_0) = 0$ of (2.10i) is replaced by $\dot{\bar{x}}(\bar{s}_0) = e_0$. Similarly if $\dot{\bar{x}}(\bar{s}_m) = e_m$ is a constraint then this condition replaces $\ddot{\bar{x}}(\bar{s}_m) = 0$. There is no other change in the conditions of Proposition 2.2.

§3. Normal Representation of Extremals

In this section the dependent variable is an angle. Let $T(=T^1)$ denote the 1-dimensional torus. $\phi \in T$ is represented by a real number (also denoted as) ϕ , one of the set $\phi + 2k\pi$ ($k = 0, \pm 1, \pm 2, \dots$). A continuous function $\theta : (0, \bar{s}) \rightarrow T$ is represented by a continuous function (also denoted as) $\theta : (0, \bar{s}) \rightarrow \mathbb{R}$, one of the set $\theta + 2k\pi$. The derivative $\dot{\theta}$ is always a unique function $(0, \bar{s}) \rightarrow \mathbb{R}$. Let $\dot{H}_1(0, \bar{s})$ denote the class of absolutely continuous functions $\theta : (0, \bar{s}) \rightarrow \mathbb{R}$ for which $\dot{\theta} \in L_2(0, \bar{s})$. Then the function $x = x_\theta : (0, \bar{s}) \rightarrow \mathbb{R}^2$, defined by

$$(3.1) \quad x_\theta^1(s) = \int_0^s \cos \theta, \quad x_\theta^2(s) = \int_0^s \sin \theta$$

is in $H_2(0, \bar{s})$ and represents an oriented curve C in the x^1x^2 -plane, parametrized with respect to arc length, $x(0) = 0$, $\dot{x}_2/\dot{x}_1 = \tan \theta$, $\theta(s)$ is the angle which the curve C makes with the x^1 -axis at arc length s , and $\dot{\theta}_s$ is the curvature of C at s . Conversely, given an oriented curve C with cartesian representation $x \in H_2(0, \bar{s})$, parametrized with respect to arc length, there is a unique function $\theta_x \in \dot{H}_1(0, \bar{s})$ such that

$$x^1(s) = x^1(0) + \int_0^s \cos \theta_x, \quad x^2(s) = x^2(0) + \int_0^s \sin \theta_x.$$

We say that θ_x is the normal representation (n.r.) of C .

Curves C that differ by a translation have the same normal representation. If θ_x is the n.r. of C then $\theta_x + \text{const.}$ is the n.r. of a curve obtained from C by a rotation, and $-\theta_x$ is the n.r. of a curve obtained from C by a reflection at the x^1 -axis. In a geometric setting we may identify curves C which differ only by a congruence, and each congruence class is represented by a single function θ with the specification $\theta(0) = 0$, $\dot{\theta}(0) > 0$ (or $\ddot{\theta}(0) > 0$ if $\dot{\theta}(0) = 0$).

In many cases it will be convenient to characterize extremal P-interpolants by their normal representation. For this purpose we replace Propositions 2.2 and 2.3 by the following propositions whose proofs we omit.

Proposition 3.1. The function $\bar{\theta} \in \dot{H}_1(0, \bar{s})$ is the normal representation of an extremal P-interpolant with knots p_i ($i = 0, 1, \dots, m$) at $0 = \bar{s}_0 < \bar{s}_1 < \dots < \bar{s}_m$ and length $\bar{s} = \bar{s}_m$, if and only if the conditions

$$\begin{aligned}
 (i) \quad & \bar{\theta} \in C^1[0, \bar{s}], \quad \bar{\theta}(0), \quad \dot{\bar{\theta}}(\bar{s}) = 0, \\
 (ii) \quad & \dot{\bar{\theta}}^2(s) = c_i^1 \cos \bar{\theta}(s) + c_i^2 \sin \bar{\theta}(s) \quad \text{for } s \in (\bar{s}_{i-1}, \bar{s}_i), \\
 (iii) \quad & \int_0^{\bar{s}_i} \cos \bar{\theta}(s) ds = p_i^1, \quad \int_0^{\bar{s}_i} \sin \bar{\theta}(s) ds = p_i^2, \quad i = 1, \dots, m,
 \end{aligned}
 \tag{3.3}$$

$$c_i \in \mathbb{R}^2, \quad i = 1, \dots, m,$$

hold with $\bar{\theta}|_{(\bar{s}_{i-1}, \bar{s}_i)} \in C^\infty(\bar{s}_{i-1}, \bar{s}_i)$ ($i = 1, \dots, m$).

Proposition 3.2. The function $\bar{\theta} \in \dot{H}_1(0, \bar{s})$ is the normal representation of a length-prescribed extremal P-interpolant with knots p_i ($i = 0, 1, \dots, m$) at $0 = \bar{s}_0 < \bar{s}_2 < \dots < \bar{s}_m$ and length $\bar{s} = \bar{s}_m$ if and only if the conditions

$$\begin{aligned}
 (i) \quad & \bar{\theta} \in C^1[0, \bar{s}], \quad \bar{\theta}(0) = \dot{\bar{\theta}}(\bar{s}) = 0, \\
 (ii) \quad & \dot{\bar{\theta}}^2(s) = c_i^1 \cos \bar{\theta}(s) + c_i^2 \sin \bar{\theta}(s) + \lambda \quad \text{for } s \in (\bar{s}_{i-1}, \bar{s}_i), \\
 (iii) \quad & \int_0^{\bar{s}_i} \cos \bar{\theta}(s) ds = p_i^1, \quad \int_0^{\bar{s}_i} \sin \bar{\theta}(s) ds = p_i^2, \quad i = 1, \dots, m, \\
 (iv) \quad & \lambda \bar{s} = \int_0^{\bar{s}} \dot{\bar{\theta}}^2 - \sum_{i=1}^m c_i (p_i^1 - p_{i-1}^1),
 \end{aligned}
 \tag{3.6}$$

hold with $\bar{\theta}|_{(\bar{s}_{i-1}, \bar{s}_i)} \in C^\infty(\bar{s}_{i-1}, \bar{s}_i)$ ($i = 1, \dots, m$).

Remark 3.1. The conditions $\dot{\bar{\theta}}(\bar{s}_i - 0) = \dot{\bar{\theta}}(\bar{s}_i + 0)$ ($i = 1, \dots, m-1$), $\dot{\bar{\theta}}(0) = 0$, and $\dot{\bar{\theta}}(\bar{s}_m) = 0$ result in $m + 1$ conditions on the vector constants $c_1, \dots, c_m$. Both in Proposition 3.1 and 3.2 we have

$$(i) \quad (c_{i+1}^1 - c_i^1) \cos \bar{\theta}(\bar{s}_i) + (c_{i+1}^2 - c_i^2) \sin \bar{\theta}(\bar{s}_i) = 0, \\ i = 1, \dots, m-1$$

(3.7)

$$(ii) \quad c_1^1 \cos \bar{\theta}(0) + c_1^2 \sin \bar{\theta}(0) + \lambda = 0,$$

$$(iii) \quad c_m^1 \cos \bar{\theta}(\bar{s}_m) + c_m^2 \sin \bar{\theta}(\bar{s}_m) + \lambda = 0,$$

where λ is to be taken as 0 in the case of Proposition 3.1.

For use in Sections 4 and 5 we state and prove

Proposition 3.3. The function $\bar{\theta} \in \tilde{H}_1(0, \bar{s})$ is the normal representation of an extremal P-interpolant with knots p_i ($i = 0, \dots, m$) at $0 = \bar{s}_0 < \dots < \bar{s}_m$ and length $\bar{s} = \bar{s}_m$ if and only if the conditions

$$(i) \quad 2\ddot{\bar{\theta}}(s) + c_1^1 \sin \bar{\theta}(s) - c_1^2 \cos \bar{\theta}(s) = 0 \text{ for} \\ s \in (\bar{s}_{i-1}, \bar{s}_i), \quad c_i \in \mathbb{R}^2. \\ (3.8)$$

$$(ii) \quad (c_{i+1}^1 - c_i^1) \cos \bar{\theta}(\bar{s}_i) + (c_{i+1}^2 - c_i^2) \sin \bar{\theta}(\bar{s}_i) = 0, \\ i = 1, \dots, m,$$

$$(iii) \quad \dot{\bar{\theta}}(0) = 0, \dot{\bar{\theta}}(\bar{s}) = 0,$$

hold with $\bar{\theta}|_{(\bar{s}_{i-1}, \bar{s}_i)} \in C^m(\bar{s}_{i-1}, \bar{s}_i)$ ($i = 1, \dots, m$) and $c_{m+1} = 0$.

Proof. The forward implication follows directly from Propositions 2.3 and 2.4. The converse implication follows upon multiplying (3.8i) by $\dot{\bar{\theta}}(s)$ and integrating; if the integrated equation is evaluated at $s = \bar{s}_m$ and (3.8ii, iii) is used, the constant of integration is seen to be 0. Thus, (3.8) implies

$$\ddot{\bar{\theta}}^2(s) - c_1^1 \cos \bar{\theta}(s) - c_1^2 \sin \bar{\theta}(s) = 0 \quad \text{for } s \in (\bar{s}_{i-1}, \bar{s}_i) ,$$

$$\bar{\theta} \in C^1[0, \bar{s}], \quad i = 1, \dots, m ,$$

and the result now follows from Proposition 3.1.

§4. Manifold of Extremals

As noted in the previous section, we may consider equivalence classes of curves differing by a congruence, with representer satisfying $x(0) = 0$, $\theta(0) = 0$, $\dot{\theta}(0) > 0$; or $\dot{\theta}(0) < 0$ if $\dot{\theta}(0) \neq 0$.

Definition 4.1. The extremal interpolant E is proper if:

- (i) E has nonzero curvature κ_i at each internal interpolation node s_i , $i = 1, \dots, m-1$.
- (ii) Each internal interpolation node is a genuine knot, i.e., there is a discontinuity $\Delta_i \dot{\kappa} \neq 0$ in the derivative of the curvature at s_i , $i = 1, \dots, m-1$.

Definition 4.2. The m -tuple $(k_1, \dots, k_m)$ of nonnegative integers is the mode of E if there are k_j inflection points strictly between the $(j-1)$ th and j th interpolation nodes; here, an inflection point denotes a point of zero curvature and we note that $\dot{\theta}$ must change sign. For a fixed mode $(k_1, \dots, k_m)$, $\bar{E} = \bar{E}_{(k_1, \dots, k_m)}$ will denote the class of proper $(m+1)$ -extremal interpolants E in the mode $(k_1, \dots, k_m)$.

Proposition 4.1. $\bar{E}$ is a (finite-dimensional) metric space under the metric

$$(4.2) \quad d(E_1, E_2) = \max_{0 \leq t \leq 1} \left| \frac{d}{dt} x_1(s_{E_1} t) - \frac{d}{dt} x_2(s_{E_2} t) \right|.$$

Here s_{E_j} and x_j represent the lengths and Cartesian representations (parametrized w.r.t. arc length) of E_j , $j = 1, 2$, respectively, where $x_j(0) = 0$, $j = 1, 2$.

Remark 4.1. We omit the routine proof. We observe that $\bar{E}$ is not complete. Indeed, if each $\bar{E}_{(k_1, \dots, k_m)}$ (m fixed) is embedded in the space of all extremal interpolants, with metric described by (4.2), then the boundary $\partial \bar{E}$ of $\bar{E}$ may contain an extremal interpolant

which is not proper. We also observe that if E_1 is close to E_2 , then the configuration interpolated by E_1 is close to that interpolated by E_2 . This would not be true if E_1, E_2 were not restricted to a class $E_{(k_1, \dots, k_m)}$ (see Example 4.2 at the end of this section).

Now let $\theta_E \in C^1[0, s_E]$ be the normal representation of some $E \in E$. If $0 = s_0 < s_1 < \dots < s_m = s_E$ are the interpolation nodes of E , put $\theta_E(s_i) = \alpha_i$, $i = 0, \dots, m$ and $\alpha_E = (\alpha_1, \dots, \alpha_{m-1}) \in T^{m-1}$. Setting $c_i = -2\mu_i$ we have from Propositions 3.1 and 3.3 the existence of a unique multiplier $\mu_E = (\mu_1, \dots, \mu_m) \in (\mathbb{R}^2)^m$ such that, for $k = 1, \dots, m$,

$$(4.3) \quad \begin{aligned} (i) \quad & \frac{1}{2} \dot{\theta}_E^2(s) + \mu_k^1 \cos \theta_E(s) + \mu_k^2 \sin \theta_E(s) = 0, \quad s_{k-1} < s < s_k, \\ (ii) \quad & \ddot{\theta}_E(s) - \mu_k^1 \sin \theta_E(s) + \mu_k^2 \cos \theta_E(s) = 0, \quad s_{k-1} < s < s_k; \\ \text{and,} \\ (iii) \quad & \dot{\theta}_E(0) = 0, \quad \dot{\theta}_E(s_E) = 0. \end{aligned}$$

By introducing the more convenient notation

$$B_k = (A_k, \beta_k), \quad A_k > 0, \quad \beta_k \in T^1, \quad k = 1, \dots, m,$$

where

$$\mu_k^1 = -A_k \sin \beta_k, \quad \mu_k^2 = A_k \cos \beta_k,$$

we may rewrite (4.3) in terms of the multipliers B_k for $k = 1, \dots, m$:

$$(4.4) \quad \begin{aligned} (i) \quad & \frac{1}{2} \dot{\theta}_E^2(s) + A_k \sin(\theta_E(s) - \beta_k) = 0, \quad s_{k-1} < s < s_k, \\ (ii) \quad & \ddot{\theta}_E(s) + A_k \cos(\theta_E(s) - \beta_k) = 0, \quad s_{k-1} < s < s_k; \\ \text{and,} \\ (iii) \quad & \dot{\theta}_E(0) = 0, \quad \dot{\theta}_E(s_E) = 0. \end{aligned}$$

Since $\theta = 0$ is not a proper extremal we must have $\ddot{\theta}_E(0+) \neq 0$. We consider it as part of the definition of E that

$$(4.4iv) \quad \ddot{\theta}_E(0+) > 0$$

for all $E \in \bar{E}$. Geometrically speaking, $\bar{E}$ contains only extremals which turn counterclockwise near the initial point.

Proposition 4.2. $\theta_E \in C^1[0, s_E]$ is the n.r. of an extremal $E \in \bar{E}_{(k_1, \dots, k_m)}$ with interpolation nodes at $0 = s_0 < s_1 < \dots < s_m = s_E$ if and only if

- A. θ_E satisfies (4.4i - iv) for some $A_k > 0$, $\beta_k \in T^1$.
- B. $\sin(\beta_i - \beta_{i+1}) \neq 0$
- C. $\ddot{\theta}_E(0+) > 0$, $\text{sgn } \dot{\theta}_E(s_i) = (-1)^{k_1 + \dots + k_i}$ $i = 1, \dots, m-1$.

Proof: Since $\theta_E \in C^1$ we have, by (4.4i)

$$(4.5i) \quad A_i \sin(\alpha_i - \beta_i) - A_{i+1} \sin(\alpha_i - \beta_{i+1}) = 0, \quad i = 1, \dots, m-1,$$

where $\alpha_i = \theta_E(s_i)$. $\ddot{\theta}_E(s_i - 0) = \ddot{\theta}_E(s_i + 0)$ if and only if

$$(4.5ii) \quad A_i \cos(\alpha_i - \beta_i) - A_{i+1} \cos(\alpha_i - \beta_{i+1}) = 0.$$

The two equations (4.5i, ii) are equivalent to (4.5i) and $\sin(\beta_i - \beta_{i+1}) = 0$. Thus, the above condition B. is equivalent to the condition that each interpolation node of E be a genuine knot (see 4.1ii). There are k_i inflection points of E between the $(i-1)$ th and i -th interpolation nodes if and only if $\dot{\theta}_E$ changes sign k_i times between s_{i-1} and s_i , i.e.

$$(4.6) \quad \text{sgn } \kappa_{i-1} \cdot \text{sgn } \kappa_i = (-1)^{k_i}, \quad \kappa_i = \dot{\theta}_E(s_i).$$

Since by (4.4iv) $\dot{\theta}_E(s) > 0$ for all sufficiently small s , (4.6) is equivalent to the above condition C. Condition C. also implies that $\kappa_i \neq 0$ for $i = 1, \dots, m-1$, thus condition (4.1i) is also satisfied.

