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METRICALLY UPPER SEMICONTINUOUS
MULTIFUNCTIONS AND THEIR
INTERSECTIONS

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METRICALLY UPPER SEMICONTINUOUS MULTIFUNCTIONS

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Szymon Dolecki[†]

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ABSTRACT

We present a variety of methods for establishing metric

upper semicontinuity. We give conditions for the intersection

of metric upper semicontinuous multifunctions to be metric upper

semicontinuous and discuss their applicability.

AMS(MOS) Subject Classification: 90C30, 52A99, 54C60

Key Words: semicontinuity, stability, Lusternik theorem,
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SIGNIFICANCE AND EXPLANATION

The study of metric upper semicontinuity (stability) is of
importance in optimization theory. The report discusses classical
and recent techniques of establishing metric upper semicontinuity
and provides their extensions. The metric upper semicontinuity of
intersections, the importance of which has been recognized only
recently, and several applications to optimization problems are
discussed.

METRICALLY UPPER SEMICONTINUOUS MULTIFUNCTIONS
AND THEIR INTERSECTIONS

Szymon Dolecki[+]

A multifunction Γ from a topological space Y into a metric space (X, ρ) is called <u>metrically upper semicontinuous at</u> y_0, if for each $r > 0$ there is a neighborhood W of y_0 such that

$$(0.1) \qquad \Gamma W \subset B(\Gamma y_0, r)$$

where for $A \subset X$, $B(A, r) = \{x : B(x, r) \cap A \neq \emptyset\}$, $B(x, r) = \{v : \rho(x, v) < r\}$ and $\Gamma W = \bigcup_{y \in W} \Gamma y$. Such multifunctions are frequently called <u>upper Hausdorff semi-continuous (u.H.s.c.)</u>

Classical methods of establishing metric upper semicontinuity and their recent extensions (Sections 2 and 3) apply to certain classes of multi-functions. But, what is really needed in applications (e.g., duality in optimization theory, exact penalty methods, sensitivity) is metric upper semicontinuity of the intersections of some multifunctions from the above mentioned classes.

A dramatic suspense is caused by the fact that the intersection of two metrically upper semicontinuous multifunctions need not be itself metrically upper semicontinuous. This fact was recognized in [2] and a way of getting around this difficulty in some applications was proposed in [7]. In [15] Rolewicz gave a geometrical sufficient condition (for the inter-section of u.H.s.c. multifunctions to be u.H.s.c) and applied it to some open problems concerning continuously differentiable maps. His sufficient condition however does not apply to some other important problems.

In Section 4 we provide a general necessary and sufficient condition for metric upper semicontinuity of intersections. As a special case it yields the Rolewicz condition and enables us to refine and to extend the results of [15] (Section 5).

Section 6 provides an example of applications of our general condition to problems of different nature than those at Section 5.

1. Some basic facts about metric upper semicontinuity.

Let (Y, δ) be a metric space. A function $q: \mathbb{R}_+ \to \mathbb{R}_+ \cup \{\infty\}$ is called a <u>rate of semicontinuity at</u> y_0, if

$$(1.1) \qquad \Gamma B(y_0, q(r)) \subseteq B(\Gamma y_0, r) \qquad\qquad r > 0$$

A function $\beta: \mathbb{R}_+ \to \mathbb{R}_+ \cup \{\infty\}$ is called a <u>modulus of semicontinuity at</u> $\cdot$ y_0, if

$$(1.2) \qquad \Gamma B(y_0, r) \subseteq B(\Gamma y_0, \beta(r)) \qquad\qquad r > 0.$$

Γ is u.H.s.c at y_0, if and only if there is a rate q strictly positive (there is a modulus β such that $\lim_{r \to 0} \beta(r) = 0$).

We say that Γ is <u>graph-closed at</u> y_0, if

$$(1.3) \qquad \Gamma y_0 = \bigcap_{W \in \mathcal{B}(y_0)} c\ell \ \Gamma W$$

where $\mathcal{B}(y_0)$ is a neighborhood basis at y_0 and $c\ell$ stands for the closure. Note that Γ is graph-closed at y_0 for each y_0 in Y, if and only if the graph of Γ $\mathcal{G}(\Gamma) = \{(y, x): x \in \Gamma y\}$ is closed. It is a simple observation that

1.1 Proposition

If Γy_0 is a closed set and Γ is u.H.s.c at y_0, then Γ is graph-closed at y_0.

(Topological) upper semicontinuity implies metric upper semicontinuity, the converse statement being true under some auxiliary assumptions (see [6]).

A multifunction Γ is said to be _metrically continuous_ (Hausdorff continuous) _at_ y_0, if for each $r > 0$ there is a neighborhood W of y_0 such that (0.1) holds and for y in W

$$(1.4) \qquad\qquad \Gamma y_0 \subset B(\Gamma y, r)$$

Formulas (0.1) and (1.4) may be rephrased: $h(\Gamma y_0, \Gamma y) < r$, where h stands for the Hausdorff distance: $h(A_1, A_2) = \inf \{r : A_1 \subset B(A_2, r), A_2 \subset B(A_1, r)\}$.

If there are a positive function β tending to 0 with its argument and a $\delta > 0$ such that β is a modulus of semicontinuity of Γ at each y in $B(y_0, \delta)$, then Γ is metrically continuous about y_0. In other words,

$$(1.5) \qquad\qquad h(\Gamma y_1, \Gamma y_2) < \beta (\rho(y_1, y_2)), \; y_1, y_2 \in B(y_0, \delta)$$

If β is linear about 0 and (1.5) holds we say that Γ is (locally) Lipschitz at y_0. If (0.1) holds and, (1.4) holds for all y in $W \cap \Gamma^{-1}X$, then we say that Γ is domain continuous at y_0.

We say that Γ is _lower semicontinuous at_ (y, x) _at a rate_ q, if

$$(1.6) \qquad\qquad \Gamma^{-1}B(x, r) \supset B(y, q(r)) \; .$$

Γ is uniformly lower semicontinuous at (y_0, x_0) if there is a neighborhood Q of (y_0, x_0) and a (strictly) positive function q such that for each (y, x) in $Q \cap \mathcal{G}(\Gamma)$, Γ is lower semicontinuous at (y, x) at the rate q.

1.2 Proposition [2]

Γ is uniformly (at a rate q) lower semicontinuous at (y_0, x_0), if and only if there are neighborhoods V of x_0 and W of y_0 such that for each y in W the multifunction $\Gamma_{(y)}$ defined by

$$(1.7) \qquad \Gamma_{(y)}y = \Gamma y \qquad \Gamma_{(y)}z = V \cap \Gamma z, \quad z \neq y$$

is u.H.s.c at y at a rate $\tilde{q}$, where for some $r_0 > 0$ $\tilde{q}(r) = q(r)$ as $r \leq r_0$.

A simple but important generalization of this proposition we obtain by restricting the multifunction Γ to its effective domain $\Gamma^{-1}X = \{y: \Gamma y \neq \emptyset\}$. Then, all the notions may be related to the metric space $\Gamma^{-1}X$: for example, Γ is domain lower semicontinuous at (y, x) at a rate q, if

$$\Gamma^{-1}B(x, r) \supset B(y, q(r)) \cap \Gamma^{-1}X.$$

Observe, however, that metric upper semicontinuity when related to its domain remains unchanged. The resulting theorem is obtained from Proposition 1.2 by replacing lower semicontinuity by domain lower semi-continuity and "for each y in W" by "for each y in $W \cap \Gamma^{-1}X$".

The importance of domain semicontinuities was recognized by Levine and Pomerol [11].

1.3 Example

Let X be a Banach space and let f be a nonzero continuous linear form on X. Let $\Gamma : \mathbb{R}^2 \to 2^X$ be given by: $\Gamma(r_1, r_2) = \{x : f(x) = r_1, f(x) \cdots$

Γ is not lower semicontinuous, but it is uniformly u.H.s.c at each point of its domain $\{(r_1, r_2) : r_1 = r_2\}$.

