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TWO DIMENSIONAL WINDOW FUNCTIONS.(U)

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THESIS

TWO DIMENSIONAL WINDOW FUNCTIONS

by

Wulang Widada

December 1979

Thesis Advisor: S. R. Parker

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TWO DIMENSIONAL WINDOW FUNCTIONS

by

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I. INTRODUCTION

Windowing is essential to signal processing and to digital filter design. For example, the weighting functions of the window are used in harmonic analysis of signals [1] and in design techniques for linear-phase digital FIR filter [2].

In fact, if an infinite sequence of numbers is truncated, then its Fourier transform will be subject to the well-known Gibb's phenomenon; which will cause over shoot and ripples before and after any point of discontinuity in the frequency domain. The overshoot and ripples will not disappear no matter how long the sequence is.

To overcome this undesirable effect in the frequency domain, the finite sequence should be modified using a weighting function, called a window. For example, if w(n) is a weighting sequence and h(n) is an infinite sequence of number, then the windowed finite sequence is:

$$h_w(n) = w(n) . h(n) \tag{1.1}$$

where:

$$w(n) = \begin{cases} f(n) & |n|<N \\ 0 & \text{elsewhere} \end{cases} \tag{1.2}$$

In this example the weighting functions performs both truncation and modification of the signal at the same time. Multiplication of two sequences in time domain is equivalent to the convolution of their transforms in the frequency domain. Consequently the windowing method has the effect of smoothing out the rough points - moderating the overshoot and ripples - in the original frequency response.

The technique of windowing is shown graphically in Fig. 1.1.

There are many window functions available in many pieces of literature. [1, 3] According to the way in which the window function is formed, they can be separated into two categories; basic windows and constructed windows. A constructed window is formed from a product, a sum, or a convolution of two or more basic windows.

Seven best-known basic windows are listed in Table 1.1, and they are: Rectangular, Barlett, Cosine, Hann, Hamming, Blackman and Kaiser window. Most of the windows are named after individuals. These seven basic windows will be discussed thoroughly in later chapters.

Although Huang [4] shows that a good two dimensional window can be derived from one dimensional window, most windows found in the literature are given in only one-dimensional form. It is the purpose of this thesis to present and discuss two-dimensional windows derived from those one dimensional form, using Huang's theorem.

The easiest way to compare various windows is to create some appropriate figures of merit for windows. This thesis presents those figures of merit that are used to compare the given window functions.

A. TECHNIQUE OF WINDOWING [2]

A graphical interpretation of windowing is given in Fig. 1.1 (a non-recursive filter design using the window method).

The first figure shows an infinite impulse response of a digital filter, h(n), and its periodic frequency response H(w).

The next figure shows a rectangular window r(n), with its frequency response, R(w).

The third figure consists of multiplication of h(n) and r(n), which produces a truncation of the filter impulse response, $h_t(n)$. Multiplication in the time domain is equivalent to convolution in the frequency domain. The convolution of H(w) and R(w) is $H_t(w)$ containing overshoot and ripples in its frequency response of the corresponding digital filter as shown in the figure.

Truncation of impulse response is required for a non-recursive digital filter design, since the filter produces an output which depends only upon the present and a finite number of past inputs.

The fourth figure shows the non-rectangular window w(n), and its periodic Fourier Transform W(w). The purpose of this window is to alter the truncated filters impulse respons $h_t(n)$, slightly in order to minimize the overshoot and ripple in its frequency response $H_t(w)$.

The last figure shows the finite impulse response $h_w(n)$, which consists of multiplication of $h_t(n)$ and w(n). The frequency response of $h_w(n)$ is produced by convolving $H_t(w)$ with $W(w)$, resulting in a smooth frequency response $H_w(w)$.

The technique of windowing applied to a filter design, results in an improved filter with a smooth frequency response.

B. THESIS ORGANIZATION

The following discussion presents the organization of this thesis.

As mentioned before, the best way to discuss windows is to define parameters which can be used to characterize a window. Logically, the window figures of merit are first discussed in the next chapter, Chapter II.

Chapter III gives an overview of one-dimensional windows and specifies their characteristics, using parameters defined in Chapter II. Table III.1, given at the end of this chapter, tabulates one-dimensional windows and their parameters.

The transformation from one-dimensional to two-dimensional is given in Chapter IV, where we present three major discussions. First we present the expansion of the Fourier Transform from one-dimension to two-dimensions. Then we present how Huang [4] expanded a good one-dimensional window into a good two-dimensional window. Finally, the definition of one-dimensional figures of merit are expanded into two-dimensional window figures of merit.

Chapter V presents two-dimensional windows in detail, and specifies their parameters as defined in Chapter IV. Table V.1, given at the end of this chapter, tabulates two-dimensional windows and their parameters.

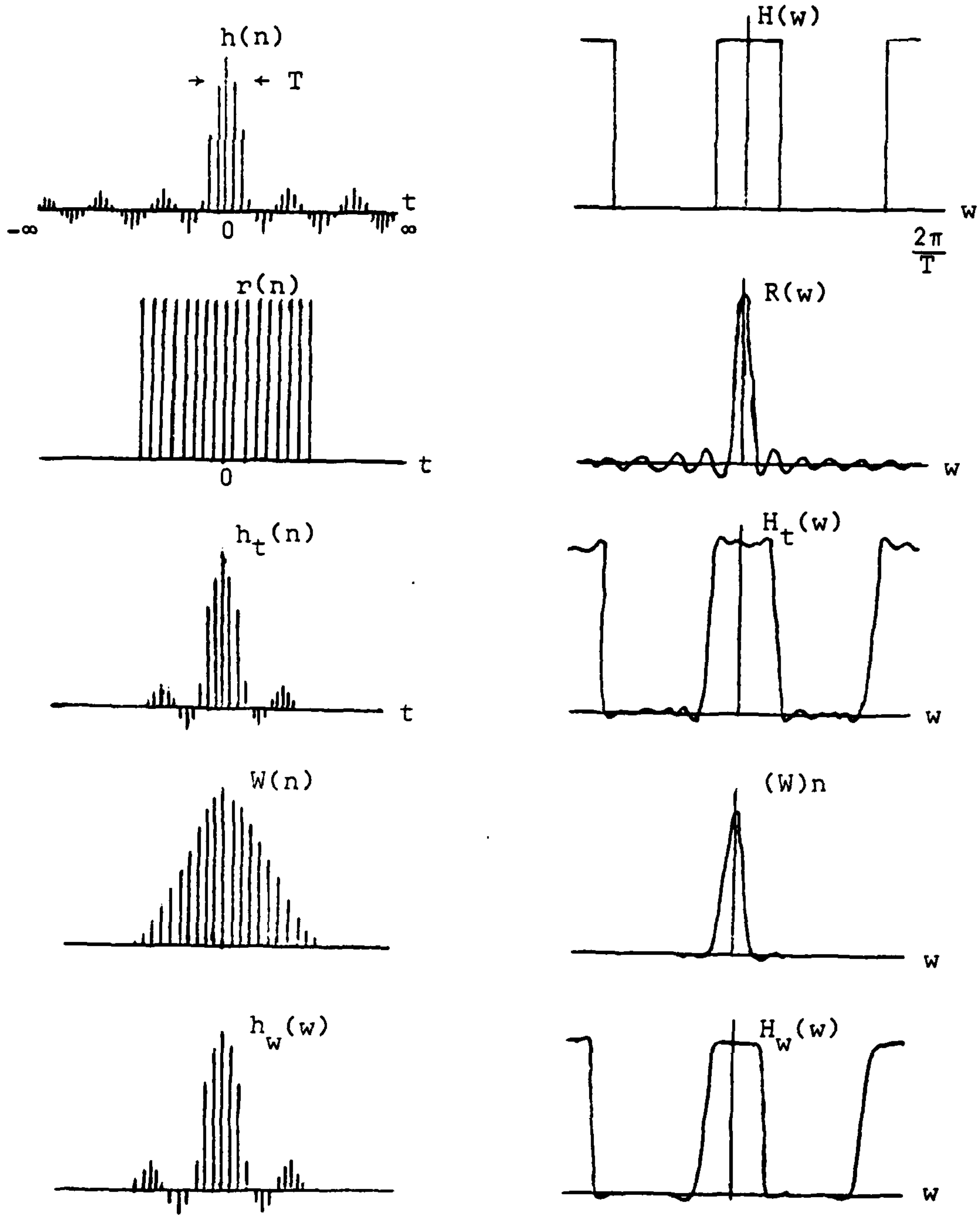


Figure 1.1

Application of Window Function To FIR Filter Design

Two-dimensional Fourier Transform of window functions are hard to present in a closed form. Fortunately a closed form Fourier Transform is not required in studying windows, so two-dimensional Fourier Transform of the windows are presented in integral form. The effort to present a two-dimensional Fourier Transform of a window in a closed form is given in the Appendix.

TABLE I.1

Seven Basic One-Dimensional Window Functions

No.	Window	Function	
1	Rectangular	$W(x) = 1.0$	$\|x\| \leq a$
2	Barlett	$W(x) = 1.0 + \frac{\|x\|}{a}$	$\|x\| \leq a$
3	Cosine	$W(x) = \cos \frac{\pi}{2a} x$	$\|x\| \leq a$
4	Hann	$W(x) = \frac{1}{2} + \frac{1}{2} \cos \frac{\pi}{a} x$	$\|x\| \leq a$
5	Hamming	$W(x) = 0.54 + 0.46 \cos \frac{\pi}{a} x$	$\|x\| \leq a$
6	Blackman	$W(x) = 0.42+0.5 \cos \frac{\pi}{2} x + 0.08 \cos \frac{\pi}{a} x$	$\|x\| \leq a$
7	Kaiser	$W(x) = \frac{I_o\{\beta \sqrt{1.0-(\frac{x}{a})^2}\}}{I_0(\beta)}$	$\|x\| \leq a$

II. WINDOW FIGURES OF MERIT [1, 3]

Since windows are used in frequency response analysis to reduce undesirable effects - overshoots and ripples in the frequency domain - it is very important to find the parameters or figures of merit of the windows that can be used to identify their characteristics and to indicate their performance.

There are many convenient measures that can be used as parameters.

The common desirable characteristics of a window are:

1. A narrow main lobe width of the frequency response of the window, containing as much of the total energy as possible.

2. Side lobes of the frequency response decreasing in energy rapidly as frequency increases.

These two common characteristics certainly can be used as parameters to identify the window's performance.

Harris [1] and Geckinli [3] define one-dimensional window performance parameters which will be repeated here. These parameters are used throughout this thesis and will be expanded for the two-dimensional case in later chapters. The parameters are: equivalent noise BW, coherent gain, and trade-off parameters.

A. EQUIVALENT NOISE BANDWIDTH (ENBW)

The ENBW parameter is the width of a rectangular filter in the frequency domain with the same amplitude as the peak power gain of the window, and with a width such that the total noise power in the rectangular filter is equal to the total window noise power.

A graphical explanation of the parameter is shown in Fig. 2.1. The first figure shows a window frequency response which consists of a main lobe and side lobes. The second figure shows a discrete frequency component of a signal, together with white noise. The third figure shows the result of windowing.

The window's frequency response convolves with the discrete component of the signal plus noise. In this case the window acts as a filter, gathering information over its bandwidth. The last figure represents the area of the windowed discrete frequency response of the signal plus noise, at discrete frequency W_o.