Remark 4.2. Condition B. also implies

$$(4.7) \quad A_i - A_{i+1} \neq 0, \quad i = 1, \dots, m-1.$$

Indeed if $A_i - A_{i+1} = 0$ then by (4.5i) $\sin(\alpha_i - \beta_i) = \sin(\alpha_i - \beta_{i+1})$, hence $\beta_i = \beta_{i+1}$ or $\beta_i = \beta_{i+1} + \pi \pmod{2\pi}$, which contradicts B.

We now define

$$(4.8) \quad \begin{aligned} \tilde{R}_+^m &= \{A \in \mathbb{R}_+^m : A_i - A_{i+1} \neq 0, \quad i = 1, \dots, m-1\} \\ \tilde{T}^m &= \{\beta \in T^m : \sin(\beta_i - \beta_{i+1}) \neq 0, \quad i = 1, \dots, m-1\} \\ B = B_E &= (B_1, \dots, B_m), \quad \alpha = \alpha_E = (\alpha_1, \dots, \alpha_m). \end{aligned}$$

By Proposition 4.2 each extremal $E \in E_{(k_1, \dots, k_m)}$ determines a unique point $B_E \in \tilde{R}_+^m \times \tilde{T}^m$.

In the next three propositions we shall describe the mapping $E \rightarrow B_E$ via the composition of two mappings: the homeomorphism

$$(4.9i) \quad J : E \rightarrow (\alpha_E, B_E)$$

of E onto $J(E) \subset T^{m-1} \times (\tilde{R}_+^m \times \tilde{T}^m)$; and the projection

$$(4.9ii) \quad M : (\alpha, B) \rightarrow B$$

of $J(E)$ into $\tilde{R}_+^m \times \tilde{T}^m$, which is a local diffeomorphism. The composition $M \circ J$ is a global homeomorphism.

Proposition 4.3. The mapping J is a homeomorphism of $E_{(k_1, \dots, k_m)}$ onto its image.

Proof: The continuity of J follows directly from (4.2) and (4.4); note that $B_1, \dots, B_m$ can be expressed via (4.4) in terms of $\alpha_1, \kappa_1, \Delta_1 \dot{\kappa}$, thus also in terms of

$\theta_E = \arctan \dot{x}_{E,2} / \dot{x}_{E,1}$. Suppose now that $J(E) = (\alpha_E, B_E)$. We show that $E \in E$ is uniquely and continuously determined by its map (α_E, B_E) . By (4.4)

$$\sin(\theta_E(0) - \beta_1) = 0, \quad \cos(\theta_E(0) - \beta_1) < 0,$$

hence $\alpha_0 = \theta_E(0) = \beta_1 + \pi \pmod{2\pi}$. The restriction of θ_E to $[s_0, s_1]$ is now uniquely determined from

$$(i) \quad \dot{\theta}_E(s) = [-2A_1 \sin(\theta_E(s) - \beta_1)]^{\frac{1}{2}}$$

$$(4.10) \quad (ii) \quad \theta_E(0) = \alpha_0,$$

with s_1 uniquely determined from

$$(iii) \quad \theta_E(s_1) = \alpha_1, \quad \dot{\theta}_E(s) = 0 \text{ for } k_1 \text{ values of } s \text{ in } (0, s_1).$$

Indeed if there is an $s'_1 > s_1$ for which $\theta_E(s_1) = \theta_E(s'_1) = \alpha_1$ then $\dot{\theta}_E(\bar{s}) = 0$ for some $s_1 < \bar{s} < s'_1$, hence $\dot{\theta}_E(s) = 0$ for more than k_1 values of s in $(0, s'_1)$. This clearly leads to an inductive process; indeed if θ_E is defined on $[0, s_i]$, one obtains the restriction of θ_E to $[s_i, s_{i+1}]$ from the initial value problem defined by (4.4ii) with initial values $\theta_E(s_i)$ and $\dot{\theta}_E(s_i)$. s_{i+1} is uniquely determined from

$$\theta_E(s_{i+1}) = \alpha_{i+1}, \quad \dot{\theta}_E(s) = 0 \text{ for } k_{i+1} \text{ values of } s \text{ in } (s_i, s_{i+1}).$$

The process is terminated at $i = m-1$ by replacing the condition $\theta_E(s_{i+1}) = \alpha_{i+1}$ by $\dot{\theta}_E(s_m) = 0$. Since the continuity of J^{-1} is an easy consequence of (4.4) the proof is complete.

We determine now the image set

$$(4.11) \quad S = S_{(k_1, \dots, k_m)} = J(E_{(k_1, \dots, k_m)}).$$

The following are necessary conditions for $(\alpha, B) \in S$:

$$(i) \quad \sin(\alpha_i - \beta_i) < 0 \quad i = 1, \dots, m-1.$$

$$(4.12) \quad (ii) \quad \text{If } k_\ell = 0 \text{ for some } 2 \leq \ell \leq m-1 \text{ then } (-1)^{k_1 + \dots + k_{\ell-1}} \sin(\alpha_\ell - \alpha_{\ell-1}) > 0.$$

$$(iii) \quad A_i \sin(\alpha_i - \beta_i) = A_{i+1} \sin(\alpha_i - \beta_{i+1}), \quad i = 1, \dots, m-1.$$

Conditions (4.9i and iii) express that $\dot{\theta}_E^2$ is positive and continuous at $s_1, \dots, s_{m-1}$. If some $k_i = 0$ then there must be no inflection point between s_{i-1} and s_i , hence $\sin(\theta_E - \alpha_{i-1})$ does not change sign, or

$$\sin(\theta_E(s) - \alpha_{i-1}) \cdot \dot{\theta}_E(s_{i-1}) > 0 \text{ for } s_{i-1} < s \leq s_i.$$

Using C. of Proposition 4.2, we obtain (4.12ii).

We now show that conditions (4.12) characterize the image set S completely. We observe that (4.12i, ii) define an open set in the $(3m-1)$ -dimensional space $T^{m-1} \times (\tilde{R}_+^m \times \tilde{T}^m)$, while equations (4.9iii) single out a $2m$ -dimensional surface in the open set.

Proposition 4.4. The image set $J(E_{(k_1, \dots, k_m)})$ is

$$S_{(k_1, \dots, k_m)} = \{(\alpha, B) \in T^{m-1} \times (\tilde{R}_+^m \times \tilde{T}^m) : \text{conditions (4.12i, ii, iii) hold}\}.$$

Proof: We need to show that if (α, B) is such that (4.12) holds, then there are numbers

$$0 = s_0 < s_1 < \dots < s_m = s_E \text{ and a function } \theta_E \in C^1[0, s_E]$$

such that conditions A., C. of Proposition 4.2 are satisfied and moreover,

$$(4.13) \quad \theta_E(s_i) = \alpha_i, \quad i = 1, \dots, m-1$$

(condition B. follows from the definition of $\tilde{T}^m$). s_1 and the restriction of θ_E to $[s_0, s_1]$ are determined as in the proof of Proposition 4.3. Next, the restriction of θ_E to $[s_1, s_2]$ (s_2 as yet unknown) is determined from the initial value problem

$$(4.14) \quad \begin{aligned} \ddot{\theta}(s) + (-1)^{k_1} [-2\lambda_2 \sin(\theta(s) - \beta_2)]^{\frac{1}{2}} &= 0 \\ \theta(s_1) &= \alpha_1. \end{aligned}$$

The solution is the n.r. of a simple elastica with inflection points at equally spaced abscissas σ_k where $\theta(\sigma_k) = \beta_2$ or $\beta_2 + \pi \pmod{2\pi}$. By (4.12i and iii) we have

$$\sin(\alpha_1 - \beta_2) < 0, \quad \sin(\alpha_2 - \beta_2) < 0,$$

and, therefore, $\theta(s)$ attains the values α_1 and α_2 exactly once between any two consecutive α_k . Thus if there are $k_2 \geq 1$ inflection points between s_1 and s_2 there is exactly one s_2 for which $\theta(s_2) = \alpha_2$. If $k_2 = 0$ and, say $\dot{\theta}(s_1) > 0$, then by condition (4.12ii) $\sin(\alpha_2 - \alpha_1) > 0$, which implies that $\theta(s)$ attains the value α_2 for $s_2 > s_1$, with no inflection point between s_1 and s_2 . By the same arguments the values of $s_3, \dots, s_{m-1}$ and the restriction of θ_E to $[s_3, s_4], \dots, [s_{m-2}, s_{m-1}]$ are determined. $s_m = s_E$ and θ_E on $[s_{m-1}, s_m]$ are similarly obtained, except that the condition $\theta(s_m) = \alpha_m$ is replaced by $\dot{\theta}(s_m) = 0$. The obtained function θ_E is in $C^1[0, s_E]$ because of condition (4.12iii), and it satisfies (4.13) and conditions A. and C. of Proposition 4.2 by construction.

Proposition 4.5. The projection $M|_S$ is a local diffeomorphism onto an open subset $M_{(k_1, \dots, k_m)}$ of $\mathbb{R}_+^m \times \tilde{T}^m$. Thus S is a $2m$ -dimensional smooth (even analytic) manifold (S is not connected if $m \geq 1$). The composition map $M \circ J$ is a (global) homeomorphism of $E_{(k_1, \dots, k_m)}$ onto $M_{(k_1, \dots, k_m)}$.

Proof: Choose any $(\alpha^0, B^0) \in S$, $\alpha^0 = (\alpha_1^0, \dots, \alpha_m^0)$ and $B^0 = (A_1^0, \dots, A_m^0, \beta_1^0, \dots, \beta_m^0)$. Let U denote the open subset of $T^{m-1} \times (\mathbb{R}_+^m \times \tilde{T}^m)$ satisfying conditions (4.12i, ii). Define a mapping $\varphi : U \rightarrow \mathbb{R}^{m-1}$ by

$$(4.15) \quad \varphi_i(\alpha, B) = A_i \sin(\alpha_i - \beta_i) - A_{i+1} \sin(\alpha_i - \beta_{i+1}), \quad i = 1, \dots, m-1.$$

With this notation, $(\alpha, B) \in U$ is in S if and only if $\varphi_i(\alpha, B) = 0$ ($i = 1, \dots, m-1$). Now $\varphi_i(\alpha^0, B^0) = 0$ and the Jacobian $[\partial \varphi_i / \partial \alpha_j]$ is nonsingular at (α^0, B^0) , since it is a diagonal matrix with diagonal entries

$$(4.16) \quad \begin{aligned} \frac{\partial \varphi_i}{\partial \alpha_i}(\alpha^0, B^0) &= A_i^0 \cos(\alpha_i^0 - \beta_i^0) - A_{i+1}^0 \cos(\alpha_i^0 - \beta_{i+1}^0) \\ &= A_i^0 \neq 0. \end{aligned}$$

We conclude there is, for every neighborhood $U_0 \subset U$ of (α^0, B^0) a neighborhood N_0 of B^0 in $\mathbb{R}_+^m \times \tilde{T}^m$ and a C^1 -mapping α of N_0 such that $\alpha(B^0) = \alpha^0$ and $\varphi_i(\alpha(B), B) = 0$

for all $B \in N_0$. This proves that M is a local diffeomorphism.

By Proposition 4.3 the mapping J is a homeomorphism of $E_{(k_1, \dots, k_m)}$ onto S . If the composite map $M \circ J$ is not a homeomorphism, there must be $(\alpha, B), (\alpha', B)$ in S with $\alpha \neq \alpha'$, say $\alpha_1 \neq \alpha'_1 \pmod{2\pi}$. By (4.12iii) we have $A_1 \sin(\alpha_1 - \beta_1) - A_{i+1} \sin(\alpha_i - \beta_{i+1}) = 0$ and $A_1 \sin(\alpha'_1 - \beta_1) - A_{i+1} \sin(\alpha'_i - \beta_{i+1}) = 0$. These equations imply $\alpha'_i = \alpha_i + \pi \pmod{2\pi}$. But by (4.12i), $\sin(\alpha_1 - \beta_1) < 0$ and $\sin(\alpha'_1 - \beta_1) < 0$, which contradicts the previous conclusion. Thus Proposition 4.5 is completely proved.

Remark 4.3. It seems to be difficult to give an intrinsic characterization of the set $M_{(k_1, \dots, k_m)}$. Examples show that it does not coincide with $\mathbb{R}_+^m \times \mathbb{T}^m$. It certainly contains points $B = (\alpha, \beta)$ for each combination of the inequalities $A_1 \geq A_2, A_2 \geq A_3, \dots, A_{m-1} \geq A_m$. Thus, $M_{(k_1, \dots, k_m)}$ and $S_{(k_1, \dots, k_m)}$ have at least 2^{m-1} disjoint components.

Remark 4.4. The following examples show that the results of this section fail if in Definition 4.1 either (4.1i) or (4.1ii) is omitted.

Example 4.1. Consider the 3-point interpolant E_0 with n.r.

$$\dot{\theta}_0(s) - [-2 \sin \theta_0(s)]^{\frac{1}{2}} = 0, \quad 0 \leq s \leq s_1 = \int_0^\pi \frac{du}{\sqrt{2 \sin u}}$$

$$\dot{\theta}_0(s) + [-\sin \theta_0(s)]^{\frac{1}{2}} = 0, \quad s_1 \leq s \leq s_1 + \int_0^\pi \frac{du}{\sqrt{\sin u}}.$$

Here $B_{E_0} = (1, 0, \frac{1}{2}, 0)$, $\alpha_0 = \pi$, $\alpha_1 = 2\pi$, $\alpha_2 = \pi$. The mode is $(0, 0)$. E_0 violates (4.1i) since $\dot{\theta}_0(s_1) = 0$. For $\epsilon > 0$ let the extremal E_ϵ of the same mode $(0, 0)$ be given by $B_{E_\epsilon} = (1, 0, \frac{1}{2}, \epsilon)$, so that

$$\dot{\theta}_\epsilon(s) - [-2 \sin \theta_\epsilon(s)]^{\frac{1}{2}} = 0, \quad 0 \leq s \leq s_{1,\epsilon},$$

$$\dot{\theta}_\epsilon(s) + [-\sin(\theta_\epsilon(s) - \epsilon)]^{\frac{1}{2}} = 0, \quad s_{1,\epsilon} \leq s \leq s_{2,\epsilon}.$$

If $\theta_\epsilon(s_{1,\epsilon})$ is close to $\alpha_1 = 2\pi$ then $\theta_\epsilon(s_{1,\epsilon}) = 2\pi - \delta$ for some $\delta > 0$. Thus, for $s = s_{1,\epsilon}$,

$$\dot{\theta}_\epsilon(s_{1,\epsilon}) = [2 \sin \delta]^{\frac{1}{2}} = -[\sin(\delta + \epsilon)]^{\frac{1}{2}},$$

which is impossible. Thus, $\alpha_1(B)$ cannot be defined as a continuous function in a full neighborhood of B_0 .