1.4 Corollary

Γ is uniformly (domain) lower semicontinuous at (y_0, x_0) at a rate q, if and only if there are neighborhoods U of x_0 and W of y_0 and a number $r_0 > 0$ such that for x in V and for y in W (in $W \cap \Gamma^{-1}x$)

$$(1.8) \qquad \operatorname{dist}(y, \Gamma^{-1}x) < q(r) \Longrightarrow \operatorname{dist}(x, \Gamma y) < r, r < r_0.$$

Proof   It is enough to prove "non domain" part of the corollary. We rephrase the conclusion of Proposition 1.2 for each y in W for $r < r_0$,

$$(1.9) \qquad \Gamma B(y, q(r)) \cap V \subset B(\Gamma y, r).$$

Let x be in $\Gamma B(y, q(r)) \cap V$. Equivalently, x is in V and $\Gamma^{-1}x$ intersects $B(y, q(r))$. In other words $\operatorname{dist}(y, \Gamma^{-1}x) < q(r)$. On the other hand x is in $B(\Gamma y, r)$, whenever $B(x, r)$ meets Γy, or $\operatorname{dist}(x, \Gamma_y) < r$.

2. Where does one encounter metric upper semicontinuity

The Banach open mapping theorem states that if a continuous linear operator F maps a Banach space X onto a Banach space Y, then the multifunction $\Gamma^{-1}: Y \to 2^X$ is lower semicontinuous at $(0,0)$ at a linear rate. Linearity implies uniform lower semicontinuity everywhere, hence Γ^{-1} is metrically continuous, in fact, Lipschitz. More generally, if $\Gamma: Y \to 2^X$ is a multifunction with closed convex graph such that $y_0 \in \text{Int } \Gamma^{-1}X$, then for every x_0 in Γy_0, Γ is lower semicontinuous at (y_0, x_0), thus, by convexity, uniformly lower semicontinuous at $(y_0, x_0)([16])$.

Consequently, in view of Proposition 1.2, the multifunctions $\Gamma_{(y)}$ defined in (1.7) are u.H.s.c (this property we call sometimes δ - semicontinuity). Moreover, if $y_0 \in \text{Int} \Gamma^{-1}X$, then all graph closed convex multifunctions which are "close" to Γ are lower semicontinuous at (y_0, x_0) at a uniform universal rate (a perturbation theorem [2], see [10] for a special case). This nice property of families of graph-convex multifunctions was used in proving uniform lower semicontinuity[*]of non-convex multifunctions (that can be approximated by families of graph-convex multifunctions) by Lusternik [12]: let F be a continuously differentiable (about x_0) mapping into a Banach space Y such that $F'(x_0)X = Y$; then F^{-1} is uniformly lower semicontinuous (at a piecewise linear rate) at $(F(x_0), x_0)$. Extensions of the Lusternik theorem were proposed by Ioffe-Tikhomirov [9], Robinson [14], Ioffe [8] and the present auther [2][3].

In this type of results, an approximation theorem [2] (an extension of the Pták theorem [13]) is very useful. We shall need only its special form:

[*) Actually Lusternik proved a special consequence of uniform lower semi-continuity.

2.1 Theorem [2]

Let Γ be a multifunction from a metric space into a complete metric space X. Suppose that there are a neighborhood Q of $(y_0 x_0)$, numbers r_0 0, $0 < \vartheta < b$ such that for (y, x) in $Q \cap G(\Gamma)$

$$(2.1) \qquad B(\Gamma^{-1}B(x, r), \vartheta r) \supset B(y, br), \ 0 < r \leq r_0 \ .$$

Then Γ is uniformly lower semicontinuous at (y_0, x_0) at a rate q such that $q(r) \ (b - \vartheta)r, r \leq r_1$.

Theorem 2.1 may be related to the domain of Γ. Then (2.1) becomes

$$(2.1') \qquad B(\Gamma^{-1}B(x, r), \vartheta r) \supset B(y, br) \cap \Gamma^{-1}X, \ 0 < r \leq r_0$$

and the thesis is that of uniform domain lower semicontinuity. The applicability of Theorem 2.1 appears through the following scheme: Let X be a complete metric space, Y, Z Banach spaces, $\Gamma: Y \to 2^X$ a multifunction, $\{\Gamma'(y, x)\}$ a family of (graph -) convex, closed multifunctions from Y into Z (such that $0 \in \Gamma'(y, x)0$ for each (y, x)).
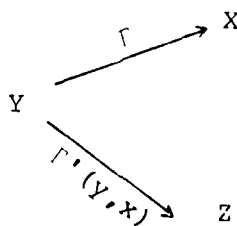
Consider the following assumptions: there are numbers $0 < a < 1$ and $c > 0$ such that for each $\vartheta > 0$ there is an $r_0 > 0$ such that

$$(2.2) \qquad B(\Gamma^{-1}B(x, r), \vartheta r) \supset y + \Gamma'(y, x)^{-1} B(0, ar) \cap B(0, cr)$$

for $\|x - x_0\| < r_0$, $\|y - y_0\| < r_0$ $y \in \Gamma^{-1}x$, $r \leq r_0$ and for each ϵ there are a neighborhood V of (y_0, x_0) and v_1 such that

$$(2.3) \quad B(\Gamma'(y, x)^{-1}B(0, ar) \cap B(0, cr), \epsilon r) \supset \Gamma'(y_0, x_0)^{-1}B(0, ar) \cap B(0, cr) \quad r \leq$$

Such a family $\{\Gamma'(y, x)\}$ is called an _image nearly inner approximation_ of Γ at (y_0, x_0) (_inia_) (see [2] [7] [3] for special cases)

2.2 Proposition

Let $\{\Gamma'(y, x)\}$ be an inia of Γ at (y_0, x_0). If

$$(2.4) \qquad 0 \in \text{Int } \Gamma'(y_0, x_0)^{-1}X,$$

then Γ is uniformly lower semicontinuous at (y_0, x_0) at a piecewise linear rate.

Proof

It follows from the Baire category theorem and from the convexity of $\Gamma'(y_0, x_0)$ that there are b_0 and r_1 such that

$$\Gamma'(y_0, x_0)^{-1}B(0, ar) \supset B(0, b_0 r), \qquad r \leq r_1$$

Let $b_1 > \epsilon > 0$. On setting $b_1 = \min(b_0, c)$ $r_2 = \min(r_0, r_1)$ we have, in view of (2.3), that

$$B(\Gamma'(y, x)^{-1}B(0, ar) \cap B(0, cr), \epsilon r) \supset B(0, b_1 r), r \leq r_2$$

for (y, r) in V.

Now, we choose ϑ such that $\vartheta + \varepsilon < b_1$ and in view of
(2.2) we obtain

$$B(\Gamma^{-1}B(x, r), (\vartheta + \varepsilon)r) \supset B(y, b_1 r)$$

for $||x - x_0|| < r_2, ||y - y_0|| < r_2, y \in \Gamma^{-1}x, r \leq r_2$. In virtue of Theorem 2.1 the proof is complete.

The above scheme embraces various convex approximations of multi-functions (continuous differentiability and more generally strict differentia-bility, Levitin – Milyutin – Osmolovskii approximation (see [2]), their combinations and extensions [3]. We are going to present two concrete results, not most general, but having some important implications.

2.3 Example

Let $G: X \rightarrow Y$ be a continuously differentiable mapping about a point x_0 of a Banach space X and let D be a closed convex cone in a Banach space Y. Consequently for each $\vartheta > 0$ there is $r_0 > 0$ such that

$$(2.5) \qquad ||G(x + h) - G(x) - G'(x)h|| < \vartheta ||h||,$$

for $||x - x_0|| < r_0, ||h|| < r_0$. Define

$$(2.6) \qquad \Gamma y = \{x : y \in G(x) + D\}$$

and note that

$$(2.7) \qquad \Gamma'(y, x)z = \{h : z \in G'(x)h + D\} \qquad y \in \Gamma^{-1}x$$

constitutes an image nearly inner approximation of Γ at $(G(x_0), x_0)$

As a corollary of Proposition 2.2 one has

2.4 Proposition ([14])

If

$$(2.8) \qquad G'(x_0)X + D = Y,$$

then Γ of (2.6) is uniformly lower semicontinuous at $(G(x_0), x_0)$ at a piecewise linear rate.

Let G be a family of subsets of a Banach space X. A family G' of closed convex cones is said to be <u>uniformly tangent to</u> $\{G\}$ at x_0, if for each $\varepsilon > 0$ there is an $r_0 > 0$ such that for $x \in \partial A$ (boundary of A), $\|x - x_0\| < r_0$, $A \in G$ there is an element $A'(x)$ of G' such that for $\|v - x\| < r_0$, one has

(a) if v is in $A'(x)$, then

$$dist(v, A) < \varepsilon \|v - x\|$$

(2.9)

(b) if v is in A, then

$$dist(v, A'(x)) < \varepsilon \|v - x\| .$$

2.5 Proposition (see [3])

Let Γ be given by (2.6), $\Gamma'(x, y)$ by (2.7) and suppose that (2.5) holds.

