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CONFIDENCE INTERVALS ON A RATIO OF VARIANCES IN THE TWO-FACTOR NESTED COMPONENTS OF VARIANCE MODEL

by

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ABSTRACT

Consider the two-factor nested components of variance model $Y_{ijk} = \mu + A_i + B_{ij} + C_{ijk}$, where $\mathrm{Var}[A_i] = \sigma_A^2$, $\mathrm{Var}[B_{ij}] = \sigma_B^2$, $\mathrm{Var}[C_{ijk}] = \sigma_C^2$.

Confidence intervals are derived for σ_A^2/σ_C^2, $\sigma_A^2/(\sigma_A^2 + \sigma_C^2)$ and $\sigma_C^2/(\sigma_A^2 + \sigma_C^2)$.

KEY WORDS: Confidence intervals on ratios of variances.

1. Introduction

Consider the two-factor nested components-of-variance model given by

$$Y_{ijk} = \mu + A_i + B_{ij} + C_{ijk} \quad \text{for}$$

$i = 1, 2, \ldots, I > 1$; $j = 1, 2, \ldots, J > 1$; and $k = 1, 2, \ldots, K > 1$; where $E[A_i] = 0$; $Var[A_i] = \sigma_A^2$; $E[B_{ij}] = 0$; $Var[B_{ij}] = \sigma_B^2$; $E[C_{ijk}] = 0$; and $Var[C_{ijk}] = \sigma_C^2$. The random variables Y_{ijk} are observable; the random variables $A_1, \ldots, A_I$; $B_{11}, \ldots, B_{IJ}$; $C_{111}, \ldots, C_{IJK}$ are pairwise uncorrelated and unobservable and are jointly normally distributed; μ, σ_A^2, σ_B^2, and σ_C^2 are unobservable parameters. The parameter space Ω is defined by

$$\Omega = \{(\mu, \sigma_A^2, \sigma_B^2, \sigma_C^2): -\infty < \mu < \infty, \sigma_A^2 \geq 0, \sigma_B^2 \geq 0, \sigma_C^2 \geq 0\}.$$

These specifications define a two-factor nested components-of-variance model with equal numbers in the subclasses and the ANOVA table is displayed in Table 1.

Table 1.

ANOVA table for two-factor nested components-of-variance model with equal numbers in the subclasses

Source	d.f.	S.S.	M.S.	E.M.S.
Total	IJK	$\Sigma\Sigma\Sigma Y_{ijk}^2$		
Mean	1	$IJK\bar{Y}_{\ldots}^2$		
Factor A	$n_1 = I-1$	$\Sigma\Sigma\Sigma(\bar{Y}_{i..}-\bar{Y}_{\ldots})^2$	S_1^2	$\theta_1 = \sigma_C^2+K\sigma_B^2+JK\sigma_A^2$
B within A	$n_2 = I(J-1)$	$\Sigma\Sigma\Sigma(\bar{Y}_{ij.}-\bar{Y}_{i..})^2$	S_2^2	$\theta_2 = \sigma_C^2+K\sigma_B^2$
Error	$n_3 = IJ(K-1)$	$\Sigma\Sigma\Sigma(Y_{ijk}-\bar{Y}_{ij.})^2$	S_3^2	$\theta_3 = \sigma_C^2$

In this model there are several functions of the variance components that may be of interest in applied problems. These include σ_A^2, σ_B^2, σ_C^2, $\sigma_C^2/(\sigma_C^2 + \sigma_B^2)$, $\sigma_C^2/(\sigma_C^2 + \sigma_A^2)$, $\sigma_A^2/(\sigma_A^2 + \sigma_B^2 + \sigma_C^2)$, $\sigma_B^2/(\sigma_A^2 + \sigma_B^2 + \sigma_C^2)$, and $\sigma_C^2/(\sigma_A^2 + \sigma_B^2 + \sigma_C^2)$. The only functions of σ_A^2, σ_B^2, σ_C^2 given above for which an exact size confidence interval exists is σ_C^2 and $\sigma_C^2/(\sigma_C^2 + \sigma_B^2)$. Approximate size confidence intervals for σ_A^2 and σ_B^2 have been given by Moriguti (1954), Bulmer (1956) and Howe (1974). Approximate size confidence intervals for $\sigma_A^2/(\sigma_A^2 + \sigma_B^2 + \sigma_C^2)$, $\sigma_B^2/(\sigma_A^2 + \sigma_B^2 + \sigma_C^2)$ and $\sigma_C^2/(\sigma_A^2 + \sigma_B^2 + \sigma_C^2)$ have been given by Graybill and Wang (1979). In this paper we give approximate size confidence intervals for $\sigma_C^2/(\sigma_A^2 + \sigma_C^2)$, $\sigma_A^2/(\sigma_A^2 + \sigma_C^2)$, σ_A^2/σ_C^2 and σ_C^2/σ_A^2.

