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OPTIMAL STATIONARY LINEAR CONTROL OF THE WIENER PROCESS

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by

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The cutoff point b and the performance rate of the optimal law u^* are simultaneously determined in terms of the function $\phi(\cdot)$ through a simple system of integrotranscendental equations.

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OPTIMAL STATIONARY LINEAR CONTROL OF THE WIENER PROCESS

Václav E. Beneš[✓] and Ioannis Karatzas

ABSTRACT

In the present paper we consider the following stochastic control problem: minimize the average expected total cost

$$J(x,u) = \liminf_{T \rightarrow \infty} \frac{1}{T} E_x^u \int_0^T [\phi(\xi_t) + |u_t(\xi)|] dt$$

subject to $d\xi_t = u_t(\xi)dt + dw_t$, $\xi_0 = x$; $|u| \leq 1$, w_t Wiener, with all bounded by unity and measurable functionals on the past of the state process $\{\xi_s; s \leq t\}$ admissible as controls. It is proved that under very mild conditions on the running cost function $\phi(\cdot)$ the optimal law is of the form

$$\begin{aligned} u_t^*(\xi) &= -1, \xi_t > b \\ &= 0, |\xi_t| \leq b \\ &= 1, \xi_t < -b. \end{aligned}$$

The cutoff point b and the performance rate of the optimal law u^* are simultaneously determined in terms of the function $\phi(\cdot)$ through a simple system of integrotranscendental equations.

OPTIMAL STATIONARY LINEAR CONTROL OF THE WIENER PROCESS

1. INTRODUCTION

➤ In this paper we consider the problem of stationary control of the stochastic differential equation $d\xi_t = u_t(\xi)dt + dw_t$; $\xi_0 = x$, where $(w_t) = \{w_t; t \geq 0\}$ is a Wiener process on an underlying probability space, $(\Omega, \mathcal{F}, P)$.

(w sub t) Two kinds of cost are involved in this problem. First, one pays $\phi(\xi_t)$ per unit time for being in the wrong state ξ_t , where $\phi(\cdot)$ is a suitable cost function to be described later; secondly, one pays u_t per unit time for using the control law u_t . The control problem is to choose a law $u_t(\xi)$ as a non-anticipative functional of the solution process (ξ_t) with values in the bounded interval $[-1, 1]$, so as to minimize the average expected total cost.

It is proved that the optimal law can be explicitly described and its performance characterized in terms of the cost function $\phi(\cdot)$. The method consists in first restricting attention to an important subclass of admissible control laws, namely those giving rise to an ergodic solution process (ξ_t) . A process is said to be ergodic if it admits a unique invariant distribution. The optimal law u^* in this subclass can be obtained by using a dynamic programming approach, similar to that of Wonham [11]; it turns out that u^* is of the form

$$\begin{aligned} u_t^*(\xi) &= -\text{sgn } \xi_t, \quad |\xi_t| > b \\ &= 0, \quad |\xi_t| \leq b \end{aligned} \quad (1.1)$$

where b is a positive constant that can be characterized in terms of the function $\phi(\cdot)$. Secondly, the law u^* is proved optimal against any possible nonanticipative law u whatsoever.

The result (1.1) is the natural and expected one; it says that the best policy is to push ξ_t with full force in the negative direction if it is too positive and in the positive direction if it is too negative, while refraining from any action if ξ_t is inside a "dead-zone" $[-b, b]$. The appearance of the latter is a consequence of the running cost $|u|$ on the control, of the fact that the control is "expensive". Were such a cost absent, it is fairly obvious - and easily provable by using the methods of the present paper - that the optimal policy would be described by the "bang-bang" law: $-\text{sgn } \xi_t$.

Among previous works on the topic of stationary control of systems driven by a Wiener process we cite those of Wonham [11] and Kushner [8]. The scope of both was severely restricted, however, in that they allowed only those laws that generate an ergodic solution process (actually, only a subclass of these was considered).

2. FORMULATION

Consider the space $\Omega = C_{[0, T]}$ of real-valued, continuous functions on $[0, T]$, for some $T > 0$. Let (ξ_t) denote the family of evaluation functionals on $C_{[0, T]}$ and $\mathcal{F}_t, 0 \leq t \leq T$ the σ -field of subsets of $C_{[0, T]}$ generated by $\{\xi_s; s \leq t\}$.

Consider also the σ -field $\mathcal{M}$ of subsets M of $[0, T] \times C_{[0, T]}$ having the property that, for any $t \in [0, T]$, M_t belongs to $\mathcal{F}_t$ and that each ξ -section M_ξ of M , $\xi \in C_{[0, T]}$, is Lebesgue measurable.

A function g defined on $[0, T] \times C_{[0, T]}$ is $\mathcal{M}$ -measurable if

and only if $g(t, \cdot)$ is $\mathcal{M}_t$ -measurable, for each t , and $g(\cdot, \xi)$ is Lebesgue measurable, for each ξ .

Definition 2.1: Let the control measure space be the interval $[-1, 1]$ with its Borel sets. An admissible nonanticipative control law u is a measurable function $u: ([0, T] \times C_{[0, T]}, \mathcal{M}) \rightarrow [-1, 1]$. The class of all such control laws is denoted by $\mathcal{U}$.

For any control law $u \in \mathcal{U}$ and any $x \in \mathbb{R}$, a weak solution (ξ_t) to the stochastic differential equation

$$(2.1) \quad d\xi_t = u_t(\xi)dt + dw_t; \quad 0 \leq t \leq T$$

$$(2.2) \quad \xi_0 = x$$

is constructed as follows: one starts with the probability space $(\Omega, \mathcal{F}_T, P)$, where P is Wiener measure on $\Omega = C_{[0, T]}$. Corresponding to each law $u \in \mathcal{U}$ and each initial position $x \in \mathbb{R}$, the new measure

$$(2.3) \quad P_x^u(d\omega) = \exp \left[\int_0^T u_t(\xi)dw_t - \frac{1}{2} \int_0^T u_t^2(\xi)dt \right] \cdot P(d\omega)$$

is constructed on $(\Omega, \mathcal{F}_T)$, where (ξ_t) is the process defined by $\xi_t = x + w_t$; $0 \leq t \leq T$. According to Girsanov [5], P_x^u is a probability measure on $(\Omega, \mathcal{F}_T)$ and the process

$$(2.4) \quad \tilde{w}_t \triangleq w_t - \int_0^t u_s(\xi)ds = \xi_t - x - \int_0^t u_s(\xi)ds$$