The family $\{x + \Gamma'(y, x)(0)\}$ is uniformly tangent to $\{\Gamma(y)\}$ at x_0.

Proof

It follows from Proposition 2.4 and Corollary 1.3 that there are numbers $k > 0$, $r_0 > 0$ such that for $||v - x_0|| < r_0$, $||z - y_0|| < r_0$

$$(2.11) \qquad \text{dist}(v, \Gamma z) \leq k \, \text{dist}(z, \Gamma^{-1} v) \, .$$

Let $\varepsilon = k\vartheta$ and let v be in $x + \Gamma'(y, x)(0)$, $(y \in G(x) + D)$ equivalently $0 \in \Gamma'(y, x)^{-1}(v - x) = G'(x)(v - x) + D$ thus there is d in D such that $0 = G'(x)(v - x) + d$ and in view of (2.5) and (2.6), $G(x)$ is in $B(\Gamma^{-1} v, \vartheta || v - x||)$, hence recalling (2.11)

$$\text{dist}(v, \Gamma(G(x))) \leq k\vartheta ||v - x||$$

Let v be in $\Gamma G(x)$. Thus there is d in D such that $G(v) + d = G(x)$ and by (2.5)

$$||G'(x)(v - x) + d|| < \vartheta ||v - x||$$

By (2.8) and the continuity of $G'(x)$ there is an element w of X such that $G(x)(v - w) = 0$ and $||w - x|| \leq k ||G'(x)(v - x) + d|| < k\vartheta ||v - x||$. (In fact k may be taken the same as in (2.11)).

The following result is that of uniform lower semicontinuity of a multifunction $\Gamma: Y \to 2^S$, where Y is a Banach space and S is a closed subset of a Banach space admitting a uniform tangent family. This

result generalizes Proposition 2.4 and is very close to that of Ioffe [8, Theorem 2].

In our case (in comparison to [8]) the set S will be more specific, the multifunction Γ more general, the sufficient condition (controllability) much easier to verify and the conclusion stronger (uniformity). Let G be a continuously differentiable mapping from X into a Banach space Z, D a closed convex cone in Z such that

$$(2.12) \qquad G'(x_0)X + D = Z$$

Set

$$(2.13) \qquad S = \{x : 0 \in G(x) + D\}$$

Denote by

$$(2.14) \qquad T(x) = \{v : 0 \in G'(x)v + D\}$$

Let F be a continuously differentiable mapping of X into a Banach space Y, C a closed convex cone in Y. Consider the multifunction $\Gamma : Y \to 2^S$ defined by

$$(2.15) \qquad \Gamma y = \{x \in S : y \in F(x) + C\}$$

2.6 Theorem

If (2.12) holds and

$$(2.16) \qquad F'(x_0)(T(x_0)) + C = Y,$$

then the multifunction Γ of (2.15) is uniformly lower semicontinuous at $(F(x_0), x_0)$ at a piecewise linear rate.

Proof

We shall show that the multifunction $\Gamma'(y, x) : Y \to 2^X$ defined by

$$(2.17) \qquad \Gamma'(y, x)z = \{h \in T(x) : z \in F'(x)h + C\} \qquad \text{for} \quad y \in \Gamma^{-1}x.$$

is an image nearly inner approximation of Γ at $(F(x_0), x_0)$. It will be then enough to apply Proposition 2.2, since our assumption in view of (2.17) implies (2.4). It follows from Proposition 2.5 that $\{x + T(x)\}$, $x \in S$, is a family uniformly tangent to S about x_0. Consequently, there is a family of functions $\{\xi_x\}$

$$(2.18) \qquad \xi_x : x + T(x) \to S$$

such that for each $1 \geq \vartheta > 0$ there is an $r_0 > 0$ such that if $||x - x_0|| < r_0$ and $||v - x|| < r_0$, then

$$||\xi_x(v) - v|| \leq \vartheta ||v - x|| .$$

Therefore, for $r \leq r_0$

$$\xi_x(B(x, r) \cap x + T(x)) \subset B(x, (1 + \vartheta)r) \cap S$$

There is an $r_1 \leq r_0$ such that in $B(x_0, 3r_1)$ the mapping F is Lipschitz

continuous with constant 1. Hence for each v in $B(x,r) \cap x + T(x)$ there is an element $\xi_x(v)$ in $S \cap B(x,(1+\vartheta)r)$ such that

$$\Gamma(v) - \Gamma(\xi_x(v)) \leq \vartheta \|v - x\| \quad \text{or}$$

(2.19) $\quad B(\Gamma(B(x,(1+\vartheta)r) \cap S),\vartheta r) \supset F(B(x,r) \cap x + T(x))$.

On the other hand, there is an $r_2 \leq r_1$ such that for

$$\|x - x_0\| \leq r_2, \quad \|v - x\| \leq r_2$$

$$\|F(v) - F(x) - F'(x)(v - x)\| < \vartheta \|v - x\|;$$

in particular this is **true** for $v - x$ in $T(x)$, thus

(2.20) $\quad B(F(B(x,r) \cap x + T(x)),\vartheta r) \supset F(x) + F'(x)(B(0,r) \cap T(x))$.

On recalling (2.15) and (2.17) we conclude that

(2.21) $\quad B(\Gamma^{-1}B(x,r), 2\vartheta r) \supset y + \Gamma'(y,x)^{-1}B(0,\frac{r}{2})$

for $r \leq \frac{r_2}{2}$, $\|x - x_0\| < \frac{r_2}{2}$, $y \in \Gamma^{-1}x$, hence (2.2) is satisfied.

We shall show now that the family $\{T(x)\}$ is continuous in the following sense. For each $\varepsilon > 0$ there is $r > 0$ such that for $\|x_1 - x_0\| < r$, $\|x_2 - x_0\| < r$ one has that if $h \in T(x_1)$, then $\text{dist}(h, T(x_2)) \leq \eta \|h\|$. In fact, for each $\varepsilon > 0$ there is $r_0 > 0$ such that, for x_1, x_2 in $B(x_0, r_0)$

$$\|G'(x_1)h - G'(x_2)h\| \leq \varepsilon \|h\| .$$

Let $h \in T(x_1)$. There is d in D such that $G'(x_1)h + d = 0$. On the other hand,

$$\text{dist}(0, \Gamma'(G(x_2), x_2)^{-1}h) \leq |G'(x_1)h + d| =$$

(2.22)
$$= |G'(x_1)h + d - G'(x_2)h - d| \leq$$

$$\leq \varepsilon ||h||$$

By the uniform lower semicontinuity of the family $\Gamma'(y, x)$ about (y_0, x_0) there are $m > 0$ and $r_0 \geq r_1 > 0$ such that

$$\text{dist}(h, \Gamma'(G(x_2), x_2)0) \leq m \, \text{dist}(0, \Gamma'(G(x_2), x_2)^{-1}h) \leq m\varepsilon ||h||;$$

(the last estimate following from (2.22). Set $m\varepsilon \leq \eta$. Consequently, $x \to B(0, r) \cap T(x)$ and $x \to F'(x)(B(0, r) \cap T(x)) + C$ are continuous multifunctions and (2.3) is satisfied.

We shall now provide another condition equivalent to (2.16) combined with (2.12). Let $\mathfrak{F}: X \to Y \times Z$ be given by

$$\mathfrak{F}(x) = (F(x), G(x))$$

and let $\mathcal{C} = C \times D$.

2.7 Lemma

Combined, conditions (2.12) and (2.16) are equivalent to

(2.23)
$$\mathfrak{F}(x_0)X + \mathcal{C} = Y \times Z$$

Proof

It follows from (2.23) that for each y in Y there is an x such that $(y, 0) \in (F'(x_0)x + C, G'(x_0)x + D)$ what amounts to (2.16). Similarly, (2.12) follows immediately from (2.23).

Suppose (2.12) and (2.16) and take $(y, z) \in Y \times Z$. By (2.12) there is an x such that $z \in G'(x_0)x + D$. By (2.16) there is an $\tilde{x}$ such that $0 \in G'(x_0)\tilde{x} + D$ and $y - F'(x_0)x \in F'(x_0)\tilde{x} + C$. Consequently $z \in G'(x_0)(x + \tilde{x}) + D$ and $y \in F'(x_0)(x + \tilde{x}) + C$.

3. On farther extensions

Proposition 2.4 may be generalized by replacing the cone D by a cone-valued multifunction.