Example 4.2. Choose $\pi < \alpha_* < 2\pi$ and consider the 3-point interpolant E_* with n.r.

$$\begin{aligned} \dot{\theta}_*(s) - [-2 \sin \theta_*(s)]^{\frac{1}{2}} &= 0, \quad 0 \leq s \leq s_{1*} = \int_0^{\alpha_*} \frac{du}{\sqrt{2 \sin u}} \\ \dot{\theta}_*(s) - [-2 \sin \theta_*(s)]^{\frac{1}{2}} &= 0, \quad s_{1*} \leq s \leq s_2 = 2 \int_0^\pi \frac{du}{\sqrt{2 \sin u}}. \end{aligned}$$

Here $B_{E_*} = (1, 0, 1, 0)$, $\alpha_0 = \pi$, $\alpha_1 = \alpha_*$, $\alpha_2 = \pi$. The mode is $(0, 1)$. E_* violates (4.1ii) since $A_1 = A_2$. For $\pi < \tilde{\alpha} < 2\pi$ let $\tilde{E}$ be defined by the n.r.:

$$\begin{aligned} \dot{\tilde{\theta}}(s) - [-2 \sin \tilde{\theta}(s)]^{\frac{1}{2}} &= 0, \quad 0 \leq s \leq \tilde{s}_1 = \int_0^{\tilde{\alpha}} \frac{du}{\sqrt{2 \sin u}} \\ \dot{\tilde{\theta}}(s) - [-2 \sin \tilde{\theta}(s)]^{\frac{1}{2}} &= 0, \quad \tilde{s}_1 \leq s \leq s_2 = 2 \int_0^\pi \frac{du}{\sqrt{2 \sin u}}. \end{aligned}$$

Here $B_{\tilde{E}} = (1, 0, 1, 0)$, $\alpha_0 = \pi$, $\alpha_1 = \tilde{\alpha}$, $\alpha_2 = \pi$, and the mode is $(0, 1)$ as before. Since there are extremals for all $\pi < \tilde{\alpha} < 2\pi$, $\tilde{E}$ cannot be defined by B and its mode.

§5. Perturbations of Configurations

In the last section it was seen that the extremal interpolants with m variable interpolation nodes (more precisely, those that belong to a fixed class $E_{(k_1, \dots, k_m)}$) form a $2m$ -dimensional manifold. One expects that an arbitrary configuration $P = \{0, p_1, \dots, p_m\}$ can be interpolated by an extremal interpolant (possibly by one from each class $E_{(k_1, \dots, k_m)}$). No solution of any kind exists for this existence problem. In this section we investigate existence in the small. Does the set of configurations P for which extremal interpolants exist have nonempty interior in $\mathbb{R}^{2m}$? More specifically, which P in $\mathbb{R}^{2m}$ are interior points of this set?

To attack this perturbation problem one is tempted to consider the mapping from the $2m$ -dimensional set $M_{(k_1, \dots, k_m)}$ that coordinatizes the elements of $E_{(k_1, \dots, k_m)}$ (see §4), or from another $2m$ -dimensional set of parameters, to the configurations in $\mathbb{R}^{2m}$ which are interpolated. However, this mapping is so complicated - it involves the elliptic integrals which are the solutions of the extremal equations - that little insight is gained from its consideration. For this reason we start with the extremals themselves, as defined by their differential equations.

Let $\bar{\theta} : [0, \bar{s}_m] \rightarrow T^1$ be the normal representation of a given extremal interpolant $\bar{E}$, which interpolates the configuration

$$\bar{P} = \{0, \bar{p}_1, \dots, \bar{p}_m\} ,$$

so that

$$(5.1) \quad \int_0^{\bar{s}_1} \cos \bar{\theta} = \bar{p}_1^1, \quad \int_0^{\bar{s}_1} \sin \bar{\theta} = \bar{p}_1^2, \quad i = 0, 1, \dots, m$$

$$0 = \bar{s}_0 < \bar{s}_1 < \dots < \bar{s}_m .$$

Since we consider only extremals E with n.r. θ near $\bar{\theta}$, hence with knots s_i near $\bar{s}_i$, we choose $\bar{\epsilon} > 0$, $\bar{\epsilon} = \frac{1}{2} \min(\bar{s}_1 - \bar{s}_{i-1})$ and extend $\bar{\theta}$ to the interval $[0, \bar{s}]$, $\bar{s} = \bar{s}_m + \bar{\epsilon}$, by setting

$$\bar{\theta}(s) = \bar{\theta}(\bar{s}_m), \quad \bar{s}_m < s \leq \bar{s} .$$

We introduce two spaces of mappings from the interval $[0, \bar{s}]$:

NBV = space of functions $\kappa : [0, \bar{s}] \rightarrow \mathbb{R}$ of bounded variation $V(\kappa)$ and continuous from the right with $\kappa(0) = \kappa(\bar{s} - 0) = 0$ and norm $V(\kappa)$.

NBV_1 = space of functions $\theta : [0, \bar{s}] \rightarrow T^1$, which are locally absolutely continuous and have derivatives $\dot{\theta} \in NBV$, with norm $\sup|\theta| + V(\dot{\theta})$.

Both NBV and NBV_1 are B-spaces. Clearly $\bar{\theta}$ as defined above is in NBV_1 and $\dot{\bar{\theta}}$ is in NBV .

If $\theta \in NBV_1$ is the normal representation of an extremal E which interpolates the configuration $P = \{0, p_1, \dots, p_m\}$ at the nodes $0 = s_0 < s_1 < \dots < s_m < \bar{s}$ (more precisely, we speak of the linear extension of E to length $\bar{s}$) then the following equations hold (see Proposition 2.4):

$$\begin{aligned} (i) \quad & \ddot{\theta}(s) + \frac{1}{2} \dot{\theta}^3(s) = 0, \text{ for } s_{i-1} < s < s_i, \quad i = 1, \dots, m \\ & \ddot{\theta}(s) = 0, \quad \text{for } s_m < s < \bar{s} . \\ (5.2) \quad (ii) \quad & \dot{\theta}(s_i - 0) - \dot{\theta}(s_i) = 0, \quad i = 1, \dots, m . \\ (iii) \quad & \int_0^{s_i} \cos \theta = p_i^1, \quad \int_0^{s_i} \sin \theta = p_i^2, \quad i = 1, \dots, m . \end{aligned}$$

It is easy to show that these equations characterize the interpolant E completely.

We rewrite equations (5.2) by using the values

$$(5.3) \quad \dot{\theta}(s_i) = a_i, \quad \ddot{\theta}(s_i + 0) = b_i, \quad i = 1, \dots, m$$

as parameters (but $a_0 = 0, b_m = 0$ always).

$$(i) \quad \begin{aligned} \dot{\theta}(s) + \frac{1}{2} \int_{s_{i-1}}^s (s-t) \dot{\theta}^3(t) dt &= a_{i-1} + b_{i-1}(s-s_{i-1}), \quad s_{i-1} \leq s < s_i, \quad i = 1, \dots, m \\ \dot{\theta}(s) &= 0, \quad s_m \leq s \leq \bar{s}. \end{aligned}$$

$$(5.4) \quad (ii) \quad a_i = a_{i-1} + b_{i-1}(s_i - s_{i-1}) - \frac{1}{2} \int_{s_{i-1}}^{s_i} (s_i - t) \dot{\theta}^3(t) dt, \quad i = 1, \dots, m.$$

$$(iii) \quad \int_0^{s_i} \cos \theta = p_i^1, \quad \int_0^{s_i} \sin \theta = p_i^2, \quad i = 1, \dots, m.$$

Equations (5.4) define implicitly a mapping G from the space $P \subset (\mathbb{R}^2)^m$ of configurations P to the space E of extremal interpolants E . To apply the Implicit Function Theorem we introduce a mapping G on the product space $E \times P$ to $NBV \times \mathbb{R}^m \times \mathbb{R}^{2m}$ as follows. We set

$$\Theta = (\theta; s_1, \dots, s_m; a_1, \dots, a_m; b_0, \dots, b_{m-1}), \quad \text{where}$$

$$\theta \in NBV_1; \quad s_i \in \mathbb{R}, \quad a_i \in \mathbb{R}, \quad b_i \in \mathbb{R}$$

$$D = NBV_1 \times \Pi(\bar{s}_1 - \bar{c}, \bar{s}_1 + \bar{c}) \times \mathbb{R}^m \times \mathbb{R}^m$$

and define a mapping $G = (g, r_i, q_i)$ with components $g \in NBV, r_i \in \mathbb{R}, q_i \in \mathbb{R}^2$, as follows:

$$(i) \quad \begin{aligned} g(s) &= \dot{\theta}(s) + \frac{1}{2} \int_{s_{i-1}}^s (s-t) \dot{\theta}^3(t) dt - a_{i-1} - b_{i-1}(s-s_{i-1}), \quad s_{i-1} \leq s < s_i \\ g(s) &= \dot{\theta}(s), \quad s_m \leq s \leq \bar{s}. \end{aligned}$$

$$(5.5) \quad (ii) \quad r_i = a_i - a_{i-1} - b_{i-1}(s_i - s_{i-1}) + \frac{1}{2} \int_{s_{i-1}}^{s_i} (s_i - t) \dot{\theta}^3(t) dt.$$

$$(iii) \quad q_i^1 = \int_0^{s_i} \cos \theta - p_i^1, \quad q_i^2 = \int_0^{s_i} \sin \theta - p_i^2.$$

Clearly, Equations (5.4) are equivalent to

$$G(\Theta, P) = 0.$$

In particular we have $G(\bar{\theta}, \bar{p}) = 0$, since we assume that $\bar{\theta}$ is the n.r. of the extremal interpolant $\bar{E}$ for the configuration $\bar{p}$. We need the Fréchet differential $G'_\theta(\theta, p)[\psi]$ where

$$\psi = (\psi; t_1, \dots, t_m; \alpha_1, \dots, \alpha_m; \beta_0, \dots, \beta_{m-1})$$

is an increment to θ . The components of $G'_\theta(\theta, p)[\psi]$ are denoted by g', r'_i, q'_i . One finds readily

$$(i) \quad g'(s) = \dot{\psi}(s) + \frac{3}{2} \int_{s_{i-1}}^s (s-t) \kappa^2(t) \dot{\psi}(t) dt - \frac{1}{2} \kappa_{i-1}^3 (s-s_{i-1}) t_{i-1} \\ - \alpha_{i-1} - \beta_{i-1} (s-s_{i-1}) + b_{i-1} t_{i-1}, \quad s_{i-1} \leq s < s_i$$

$$g'(s) = \dot{\psi}(s), \quad s_m \leq s < \bar{s}.$$

(5.6)

$$(ii) \quad r'_i = \alpha_i - \alpha_{i-1} - \beta_{i-1} \Delta_i s + (b_{i-1} - \frac{1}{2} \Delta_i s \kappa_{i-1}^3) t_{i-1} - \dot{\kappa}(s_i - 0) t_i + \frac{3}{2} \int_{s_{i-1}}^{s_i} (s_i - t) \kappa^2 \dot{\psi}.$$

$$(iii) \quad (q_i^1)' = - \int_0^{s_i} \psi \sin \theta + t_i \cos \theta_i, \quad (q_i^2)' = \int_0^{s_i} \psi \cos \theta + t_i \sin \theta_i.$$

Here we have used the notations

$$(5.7) \quad \kappa = \dot{\theta}, \quad \theta_i = \theta(s_i), \quad \kappa_i = \dot{\theta}(s_i), \quad \Delta_i s = s_i - s_{i-1},$$

and the relation

$$\frac{1}{2} \int_{s_{i-1}}^{s_i} \dot{\theta}^3 = \ddot{\theta}(s_{i-1} + 0) - \ddot{\theta}(s_i - 0) = b_{i-1} - \dot{\kappa}(s_i - 0),$$

which follows from (5.2i) and (5.3).

The continuity of G' near $(\bar{\theta}, \bar{p})$ is readily ascertained from (5.6).

We can now state the main result of this section.

Theorem 5.1. $G'_\theta(\bar{\theta}, \bar{p})$ is an isomorphism of

$$NBV_1 \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \quad \text{onto} \quad NBV \times \mathbb{R}^m \times (\mathbb{R}^2)^m$$

if and only if $(\bar{\Theta}, \bar{P})$ satisfies the following hypothesis:

(A) The system for the unknown $\psi \in NBV_1$:

$$\begin{aligned}
 & \ddot{\psi} + \frac{3}{2} \bar{\kappa}^2 \dot{\psi} = 0 \quad \text{on } (\bar{s}_{i-1}, \bar{s}_i), \quad i = 1, \dots, m \\
 (i) \quad & \dot{\psi} = 0 \quad \text{on } (\bar{s}_m, \bar{s}), \\
 (5.8) \quad & (ii) \quad \Delta_i \dot{\psi} + \Delta_i \bar{\kappa} \int_0^{\bar{s}_i} \psi \sin(\bar{\theta} - \bar{\theta}_1) = 0, \quad i = 1, \dots, m, \\
 & (iii) \quad \int_0^{\bar{s}_i} \psi \cos(\bar{\theta} - \bar{\theta}_1) = 0, \quad i = 1, \dots, m,
 \end{aligned}$$

has only the trivial solution $\psi = 0$. If (A) is satisfied then there are neighborhoods $N_{\bar{\Theta}}$ in D and $N_{\bar{P}}$ in $(\mathbb{R}^2)^m$ of $\bar{\Theta}$ and $\bar{P}$ respectively and a diffeomorphism Π ,

$$\Pi : N_{\bar{\Theta}} \xrightarrow{\text{onto}} N_{\bar{P}}$$

such that $\Theta \in N_{\bar{\Theta}}$ defines an extremal P -interpolant E for $P = \Pi(\Theta)$.

Remark 5.1. Explanation of the notations used above and in the following:

$$\begin{aligned}
 \bar{\theta}_1 &= \bar{\theta}(\bar{s}_1), \quad \bar{\kappa}_1 = \dot{\bar{\theta}}(\bar{s}_1), \quad \psi_1 = \psi(\bar{s}_1), \\
 \Delta_1 \bar{\kappa} &= \dot{\bar{\kappa}}(\bar{s}_1+0) - \dot{\bar{\kappa}}(\bar{s}_1-0), \quad \Delta_1 \dot{\psi} = \dot{\psi}(\bar{s}_1+0) - \dot{\psi}(\bar{s}_1-0),
 \end{aligned}$$

in particular $\Delta_0 \bar{\kappa} = \dot{\bar{\kappa}}(\bar{s}_0+0)$, $\Delta_m \bar{\kappa} = -\dot{\bar{\kappa}}(\bar{s}_m-0)$, etc.

Proof: We first demonstrate the injectivity of the bounded linear mapping $G'_{\bar{\Theta}}(\bar{\Theta}, \bar{P})$ under hypothesis (A). Thus, assume that for some Ψ :

$$G'_{\bar{\Theta}}(\bar{\Theta}, \bar{P})[\Psi] = 0.$$

Then we have by (5.61):

$$(5.9i) \quad \ddot{\psi} + \frac{3}{2} \kappa^{-2} \dot{\psi} = 0 \quad \text{on } (\bar{s}_{i-1}, \bar{s}_i), \quad i = 1, \dots, m$$

$$\dot{\psi} = 0 \quad \text{on } (\bar{s}_m, \bar{s}) ,$$

$$(5.10) \quad \dot{\psi}(\bar{s}_i - 0) + \frac{3}{2} \int_{\bar{s}_{i-1}}^{\bar{s}_i} (\bar{s}_i - t) \kappa^{-2} \dot{\psi} - \frac{1}{2} \kappa_{i-1}^3 \Delta_i \bar{s} t_{i-1} \\ - \alpha_{i-1} - \beta_{i-1} \Delta_i \bar{s} + b_{i-1} t_{i-1} = 0 ,$$

and using (5.6ii)

$$(5.11i) \quad \dot{\psi}(\bar{s}_i - 0) - \alpha_i + \kappa(\bar{s}_i - 0) t_i = 0, \quad i = 1, \dots, m .$$

Also by (5.6i)

$$(5.11ii) \quad \dot{\psi}(\bar{s}_i + 0) - \alpha_i + \bar{b}_i t_i = 0 .$$

Since $\bar{b}_i = \kappa(\bar{s}_i + 0)$, the last two equations yield

$$(5.12) \quad \Delta_i \dot{\psi} + t_i \Delta_i \kappa = 0, \quad i = 1, \dots, m .$$

The remaining equations $(q_i^1)' = (q_i^2)' = 0$ (see 5.6iii) are equivalent to

$$(5.13) \quad t_i = \int_0^{\bar{s}_i} \psi \sin(\bar{\theta} - \bar{\theta}_i), \quad i = 1, \dots, m .$$

$$(5.9iii) \quad \int_0^{\bar{s}_i} \psi \cos(\bar{\theta} - \bar{\theta}_i) = 0, \quad i = 1, \dots, m .$$

When (5.13) is substituted in (5.12), one obtains

$$(5.9ii) \quad \Delta_i \dot{\psi} + \Delta_i \kappa \int_0^{\bar{s}_i} \psi \sin(\bar{\theta} - \bar{\theta}_i) = 0, \quad i = 1, \dots, m .$$

By hypothesis (A) the equations (5.9i, ii, iii), which coincide with Equations (5.8), together with $\psi \in NBV_1$ imply $\psi = 0$ on $[0, \bar{s}_m]$. (5.9i) then implies $\psi = 0$ also on $[\bar{s}_m, \bar{s}]$. Then by (5.13), $t_i = 0$ ($i = 1, \dots, m$), and by (5.11), $\alpha_i = 0$ ($i = 1, \dots, m$). Finally, by (5.10),

$\beta_i = 0$ ($i = 0, \dots, m-1$), thus $\Psi = 0$, hence $G'_\Theta(\bar{\Theta}, \bar{P})$ is injective.