is a Wiener process on $(\Omega, \mathcal{F}_T, P_x^u)$. Equation (2.4) is an equivalent way of saying that the process (ξ_t) , $\xi_t = x + w_t$; $0 \leq t \leq T$ satisfies the stochastic differential equation

$$(2.1)' \quad d\xi_t = u_t(\xi)dt + d\tilde{w}_t; \quad 0 \leq t \leq T$$

$$(2.2) \quad \xi_0 = x$$

on $(\Omega, \mathcal{F}_T, P_x^u)$. All processes involved here are adapted to the underlying family $(\mathcal{F}_t)$ of sub- σ -fields of $\mathcal{F}_T$. The process (ξ_t) is called a weak solution of (2.1)'-(2.2) because by construction $\sigma(\tilde{w}_s; s \leq t) \subseteq \sigma(\xi_s; s \leq t)$, though not necessarily the other way around. Such a solution is known to be unique in the sense of the probability law; see Liptser and Shiriyayev [9].

Now consider a function $\phi: \mathbb{R} \rightarrow \mathbb{R}^+$ which is even, convex piecewise $C^{(2)}$, monotonically increasing to infinity on $x > 0$, and satisfying an exponential growth condition:

$$(2.5) \quad \phi(x) = O(e^{\alpha|x|}) \quad \text{as } |x| \rightarrow \infty, \text{ some } 0 < \alpha < 2.$$

The optimal control problem can now be formulated as follows: choose a law $u \in \mathcal{U}$ for which the limit

$$J(x, u^*) = \lim_{T \rightarrow \infty} \frac{1}{T} E_x^{u^*} \int_0^T (\phi(\xi_t) + |u_t^*(\xi)|) dt$$

exists for all $x \in \mathbb{R}$, and which minimizes the average expected total cost rate

$$(2.6) \quad J(x,u) = \liminf_{T \rightarrow \infty} \frac{1}{T} E_x^u \int_0^T [\phi(\xi_t) + |u_t(\xi)|] dt$$

of starting at place x and using control law u , for all $(x,u) \in [\mathbb{R}^n \times \mathcal{U}]$. E_x^u denotes expectation with respect to the probability measure P_x^u introduced in (2.3).

3. SUMMARY

In Section 4 we briefly study the important subclass of feedback (Markov) admissible control laws. It is pointed out (and in the special case of time-homogeneous feedback laws, proved) that for such controls the stochastic differential equation (2.1)-(2.2) of the system can be solved in the strong sense.

In Section 5 we consider a subclass of time-homogeneous feedback laws that give rise to an ergodic solution process. Asymptotic properties of those processes, such as existence of a unique invariant measure, laws of large numbers and ergodicity of their distributions are discussed.

The optimal law u^* in the abovementioned subclass is discerned in Section 6 and it is proved that u^* is of the form (1.1). Both the cutoff point b and the asymptotic performances λ of u^* are characterized in terms of the cost function $\phi(\cdot)$, through the system of integrotranscendental equations (6.3), (6.4). The method proceeds by constructing a solution to the "asymptotic" version of the Bellman equation of dynamic programming (6.2).

Finally, the asymptotic performance of the law u^* is compared against that of any admissible nonanticipative control u in $\mathcal{U}$. The result, proved in Section 7, is that u^* is actually

optimal in the (largest possible) class $\mathcal{U}$. The idea employed here is to first compare the performance of the control laws over finite time intervals $[0, T]$ and then pass to the limit as $T \rightarrow \infty$.

4. MARKOV LAWS AND STRONG SOLUTIONS

Definition 4.1. Suppose there exists a measurable function $\gamma: \mathbb{R} \times [0, T] \rightarrow [-1, 1]$ such that the nonanticipative law $u \in \mathcal{U}$ can be represented in the form

$$(4.1) \quad u_t(\xi) = \gamma(\xi_t, t), \text{ any } \xi \in C_{[0, T]}, \quad 0 \leq t \leq T.$$

Then u is called an admissible Markov law. The class of all such laws will henceforth be denoted by $\mathcal{A}$; obviously $\mathcal{A} \subseteq \mathcal{U}$.

For laws in $\mathcal{A}$ the stochastic differential equation

$$(4.2) \quad d\xi_t = \gamma(\xi_t, t)dt + dw_t, \quad \xi_0 = x$$

is known to possess a pathwise unique, strong nonanticipative solution, in the sense that the solution is adapted to the Wiener process: $\sigma(\xi_s; s \leq t) \subseteq \sigma(w_s; s \leq t)$, $0 \leq t \leq T$; see Zvonkin [12].

Definition 4.2. Consider the subclass of $\mathcal{A}$ consisting of those admissible nonanticipative laws u for which there exists a measurable function $a: \mathbb{R} \rightarrow [-1, 1]$, such that

$$(4.3) \quad u_t(\xi) = a(\xi_t), \text{ any } \xi \in C_{[0, T]}, \quad 0 \leq t \leq T.$$

Such laws u are called admissible time-homogeneous Markov laws and their class is denoted by $\mathcal{A}$.

For laws in $\mathcal{A}$ one can easily construct the (pathwise unique) strong solution to the stochastic differential equation

$$\begin{aligned} d\xi_t &= a(\xi_t)dt + dw_t, \quad 0 \leq t \leq T \\ (4.4) \quad \xi_0 &= x. \end{aligned}$$

Indeed, consider the function

$$(4.5) \quad \beta(x) = \int_0^x \exp\{-2 \int_0^y a(z)dz\} dy; \quad x \in \mathbb{R}$$

which is continuous, strictly increasing and satisfies the equation $\beta'' + 2a\beta' = 0$. The function

$$(4.6) \quad \sigma(x) = \beta'(\beta^{-1}(x)); \quad x \in \mathbb{R}$$

is Lipschitz continuous, as can be checked by simple calculus. Therefore the stochastic differential equation

$$(4.7) \quad d\zeta_t = \sigma(\zeta_t)dw_t; \quad 0 \leq t \leq T$$

$$(4.8) \quad \zeta_0 = \beta(x)$$

has for any $x \in \mathbb{R}$ a pathwise unique solution (ζ_t) on the probability space $(\Omega, \mathcal{F}_T, P)$, strong in the sense that

$\sigma(\zeta_s; s \leq t) \subseteq \sigma(w_s; s \leq t)$, any $0 \leq t \leq T$, according to Itô's classical theory; see for instance Gihman and Skorohod [4]. Denote by $\{\Omega, \mathcal{F}_T, \mathcal{F}_t, \zeta_t, P_{\beta(x)}^u\}$ the corresponding time-homogeneous Markov process