Let G be a continuously differentiable (about x_0) mapping from a Banach space X valued in a Banach space Y. Consider, as well, a multifunction

$$C : X \to 2^Y,$$

such that for each x, Cx is a closed convex cone in Y with the vertex at 0. We assume that for each $\varepsilon > 0$ there is a neighborhood Q at x_0 such that for every x from Q and for each h from Cx_0 there exists an $h(x), ||h(x)|| = ||h||$ in Cx such that

(3.1) $$||h(x) - h|| \leq \varepsilon ||h|| .$$

Rephrasing, for each $\epsilon > 0$ there is a neighborhood Q of x_0 such that for each x in Q

(3.2) $B(Cx \cap B(0,r), \epsilon r) \supset Cx_0 \cap B(0,r),$ $r > 0$

Define

(3.3) $\Gamma y = \{x : y \in G(x) + Cx\}$

The multifunction

(3.4) $\Gamma'(x,y)z = \{h : z \in G'(x)h + Cx\},$ $y \in \Gamma^{-1}x$

is an image nearly inner approximation of Γ at $(x_0, G(x_0))$.

Indeed, (2.2) (with $c = +\infty$) follows directly from the definitions. In order to check (2.3) let $v \in \Gamma'(y_0, x_0)^{-1}B(0,r) \cap B(0,r)$; there are g in $B(0,r)$ and h in Cx_0 such that $v = G'(x_0)g + h$. Certainly $||h|| \leq ||v|| + ||G'(x_0)|| \, ||g||$, thus h is in $B(0, (1+k)r)$ where $k > ||G'(x_0)||$.   There exists $r_0 > 0$ such that if $||x_0 - x|| < r_0$ then there is an $h(x)$ in Cx such that (3.1) holds. On the other hand, by the continuity of $G'(\cdot)$, there is an $r_1 < r_0$ such that

$$||G'(x)g - G'(x_0)g|| < \epsilon ||g|| \quad \text{as} \quad ||x - x_0|| < r_1 .$$

Therefore the element $w = G'(x)g + h(x)$ of $\Gamma'(y,x)^{-1}B(0,r)$ satisfies $||v - w|| < \epsilon(2+k)r$ and by convexity there is z in $\Gamma'(y,x)B(0,r) \cap B(0,r)$

such that $\quad$ v - $\quad$ 2: $(2 + k)r$. Therefore for the discussed multi-
functions (2.3) holds where $a = 1$, $c = 1$, ϵ is replaced by $2\cdot(2 + k)$.
As an immediate conclusion we formulate

Theorem 3.1

If

$$(3.5) \qquad\qquad G'(x_0)X + Cx_0 = Y,$$

then the multifunction (3.3) is uniformly lower semicontinuous at $(F(x_0), x_0)$
at a piecewise linear rate.

Another way of generalizing results of type of Proposition 2.4 or

Theorem 2.6 may become a temptation, when one observes that if for a

continuous linear operator F (from a Banach space X into a Banach

space Y) we have that FX is closed, then the multifunction F^{-1} is

u.H.s.c (domain Lipschitz continuous). This is due to the fact that F^{-1}

is uniformly domain lower semicontinuous (its domain FX is itself a

Banach space, hence we may apply the Banach open mapping theorem).

However, an attempt to replace, say (2.8) in Proposition 2.4 by

an assumption that $G'(x_0)X + D$ is closed, may be discarded quickly by

noting, that even for $D = \{0\}$ the property "$G'(x)X$ is closed" is

unstable (see e.g. [10] p. 57).

There remains however a possibility of generalization, when

$G'(x_0)X$ is of finite codimension (the property which is stable) we eliminate

this possibility too.

Example 3.2

Let X be a Hilbert space which orthonormal basis is denoted by $\{e_i\}_{i-1}^{\infty}$. Let $G : X \to \mathbb{R}^2$ be given by

$$G(x) = (g_1(x), g_2(x)),$$

where $g_1(x) = x_1$ (where $x = \sum_{i=1}^{\infty} x_i e_i$) and $g_2(x) = x_1 + \sum_{i=2}^{\infty} \frac{x_i^2}{i}$. Of course, G is continuously differentiable and $G'(0)X$ has finite co-dimension. But the multifunction G^{-1} is not domain lower semicontinuous. To see this observe that $G^{-1}(0, 0) = \{0\}$ and for $r < 0$

$$G^{-1}(r, 0) = \{x : x_1 = r = -\sum_{i=2}^{\infty} \frac{x_i^2}{i}\}$$ thus for $|r|$ arbitrarily small there are x in $G^{-1}(r, 0)$ that do not belong to $B(0, 1)$.

3.3 Example

Modifying Example 3.2 set $g_2(x) = x_1 + \sum_{i=2}^{\infty} x_i^2$. In this case the dimension of $G'(0)X$ is one, but G^{-1} is uniformly domain lower semicontinuous (at 0) (for details see Example 4.6).

4. General conditions on metric upper semicontinuity of intersections

Let A_1, A_2 be subsets of a metric space (X, ρ). We say that A_1 and A_2 __separate decisively__, if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$(4.1) \qquad B(A_1 \cap A_2, \varepsilon) \supset B(A_1, \delta) \cap B(A_2, \delta)$$

When A_1 and A_2 do not intersect, we understand that $B(A_1 \cap A_2, \varepsilon)$ is empty for each ε; then condition (4.1) becomes that there is a $\delta > 0$ such that $B(A_1, \delta) \cap B(A_2, \delta) = \emptyset$.

When A_1 is a subset of A_2, then the sets separate decisively and δ may be taken equal to ε.

4.1 Lemma

Let X and Z be two metric spaces, p a mapping of X onto Z such that p and p^{-1} are uniformly continuous. If A_1, A_2 separate decisively, then $p(A_1), p(A_2)$ separate decisively.

Proof

Let $\vartheta > 0$. There is $\varepsilon > 0$ such that for each x, $B(p(x), \vartheta) \supset p(B(x, \varepsilon))$. Consequently

$$B(p(A_1) \cap p(A_2), \vartheta) = B(p(A_1 \cap A_2), \vartheta) \supset pB(A_1 \cap A_2, \varepsilon) .$$

In view of (4.1) and the above inclusion, there is δ such that

$$B(p(A_1) \cap p(A_2), \vartheta) \supset p(B(A_1, \delta)) \cap p(B(A_2, \delta)) .$$

By the uniform continuity of p^{-1} there is a ξ such that

$$B(A_1, \delta) \supset p^{-1}B(p(A_1), \xi), \; p(A_2, \delta) \supset p^{-1}B(p(A_2), \xi)$$

thus

$$B(p(A_1) \cap p(A_2), \vartheta) \supset B(p(A_1), \xi) \cap B(p(A_2), \xi)) \; .$$

4.2 Lemma

The sets A_1 and A_2 separate decisively, if and only if there exists a function $h : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+ \cup \{+\infty\}$ continuous at $(0, 0)$ and $h(0, 0) = 0$ such that

$$(4.2) \qquad \text{dist}(x, A_1 \cap A_2) \leq h(\text{dist}(x, A_1), \text{dist}(x, A_2))$$

Such a function h is called a <u>modulus of separation</u>.

Proof

Assume that such a function exists and set $\varepsilon > 0$. By the continuity of h at $(0, 0)$, there is a $\delta > 0$ such that $h(r_1, r_2) < \varepsilon$, if $r_1 < \delta$ and $r_2 < \delta$. If an element x is in $B(A_1, \delta) \cap B(A_2, \delta)$ (or equivalently if $\text{dist}(x, A_1) < \delta$ and $\text{dist}(x, A_2) < \delta$), then by (4.2) $\text{dist}(x, A_1 \cap A_2) < \varepsilon$, hence it belongs to $B(A_1 \cap A_2, \varepsilon)$.

On the other hand, if A_1 and A_2 separate decisively, then for each $n \geq 1$ there is a $\delta_n > 0$ such that

$$(4.3) \qquad B(A_1 \cap A_2, \frac{1}{n}) \supset B(A_1, \delta_n) \cap B(A_2, \delta_n) \; .$$

We define the functions $h_n : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\}$ by setting

(4.4)
$$h_n(r_1, r_2) = \begin{cases} \frac{1}{n}, & \text{if } r_1 < \delta_n, r_2 < \delta_n \\ +\infty, & \text{otherwise} \end{cases}$$

and define

(4.5)
$$h(r_1, r_2) = \inf_n h_n(r_1, r_2) .$$

The function h of (4.5) satisfies the listed properties. (It is obviously continuous and 0 at $(0, 0)$.)