It is clear from the figure that to minimize the accumulated noise collected by the window, it is required that the window has a narrow bandwidth.

A convenient measure of window's bandwidth is called equivalent noise bandwidth.

Noise power accumulated by the window is defined as:

$$\text{Noise Power} = \frac{N_o}{2} \int_{-\infty}^{\infty} |W(w)|^2 \; dw \qquad (2.1)$$

where N_o = noise power per unit BW.

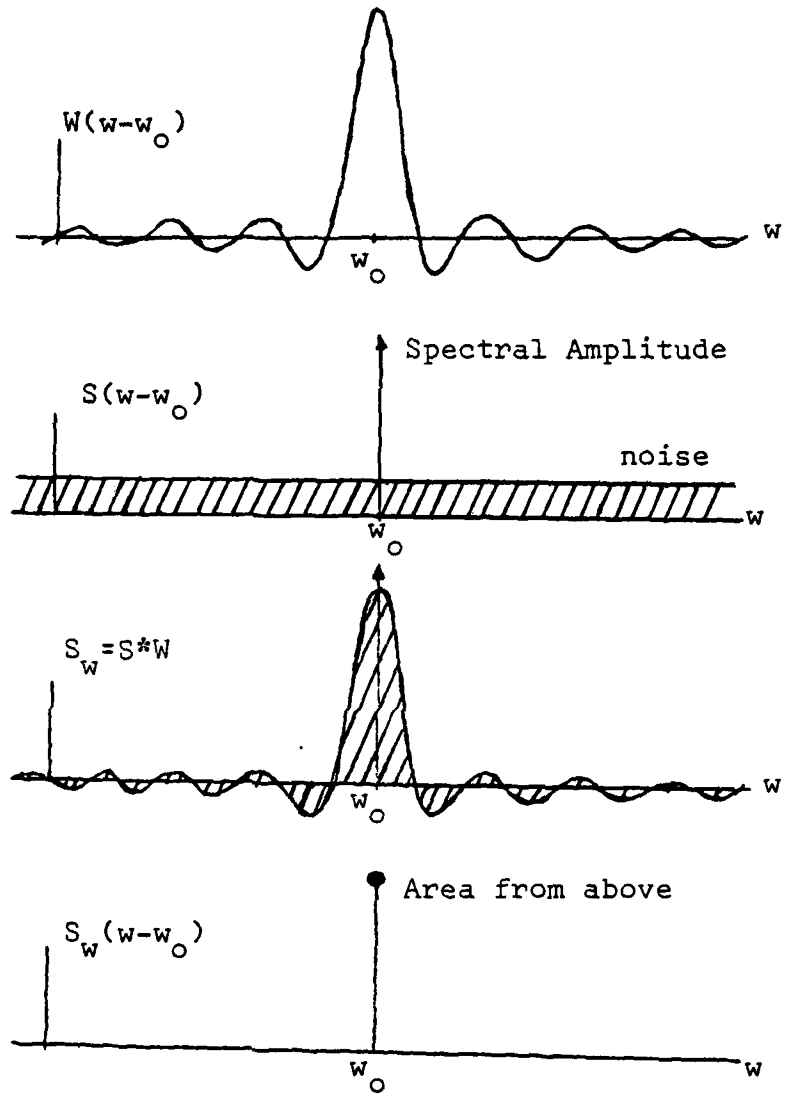


Figure 2.1

Graphical Interpretation Of Convolution Of A Window And Spectral Amplitude Of A Signal, At $w=w_o$

Parseval's theorem allows equation (2.1) to be computed as:

$$\text{Noise Power} = \frac{N_o}{2T} \int_{-T}^{T} W^2(t)\,dt \qquad (2.2)$$

The integral in (2.1) and (2.2) represents the area under the window curve. If the area is divided by height, then the result is the width of a rectangle that has the same area with the window.

The peak gain of signal occurs at w=0, and is defined as:

$$\text{Peak signal gain} = w(0) = \int_{-T}^{T} w(t)\,dt \qquad (2.3)$$

$$\text{Peak power gain} = w^2(0) = \left[\int_{-T}^{T} w(t)\,dt\right]^2 \qquad (2.4)$$

Then the ENBW is equal to the noise-power divided by peak power:

$$\text{ENBW} = \frac{\int_{-T}^{T} w^2(t)\,dt}{\left[\int_{-T}^{T} w(t)\,dt\right]^2} \qquad (2.5)$$

The above expression is a normalized ENBW, after it is divided by factor $\frac{N_o}{2T}$. This parameter will be used as a window figure of merit.

B. COHERENT GAIN

Coherent gain is best defined using a sinusoidal signal. In Fig. 2.1 a component of signal frequency response at a discrete frequency is shown. Convolving this signal with a window's frequency response can be thought as passing the

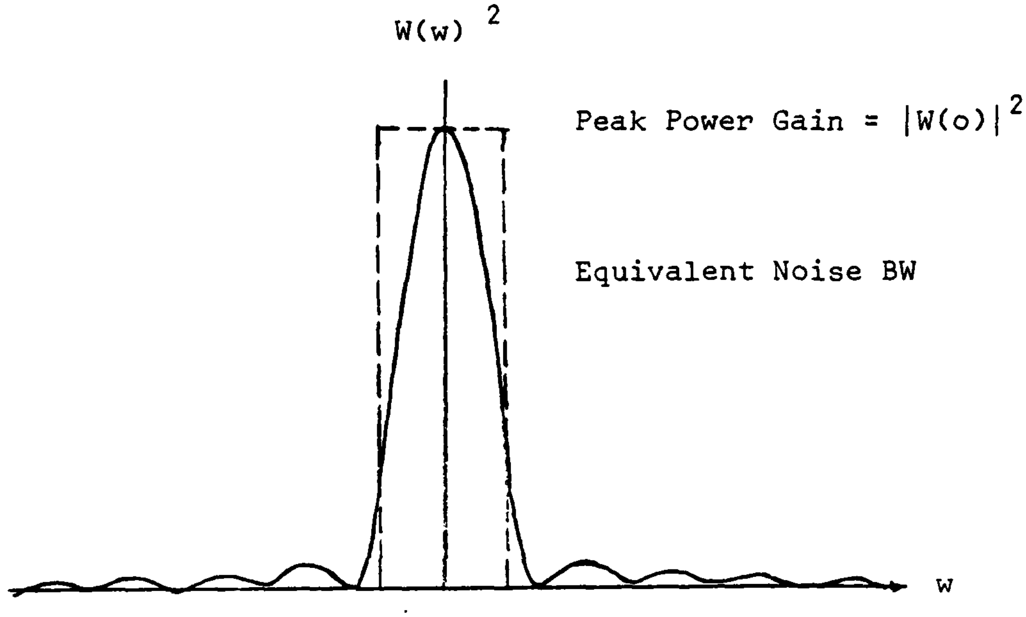


Figure 2.2

Equivalent Noise BW Of A Window

signal through a filter. Using this view, the coherent gain of a window can be examined by passing a sinusoidal signal through a filter.

Let the sinusoidal signal be defined as:

$$f(x) = Ae^{jwx} + n(x) \qquad (2.6)$$

where n(x) = white noise.

The filter output on the windowed signal is given as:

$$f_w(x) = A\,w(x)e^{jwx} + n(x)\,w(x) \quad -a<x<a \qquad (2.7)$$

The frequency response of the windowed signal only (less noise) is given as:

$$F_w(f) = \int_{-a}^{a} A\,W(x)e^{jwx}\,e^{-jwx}\,dx =$$

$$= A\int_{-a}^{a} W(x)\,dx \qquad (2.8)$$

The windowed frequency response is proportional to the input amplitude A. The proportionality factor is the integral of the window function.

For a rectangular window this factor is 2a. For any other window the factor is reduced due to the window smoothly going to zero near the boundaries.

The coherent gain is defined as the proportinality factor divided by factor 2a, the extension of the window (normalized).

$$\text{COH.GAIN} = \frac{1}{2a}\int_{-a}^{a} W(x)\,dx \qquad (2.9)$$

The coherent gain is the second parameter that will be used as window figure of merit.

C. TRADE-OFF PARAMETERS

Trade-off parameters are best explained using Fig. 2.3. The top figure shows the Hamming window in the time or spatial domain, and the bottom figure shows its Fourier transform, which is given in log frequency versus normalized log magnitude (20 log W(f)/W(0)).

The window's characteristics in frequency domain are largely determined using these trade-off parameters.

From Fig. 2.3(b) these parameters are:

1. a_1 : highest side lobe level in dB.
2. a_2 : side lobe level at a higher frequency, where asymptotic assumption is valid for the window behavior.
3. b : frequency at which main lobe drops to the peak side lobe.
4. d : side lobe fall-off rate in dB/octave.

Clearly from the above definitions, the smaller the value for b, the better the window performance; the smaller the value for a_1, the smaller the leakage through the near side lobes; the smaller the value for a_2, and the larger the value of d, the smaller the leadage through the far side lobes. These parameters are expanded for two-dimensional case in a later chapter.

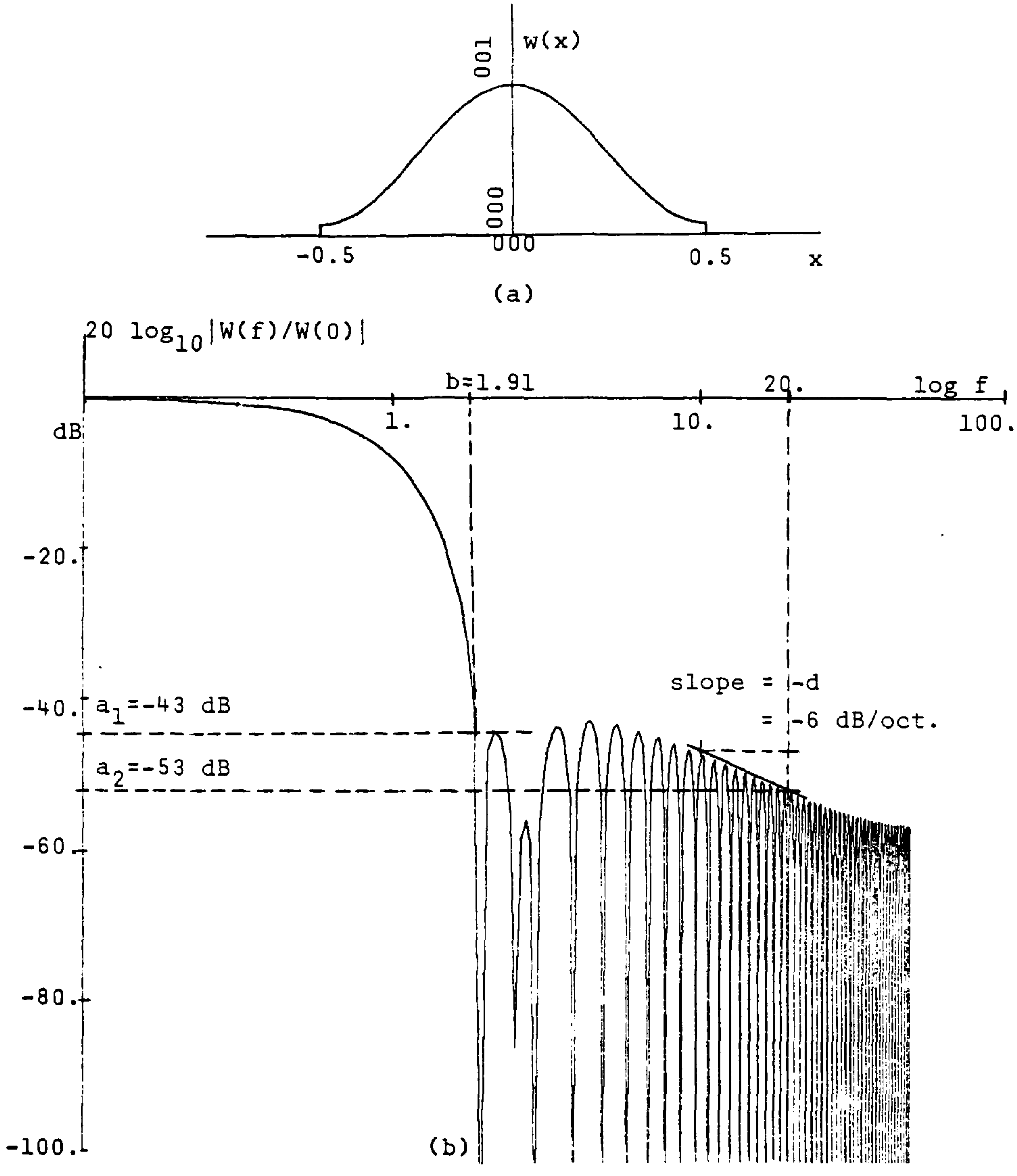


Figure 2.3

Plot Of Hamming Window, Is Used To Illustrate The Definition Of Trade-Off Parameters a_1, a_2, b and d.