The process

$$(4.9) \quad \xi_t = \beta^{-1}(\zeta_t)$$

is now well defined, and an application of Itô's rule gives

$$\begin{aligned} d\xi_t &= \frac{1}{\beta'(\beta^{-1}(\zeta_t))} d\zeta_t - \frac{1}{2} \frac{\beta''(\beta^{-1}(\zeta_t))}{(\beta'(\beta^{-1}(\zeta_t)))^3} \sigma^2(\zeta_t) dt \\ &= a(\xi_t) dt + dw_t. \end{aligned}$$

So (ξ_t) satisfies both the equation and the initial condition in (4.4) and because it is a bijection of (ζ_t) pointwise in time:

$$\sigma\{\zeta_s; s \leq t\} = \sigma\{\xi_s; s \leq t\} \subseteq \sigma\{w_s; s \leq t\}, \quad 0 \leq t \leq T$$

i.e. (ξ_t) is a strong solution to (4.4). The corresponding time-homogeneous Markov process is denoted by $\{\Omega, \mathcal{F}_T, \mathcal{F}_t, \xi_t, P_x^u\}$.

5. SOME ERGODIC THEOREMS

Introduce the function $G(x) \triangleq \int_{-\infty}^x \frac{dz}{\sigma^2(z)}$, $\sigma(\cdot)$ as in (4.6),

and consider the subclass $\mathcal{E}'$ of $\mathcal{E}$, consisting of those laws

$u, u_t(\xi) = a(\xi_t)$ for which

$$(5.1) \quad G(\infty) = \int_{-\infty}^{\infty} \frac{dz}{\sigma^2(z)} = \int_{-\infty}^{\infty} \exp\{2 \int_0^y a(z) dz\} dy < \infty;$$

recall also the processes $(\xi_t), (\zeta_t)$ of the preceding section, corresponding to this law. According to Gihman and Skorohod [4; §18], the probability distribution $\frac{G(\cdot)}{G(\infty)}$ is ergodic for the Markov process $\{\Omega, \mathcal{F}_T, \mathcal{F}_t, \zeta_t, P_z\}$ in the sense that the following are true:

Fact 1. Positive Recurrence: The stopping times $\tau_{zy} = \inf\{t: \zeta_t = y\}$ are well defined and a.s. finite for any $z, y \in \mathbb{R}$; besides,

$$(5.2) \quad E_z^u(\tau_{zy}) < G(\infty)(2 + |z-y|)|z-y|.$$

Fact 2. Invariance of the Probability Distribution Function $G(\cdot)/G(\infty)$:
For any $0 \leq t \leq T$,

$$(5.3) \quad \int_{-\infty}^{\infty} P_z^u\{\zeta_t \leq y\} dG(z) = G(y), \quad y \in \mathbb{R}.$$

Fact 3. Law of Large Numbers: For any Borel function $f(\cdot)$ such that $\int_{-\infty}^{\infty} |f(y)| dG(y) < \infty$, we have

$$(5.4) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(\zeta_t) dt = \frac{1}{G(\infty)} \int_{-\infty}^{\infty} f(y) dG(y);$$

a.s. (P_z^u) and $L^1(E_z^u)$, any $z = \zeta_0 \in \mathbb{R}$.

Fact 4. Ergodicity of the Distributions: For any function $f(\cdot)$ as above,

$$(5.5) \quad \lim_{t \rightarrow \infty} E_z^u f(\zeta_t) = \frac{1}{G(\infty)} \int_{-\infty}^{\infty} f(y) dG(y), \text{ any } z = \zeta_0.$$

As a consequence:

$$\lim_{t \rightarrow \infty} P_z^u \{\zeta_t \leq y\} = \frac{G(y)}{G(\infty)}, \quad y \in \mathbb{R}.$$

It follows from the properties of the function $\beta(\cdot)$ introduced in (4.5) that the limiting distributions of the processes (ζ_t) and (ξ_t) exist simultaneously. Consequently, the probability distribution function $\frac{F(\cdot)}{F(\infty)}$, where

$$(5.6) \quad F(x) \triangleq G(\beta(x)) = \int_{-\infty}^x \frac{dy}{\beta'(y)} = \int_{-\infty}^x \exp\{2 \int_0^y a(z) dz\} dy,$$

is invariant for the Markov process $\{\Omega, \mathcal{F}_T, \mathcal{F}_t, \xi_t, P_x^u\}$. The ergodic properties of the latter can be read off from those of the (ζ_t) process:

$$(5.3)' \quad \int_{-\infty}^{\infty} P_x^u \{\xi_t \leq y\} dF(x) = F(y), \text{ any } 0 \leq t \leq T, y \in \mathbb{R}.$$

$$(5.4)' \quad \frac{1}{T} \int_0^T f(\xi_t) dt \xrightarrow{T \rightarrow \infty} \frac{1}{F(\infty)} \int_{-\infty}^{\infty} f(y) dF(y), \text{ a.s. } (P_x^u) \text{ and } L^1(E')$$

$$(5.5)' \quad \lim_{t \rightarrow \infty} E_x^u f(\xi_t) = \frac{1}{F(\infty)} \int_{-\infty}^{\infty} f(y) dF(y),$$

any Borel function $f(\cdot)$ such that:

$$\int_{-\infty}^{\infty} |f(y)| dF(y) < \infty, \text{ any } x = \xi_0 \in \mathbb{R}.$$

Proposition 5.1: For any law $u \in \mathcal{K}$, $u_t(\xi) = a(\xi_t)$, the corresponding solution process $\xi_t^u = \xi_t$ of the system equation (4.4) is a strongly Feller process, possessing a unique invariant probability distribution $F^u(\cdot)/F^u(\infty)$, $F^u(x) = F(x)$ as in (5.6), for which (5.3)'-(5.5)' hold.

Proof. All that remains to be proven is the strong Feller property and the uniqueness of the invariant distribution, and it suffices to do both on the (ξ_t) process. The latter is indeed strongly Feller, since (5.1) implies a fortiori: $\sigma^2(z) \geq \sigma^2$, all $z \in \mathbb{R}$, some $\sigma^2 > 0$; see Wonham [10]. On the other hand (ξ_t) is recurrent and positive, by (5.2).. For such processes, Khas'minskii [7] proves the existence of a unique invariant distribution, Q.E.D.