Take an arbitrary x. If $\max(\text{dist}(x, A_1), \text{dist}(x, A_2)) \geq 1$, then by (4.4)

(4.5)
$$h(\text{dist}(x, A_1), \text{dist}(x, A_2)) = +\infty$$

and (4.2) is fulfulled. If

$$\delta_{n+1} \leq \max(\text{dist}(x, A_1), \text{dist}(x, A_2)) < \delta_n ,$$

then by (4.3)

$$\text{dist}(x, A_1 \cap A_2) < \frac{1}{n} ,$$

thus h satisfies (4.2).

Let us give some attention to those pairs of sets A_1, A_2 that do not separate decisively. This means, by definition, that there exists an $r_0 > 0$ and a sequence $\{x_n\}$ such that

$$\text{dist} (x_n, A_1 \cap A_2) \geq \varepsilon_0$$

and

(4.7)
$$\lim_n \text{dist} (x_n, A_1) = \lim_n \text{dist} (x_n, A_2) = 0$$

4.3 Example

Let $X = \mathbb{R}^2$, $A_1 = \{(x, y) : y = 0\}$, $A_2 = \{(x, y) : y = e^{-x}\}$. The intersection of these sets is empty but for each δ the sets $B(A_1, \delta)$ and $B(A_2, \delta)$ meet. The sequence $\{(n, 0)\}$ satisfies (4.6), (4.7).

4.4 Example (compare [7])

Let X be a Hilbert space which orthonormal basis is denoted by $\{e_n\}_{n=1}^\infty$. Let

$$A_1 = \{\sum_{n=1}^\infty t_n e_n \in X : \sum_{n=1}^\infty \frac{t_n^2}{n} \leq 1\}$$

and

$$A_2 = \{\sum_{n=1}^\infty t_n e_n \in X : t_1 = 1\} .$$

The only common element of these sets is $\{e_1\}$. The sequence $\{e_1 + e_n\}_{n=1}^\infty$ is distant from e_1 by one. On the other hand, it is a subset of A_2 and for each n $e_1 + e_n$ is distant from the element $1 - \frac{1}{n} e_1 + e_n$ of A_1 by $1 - \sqrt{1 - \frac{1}{n}}$.

In first of the two examples the sequence satisfying (4.6) (4.7) is unbounded; in the latter example it is bounded. Nevertheless in both cases,

it is not compact. Compactness implies decisive separation, but is not necessary.

4.5 Lemma

If one of the sets A_1, A_2 is compact and the other closed, then th. sets separate decisively.

Proof

Since X is metric it is enough to consider sequential compactness. We shall prove that no sequence can satisfy (4.6) and (4.7). Indeed, assume that A_1 is compact and there are an ε_0 and a sequence $\{x_n\}$ satisfying (4.6) (4.7). Consequently there are sequences $\{y_n\}$ in A_1, $\{z_n\}$ in A_2 and $\{\delta_n\}$, $\delta_n \downarrow 0$ such that $\rho(x_n, y_n) < \delta_n$ and $\rho(x_n, z_n) < \delta_n$. There is a subsequence $\{y_{n_k}\}$ convergent to an element, say y_∞, of A_1, thus $\{x_{n_k}\} \{z_{n_k}\}$ converge to y_∞, too. On the other hand, y_∞ is in A_2 (in view of the closedness of A_2). This contradicts (4.(..

4.6 Example

X stands for a Hilbert space, like in Example 4.4. Put

$$A_1 = \{\sum_{n=1}^{\infty} t_n e_n \in X : \sum_{n=1}^{\infty} t_n^2 \leq 1\}$$

and

$$A_2 = \{\sum_{n=1}^{\infty} t_n e_n \in X : t_1 = 1\}$$

The only common point is $\{e_1\}$. These sets separate decisively. To show

that, denote by $r^2 = \sum_{n=2}^{\infty} t_n^2$. Then A_1 is included in

$$\{\sum_{n=1}^{\infty} t_n e_n \in X : r \leq 1, t_1 \leq \sqrt{1 - r^2}\}$$
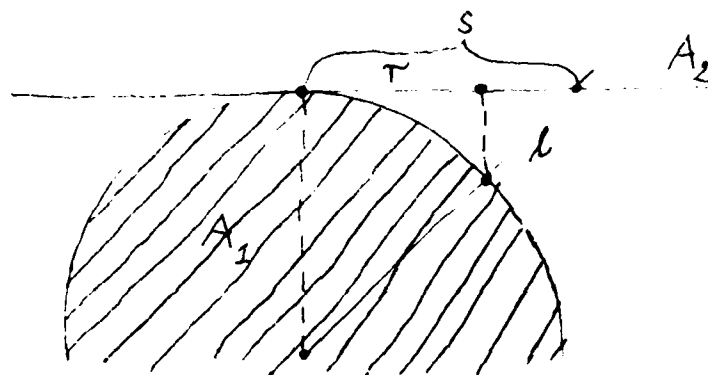
Fig. 1

We infer, that if $\operatorname{dist}(x, A_1) + \operatorname{dist}(x, A_2) < \ell$, then $\operatorname{dist}(x, A_1 \cap A_2) < s$. In order to relate s to ℓ we observe that

$$\frac{s}{1 + \ell} = \frac{s - r}{\ell}$$

$$\frac{1 - \sqrt{1 - r^2}}{\ell} = \frac{1}{1 + \ell} \quad ,$$

thus

(4.8) $$s = \sqrt{(2 + \ell)\ell} \ .$$

Therefore for $\varepsilon > 0$ we may pick $\delta = (\frac{\varepsilon}{2})^2$ if it is less than 1.

The following theorem shows how decisive separation characterizes the metrically upper semicontinuous multifunctions, the intersection of which is also metrically upper semicontinuous.

4.7 Theorem

a)   Let Γ_1, Γ_2 be multifunctions from a topological space Y into subsets of X, u.H.s.c at y_0. If $\Gamma_1 y_0$ and $\Gamma_2 y_0$ separate decisively, then the intersection $\Gamma_1 \cap \Gamma_2$ is u.H.s.c at y_0. Moreover, there is a modulus of semicontinuity β of $\Gamma_1 \cap \Gamma_2$ such that

$$(4.9) \qquad\qquad \beta(r) = h(\beta_1(r), \beta_2(r)),$$

where h is a modulus of separation of $\Gamma_1 y_0$ and $\Gamma_2 y_0$, β_i is a modulus of semicontinuity of Γ_i (at y_0), $i = 1, 2$.

b)   If the sets A_1 and A_2 do not separate decisively, then there are a (metric) space Y and u.H.s.c (at y_0) multifunctions Γ_1, Γ_2:
$Y \to 2^X$, such that

$$\Gamma_1 y_0 = A_1, \ \Gamma_2 y_0 = A_2 ,$$

the intersection of which is not u.H.s.c at y_0.

Proof

a)   Let $\varepsilon > 0$. There is a $\delta > 0$ such that $B(\Gamma_1 y_0 \cap \Gamma_2 y_0, \varepsilon)$
$\supset B(\Gamma_1 y_0, \delta) \cap B(\Gamma_2 y_0, \delta)$. Since Γ_1, Γ_2 are u.H.s.c. at y_0, there are neighborhoods W_1, W_2 of y_0 such that $\Gamma_1 W_1 \subset B(\Gamma_1 y_0, \delta)$ and
$\Gamma_2 W_2 \subset B(\Gamma_2 y_0, \delta)$. Consequently $(\Gamma_1 \cap \Gamma_2)(W_1 \cap W_2) \subset B(\Gamma_1 y_0 \cap \Gamma_2 y_0, \varepsilon)$.
To show (4.9) assume that x is in $(\Gamma_1 \cap \Gamma_2) B(y_0, r)$. Thus, by semicontinuity $\text{dist}(x, \Gamma_1 y_0) < \beta_1(r)$ and $\text{dist}(x, \Gamma_2 y_0) < \beta_2(r)$. In view of (4.2) we may set $\beta(r) = h(\beta_1(r), \beta_2(r))$.

b)   Suppose that A_1 and A_2 do not separate decisively and let $\{x_n\}$ be a sequence satisfying (4.6) (4.7). Define $Y = [0, -\infty)$ and

$$\Gamma_1 \frac{1}{n} = \Gamma_2 \frac{1}{n} = \{x_n\} \qquad\qquad , n = 1, 2, \ldots$$

$$\Gamma_1 0 = A_1, \ \Gamma_2 0 = A_2$$

$$\Gamma_1 y = \Gamma_2 y = \emptyset, \quad \text{otherwise.}$$

So defined multifunctions are u.H.s.c. at 0, but their intersection is not.