Conversely if hypothesis (A) is not satisfied, i.e. system (5.9i,ii,iii) has a nontrivial solution $\psi \in NBV_1$, then determine the t_i from (5.13), the α_i from (5.11) and the β_i from (5.10). $\Psi = (\psi; t_1, \dots, t_m; \alpha_1, \dots, \alpha_m; \beta_0, \dots, \beta_{m-1})$ is then a nontrivial solution of $G'_\Theta(\bar{\Theta}, \bar{P})[\Psi] = 0$, thus $G'_\Theta(\bar{\Theta}, \bar{P})$ is not an isomorphism.

To show surjectivity of $G'_\Theta(\bar{\Theta}, \bar{P})$ onto $NBV \times \mathbb{R}^m \times \mathbb{R}^{2m}$, assume we are given $h \in NBV$, $u_i \in \mathbb{R}$, $v_i \in \mathbb{R}^2$ ($i = 1, \dots, m$). We must find $\psi \in NBV_1$, $t_i \in \mathbb{R}$, $\alpha_i \in \mathbb{R}$, $\beta_i \in \mathbb{R}$, such that (see 5.6)

$$(5.14) \quad \begin{aligned} (i) \quad & g'(s) = h(s), \quad 0 \leq s < \bar{s} \\ (ii) \quad & r'_i = u_i, \quad i = 1, \dots, m \\ (iii) \quad & (q_i^1)' = v_i^1, (q_i^2)' = v_i^2, \quad i = 1, \dots, m \end{aligned}$$

In particular, (5.14i) requires (see 5.6i)

$$(5.15i) \quad \begin{aligned} & \dot{\psi}(\bar{s}_i - 0) + \frac{3}{2} \int_{\bar{s}_{i-1}}^{\bar{s}_i} (\bar{s}_i - t) \bar{\kappa}^2 \dot{\psi} - \frac{1}{2} \bar{\kappa}_{i-1}^3 \Delta_i \bar{s} t_{i-1} \\ & - \alpha_{i-1} - \beta_{i-1} \Delta_i \bar{s} + \bar{b}_{i-1} t_{i-1} = h(\bar{s}_i - 0) \end{aligned}$$

and (5.14ii) requires (see 5.6ii)

$$\begin{aligned} \alpha_i - \alpha_{i-1} - \beta_{i-1} \Delta_i \bar{s} + (\bar{b}_{i-1} - \frac{1}{2} \Delta_i \bar{\kappa}_{i-1}^3) t_{i-1} \\ - \dot{\kappa}(\bar{s}_i - 0) t_i + \frac{3}{2} \int_{\bar{s}_{i-1}}^{\bar{s}_i} (\bar{s} - t) \bar{\kappa}^2 \dot{\psi} = u_i \end{aligned}$$

The last two equations imply

$$(5.16i) \quad \dot{\psi}(\bar{s}_i - 0) - \alpha_i + \dot{\kappa}(\bar{s}_i - 0) t_i = h(\bar{s}_i - 0) - u_i$$

(5.14i) also requires

$$(5.16ii) \quad \dot{\psi}(\bar{s}_1+0) - \alpha_1 + \frac{\Delta}{k}(\bar{s}_1+0)t_1 = h(\bar{s}_1+0) .$$

We therefore must have

$$(5.17i) \quad \dot{\psi}(\bar{s}_1+0) = \dot{\psi}(\bar{s}_1-0) - \Delta_1 \frac{\Delta}{k} t_1 + \Delta_1 h + u_1, \quad i = 1, \dots, m .$$

The last equations (5.14iii) are (see 5.6iii)

$$(5.17ii) \quad - \int_0^{\bar{s}_1} \psi \sin \bar{\theta} + t_1 \cos \bar{\theta}_1 = v_1^1, \quad \int_0^{\bar{s}_1} \psi \cos \bar{\theta} + t_1 \sin \bar{\theta}_1 = v_1^2 .$$

The general solution $\psi \in NBV_1$ of (5.14i) is the sum of a particular solution and the linear combination of $3m$ functions with the coefficients $\alpha_1, \dots, \alpha_m, \beta_0, \dots, \beta_{m-1}, t_1, \dots, t_m$. When this ψ is substituted in (5.17i and ii) a nonhomogeneous system of $3m$ equations for the unknowns α_1, β_1, t_1 is obtained. The homogeneous part of this system corresponds to the case $h = 0, u_1 = 0, v_1 = 0$, and it has only the trivial solution $\alpha_1 = 0, \beta_1 = 0, t_1 = 0$, as shown in the first part of the proof. This demonstrates the surjectivity of $G_{\theta}^1(\bar{\theta}, \bar{P})$ and finishes the proof of Theorem 5.1.

The utility of this theorem is illustrated by the fact that it readily implies the following important result.

Corollary 5.2. Suppose $\bar{P} = (0, \bar{p}_1, \dots, \bar{p}_m)$ is the ray configuration $\bar{p}_i = (\bar{s}_i, 0)$ ($i = 1, \dots, m$) with the trivial interpolant $\bar{E}$. Then hypothesis (A) is satisfied, hence the conclusion of Theorem 5.1 holds.

Proof: $\bar{\theta} = 0$ in this case. If we put

$$(5.18) \quad s = x, \quad \bar{s}_1 = x_1, \quad \bar{s} = \bar{x}, \quad y(x) = \int_0^x \psi$$

then $y(0) = 0$ and Equations (5.8) become

$$(5.19) \quad \begin{aligned} y^{(4)} &= 0 \quad \text{on } (x_{i-1}, x_i), \quad i = 1, \dots, m \\ y'' &= 0 \quad \text{on } (x_m, \bar{x}) \\ y''(x_i+0) - y''(x_i-0) &= 0, \quad i = 1, \dots, m \\ y(x_i) &= 0 \quad i = 1, \dots, m \end{aligned}$$

while $y' \in NBV_1$, i.e. $y'' \in NBV$, in particular $y''(0) = 0$. These are exactly the equations for a natural cubic spline that interpolates the points $(x_i, 0)$ ($i = 0, \dots, m$). It follows that $y(x) \equiv 0$, and the corollary is proved.

Remark 5.2. We briefly draw a connection between the above corollary and natural cubic spline interpolation. The mapping Γ from the space of configurations P to the space of extremal interpolants E is implicitly defined by $G(\Theta, P) = 0$. Perturbation theory looks for a pair $\Theta = \bar{\Theta} + \psi$, $P = \bar{P} + Z$ ($Z = \{z_1, \dots, z_m\}$) close to the initial pair $\bar{\Theta}, \bar{P}$ for which

$$(5.20) \quad G'_\Theta(\bar{\Theta}, \bar{P})[\psi] + G'_P(\bar{\Theta}, \bar{P})[Z] = 0.$$

It is readily seen that this is system (5.9i, ii, iii) except that (5.9ii) is replaced by

$$(5.21i) \quad \int_0^{\bar{s}_1} \psi \cos(\bar{\theta} - \bar{\theta}_1) = z_1^2 \cos \bar{\theta}_1 - z_1^1 \sin \bar{\theta}_1.$$

Also, Equation (5.13) is replaced by

$$(5.21ii) \quad t_1 = \int_0^{\bar{s}_1} \psi \sin(\bar{\theta} - \bar{\theta}_1) + z_1^1 \cos \bar{\theta}_1 + z_1^2 \sin \bar{\theta}_1.$$

Now suppose the initial configuration $\bar{P}$ and the interpolant $\bar{E}$ are the ray configuration and trivial interpolant, as in Corollary 5.2. Further suppose P is the configuration

$$P = \{(x_0, y_0), (x_1, y_1), \dots, (x_m, y_m)\}, \quad x_0 = y_0 = 0$$

with the $x_1 - \bar{s}_1 = z_1^1$ and $y_1 = z_1^2$ small. Then (5.21i, ii) become (in the perturbation approximation)

$$(5.22) \quad \int_0^{x_1} \psi \approx y_1, \quad \bar{s}_1 + t_1 \approx x_1,$$

moreover

$$(5.23) \quad x(s) = \int_0^s \cos \theta = \int_0^s \cos \psi \approx s, \quad y(s) = \int_0^s \sin \theta = \int_0^s \sin \psi \approx \int_0^s \psi.$$

The perturbed extremal E is now the graph of the function $x + y(x)$, which satisfies Equations (5.19), except that the last equation is replaced by $y(x_1) = y_1$. Thus y is the natural cubic spline that interpolates the data (x_i, y_i) ($i = 0, 1, \dots, m$).

We have shown that the cubic spline interpolant is the result of linearization of extremal interpolation (in the sense of making $\int \kappa^2 ds$ stationary) near the trivial interpolant for the ray configuration.

Remark 5.3. The differential equation that appears in hypothesis (A)

$$(5.24i) \quad \ddot{\psi} + \frac{3}{2} \kappa^2 \psi = 0$$

where $\kappa = \dot{\theta}$ is defined by the equation

$$(5.24ii) \quad \ddot{\theta} + \frac{1}{2} \dot{\theta}^3 = 0$$

can be completely integrated by quadratures alone. Indeed $\psi = 1$ is clearly one integral;

$$(5.25i) \quad \psi_1 = \kappa$$

is another; for $\ddot{\psi}_1 = \theta^{(4)}$ and $\theta^{(4)} + (3/2)\dot{\theta}^2\ddot{\theta} = 0$. The third one is

$$(5.25ii) \quad \psi_2(s) = s\kappa(s) .$$

In the special case where $\kappa(s) \equiv cs$, three linearly independent integrals are

$$(5.25iii) \quad 1, s, s^2 .$$

In the remainder of this section we discuss the replacement of hypothesis (A) by a simpler condition, which requires only that the value of an explicitly given function (involving many quadratures) at $\bar{s}_m$ be $\neq 0$. For this purpose we introduce a condition on arcs of simple elastica. The arc of $\bar{E}$ from $\bar{s}_{i-1}$ to $\bar{s}_i$ is said to be ordinary if it is either straight or

$$(5.26) \quad \begin{aligned} \delta_i : &= (\bar{p}_i^1 - \bar{p}_{i-1}^1) [(\bar{\kappa}_{i-1})^2 \cos \bar{\theta}_i + \bar{\kappa}_{i-1} \sin \bar{\theta}_i] \\ &+ (\bar{p}_i^2 - \bar{p}_{i-1}^2) [(\bar{\kappa}_{i-1})^2 \sin \bar{\theta}_i - \bar{\kappa}_{i-1} \cos \bar{\theta}_i] + \bar{\kappa}_{i-1} \sin(\bar{\theta}_{i-1} - \bar{\theta}_i) \neq 0 . \end{aligned}$$

One sees readily that only exceptionally is such an arc not ordinary. For example, the arc from $\bar{s}_0 = 0$ to $\bar{s}_1$, where $\bar{\kappa}_0 = 0$, $\dot{\bar{\kappa}}_0 \neq 0$, is not ordinary if and only if

$$\bar{p}_1^1 \sin \bar{\theta}_1 - \bar{p}_1^2 \cos \bar{\theta}_1 = 0,$$

i.e., the chord $\overline{p_0 p_1}$ is tangent to E at $\bar{p}_1$.

Theorem 5.3. Suppose the extremal $\bar{E}$ consists of ordinary arcs only. Let $\psi_0, \psi_0(0) = 1$, be the solution of system (5.8), exclusive of the condition on $\Delta_m \dot{\psi}$ (ψ_0 is explicitly constructed below). Then hypothesis (A) of Theorem 5.1 is satisfied if and only if

$$(5.27) \quad \dot{\psi}_0(\bar{s}_m - 0) + \dot{\bar{\kappa}}(\bar{s}_m - 0) \int_0^{\bar{s}_m} \psi_0 \sin(\bar{\theta} - \bar{\theta}_m) \neq 0.$$

Proof: Let ϕ_1, χ_1 be the integrals of Equation $\ddot{\psi} + (3/2)\bar{\kappa}^2 \dot{\psi} = 0$ (case $\bar{\kappa} \neq 0$) for which

$$(5.28) \quad \begin{aligned} \phi_1(\bar{s}_{i-1}) &= \dot{\phi}_1(\bar{s}_{i-1} + 0) = 0, \quad \ddot{\phi}_1(\bar{s}_{i-1} + 0) = 1 \\ \chi_1(\bar{s}_{i-1}) &= \dot{\chi}_1(\bar{s}_{i-1} + 0) = 0, \quad \ddot{\chi}_1(\bar{s}_{i-1} + 0) = 1. \end{aligned}$$

These are linear combinations of $1, \psi_1, \psi_2$. One finds easily

$$(5.29i) \quad \begin{aligned} \phi_1 &= \rho_1 [(\bar{\kappa}_{i-1})^2 - \bar{\kappa}_{i-1} \bar{\kappa} + \dot{\bar{\kappa}}_{i-1} (s - \bar{s}_{i-1}) \bar{\kappa}] \\ \chi_1 &= \rho_1 [-2\bar{\kappa}_{i-1} \dot{\bar{\kappa}}_{i-1} + 2\dot{\bar{\kappa}}_{i-1} \bar{\kappa} + \frac{1}{2}(\bar{\kappa}_{i-1})^3 (s - \bar{s}_{i-1}) \bar{\kappa}] \end{aligned}$$

where we have used the abbreviation

$$(5.29ii) \quad 1/\rho_1 = 2(\dot{\bar{\kappa}}_{i-1})^2 + \frac{1}{2}(\bar{\kappa}_{i-1})^4.$$

We construct ψ_0 successively on the intervals $[\bar{s}_{i-1}, \bar{s}_i]$ ($i = 1, \dots, m$). On $[0, \bar{s}_1]$ we have since $\psi_0(0) = 1, \dot{\psi}_0(0) = 0$:

$$(5.30i) \quad \psi_0 = 1 + \ddot{\psi}_0(0) \phi_1$$

with ψ_0 satisfying condition (5.9iii):

$$(5.30ii) \quad \int_0^{\bar{s}_1} \cos(\bar{\theta} - \bar{\theta}_0) + \ddot{\psi}_0(0) \int_0^{\bar{s}_1} \phi_1 \cos(\bar{\theta} - \bar{\theta}_0) = 0 .$$

One finds the last integral to be $\rho_1 \delta_1$. Thus, $\ddot{\psi}_0(0)$ is uniquely determined from (5.30ii), and (5.30i) gives ψ_0 on $(0, \bar{s}_1)$. (If $\bar{\kappa} = 0$ on $[0, \bar{s}_1]$ then one finds $\psi_0(s) = 1 - 3(s/\bar{s}_1)^2$). Assume ψ_0 has been determined on $[0, \bar{s}_{i-1}]$ ($i < m$). Then $\psi_0(\bar{s}_{i-1})$ and $\dot{\psi}_0(\bar{s}_{i-1} - 0)$ are known, hence $\dot{\psi}_0(\bar{s}_{i-1})$ can be found from condition (5.8ii).

$$(5.31) \quad \dot{\psi}_0(\bar{s}_{i-1}) = \dot{\psi}_0(\bar{s}_{i-1} - 0) + \Delta_{i-1} \bar{\kappa} \int_0^{\bar{s}_{i-1}} \psi_0 \sin(\bar{\theta} - \bar{\theta}_{i-1}) .$$

Then if $\bar{\kappa} \neq 0$ on $[\bar{s}_{i-1}, \bar{s}_i]$ we have

$$(5.32i) \quad \psi_0 = \psi_0(\bar{s}_{i-1}) + \dot{\psi}_0(\bar{s}_{i-1}) \chi_i + \ddot{\psi}_0(\bar{s}_{i-1}) \phi_i ,$$

and to satisfy condition (5.9iii):

$$(5.32ii) \quad \int_0^{\bar{s}_{i-1}} \psi_0 \cos(\bar{\theta} - \bar{\theta}_i) + \int_{\bar{s}_{i-1}}^{\bar{s}_i} [\psi_0(\bar{s}_{i-1}) + \dot{\psi}_0(\bar{s}_{i-1}) \chi_i] \cos(\bar{\theta} - \bar{\theta}_i) + \ddot{\psi}_0(\bar{s}_{i-1}) \int_{\bar{s}_{i-1}}^{\bar{s}_i} \phi_i \cos(\bar{\theta} - \bar{\theta}_i) = 0 .$$

One finds, using (5.29i) and

$$\int_{\bar{s}_{i-1}}^{\bar{s}_i} \cos \bar{\theta} = \bar{p}_i^1 - \bar{p}_{i-1}^1, \quad \int_{\bar{s}_{i-1}}^{\bar{s}_i} \sin \bar{\theta} = \bar{p}_i^2 - \bar{p}_{i-1}^2$$

that the last integral in (5.32ii) is $\rho_1 \delta_1 \neq 0$. Thus $\ddot{\psi}_0(\bar{s}_{i-1})$ is uniquely determined from (5.32ii), and (5.32i) gives ψ_0 on $[\bar{s}_{i-1}, \bar{s}_i]$.