Definition 5.2. For the constant α of (2.5), $0 < \alpha < 2$, let $\mathcal{K}_\alpha$ be the subclass of $\mathcal{K}$ consisting of those laws u , $u_t(\xi) = a(\xi_t)$ for which

$$(5.7) \quad \int_{-\infty}^{\infty} e^{\alpha|x|} dF^u(x) = \int_{-\infty}^{\infty} e^{\alpha|x|} \exp\{2 \int_0^x a(z) dz\} dx < \infty.$$

It is evident from (5.4)' and the assumption (2.5) that, for any $u \in \mathcal{K}_\alpha$:

$$\begin{aligned}
 J(u) &= J(x, u) = \lim_{T \rightarrow \infty} \frac{1}{T} E_x^u \int_0^T [\phi(\xi_t) + |u_t(\xi)|] dt \\
 (5.8) \quad &= \frac{1}{F^u(\infty)} \int_{-\infty}^{\infty} [\phi(y) + |a(y)|] dF^u(y),
 \end{aligned}$$

any $x = \xi_0 \in \mathbb{R}$.

6. THE OPTIMAL LAW IN $\mathcal{L}_\alpha$

Introduce the function

$$\begin{aligned}
 c(p) &\triangleq \min_{|u| \leq 1} (up + |u|) = 1 - |p|, \quad |p| \geq 1 \\
 (6.1) \quad &= 0, \quad |p| < 1.
 \end{aligned}$$

Our objective is to find a positive constant λ and a function $v(x)$, twice continuously differentiable on $\mathbb{R}$ and $O(e^{\alpha|x|})$ as $|x| \rightarrow \infty$, with $0 < \alpha < 2$ as in (2.5), satisfying the Dynamic Programming equation

$$(6.2) \quad \lambda = \frac{1}{2} v_{xx}(x) + c(v_x(x)) + \phi(x), \quad x \in \mathbb{R}.$$

We start with a preliminary result.

Lemma 6.1. Under the assumptions on the running cost function $\phi(\cdot)$ made in section 2, there exists a unique solution (λ, b) to the pair of equations

$$(6.3) \quad \lambda b - \int_0^b \phi(s) ds = \frac{1}{2}$$

$$(6.4) \quad \lambda = 2 \int_0^{\infty} e^{-2s} \phi(b+s) ds.$$

Proof. It suffices to prove that equation $H(x) = 0$,

$$(6.5) \quad H(x) = 2x \int_0^{\infty} e^{-2s} \phi(x+s) ds - \int_0^x \phi(s) ds - \frac{1}{2},$$

has a unique solution b on $\mathbb{R}^+$. Indeed, $H(0) = -1/2$ and

$$H'(x) = 2 \int_0^{\infty} e^{-2s} [\phi(x+s) - \phi(x)] ds + 2x \int_0^{\infty} e^{-2s} \phi'(x+s) ds \geq \lambda \phi'(x), \quad x > 0.$$

Clearly, $H(x) \rightarrow \infty$ as $x \rightarrow \infty$, so there exists a unique number $b > 0$, such that $H(b) = 0$, Q.E.D.

The constants (λ, b) being as in the previous Lemma, consider the function $v(x)$ with $v(0) = 0$ and derivative given by

$$\begin{aligned} v_x(x) &= 2\lambda x - 2 \int_0^x \phi(s) ds && ; \quad 0 \leq x \leq b \\ (6.6) \quad &= 1 + \lambda [e^{2(x-b)} - 1] - 2 \int_b^x e^{2(x-s)} \phi(s) ds; && x > b \\ &= -v_x(-x) && ; \quad x < b. \end{aligned}$$

Proposition 6.2. The function $v(x)$ defined above is the unique (up to an additive constant) solution of (6.2) in $C^{(2)}(\mathbb{R})$, with λ determined along with the constant b through (6.3)-(6.4). $v(x)$ is also the smallest nonnegative function satisfying equation (6.2).

Proof. From (6.6), $v_x(b+) = 1$ while $v_x(b-) = 2\lambda b - 2 \int_0^b \phi(s) ds$ by (6.3). Therefore $v_x(x)$ is continuous on $\mathbb{R}$. On the other hand

$$\begin{aligned}
 v_{xx}(x) &= 2(\lambda - \phi(x)) & ; 0 \leq x \leq b \\
 (6.7) \quad &= 2 \left[\lambda e^{2(x-b)} - \phi(x) - 2 \int_b^x e^{2(x-s)} \phi(s) ds \right]; & x > b \\
 &= v_{xx}(-x) & ; x < 0
 \end{aligned}$$

is clearly continuous on $\mathbb{R}$. From (6.4) and the fact that $\phi(\cdot)$ is strictly increasing on $\mathbb{R}^+$ one gets: $v_{xx}(x) \geq 2(\lambda - \phi(b)) > 0$, on $0 \leq x \leq b$, as well as

$$v_{xx}(x) = 2 \left[2 \int_x^\infty e^{-2(s-x)} \phi(s) ds - \phi(x) \right] > 0, \text{ on } x > b.$$

The function $v(x)$ is even and strictly convex, therefore minimal among nonnegative solutions of (6.2). By strict convexity, $0 < v_x(x) < 1$, on $0 < x < b$ and $v_x(x) > 1$, on $x > b$. It remains to verify (6.2), which in the present case becomes

$$\begin{aligned}
 \lambda &= \frac{1}{2} v_{xx}(x) + \phi(x) & ; |x| \leq b \\
 (6.2)' \quad &= \frac{1}{2} v_{xx}(x) + 1 - v_x(x) + \phi(x); & x > b \\
 &= \frac{1}{2} v_{xx}(x) + 1 + v_x(x) + \phi(x); & x < -b.
 \end{aligned}$$

(6.2)' is readily verified, by substitution. Uniqueness of $v_x(x)$ is a consequence of Lipschitz continuity of the function $c(p)$ defined in (6.1).

Proposition 6.3. Suppose that $\tilde{\lambda}, \tilde{v}(x)$ are a constant a $C^{(2)}(\mathbb{R})$ function, respectively, for which (6.2) is satisfied, and such that

- (i) $0 < \tilde{v}_x(x) < 1, 0 < x < \tilde{b}$
- (ii) $\tilde{v}_x(x) = 1,$
- (iii) $\tilde{v}_x(x) > 1, x > \tilde{b},$

for some positive constant $\tilde{b}$.

