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ABSTRACT

In the main body of our report, we developed an asymptotic ray theory that predicts correctly the exterior resonant frequencies of a sphere. A logical extension of this work is to spheroids where, unlike the sphere, the polar ray orbit resonance condition is nontrivial. The idea is thus to extend the results to separable coordinate systems, so that the WKB resonance condition can be applied independently to the rays along the three coordinate paths. The radial ray path remains trapped between a scattering surface and the turning point caustic.

At this point the results of this addendum are only formal and tentative. No numerical work has yet been carried out.

INTRODUCTION

As a further development of our ideas, let us set up the exterior resonance conditions on a prolate spheroid. Since natural frequencies are available in the literature for this body, this computation can be checked against existing theory [1].

Thus we introduce the spheroidal coordinates (u, θ, ϕ) by stating their relation to the Cartesian coordinates (x, y, z).

$$
\begin{aligned}
x &= a \sinh u \sin\theta \cos\phi \\
y &= a \sinh u \sin\theta \sin\phi \\
z &= a \cosh u \cos\theta
\end{aligned}
\tag{A.1}
$$

The u = constant surfaces are confocal prolate spheroids rotationally symmetric about the z axis. In the limit $u \to \infty$, and $a \to 0$ such that $r = a \sinh u$ is finite, the system (A.1) returns to the usual spherical system (r, θ, ϕ). Let us define the contravariant vectors

$$
\begin{aligned}
u^1 &= \cosh u \quad , & x^1 &= x \\
u^2 &= \cos\theta & x^2 &= y \\
u^3 &= \phi & x^3 &= z
\end{aligned}
\tag{A.2}
$$

Then the fundamental arc length formula is defined to be

$$
ds^2 = \sum_{i,j=1}^{3} g_{ij} \, du^i \, du^j \tag{A.3}
$$

where

$$g_{ij} = \sum_{\ell=1}^{3} \frac{\partial x^\ell}{\partial u^i} \frac{\partial x^\ell}{\partial u^j}$$

$$= h^2{}_i \, \delta_{ij}$$

$$h^2{}_1 = \frac{a^2((u^1)^2 - (u^2)^2)}{(u^1)^2 - 1} \quad , \quad h^2_2 = \frac{a^2((u^1)^2 - (u^2)^2)}{1 - (u^2)^2} \quad ,$$

$$h^2{}_3 = a^2((u^1)^2 - 1)\ (1 - (u^2)^2)$$

Since the metric tensor g_{ij} is diagonal, the prolate spheroidal coordinate system is seen to be orthogonal. From the scale factors h_i as given by (A.3) one computes the Laplacian operator according to the prescription

$$\nabla^2 \psi = \frac{1}{\sqrt{g}} \sum_{i=1}^{3} \frac{\partial}{\partial u^i} \left(q_i \frac{\partial \psi}{\partial u^i}\right) \tag{A.5}$$

where

$$q_i = \frac{\sqrt{g}}{h^2{}_i} \quad , \quad g = h_1^2\, h_2^2\, h_3^2 = \det(g_{ij})$$

The scalar wave equation $(\nabla^2 + k^2)\psi = 0$

takes the form

$$\frac{1}{(u^1)^2 - (u^2)^2} \left\{ \frac{\partial}{\partial u^1} [((u^1)^2 - 1)] + \frac{\partial}{\partial u^2} [(1-(u^2)^2) \frac{\partial \psi}{\partial u^2}]\right\} + \frac{1}{((u^1)^2 - 1)(1-(u^3)^2} \frac{\partial^2 \psi}{\partial (u^3)^2} + (ka)^2\, \psi = 0. \tag{A.6}$$

Using the usual separation of variables techniques we let

$$\psi = R(u^1)\ \Theta\ (u^2)\ \Phi\ (u^3). \tag{A.7}$$

Substitution of (A.7) into (A.6) eventually leads to the following three separated ordinary differential equations for R, Θ and Φ.

$$(((u^1)^2 - 1)\ R^1(u^1))' + ((ka)^2\ (u^1)^2 - \frac{m^2}{(u')^2-1} + \lambda)R(u^1) = 0$$

$$(1 - (u^2)^2)\ \Theta\ '(u^2))' - ((ka)^2\ (u^2)^2 + \frac{m^2}{1-(u^2)^2} + \lambda)\ \Theta\ (u^2) = 0$$

$$\Phi''\ (u^3) + m^2\ \Phi\ (u^3) = 0 \tag{A.8}$$

In (A.8), m^2 and λ are the separation constants. The equation for $\Phi(u^3)$ requires that m be an integer to satisfy physical constraint that $u^3 = 0$ be the same point as $u^3 = 2\pi$.

The rotational symmetry of the prolate spheroid about the z axis as follows from equation (A.1) determines the separation constant m. Consequently, the two remaining unknowns for the geometric computation of resonant frequencies are the wave number k and the separation constant λ.