In the omitted case $\bar{\kappa} = 0$ on $[\bar{s}_{i-1}, \bar{s}_i]$, (5.32ii) are replaced by

$$\begin{aligned}
(5.33) \quad (i) \quad \psi_0(s) &= \psi_0(\bar{s}_{i-1}) + \dot{\psi}_0(\bar{s}_{i-1})(s-\bar{s}_{i-1}) + \frac{1}{2} \ddot{\psi}_0(\bar{s}_{i-1})(s-\bar{s}_{i-1})^2 \\
(ii) \quad \int_0^{\bar{s}_{i-1}} \psi_0 \cos(\bar{\theta}-\bar{\theta}_1) &+ \psi_0(\bar{s}_{i-1})(\bar{s}_1-\bar{s}_{i-1}) + \frac{1}{2} \dot{\psi}_0(\bar{s}_{i-1})(\bar{s}_1-\bar{s}_{i-1})^2 \\
&+ \frac{1}{6} \ddot{\psi}_0(\bar{s}_{i-1})(\bar{s}_1-\bar{s}_{i-1})^3 = 0 .
\end{aligned}$$

The conclusions remain the same as before.

With ψ_0 found on $[0, \bar{s}_m]$ there remains condition (5.9ii) to be satisfied:

$$(5.34) \quad \dot{\psi}_0(\bar{s}_m-0) + \kappa(\bar{s}_m-0) \int_0^{\bar{s}_m} \psi_0 \sin(\bar{\theta}-\bar{\theta}_m) = 0 .$$

System (5.6) has a nontrivial solution if and only if the constructed integral ψ_0 satisfies (5.34). This proves the theorem.

Example 5.1. To illustrate the utility of the preceding theorem consider the configuration

$$P = \{(0,0), (a,0), (a,b)\}$$

with the extremal interpolant $\bar{E}$ whose n.r. $\bar{\theta}$ is defined by:

$$\begin{aligned}
(5.35) \quad \bar{\theta}(s) &= 0 & 0 \leq s \leq a , \\
\bar{\theta}(s) &= (\beta/b) [\sin \bar{\theta}(s)]^{1/2}, & a \leq s \leq \bar{s} .
\end{aligned}$$

Here β and $\bar{s}$ are the definite integrals.

$$\beta = \int_0^{\pi} \sin^{-1/2} t, \quad \bar{s} = a + (b/\beta) \int_0^{\pi} \sin^{-1/2} t .$$

That $\bar{E}$ does indeed interpolate the point (a,b) follows from

$$\begin{aligned}
\int_a^{\bar{s}} \cos \bar{\theta} \, ds &= \int_0^{\pi} (1/\bar{\theta}) \cos t \, dt = (b/\beta) \int_0^{\pi} \sin^{-1/2} t \cos t \, dt = 0 , \\
\int_a^{\bar{s}} \sin \bar{\theta} \, ds &= (b/\beta) \int_0^{\pi} \sin t \cdot \sin^{-1/2} t \, dt = b .
\end{aligned}$$

Using the construction of the preceding theorem, one finds by straight forward computation:

$$\psi_0(s) = 1 - 3(s/a)^2, \quad 0 \leq s \leq a,$$

$$\phi_2(s) = (b/\beta)^2 (s-a) \dot{\bar{\theta}}(s), \quad \chi_2(s) = 2(b/\beta)^2 \dot{\bar{\theta}}(s),$$

$$\psi_0(s) = -2 - (12/a)(b/\beta)^2 \dot{\bar{\theta}}(s), \quad a \leq s \leq \bar{s},$$

$$\dot{\psi}_0(\bar{s}-0) + \dot{\kappa}(\bar{s}-0) \int_0^{\bar{s}} \psi_0 \sin(\bar{\theta}-\bar{\theta}_2) = \frac{6}{ab} + \frac{\beta^2}{b} > 0.$$

By Theorem 5.2 unrestricted perturbation of the 3-point rectangular configuration $\bar{P}$ is possible.

Even for this simple example we were not able to prove this result in the more direct way, by expressing the parameters of the elastica spline in terms of the coordinates of the interpolated configuration.

Example 5.2. Let $\bar{P}$ be the "hair pin configuration"

$$\bar{P} = \{(0,0), (0,\delta), (0,0)\}, \quad \delta = \int_0^\pi (2 \sin)^{1/2},$$

with the extremal interpolant $\bar{E}$ whose n.r. $\bar{\theta}$ is defined by:

$$\bar{\theta}(0) = 0$$

$$\dot{\bar{\theta}}(s) = [2 \sin \bar{\theta}(s)]^{1/2}, \quad 0 \leq s \leq \sigma := \int_0^\pi (2 \sin)^{-1/2}$$

$$= [-2 \sin \bar{\theta}(s)]^{1/2}, \quad \sigma \leq s \leq 2\sigma.$$

Then $\bar{\kappa}(0) = \bar{\kappa}(\sigma) = \bar{\kappa}(2\sigma) = 0$, $\Delta \dot{\bar{\kappa}}_1 = \dot{\bar{\kappa}}(\sigma+0) - \dot{\bar{\kappa}}(\sigma-0) = 2$. One finds readily:

$$\phi_1(s) = \frac{s}{2} \bar{\kappa}(s), \quad \chi_1(s) = \bar{\kappa}(s), \quad 0 \leq s \leq \sigma,$$

$$\phi_2(s) = \frac{s-\sigma}{2} \bar{\kappa}(s), \quad \chi_2(s) = \bar{\kappa}(s), \quad \sigma \leq s \leq 2\sigma.$$

$$\psi_0(s) = \begin{cases} 1, & 0 \leq s \leq \sigma, \\ 1 + 2\delta \bar{\kappa}(s), & \sigma \leq s \leq 2\sigma. \end{cases}$$

Then

$$\begin{aligned}
 & \dot{\psi}_0(2\sigma-0) + \dot{\kappa}(2\sigma-0) \int_0^{2\sigma} \psi_0 \sin(\tilde{\theta}-2\pi) \\
 &= -2\delta - \left[\int_0^\delta \sin \tilde{\theta} + \int_\sigma^{2\sigma} (1 + 2\delta\tilde{\kappa}) \sin \tilde{\theta} \right] \\
 &= -2\delta - [\delta - \delta - 2\delta] = 0.
 \end{aligned}$$

By Theorem 5.3 hypothesis (A) is not satisfied, thus it cannot be concluded that the hair pin configuration with the extremal interpolant $\tilde{E}$ permits perturbation. Indeed, one can show directly that if $\tilde{P}$ is replaced by the perturbed configuration $P_\epsilon = \{(-\epsilon, 0), (0, \delta), (\epsilon, 0)\}$ there exists no extremal interpolant close to $\tilde{E}$ no matter how small $\epsilon \neq 0$ is. On the other hand, if $\tilde{P}$ is replaced by ${}_eP = \{(0, 0), (0, \delta), (0, \epsilon)\}$, which is also close to $\tilde{P}$, then there is an extremal interpolant ${}_eE$ near $\tilde{E}$, ${}_eE$ coincides with E for the arc from $(0, 0)$ to $(0, \delta)$, the remaining arc is the simple elastica joining $(0, \delta)$ and $(0, \epsilon)$. Thus, we have an example of singular behavior taking place in the perturbation from $\tilde{P}$ to ${}_eP$.

§6. Special Cases of Open Extremals

We study in some detail in this section extremal P-interpolants for some special configurations.

A. Two Point Extremal Interpolants

Let P be the configuration $P = \{p_0, p_1\}$, $p_0 = 0$. For an extremal E with normal representation $\bar{\theta}, \bar{\theta}(0) = 0$, we have by (3.3):

$$(6.1) \quad \begin{aligned} (i) \quad & \bar{\theta} \in C^1[0, \bar{s}], \bar{\theta}(0) = \dot{\bar{\theta}}(0) = \dot{\bar{\theta}}(\bar{s}) = 0, \\ (ii) \quad & \ddot{\bar{\theta}}^2(s) = c^1 \cos \bar{\theta}(s) + c^2 \sin \bar{\theta}(s), 0 < s < \bar{s}, \\ (iii) \quad & \int_0^{\bar{s}} \cos \bar{\theta} = p_1^1, \int_0^{\bar{s}} \sin \bar{\theta} = p_1^2. \end{aligned}$$

(6.1i) implies $c^1 = 0$ and $c^2 \sin \bar{\theta}(\bar{s}) = 0$. Clearly, either $c^2 = 0$, yielding the extremal $\bar{\theta}(s) \equiv 0$, $p_0 = 0$, $p_1 = (\bar{s}, 0)$, or, $c^2 \neq 0$. In this case we write $c^2 = -2/l$. Differentiation of (6.1ii) gives by Proposition 2.4 the differential equation,

$$(6.2) \quad l\ddot{\bar{\theta}}(s) + \cos \bar{\theta}(s) = 0, 0 < s < \bar{s}.$$

When this equation is integrated over $(0, \bar{s})$ and $\dot{\bar{\theta}}(0) = \dot{\bar{\theta}}(\bar{s}) = 0$ is used, one obtains the first of equations (6.1iii) with $p_1^1 = 0$. From the equation,

$$(l/2)\dot{\bar{\theta}}^2 = -\sin \bar{\theta},$$

it follows that $-\pi \leq \bar{\theta} \leq 0$ if $l > 0$ and $0 \leq \bar{\theta} \leq \pi$ if $l < 0$. Since the choice $l < 0$ amounts to a rotation through π of the extremal corresponding to $l > 0$ or, to a change in orientation transforming $\bar{\theta}$ into $-\bar{\theta}$, we may assume $l > 0$ and $-\pi \leq \bar{\theta} \leq 0$. In this case, $p_1^2 = -d: = -|p_0 - p_1|$ in the second equation of (6.1iii). We may rewrite (6.1iii):

$$(6.1iii') \quad \int_0^{\bar{s}} \cos \bar{\theta} = 0, \int_0^{\bar{s}} \sin \bar{\theta} = -d.$$

From (6.1ii), $\dot{\bar{\theta}}(\bar{s}) = 0$ is equivalent to $\sin \bar{\theta}(\bar{s}) = 0$. Since $\sin \bar{\theta}(s) \leq 0$ for $0 \leq s \leq \bar{s}$ and $\bar{\theta}$ is continuous, there are only two possibilities: $\bar{\theta}(\bar{s}) = 0$ or $\bar{\theta}(\bar{s}) = -\pi$.

Altogether we have shown that (6.1) may be replaced by the simpler system

$$(6.3) \quad \begin{aligned} (i) \quad & \frac{l}{2} \dot{\bar{\theta}}^2(s) = -\sin \bar{\theta}(s), \quad 0 \leq s \leq \bar{s}; \quad \bar{\theta}(0) = 0, \quad \bar{\theta}(\bar{s}) = 0 \\ & \text{or } -\pi, \\ (ii) \quad & \int_0^{\bar{s}} \sin \bar{\theta} = -d. \end{aligned}$$

If s is interpreted as physical time, $l\bar{\theta}$ as the displacement along a circle of radius l , then (6.3i) represents the pendulum equation with pendulum length l , unit mass and unit force downward, starting from horizontal position with velocity 0 at $s = 0$ and reaching velocity 0 again at $s = \bar{s}$ when $\bar{\theta} = 0$ or $-\pi$. The pendulum swings from horizontal position $\bar{\theta} = 0$ at $s = 0$ through one or more half-swings to horizontal position at $s = \bar{s}$. The kinetic analogue of the elastica equation was discovered by G. Kirchhoff; see [8, p. 399].

One interpretation of the interpolation condition (6.3ii) is that the time integral of the kinetic energy $\frac{1}{2}(l\dot{\bar{\theta}})^2$ divided by the maximum kinetic energy, l , is the prescribed "minimum time" d . The length of the pendulum is the main unknown of the problem.

The solution of (6.3i) for various values of l can be derived from the solutions of the same system for $l = 2$. Indeed, the transformation

$$(6.4) \quad \bar{\theta}(s) = \bar{\theta}(\sqrt{2/l} s), \quad \bar{s} = \sqrt{l/2} \tilde{s}$$

converts (5.3i) to the "normalized system":

$$(6.5) \quad \dot{\bar{\theta}}^2(s) = -\sin \bar{\theta}(s), \quad 0 \leq s \leq \tilde{s}; \quad \bar{\theta}(0) = 0, \quad \bar{\theta}(\tilde{s}) = 0 \text{ or } -\pi.$$

The solution of the pendulum equation (6.5) with $\bar{\theta}(\tilde{s}) = -\pi$ is well known. It is explicitly given by

$$(6.6) \quad \bar{\theta}(s) = -\frac{\pi}{2} - 2 \arcsin[2^{-1/2} \operatorname{sn}(2^{-1/2}(s - \frac{\tilde{s}}{2}))], \quad 0 \leq s \leq \tilde{s},$$

where $2^{-1/2}\tilde{s}$ is the half-period of the Jacobi function $\operatorname{sn}(u) = \operatorname{sn}(u; 2^{-1/2})$ and $\arcsin$ is the branch of the inverse of $\sin$ with range $[-\frac{\pi}{2}, \frac{\pi}{2}]$. For $\tilde{s}$ and $\tilde{u} = u(\tilde{\theta}) = \int_0^{\tilde{s}} \dot{\bar{\theta}}^2$

we find in terms of the complete elliptic integrals of the first and second kind [13]:

$$(6.7)(i) \quad \bar{s} = \int_0^{\pi} \frac{d\theta}{\sqrt{\sin \theta}} = 2\sqrt{2} \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-t^2/2)}} = 2\sqrt{2} K(2^{-1/2}),$$

and

$$(ii) \quad \bar{U} = \int_0^{\pi} \sqrt{\sin \theta} d\theta = 2\sqrt{2} \int_0^1 \sqrt{\frac{1-t^2}{1-t^2/2}} dt = 2\sqrt{2} [2E(2^{-1/2}) - K(2^{-1/2})].$$

The analytic continuation of $\bar{\theta}$ (also denoted as $\bar{\theta}$) to all of $\mathbb{R}$ is given by $\bar{\theta}(s) = -\bar{\theta}(-s)$ for $s < 0$ and

$$(6.8) \quad \bar{\theta}(s) = \begin{cases} \bar{\theta}(2\bar{s}-s) & \text{for } \bar{s} \leq s \leq 2\bar{s} \\ \bar{\theta}(s-2k\bar{s}) & \text{for } 2k\bar{s} \leq s \leq 2(k+1)\bar{s}, \quad k = 1, 2, \dots \end{cases}$$

It is seen that $\bar{\theta}_{[0, 2\bar{s}]}$ also solves (6.5), the value at the new boundary point $2\bar{s}$ being 0. In general $\bar{\theta}_{[0, k\bar{s}]}$ ($k = 1, 2, \dots$) solves (6.5), with $\bar{\theta}(k\bar{s}) = -\pi$ or 0 depending on whether k is odd or even. These are all the solutions of (6.5) with free right end-point.