4.8 Remark

Let A_1, A_2 be given. If A_1 is a subset of a ball, say $B(x_0, r_0)$ and for $r_1 > r_0$ the sets A_1 and $A_2 \cap B(x_0, r_1)$ separate decisively then A_1 and A_2 separate decisively. Moreover there is a modulus of separation of the latter equal to that of the former for small r_1, r_2.

4.9 Remark

Let $C_1 \subset A_1$, $C_2 \subset A_2$ be such that $C_1 \cap C_2 = A_1 \cap A_2$. If A_1 and A_2 separate decisively, then C_1 and C_2 too (with the same modulus).

4.10 Remark

A decisive separation property has itself a semicontinuity character. For two given subsets A_1, A_2 of X define the multifunction $\Delta : \mathbb{R}_+ \to 2^X$ by

$$\Delta 0 = A_1 \cap A_2$$

$$\Delta r = B(A_1, r) \cap B(A_2, r), \ r > 0$$

A_1, A_2 separate decisively, if and only if the multifunction Δ is metrically upper semicontinuous at 0.

5. Rolewicz theorem and localization of metric upper semicontinuity.

In [15] Rolewicz introduced, what we call, c-stars and d-convex sets, and proved that for two u.H.s.c multifunctions Γ_1, Γ_2 for which $\Gamma_1 y_0$ is a c-star and $\Gamma_2 y_0$ is d-convex $(c > d)$, $\Gamma_1 \cap \Gamma_2$ is u.H... at y_0. His proof amounts, in practice, to demonstrating that $\Gamma_1 y_0$ and $\Gamma_2 y_0$ separate decisively.

Let X be a normed space. A subset A of X is called c-convex at x_0 ($\in A$), if for every x in A and for each $0 < \alpha < 1$ there exists x_α in A such that

$$(5.1) \qquad ||x_\alpha - (\alpha x + (1 - \alpha)x_0)|| \le (1 - \alpha)c\,||x - x_0||$$

Of course, every convex set is c-convex for each $c > 0$ at each point. The union of c-convex (at x_0) sets is c-convex at x_0.

5.1 Lemma ([15])

Let $F : X \to Y$ be continuously differentiable about x_0 and such that $F'(x_0)X = Y$. Then for each $c > 0$, there exists a ball Q (centered at x_0) such that the set

$$\{x : F(x) = F(x_0)\} \cap Q$$

is c-convex at x_0.

A subset A of X is called a c-star at x_0, if for each x in A the convex hull of

$$(5.2) \qquad B(x_0, c||x - x_0||) \cup \{x\}$$

is included in A.

It is a simple observation that A is a c-star at x_0, if and only if for each x in A

$$(5.3) \qquad \bigcup_{0 < \alpha < 1} B(\alpha x_0 + (1 - \alpha)x, \alpha c||x - x_0||) \subset A$$

Certainly, each c-star at x_0 is c-convex at x_0.

5.2 Proposition

Every c-star at x_0 not equal to the whole space is bounded.

Proof

Suppose that A is a c-star at x_0 and there is a sequence $\{x_n\}$ such that $||x_n - x_0|| \geq n$.

Then by (5.3) for each $n = 1, 2, \ldots$

$$B(x_0, cn) \subset B(x_0, c||x_n - x_0||) \subset A,$$

hence A = X.

We shall give a simpler proof of

5.3 Proposition [15]

Every bounded convex set A for which x_0 is an interior point, is a c-star at x_0.

Proof

By assumptions there are numbers $0 < r < M$ such that $B(x_0, r) \subset A \subset B(x_0, M)$. We set $c = \frac{r}{M}$ and observe that for each x in A $B(x_0, c|x - x_0|) \subset B(x_0, cM) \subset A$. By convexity of A the convex hull (5.2) is a subset of A.

We note that for a family $\{A_i\}_{i \in I}$ of c-stars at x_0, $\bigcup_{i \in I} A_i$ and $\bigcap_{i \in I} A_i$ are c-stars at x_0. Consequently, if $\{A_i\}$ is a family of convex sets such that $B(x_0, r) \subset A_i \subset B(x_0, M)$ for each i, then $\bigcup_{i \in I} A_i$ is a $\frac{r}{M}$ - star at x_0.

5.4 Theorem (Rolewicz [15])

Let $0 < d < c \leq 1$. If A_1 is a d-convex set at x_0 and A_2 is a c-star at x_0, then A_1 and A_2 separate decisively and there is a modulus of separation of the form

$$(5.4) \qquad h(r_1, r_2) = m \cdot (r_1 + r_2),$$

where m depends only on d and c.

The Rolewicz theorem is especially useful in localization of metric upper semicontinuity. It is known [7] that if Γ is u.H.s.c at y_0 (with modulus β) then for each $\varepsilon > 0$ and each neighborhood Q of $x_0(\varepsilon \Gamma y_0)$, there is a neighborhood $Q_0 \subset Q$ of x_0 such that $Q_0 \cap \Gamma$ is u.H.s.c at y_0 with modulus $(1 + \varepsilon)\beta$. The following example shows that Q_0 cannot in general be replaced by a ball about x_0.

5.5 Example

Let E be a (nonseparable) Hilbert space, $\{e_r\}_{r \in \mathbf{R}}$ a family of orthonormal vectors in E, χ a characteristic function of $\{r e_r\}_{r \in \mathbf{R}}$,

$$A = \{(e, r) \in E \times \mathbf{R} : 0 \le r \le \chi(e)\} \ .$$

Equip $E \times \mathbf{R}$ with the norm

$$||(e, r)|| = \sup(||e||, |r|)$$

and define $\Gamma : E \to 2^{E \times \mathbf{R}}$ by

$$\Gamma y = A + (y, 0) \ .$$

Of course Γ is closed-valued continuous multifunction, but no multifunction Δ of form

$$c\ell \, B(0, R) \cap \Gamma \qquad\qquad B(0, R) \cap \Gamma$$

is u.H.s.c at 0. We shall show this fact for $R \le 1$. Let $0 < \varepsilon < \frac{R}{2}$ and pick an r such that $R < r < R + \frac{\varepsilon}{2}$. The element $x = ((r - \varepsilon)e_r, R)$ is in $\Delta(-\varepsilon e_r)$ and $x \notin B(\Delta 0, \frac{R}{2})$. As $|| - \varepsilon e_r|| = \varepsilon$ and ε was chosen arbitrarily Δ is not u.H.s.c at 0.

It follows immediately from the definition that for each $r > 0$ the ball $B(x_0, r)$(or $c\ell \, B(x_0, r)$) is a 1-star at x_0. Therefore, if Γ is an u.H.s.c multifunction (at y_0) and Γy_0 is d-convex at x_0 with $d < 1$, then $B(x_0, r) \cap \Gamma$ (and $c\ell \, B(x_0, r) \cap \Gamma$) is u.H.s.c at y_0 in virtue of Theorems 5.4 and 4.7.

5.6 Lemma

Let $\{A'(x)\}_{x \in X}$ be uniformly tangent to a family G at x_0.

For each $d > 0$ and every neighborhood Q of x_0 there are neighborhoods $Q_0 \subset Q_1 \subset Q$ of x_0 such that for each A of G such that $A \cap Q_0 \neq \emptyset$ the set $A \cap Q_1$ is d-convex at each v in $A \cap Q_0$.

Proof

Fix $d > 0$ and choose a neighborhood Q of x_0. Let $\varepsilon < \dfrac{d}{3}$ and let r_0 correspond to ε in (2.9)(2.10) and be such that $B(x_0, r_0) \subset Q$. Set $Q_0 = B(x_0, \dfrac{r_0}{3})$, $Q_1 = B(x_0, \dfrac{2r_0}{3})$.

Let v be in $Q_0 \cap A$ for A in G and let x be in $A \cap Q_1$. By (2.10) there is v_1 in $A'(x)$ such that

$$(5.5) \qquad ||v_1 - v|| \leq \varepsilon ||v - x||$$

The element $(1 - \alpha)v_1 + \alpha x$ lies in $A'(x)$, thus there is an x_α in A such that

$$||x_\alpha - (\alpha x + (1 - \alpha)v_1)|| \leq \varepsilon(1 - \alpha)||v_1 - x||$$

what combined with (5.5) implies

$$(5.6) \qquad ||x_\alpha - (\alpha x + (1 - \alpha)v_1)|| \leq (1 - \alpha)\varepsilon(1 + \varepsilon)||v - x||$$

We estimate, taking into account (5.5) and (5.6)

$$||x_\alpha - (\alpha x + (1 - \alpha)v|| \leq (1 - \alpha)\varepsilon(2 + \varepsilon)||v - x||$$

which is less than $(1 - \alpha)d||v - x||$, if we assume that $d \leq 3$. The proof is complete.