III. ONE DIMENTIONAL WINDOW FUNCTIONS

[1, 3, 5, 6, 7]

The seven common window functions listed in Table I.1, are reviewed in detail in this chapter. Figures of merit for each window are presented using the definitions given in the preceding chapter. The window functions treated here are in continuous form rather than in discrete form.

Without loss of generality, let the window function, w(x), be unity at the origin and limited to the interval $|x| \leq a$. Presented in mathematical form this means:

$$w(o) = \int_{-\infty}^{\infty} W(f)\, df = 1 \qquad (3.1)$$

$$w(x) = 0 \qquad \text{for } |x| \geq a \qquad (3.2)$$

The Fourier Transform of the above window is given as:

$$W(f) = \int_{-a}^{a} w(x)\, e^{-jwx}\, dx \qquad |x| \leq a \qquad (3.3)$$

The transforms of the windows generally consists of a main lobe with a large peak amplitude and side lobe with comparatively small peak amplitudes. To better display the presence of the side lobes, logarithmis scales are used for plotting in place of linear scales.

A. RECTANGULAR WINDOW [1]

The rectangular window is commonly called a boxcar or Dirichlet window. Its main purpose is to truncate the length of a function to a finite extent, without affecting its amplitude.

The expression for this window is given as:

$$w(x) = 1.0 \qquad |x| \leq a \tag{3.4}$$

The Fourier Transform of the window is given as:

$$W(f) = \frac{1}{2a} \int_{-a}^{a} e^{-jwx}\, dx =$$

$$= \frac{\sin aw}{aw} = \mathrm{sinc}(aw) \tag{3.5}$$

The transform is seen to be a Dirichlet kernel. In this report the extent of the window is taken as a=0.5, and the ENBW is given as:

$$\mathrm{ENBW} = \frac{\int_{-0.5}^{0.5} dx}{\left[\int_{-0.5}^{0.5} dx\right]^2} = 1.0 \ .$$

The coherent gain is given as:

$$\mathrm{COH.GAIN} = \int_{-0.5}^{0.5} dx = 1.0$$

Plots of the window function and its tranform are given in Fig. 3.1(a) and (b) respectively. The main lobe width (the width between origin and the first zero crossing) is 1.0, and the first side lobe peak is approximately 13 dB down from the main lobe peak. The side lobe fall off is 6.0 dB/octave.

B. BARTLETT WINDOW

The Bartlett window is also called Fejer or triangular window, and can be thought of as a convolution of rectangular window with itself. This window is the simpliest of all windows that changes the data values by multiplication by a number other than unity. The mathematical expression of this window is given as:

$$w(x) = 1.0 - \frac{|x|}{a} \qquad |x| \leq a \qquad (3.6)$$

The transform of this window is given as:

$$W(f) = \frac{1}{a} \int_{-a}^{a} (1.0 - \frac{|x|}{a}) e^{-jwx}\, dx =$$

$$= \frac{1}{a} \int_{-a}^{o} (1.0 + \frac{x}{a}) e^{-jwx}\, dx + \frac{1}{a} \int_{o}^{a} (1.0 - \frac{x}{a}) e^{-jwx}\, dx =$$

$$= \frac{1}{a} \int_{a}^{o} e^{-jwx}\, dx + \frac{1}{a^2} \int_{-a}^{o} x\, e^{-jwx}\, dx + \frac{1}{a} \int_{o}^{a} e^{-jwx}\, dx -$$

$$- \frac{1}{2a^2} \int_{o}^{a} x\, e^{-jwx}\, dx =$$

$$= \frac{1}{a} \left(\frac{e^{-jwx}}{-jw} \Big|_{-a}^{o} \right) + \frac{1}{a^2} \int_{-a}^{o} x\, \frac{de^{-jwx}}{-jw} + \frac{1}{a} \left(\frac{e^{-jwx}}{-jw} \Big|_{o}^{a} \right) -$$

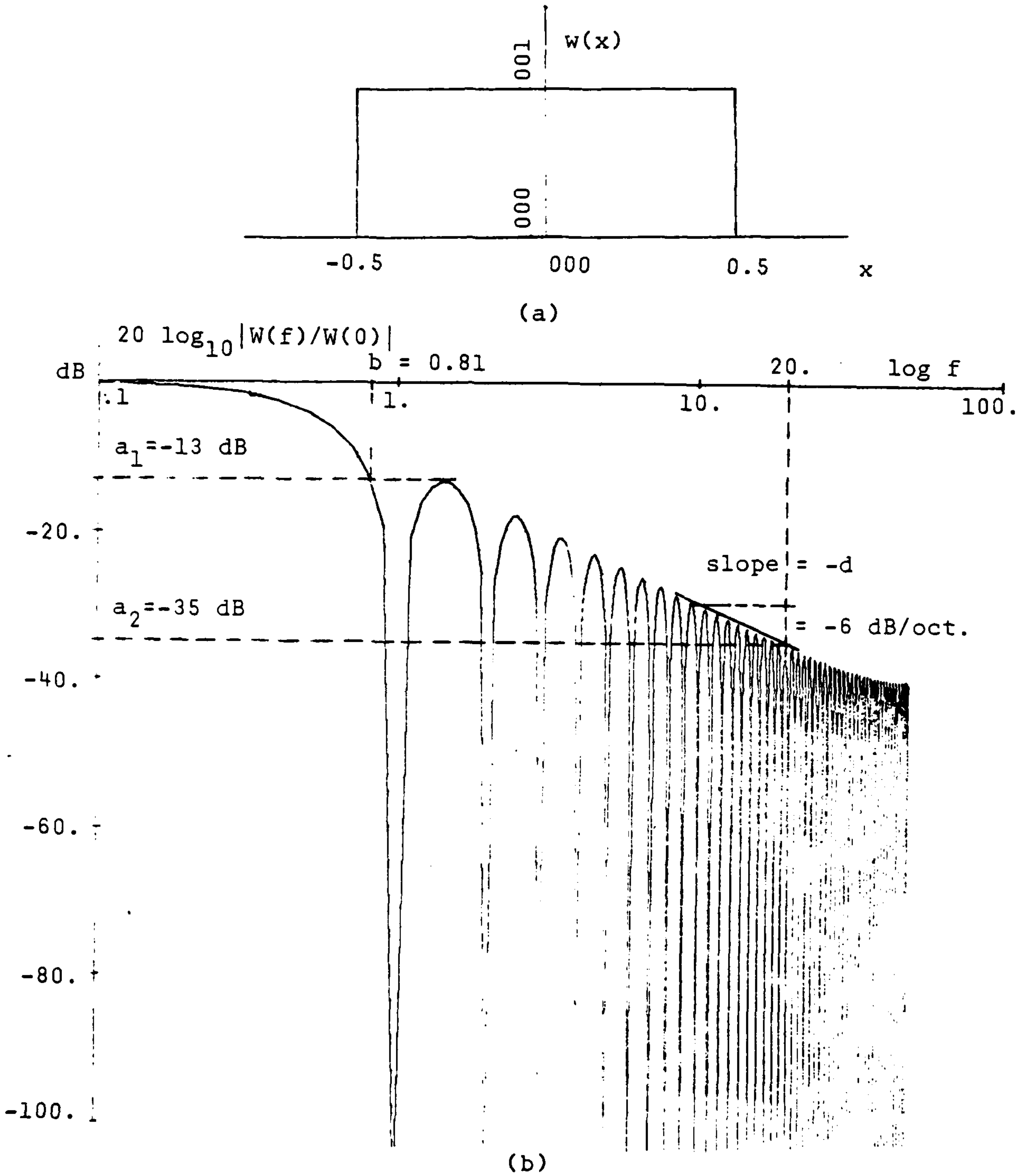


Figure 3.1

(a) Rectangular Window, (b) Spectral Window

$$- \frac{1}{a^2} \int_0^a x \, \frac{de^{-jwx}}{-jw} =$$

$$= \frac{1}{a} \left(\frac{1-e^{jaw}}{-jw}\right) + \frac{1}{a^2} \left\{\frac{xe^{-jwx}}{-jw} \Big|_{-a}^{0} - \int_{-a}^{0} \frac{e^{-jwx}dx}{-jw}\right\} + \frac{1}{a}\left(\frac{e^{-jaw}-1}{-jw}\right) -$$

$$- \frac{1}{a^2} \left\{\frac{xe^{-jwx}}{-jw} \Big|_0^a - \int_0^a \frac{e^{-jwx}}{-jw} dx\right\} =$$

$$= \frac{1}{a}\left(\frac{e^{jaw}-e^{-jaw}}{}\right) - \frac{1}{a} \left(\frac{e^{jaw}-e^{-jaw}}{jw}\right) - \frac{1}{a^2} \int_{-a}^{0} \frac{e^{-jwx}}{-jw} dx +$$

$$+ \frac{1}{a^2} \int_0^a \frac{e^{-jwx}}{-jw} dx =$$

$$= - \frac{1}{a^2} \left(\frac{1}{-jw}\right)^2 e^{-jwx} \Big|_a^0 + \frac{1}{a^2} \left(\frac{1}{-jw}\right)^2 e^{-jwx} \Big|_0^a =$$

$$= + \frac{1}{a^2w^2} (1 - e^{+jaw} - e^{-jaw} + 1) = \frac{-1}{a^2w^2} (e^{jaw} - 2 + e^{-jaw}) =$$

$$= \left(\frac{1}{aw/2}\right)^2 \left(\frac{e^{jaw/2}-e^{-jaw/2}}{2j}\right)^2 .$$

$$= \left(\frac{\sin aw/2}{aw/2}\right)^2 = (\text{sinc } aw/2)^2 \qquad (3.7)$$

The transform is seen to be the squared of Dirichlet Kernel. If the extent of the window is taken as a=0.5, then the parameters ENBW and COH.GAIN are:

$$\text{ENBW} = \frac{\int_{-0.5}^{0.5} (1-\frac{x}{0.5})^2 \, dx}{\left[\int_{-0.5}^{0.5} (1-\frac{x}{0.5}) \, dx\right]^2} = \frac{0.33}{0.25} = 1.33$$

$$\text{COH.GAIN} = \int_{-0.5}^{0.5} (1-\frac{x}{0.5}) \, dx = 0.5$$

The window function and its transform are shown in Fig. 3.2(a) and (b) respectively. Its main lobe width is 2.0, twice that of the rectangular window, and the side lobe fall-off is 12 dB/octave.