We now use the solution $\bar{\theta}$ of the normalized problem to express the general solution of the boundary value problem (6.3i) for fixed $\bar{s} > 0$. It is given by

$$(6.9) \quad \begin{aligned} (i) \quad \bar{\theta}(s) &= \bar{\theta}(k \frac{\bar{s}}{s} s), \quad 0 \leq s \leq \bar{s}, \\ (ii) \quad l &= 2(\bar{s}/k\bar{s})^2 \end{aligned}$$

where k is any positive integer. To satisfy the remaining condition (6.3ii) we must have

$$\begin{aligned} -d &= \int_0^{\bar{s}} \sin \bar{\theta}(s) ds = \int_0^{\bar{s}} \sin \bar{\theta}(k \frac{\bar{s}}{s} s) ds = \\ &= \frac{\bar{s}}{k\bar{s}} \int_0^{k\bar{s}} \sin \bar{\theta}(s) ds = -\frac{\bar{s}}{s} \bar{U}, \end{aligned}$$

hence

$$(6.10) \quad \bar{s} = (\bar{s}/\bar{U})d,$$

independent of k .

We write $\bar{\theta}_k$ for the solution corresponding to $k (= 1, 2, \dots)$, l_k for the corresponding parameter value in (6.3i) and $\bar{U}_k$ for the value of the curvature functional for $\bar{\theta}_k$. Thus,

$$\begin{aligned}
 (i) \quad & \bar{\theta}_k(s) = \bar{\theta}(k \frac{\bar{U}}{d} s), \quad 0 \leq s \leq \bar{s} = (\bar{s}/\bar{U})d, \\
 (6.11) \quad & (ii) \quad l_k = 2(d/k\bar{U})^2 = 2(\bar{s}/k\bar{s})^2, \quad k = 1, 2, \dots, \\
 & (iii) \quad \bar{U}_k = \int_0^{\bar{s}} \bar{\theta}_k^2(s) ds = k^2 \bar{U}^2/d = k^2 (\bar{s}/\bar{s}) \bar{U}.
 \end{aligned}$$

All these solutions have the same arc length $\bar{s} = d\bar{s}/\bar{U} \approx 2.2d$. Also $\bar{\theta}_k(s) = \bar{\theta}_1(ks)$ for $0 \leq s \leq \bar{s}/k$ and $\bar{\theta}_k(s) = \bar{\theta}_1(k\bar{s} - s)$ for $\bar{s}/k \leq s \leq 2\bar{s}/k$, etc. Thus the curve represented by $\bar{\theta}_k$ consists of k congruent arcs, all similar (contracted by the factor $1/k$) to $\bar{\theta}_1$. The curve whose normal representation is $\bar{\theta}_k$ is characterized as that arc of the simple elastica whose endpoints are inflection points and which has $k - 1$ internal inflection points (it belongs to class $E_{(k-1)}$ in the notation of §4). For the above physical interpretation the result means: If the ratio of the time integral of the kinetic energy to the maximum kinetic energy (in a motion from the horizontal to the horizontal position) is to be the fixed number d then the pendulum must have one of the lengths $l_k = 2(d/k\bar{U})^2$ ($k = 1, 2, \dots$) and the pendulum makes k half-swings in total time $\bar{s} = (\bar{s}/\bar{U})d$.

We summarize the results in

Proposition 6.1. There are countably many extremal P-interpolants E_k interpolating a two-point configuration with $|p_0 - p_1| = d > 0$, one for each $k = 0, 1, 2, \dots$. E_0 is the trivial ray interpolant; E_k is an arc of the simple elastica with inflection points at the terminals and $k - 1$ internal inflection points. The normal representation $\bar{\theta}_k$ of E_k is given by (6.11), where $\bar{\theta}$ is the elliptic function (6.6). Each of the E_k ($k = 1, 2, \dots$) has the same length $\bar{s} = (\bar{s}/\bar{U})d$, where $\bar{s}$ and $\bar{U}$ are given by (5.7). The extremal value of the curvature functional for E_k is $\bar{U}_k = k^2 \bar{U}_1$ ($k = 0, 1, 2, \dots$), where $\bar{U}_1 = \bar{U}^2/d$.

Remark 6.1. It is shown in [5] that none of the extremals $E_1, E_2, \dots$ provides a local minimum.

B. Ray and Rectangular Configurations.

Corollary 6.2. Suppose $P = \{0, p_1, \dots, p_m\}$ is the "ray" configuration where $p_i = (s_i, 0)$ and $0 < s_1 < \dots < s_m$. Countably many nontrivial extremal P -interpolants are obtained from the 2-point extremals of Proposition 6.1 as follows. Let $\bar{x}_{[0, \bar{s}_1]}$ represent any of the nontrivial extremals for the configuration $\{0, p_1\}$. Define $\bar{x}_{[\bar{s}_1, \bar{s}_2]}$ as one of the $\{p_1, p_2\}$ interpolants of Proposition 6.1 or the negative of it so that $\dot{\bar{x}}(\bar{s}_1 - 0) = \dot{\bar{x}}(\bar{s}_1 + 0)$. Continue in this way to the intervals $[\bar{s}_2, \bar{s}_3], \dots, [\bar{s}_{m-1}, \bar{s}_m]$. The obtained curve represented by $\bar{x}$ is an extremal P -interpolant.

Corollary 6.3. Suppose $P = \{p_0, p_1, \dots, p_m\}$ is the "rectangular" configuration where the angle between $\overline{p_{i-1}p_i}$ and $\overline{p_i p_{i+1}}$ ($i = 1, \dots, m-1$) is either 0 or $\pm \pi/2$. Let $\bar{x}$ represent a P -interpolant such that the segment from p_i to p_{i+1} ($i = 0, \dots, m-1$) is any of the extremals of Proposition 6.1 (including the trivial one) and so that $\dot{\bar{x}}$ is continuous. In this way countably many extremal P -interpolants are obtained for any rectangular configuration P .

C. Angle-Constrained Two Point Extremal Interpolants

The boundary conditions

$$(6.12) \quad \dot{\bar{x}}(0) \cdot (p_1 - p_0) = \gamma_0, \quad \dot{\bar{x}}(\bar{s}) \cdot (p_1 - p_0) = \gamma_1, \quad 0 \leq \gamma_i \leq |p_1 - p_0|$$

are added to the problem of Section A, replacing the zero curvature endpoint conditions.

Proposition 6.2. There are countably many extremal $\{p_0, p_1\}$ -interpolants constrained by conditions (6.12) if $\gamma_0 = \gamma_1$.

Proof. Suppose $\bar{\theta}$ is the normal representation of the sought extremal. It is easily seen that one can normalize $\bar{\theta}$ so that $\bar{\theta}$ satisfies equation (6.31), except that the values of $\bar{\theta}$ at 0 and $\bar{s}$ are not 0 and $-\pi$, but given numbers θ_0, θ_1 , $0 \geq \theta_0 \geq -\frac{\pi}{2}$, $\theta_1 = \theta_0$ or $\theta_1 = \theta_0 - \pi$. It follows that the solution of the problem is a symmetric arc of the curve (6.9i), with k even if $\theta_0 = \theta_1$, k odd if $\theta_0 = \theta_1 + \pi$, scaled and moved so that the terminals are p_0, p_1 .

D. Regular Configurations.

Suppose $P = \{p_0, p_1, \dots, p_m\}$ is a "regular" configuration, by which we mean each segment $\overline{p_i p_{i+1}}$ ($i = 0, \dots, m-1$) is of the length d and makes the same (exterior) angle α , $\pi < \alpha \leq 2\pi$, with the following segment $\overline{p_{i+1} p_{i+2}}$. We seek an angle-constrained P -interpolant $\bar{x}$ for which

$$(6.13) \quad \dot{\bar{x}}(0) \cdot (p_1 - p_0) = \dot{\bar{x}}(\bar{s}_m) \cdot (p_m - p_{m-1}) = d \sin \frac{\alpha}{2}.$$

Corollary 6.4. For each regular configuration $P = \{p_0, p_1, \dots, p_m\}$ there are infinitely many extremal P -interpolants constrained by the condition (6.13), which consist of m congruent segments.

Proof. Let $\bar{z}$ represent one of the infinitely many extremal $\{p_0, p_1\}$ -interpolants E_0 of length $\bar{s}_1$, constrained by

$$\dot{\bar{z}}(0) \cdot (p_1 - p_0) = \dot{\bar{z}}(\bar{s}_1) \cdot (p_1 - p_0) = d \sin \frac{\alpha}{2}$$

and such that $\dot{\bar{z}}(0) \times (p_1 - p_0) = -\dot{\bar{z}}(\bar{s}_1) \times (p_1 - p_0)$. Let $\bar{x}$ represent the uniquely defined P -interpolant E that extends E_0 by congruent pieces. Then $\bar{x}$ has continuous slope and curvature and satisfies (6.13); hence E is one of the sought extremals.

E. Length-Prescribed Two Point Extremal Interpolants.

We assume now the length $\bar{s}$ of the extremal $\{p_0, p_1\}$ -interpolant $\bar{E}$ is prescribed. Suppose $\bar{\theta}$ is the normal representation of $\bar{E}$. With the proper choice of the coordinate system we conclude, using Proposition 3.2, that $\bar{\theta}$ must satisfy the following conditions (assuming $\bar{s} > d$)

$$(6.14) \quad \begin{aligned} & \text{(i)} \quad \dot{\bar{\theta}}(0) = \dot{\bar{\theta}}(\bar{s}) = 0, \\ & \text{(ii)} \quad \frac{1}{2} \dot{\bar{\theta}}^2(s) = -\sin \bar{\theta}(s) + \lambda, \quad 0 \leq s \leq \bar{s}, \quad \lambda > 0, \quad \lambda \in \mathbb{R}, \\ & \text{(iii)} \quad \int_0^{\bar{s}} \cos \bar{\theta}(s) ds = 0, \quad \int_0^{\bar{s}} \sin \bar{\theta}(s) ds = -d. \end{aligned}$$

Set $\bar{\theta}(0) = \theta_0$ ($-\frac{\pi}{2} \leq \theta_0 \leq \frac{\pi}{2}$); then

$$(6.15) \quad \lambda = \sin \theta_0 .$$

In particular, $-1 \leq \lambda \leq 1$. We may again interpret Equations (6.14) as describing the motion of a pendulum swinging from some position θ_0 where $\dot{\theta} = 0$ in fixed time $\bar{s}$ to another position where $\dot{\theta} = 0$ and so that the time integral of the kinetic energy satisfies a certain condition.

By (6.14ii), $\dot{\bar{\theta}}(\bar{s}) = 0$ if and only if $\bar{\theta}(\bar{s})$ is θ_0 or $-\pi - \theta_0$. The condition $\int_0^{\bar{s}} \cos \bar{\theta} = 0$ follows from (6.14ii). Thus, (6.14) may be replaced by the simpler system

$$(6.16) \quad \begin{aligned} (i) \quad & \frac{g}{2} \bar{\theta}^2(s) = \sin \theta_0 - \sin \bar{\theta}(s), \quad 0 \leq s \leq \bar{s}; \quad \bar{\theta}(0) = \theta_0, \\ & \bar{\theta}(\bar{s}) = \theta_0 \text{ or } -\pi - \theta_0, \\ (ii) \quad & \int_0^{\bar{s}} \sin \bar{\theta} = -d. \end{aligned}$$

We make the transformation (6.4) again, to obtain the normalized system:

$$(6.17) \quad \bar{\theta}^2(s) = \sin \theta_0 - \sin \bar{\theta}(s), \quad 0 \leq s \leq \bar{s}, \quad \bar{\theta}(0) = \theta_0, \quad \bar{\theta}(\bar{s}) = -\pi - \theta_0.$$

The solution is given by

$$(6.18i) \quad \bar{\theta}(s) = \bar{\theta}(s; \theta_0) = -\frac{\pi}{2} - 2 \arcsin[q \operatorname{sn}(2^{-1/2}(s-\bar{s}/2); q^2)] ,$$

$$q = |\sin(\pi/4 + \theta_0/2)| ,$$

where $2^{-1/2}\bar{s}$ is the half-period of the Jacobi function $\operatorname{sn}(u; q)$:

$$(6.18ii) \quad \bar{s} = \bar{s}(\theta_0) = - \int_{\theta_0}^{-\pi-\theta_0} \frac{d\theta}{\sqrt{\sin \theta_0 - \sin \theta}} = \int_{-\theta_0}^{\pi+\theta_0} \frac{d\theta}{\sqrt{\sin \theta_0 + \sin \theta}} =$$

$$2 \int_{-\theta_0}^{\pi/2} \frac{d\theta}{\sqrt{\sin \theta_0 + \sin \theta}} .$$

We observe that $\tilde{s}(\theta_0)$ increases monotonically from 0 to ∞ as θ_0 varies from $-\pi/2$ to $\pi/2$. For $\tilde{d} = - \int_0^{\tilde{s}} \sin \tilde{\theta}$ and $\tilde{U} = \int_0^{\tilde{s}} \tilde{\theta}^2$ we have

$$(6.18iii) \quad \tilde{d} = \tilde{d}(\theta_0) = 2 \int_{-\theta_0}^{\pi/2} \frac{\sin \theta}{\sqrt{\sin \theta_0 + \sin \theta}} d\theta ,$$

$$(6.18iv) \quad \tilde{U} = \tilde{U}(\theta_0) = 2 \int_{-\theta_0}^{\pi/2} \sqrt{\sin \theta_0 + \sin \theta} d\theta .$$

We also observe the identity

$$(6.19) \quad \tilde{s}(\theta_0) \sin \theta_0 + \tilde{d}(\theta_0) = \tilde{U}(\theta_0) .$$

The analytic continuation of $\tilde{\theta}$ (also denoted as $\tilde{\theta}$) to all of $\mathbb{R}$ is given, as before, by (6.8). Then we put

$$(6.20) \quad \begin{aligned} (i) \quad \tilde{\theta}_k(s) &= \tilde{\theta}(k \frac{\tilde{s}}{\tilde{s}} s), \quad 0 \leq s \leq \tilde{s}, \quad k = 1, 2, \dots, \\ (ii) \quad l_k &= 2(\tilde{s}/k\tilde{s})^2 . \end{aligned}$$

These quantities still depend on θ_0 . θ_0 must be determined so that the last condition

$$(6.20iii) \quad \begin{aligned} -d &= \int_0^{\tilde{s}} \sin \tilde{\theta}_k(s) ds = \int_0^{\tilde{s}} \sin \tilde{\theta}(k\tilde{s}s/\tilde{s}) ds \\ &= \frac{\tilde{s}}{k} \int_0^{\tilde{s}} \sin \tilde{\theta}(s) ds = - \frac{\tilde{s}}{\tilde{s}(\theta_0)} \tilde{d}(\theta_0) \end{aligned}$$

is satisfied. Thus θ_0 is determined from the equation

$$(6.21) \quad \frac{\tilde{d}(\theta_0)}{\tilde{s}(\theta_0)} = \frac{d}{\tilde{s}} ,$$

where $\tilde{d}$ and $\tilde{s}$ are given in (6.18). As θ_0 varies from $-\pi/2$ to 0, $\tilde{d}(\theta_0)$ increases from 0 to $2 \int_0^{\pi/2} \sin^{1/2} \theta d\theta$ and then decreases from this value to $-\infty$ as θ_0 varies from

0 to $\pi/2$. Clearly $\bar{d}(\theta_*) = 0$ for some θ_* between 0 and $\pi/2$ ($\theta_* \approx 40^\circ$). The ratio $\bar{d}(\theta_0)/\bar{s}(\theta_0)$ can be seen to decrease monotonically on $[-\pi/2, \theta_*]$ with values in the entire interval $[0, 1]$. Therefore, for any $0 < d < \bar{s}$, there is a unique θ_0 in $[-\pi/2, \theta_*)$ such that (6.21) holds. For this unique θ_0 , $\bar{s} = \bar{s}(\theta_0)$ is determined according to (6.18ii), then $\bar{\theta}_k$ and l_k from (6.20). Together with $\lambda = \sin \theta_0$, these quantities satisfy the original system. $\bar{U}(\theta_0)$ is given by (6.18iv) and the value of U for $\bar{\theta}_k$ by

$$(6.22) \quad \bar{U}_k = \int_0^{\bar{s}} \bar{\theta}_k^2(s) ds = k^2 \frac{\bar{s}(\theta_0)}{\bar{s}} \bar{U}(\theta_0).$$

Altogether we have proved

Proposition 6.3. There are countably many extremal $\{p_0, p_1\}$ -interpolants $\bar{E}_1, \bar{E}_2, \dots$ of prescribed length $\bar{s} > |p_0 - p_1|$. Their normal representations are given explicitly by equations (6.18) and (6.20), with the angle of inclination θ_0 at p_0 determined from (6.21). For the value of the functional U , the relation $\bar{U}_k = k^2 \bar{U}_1$ ($k = 1, 2, \dots$) holds.

