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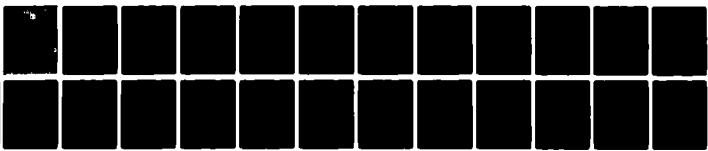
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MATHEMATICAL THEORY OF LAMINAR COMBUSTION, VIII.

Three-Dimensional Flames

9 Technical Report No. 110

10 J.D. Buckmaster & G.S.S. Ludford

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Foreward

This report is Chapter VIII of the twelve in a forthcoming research monograph on the mathematical theory of laminar combustion. Chapters I-IV originally appeared as Technical Reports Nos. 77, 80, 82 & 85; these were later extensively revised and then issued as Technical Summary Reports No's 1803, 1818, 1819 & 1888 of the Mathematics Research Center, University of Wisconsin-Madison. References to I-IV mean the MRC reports.

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Chapter VIII

Three-Dimensional Flames

1. The Flame as a Hydrodynamic Discontinuity.

The plane premixed flame discussed in Chapters II & III is an idealization seldom approximated, since in practice the flame is usually curved. A bunsen flame is quite unavoidably so, but even under circumstances carefully chosen to nurture a plane state, instabilities can lead to a three-dimensional structure. Such flames have been extensively studied (Markstein 1964, p. 7) using what may be called a hydrodynamical approach, a brief description of which will provide an appropriate introduction to our subject.

On a scale that is large compared to its nominal thickness $\lambda/c_p M$, the flame is simply a sheet across which there is a jump in temperature and density subject to Charles' Law, as is appropriate for an isobaric process. Deformation of the sheet from a plane is associated with pressure variations, of the order of the square of the Mach numbers (cf. Sec. I.5), in the hydrodynamic fields. These small pressures jump across the sheet in order to conserve normal momentum flux. Because Euler's equation for small Mach number hold outside the flame (cf. end of Sec. 3), the temperature and density do not change along particle paths; so that for a flame travelling into a uniform gas the temperature and density ahead are constant and the flow is irrotational. The flow behind is stratified, however, since flame curvature generates both vorticity and non-uniform temperature jumps. Nevertheless variations in temperature from the adiabatic flame temperature are usually neglected everywhere, a matter to which we shall return later. On the other hand, vorticity generation cannot

be ignored so that, although Euler's equations hold behind the flame, the flow is not potential there.

The flow fields on the two sides of the flame are coupled by conservation of mass and momentum fluxes through the sheet. Indeed, if V is the speed of the discontinuity back along its normal $\underline{n}$ (Fig. 1) we have

$$(1,2,3) \quad p_1(v_{n1} + V) = p_2(v_{n2} + V), \quad v_{11} = v_{12}, \quad p_1 + \rho_1(v_{n1} + V)^2 = p_2 + \rho_2(v_{n2} + V)^2$$

where $v_n = \underline{v} \cdot \underline{n}$ and $\underline{v}_\perp = \underline{v} - v_n \underline{n}$. The flame speed is defined to be the normal gas speed immediately ahead, as measured in a local frame fixed at the discontinuity, i.e. $v_{n1} + V$. If this is specified the hydrodynamic problem can in principle be solved and the locus of the flame determined. [Equations (1), (2), (3) become four scalar equations for v_2, p_2 once ρ_2 is known, and that follows from the flame temperature T_2 .

Evaluation of the flame speed from a combustion analysis of the sheet has often been sidestepped, and replaced by one of several hypotheses. The simplest is that the speed is constant. In this way Landau (1944) considered infinitesimal disturbances of a plane flame and concluded it is always unstable, a result that will be examined in Ch. XI.

The already nonlinear flow field is only complicated by the combustion, even for a constant-speed flame. Accordingly an additional simplification, often adopted, is that the flow is not affected by the flame. This can be justified only when the heat released by the combustion is small compared to the thermal energy of the fresh mixture (Sec. I.5), so that the temperature jump across the sheet is negligible. Such is not a common characteristic of actual flames, but the approach can provide insight into their qualitative nature.

Consider, for example, a stationary flame located in the shear flow of Fig. 2, representative of that from a bunsen tube. Since v_n is equal to the flame speed, assumed constant, the angle α decreases as u increases and there is a cusp on the centerline. The resulting flame shape has some resemblance to that of the inner cone of a bunsen flame. (Even for a parabolic velocity distribution, a good approximation to the flow from the tube when that is unaffected by the combustion, the shape of the flame can only be obtained numerically.)

In practice the tip is usually rounded so that the flame speed at the centerline is equal to $v_{\max}$ and must have a smaller value elsewhere. (A detailed study of flame tips is presented in Ch. IX.) Observations of this nature and dissatisfaction with the stability prediction of Landau have encouraged more sophisticated hypotheses, in particular that of Markstein discussed at the end of Sec. III.6. We shall not pursue these here since rational analysis of the burning zone (which is embedded within the hydrodynamic discontinuity) determines the actual flame speed from first principles.

2. Slow Variation and Near Equidiffusion.

As discussed in Sec. II.7, the combustion of a two-component mixture that is well removed from stoichiometry can be characterized by a single mass fraction Y related to the component which is consumed at the sheet. To be sure, the argument was given for plane flames, equal molecular masses, and unit Lewis numbers and stoichiometric coefficients; but all these restrictions are readily seen to be inessential (as also is the number of components). The single Lewis number is then properly interpreted as that of the deficient component - oxidant in a rich flame, fuel for a lean one.