The two conditions needed to determine k and λ are the polar resonance orbit and the radial resonance orbit. In the latter condition the "orbit" is between the turning point caustic and the surface of the spheroid.

POLAR ORBIT RESONANCE CONDITION

Because of the analytic nature of the phase function, there is no loss of generality to simplify the polar resonance condition by computing it for $m = 0$ and also just inside the caustic surface. By inspection of the radial differential equation for R given in (A.8), the turning point caustic satisfies the equation

$$\lambda = -\Omega^2 \cosh^2(u) \quad , \quad \Omega = ka \tag{A.9}$$

where from (A.1) u is seen to be the radial coordinate.

In (A.9) we have introduced Ω, the large parameter of this asymptotic calculation. The differential equation for Θ as given by (A.8) under these conditions becomes

$$((1-x^2)\ T'(x))' + \Omega^2 (\cosh^2 u - x^2)\ T(x) = 0 \tag{A.10}$$

where $x = u^2 \equiv \cos\theta$ and $T(x) = \Theta(\cos\theta)$. To utilize the WKB approach, we make the further change of variables

$$y = \tanh^{-1}(x)\ ,\ \psi(y) = T(x) \tag{A.11}$$

in (A.10) to obtain

$$\psi''(y) + \Omega^2 f^2(y)\, \psi(y) = 0 \tag{A.12}$$

where

$$f(y) = \left(\frac{\cosh^2 u - \tanh^2 y}{\cosh^2 y}\right)^{1/2} \tag{A.13}$$

From (A.13) it is seen that there are no finite turning points. Thus the WKB solution of (A.12) is

$$\bar{\psi}\,(y) = \exp(i\,\Omega\,S(y)) \tag{A.14}$$

where the phase function S(y) is

$$S(y) = \int_y^\infty f(y')dy' + \frac{i}{2\Omega}\ln\,(f(y)) + 0\left(\frac{1}{\Omega^2}\right) \tag{A.15}$$

Returning to the original angle variable θ through the transformation $\cos\theta = \tanh y$, the real $(S_R(\theta))$, and imaginary $(S_I(\theta))$, parts of the phase function $S(\theta)$ become

$$S_R(\theta) = \int_0^\theta (\cosh^2 u - \cos^2\theta')^{1/2}\,d\theta' \tag{A.16}$$

$$S_I(\theta) = \frac{1}{2\Omega}\ln\,[(\cosh^2 u - \cos^2\theta)^{1/2}\sin\theta] \tag{A.17}$$

Let us interpret this result geometrically by computing a geometric optics solution of a ray traveling along a polar orbit on the spheroid. The geometric optics ray solution $\psi_{G.O.}(\theta)$ takes the form

$$\psi_{G.O.}\,(\theta) = e^{i\,kl(\theta)} \tag{A.18}$$

$$\text{where } l(\theta) = \int_0^\theta ds\ , \quad ds = \sqrt{(dx)^2 + (dz)^2} \tag{A.19}$$

From (A.1) the arc length increment ds is easily seen to be

$$ds = a(\cosh^2 u - \cos^2\theta)^{1/2}\,d\theta \tag{A.20}$$

Thus from (A.16) and (A.18) through (A.20)

$$\psi_{G.O.}(\theta) = \exp\left(i\,\Omega\, S_R(\theta)\right) \tag{A.21}$$

Comparing the geometric optics field (A.21) with the WKB solution (A.14), it is seen they differ by terms of order 0 $(1/\Omega)$. The attenuation factor $S_I(\theta)$ is of order 0 $(1/\Omega)$ and accounts for ray tube spreading on the curved convex ray path.

The polar resonance condition consists of requiring that the polar phase $S(\theta)$ satisfy the global condition

$$\Omega \int_0^{2\pi} S(\theta)\, d\theta = 2 n_\theta \pi$$

$$n_\theta = 1,2,3....$$

Because of four-fold symmetry this can also be written as

$$\Omega \int_0^{\pi/2} S(\theta)\, d\theta = n_\theta\, \pi/2 \tag{A.22}$$

It remains to evaluate the integral in (A.22). Separating the phase function into its real and imaginary parts according to (A.16) and (A.17) produces two integrals. The imaginary part can be evaluated in terms of elementary functions. One determines

$$\int_0^{\pi/2} S_I(\theta)\, d\theta = \frac{\pi}{4\Omega}\,[u - 2 \ln 2] \tag{A.23}$$

The corresponding integral of $S_R(\theta)$ is expressible in terms of elliptic integrals of the second kind. This is not surprising. Elliptic integrals were originally encountered in the calculation of the perimeter of an ellipse [4]! Manipulation of (A.16) results in

$$S_R(\theta) = \csc\alpha \{E(\sin^2\alpha) - E(\pi/2 - \theta\backslash\alpha)\} \tag{A.24}$$

where

$$\alpha = \mathrm{Sin}^{-1} \left(\frac{1}{\cosh u}\right)$$

and the elliptic integrals of the second kind are defined as [5]:

$$E(\phi\backslash\alpha) = \int_0^{\phi} (1 - \sin^2 \alpha \sin^2 \theta)^{1/2} d\theta \tag{A.25}$$

and the complete integral $E(\sin \alpha)$ is defined as

$$E(\sin\backslash\alpha) = E(\pi/2\ \alpha) = \int_0^{\pi/2} (1 - \sin^2 \alpha \sin^2 \theta)^{1/2} d\theta \tag{A.26}$$