C. COSINE WINDOW

Cosine window is actually one member of a family window with a general form given as:

$$w(x) = \cos^{\alpha}\left(\frac{\Pi}{2a}x\right) \qquad |x| \leq a \qquad (3.8)$$

Type of this window depends upon parameter α, which usually has a value between 1 and 4. Of particular interest in this family window is Hann window, where α=2, which will be considered later.

The cosine window is defined with α=1, as:

$$w(x) = \cos\left(\frac{\pi}{2a}x\right) \qquad |x| \leq a \qquad (3.9)$$

The Fourier Transform of this window is given as:

$$W(f) = \frac{1}{2a} \int_{-a}^{a} \cos \frac{\pi}{2a}x \, e^{-jwx} \, dx$$

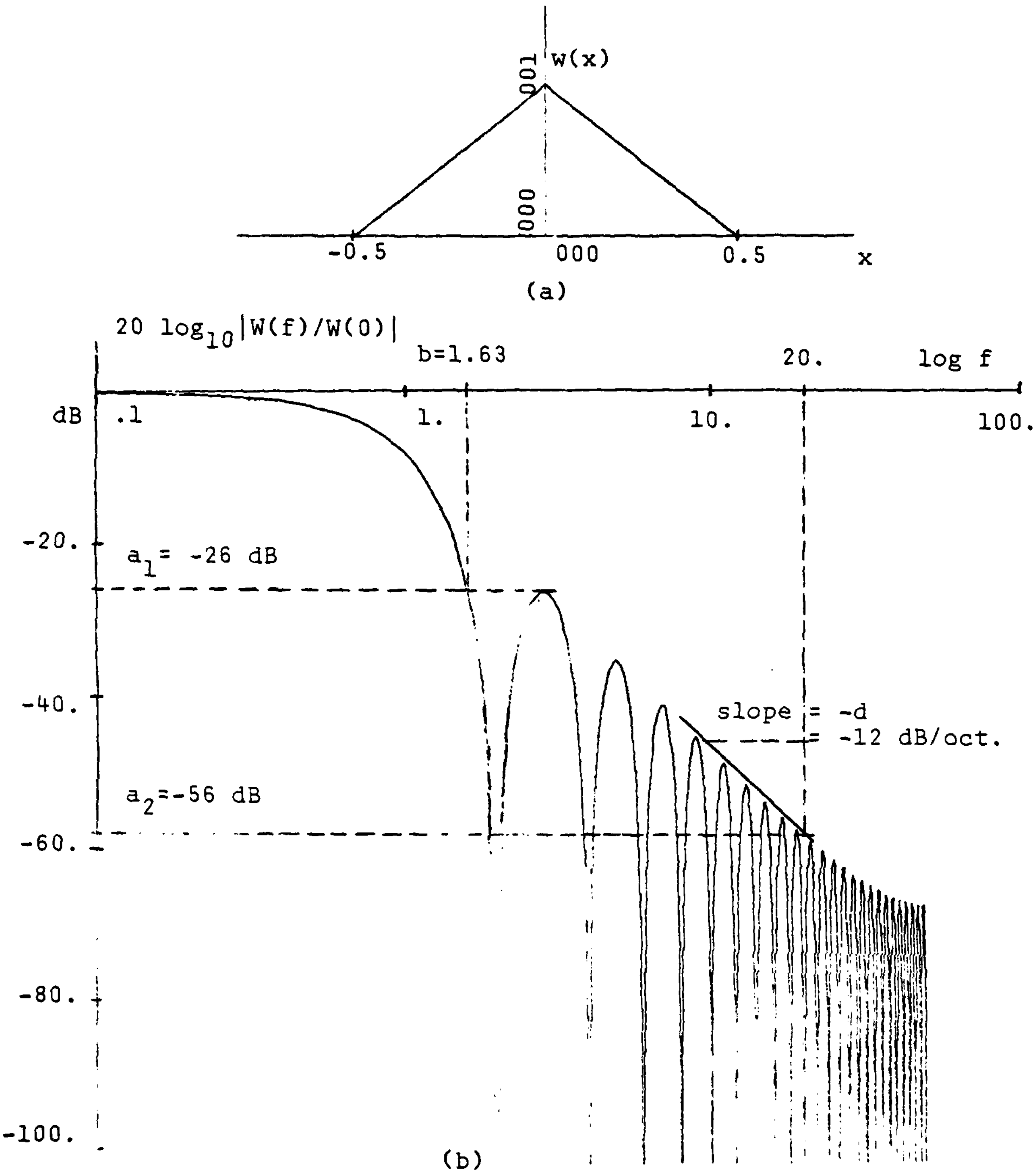


Figure 3.2

(a) Barlett Window, (b) Spectral Window

$$= \frac{1}{2a} \int_{-a}^{a} e^{-j(w-\frac{\pi}{2a})x}\, dx + \frac{1}{2a} \int_{-a}^{a} e^{-j(w+\frac{\pi}{2a})x}\, dx =$$

$$= \frac{1}{2} \frac{\sin a(w-\frac{\pi}{2a})}{a(w-\frac{\pi}{2a})} + \frac{1}{2} \frac{\sin a(w+\frac{\pi}{2a})}{a(2+\frac{\pi}{2a})} =$$

$$= \frac{1}{2} \text{sinc } a(w-\frac{\pi}{2a}) + \frac{1}{2} \text{sinc } a(w+\frac{\pi}{2a}) \qquad (3.10)$$

The transform of this window is a summation of two shifted Dirichlet Kernel.

If the extend of the window is taken as a=0.5, then the parameters ENBW and COH.GAIN are given as:

$$\text{ENBW} = \frac{\int_{-0.5}^{0.5} (\cos\pi x)^2\, dx}{\left[\int_{-0.5}^{0.5} \cos\pi x\, dx\right]^2} = \frac{0.5}{0.404} = 1.23$$

$$\text{COH.GAIN} = \int_{-0.5}^{0.5} \cos\pi x\, dx = \frac{2}{\pi} = 0.64$$

Plots of the window and its transform are given in Fig. 3.3(a) and (b) respectively. From the transform plot, the main lobe width is 1.5 times that of the rectangular window. The first side lobe peak is approximately 23 dB down from the main lobe peak, while the side lobe fall off rate is 12 dB/octave.

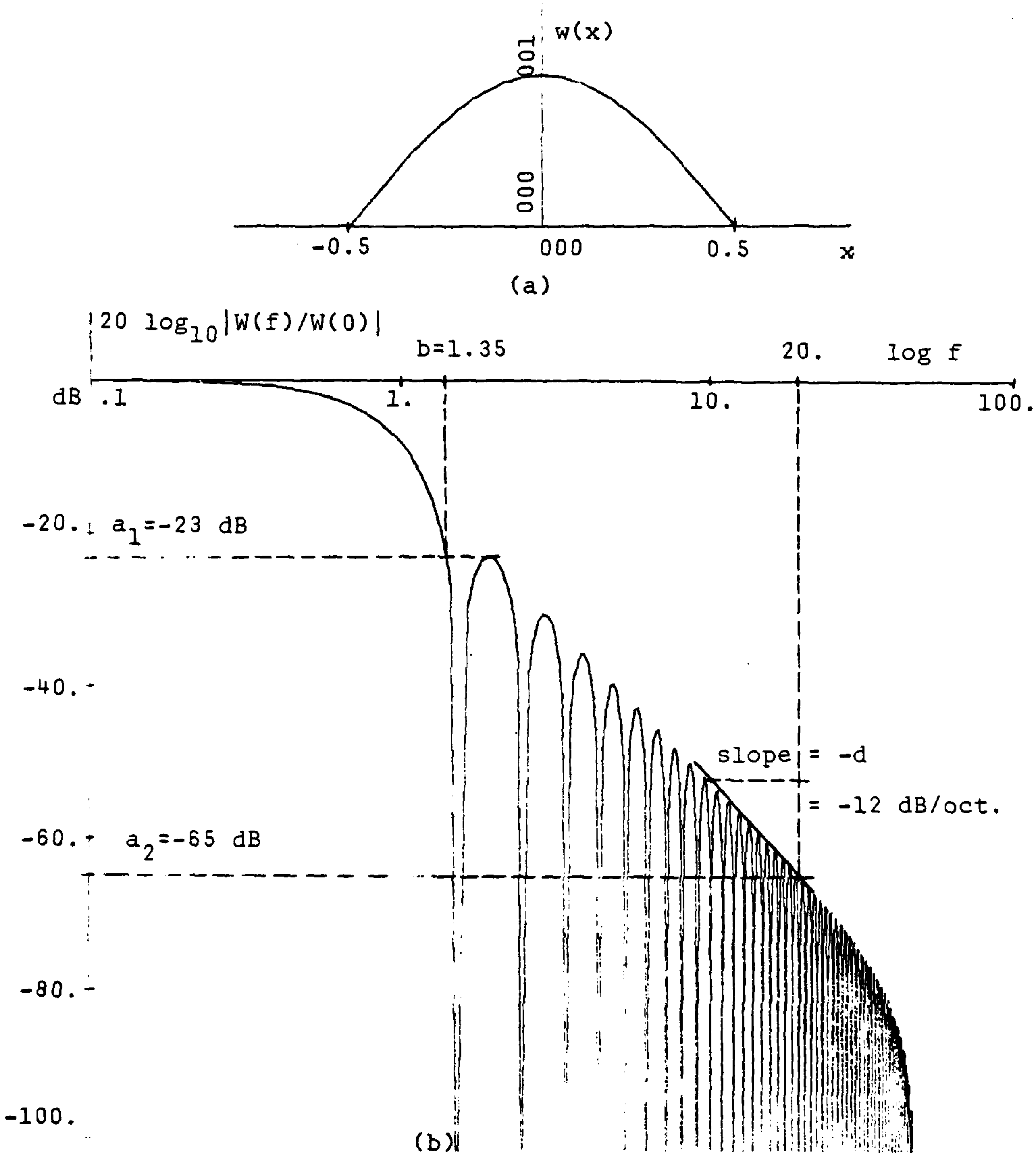


Figure 3.3

(a) Cosine Window, (b) Spectral Window

D. HANN WINDOW

The Hann window is one member of the family windows given by $\cos^{\alpha} x$, where $\alpha=2$. The window equation is written as:

$$w(x) = \cos^2\frac{\pi x}{2a} = \frac{1}{2} + \frac{1}{2}\cos\frac{\pi x}{a} \qquad |x| \le a \tag{3.11}$$

The Fourier Transform of this window is given as:

$$W(f) = \frac{1}{2a}\int_{-a}^{a} (\tfrac{1}{2} + \tfrac{1}{2}\cos\frac{\pi}{a}x)\, e^{-jwx}\, dx =$$

$$= \frac{1}{4a}\int_{-a}^{a} e^{-jwx}\, dx + \frac{1}{4a}\int_{-a}^{a} \cos\frac{\pi}{a}x\, e^{-jwx}\, dx =$$

$$= \frac{1}{2}\frac{\sin aw}{aw} + \frac{1}{4}\frac{\sin a(w-\frac{\pi}{a})}{a(w-\frac{\pi}{a})} + \frac{1}{4}\frac{\sin a(2+\frac{\pi}{a})}{a(w+\frac{\pi}{a})} \tag{3.12}$$

The Transform of this window consists of a summation of three Dirichlet Kernels, one at the origin and the two others shifted $\frac{\pi}{a}$ radians from the origin.