Appropriate equations for our discussion of three-dimensional unsteady motion are therefore

$$(4,5) \quad \rho T = 1, \quad \partial \rho / \partial t + \nabla \cdot (\rho \mathbf{v}) = 0,$$

$$(6) \quad \rho (\partial \mathbf{Y} / \partial t + \mathbf{v} \cdot \nabla \mathbf{Y}) - \mathcal{L}^{-1} \nabla^2 \mathbf{Y} = -\Omega,$$

$$(7) \quad \rho (\partial \mathbf{v} / \partial t + \mathbf{v} \cdot \nabla \mathbf{v}) = -\nabla p + \rho [\nabla^2 \mathbf{v} + \nabla (\nabla \cdot \mathbf{v}) / 3],$$

$$(8) \quad \rho (\partial T / \partial t + \mathbf{v} \cdot \nabla T) - \nabla^2 T = \Omega$$

where

$$(9) \quad \Omega = \Lambda Y e^{-\theta/T}.$$

These have been rendered dimensionless as in Sec. I.6 and the undisturbed pressure has been taken constant (i.e. $p_c = 1$), with p the variations from it.

On the length scale $\lambda/c_p M$ now being used the flame is no longer a sheet but, for large activation energy, the reaction zone within it can be considered so. We now take the x -axis instantaneously parallel to the normal to this sheet (and therefore to the hydrodynamic discontinuity) at the point of interest and introduce the new variable

$$(10) \quad n = x - x_*(t),$$

as was done in the discussion of plane deflagration waves in Sec. III.3. As usual x_* denotes the position of the sheet. Everywhere in the above equations $\partial/\partial t$ and $\partial/\partial x$ should now be replaced by

$$(11) \quad \partial/\partial t - \dot{x}_* \partial/\partial n \quad \text{and} \quad \partial/\partial n,$$

respectively.

In analysing the resulting equations for the limit $\theta \rightarrow \infty$ some thought must be given to the flame temperature, i.e. the temperature immediately behind the sheet. It is clear from the results of Chs. II & III (cf. Sec. III.1) that there must be a general correlation between this temperature and the flame speed in the sense that $O(1/\theta)$ changes in the former are associated with $O(1)$ changes in the latter. We may therefore expect that if the flame temperature varies by larger amounts, either temporally or spatially, then the flame will vary in an extreme fashion through large acceleration or curvature; and such behavior will introduce serious difficulties into the analysis. Only in the discussion of explosions in Ch. XII will these difficulties be confronted, and then only in relatively simple circumstances. Here, as in Ch. III, we shall restrict ourselves to physical situations where the temperature of the burnt mixture in the neighborhood of the flame is everywhere, and at all times, within $O(1/\theta)$ of the adiabatic flame temperature. Necessary restrictions are established by the following argument (Buckmaster 1979a).

As in Sec. III.3, we can calculate the flame temperature from the overall change in partial enthalpy $H = Y + T$ in the flame up to and including the reaction zone. Thus equations (6) and (8), with the modifications (11), are added to eliminate the reaction terms and then integrated from $n = -\infty$ to $0+$ to obtain

$$(12) \int_{-\infty}^{0+} \rho \left[\frac{\partial}{\partial t} + (v_n - \dot{x}_*) \frac{\partial}{\partial n} + v_{\perp} \cdot \nabla_{\perp} \right] H \, dn = \frac{\partial T}{\partial n} \Big|_{0+} + \int_{-\infty}^{0+} v_{\perp}^2 (\mathcal{L}^{-1} Y + T) \, dn,$$

where as before the subscript $\perp$ denotes the component perpendicular to n , i.e. in the flame sheet, and we have used the fact that Y vanishes behind the sheet. The middle term on the left-hand side can be integrated by parts to yield

$$\rho H(v_n - \dot{x}_*) \Big|_{-\infty}^{0+} - \int_{-\infty}^{0+} H \frac{\partial}{\partial n} [\rho(v_n - \dot{x}_*)] dn,$$

in which

$$\frac{\partial}{\partial n} [\rho(v_n - \dot{x}_*)] = - \frac{\partial \rho}{\partial t} - \nabla_{\perp} \cdot (\rho \mathbf{v}_{\perp}), \quad \rho(v_n - \dot{x}_*) \Big|_{0+} = \rho_1(v_{n1} - \dot{x}_*) - \int_{-\infty}^{0+} \left[\frac{\partial \rho}{\partial t} + \nabla_{\perp} \cdot (\rho \mathbf{v}_{\perp}) \right] dn$$

according to the equation of continuity (5). Equation (12) may therefore be rewritten

$$(13) \quad \rho_1(v_{n1} - \dot{x}_*)(T_* - H_1) - \frac{\partial T}{\partial n} \Big|_{0+} = \int_{-\infty}^{0+} [\nabla_{\perp}^2 \mathcal{L}^{-1} Y + T] + T_* \left(\frac{\partial \rho}{\partial t} + \nabla_{\perp} \cdot \rho \mathbf{v}_{\perp} \right) - \frac{\partial}{\partial t} (\rho H) - \nabla_{\perp} \cdot (\rho H \mathbf{v}_{\perp}) \Big|_{0+} dn$$

where, as usual, T_* denotes the temperature at the flame sheet.