The above lemma enables us to prove the following result concerning the Lipschitz continuity of the multifunction Γ (2.15), $\Gamma : Y \to 2^S$ where where S is defined by (2.13) under condition (2.12).

5.7 Theorem

If (2.16) holds, then there are a neighborhood W of y_0, a neighborhood Q of x_0 and numbers $c, r_0 > 0$ such that for x in Q, for y in $\Gamma^{-1}x \cap W$ and $r \leq r_0$ the multifunction

$$B(x, r) \cap \Gamma$$

is Lipschitz continuous about y with constant c.

Proof

Let $d < 1$. In view of Lemma 2.7 and Proposition 2.5 applied to the multifunction

$$\Delta(y, z) = \{x : (y, z) \in \mathfrak{F}(x) + C\}$$

the family $\{\Delta(y, z)\}_{(y, z)}$ possesses a uniformly tangent family at x_0. A fortiori, the family $\Gamma y = \Delta(y, 0)$ has a uniformly tangent family at x_0. In virtue of Lemma 5.6 in every neighborhood V of x_0 there are neighborhoods $Q_0 \subset Q_1 \subset V$ of x_0 such that for each y and x in $\Gamma y \cap Q_0$, $\Gamma y \cap Q_1$ is d-convex at x.

Choose positive numbers r_0 and s_0 such that $B(x_0, 2s_0) \subset Q_0$,

$B(x_0, 2r_0) \quad Q_1 \quad$ and

$$c = -\frac{r_0 - s_0}{r_0} > d$$

Let x be in $Q = B(x_0, s_0)$ and let $r \leq r_0$. The ball $B(x, r)$ is a $\frac{r - s}{r}$ - star at each v in $B(x, s)$. We choose s so that $\frac{s}{r} = \frac{s_0}{r_0}$ to guarantee that $B(x, r)$ is a c-star at such v.

On the other hand, by Theorem 2.6 there is a neighborhood W of y_0 such that for every y in $\Gamma^{-1} x \cap W$ there is a neighborhood W_y of y with $\Gamma^{-1} G(x, s) \supset W_y$, or equivalently such that for each z in W_y, $\Gamma z \cap B(x, s) \neq \emptyset$. From the first part of the proof we know that $\Gamma z \cap Q_1$ is d-convex at each v in $\Gamma z \cap B(x, s)$.

Now each ball $B(x, s)$ in consideration has the property that $B(x, s) + B(0, r_0 - s_0) \subset Q_1$, thus we conclude on recalling Theorem 5.4 and Remark **4.8** that there is a function h of the form (5.4) for $r < r_0 - s_0$ which is a modulus of separation for each $B(x, s)$ and Γz described above).

We conclude that the same is true about the sets $B(x, s) \cap S$ and Γz (Remark 4.9)

From the assumptions of our theorem in view of Theorem 2.6 and Proposition 1.2 there is a neighborhood V of x_0 and a neighborhood W of y_0 such that for y in W the multifunctions **(1.7)** are metrically upper semicontinuous (at a universal piecewise linear rate). We may assume that V that appears at the beginning of the proof is equal to this just introduced.

On the other hand, $y \rightarrow B(x, s) \cap S$ is a constant, thus metrically upper semicontinuous multifunction (with the rate $q(r) = \infty$).

Therefore, by Theorem 4.7 (a), the multifunction $\Gamma \cap B(x, r)$ is u.H.s.c at each element of W_y.

The uniform linear rate of semicontinuity (of all these multifunctions) at all points implies Lipschitz continuity.

It is of great importance in optimization to establish the Lipschitz continuity of so-called primal functionals [2] [15] [7]. Let f be a real-valued function on S (2.13) locally Lipschitz continuous about x_0. The primal functional of (f, Γ) restricted to Q is the real-valued function on Y:

$$f\Gamma_Q(y) = \inf_{x \in \Gamma y \cap Q} (f(x)$$

5.8 Corollary

Under the assumptions of Theorem 2.6, there are neighborhoods W of y_0, Q of x_0 and a number r_0 such that for each x in Q for every y in $\Gamma^{-1}x \cap W$ and for $r \leqslant r_0$, the primal functional of (f, Γ) restricted to $B(x, r)$ is locally Lipschitz at y. There is a universal Lipschitz constant for all such primal functionals.

Proof

Apply Theorem 5.7 together with [5].

6. Decisive separation of weakly separated sets

The sets A_1, A_2, the decisive separation of which we discuss in this section, have the property

$$(6.1) \qquad x \in A_1 \cap A_2 \Rightarrow x \in \partial A_1 \cap \partial A_2$$

where ∂ stands for the topological boundary. We shall consider sets of the form

$$(6.2) \qquad A_1 = \{x : f_1(x) \leqslant 0\}, \; A_2 = \{x : f_2(x) \leqslant 0\},$$

where f_1, f_2 are real-valued functions on a Banach space X .

Property (6.1) links the study with optimization theory; decisive separation of sets satisfying (6.1) is crucial in sensitivity theory, a branch of optimization ([4]).

6.1 Proposition

Suppose that the functions f_1, f_2 are continuous and the sets (6.2) satisfy (6.1). Then if the set

$$A_1 \cap A_2$$

is nonempty it is the set of all the global solutions of the problem

$$(6.3) \qquad f_1(x) \to \inf, \; f_2(x) \leqslant 0$$

Proof

If x is in A_2, then $f_1(x) \geq 0$, because otherwise by the continuity of f_1 there would be a neighborhood Q of x such that $Q \subset \{v : f_1(v) < 0\} \subset A_1$, contradicting (6.1). Since $A_1 \cap A_2$ is nonempty, there is an element $\hat{x}$ of A_2 such that $f_1(\hat{x}) \leq 0$; consequently $\inf\limits_{f_2(x) \leq 0} f_1(x) = 0$ and every element of $A_1 \cap A_2$ is a solution of (6.3). On the other hand every solution x of (6.3) satisfies $f_1(x) \leq 0$ and $f_2(x) \leq 0$.

6.2 Corollary

If the functions f_1, f_2 are differentiable and the sets (6.2) satisfy $\ldots$, then for each x in $A_1 \cap A_2$ there are positive numbers λ_1, λ_2 not both zero such that

$$(6.4) \qquad \lambda_1 f_1'(x) + \lambda_2 f_2'(x) = 0 .$$

If besides f_1, f_2 are twice differentiable, then

$$(6.5) \qquad \lambda_1 f_1''(x) + \lambda_2 f_2''(x) \geq 0 .$$

Formulas (6.4) and (6.5) follow from well-known necessary conditions for an x to be a local minimum at (6.3). Under additional assumptions the sets of type (6.2) associated with (6.3) fulfill (6.1) ([4]).

In the sequel, we shall assume that for an x_0 in $A_1 \cap A_2$,

(6.6) $f_1'(x_0) \neq 0, \ f_2'(x_0) \neq 0$.

Then there will be a strictly positive λ such that

(6.7) $f_1'(x_0) + \lambda f_2'(x_0) = 0$

6.3 Theorem

Let A_1, A_2 satisfy (6.1) and be of the form (6.2) where f_1, f_2 are continuously differentiable.

Assume that an x_0 in $A_1 \cap A_2$ satisfies (6.6).

Let ι be an isomorphism of $\ker f_1'(x_0) \times \mathbb{R}$ onto X such that $\iota(0, 0) = x_0$. Then there are neighborhoods W_1 of 0 in $\ker f_1'(x_0)$, W_2 of 0 in $\mathbb{R}$ and Q of x_0 in X and real-valued functions φ_1, φ_2 on $\ker f_1'(x_0)$ such that $Q = \iota(W_1 \times W_2)$ and

$$\{x : f_i(x) = 0\} \cap Q = \iota\{(y, r) : r = \varphi_i(y), \ y \in W_1\}$$

The sets $A_1 \cap Q, A_2 \cap Q$ separate decisively, if and only if the multifunction Δ:

(6.8) $\Delta r = W_1 \cap \{y : \varphi_1(y) - \varphi_2(y) = r\}$

is metrically upper semicontinuous at 0.

Proof

It follows from Proposition 2.5 and from (6.6) that

$$Y = \ker f_1'(x_0) = \ker f_2'(x_0)$$

is tangent (at x_0) to $\{x : f_1(x) = 0\}$ and to $\{x : f_2(x) = 0\}$. We shall represent these two sets locally about x_0 as functions on Y.