The three terms on the transform of this window are shown in Fig. 3.4, where the shifted Dirichlet Kernels are located at the first zeroes of the center kernel. The peak amplitudes of the shifted kernels are half that of the center kernel; also the side lobes of the translated kernels are about half the size of, and of opposite phase from those of the center kernel. The summation of the three side lobes in phase opposition, tends to cancel the overall side lobe structure.

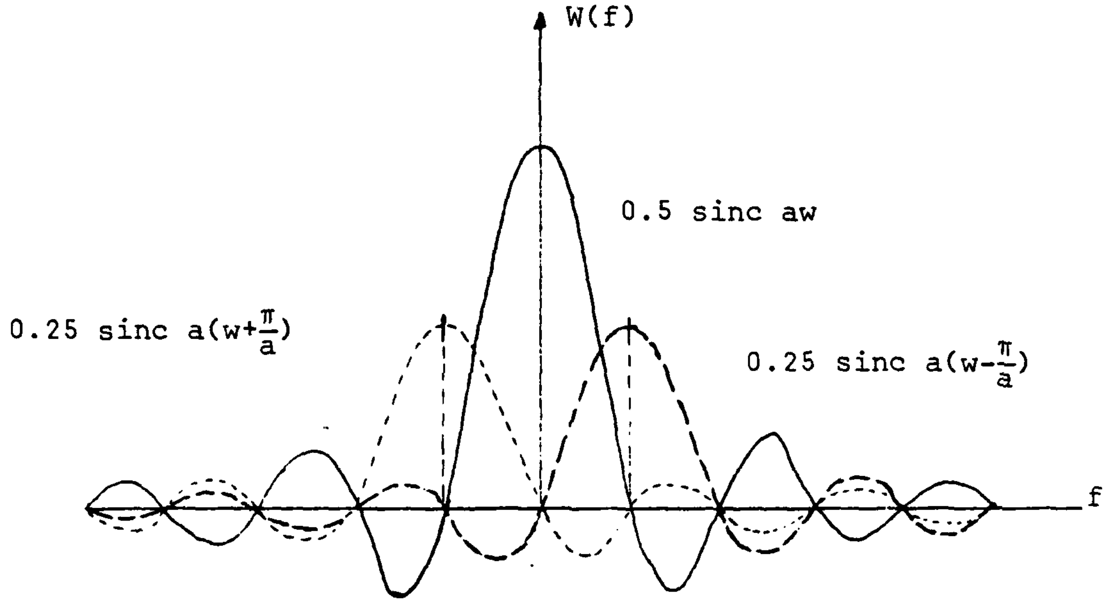


Figure 3.4

Transform Of Hann Window As A Sum
Of Three Dirichlet Kernel

The cancellation of the side lobes structure suggests a constructive technique to define new windows. The best-known windows with this type of side lobes cancellation are Hamming and Blackman windows, which are given in the next section.

If the extent of the Hann window is taken as a=0.5, the parameters ENBW and COH.GAIN are given by:

$$\text{ENBW} = \frac{\int_{-0.5}^{0.5} \left(\frac{1}{2}+\frac{1}{2}\cos\frac{\pi x}{0.5}\right)^2 dx}{\left[\int_{-0.5}^{0.5} \left(\frac{1}{2}+\frac{1}{2}\cos\frac{\pi}{0.5}x\right) dx\right]^2} = \frac{0.375}{0.25} = 1.5$$

$$\text{COH.GAIN} = \int_{-0.5}^{0.5} \left(\frac{1}{2}+\frac{1}{2}\cos\frac{\pi x}{0.5}\right) dx = 0.5$$

The window and its Fourier Transform are plotted in Fig. 3.5. The main lobe width is 2.0, twice that of the rectangular window and the first side lobe level is 23 dB down from the main lobe. The side lobe level fall off rate is 18 dB/octave.

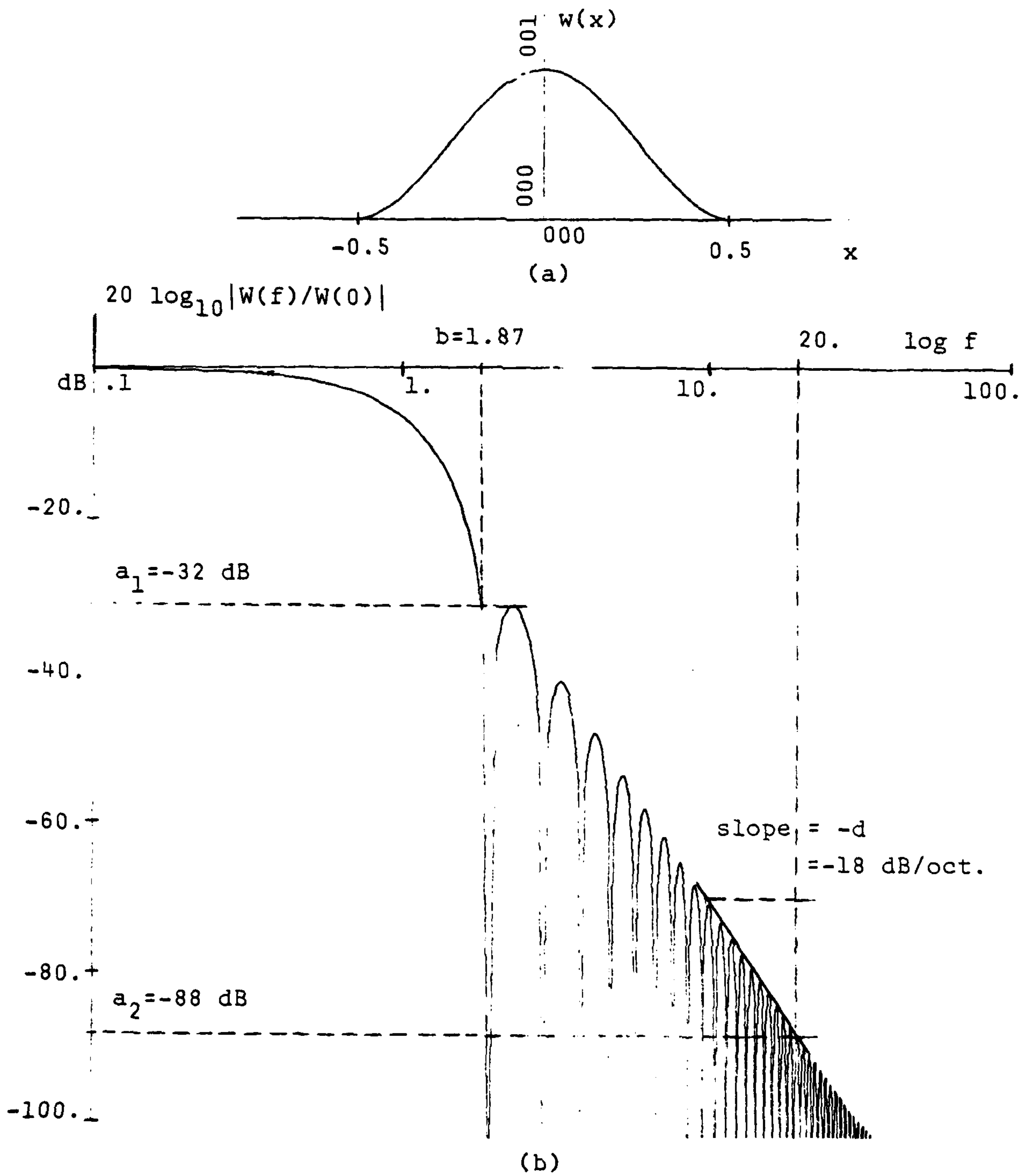


Figure 3.5

(a) Hann Window, (b) Spectral Window

E. HAMMING WINDOW [5]

Hamming window can be thought an an improvement of Hann window, in a sense that the first side lobe cancellation is even greater. From Fig. 3.4, the summation of the three side lobes produce an inexact cancellation. The cancellation can be made more pronounced by adjusting relative size of each term.

Exact cancellation of the side lobes was first shown by Hamming using a window given in general form as:

$$w(x) = \alpha + (1-\alpha)\cos\frac{\pi}{a}x \qquad |x| \leq a \qquad (3.13)$$

where α is a coefficient that will be defined later. The Fourier transform of the window is given as:

$$W(f) = \alpha \text{sinc } aw + 0.5(1-\alpha)\text{sinc } a(w-\frac{\pi}{a})$$
$$+ 0.5(1-\alpha)\text{sinc } a(w+\frac{\pi}{a}) \qquad (3.14)$$

From (3.14) the best cancellation of the first side lobe occurs when $\alpha = \frac{25}{26}$, ($\frac{25}{26} = 0.543478261$). If α is selected as 0.54, an approximation of $\frac{25}{26}$, then an improvement in cancellation of the side lobes occurs. The value α=0.54, gives rise to a window called the Hamming window, which is written as:

$$w(x) = 0.54 + 0.46\cos\frac{\pi x}{a} \quad . \qquad |x| \leq a \qquad (3.15)$$

The Fourier transform then is given as:

$$W(f) = 0.54 \text{ sinc } aw + 0.23 \text{ sinc } a(w-\frac{\pi}{a}) +$$

$$+ 0.23 \text{ sinc } a(w+\frac{\pi}{a}) \quad . \qquad (3.16)$$

If the extent of the window is a = 0.5, then the parameters ENBW and COH.GAIN are given as:

$$ENBW = \frac{\int_{-0.5}^{0.5} (0.54+0.46\cos \frac{\pi}{0.5}x)^2 \, dx}{[\int_{-0.5}^{0.5} (0.54+0.46\cos\frac{\pi}{0.5}x) \, dx]^2} = \frac{0.3974}{0.2916} = 1.36$$

$$COH.GAIN = \frac{\int_{-0.5}^{0.5} (0.54+0.46\cos\frac{\pi}{0.5}x)^2 \, dx}{[\int_{-0.5}^{0.5} (0.54+0.46\cos\frac{\pi}{0.5}x) \, dx]^2} = \frac{0.3974}{0.2916} = 1.36$$

The Hamming window and its transform are ploted in Fig. 3.6(a) and (b) respectively. From its transform plot, the main lobe width is twice that of the rectangular window, and the first side lobe level is 43 dB down from the main lobe. The side lobe fall of rate is 6 dB/octave.

F. BLACKMAN WINDOW [6]

Hann and Hamming windows are examples of windows constructed as the summation of shifted sinc(x) functions.