By assumption the left-hand side of equation (13) is at most $O(1/\theta)$ so that the right side must be similarly bounded. The only way to guarantee this within a general framework is to make the terms in $\nabla_{\perp}^2, \nabla_{\perp}$ and $\partial/\partial t$ separately small, which has been accomplished in two ways. We confine attention to disturbances which vary over times and distances of order θ . Such slowly varying flames are a generalization of those introduced in Sec. III.5. Alternatively, we restrict the Lewis number to values close to one, i.e. write

$$(14) \quad \mathcal{L}^{-1} = 1 - \lambda/\theta \quad \text{with } \lambda = O(1),$$

insist that the state 1 does not vary along the flame and impose the integral

$$(15) \quad H = H_1 + O(1/\theta)$$

which is then a possibility. Such near-equidiffusional flames are a generalization of those introduced in Sec. III.3.

The full generality of the above analysis has not appeared elsewhere although its essence, for shear flows, is contained in Buckmaster (1979a). The rest of the chapter is concerned with the detailed formulation of each of these two kinds of flame.

3. The Basic Equation of Slowly Varying Flames.

Slowly varying flames can be characterized by the scaling

$$(16) \quad (y, z, t) = \theta(\eta, \zeta, \tau),$$

but it should be recognized from the outset that the actual lengths involved are quite moderate. In Sec. II.4 we saw that the preheat zone typically has a thickness of less than 0.1 mm. So that, even for θ as large as 100, appreciable changes occur in less than 1 cm.

Flame-sheet disturbances on the θ -scale will be associated with field disturbances at distances $n = O(\theta)$, where the governing equations are fundamentally different from those where $n = O(1)$. The limit $\theta \rightarrow \infty$ therefore generates a structure with at least three zones: the flame sheet described on the scale $n = O(1/\theta)$, to which all the reaction is confined; a diffusion zone described on a scale $n = O(1)$, called a preheat zone in front of the flame; and an ideal-fluid region described on the scale $n = O(\theta)$. Description of the latter is the hydrodynamic problem discussed in Sec. 1. The stratification behind the flame which is an essential feature of this description must eventually be smoothed out by diffusion, so that additional structure is described in a scale much larger than $n = O(\theta)$; but that will not be considered.

Anticipation of the matching conditions imposed by the ideal-fluid region enables the diffusion zone to be analyzed. These conditions are

$$(17) \quad n \rightarrow +\infty: \quad (\rho, T, Y, v, p) \rightarrow (\rho_1, T_1, Y_1, v_1, p_1);$$

$$(18) \quad n \rightarrow -\infty: \quad \text{exponential growth not permitted.}$$

The state 1 is that immediately ahead of the hydrodynamic discontinuity, and is to be considered a known function of η, ζ, τ . Our aim is the same as that in Sec. III.5, namely to calculate various quantities in the overall enthalpy balance (13) and hence derive an expression, in terms of the state 1, for the perturbation of the flame temperature. (The change of units introduced in Sec. III.3 is not appropriate here since ρ_1 , which replaces ρ_∞ there, is in general variable; hence the differences in some of the subsequent formulas.)

To leading order the governing equations become

$$(19, 20) \quad \rho T = 1, \quad \partial[\rho(V + v_n)]/\partial n = 0,$$

$$(21) \quad \rho(V + v_n)\partial Y/\partial n - \mathcal{L}^{-1}\partial^2 Y/\partial n^2 = -\Omega,$$

$$(22, 23) \quad \rho(V + v_n)\partial v_n/\partial n = -\partial p/\partial n + (4\mathcal{P}/3)\partial^2 v_n/\partial n^2, \quad \rho(V + v_n)\partial v_\perp/\partial n = \partial^2 v_\perp/\partial n^2,$$

$$(24) \quad \rho(V + v_n)\partial T/\partial n - \partial^2 T/\partial n^2 = \Omega$$

where $-\dot{x}_*$ has now been identified with V , the speed of the hydrodynamic discontinuity. Apart from notation, equations (19), (20), (21) and (24) are identical to those for a plane flame so that we may immediately write (cf. Sec. III.5)

$$(25) \quad \rho(V + v_n) = \rho_1(V + v_{n1}) = M_n \quad (\text{say}),$$

$$(26) \quad Y = Y_1(1 - e^{\frac{M_n}{M_n}}), \quad T = T_1 + Y_1 e^{\frac{M_n}{M_n}}, \quad \rho = (T_1 + Y_1 e^{\frac{M_n}{M_n}})^{-1} \quad \text{for } n < 0,$$

$$(27) \quad Y = 0, \quad T = H_1 + \theta^{-1} t_\infty(\eta, \zeta, \tau), \quad \rho = H_1^{-1} \quad \text{for } n > 0.$$

The mass flux M_n may also be called the burning rate. Expressions for v_n and p can also be written, but we shall not need them. On the other hand, $v_\perp$ is now needed; since it does not jump across the hydrodynamic discontinuity, it can be expected to stay constant over the whole range of n :

$$(28) \quad v_\perp = v_{\perp 1}.$$

Formal proof comes from equation (23), which holds even in the flame sheet and possesses only constant vectors as bounded solutions.

The perturbation t_∞ in the flame temperature may now be calculated from the enthalpy balance (13) since all other quantities have been determined, in terms of V , to sufficient accuracy. The result is quite complicated, involving terms in $\partial/\partial\tau$ and $v_\perp$ of ρ_1, T_1, Y_1 and v_1 . Considerable simplification occurs when the upstream temperature and composition of the mixture (on the hydrodynamic scale) are uniform, since then ρ_1, T_1 and Y_1 are constant, as we shall see at the end of this section. Only the simpler result will be written here.