Actually we obtain approximate size confidence intervals for σ_A^2/σ_C^2 only since σ_C^2/σ_A^2, $\sigma_A^2/(\sigma_A^2 + \sigma_C^2)$, and $\sigma_C^2/(\sigma_A^2 + \sigma_C^2)$ can be obtained from these.

In Section 2 the lower limit of the upper confidence interval is derived, in Section 3 the upper limit of the lower confidence interval is given, and in Section 4 is a short discussion of other methods that could possibly be used for confidence intervals on σ_A^2/σ_C^2.

2. Lower Limit of the Upper Confidence Interval on σ_A^2/σ_C^2

Since $\overline{Y}$ ..., S_1^2, S_2^2, and S_3^2 are complete sufficient statistics for this problem, we will require the upper confidence interval to be a function of them. Write

$$g(\overline{Y} \ldots, S_1^2, S_2^2, S_3^2) < \frac{\theta_1 - \theta_2}{\theta_3} < \infty$$

for the $1 - \alpha$ upper confidence interval where the function $g(\overline{Y} \ldots, S_1^2, S_2^2, S_3^2)$, the lower confidence point, is to be determined.

Using the notation in Table 1 observe that $\frac{\theta_1 - \theta_2}{\theta_3} = \frac{JK\sigma_A^2}{\sigma_C^2}$, so an upper confidence interval on $\frac{\theta_1 - \theta_2}{\theta_3}$ is equivalent to an upper confidence interval on σ_A^2/σ_C^2.

Since $\sigma_A^2/(\sigma_A^2 + \sigma_C^2)$ is a function of θ_1, θ_2, θ_3 only, this is unchanged if any constant c is added to Y_{ijk} in the model given in Section 1. Thus the lower confidence point $g(\overline{Y} \ldots S_1^2, S_2^2, S_3^2)$ should also be unchanged if c is added to Y_{ijk}. Let $c = -\overline{Y} \ldots$; thus $\overline{Y} \ldots + c$ is zero and S_1^2, S_2^2, S_3^2 are unchanged when Y_{ijk} is replaced by $Y_{ijk} + c$ (or specifically by $Y_{ijk} - \overline{Y} \ldots$). Hence $g(\overline{Y} \ldots, S_1^2, S_2^2, S_3^2)$ becomes $g(0, S_1^2, S_2^2, S_3^2)$ and the lower confidence point is a function of S_1^2, S_2^2, and S_3^2 only. So the objective is to find a function of S_1^2, S_2^2, S_3^2, say $f(S_1^2, S_2^2, S_3^2)$ such that

$$P[f(S_1^2, S_2^2, S_3^2) \leq (\theta_1 - \theta_2)/\theta_3]$$

is approximately (and very close to) equal to a specified number $1 - \alpha$.

If Y_{ijk} is replaced by cY_{ijk} for $c \neq 0$, then $(\theta_1 - \theta_2)/\theta_3$ is unchanged. Thus we require $f(c^2S_1^2, c^2S_2^2, c^2S_3^2) = f(S_1^2, S_2^2, S_3^2)$. Let $c^2 = 1/S_2^2$, then $f(S_1^2, S_2^2, S_3^2) = f(S_1^2/S_2^2, 1, S_3^2/S_2^2) = h(S_1^2/S_2^2, S_3^2/S_2^2)$, so the lower confidence point of $(\theta_1 - \theta_2)/\theta_3$ is a function of S_1^2/S_2^2 and S_3^2/S_2^2 only.

Since the maximum likelihood estimator of $\dfrac{\theta_1 - \theta_2}{\theta_3} = \dfrac{\theta_1/\theta_2 - 1}{\theta_3/\theta_2}$ is of the form $\dfrac{S_1^2 - S_2^2}{S_3^2} = \dfrac{S_1^2/S_2^2 - 1}{S_3^2/S_2^2}$, we require $h(S_1^2/S_2^2, S_3^2/S_2^2)$ to be

(a) monotonic increasing in S_1^2/S_2^2; (b) monotonic decreasing in S_3^2/S_2^2.