Then the function $\tilde{v}(x)$ is necessarily strictly convex, therefore $\tilde{v}_x(x)$ is strictly increasing, $\tilde{b} \leq b$ and

$$(6.8) \quad \tilde{\lambda} \geq \lambda.$$

Proof. It is a straightforward exercise to verify that $\tilde{v}_x(x)$ will necessarily be of the form (6.6), with $(\tilde{\lambda}, \tilde{b})$ replacing (λ, b) . A necessary and sufficient condition for continuity of $\tilde{v}_x(x)$ is then

$$(6.3) \quad \tilde{\lambda} \tilde{b} - \int_0^{\tilde{b}} \phi(s) ds = 1/2,$$

while (iii) implies

$$(6.9) \quad \tilde{\lambda} > \frac{2 \int_b^x e^{2(x-y)} \phi(y) dy}{e^{2(x-\tilde{b})} - 1} = \frac{2 \int_0^{x-\tilde{b}} e^{-2s} \phi(\tilde{b}+s) ds}{1 - e^{-2(x-\tilde{b})}}, \text{ all } x > \tilde{b}.$$

A necessary and sufficient condition for (6.9) is (6.10) below:

$$(6.10) \quad \tilde{\lambda} \geq 2 \int_0^{\infty} e^{-2s} \phi(\tilde{b}+s) ds.$$

Indeed, if $\tilde{\lambda} < 2 \int_0^{\infty} e^{-2s} \phi(\tilde{b}+s) ds$ holds, then (6.9) is eventually false as $x \rightarrow \infty$. On the other hand, suppose that (6.10) is true; to prove (6.9) it suffices to prove

$$(6.11) \quad (1-e^{-2t}) \int_0^{\infty} \phi(\tilde{b}+s) e^{-2s} ds > \int_0^t \phi(\tilde{b}+s) e^{-2s} ds, \text{ all } t > 0$$

where $t = x - \tilde{b}$. But (6.11) is equivalent to:

$$\int_t^{\infty} e^{-2s} [\phi(\tilde{b}+s) - \phi(\tilde{b}+s-t)] ds > 0, \text{ all } t > 0,$$

which is obviously true since $\phi(\cdot)$ is strictly increasing.

Relations $(\widetilde{6.3})$, (6.10) are therefore necessary and sufficient conditions for the feasibility of (i)-(iii). They imply that $H(\tilde{b}) \leq 0$, $H(\cdot)$ being the function introduced in (6.5). But $H(\cdot)$ is strictly increasing so $\tilde{b} \leq b$ and therefore $\tilde{\lambda} \geq \lambda$, from (6.3) and (6.10). Strict convexity of $\tilde{v}(x)$ is proven as in Proposition 6.2, Q.E.D.

Once the solution of the dynamic programming equation (6.2) corresponding to the smallest possible value of the constant λ has been constructed, we proceed to prove the main result of this section, namely the optimality in the class $\mathcal{G}_\alpha$ (Definition 5.2) of the law $u_t^*(\xi) = a^*(\xi_t)$, with

$$(6.12) \quad \begin{aligned} a^*(x) &= -\operatorname{sgn} x, \quad |x| > b \\ &= 0, \quad |x| \leq b \end{aligned}$$

obtained through the minimization

$$(6.13) \quad a^*(x) \cdot v_x(x) + |a^*(x)| \equiv \min_{|u| \leq 1} [u \cdot v_x(x) + |u|] = c(v_x(x)), \text{ all } x \in \mathbb{R}$$

Lemma 6.4. $v(x) = O(e^{\alpha|x|})$, as $|x| \rightarrow \infty$.

Proof. It is checked that for all x large enough

$$v_x(x) = 1 - \lambda + 2c^{2x} \int_x^\infty e^{-2y} \phi(y) dy \leq 1 - \lambda + \frac{2c}{2-\alpha} e^{\alpha x},$$

some $c > 0$. The result follows readily.

Remark. Dr. Martin Day has noted that, for any other pair $(\tilde{\lambda}, \tilde{b})$ as in Proposition 6.3, the functions $v_x(x), v(x)$ have a growth of the order $e^{2|x|}$, as $|x| \rightarrow \infty$.

Theorem 6.5. The law $u^* \in \mathcal{K}_\alpha$, defined through

$$u_t^*(\xi) = a^*(\xi_t), \text{ all } \xi \in C_{[0,T]}, \quad 0 \leq t \leq T,$$

with $a^*(\cdot)$ as in (6.12), is optimal in $\mathcal{K}_\alpha$. Furthermore:

$$J(u^*) = \lambda.$$

Proof. Consider any law $u \in \mathcal{E}_u$ and the Markov process $\{\Omega, \mathcal{F}_T, \mathcal{F}_t, \xi_t^u, p_x^u\}$ -solution to the stochastic differential equation (4.4). An application of Itô's rule to the process $v(\xi_t^u)$, along with (6.13) and equation (6.2), yields

$$\begin{aligned} v(\xi_t^u) - v(x) &= \int_0^t \left[\frac{1}{2} v_{xx}(\xi_s^u) + u_s(\xi_s^u) v_x(\xi_s^u) \right] ds + \int_0^t v_x(\xi_s^u) dw_s \\ &\geq \int_0^t \left[\frac{1}{2} v_{xx}(\xi_s^u) + c(v_x(\xi_s^u)) - |u_s(\xi_s^u)| \right] ds + \int_0^t v_x(\xi_s^u) dw_s \\ &\geq \lambda t - \int_0^t \left[\phi(\xi_s^u) + |u_s(\xi_s^u)| \right] ds + \int_0^t v_x(\xi_s^u) dw_s, \text{ a.s. } (p_x^u). \end{aligned}$$

Taking expectations, and noting that

$$E_x^u \int_0^t v_x^2(\xi_s^u) ds \leq \text{Const.} e^{2\alpha(|x|+t)} E_x^u \int_0^t e^{2\alpha|w_s|} ds < \infty,$$

one gets:

$$(6.14) \quad \frac{E_x^u v(\xi_t^u)}{t} - \frac{v(x)}{t} + \frac{1}{t} E_x^u \int_0^t \left[\phi(\xi_s^u) + |u_s(\xi_s^u)| \right] ds \geq \lambda, \quad \text{all } x.$$