The $O(\theta^{-1})$ terms in the integrand (13) are now

$$\begin{aligned} & \frac{\partial}{\partial\tau} [\rho(T_* - H)] + v_\perp [\rho v_\perp (T_* - H)] \\ &= \left\{ \left[\dot{M}_n + v_\perp \cdot v_\perp \frac{M_n}{M_n} \right] \frac{n}{M_n} \frac{\partial}{\partial n} + v_\perp \cdot v_\perp \right\} \frac{Y_1 (e^{\frac{M_n}{M_n}} - e^{\frac{M_n}{M_n}})}{T_1 + Y_1 e^{\frac{M_n}{M_n}}}, \end{aligned}$$

which may be replaced by

$$\left\{ -\frac{1}{M_n} [\dot{M}_n + v_{11} \cdot \nabla_{11} M_n] + \nabla_{11} \cdot v_{11} \right\} \frac{Y_1 (e^{\frac{M_n}{T_1 + Y_1 e^{M_n}}} - e^{\frac{M_n}{T_1}})}{T_1 + Y_1 e^{M_n}}$$

under integration by parts. Here the dot denotes differentiation with respect to τ and ∇_{11} is the gradient operator with respect to η, ζ . For $v_{11} = 0$ the calculation is the same as for a plane flame, so that it is not surprising to find the generalization

$$(29) \quad t_\infty = -b M_n^{-3} \left[\dot{M}_n + v_{11} \cdot \nabla_{11} M_n - M_n \nabla_{11} \cdot v_{11} \right] \text{ where } b = Y_1 \int_0^1 \sigma \frac{\xi^{-1} - 1}{T_1 + Y_1 \sigma} d\sigma$$

of the result (III.39) from the integration (13). (A factor T_1 is now missing from b because of the different unit for ρ_1 in M_n and the use of $1/\theta$ as expansion parameter.)

Another relation between the burning rate M_n and the perturbation t_∞ of the flame temperature follows from the structure of the flame sheet, which is described on the scale $n = O(1/\theta)$. The analysis is the same as in Ch. 3 since, to leading order, the equations for ρ, Y and T are those for a plane flame. We conclude that

$$(30) \quad M_n = e^{t_\infty / 2T_*^2} \text{ with } T_* = H_1,$$

cf. equation (III.26), if the mass flux M on which units have been based is chosen to be the burning rate of a plane flame moving into the state Y_1, T_1 , i.e.

$$(31) \quad M = \sqrt{2\mathcal{L}D} T_*^2 \exp(-\theta/2T_*) / Y_1 \theta,$$

cf. equation (III.16). The two results (29), (30) now give us the basic equation

$$(32) \quad b[\dot{M}_n + \mathbf{v}_{11} \cdot \nabla_{11} M_n - M_n \nabla_{11} \cdot \mathbf{v}_{11}] + 2T_n^2 M_n^3 \ln M_n = 0$$

of slowly varying flames, which should be compared with its plane from (III.4C). The equation is due to Buckmaster (1977), whose derivation has been simplified. Restricted forms of it have been given by Sivashinsky (1974, 1975) and, in a linearized context, by Eckhaus (1961). Interpretation involves the concept of flame stretch to be introduced in the next section. We shall finish the present section with a brief discussion of the ideal-fluid region where $n = O(\epsilon)$.

To the slow variables η, ζ, τ we now add

$$(33) \quad \xi = x/\theta.$$

Then $\rho (= 1/T)$, T and Y are found to be constant on particle lines, to leading order, while the velocity and pressure fields satisfy

$$(34) \quad \nabla \cdot \mathbf{v} = 0, \quad \rho(\partial \mathbf{v} / \partial \tau + \mathbf{v} \cdot \nabla \mathbf{v}) = - \nabla p,$$

where ∇ is now the gradient operator with respect to ξ, η, ζ . These are Euler's equations for an incompressible (but not necessarily constant-density) ideal fluid, i.e. one devoid of viscosity and heat conduction. Particles retain the values ρ_1, T_1 and Y_1 they acquire far upstream until they reach the hydrodynamic discontinuity, where they exchange them for the values

$$(35) \quad \rho_2 = H_1^{-1}, \quad T_2 = H_1, \quad Y_2 = 0,$$

which they then carry downstream. The ideal-fluid region is coupled to the flame through the velocity and pressure, in a manner that will now be seen in detail for uniform temperature and composition upstream.

ρ , T and Y are all constant on either side of the hydrodynamic discontinuity. In particular there is no thermal stratification on the downstream side, in accordance with a common assumption mentioned in Sec. 1. The problem is to find y and p subject to the jump conditions (1), (2), (3) and the evolution equation (32), which may be thought of as five equations for y_2 , p_2 and V in terms of the state 1. Clearly its complete analysis would be formidable.

4. Flame Stretch.

In view of the difficulties in giving a complete analysis of a three-dimensional flame (even when it is slowly varying), it is natural to try to identify special characteristics that play particularly important roles in the understanding of flame behavior. Flame speed and temperature are elementary examples of such characteristics that have already been identified. A more subtle characteristic, attributed to Karlovitz (1953), is flame stretch.

In order to define this in an unambiguous fashion it is first necessary to define a flame surface, i.e. a sheet that characterizes the location of the reaction. For large activation energy the reaction zone is such a surface when viewed on the scale of the preheat zone. Lewis & von Elbe (1961, p. 225) use the locus of inflection points of the temperature field near the burning zone, a definition that coincides with the preceding one in the limit $\theta \rightarrow \infty$. If the flame can be viewed as a hydrodynamic discontinuity, as in Sec. 1, then the discontinuity itself is a flame surface. In all these examples and any other for which the present discussion is meaningful, a flow velocity is defined on each side of the surface such that $\underline{v}_1$ is continuous across the surface.