Let $\hat{\theta} = \dfrac{S_1^2 - S_2^2}{S_3^2}$, then from Mood et al. (1974, p. 180).

$$\mathrm{Var}(\hat{\theta}) = \mathrm{Var}\left(\frac{S_1^2 - S_2^2}{S_3^2}\right) = \frac{2n_3^2}{n_1(n_3 - 2)^2} \frac{\theta_1^2}{\theta_3^2} + \frac{2n_3^2}{n_2(n_3 - 2)^2} \frac{\theta_2^2}{\theta_3^2}$$

$$+ \frac{2n_3^2}{(n_3 - 4)(n_3 - 2)^2} \left(\frac{\theta_1}{\theta_3} - \frac{\theta_2}{\theta_3}\right)^2 + \frac{4n_3^2}{n_1(n_3 - 4)(n_3 - 2)^2} \frac{\theta_1^2}{\theta_3^2}$$

$$+ \frac{4n_3^2}{n_2(n_3 - 4)(n_3 - 2)^2} \frac{\theta_2^2}{\theta_3^2}$$

If we replace the θ_i by UMVU estimators and denote the resulting $\mathrm{Var}(\hat{\theta})$ by $\hat{\mathrm{Var}}(\hat{\theta})$, then $\hat{\mathrm{Var}}(\hat{\theta}) = c_1 S_1^4/S_3^4 + c_2 S_2^4/S_3^4 + (c_3 S_1^2/S_3^2 - c_4 S_2^2/S_3^2)^2$ where c_1, c_2, c_3 and c_4 are appropriate constants which are functions of n_1, n_2, and n_3.

So a large sample lower confidence point for $\frac{\theta_1 - \theta_2}{\theta_3}$ is

$$\hat{\theta} - N_\alpha \sqrt{\hat{Var}(\hat{\theta})} = \frac{S_1^2 \quad S_2^2}{S_3^2} - N_\alpha\{c_1 S_1^4/S_3^4 + c_2 S_2^4/S_3^4 + (c_3 S_1^2/S_3^2 - c_4 S_2^2/S_3^2)^2\}^{\frac{1}{2}}$$

$$= \frac{S_2^2}{S_3^2}\left[\frac{S_1^2}{S_2^2} - 1 - N_\alpha\{c_1(S_1^2/S_2^2)^2 + c_2 + (c_3 S_1^2/S_2^2 - c_4)^2\}^{\frac{1}{2}}\right]$$

$$= \frac{S_2^2}{S_3^2}\, q(S_1^2/S_2^2)$$

where N_α is the upper α probability point of a standard normal p.d.f.

Therefore, in general we require the lower confidence point, $h(S_1^2/S_2^2,\ S_3^2/S_2^2)$, of $\frac{\theta_1 - \theta_2}{\theta_3}$ to be of the form $\frac{S_2^2}{S_3^2} q(S_1^2/S_2^2)$, and we determine the function $q(S_1^2/S_2^2)$ such that

$$P\left[\frac{S_2^2}{S_3^2}\ q(S_1^2/S_2^2) \le \frac{\theta_1 - \theta_2}{\theta_3}\right] \tag{2.1}$$

is close to $1 - \alpha$. We require $q(S_1^2/S_2^2)$ to satisfy (1), (2), (3) below.

(1) When the hypothesis H_0: $\sigma_A^2 = 0$ vs. H_a: $\sigma_A^2 > 0$ is accepted for a size α test the confidence interval should include zero, and when H_0 is rejected, $h(S_1^2/S_2^2,\ S_3^2/S_2^2)$ should be an increasing function of S_1^2/S_2^2. To test H_0: $\sigma_A^2 = 0$ vs. H_a: $\sigma_A^2 > 0$ the hypothesis H_0 is accepted if and only if $S_1^2/S_2^2 < F_{\alpha:\ n_1,n_2}$ (This test is uniformly most powerful unbiased). Thus