Blackman suggested a general rule by which window functions could be constructed as follows:

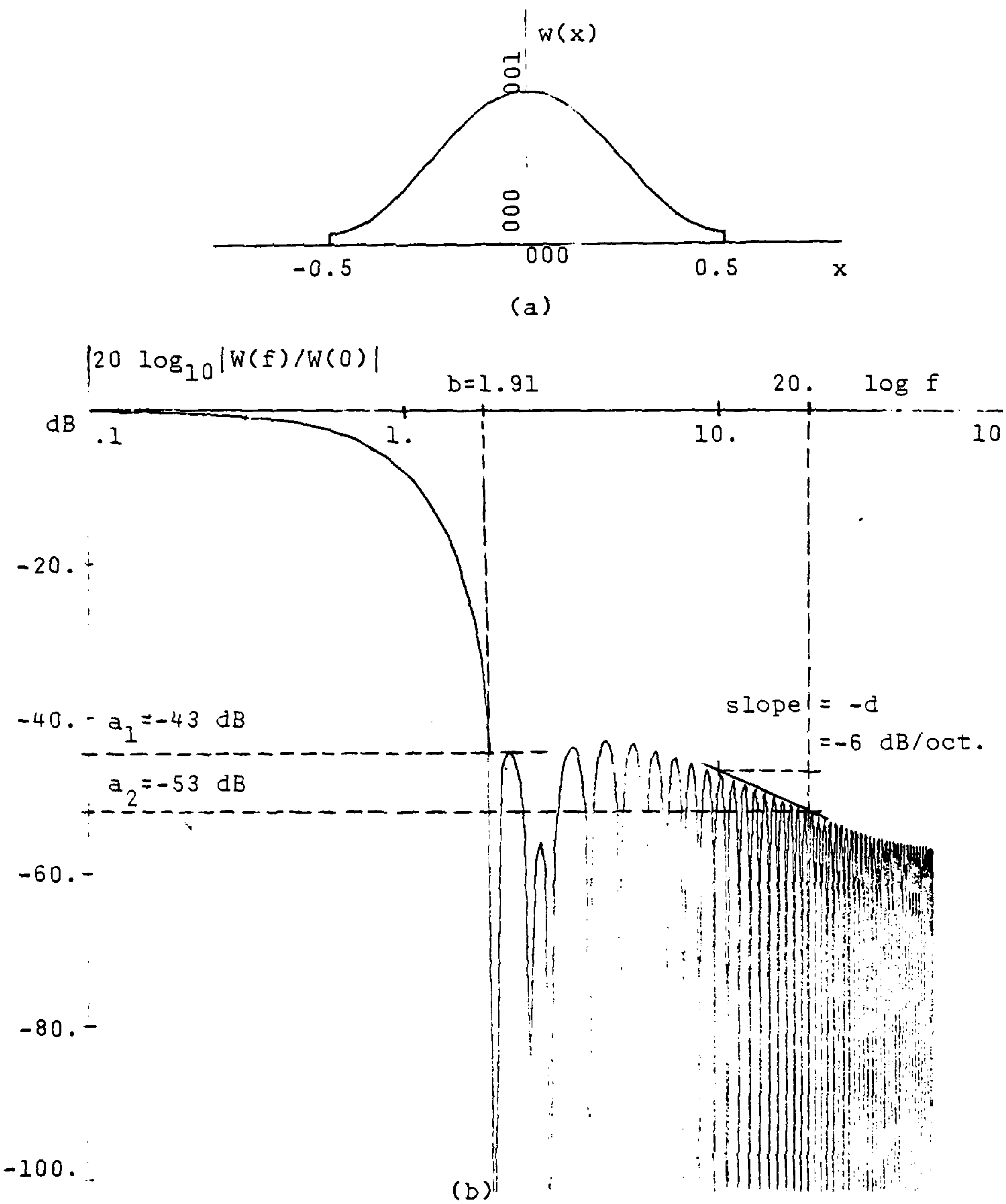


Figure 3.6

(a) Hamming Window, (b) Spectral Window

$$w(x) = \sum_{m=0}^{k} a_m \cos m \left(\frac{\pi x}{a}\right) \qquad |x| \leq a \qquad (3.17)$$

and subject to the constraint:

$$\sum_{m=0}^{k} a_m = 1.0 \qquad (3.18)$$

The Fourier transform of this window becomes:

$$W(f) = \sum_{m=0}^{k} (-1)^m \frac{a_m}{2} \{\text{sinc } a(w-\frac{\pi}{a}m) + \text{sinc } a(w+\frac{\pi}{a}m)\} \qquad (3.19)$$

Hann and Hamming windows are of this form with a_0, and a_1, being non zero. Their spectral windows are summations of three shifted kernels.

Any value of k will produce a new window with a summation of (2k-1) kernels in its tranform, and k non zero corficients, however one way to achieve windows with a narrow main lobe is to restrict k to a small interger.

Blackman examined this window with k=3, and found best cancellation of the third and fourth side lobes occurs when:

$$a_0 = \frac{7938}{18608} = 0.425\ 590\ 71 \cong 0.42$$

$$a_1 = \frac{9240}{18608} = 0.496\ 560\ 62 \cong 0.50$$

$$a_2 = \frac{1430}{18608} = 0.076\ 848\ 67 \cong 0.08$$

The window function which uses these three coeficients is known as the Blackman window, and is given as:

$$w(x) = 0.42 + 0.5 \cos\frac{\pi x}{a} + 0.08 \cos\frac{\pi x}{a}x \qquad |x| \leq a \qquad (3.20)$$

The Fourier transform of this window is:

$$W(f) = 0.42 \text{ sinc } aw + 0.25 \text{ sinc } a(w-\frac{\pi}{a}) + 0.25 \text{ sinc } a(w+\frac{\pi}{a})$$

$$+ 0.04 \text{ sinc } a(w-\frac{2\pi}{a})+0.04 \text{ sinc } a(w+\frac{2\pi}{a})$$

$$(3.21)$$

If the extent of this window is a=0.5, then the ENBW and COH.GAIN parameters are given by:

$$ENBW = \frac{\int_{-0.5}^{0.5} (0.42 + 0.5 \cos \frac{\pi}{a}x + 0.08 \cos \frac{2\pi}{a}x)^2 \, dx}{[\int_{-0.5}^{0.5} (0.42 + 0.5 \cos\frac{\pi}{a}x + 0.08 \cos \frac{2\pi}{a}x) \, dx]^2} =$$

$$= \frac{0.2646}{0.1764} = 1.57$$

$$COH.GAIN = \int_{-0.5}^{0.5} (0.42 + 0.5 \cos \frac{\pi}{a}s + 0.08 \cos \frac{2\pi}{a}x) \, dx = 0.46$$

The window and its transform are shown in Fig. 3.7(a) and (b) respectively. The main lobe width is 3.0, three times that of the rectangular window, and its first side lobe level is 41 dB down from the main lobe. The side lobe fall off rate is 18 dB/octave.

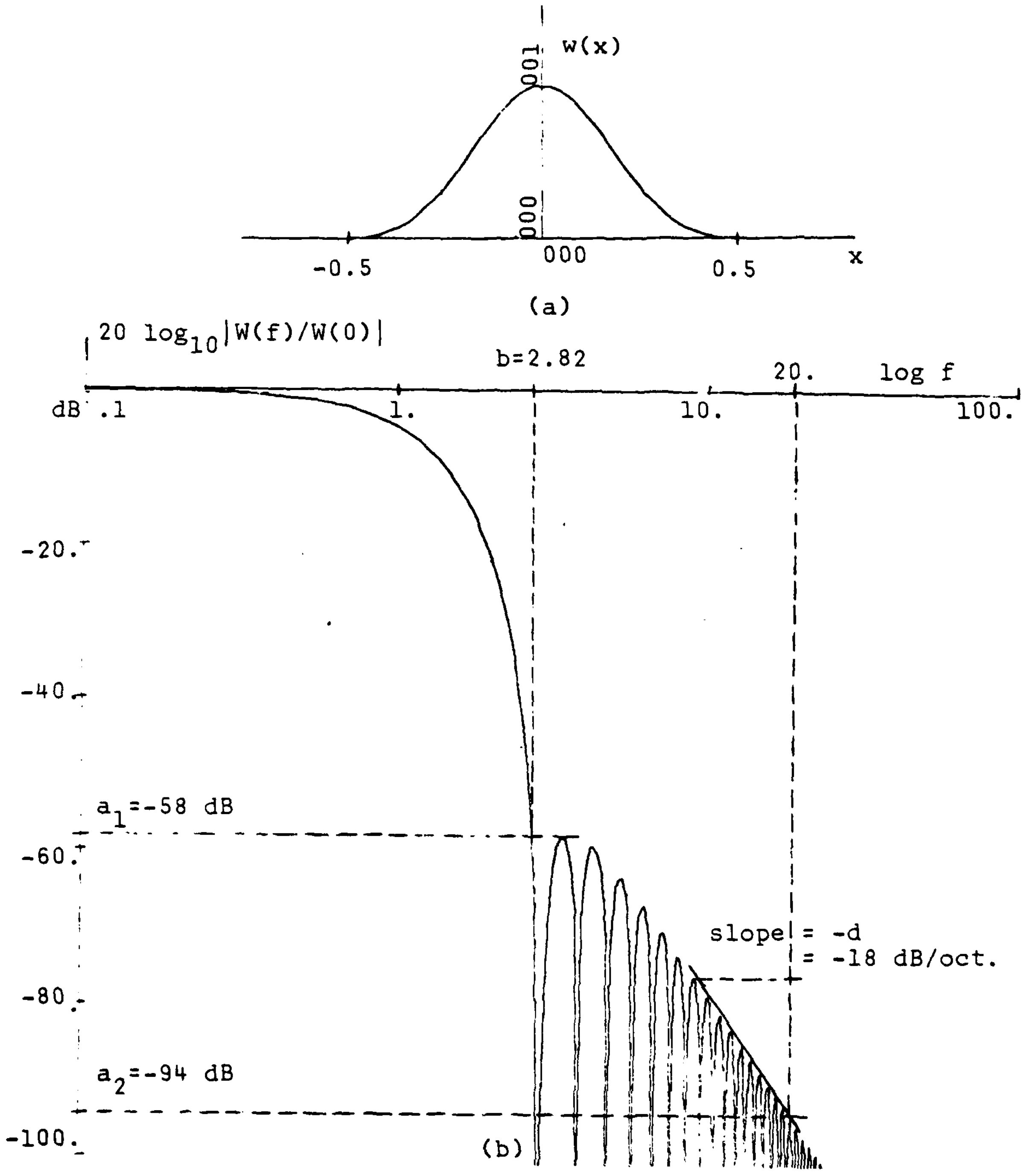


Figure 3.7

(a) Blackman Window, (b) Spectral Window

G. KAISER WINDOW [7]

The Kaiser window is also called a Kaiser-Bessel window and its mathematical expression is given as:

$$w(k) = \frac{I_o[\beta\sqrt{1.0-(\frac{x}{a})^2}]}{I_o[\beta]} \qquad |x| \leq a \qquad (3.22)$$

where $I_o(x)$ = zeroth order modified Bessel-function, and is defined as:

$$I_o(x) = 1 + \sum_{n=1}^{\infty} [\frac{(\frac{x}{2})^n}{n^1}]^2 \qquad (3.23)$$

The Fourier transform of this window is given as:

$$W(f) = \frac{2}{I_o(\beta)} \frac{\sinh[\beta\sqrt{1-(\frac{f}{fa})^2}]}{\beta\sqrt{1-(\frac{f}{fa})^2}} \qquad f < fa \qquad (3.24)$$

where $fa = \beta/\pi$.