Consider a point on the flame surface which moves in it with the velocity $v_{\perp}$. A set of such points forming an infinitesimal surface element of area S will, in general, deform during the motion, so that S will vary with time. If S increases the flame is said to be stretched, whereas if S decreases the flame experiences negative stretch and is said to be compressed. A precise measure of the stretch is the Karlovitz number

$$(36) \quad K \equiv \frac{1}{S} \frac{dS}{dt}.$$

Certain configurations for which stretch is an important characteristic will be discussed in detail in Ch. X. Here we shall give two elementary examples.

Consider a stationary flame surface which, for simplicity, is locally plane, inclined at an angle α to the horizontal (Fig. 3). If the flame is located in a horizontal shear flow $U(y)$, then

$$(37) \quad \frac{1}{S} \frac{dS}{dt} = \sin \alpha \cos \alpha U'(y).$$

Note that if $\alpha = 0$ or $\pi/2$ there is no stretch because then the tangential velocity does not change along the surface. The second example is an expanding spherical flame. If the radius is $R(t)$, then

$$(38) \quad \frac{1}{S} \frac{dS}{dt} = \frac{2}{R} \dot{R}.$$

Stretch of this type is discussed under somewhat more general circumstances in connection with the quasi-plane flames of Sec. III.8.

In these two examples the stretch could be determined by inspection, but we now need a general expression for it. That comes from connecting it with stretch in a plane. Thus, as an element of the flame surface passes through a designated point it experiences the same stretch as its projection

on the tangent plane at that point, which is moving with velocity $v_{\perp}$. The plane element, however, has stretch

$$(39) \quad \frac{1}{S} \frac{dS}{dt} = v_{\perp} \cdot v_{\perp},$$

as is easily seen from the divergence theorem, so that this formula holds also for the surface element.

In the first example above the flame surface is its own tangent plane at every point and we have

$$v_{\perp} = (U \cos^2 \alpha, U \sin \alpha \cos \alpha, 0),$$

from which the result (37) follows. In the second example

$$v_{\perp} = (0, \dot{R}_y/R, \dot{R}_z/R),$$

if the x-axis is taken along the normal at the point considered, and the result (38) follows.

Stretch is a kinematical concept. For a slowly varying flame with uniform temperature upstream it may be related to the flame speed. The burning rate M_n is then essentially the flame speed $v_{n1} + V$, differing from it only by a constant factor ρ_1 . The basic equation (32), which now reads

$$(40) \quad \frac{1}{S} \frac{dS}{dt} = \frac{1}{M_n} \frac{dM_n}{dt} + 2T_{*b}^{-1} M_n^2 \ln M_n,$$

therefore expresses the stretch in terms of the flame speed. Note that stretch is the same for the hydrodynamic discontinuity as for the reaction zone, but flame speed relates to the former and not the latter.

Flame thickness is another concept that can usefully characterize a flame. Although a plane deflagration wave is not confined to a zone of finite thickness,

to say that it has a thickness $\lambda/c_p M$ obviously has meaning. The implication is that while an infinite distance is required for the temperature to rise from the cold value to the hot, most of the change takes place over a small multiple of $\lambda/c_p M$.

For a general flame, it is natural to define the local thickness as $\lambda/c_p M_n$, where M_n is the normal mass flux. Indeed, for the slowly varying flame the dimensionless M_n is the decay coefficient for T in the preheat zone, as the second of formulas (26) shows, so that

$$(41) \quad \ell = M_n^{-1}$$

is properly called the non-dimensional thickness. If now the elemental volume

$$(42) \quad \Delta = \ell S$$

generated by S is introduced, the basic equation becomes

$$(43) \quad \frac{1}{\Delta} \frac{d\Delta}{d\tau} = 2T_*^2 b^{-1} M_n^2 \ln M_n,$$

expressing what may be called voluminal stretch in terms of flame speed or thickness. The results (40) and (43) were originally derived by Buckmaster (1973a).

Note that the sign of b depends on the Lewis number $\mathcal{L}$, being positive for $\mathcal{L} < 1$ and negative for $\mathcal{L} > 1$. When $\mathcal{L} = 1 + O(1/\theta)$ so that $b = O(1/\theta)$, equation (43) suggests that changes occur on the scale of t rather than τ , although it then fails to provide a valid description. In Sec. 8 we shall develop the theory of near-equidiffusional flames, and fill the void.

We shall end the section with a few remarks about the contemporary significance of flame stretch. Although the concept leads to an elegant

description of slowly varying flames (especially when extended to voluminal stretch, about which the literature is silent), the importance attached to it derives from its far-ranging use as an intuitive tool in the prediction and explanation of flame behavior (Lewis & von Elbe, 1961). This is not the appropriate place to discuss the arguments used, which are in fact no more than speculation. We just mention here the fundamental claim (sometimes modified in an inconsistent fashion to fit the facts) that stretching a flame causes it to decelerate. The mathematical evidence for such a simple picture is mixed. Certainly the present results for slowly varying flames are not favorable: equation (32) shows that deceleration is associated with negative stretch. But there is supportive evidence from near-equidiffusional flames in certain special circumstances. These will be discussed in Ch. X.