Remark 6.2. The curves of Proposition 6.2 are subarcs of inflexional elastica (cf. Remark 2.4). For illustrations see [8, p. 404, Figures 48-53].

Remark 6.3. In the beam interpretation the joint at p_0 exerts a force R whose tangential component is $\sin \theta_0/l$ and whose normal component is $\cos \theta_0/l$. Thus R acts along the line joining p_0 to p_1 . The magnitude of the force is $1/l$. For fixed $\bar{s}$ and d , the force R_k in the mode $\bar{E}_k$ has magnitude $k^2 R_1$. For $\theta_0 < \pi/2$ the tangential force on the joints is a pressure, for $\theta_0 > \pi/2$ it is a pull.

§7. Examples of Closed Extremals

Let $P = \{p_0, p_1, \dots, p_m\}$ be a configuration as in §2. If $x \in H_2^{\text{reg}}$ is such that $x(t_i) = p_i$ ($i = 0, \dots, m$) for some $0 < t_0 < \dots < t_m < 1$, and besides $x^{(k)}(0+) = x^{(k)}(1-)$ for $k = 0, 1, 2$, we say x represents an admissible closed P-interpolant with knots p_i . Suppose $\bar{x}$ represents an extremal closed P-interpolant, i.e. $\bar{x}$ makes the curvature functional U stationary in the family of admissible closed P-interpolants. Then the graph of $\bar{x}$ is a closed curve which has continuous curvature everywhere, and the curvature is continuously differentiable at all points other than the knots.

In this paragraph we examine four classes of closed extremals:

- A. Closed extremals of prescribed length with no knots.
- B. Closed symmetric extremals with two knots.
- C. Closed extremals for rectangular configurations.
- D. Closed extremals for regular polygons.

A. If $\bar{x}$ represents a closed extremal with no knots, of prescribed length $\bar{s} > 0$, parametrized with respect to arc length, $\kappa(s)$ its curvature at s , then one finds, as in Proposition 2.3, that

$$(7.1) \quad \begin{aligned} (i) \quad & \bar{x} \in C^\infty[0, \bar{s}], \quad x^{(k)}(0) = x^{(k)}(\bar{s}), \quad k = 0, 1, 2, \\ (ii) \quad & 2\ddot{\bar{x}} + 3\kappa^2\bar{x} - \lambda\bar{x} = c, \quad c \in \mathbb{R}^2, \\ (iii) \quad & \lambda = \frac{1}{\bar{s}} \int_0^{\bar{s}} \kappa^2. \end{aligned}$$

Conversely if $\bar{x}$ satisfies (7.1) then $\bar{x}$ represents a closed extremal with no knots, of prescribed length $\bar{s}$, parametrized by arc length. For the normal representation $\bar{\theta}$, where

$$\bar{x}^1(s) = \int_0^s \cos \bar{\theta}, \quad \bar{x}^2(s) = \int_0^s \sin \bar{\theta}, \quad (7.1) \text{ gives}$$

$$\begin{aligned}
(7.2) \quad & (i) \quad \bar{\theta} \in C^\infty[0, \bar{s}], \quad \int_0^{\bar{s}} \cos \bar{\theta} = \int_0^{\bar{s}} \sin \bar{\theta} = 0, \\
& \bar{\theta}(\bar{s}) = \bar{\theta}(0) + 2k\pi \quad (k = 0, \pm 1, \dots); \quad \dot{\bar{\theta}}(0) = \dot{\bar{\theta}}(\bar{s}), \\
& (ii) \quad \dot{\bar{\theta}}^2 = c^1 \cos \bar{\theta} + c^2 \sin \bar{\theta} + \lambda, \\
& (iii) \quad \lambda = \frac{1}{\bar{s}} \int_0^{\bar{s}} \dot{\bar{\theta}}^2.
\end{aligned}$$

Proposition 7.1. For each $k = 1, 2, \dots$ there exist exactly two closed extremals with no knots of prescribed length $\bar{s} > 0$. These are the circle of radius $\bar{s}/(2k\pi)$ transversed k times and a contracted figure eight configuration traversed k times.

Proof. We can omit (iii) in (7.2) since it follows from (i) and (ii). We write $A \sin(\bar{\theta} + \alpha)$ with $A \geq 0, \alpha \in T$ for $c^1 \cos \bar{\theta} + c^2 \sin \bar{\theta}$. If $\bar{\theta}$ represents an extremal then $\bar{\theta} - \alpha$ represents the same extremal rotated by the angle α . Therefore, (7.2ii) may be replaced by $\dot{\bar{\theta}}^2 = A \sin \bar{\theta} + \omega_0^2$, where $\omega_0^2 > 0$. ($\omega_0^2 = 0$ means $\lambda = 0$, which is impossible by (7.2iii)). We may also assume $k = 0, 1, \dots$, in (7.2i). Thus, (7.2) is replaced by

$$\begin{aligned}
(7.3) \quad & (i) \quad \bar{\theta} \in C^\infty[0, \bar{s}], \quad \int_0^{\bar{s}} \cos \bar{\theta} = \int_0^{\bar{s}} \sin \bar{\theta} = 0, \\
& \bar{\theta}(\bar{s}) = \bar{\theta}(0) + 2k\pi \quad (k = 0, 1, 2, \dots); \quad \dot{\bar{\theta}}(0) = \dot{\bar{\theta}}(\bar{s}), \\
& (ii) \quad \dot{\bar{\theta}}^2 = A \sin \bar{\theta} + \omega_0^2, \quad A \geq 0, \omega_0 > 0.
\end{aligned}$$

Case 1. $A = 0$. In this case we may assume $\bar{\theta}_0 = 0$. Then $\bar{\theta}(s) = \omega_0 s$ and $\omega_0 \bar{s} = 2k\pi$, hence $\omega_0 = 2k\pi/\bar{s}$. For $k = 1, 2, \dots, \bar{\theta}_k(s) = 2k\pi s/\bar{s}$ satisfies all conditions. $\bar{\theta}_k$ represents a circle of radius $\bar{s}/2k\pi$, traversed k times. The value of U for $\bar{\theta}_k$ is $(2k\pi/\bar{s})^2$.

Case 2. $A > 0, \omega_0^2 > A$. We may assume $A = 1$ since if $\bar{\theta}$ is a solution of (7.3) for $A > 0$, of length $\bar{s} > 0$, then $\bar{\theta}$ defined by $\bar{\theta}(A^{-1/2}s)$ is a solution for $A = 1, \omega_0^2 = A^{-1}\omega_0^2 > 1$, of length $\bar{s} = A^{1/2}\bar{s}$. If $\bar{\theta}$ satisfies (7.3) then $\bar{\theta}(s)$ is uniquely defined by

$$(7.4) \quad s = \int_0^{\bar{\theta}(s)} \frac{d\psi}{\sqrt{\omega_0^2 + \sin \psi}},$$

k in (7.3i) must be positive, and ω_0^2 is uniquely defined by

$$\bar{s} = \int_0^{2k\pi} \frac{d\varphi}{\sqrt{\omega_0^2 + \sin \varphi}} .$$

For $\bar{\theta}$ defined in this way we have, after a change of variable,

$$\begin{aligned} \int_0^{\bar{s}} \sin \bar{\theta}(s) ds &= \int_{-k\pi}^{k\pi} \frac{\sin \varphi}{\sqrt{\omega_0^2 + \sin \varphi}} d\varphi \\ &= k \left[\int_0^{\pi} \frac{\sin \varphi}{\sqrt{\omega_0^2 + \sin \varphi}} d\varphi - \int_0^{\pi} \frac{\sin \varphi}{\sqrt{\omega_0^2 - \sin \varphi}} d\varphi \right] < 0 , \end{aligned}$$

which contradicts (7.3i). Thus no solution exists for $\omega_0^2 > A$.

Case 3. $A > 0$, $\omega_0^2 < A$. We may again assume $A = 1$, thus $\omega_0 < 1$. Conveniently replace θ by $\theta + \pi$, and write $\sin \theta_*$ for ω_0^2 , with $0 < \theta_* < \frac{\pi}{2}$. Thus (7.3ii) is replaced by

$$(7.3ii') \quad \dot{\bar{\theta}}^2 = \sin \theta_* - \sin \bar{\theta} .$$

In this case $-\pi - \theta_* \leq \bar{\theta}(s) \leq \theta_*$, thus we must have $k = 0$ in (7.3i). Since $\bar{\theta}(s)$ cannot be monotone, we must have $\dot{\bar{\theta}}(s) = 0$ and $\sin \bar{\theta}(s) = \sin \theta_*$ for some s ; it is no restriction to assume that this happens for $s = 0$, and $\bar{\theta}(0) = \theta_*$, $\dot{\bar{\theta}}(0) = 0$. As s increases from 0 to some s_* , $\bar{\theta}(s)$ decreases from θ_* to $-\pi - \theta_*$, when $\dot{\bar{\theta}}(s_*) = 0$. Since $-\pi - \theta_* \neq \theta_*$ the curve cannot be closed yet, and as s increases further, $\bar{\theta}(s)$ increases up to the value θ_* , which is attained for some $s = s_{**}$, and $\dot{\bar{\theta}}(s_{**}) = 0$. If we set $\theta_1(t) = \bar{\theta}(2s_* - t)$, we see that θ_1 satisfies (7.3ii') and $\theta_1(s_*) = \bar{\theta}(s_*)$, hence $\theta_1 = \bar{\theta}$, i.e.

$$\bar{\theta}(s_* + t) = \bar{\theta}(s_* - t)$$

and, in particular, $s_{**} = 2s_*$. Thus the curve obtained is symmetric w.r.t. the point $s = 0$.

If we put $\theta_2(s) = -\pi - \bar{\theta}(s_* - s)$ then we see that θ_2 satisfies (7.3ii') and

$$\theta_2(0) = \bar{\theta}(0) = \theta_* , \text{ hence}$$

$$\bar{\theta}(s_* - t) = -\pi - \bar{\theta}(t) ,$$

the curve obtained is symmetric with respect to the line $\theta = 0$ and $\bar{\theta}(s_*/2) = -\pi/2$.

The curve will be closed iff the conditions $\int_0^{s_{**}} \cos \bar{\theta} = 0$, $\int_0^{s_{**}} \sin \bar{\theta} = 0$ are satisfied. The first equation follows directly from the symmetry of the curve. We are left with

$$(7.31'') \quad 0 = \int_0^{s_{**}} \sin \bar{\theta}(s) ds = 4 \int_{-\pi/2}^{\theta_*} \frac{\sin \varphi}{\sqrt{\sin \theta_* - \sin \varphi}} d\varphi = 0.$$

Let the last integral be denoted as $I(\theta_*)$. Clearly $I(0) = -\frac{1}{2} \tilde{U} < 0$ and $I(\frac{\pi}{2}) = +\infty$. Thus, there is a value θ_* between 0 and $\pi/2$ for which (7.31'') holds, and it is easily seen that there is only one such value (approximately $\theta_* = 40^\circ$). With this value of θ_* we have obtained a closed extremal E^* of length s_{**} . It is an analytic curve, crossing itself at $s = \frac{1}{2} s_{**}$, and consists of 2 congruent loops, each symmetric w.r.t. the same axis (an illustration appears in [8, p. 404] as an example of an inflexional elastica). By proper scaling the curve will have the prescribed length $\bar{s}$. The differential equation for the normal representation of the curve E_1^* is

$$s^{2\theta_1^{*2}} = \sin \theta_* - \sin \theta_1^*, \quad \theta_1^*(0) = \theta_*, \quad \theta_1^*\left(\frac{\bar{s}}{4}\right) = \frac{-\pi}{2}.$$

Thus, the inverse function $\theta \rightarrow s(\theta)$ is given by

$$s(\theta) = a \int_{\theta}^{\theta_*} \frac{d\varphi}{\sqrt{\sin \theta_* - \sin \varphi}}, \quad \frac{-\pi}{2} \leq \theta \leq \theta_*$$

where the constant a is determined from

$$\frac{\bar{s}}{4} = a \int_{-\pi/2}^{\theta_*} \frac{d\varphi}{\sqrt{\sin \theta_* - \sin \varphi}}.$$

The other extremals $E_2^*, E_3^*, \dots$ in this sequence are obtained by traversing E_1^* 2, 3, ... times with scale factor $\frac{1}{2}, \frac{1}{3}, \dots$, thus their normal representations satisfy

$$\theta_k^*(s) = \theta_1^*(ks), \quad 0 \leq s \leq \bar{s}.$$

Case 4. $A_0 > 0$, $\omega_0^2 = A$. In this case the solution of (7.3i) is monotonically increasing in s but does not attain $\theta_0 + 2\pi$ for finite s .

Remark 7.1. The restriction of $\theta_1^*(s)$ to $[0, \frac{\bar{s}}{2}]$ represents a length-prescribed extremal (length = $\bar{s}/2$) interpolating the "loop configuration" $\{p_0, p_1\}$ with $p_0 = p_1$. This is not a closed extremal, although it is a closed curve. Each $\theta_1^*(ks)$ ($k = 1, 2, \dots; 0 \leq s \leq \frac{\bar{s}}{2}$) can also be considered as then.r. of such an interpolating extremal. The curvature functional (potential energy) for this extremal is seen to have the value

$$\frac{4k^2}{\bar{s}} \int_{-\pi/2}^{\theta_*} \frac{d\varphi}{\sqrt{\sin \theta_* - \sin \varphi}}^2 \sin \theta_* .$$

Clearly this function of $\bar{s}$ has no stationary value. It follows that there exists no extremal interpolant (unconstrained) for the loop configuration.