From (3.24) if f=fa, the term inside the square root becomes zero and $W(fa) = \frac{2}{I_0(\beta)}$. If f>fa, the term becomes $i\sqrt{(\frac{f}{fa})^2-1}$, and using the identity sinh ix = i sin x, (3.24) becomes:

$$W(f) = \frac{2}{I_o(\beta)} \frac{\sin[\beta\sqrt{(\frac{f}{fa})^2-1}]}{\beta\sqrt{(\frac{f}{fa})^2-1}} \qquad f > fa \qquad (3.25)$$

The usual range of β is between 4 and 9, corresponding to a range of relative peak heights from 3.1% down to 0.047%. As β increases, the side lobes decrease in amplitude while the main lobe widths increase. By varying β the proportionality between the width of the main lobe and the height of the side lobe can be adjusted as desired.

When it is given $\beta=6$ and a=0.5 then the parameters ENBW and COH.GAIN are given as:

$$\mathrm{ENBW} = \frac{\int_{-0.5}^{0.5} \left(\frac{I_o[6\sqrt{1-(\frac{x}{0.5})^2}]}{I_o[6]} \right)^2 dx}{\left[\int_{-0.5}^{0.5} \frac{I_o[6\sqrt{1-(\frac{x}{0.5})^2}]}{I_o[6]} dx \right]^2} = \frac{0.36674}{0.25028} = 1.5$$

$$\mathrm{COH.GAIN} = \int_{-0.5}^{0.5} \frac{I_o[6\sqrt{1-(\frac{x}{0.5})^2}]}{I_o[6]} dx = 0.5$$

Fig. 3.8(a) and (b) shows the window and its transform respectively. The main lobe width is 2.0, twice that of the rectangular window, and the first side lobe level is 43 dB down from the main lobe. The side lobe fall off rate is about 6 dB/octave.

For $\beta=6$ and $\beta=9$ the window resembles to Hamming and Blackman window respectively.

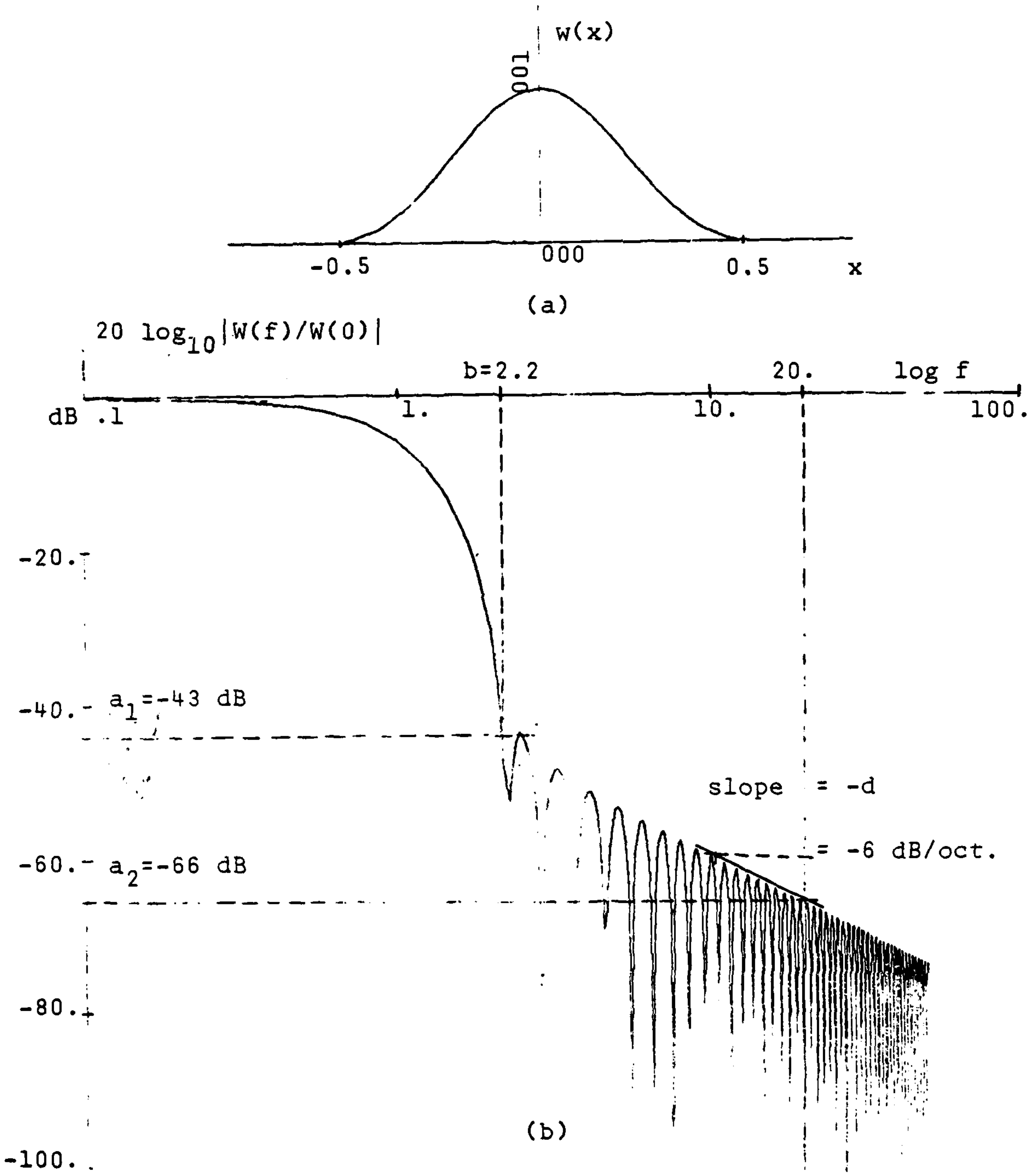


Figure 3.8

(a) Kaiser Window, (b) Spectral Window

TABLE III.1

ONE-DIMENSIONAL WINDOW AND FIGURES OF MERIT

WINDOW	ENBW	COH.GAIN	a_1	a_2	b	d
			dB	dB	Hz	dB/octave
Rectangular	1.0	1.0	-13	-35	0.81	-6
Barlett	1.33	0.5	-26	-58	1.63	-12
Cosine	1.23	0.64	-23	-65	1.35	-12
Hann	1.5	0.5	-32	-88	1.87	-18
Hamming	1.36	0.54	-43	-53	1.91	-6
Blackman	1.57	0.46	-58	-94	2.82	-18
Kaiser	1.5	0.49	-43	-66	2.2	-6

IV. TRANSFORMATION FROM ONE-DIMENSION TO TWO-DIMENSION [4, 8, 9]

The two-dimensional Fourier transform, two-dimensional window functions, and window figures of merit are discussed in this chapter.

A. TWO-DIMENSIONAL FOURIER TRANSFORM [8]

The Fourier transform of f(x,y), a function of two independent variables x and y, is represented by $F\{f(x,y)\}$ or F(u,v), as is defined as:

$$F\{f(x,y)\} = F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j(ux+vy)}dxdy \qquad (4.1)$$

The transform is itself a function of two independent variables u and v, which are generally referred to as frequencies.

Similarly the inverse Fourier transform of a function F(u,v) is represented by $F^{-1}\{F(u,v)\}$ or f(x,y) and is defined as:

$$F^{-1}\{F(u,v)\} = f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{j(ux+vy)}dudv \qquad (4.2)$$

As mathematical operations, the transform and its inverse are very similar differing only in the sign of the exponent appearing in the integral.

In order that the integral exist, the constraints on the function f(x,y) are given as:

1. f(x,y) must be absolutely integrable over the infinite x,y plane.
2. f(x,y) must have only a finite numbers of discontinuities and a finite number of maxima and minima in any finite extent.
3. f(x,y) must have no infinite discontinuites.

The basic definition of the Fourier transform pair leads to a rich mathematical structure associated with the transform operation. Considered below, a few of the basic mathematical properties that will find use in later discussion. These properties are presented as mathematical theorems, followed by a brief statement of their physical significance.

1. Linearity theorem:

$$F\{\alpha f+\beta g\} = \alpha F\{f\} + \beta F\{g\} \qquad (4.3)$$

The transform of a sum of two functions is simply the sum of their individual transforms.

2. Sealing theorem:

$$F\{g(ax,by)\} = \frac{1}{|ab|} G\left(\frac{u}{a}, \frac{v}{b}\right) \qquad (4.4)$$

A stretching of the coordinates in the space domain (x,y) results in a construction of coordinates in the frequency domain (u,v), plus a change in the overall amplitude of the spectrum.

3. Shifting theorem:

$$F\{f(x-a,\ y-b)\} = F(u,v)e^{-j(ua+vb)} \qquad (4.5)$$

A translation of a function in the space domain introduces a linear phase-shift in the frequency domain.

4. Parseval's theorem:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x,y)^{2}\, dxdy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(u,v)|^{2}\, dudv \qquad (4.6)$$

This theorem is interpretable as a statement of conservation of energy.

5. Convolution theorem:

$$F\ \{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(z,n)g(x-z,y-n)dzdn\} = F(u,v)\ G(u,v) \qquad (4.7)$$

The convolution of two functions in the space domain is entirely equivalent to the simpler operation of multiplying their individual transforms.

6. Autocorrelation theorem:

$$F\ \{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(z,n)f^{*}(x-z,y-n)dzdn\} \quad |F(u,v)|^{2} \qquad (4.8)$$

similarly:

$$F\{|f(x,y)|^2\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(z,n)F^*(z+u,n+v)dudv \qquad (4.9)$$

These theorems are special cases of the convolution theorem.

7. Fourier identity:

At each point of continuity of f(x,y):

$$F^{-1} \tau e \{f(x,y)\} = F \; F^{-1}\{f(x,y)\} = f(x,y) \qquad (4.10)$$

The successfive transformation and inverse transformation of a function yields that function again, except at point of discontinuity.

1. Separable and Circularly Symmetric Functions [9]

A function of two independent variable is called separable with respect to a specific coordinate system if it can be written as a product of two functions.

A function f(x,y) is separable in the rectangular coordinate x,y if:

$$f(x,y) = f_1(x)\; f_2(y) \qquad (4.11)$$

A function $f(r,\theta)$ is separable in the polar coordinate r,θ if:

$$f(r,\theta) = f_1(r)\; f_2(\theta) \qquad (4.12)$$

Separable functions are more convenient to deal with than more general functions. Separability allows complicated two dimensional manipulations to be reduced to simpler one dimensional manipulations.

A function separable in rectangular coordinates has the simple property that its Fourier transform is obtained as a product of two one dimensional transforms

$$F\{f(x,y)\} = \int_{-\infty}^{\infty} \int_{\infty}^{\infty} f(x,y)e^{-j(xu+vy)}dxdy$$

$$= \int_{-\infty}^{\infty} f_1(x)e^{-jux}dx \int_{-\infty}^{\infty} f_2(y)e^{-jvy}dy$$

$$= F\{f_1(x)\} \; F\{f_2(y)\} \qquad (4.13)$$

Thus the transform of f(x,y) is itself separable into a product of two factors. The process simplifies to a familiar one dimension manipulation with a function of u, and a function of v.

Functions separable in polar coordinates are not so easily handled as those separable in rectangular coordinates. But it is generally possible to demonstrate that two dimensional manipulations can be performed by means of a series of one dimensional manipulations. The simplest class of functions separable in polar coordinates are those functions which exhibit circular symetry.

A function is said to be a circularly symmetric if it can be written as a function of radius r, independent of the variable θ.

$$f(r,\theta) = f(r) \tag{4.14}$$

Such functions are very important, since a window function can be extended into a two dimensions if it is given a circularly symmetric form. Accordingly, special attention should be paid to these functions.