5. Near-Equidiffusional Flames.

The nonuniformity as $\lambda \rightarrow 1$ revealed by equation (32) makes it clear that a different formulation, distinct from that for slowly varying flames, is needed when heat and reactant diffuse at comparable rates. The essential characteristics of such a formulation have been identified in Sec. 2, in particular equations (14) and (15); we shall now present a detailed mathematical treatment. The starting point is the system of equations (3-9) with

$$(44) \quad \Lambda = (Y_1^2 \theta^2 / 2T_*^4) \exp(\theta/T_*) \quad \text{with} \quad T_* = H_1,$$

corresponding to the choice (31), with $\lambda = 1$, for M. Recall that the state 1 does not vary along the flame.

We seek an asymptotic development in which spatial and temporal variations in the flame-sheet location are $O(1)$. As discussed in Sec. 2, limitation of

the allowable variation in flame temperature then imposes the restriction (14) and leads to the integral (15), which we now write

$$(45) \quad H = T_* + \theta^{-1}h + o(\theta^{-2})$$

outside the reaction zone. Equations (6), (8) may then be written

$$(46,47) \quad \rho\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right)T = \nabla^2 T, \quad \rho\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right)h = \nabla^2 h + \lambda \nabla^2 T$$

there, where ρ , T , $\mathbf{v}$ now stand for the leading terms in their own expansions for θ large. The same convention (including p) in the remaining equations (4), (5), (7) completes the system. The relevant solution of equation (46) behind the reaction zone is

$$(48) \quad T = T_*.$$

Note that the equations are essentially multi-dimensional. Non-planar effects are no longer removed to distances $O(\theta)$, where diffusion is negligible; and the problem is correspondingly more difficult. Note also that the actual distances over which changes take place are very small, being measured on the scale of the preheat zone (typically less than 0.1 mm.).

To complete the formulation we need jump conditions across the reaction zone. While some of these could be deduced from Sec. II.4, the generalization in notation here makes it more convenient to obtain all of them ab initio. Introduce the coordinate (10) at the point of interest and set

$$(49) \quad n = \theta^{-1}\xi.$$

The temperature within the flame zone is now written

$$(50) \quad T = T_* + \theta^{-1} t_1(\xi, y, z, t) + O(\theta^{-2})$$

so that

$$(51) \quad \rho = T_*^{-1} [1 - \theta^{-1} T_*^{-1} t_1 + O(\theta^{-2})];$$

likewise we set

$$(52) \quad Y = \theta^{-1} y_1(\xi, y, z, t) + O(\theta^{-2}),$$

$$(53) \quad v = v_*(y, z, t) + \theta^{-1} y_1(\xi, y, z, t) + O(\theta^{-2}),$$

and

$$(54) \quad p = p_0(\xi, y, z, t) + O(\theta^{-1}).$$

As the arguments of the various functions imply, the temperature, density, mass fraction and velocity are continuous to leading order across the flame sheet but the pressure jumps. The goal of the reaction-zone analysis is to find explicit expressions for this jump and those in the normal derivatives of the other variables.

The variables y_1 and t_1 satisfy

$$(55) \quad \partial^2 y_1 / \partial \xi^2 = -\partial^2 t_1 / \partial \xi^2 = (Y_1^2 / 2T_*^4) y_1 \exp(t_1 / T_*^2)$$

so that, in particular, $y_1 + t_1$ is a linear function of ξ which matching shows to be in fact independent of ξ (to leading order $\partial H / \partial n$ is zero).

Hence h is continuous across the flame sheet and we may write

$$(56) \quad y_1 + t_1 = h_*(y, z, t).$$

In the usual way, integration of the equation for t_1 now yields an expression for the temperature gradient immediately ahead of the reaction

zone (coming from $\partial t_1 / \partial \xi$ as $\xi \rightarrow -\infty$), which may be written

$$(57) \quad \left[\frac{\partial T}{\partial n} \right] = -Y_1 \exp(h_*/2T_*^2).$$

[] denotes the jump, i.e. the difference between conditions on the hot and cold sides of the flame sheet.

We turn now to the continuity and momentum equations, and immediately note that to leading order the former confirms the continuity of v_n across the flame sheet while the latter confirms that of $v_{\perp}$. To next order the continuity equation yields

$$-\frac{(V + v_{n*})}{T_*^2} \frac{\partial t_1}{\partial \xi} + \frac{1}{T_*} \left(\frac{\partial v_{n1}}{\partial \xi} + v_{\perp} \cdot v_{\perp*} \right) = 0$$

so that

$$(58) \quad \left[\frac{\partial v_n}{\partial n} \right] = \frac{V + v_{n*}}{T_*} \left[\frac{\partial T}{\partial n} \right].$$

Similarly, the normal component of the momentum equation yields

$$0 = -\frac{\partial p_0}{\partial \xi} + \frac{4\rho}{3} \frac{\partial^2 v_{n1}}{\partial \xi^2}$$

so that

$$(59) \quad [p] = \frac{4\rho}{3} \left[\frac{\partial v_n}{\partial n} \right].$$

Finally, the other components of the momentum equation yield

$$\frac{\partial^2 v_{\perp 1}}{\partial \xi^2} = 0$$

so that

$$(60) \quad \left[\frac{\partial v_{\perp}}{\partial n} \right] = 0.$$

The jump conditions are completed by noting that

$$\rho \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) H = (1 - \lambda \theta^{-1}) \nabla^2 H + \lambda \theta^{-1} \nabla^2 T$$

is an exact equation everywhere, so that

$$(61) \quad \left[\frac{\partial h}{\partial n} \right] = -\lambda \left[\frac{\partial T}{\partial n} \right].$$

In particular, we see that when the Lewis number is exactly one ($\lambda = 0$) the function h will vanish identically in the absence of inhomogeneous initial or boundary data.