From (5.5)', (5.7) and Lemma (6.4) one gets

$$\lim_{t \rightarrow \infty} E_x^u v(\xi_t^u) = \frac{1}{F^u(\infty)} \int_{-\infty}^{\infty} v(y) dF^u(y), \quad \text{any } x \in \mathbb{R}.$$

while taking (5.8) into account and letting $t \rightarrow \infty$ in (6.14):

$$J(u) \geq \lambda, \quad \text{any } u \in \mathcal{E}_\alpha.$$

On the other hand, (6.14) holds as an equality if $u = u^*$. Therefore

$$J(u^*) = \lambda.$$

The last two relations prove optimality of u^* in $\mathcal{E}_\alpha$. The density of $\mathcal{P}^{u^*}(\cdot)$ is given by

$$\begin{aligned} p_*(y) &= (1+2b)^{-1}, \quad |y| \leq b \\ (6.15) \quad &= (1+2b)^{-1} \exp[-2(|y|-b)], \quad |y| > b. \end{aligned}$$

7. OPTIMALITY OF THE LAW u^* in $\mathcal{U}$

In this section the performance of the law u^* of Theorem 6.5 is compared against the performance of any admissible nonanticipative control law u , and u^* is proven optimal in the class $\mathcal{U}$.

The method consists in considering the finite-horizon optimization problem: minimize

$$E_x^u \int_0^T [\phi(\xi_s) + |u_s(\xi)|] ds$$

subject to $d\xi_t = u_t(\xi)dt + dw_t$, $\xi_0 = x$ and $u \in \mathcal{U}$. Roughly speaking, the value function

$$V(x, \tau) = \inf_{u \in \mathcal{U}} E_x^u \int_{T-\tau}^T [\phi(\xi_s) + |u_s(\xi)|] ds; \quad (x, \tau) \in \mathbb{R} \times [0, T]$$

solves the Cauchy problem

$$(7.1) \quad V_\tau = \frac{1}{2} V_{xx} + c(V_x) + \phi(x); \quad (x, \tau) \in \mathbb{R} \times (0, T].$$

$$(7.2) \quad V(x, 0) = 0; \quad x \in \mathbb{R},$$

where $c(\cdot)$ is the function defined in (6.1).

For any law $u \in \mathcal{U}$, Itô's rule gives

$$E_x^u \int_0^T [\phi(\xi_s) + |u_s(\xi)|] ds \geq V(x, T),$$

and optimality of u^* would follow if it were proved that:

$$\lim_{T \rightarrow \infty} \frac{V(x, T)}{T} = \lambda, \quad \text{all } x \in \mathbb{R}.$$

In the remaining of this section we justify the method and substantiate the above heuristics.

Lemma 3.1: A priori bounds on the solution of the Bellman equation and its gradient. Suppose that the Cauchy problem (7.1), (7.2) has a $C^{2,1}$ solution $V(x, \tau)$ on $\mathbb{R} \times (0, T]$, with $V(x, \tau)$ continuous on $\mathbb{R} \times [0, T]$. Then the following inequalities hold:

$$(7.3) \quad V(x, \tau) \leq v(x) + \lambda \tau, \text{ on } \mathbb{R} \times [0, T],$$

$$(7.4) \quad |V_x(x, \tau)| \leq v_x(|x|), \text{ on } \mathbb{R} \times [0, T].$$

Proof. It is immediately verified that the function $M(x, \tau) = v(x) + \lambda \tau$ is a $C^{2,1}$ solution in $\mathbb{R} \times (0, T]$ of the Cauchy problem

$$(7.5) \quad M_\tau = \frac{1}{2} M_{xx} + c(M_x) + \phi(x); \text{ on } \mathbb{R} \times (0, T]$$

$$(7.6) \quad M(x, 0) = v(x), \text{ on } \mathbb{R}$$

and that, if $\mathcal{L}$ is the parabolic operator

$$\mathcal{L} = \frac{1}{2} \frac{\partial^2}{\partial x^2} + \frac{c(v_x) - c(V_x)}{v_x - V_x} \frac{\partial}{\partial x} - \frac{\partial}{\partial \tau}.$$

then

$$\begin{aligned} \mathcal{L}(M-N) &= 0, \quad \text{in } \mathbb{R} \times (0, T] \\ M(x, 0) - V(x, 0) &= v(x) \geq 0, \quad \text{on } \mathbb{R}. \end{aligned}$$

By the maximum principle (see [3]) one obtains (7.3).

Now consider a sequence $\{c_n(p), n \in \mathbb{N}\}$ of smooth (piecewise C^2) approximations to the function $c(p)$, with $\ddot{c}_n(p) \leq 0$ a.e. on along with the functions $v^{(n)}(x, \tau)$, $M^{(n)}(x, \tau)$ satisfying (7.1), (7.2) and (7.5), (7.6) respectively, with $c(\cdot)$ replaced by $c_n(\cdot)$. Under such an approximating scheme, $v^{(n)}(x, \tau)$, $v_x^{(n)}(x, \tau)$, $v_{xx}^{(n)}(x, \tau)$ converge as $n \rightarrow \infty$ to $v(x, \tau)$, $v_x(x, \tau)$, $v_{xx}(x, \tau)$ respectively, uniformly on compact (x, τ) sets. Similarly for the function $M(x, \tau)$ and its approximations.

It is easily checked that if $\mathcal{L}_1$ is the parabolic operator

$$(7.7) \quad \mathcal{L}_1 = \frac{1}{2} \frac{\partial^2}{\partial x^2} - \dot{c}_n(M_x^{(n)}) \frac{\partial}{\partial x} + \frac{\dot{c}_n(M_x^{(n)}) - \dot{c}_n(v_x^{(n)})}{M_x^{(n)} - v_x^{(n)}} v_{xx}^{(n)} - \frac{\partial}{\partial \tau},$$

$$\mathcal{L}_1(M_x^{(n)} - v_x^{(n)}) = 0, \quad \text{on } \mathbb{R}^+ \times (0, T]$$

then:

$$M_x^{(n)}(x, 0) - v_x^{(n)}(x, 0) = v_x(x) \geq 0, \quad \text{on } \mathbb{R}^+.$$

It can be shown by yet another application of the maximum principle

that $V_{xx}^{(n)}(x, \tau) \geq 0$ on $\mathbb{R} \times [0, T]$. Therefore, the potential term

$\frac{\dot{c}_n(M_x^{(n)}) - \dot{c}_n(V_x^{(n)})}{M_x^{(n)} - V_x^{(n)}} V_{xx}^{(n)}$ is nonpositive on $\mathbb{R}^+ \times [0, T]$, so that the

the strong maximum principle is applicable (see [3]) and gives

$V_x^{(n)}(x, \tau) \leq M_x^{(n)}(x, \tau)$, or $V_x(x, \tau) \leq M_x(x, \tau) = v_x(x)$ on $\mathbb{R}^+ \times [0, T]$

in the limit as $n \rightarrow \infty$. (7.4) follows since $V_x(\cdot, \tau)$ is

odd, Q.E.D.