$h(S_1^2/S_2^2,\ S_3^2/S_2^2) = 0$ when $S_1/S_2 \le F_{\alpha:\ n_1,n_2}$

$h(S_1^2/S_2^2,\ S_3^2/S_2^2) > 0$ and increasing in S_1^2/S_2^2 when $S_1^2/S_2^2 > F_{\alpha:\ n_1,n_2}$

Since $h(S_1^2/S_2^2,\ S_3^2/S_2^2) = \dfrac{S_2^2}{S_3^2}\ q(S_1^2/S_2^2)$ we obtain

$q(S_1^2/S_2^2) = 0$ when $S_1^2/S_2^2 \leq F_{\alpha:\ n_1,n_2}$

$q(S_1^2/S_2^2) > 0$ and increasing in S_1^2/S_2^2 when $S_1^2/S_2^2 > F_{\alpha:\ n_1,n_2}$

(2) When $J \to \infty$ (hence $n_2 \to \infty$ and $n_3 \to \infty$) the confidence interval will be required to have an "exact" confidence coefficient $1 - \alpha$. When $J \to \infty$ it follows that $n_2 \to \infty$ and $n_3 \to \infty$ and from this it follows that $S_2^2 \to \theta_2$ in probability and $S_3^2 \to \theta_3$ in probability. Start with

$$P\left[\frac{S_1^2}{F_{\alpha:n_1,\infty}} \leq \theta_1\right] = 1 - \alpha$$

and use the result of $J \to \infty$, i.e. replace S_2^2 and S_3^2 by their "equivalent" values θ_2 and θ_3 respectively, to obtain

$$P\left[\frac{S_2^2}{S_3^2}\left(\frac{S_1^2}{S_2^2 F_{\alpha:\ n_1,\ \infty}} - 1\right) \leq \frac{\theta_1 - \theta_2}{\theta_3}\right] = 1 - \alpha$$

Hence when $J \to \infty$

$q(S_1^2/S_2^2) = 0$ when $S_1^2/S_2^2 \leq F_{\alpha:\ n_1,\infty}$

$q(S_1^2/S_2^2) = \dfrac{S_1^2}{S_2^2 F_{\alpha:\ n_1,\ \infty}} - 1$ when $S_1^2/S_2^2 > F_{\alpha:\ n_1,\infty}$

(3) If $\sigma_A^2 \to \infty$, the quantity $\dfrac{\theta_1 - \theta_2}{\theta_3}$ is dominated by θ_1/θ_3, and we want

$$P\left[\frac{S_2^2}{S_3^2}\ \frac{S_1^2}{S_2^2 F_{\alpha:\ n_1,\ n_3}} \leq \frac{\theta_1}{\theta_3}\right]$$

to be equal to $1 - \alpha$. This requires $q(S_1^2/S_2^2)$ to behave like $S_1^2/S_2^2F_{\alpha:\ n_1,\ n_3}$ for large S_1^2/S_2^2 in the sense that

$$q(S_1^2/S_2^2) = \frac{S_1^2}{S_2^2 F_{\alpha:\ n_1,\ n_3}}\ \{1 + \ell(S_1^2/S_2^2)\}\ \text{ where}$$

$$\ell(S_1^2/S_2^2) \to 0 \text{ as } S_1^2/S_2^2 \to \infty$$

Any function $q(S_1^2/S_2^2)$ satisfying conditions (1), (2), (3) will give an exact confidence coefficient in the three limiting cases $\theta_1/\theta_2 = 1$, $\theta_1/\theta_2 = \infty$ and $J \to \infty$..

The simplest function satisfying those conditions is the linear function $q_1(S_1^2/S_2^2) = a_1 S_1^2/S_2^2 + b_1$ where a_1 and b_1 are functions of n_1, n_2, n_3, and α and are determined by the conditions (1), (2), and (3). However, this did not give results as good as desired so a more general function was used, namely

$$q(S_1^2/S_2^2) = [a_1 S_1^2/S_2^2 + b_1 + c_1(S_1^2/S_2^2)^{-1}]/F_{\alpha:\ n_1,\ n_3} \tag{2.2}$$

From condition (3) $a_1 = 1$.