Let ι be an isomorphism of $Y \times \mathbb{R}$ and X such that $\iota(0, 0) = x_0$. For a function f on X define $\tilde{f} : Y \times \mathbb{R} \to \mathbb{R}$ by

$$(6.9) \qquad \tilde{f}(y, r) = f \circ \iota(y, r) .$$

The partial derivatives of f are

$$(6.10) \qquad \tilde{f}'_y(y, r)h = f'(\iota(y, r)) \circ \iota(h, 0)$$

$$\tilde{f}'_r(y, r)s = f'(\iota(y, r)) \circ \iota(0, s)$$

If $\ker f'(x_0) = Y$, then $\tilde{f}'_y(0, 0) = 0$ and $\tilde{f}'_r(0, 0)\mathbb{R} = \mathbb{R}$. Consequently ([1]), there are neighborhoods W_1 of 0 in Y and W_2 of 0 in $\mathbb{R}$ and a function $\varphi : W_1 \to W_2$ such that

$$(6.11) \quad \{(y, r) : r = \varphi(y), \ y \in W_1\} = \{(y, r) : \tilde{f}(y, r) = 0\} \cap (W_1 \times W_2)$$

The set (6.11) is the preimage by ι of

$$\{x : f(x) = 0\} \cap Q$$

where Q is a neighborhood of x_0. If now φ_1, φ_2 correspond to f_1 and f_2, then we may pick W_1, W_2 and Q good for both the functions.

By Lemma 4.1 the decisive separation of the sets $A_1 \cap Q$ and

and $A_2 \cap Q$ is equivalent to the same property of $\iota^{-1}(A_1 \cap Q)$, $\iota^{-1}(A_2 \cap Q)$.
We may assume without loss of generality that $\varphi_1(y) \geq \varphi_2(y)$ for $y \in W_1$
and consider the decisive separation of the sets

$$C_1 = \{(y, r) : r \geq \varphi_1(y)\} \cap W_1 \times W_2, \quad G_2 = \{(y, r) : r \leq \varphi_2(y)\} \cap W_1 \times W_2 .$$

The neighborhoods Q, W_1, W_2 might be chosen so that for $(y, r) \in W_1 \times W_2$

(6.12)
$$\varphi_1(y) - r \leq 2 \operatorname{dist}((y, r), C_1)$$

$$r - \varphi_2(y) \leq 2 \operatorname{dist}((y, r), C_2) .$$

Indeed, for (y, r) in $W_1 \times W_2$ we have that

$$\operatorname{dist}((y, r), C_1) \leq \varphi_1(y) - r$$

$$\operatorname{dist}((y, r), C_2) \leq r - \varphi_2(y)$$

Thus

(6.13)
$$\operatorname{dist}((y, r), C_1) + \operatorname{dist}((y, r), C_2) \leq \varphi_1(y) - \varphi_2(y)$$

The derivative of the function φ_1 is

(6.14)
$$\varphi_1'(y) = - \tilde{f}_{1r}'(y, \varphi_1(y))^{-1} \cdot \tilde{f}_{1y}'(y, \varphi_1(y))$$

thus is continuous and vanishes at 0.

It follows that for all $y, \tilde{y}$ in a neighborhood of 0 (say W_1)

$$\varphi_1(y) \leq \varphi_1(\tilde{y}) + ||\tilde{y} - y||$$

Consequently, for $y, \tilde{y}$ in W_1

$$\varphi_1(y) - r \leq |\varphi_1(\tilde{y}) - r| + ||\tilde{y} - y||$$

$$\leq 2\sqrt{(\varphi_1(\tilde{y}) - r)^2 + ||\tilde{y} - y||^2}$$

Therefore, if y is in W_1 and $\varphi_2(y) \leq r \leq \varphi_1(y)$, then

(6.15) $$\varphi_1(y) - \varphi_2(y) \leq 2\left(\text{dist}((y, r), C_1) + \text{dist}((y, r), C_2)\right)$$

On the other hand, (y, r) is in $C_1 \cap C_2$, whenever y is in $A = \{z : \varphi_1(z) - \varphi_2(z) = 0\}$ and $r = \varphi_1(z)$. We have the estimates $\text{dist}(y, A) \leq \text{dist}((y, r), C_1 \cap C_2) \leq \sqrt{\text{dist}^2(y, A) + (\varphi_1(y) - \varphi_2(y))^2}$, which together with (6.13) and (6.15) complete the proof.

6.4 Example

Assume now that, in addition to hypotheses of Theorem 6.3, the functions f_1, f_2 are twice continuously differentiable. In view of Corollary 6.2 and Formula (6.6) there is $\lambda > 0$ such that

$$f_1'(x_0) = -\lambda f_2'(x_0)$$

$$f_1''(x_0) + \lambda f_2''(x_0) \geq 0$$

We shall assume that there is a $k > 0$ such that

(6.16) $\qquad f_1''(x_0)(h, h) + \lambda f_2''(x_0)(h, h) \geq k \cdot h^2 \qquad h \in \ker f'(x_0)$

6.5 Proposition

If formulae (6.1), (6.6) and (6.16) hold, then there is a neighborhood Q of x_0 such that the sets $A_1 \cap Q$ and $A_2 \cap Q$ separate decisively.

Proof

In view of Theorem 6.3 we should prove that the multifunction (6.8) is u.H.s.c at 0. Denote: $\psi(y) = \varphi_1(y) - \varphi_2(y)$, where φ_1, φ_2 are those introduced in the previous proof. Using formula (6.14) we may compute the second derivative of $\varphi (= \varphi_1, \varphi_2)$

$$\varphi''(y)(h, h) =$$

$$(6.17) \quad = \frac{1}{\tilde{f}_r'(y, \varphi(y))^2} \left(\tilde{f}_{ry}''(y, \varphi(y))h + \tilde{f}_{rr}''(y, \varphi(y)) \cdot \varphi'(y)h \right) \cdot \tilde{f}_y'(y, \varphi(y))h$$

$$- \frac{1}{\tilde{f}_r'(y, \varphi(y))} (\tilde{f}_{yy}''(y, \varphi(y))(h, h) + \tilde{f}_{yr}''(y, \varphi(y))h \cdot \varphi'(y)h)$$

Since $Y = \ker f'(x_0)$ we have

$$(6.18) \quad \varphi''(0)(h, h) = - \frac{1}{f'(x_0)i(0, 1)} f''(x_0)(i(h, 0), i(h, 0))$$

Therefore, by (6.7)

(6.16) $\psi''(0)(h, h) = -\dfrac{1}{f_1'(x_0)(0, 1)} (f_1''(x_0)(h, h) + \lambda f_2''(x_0)(h, h))$ $h \in \ker f'(x_0)$

(We identify $(h, 0)$ with h). In view of our choice

$(c_1 \geq c_2)$, $f_1'(x_0)(0, 1) < 0$, hence by (6.16)

$$\psi''(0)(h, h) \geq k_1 ||h||^2$$

Since $\psi(0) = \psi'(0) = 0$, there is a neighborhood V of 0 in which

$$\psi(h) \geq \dfrac{k_1}{2} ||h||^2 .$$

The proof is complete.

7. Conclusion

Proposition 6.5 constitutes a simple example of higher order sufficient condition for uniform lower semicontinuity. It applies to the multifunction ψ^{-1}, when the first derivative vanishes (critical point). In similar circumstances first order conditions (by which we understand the Lusternik-type conditions presented in Sections 2 and 3) cannot be used and higher derivatives should be taken into account in establishment of semicontinuity properties of multifunctions. It was pointed out in Section 6 that higher order sufficient conditions are crucial in sensitivity theory, where the nature of problems excludes applicability of first order conditions.

Another failure of first order conditions is illustrated in Example 3.3. Note that the discussed multifunction G^{-1} may be represented as the intersection of two u.H.s.c multifunctions defined on $\mathbb{R}^2$ and valued in X, namely

$$G^{-1}(r_1, r_2) = \{x : g_1(x) = r_1\} \cap \{x : g_2(x) = r_2\} .$$

Again we face a problem of the metric upper semicontinuity of an intersection. In a similar context a use of higher order conditions may turn out of great value, when the usual constraint qualifications fail. This conclusion may sound like an introduction to a study of higher order conditions for semicontinuity. I hope to carry out such a study one day.

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We present a variety of methods for establishing metric upper semicontinuity. We give conditions for the intersection of metric upper semicontinuous multifunctions to be metric upper semicontinuous and discuss their applicability.

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