For slowly varying flames it does not matter, so far as stretch is concerned, whether the flame surface is taken to be the hydrodynamic discontinuity or the reaction zone since $v_{\perp}$ does not change across the preheat zone. On the other hand, flame speed differs for these two possibilities since v_n does change across the preheat zone. For near-equidiffusional flames the stretch is different also, because of changes in $v_{\perp}$. Nevertheless, it is natural to select the flame sheet rather than the hydrodynamic discontinuity, which is consistent with slowly varying flames for stretch but not for speed. The determinations of speed would be consistent if calculated relative to the burnt gas, but that would flout long-established convention.

6. Reduction to Stefan Problems.

The problem described in Sec. 5, for the first time in full generality, presents formidable mathematical difficulties. In general a numerical treatment would be needed, involving the simultaneous solution of six first- or second-order partial differential equations (some nonlinear) on each side of a (possibly) moving discontinuity which has to be found. Even the description

of small perturbations of a plane flame (Sec. III.8) demands the solution of equations with complicated variable coefficients.

These difficulties are significantly reduced when the fluid mechanics is eliminated, in the spirit of Fig. 2 (Buckmaster 1979a, b). Then only equations (46), (47) remain with ρ and y prescribed, the first as a constant and the second as a solution of the continuity and momentum equations. This is the constant-density approximation of Sec. I.5, leading to the system

$$(62, 63) \quad \left(\frac{\partial}{\partial t} + y \cdot \nabla\right)T = \nabla^2 T, \quad \left(\frac{\partial}{\partial t} + y \cdot \nabla\right)h = \nabla^2 h + \lambda \nabla^2 T$$

when the density is absorbed by the distance and time units. The accompanying jump conditions are

$$(64) \quad \left[\frac{\partial h}{\partial n}\right] = -\lambda \left[\frac{\partial T}{\partial n}\right] = \lambda Y_1 \exp(h_*/2T_*^2),$$

h and T being continuous across the flame sheet. An attractive feature is that the field equations are now linear; but we are still faced with a spatially elliptic free-boundary problem, even for the simple case of a uniform flow

$$(65) \quad y = (U, 0, 0) \quad \text{with} \quad U = \text{const.}$$

For simplicity, consider steady plane flow. If U is large, i.e. the gas speed is much greater than the flame speed, then a flame sheet would have to be almost horizontal (with slope of order U^{-1}); under the right circumstances, changes in the x -direction would then be much smaller than those in the y -direction. An example, discussed in detail in Ch. IX, is that of a flame next to an adiabatic wall parallel to the x -axis. Under such circumstances the field equations become parabolic; for, by writing

$$(66) \quad x = U\chi$$

in equations (), we obtain

$$(67, 68) \quad \frac{\partial T}{\partial x} = \frac{\partial^2 T}{\partial y^2}, \quad \frac{\partial h}{\partial x} = \frac{\partial^2 h}{\partial x^2} + \lambda \frac{\partial^2 T}{\partial y^2}$$

in the limit $U \rightarrow \infty$. Moreover, the normal n is almost vertical so that the n -derivatives in the jump conditions (64) may be replaced by y -derivatives. The result is a generalized Stefan problem, readily amenable to numerical analysis (Buckmaster 1979a). Clearly the argument is not affected in substance if U is a function of y ; the x -derivatives in equations (67), (68) acquire coefficients depending on y , the same for each. They will form the basis of the discussion of shear flows presented in Ch. X.

A similar reduction is possible for straining flows, i.e. flows in which the speed varies significantly along streamlines (Buckmaster 1979b). When the speed is large, the angle between the flame sheet and intersecting streamline is small. Then changes along the streamlines occur over a characteristic length in the flow field, whereas changes normal to them occur over a flame thickness in the neighborhood of the sheet. If the ratio of these lengths is large, a parabolic description results.

To see this in detail, consider a curvilinear system of coordinates (s, n) in which s measures distance along one of the intersecting streamlines and n distance from it. The (divergenceless) velocity field is supposed to have the form

$$(69) \quad \underline{v} = Uq(\sigma, v) \quad \text{with } \sigma = s/U, \quad v = n/U,$$

so that U characterizes both the magnitude of $\underline{v}$ and the distance over which it varies. Letting $U \rightarrow \infty$ gives a distinguished limit in which $\underline{v}$ is $O(1)$.

The combustion field is described in terms of the variables σ and n so that, in the limit

$$v_s = Uq_0, \quad v_n = -nq'_0 \quad \text{with} \quad q_0 = q_s(\sigma, 0)$$

is the velocity field experienced by the flame. Equations (46), (47) therefore simplify to

$$(70) \quad (q_0 \frac{\partial}{\partial \sigma} - nq'_0 \frac{\partial}{\partial n})T = \frac{\partial^2 T}{\partial n^2},$$

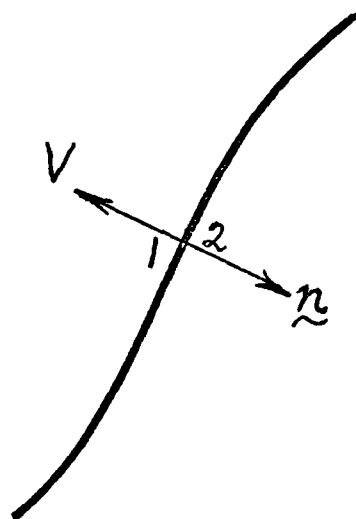
$$(71) \quad (q_0 \frac{\partial}{\partial \sigma} - nq'_0 \frac{\partial}{\partial n})h = \frac{\partial^2 h}{\partial n^2} + \lambda \frac{\partial^2 T}{\partial n^2}.$$

These are the basis of the discussion of straining flows in Ch. X. The previous equations (67), (68) are, of course, obtained on setting $q_0 \equiv 1$, $\sigma = x$, $n = y$.