From condition (2) $b_1(n_1, \infty, \infty) = -F_{\alpha:\ n_1,\ \infty}$: $c_1(n_1, \infty, \infty) = 0$

From condition (1) $F_{\alpha:\ n_1,\ n_2} + b_1 + c_1/F_{\alpha:\ n_1,\ n_2} = 0$ or

$$c_1 = -F_{\alpha:\ n_1,\ n_2}(F_{\alpha:\ n_1,\ n_2} + b_1).$$

Let $b_1(n_1, n_2, n_3) = -F_{\alpha:\ n_1,\ \infty}$ for all n_2 and n_3, then

$$c_1 = F_{\alpha:\ n_1,\ n_2}(F_{\alpha:\ n_1,\ \infty} - F_{\alpha:\ n_1,\ n_2}), \text{ and}$$

$$q(S_1^2/S_2^2) = [S_1^2/S_2^2 - F_{\alpha:n_1,\infty} + F_{\alpha:n_1,n_2}(F_{\alpha:n_1,\infty} - F_{\alpha:n_1,n_2})\ S_2^2/S_1^2]/F_{\alpha:n_1,n_3}$$

Thus a $1-\alpha$ upper confidence interval on $(\theta_1 - \theta_2)/\theta_3$ is $L_2 \leq (\theta_1 - \theta_2)/\theta_3 < \infty$ where L_2 is defined by

$$L_2 = 0 \qquad \text{if } S_1^2/S_2^2 \leq F_{\alpha:\ n_1,\ n_2} \tag{2.3}$$

$$L_2 = \frac{S_2^2}{S_3^2 F_{\alpha:n_1,n_3}}[S_1^2/S_2^2 - F_{\alpha:n_1,\infty} + F_{\alpha:n_1,n_2}\ (F_{\alpha:n_1,\infty} - F_{\alpha:n_1,n_2})\ S_2^2/S_3^2]$$

$$\text{if } S_1^2/S_2^2 > F_{\alpha:n_1,n_2}$$

Note that $L_2 = 0$ if and only if the α level test of H_0: $\sigma_A^2 = 0$ is accepted, so $P[L_2 = 0] = P[S_1^2/S_2^2 \le F_{\alpha:\, n_1,\, n_2}] \le 1 - \alpha$ and $P[L_2 = 0] = 1 - \alpha$ if and only if $\sigma_A^2 = 0$. The probability associated with Equation (2.3) is a function of the unknown parameter $\rho = \theta_1/\theta_2$ and is exactly equal to $1 - \alpha$ when ρ is one or infinity or when J is infinite.

The excellence of this approximation is indicated by Table 2, calculated by simulation. Columns 7, 8, and 9 of Table 2 contain the range of probabilities of $L_2 \le (\theta_1 - \theta_2)/\theta_3$ as the unknown parameter θ_1/θ_2 varies from 1 to ∞. The approximation appears to be quite satisfactory even for small sample sizes.

The remainder of this section is devoted to the study of the behavior of $P = P[\frac{S_2^2}{S_3^2} q(S_1^2/S_2^2) \le \frac{\theta_1 - \theta_2}{\theta_3}]$ for all values of n_1, n_2 and n_3. From Table 2 P appears to get closer to $1 - \alpha$ as the value of K (hence n_3) increases. In fact as $K \to \infty$ (hence $n_3 \to \infty$) the problem is reduced to the interval estimation of σ_A^2 in the one-factor model and the method discussed in this section is equivalent to Moriguti's method (1954). From this one knows that the error in P is of the order n_2^{-2}, i.e. $P = 1 - \alpha + 0(n_2^{-2})$. Another way to examine the behavior of P is to expand P in powers of n_2^{-1} and n_3^{-1}. The algebraic details of this work are heavy (see Bulmer (1957)). The resulting expansion is

$$P = 1 - \alpha + \alpha_0 + \alpha_{12}/n_2 + \alpha_{13}/n_3 + \alpha_{22}/n_2^2 + \alpha_{33}/n_3$$
$$+ \alpha_{23}/n_2 n_3 + 0(n_2^{-2}, n_3^{-3}).$$

This assures that as the values of J and K (hence n_2 and n_3) increase the accuracy of the approximation gets better.

In Table 2, P is between 0.9500 and 0.9597 when I = 3, J = 3, K = 3, and $1 - \alpha = 0.95$ and when I = 7, J = 3, K = 3, P is between 0.9500 and 0.9581. A study of the values of P when I is large (n_1, n_2, and n_3 are large, but $R_1 = n_1/n_2$, $R_2 = n_1/n_3$ remain constant) is in Wang (1979).