Remark 7.2. The length-prescribed extremal interpolating the loop configuration can also be obtained as the limiting case of the extremal of Sec. 6.E as $d \rightarrow 0$. There it was pointed out that $\tilde{d}(\theta_*) = 0$ for θ_* satisfying (see 6.18iii)

$$0 = \int_{-\theta_*}^{\pi/2} \frac{\sin \theta}{\sqrt{\sin \theta_* + \sin \theta}} d\theta .$$

Clearly, this is the above condition (7.3i').

B. We turn to the problem of closed extremals $\tilde{E}$ with two knots. We consider only extremals that are symmetric with respect to the line joining the two given knots. We assume $p_0 = (0, 0)$, $p_1 = (0, -d)$ with $d > 0$ are the knots and that $\dot{x} = (\dot{x}^1, \dot{x}^2)$ represents the extremal $\tilde{E}$, parametrized by arc length. If $\tilde{s}$ is the length of $\tilde{E}$ and $\dot{x}(0) = p_0$, then $\dot{x}^{(k)}(\tilde{s}) = \dot{x}^{(k)}(0)$ for $k = 0, 1, 2$, and because of the symmetry, $\dot{x}(\tilde{s}/2) = p_1$. Thus, we may assume

$$\begin{aligned} \dot{x}^1(s) &= -\dot{x}^1(\tilde{s}-s), \quad \dot{x}^2(s) = \dot{x}^2(\tilde{s}-s), \quad 0 \leq s \leq \tilde{s}, \\ (7.5) \quad \dot{x}^1(0) &= \dot{x}^1(\frac{\tilde{s}}{2}) = \dot{x}^1(\tilde{s}) = 0; \quad \dot{x}^2(0) = \dot{x}^2(\tilde{s}) = 0, \quad \dot{x}^2(\frac{\tilde{s}}{2}) = -d, \\ \dot{x}^1(0) &= \dot{x}^1(\tilde{s}) = 1, \quad \dot{x}^2(\frac{\tilde{s}}{2}) = \pm 1; \quad \dot{x}^2(0) = \dot{x}^2(\frac{\tilde{s}}{2}) = \dot{x}^2(\tilde{s}) = 0. \end{aligned}$$

If $\dot{\theta}$ is the normal representation of $\dot{E}$ then by (7.5) and Proposition 3.1,

$$\begin{aligned}
 (i) \quad & \cos \dot{\theta}(s) = \cos \dot{\theta}(\dot{s}-s), \quad \sin \dot{\theta}(s) = -\sin \dot{\theta}(\dot{s}-s), \\
 & 0 \leq s \leq \dot{s}, \\
 (ii) \quad & \int_0^{\dot{s}/2} \cos \dot{\theta} = 0, \quad \int_0^{\dot{s}/2} \sin \dot{\theta} = -d, \quad \int_0^{\dot{s}} \cos \dot{\theta} = 0, \\
 & \int_0^{\dot{s}} \sin \dot{\theta} = 0, \\
 (7.6) \quad (iii) \quad & \dot{\theta}(0) = 0, \quad \dot{\theta}(\frac{\dot{s}}{2}) = j\pi, \quad \dot{\theta}(\dot{s}) = 2j\pi; \quad j = 0 \text{ or } -1. \\
 (iv) \quad & \dot{\theta}(0) = \dot{\theta}(\frac{\dot{s}}{2}) = \dot{\theta}(\dot{s}) = 0, \\
 (v) \quad & \dot{\theta}^2(s) = \lambda_1^1 \cos \dot{\theta}(s) + \lambda_1^2 \sin \dot{\theta}(s), \quad 0 \leq s \leq \dot{s}/2, \\
 & = \lambda_2^1 \cos \dot{\theta}(s) + \lambda_2^2 \sin \dot{\theta}(s), \quad \dot{s}/2 \leq s \leq \dot{s}.
 \end{aligned}$$

$s = \dot{s}/2$ in (v) gives $\lambda_1^1 = \lambda_2^1 = 0$; substitution of (i) in (v) gives $\lambda_1^2 = -\lambda_2^2 = -\lambda$. Thus

(7.6v) becomes

$$\begin{aligned}
 (7.6v') \quad & \dot{\theta}^2(s) = -\lambda \sin \dot{\theta}(s), \quad 0 < s \leq \dot{s}/2, \\
 & = +\lambda \sin \dot{\theta}(s), \quad \dot{s}/2 \leq s \leq \dot{s}.
 \end{aligned}$$

When the conditions $\dot{\theta}(0) = 0$, $\dot{\theta}(\dot{s}/2) = 0$ or $-\pi$, $\dot{\theta}(0) = \dot{\theta}(\dot{s}/2) = 0$ are taken together with (7.6v'), it is seen that $\dot{\theta}_{[0, \dot{s}/2]}$ is one of the functions θ_k of §6, with $\bar{s}$ replaced by $\dot{s}/2$. The symmetry condition gives for $\dot{\theta}_{[\dot{s}/2, \dot{s}]}$:

$$\dot{\theta}(s) = -2j\pi - \dot{\theta}(\dot{s}-s), \quad \dot{s}/2 \leq s \leq \dot{s}.$$

Thus we have found all solutions of system (7.6) and have proved

Proposition 7.2. There are countably many closed extremals $\dot{E}_1, \dot{E}_2, \dots$ with n.r. $\dot{\theta}_1, \dot{\theta}_2, \dots$ with two knots which are symmetric with respect to the line joining the knots. $\dot{E}_k$ is obtained from the open extremal E_k of Proposition 6.1 by reflection at the line joining the knots. Each $\dot{E}_k$ has the same length $2\dot{s}$, where $\dot{s}$ is the length of the open E_k . The value of the

curvature functional U for the extremal E_k is $2k^2\bar{U}_1$ where $\bar{U}_1$ is the value for the open E_1 .

C. In this section we consider rectangular configurations as defined in Corollary 6.2.

Proposition 7.3. Let $P = \{p_0, \dots, p_m, p_0\}$ be a rectangular configuration as in Corollary 6.2. There exist countably many closed extremals with knots at $p_0, \dots, p_m$.

Proof. Since P is closed there must be an even number of right angles between consecutive segments $\overline{p_{i-1}p_i}$, $\overline{p_i p_{i+1}}$. To connect p_i to p_{i+1} we use either the trivial extremal or one of the 2-point open extremals of Proposition 6.1, with the proviso that we switch from one class of extremals to the other if the angle at p_i is a right angle, otherwise (if the angle is $\neq 0$) no switch is made. It is easy to see that infinitely many closed P -interpolants with continuous curvature everywhere can be obtained in this way.

D. Let $p_1, \dots, p_m$ be the vertices of a regular polygon ordered as they come when the polygon is traversed counter-clockwise. Define p_i for $i > m$ by periodicity: $p_i = p_{i-m}$. Let $P_{m,k} = \{p_1, p_{1+k}, \dots, p_{1+mk}\}$ for $k = 1, 2, \dots$. $P_{m,k}$ is a configuration of the kind that Corollary 6.4 applies to, and if the construction used there is applied to $P_{m,k}$ one obtains closed extremals. Thus we have

Proposition 7.4. For each regular configuration $P_{m,k}$ as described above there are infinitely many closed extremal P -interpolants, each composed of congruent segments. For each k , there is precisely one such extremal whose intersection with the polygonal path connecting the points of $P_{m,k}$ is precisely $P_{m,k}$. For $k = 1$, this extremal E_m^* circumscribes the polygon counter-clockwise and its representation x_m^* satisfies

$$(p_{i+1} - p_i) \cdot \dot{x}_m^*(s_i) = |p_{i+1} - p_i| \cos \frac{\pi}{m}, \quad i = 1, \dots, m.$$

If the polygon is inscribed in a unit circle, each of the m arcs of E_m^* may be expressed in terms of the inverse of its normal representation:

$$(7.7i) \quad s_m^*(\theta) = \frac{2 \sin \frac{\pi}{m} F(\frac{1}{2}\sqrt{2}, \psi)}{2E(\frac{1}{2}\sqrt{2}, \beta_m) - F(\frac{1}{2}\sqrt{2}, \beta_m)}, \quad \frac{\pi}{2} - \frac{\pi}{m} \leq \theta \leq \frac{\pi}{2} + \frac{\pi}{m},$$

where $\cos \psi = \sqrt{\sin \theta}$ and $\cos \beta_m = \sqrt{\cos \pi/m}$. In this case, the length s_m^* and the energy U_m^* of E_m^* are given explicitly by,

$$(7.7ii) \quad s_m^* = \frac{2m \sin \frac{\pi}{m} F(\frac{1}{2}\sqrt{2}, \beta_m)}{2E(\frac{1}{2}\sqrt{2}, \beta_m) - F(\frac{1}{2}\sqrt{2}, \beta_m)},$$

and

$$(7.7iii) \quad U_m^* = \frac{4m}{\sin \frac{\pi}{m}} [2E(\frac{1}{2}\sqrt{2}, \beta_m) - F(\frac{1}{2}\sqrt{2}, \beta_m)]^2.$$

Finally,

$$(7.8) \quad \frac{s_m^*(\theta)}{\theta} \rightarrow 1 \quad \text{as } m \rightarrow \infty,$$

so that the extremals have the unit circle as limit.

Proof. We sketch the verification of (7.7). Starting from the differential equation

$$\gamma_m^2 \dot{\theta}^2 = \sin \theta,$$

we obtain

$$s_m^*(\theta) = \gamma_m \int_{\alpha_m}^{\theta} \frac{d\varphi}{\sqrt{\sin \varphi}}, \quad \alpha_m \leq \theta \leq \pi - \alpha_m, \quad \alpha_m = \frac{\pi}{2} - \frac{\pi}{m}.$$

Since the distance between adjacent points of $P_{m,1}$ is $2 \cos \alpha_m$, the constant γ_m is determined from

$$\begin{aligned} \gamma_m &= \cos \alpha_m / \int_{\alpha_m}^{\pi/2} \sqrt{\sin \varphi} d\varphi \\ &= \frac{\sin \frac{\pi}{m}}{\sqrt{2} [2E(\frac{1}{2}\sqrt{2}, \beta_m) - F(\frac{1}{2}\sqrt{2}, \beta_m)]}. \end{aligned}$$

This gives (7.7i). (7.7ii) is immediate and (7.7iii) follows from,

$$U_m^* = 2m \int_0^{s_m^*/2m} \dot{\theta}^2 ds = \frac{2m}{Y_m} \int_m^{\pi/2} \sqrt{\sin \varphi} d\varphi .$$

The calculation (7.8) is routine.

Remark 7.3. It is shown in [5] that the extremals E_m^* are stable, i.e. they provide a local minimum for the curvature functional.

Appendix

Let $P = \{p_0, \dots, p_m\}$, be given and let $L_0 = \sum_{i=1}^m |p_{i+1} - p_i|$. We assume that P is not collinear.

Theorem. For every $L > L_0$ there exists a length-prescribed extremal P -interpolant of length L satisfying Definition 2.2.

Proof. The existence of a function $\bar{x}$, parametrized with respect to arc length, for which

$$U(\bar{x}) = \min\{U(x) : x \text{ is an admissible } P\text{-interpolant of length } L\}$$

is demonstrated in [6]. (The modification required for the ordering of the points in P is trivial.) If X denotes the closed subspace of $H_2[0, L]$ consisting of those functions vanishing at the knots,

$$0 \leq \bar{s}_0 < \bar{s}_1 < \dots < \bar{s}_m \leq L,$$

of $\bar{x}$, i.e., at those points $\bar{s}_i$ for which $\bar{x}(\bar{s}_i) = p_i$, $i = 0, \dots, m$, define f to be the mapping of X into $\mathbb{R}$ obtained by $f(y) = U(\bar{x} + y)$ and let H be the function such that $H(y) = S(\bar{x} + y) - L$, where S is the usual length functional (cf. (2.13ii)). Clearly,

$$f(0) = \min\{f(y) : H(y) = 0\},$$

and $H'(0)$ is surjective, since P is assumed non-collinear. The result follows from the Lagrange multiplier rule (see, e.g., [9, Theorem 1, p. 243]).

REFERENCES

1. S. Antmann, The shape of buckled nonlinearly elastic rings, ZAMP 21(1970), 422-438.
2. G. Birkhoff and C. de Boor, Piecewise polynomial interpolation and approximation, Approximation of Functions (H. L. Garabedian, editor), Elsevier, New York and Amsterdam, 1965, 164-190.
3. S. Fisher and J. Jerome, Stable and unstable elastica equilibrium and the problem of minimum curvature, J. Math. Anal. Appl. 53(1976), 367-376.
4. M. Golomb, Stability of Interpolating Elastica, Transactions of the 24th Conference of Army Mathematicians. ARO-Report 1, pp. 301-350, 1979.
5. M. Golomb, Stability of interpolating elastica, MRC Technical Summary Report #1852, May 1978.
6. J. Jerome, Smooth interpolating curves of prescribed length and minimum curvature, Proc. Amer. Math. Soc. 51(1975), 62-66.
7. E. Lee and G. Forsythe, Variational study of nonlinear spline curves, SIAM Review 15 (1973), 120-133.
8. A. Love, The Mathematical Theory of Elasticity, (4th ed.) Cambridge Univ. Press, London, 1927.
9. D. Luenberger, Optimization by Vector Space Methods, Wiley, New York, 1969.
10. M. Malcolm, Nonlinear spline functions, SIAM J. on Numer. Anal. 14(1977), 254-279.
11. L. Nirenberg, Topics in Nonlinear Functional Analysis, Courant Institute of Mathematical Sciences, New York University, 1974.
12. I. Tadjbakhsh and F. Odeh, Equilibrium states of elastic rings, J. Math. Anal. Appl. 18 (1967), 59-74.
13. Handbook of Mathematical Functions. National Bureau of Standards Applied Mathematics Series, 55 (1964).

MG/JJ/jvs

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER #2024	2. GOVT ACCESSION NO. AD-A083 867	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) EQUILIBRIA OF THE CURVATURE FUNCTIONAL AND MANIFOLDS OF NONLINEAR INTERPOLATING SPLINE CURVES		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Michael Golomb and Joseph Jerome		8. CONTRACT OR GRANT NUMBER(s) DAAG29-75-C-0024 MPS 74-02292 A01
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of 610 Walnut Street Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 6 - Spline Functions and Approximation Theory
11. CONTROLLING OFFICE NAME AND ADDRESS See Item 18 below		12. REPORT DATE November 1979
		13. NUMBER OF PAGES 64
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES U. S. Army Research Office and National Science Foundation P. O. Box 12211 Washington, D. C. 20550 Research Triangle Park North Carolina 27709		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Nonlinear spline curves, elastica, manifolds of extremals, perturbation stability, mode, ray configuration, rectangular configuration		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) A detailed global and local analysis of smooth solutions of the variational problem (ii) $\delta \int_0^s \kappa^2(s) ds = 0$ subject to position function constraints (iii) $x(s_1) = p_1, 0 \leq s_0 < s_1 < \dots < s_m \leq \bar{s}$		

ABSTRACT (continued)

is carried out. Here $\{p_i\}_0^m \subset \mathbb{R}^2$ is prescribed, x is a vector-valued function with curvature $\kappa(s)$ at arc length s and the interpolation nodes s_i are free. Problem (1) may be viewed as the mathematical formulation of the draftsman's technique of curve fitting by mechanical splines.

Although most of the basic equations satisfied by these nonlinear splines curves have been known for a very long time, calculation via elliptic integral functions has been hampered by a lack of understanding concerning what precise information must be specified for the stable determination of a smooth, unique interpolant modelling the thin elastic beam. In this report, sharp characterizations are derived for the extremal interpolants as well as structure theorems in terms of inflection point modes which guarantee uniqueness and well-posedness.

A certain type of stability is introduced and studied and shown to be related to (linearization) concepts associated with piecewise cubic spline functions, which have been studied for decades as a simplification of the nonlinear spline curves. Many examples are introduced and studied.