Once the a priori bounds (7.3), (7.4) have been established, one can apply the method of Theorem VI 6.2 of Fleming and Rishel [2] to prove the following result:

Proposition 7.2. The Cauchy problem (7.1)-(7.2) has a unique $C^{2,1}$ solution $V(x, \tau)$ on $\mathbb{R} \times (0, T]$ that is continuous on $\mathbb{R} \times [0, T]$ and even in x .

By the approximation argument used in the proof of Lemma 7.1 (or directly; see [2], Exercise VI.9) it can be shown that $V_{xx}(x, \tau) \geq 0$ in $\mathbb{R} \times [0, T]$.

Consider the optimal process (η_t^τ) for the finite horizon problem, defined on the probability space $(\Omega, \mathcal{F}, P)$ as the (strong) solution of the stochastic differential equation

$$(7.8) \quad d\eta_t^\tau = \dot{c}(V_x(\eta_t^\tau, \tau-t))dt + dw_t; \quad 0 \leq t \leq \tau$$

$$(7.9) \quad \eta_0^\tau = x > 0$$

where $\dot{c}(p) = -\text{sgn} p \cdot 1_{\{|p| > 1\}} = a^*(p)$.

Lemma 7.3. For any $x > 0$, consider the stopping time

$$\begin{aligned} S &= \inf\{t \leq \tau; \eta_t^\tau = 0\} \\ &= \tau, \text{ if } \eta_t^\tau > 0, \text{ all } 0 \leq t \leq \tau. \end{aligned}$$

Then

$$(7.10) \quad V_x(x, \tau) = E \int_0^S \dot{\phi}(\eta_t^\tau) dt.$$

Proof. The gradient V_x of the solution to the Cauchy problem (7.1)-(7.2) is not a $C^{2,1}$ function; it belongs, however, to the Sobolev space $W_p^{2,1}(D \times [0, T])$, for any $p > 1$ and any bounded subset $D \subseteq \mathbb{R}$, and satisfies in that space the equation $(V_x)_\tau = \frac{1}{2} (V_x)_{xx} + \dot{c}(V_x)(V_x)_x + \dot{\phi}(x)$ on $\mathbb{R} \times (0, T]$, derived from (7.1) by formal differentiation. For functions in the Sobolev space a generalized Itô formula holds (Zvonkin [12], Theorem 3) which, applied to $V_x(\eta_t^\tau, \tau - t)$ on $[0, S]$ along with (7.8) and the fact that $V_x(\eta_S^\tau, \tau - S) = 0$, a.s., yields (7.10), Q.E.D.

Consider now the "optimal process (ξ_t^*) for the stationary control problem":

$$(7.11) \quad d\xi_t^* = \dot{c}(v_x(\xi_t^*)) dt + dw_t, \quad t \geq 0$$

$$(7.12) \quad \xi_0^* = x,$$

defined on the same probability space $(\Omega, \mathcal{F}, P)$ and with the same initial condition as for (7.8), (7.9).

Lemma 7.4. $|\xi_t^*| \leq |\eta_t^*| \leq |x + w_t|$, $0 \leq t \leq \tau$; a.s.(P).

Proof. An easy consequence of the comparison theorem of Ikeda and Watanabe [6] and (7.4) of Lemma 7.1.

From (7.10) notice that, for any $\tau > 0$, $V_x(\cdot, \tau)$ increases to infinity as $x \rightarrow \infty$, since $\phi(\cdot)$ does. Therefore, for any $\tau > 0$,

$$(7.13) \quad s(\tau) \triangleq \max\{x > 0; V_x(x, \tau) = 1\}$$

is well-defined and finite.

Lemma 7.5. $s(\tau)$ is left continuous and decreasing on $\mathbb{R}^+$.

Proof. It can be checked that for the approximating functions introduced in the proof of Lemma 7.1: $\mathcal{L}_1(V_{x\tau}^{(n)}) = 0$ in $\mathbb{R} \times (0, T]$, $\mathcal{L}_1$ being the operator defined in (7.7), and $V_{x\tau}^{(n)}(x, 0) = \phi(x) \geq 0$, on $\mathbb{R}^+$. By a maximum principle argument: $V_{x\tau}^{(n)}(x, \tau) \geq 0$ on $\mathbb{R}^+ \times [0, T]$, and therefore $V_x(x, \tau_2) \geq V_x(x, \tau_1)$, $0 \leq \tau_1 < \tau_2$, $x \geq 0$ in the limit as $n \rightarrow \infty$. This proves the monotonicity of $s(\cdot)$. Left continuity is an easy consequence of definition (7.13) and monotonicity.

Lemma 7.6. $\lim_{\tau \rightarrow \infty} V_x(x, \tau) = v_x(x)$, uniformly on compact x -sets.

Proof. Notice that

$$\begin{aligned} (v_x - V_x)_\tau &= \frac{1}{2} (v_x - V_x)_{xx} + \dot{c}(v_x)(v_x - V_x)_x + v_{xx}(\dot{c}(v_x) - \dot{c}(V_x)) \\ &\leq \frac{1}{2} (v_x - V_x)_{xx} + \dot{c}(v_x)(v_x - V_x)_x, \end{aligned}$$

on $\mathbb{R} \times (0, T]$, by convexity of V , monotonicity of $\dot{c}$ and (7.4).

An application of the generalized Itô formula to $v_x(\xi_t^*) - V_x(\xi_t^*, \tau - t)$ gives:

$$0 \leq v_x(x) - V_x(x, \tau) \leq E v_x(\xi_R^*) = \int_{\{R=\tau\}} v_x(\xi_\tau^*) dP,$$

where:

$$\begin{aligned} R &= \inf\{t \leq \tau: \xi_t^* = 0\} \\ &= \tau, \text{ if } \xi_t^* > 0, \text{ all } 0 \leq t \leq \tau. \end{aligned}$$

We note that: $E v_x^{1+\delta}(|\xi_\tau^*|) \xrightarrow{\tau \uparrow \infty} \int_{-\infty}^{\infty} v_x^{1+\delta}(|y|) p_*(y) dy < \infty$ as long as

$0 < \delta < \frac{2}{\alpha} - 1$, by virtue of (5.5) and (6.15). So

$\sup_{\tau > 0} E v_x^{1+\delta}(|\xi_\tau^*|) < \infty$, which implies uniform integrability (and hence also absolute continuity with respect to measure P) of the family of random variables $\{v_x(|\xi_\tau^*|)\}_{\tau > 0}$. On the other hand,

$$P(R = \tau) \leq P(x + w_t > 0, \text{ all } 0 \leq t \leq \tau) = 2\Phi(x\tau^{-1/2}) - 1 \rightarrow$$

as $\tau \rightarrow \infty$, uniformly on compact x -sets; see Gihman and Skorohod [4; §1]. The result follows by uniform absolute continuity.