3. Upper Limit of the Lower Confidence Interval on σ_A^2/σ_C^2

Since $P[\frac{\theta_1 - \theta_2}{\theta_3} \leq f(S_1^2,S_2^2,S_3^2)] = 1 - P[f(S_1^2,S_2^2,S_3^2) \leq \frac{\theta_1 - \theta_2}{\theta_3}]$, we use the confidence coefficient α in the lower limit of the upper confidence interval in Equation (2.3) to obtain a lower $1 - \alpha$ confidence interval on $(\theta_1 - \theta_2)/\theta_3$ given by $0 \leq \theta_1 - \theta_2)/ \theta_3 \leq U$ where

$$U = \frac{S_2^2}{S_3^2 F_{1-\alpha:n_1,n_3}}[S_1^2/S_2^2 - F_{1-\alpha:n_1,\infty} + F_{1-\alpha:n_1,n_2}(F_{1-\alpha:n_1,\infty} - F_{1-\alpha:n_1,n_2})S_2^2/S_1^2] \quad \text{if } S_1^2/S_2^2 > F_{1-\alpha:n_1,n_2} \tag{3.1}$$

$$U = 0 \quad \text{if } S_1^2/S_2^2 \leq F_{1-\alpha:n_1,n_2}$$

We could determine how close the confidence coefficient of this confidence interval is to the nominal $1-\alpha$ by simulation. However, due to the expense of computer simulation we chose a different route. We used $q_1(S_1^2/S_2^2) = a_1 S_1^2/S_2^2 + b_1$ and conditions similar to (1), (2), (3), of Section 2 to obtain the confidence interval $0 \leq (\theta_1 - \theta_2)/\theta_3 \leq U_1$, where U_1 is given by

$$U_1 = 0 \quad \text{if } S_1^2/S_2^2 \leq F_{1-\alpha:n_1,n_2} \tag{3.2}$$

$$U_1 = \frac{S_2^2}{S_3^2}\,(S_1^2/S_2^2 F_{1-\alpha:n_1,n_3} - F_{1-\alpha:n_1,n_2}/F_{1-\alpha:n_1,n_3}) \quad \text{if } S_1^2/S_2^2 > F_{1-\alpha:n_1,n_2}$$

Note that $U_1 = 0$ and $U = 0$ if and only if the $1 - \alpha$ level test of $H_0:\sigma_A^2 = 0$ is accepted. Also note that conditions (2) and (3) of Section 2 are satisfied by the confidence intervals given in Equations (3.1) and (3.2).

The probability associated with Equation (3.2) depends on the value of $\rho = \theta_1/\theta_2$, n_1, n_2, n_3 and can be easily calculated if n_1 is even; we get

$$P[\frac{\theta_1 - \theta_2}{\theta_3} \leq \frac{S_2^2}{S_3^2} (S_1^2/S_2^2 F_{1-\alpha:n_1,n_3} - F_{1-\alpha:n_1,n_2}/F_{1-\alpha:n_1,n_3})]$$

$$= (\frac{1}{c+1})^{n_2/2} (\frac{1}{d+1})^{n_3/2} \sum_{y=0}^{n_1/2-1} \frac{1}{y!2^y} E[(\frac{c}{c+1})U_2 + (\frac{d}{d+1})U_3]^y$$

where $c = R_1 F_{1-\alpha:n_1,n_2}/\rho$, $d = (\rho-1)R_1R_2F_{1-\alpha:n_1,n_3}/\rho$ (see Wang (1979)).

The results of the probabilities of $(\theta_1-\theta_2)/\theta_3 \leq U_1$ are given in Table 3 for various values of I, J, K and for $1-\alpha = 0.09, 0.95, 0.99$. The actual probabilities are quite close to the specified probabilities even for small sample sizes. We expect the results to be even better if the more general confidence interval $0 \leq (\theta_1 - \theta_2)/\theta_3 \leq U$ is used where U is given in Equation (3.1).

4. Comparison with Other Methods.

The literature does not contain any references that have been evaluated and directly relate to confidence intervals on σ_A^2/σ_C^2. Perhaps Satterthwaite's (1946) method could be used but this procedure is extremely poor when used to place confidence intervals on the difference of expected mean squares (i.e. on $(\theta_1-\theta_2)/\theta_3 = J\ K\ \sigma_A^2/\sigma_C^2$). Broemeling (1969) presents a method for placing simultaneous confidence intervals on σ_A^2/σ_C^2 and σ_B^2/σ_C^2.. This method can be used to place confidence intervals on σ_A^2/σ_C^2.