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FRESH MIXTURE

 $\rho, T, Y, \underline{u}, p$  $\rho, T, Y, \underline{u}, p$

BURNT MIXTURE

Fig. 1 Flame as hydrodynamic discontinuity.

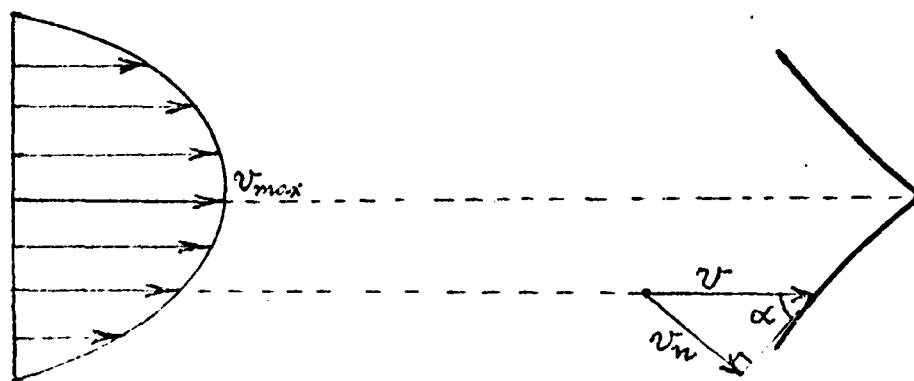


Fig. 2 Bunsen flame for v_n constant.

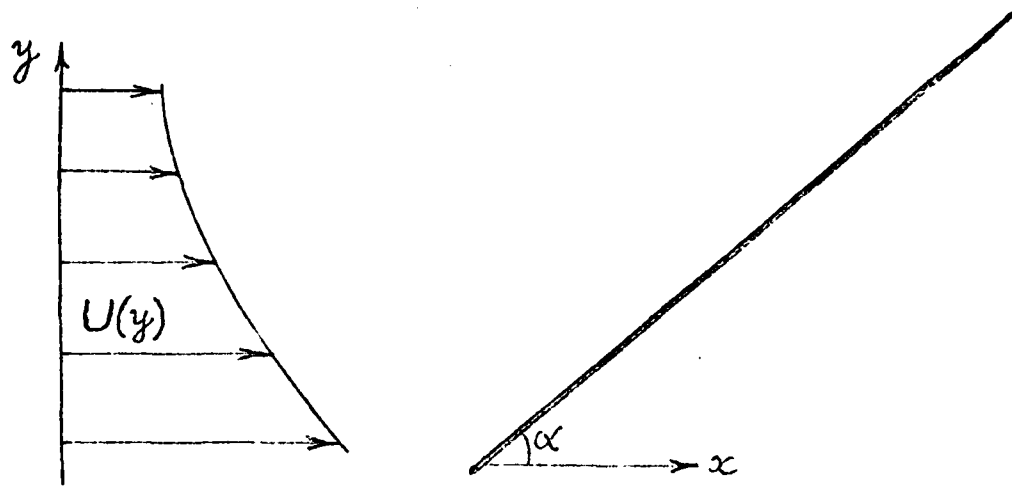


Fig. 3 Flame in a shear flow.

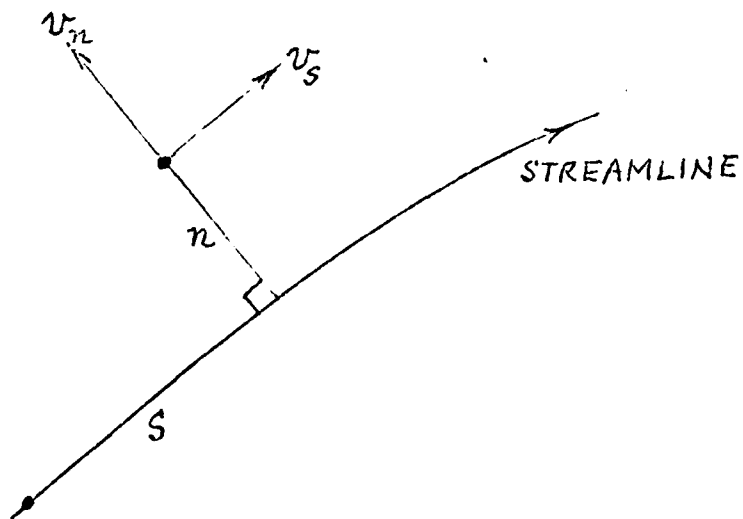


Fig. 4 Curvilinear coordinate system for straining flow.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) This report is Chapter VIII of the twelve in a forthcoming research monograph on the mathematical theory of laminar combustion. A theory of three-dimensional flames is developed, showing the key roles played by slow variation and near equidiffusion. The basic equation of slowly varying flames is developed, while near-equidiffusional flames are shown to lead to moving- boundary problems of Stefan type. The notion of flame stretch is investigated.		