Corollary. $s(\tau) \downarrow b$, as $\tau \uparrow \infty$.

Proposition 7.7. $\lim_{\tau \rightarrow \infty} \frac{V(x, \tau)}{\tau} = \lambda$, any $x \in \mathbb{R}$.

Proof. That $\limsup_{\tau \rightarrow \infty} \frac{V(x, \tau)}{\tau} \leq \lambda$, uniformly on compact x -sets, is a consequence of (7.3). To prove the opposite inequality note that, by virtue of Lemma 7.4,

$$\begin{aligned} V(x, \tau) &= E \int_0^\tau [\phi(\eta_t^\tau) + 1_{\{| \eta_t^\tau | > s(\tau-t)\}}] dt \\ &\geq E \int_0^\tau [\phi(\xi_t^*) + 1_{\{|\xi_t^*| > s(\tau-t)\}}] dt \end{aligned}$$

and therefore, for any $x \in \mathbb{R}$,

$$\begin{aligned} \frac{V(x, \tau)}{\tau} &> \frac{1}{\tau} E \int_0^\tau [\phi(\xi_t^*) + 1_{\{|\xi_t^*| > b\}}] dt - \\ (7.14) \quad &- \frac{1}{\tau} \int_0^\tau \{F_{t,x}(s(\tau-t)) - F_{t,x}(b) - \{F_{t,x}(-s(\tau-t)) - \\ &- F_{t,x}(-b)\}\} dt \end{aligned}$$

where

$$F_{t,x}(y) = P\{\xi_t^* \leq y | \xi_0^* = x\} \xrightarrow[t \rightarrow \infty]{} F^*(y) = \int_{-\infty}^y p_*(z) dz.$$

$F^*(\cdot)$ is the ergodic probability distribution function corresponding to the optimal law u^* in $\mathcal{L}_\alpha$. Now

$$(7.15) \quad \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \int_0^\tau [F_{t,x}(s(\tau-t)) - F_{t,x}(b) - \{F^*(s(\tau-t)) - F^*(b)\}] dt = 0.$$

Indeed, the integrand in (7.15) is dominated by

$2 \sup_{y \in \mathbb{R}} |F_{t,x}(y) - F^*(y)|$, which tends to zero as $t \rightarrow \infty$, because F^* is (absolutely) continuous and $F_{t,x} \xrightarrow{c} F^*$ (see [1], p. 25 Ex. 8.1.13). Hence

$$\lim_{\tau \rightarrow \infty} \frac{1}{\tau} \int_0^\tau (F_{t,x}(s(\tau-t)) - F_{t,x}(b)) dt = \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \int_0^\tau (F^*(s(\tau-t)) - F^*(b)) dt = 0$$

since $\lim_{t \rightarrow \infty} F^*(s(t)) = F^*(b)$, by the Corollary to Lemma 7.6. By the same token, the entire second term on the right hand side of (7.14) converges to zero as $\tau \rightarrow \infty$, while the first term converges to λ . Therefore, for any $x \in \mathbb{R}$:

$$\liminf_{\tau \rightarrow \infty} \frac{V(x, \tau)}{\tau} \geq \lambda, \quad \text{Q.E.D.}$$

We are now in a position to prove the main result of this section.

Theorem 7.8. The law u^* of Theorem 6.5 is optimal in the class $\mathcal{U}$ of admissible nonanticipative controls, i.e. for any $u \in \mathcal{U}$, $x \in \mathbb{R}$:

$$(7.16) \quad J(x, u) = \liminf_{T \rightarrow \infty} \frac{1}{T} E_x^u \int_0^T [\phi(\xi_t^u) + |u_t(\xi)|] dt \geq \lambda = J(u^*).$$

Proof. Take any law $u \in \mathcal{U}$ along with the Girsanov solution process (ξ_t^u) satisfying $d\xi_t^u = u_t(\xi_t^u)dt + d\tilde{w}_t$, $\xi_0^u = x$ on $(\Omega, \mathcal{F}_T, P_x^u)$ as in Section 2, and apply Itô's rule to the process $V(\xi_t^u, T-t)$, $V(x, \tau)$ being the function of Proposition 7.2:

$$\begin{aligned} V(x, T) = V(\xi_0^u, T) - V(\xi_T^u, 0) = & - \int_0^T [u_t(\xi_t^u) V_x(\xi_t^u, T-t) \\ & + \frac{1}{2} V_{xx}(\xi_t^u, T-t) - V_{\tau}(\xi_t^u, T-t)] dt - \int_0^T V_x(\xi_t^u, T-t) d\tilde{w}_t. \end{aligned}$$

Because $c(p) = \min_{|u| \leq 1} (up + |u|)$, we get

$$(7.17) \quad V(x, T) \leq \int_0^T [\phi(\xi_t^u) + |u_t(\xi_t^u)|] dt - \int_0^T V_x(\xi_t^u, T-t) d\tilde{w}_t \text{ a.s. } (P_x^u),$$

any $x \in \mathbb{R}$, $T > 0$.

The expectation of the stochastic integral on the right hand side of (7.17) is zero, because

$$E_x^u \int_0^T V_x^2(\xi_t^u, T-t) dt \leq E_x^u \int_0^T V_x^2(\xi_t^u) dt \leq \text{const. } e^{2\alpha(|x|+T)} \int_0^T E(e^{2\alpha|w_t|}) dt < \infty$$

by virtue of (7.4), and it follows from (7.17) by taking expectations that

$$\frac{V(x, T)}{T} \leq \frac{1}{T} E_x^u \int_0^T [\phi(\xi_t^u) + |u_t(\xi_t^u)|] dt; \text{ any } x \in \mathbb{R}, T > 0.$$

(7.16) is obtained by a passage to the limit as $T \rightarrow \infty$ and taking into account the assertion of Proposition 7.7.

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