We use Equation (15) in Broemeling (1969) to obtain

$$P[0 \leq K\ J\ \sigma_A^2/\sigma_C^2 \leq S_1^2/S_2^2\ F_{1-\alpha:n_1,n_3}] \geq (1-\alpha)^2 \tag{4.1}$$

which can be used for a lower confidence interval on $KJ\sigma_A^2/\sigma_C^2$ with confidence coefficient greater than or equal to $(1 - \alpha)^2$. Clearly the $1 - \alpha$ lower confidence interval in Equation (3.2) above is shorter than the $(1 - \alpha)^2$ confidence interval in Equation (4.1). Thus the confidence interval on σ_A^2/σ_C^2 derived from the procedure by Broemeling is not as good as the method presented in this paper.

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Table 2

Confidence Coefficients for an Upper Confidence Interval on $\frac{\theta_1-\theta_2}{\theta_3}$ (on σ_A^2/σ_C^2) Using Equation (2.3)

I	J	K	n_1	n_2	n_3	$1-\alpha = .90$	$1-\alpha = .95$	$1-\alpha = .99$
3	3	3	2	6	18	.90 - .9109	.95 - .9597	.99 - .9955
3	3	10	2	6	81	.90 - .9092	.95 - .9580	.99 - .9937
3	5	3	2	12	30	.90 - .9082	.95 - .9550	.99 - .9927
3	10	3	2	27	60	.90 - .9044	.95 - .9534	.99 - .9908
5	3	3	4	10	30	.90 - .9115	.95 - .9591	.99 - .9947
5	5	3	4	20	50	.90 - .9063	.95 - .9549	.99 - .9927
5	10	3	4	45	100	.90 - .9033	.95 - .9534	.99 - .9918
7	3	3	6	14	42	.90 - .9108	.95 - .9581	.99 - .9945
7	5	3	6	28	70	.90 - .9063	.95 - .9549	.99 - .9922
7	7	3	6	42	98	.90 - .9034	.95 - .9539	.99 - .9918

Table 3

Confidence Coefficients for a Lower Confidence Interval on $\frac{\theta_1-\theta_2}{\theta_3}$ (on $\frac{\sigma_A^2}{\sigma_C^2}$) Using Equation (3.2)

I	J	K	n_1	n_2	n_3	$1-\alpha = .90$	$1-\alpha = .95$	$1-\alpha = .99$
3	3	3	2	6	18	.90 - .9000	.95 - .9500	.99 - .9900
3	3	10	2	6	81	.90 - .9000	.95 - .9500	.99 - .9900
3	5	3	2	12	30	.90 - .9000	.95 - .9500	.99 - .9900
3	10	3	2	27	60	.90 - .9000	.95 - .9500	.99 - .9900
3	100	3	2	297	600	.90 - .9000	.95 - .9500	.99 - .9900
3	3	100	2	6	891	.90 - .9000	.95 - .9500	.99 - .9900
3	3	10001	2	6	90000	.90 - .9000	.95 - .9500	.99 - .9900
5	3	3	4	10	30	.90 - .9021	.95 - .9516	.99 - .9904
5	5	3	4	20	50	.90 - .9012	.95 - .9509	.99 - .9902
5	10	3	4	45	100	.90 - .9006	.95 - .9504	.99 - .9901
7	3	3	6	14	42	.90 - .9039	.95 - .9528	.99 - .9908
7	5	3	6	28	70	.90 - .9022	.95 - .9516	.99 - .9904
7	7	3	6	42	98	.90 - .9015	.95 - .9511	.99 - .9903

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ABSTRACT

Consider the two-factor nested components of variance model $Y_{ijk} = \mu + A_i + B_{ij} + C_{ijk}$, where $\mathrm{Var}(A_i) = \sigma_A^2$, $\mathrm{Var}(B_{ij}) = \sigma_B^2$, $\mathrm{Var}(C_{ijk}) = \sigma_C^2$.

sigma-squared sub A

Confidence intervals are derived for σ_A^2/σ_C^2, $\sigma_A^2/(\sigma_A^2 + \sigma_C^2)$ and $\sigma_C^2/(\sigma_A^2 + \sigma_C^2)$.

Sigma-squared sub C