Substituting (A.24) into the resonance condition (A.22) requires the integral

$$\begin{aligned}\int_0^{\pi/2} S_R(\theta)d\theta &= \csc\alpha \{\pi/2\ E(\sin^2\alpha) - \int_0^{\pi/2} \theta(1 - \sin^2 \alpha\cos^2\theta)\ d\theta\} \\ &= \csc\alpha \{ \int_0^{\pi/2} (\pi/2 - \theta)\ (1 - \sin^2 \alpha\cos^2\theta)^{1/2}\ d\theta\}\end{aligned} \tag{A.27}$$

Combining results (A.22), (A.23) and (A.27) yields the resonance condition

$$\Omega \csc\alpha \{ \int_0^{\pi/2} (\pi/2 - \theta)(1 - \sin^2 \alpha\cos^2 \theta)^{1/2}\ d\theta\} + \frac{i\pi}{4}(u - 2\ \ell n 2) = \frac{n_\theta \pi}{2}$$

where

$$\alpha = \mathrm{Sin}^{-1} \left(\frac{1}{\cosh u}\right), \quad \cosh^2 u = -\frac{\lambda}{\Omega^2} \tag{A.28}$$

Equation (A.28) contains two unknowns, λ and Ω. The required additional condition is now considered.

RADIAL RESONANCE CONDITION

As was described in the main body of the report, the radial resonance condition requires that the evanescent ray path between the scattering surface and the turning point caustic be an integer number of wavelengths. Included in this phase resonance are the discontinuous phase change at the caustic surface and, if necessary, at the surface of the scatterer.

To begin, we consider the radial differential equation for $R(u^1)$ as given in (A.8). In it let

$$s = \coth^{-1} (u^1) \ , \ \psi (s) = R (u^1) \tag{A.29}$$

This is the legimitate change since $u^1 \equiv \cosh u \geq 1$. The equation for $\psi(s)$ becomes

$$\psi''(s) + s^2 \ n^2(\Omega) \ \psi(s) = 0 \tag{A.30}$$

where

$$n^2(s) = \frac{\coth^2(s)}{\sinh^2(s)} - \frac{m^2}{\Omega^2} + \frac{\lambda}{\Omega^2} \operatorname{csch}^2 s \tag{A.31}$$

The caustic surface is defined by the positive value of s denoted by s_c such that

$$n(s_c) = 0 \tag{A.32}$$

The solution to (A.32) is the proper root of

$$s_c = \sinh^{-1}\left[\frac{\Omega^2 + \lambda \pm ((\Omega^2+\lambda)^2 + 4\Omega^2 m^2)^{1/2}}{2m^2}\right]^{1/2}$$

$$m > 0$$

$$s_c = \sinh^{-1}\left[\frac{-\Omega^2}{\Omega^2 + \lambda}\right]$$

$$m = 0$$

When $m = 0$, it is therefore necessary that $-\lambda/\Omega^2 > 1$.

For eigenvalues which correspond to bound discrete modes to exist, it is required that the effective refractive index profile $n^2(s)$ have the property that

$$\begin{aligned} n^2(s) &< 0 \qquad && s_c < s < s_b \\ n^2(s) &= 0 && s = s_c \\ n^2(s) &> 0 && s < s_c \end{aligned} \tag{A.34}$$

In (A.34) s_b defines the spheroid surface. The radial resonance condition for a perfectly conducting prolate spheroid with the electric field normal to the surface is

$$2\Omega \int_{s_b}^{s_c} n(s)ds = 2\pi\ (n_r - 1/4) \tag{A.35}$$

$$n_r = 1,2,3\ldots$$

In (A.35) s_b is the surface of the spheroid defined by $\coth(s_b) = \cosh(u_b)$ where u_b defines the spheroid through equation (A.1) when $u = u_b$. An explicit form of the radial resonance is

$$\int_{P_b}^{P_c} (m^2 - P(s^2 + \lambda) - P^2\Omega^2)^{1/2}\ \frac{dP}{P\sqrt{P+1}} = 2\pi i(n_r - 1/4) \tag{A.36}$$

where

$$P_b = \cosh^2 u_p - 1 \text{ and}$$

$$P_c = 1/2 \left[\left((1 + \lambda/\Omega^2)^2 + \frac{4m^2}{\Omega^2} \right)^{1/2} - (1 + \lambda/\Omega^2) \right]$$

The simultaneous solution of (A.28) and (A.35) for λ and Ω constitutes the formal asymptotic solution to the exterior resonant frequencies of a perfectly conducting prolate spheroid.

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