The Fourier transform of a function in rectangular coordinates is given by:

$$F(u,v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)e^{-j(ux+vy)}dxdy \tag{4.15}$$

If both x,y and u,v coordinates are transformed into a polar coordinate system then:

$$r = \sqrt{x^2+y^2} \qquad x=r\cos\theta$$

$$\theta = \tan^{-1}(\frac{x}{y}) \qquad y=r\sin\theta$$

and

$$\rho = \sqrt{u^2+v^2} \qquad u=\rho\cos\phi$$

$$\phi = \tan^{-1}(\frac{u}{v}) \qquad v=\rho\sin\phi$$

Substituting the polar coordinates into (4.15) and using the transform identity:

$$dxdy = r d\theta dr$$

$$ux+vy = \rho\cos\phi r\cos\phi + \rho\sin\phi r\sin\theta = r\cos(\theta-\phi)$$

The Fourier transform in (4.15) becomes:

$$F(\rho,\phi) = \int_0^{\infty} \int_0^{2\pi} r\, f(r,\theta) e^{-j\rho r \cos(\theta-\phi)}\, d\theta\, dr \qquad (4.16)$$

If the Bessel function identity:

$$J_o(\rho r) = \frac{1}{2\pi} \int_0^{2\pi} e^{-j\rho r\cos(\theta-\phi)}\, d\theta \qquad (4.17)$$

is used to simplify (4.16), then the transform becomes:

$$F(\rho) = 2\pi \int_0^{\infty} r\, f(r)\, J_o(\rho r)\, dr \qquad (4.18)$$

Thus the Fourier transform of a circularly symmetric function is itself circularly symmetric, and can be found by performing a one dimensional manipulation as per (4.18). The expression in (4.18) is known as the Fourier-Bessel transform or as the Hankel transform.

By identical arguments, the inverse Fourier transform of a circularly symmetric function $F(\rho)$ can be expressed as:

$$f(r) = 2\pi \int_0^{\infty} \rho F(\rho)\, J_o(r\rho)\, d\rho \qquad (4.19)$$

There is no difference between the transform and its inverse operation in the case of Hankel transform. Using the notation H{ } to represent the Hankel transform operation:

$$H\,H^{-1}\,\{f(r)\} = H\,H\,\{f(r)\} = f(r) \qquad (4.20)$$

From the Hankel transform pair (4.18) and (4.19), two dimensional Fourier transform may be found from one dimensional mainpulations.

Huang [4] has shown that a good two dimensional window function can be derived from a good one dimensional function, as long as it is circularly symmetric. This is discussed in the next section.

The Hankel transform in (4.18) will be used to calculate the Fourier transform of a two dimensional window.

B. TWO DIMENSIONAL WINDOW FUNCTION [4]

If $W_2(u,v)$ and $w_2(x,y)$ is a two dimensional Fourier transform pair, then its inverse transform relation is given as:

$$w_2(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_2(u,v)e^{j(ux+vy)}dvdu \qquad (4.21)$$

For y=0, (4.21) becomes:

$$w_2(x,o) = \int_{-\infty}^{\infty}\{\int_{-\infty}^{\infty} W_2(u,v)\ dv\}e^{jux}\ du \qquad (4.22)$$

If $w_2(x,y)$ is given as a circularly symmetric function, then $w_2(x,y) = w(\sqrt{x^2+y^2})$. For circularly symmetric function

$$w_2(x,o) = w(x) \tag{4.23}$$

And its inverse Fourier transform is given as:

$$w(x) = \int_{-\infty}^{\infty} W(u)e^{jux}\,du \tag{4.24}$$

From (4.22) and (4.23) it is evident that:

$$W(u) = \int_{-\infty}^{\infty} W_2(u,v)dv \tag{4.25}$$

The expression in (4.25) is called a profile function, and similarly the profile function in v is expressed as:

$$W(v) = \int_{-\infty}^{\infty} W_2(u,v)du \tag{4.26}$$

Now consider the following two equations:

$$H(u) = \begin{cases} 1 & \text{for } u \geq o \\ 0 & u < o \end{cases} \tag{4.27}$$

and

$$H_2(u,v) = \begin{cases} 1 & \text{for } u \geq o \text{ for all } v \\ 0 & u < o \text{ for all } v \end{cases} \tag{4.28}$$

The expressions in (4.27) and (4.28) are one-dimensional and two dimensional step function respectively.

If $W_2(u,v)$ is a two dimensional window's Fourier transform, then its convolution with the step function is given as:

$$W_2(u,v) \circledast H_2(u,v) = \int_{\infty}^{\infty} \int_{\infty}^{\infty} W_2(t,s)H(u-t,\ v-s)dtds$$

$$= \int_{0}^{u} \{ \int_{-\infty}^{\infty} W_2(t,s)ds\}dt \qquad (4.29)$$

where t,s are dummy variables.

Using the profile-function identity given in (4.25) and (4.26), it can be shown that:

$$W(t) = \int_{-\infty}^{\infty} W_2(t,s)ds \qquad (4.30)$$

Equation (4.29) becomes:

$$W_2(u,v) \circledast H_2(u,v) = \int_{0}^{u} W(t)dt \qquad (4.31)$$

Expanding the right hand side of (4.31) into a convolution form:

$$\int_{0}^{u} W(t)dt = \int_{-\infty}^{\infty} W(t)\ H(u-t)dt$$

$$= W(u) \circledast H(u) \qquad (4.32)$$

Then substituting (4.32) to (4.31):

$$W_2(u,v) \circledast H_2(u,v) = W(u) \circledast H(u) \qquad (4.33)$$

Huang states that if a two dimensional function is given in circularly symmetric form, then its Fourier transform satisfies the relation given in (4.33).

To show the benefit of the results in (4.33) consider an example of designing a low pass filter.

EXAMPLE:

The ideal Fourier transform of one dimensional Low Pass Filter is given as:

$$F(u) = \begin{cases} 1 & \text{for } |u| \leq A \\ 0 & |u| > A \end{cases} \qquad (4.34)$$

The impulse response f(x) of the filter is equal to its inverse Fourier transform, which has an infinite duration. In reality, the impulse response has to be finite. By using a window function, the infinite duration of the impulse response is truncated to become a finite impulse response as given as:

$$g(x) = f(x) \cdot w(x) \qquad (4.35)$$

where:

$$w(x) = 0 \qquad \text{for } |x| > a \qquad (4.36)$$

The windowed frequency response of the FIR low pass filter is now equal to the convolution F(u) and W(u):

$$G(u) = F(u) \circledast W(u) \quad (4.37)$$

The ideal Fourier transform of a two dimensional LPF is given as:

$$F_2(u,v) = \begin{cases} 1 & |u^2+v^2| \leq A^2 \\ 0 & |u^2+v^2| > A^2 \end{cases} \quad (4.38)$$

The infinite impulse response $f_2(x,y)$ of this filter has to be truncated using a window function $w_2(x,y)$. The finite impulse response filter is given as:

$$g_2(x,y) = f_2(x,y) \cdot w_2(x,y) \quad (4.39)$$

The windowed frequency response of this filter is given as:

$$G_2(u,v) = F_2(u,v) \circledast W_2(u,v) \quad (4.40)$$

A good window should have the width of its frequency response much smaller than the width of the filter's frequency response. W(u) and $W_2(u,v)$ should be zero for values of $\sqrt{u^2+v^2}$ much smaller than A.

Near a discontinuity in the filter's frequency response F(u), i.e., u = $\pm$A, G(u) is essentially a convolution of W(u) and a step function:

$$G(u) = H(u) \circledast W(u) \tag{4.41}$$

Similarly near a discontinuity in the filter's frequency response $F_2(u,v)$, i.e., $u^2+v^2 = A^2$, $G_2(u,v)$ is essentially a convolution of $W_2(u,v)$ and a step function:

$$G_2(u,v) = H_2(u,v) \circledast W_2(u,v) \tag{4.42}$$

From Huang's theorm given in (4.33), $G_2(u,v)=G(u)$ if only $w_2(x,y) = w(\sqrt{x^2+y^2})$.

This means that if w(x) is a good window, then, $w_2(x,y)$ is also a good two dimensional window.

If other words, a good two dimensional window can be derived from a good one dimensional window by extending it into a circularly symmetric function.

C. TWO DIMENSIONAL WINDOW FIGURE'S OF MERIT

The window's figures of merit found in the one dimensional case are now applied to the two dimensional case. The same figures of merit that may be used for comparison among windows, will be re-examined to determine if there should be any changes for the two dimensional case.

1. <u>Equivalent Noise Band Width (ENBW)</u>

The ENBW in one dimensional case is defined as the width of a rectangular filter with the same peak power gain that passes the same noise power.

A rectangular filter in two-dimension represents a cross-section of a cylindrical filter. The noise power passed by the area under the rectangular filter, becomes noise power passed by the volume under the cylindrical filter.

If the peak power gain is the same, then the volume divided by the height, results in the area covered by the base of the cylinder, which is a circle.

Two-dimensional ENBW is defined as the circumference of the horizontal cross-section of a cylindrical filter, that passes the same noise power.

The vertical cross-section of a two dimensional window is equal to a one dimensional window. Consequently the width of the equivalent rectangular window is equal to the diameter of the equivalent cylindrical window. Then the two dimensional ENBW is given as:

$$(ENBW)_{2D} = \pi(ENBW)_{1D} \tag{4.42}$$

For the two dimensional case, the definition of ENBW should be changed from the width of a rectangular to the circumference of a cylinder, and the value can be calculated directly from the one dimensional case.

2. Coherent Gain

Two dimensional coherent gain is best defined using two dimensional sinusoidal signal. The input signal is defined as:

$$f(x,y) = A_{x,y}e^{j(ux+vy)} + n(x,y) \tag{4.43}$$

where n(x,y) is two dimensional white noise.

The output signal or the windowed signal is given as:

$$f_w(x,y) = Aw(x,y)e^{j(ux+vy)}+w(x,y)n(x,y) \tag{4.44}$$

The two dimensional frequency response of the windowed signal is given as:

$$F_w(x,y) = \int_{-a}^{a}\int_{-a}^{a} Aw(x,y)e^{j(ux+vy)}e^{-j(ux+vy)}dxdy$$

$$= A\int_{-a}^{a}\int_{-a}^{a} w(x,y)dxdy. \tag{4.45}$$

The amplitude of the windowed signal frequency response is proportional to the original amplitude A. The proportionality factor is the double integral of the window function.

If the window function is given in circularly symmetric form, then the proportionality factor becomes:

$$\int_{-a}^{a}\int_{-a}^{a} w(x,y)dxdy = \int_{o}^{a}\int_{o}^{2\pi} w(r)r\, d\theta dr$$

$$= \int_{o}^{a} r\, w(r) dr \int_{o}^{2\pi} d\theta$$

$$= 2\pi \int_{o}^{a} r\, w(r) dr \qquad (4.46)$$

The proportionality factor for a rectangular window is equal to πa^2. The proportionality factor for the other window is reduced due to the window smoothly going to zero near the boundaries.

If the proportionality factor in (4.46) is normalized by factor πa^2 (that is, divided by the area covered by the base of the window), then the proportionality factor becomes coherent gain:

$$\text{COH.GAIN} = \frac{2}{a^2} \int_{o}^{a} r\, w(r) dr \qquad (4.47)$$

The coherent gain given in (4.47) above is used as window's figure of merit.

3. <u>Trade-off Parameters</u>

Two dimensional trade-off parameters are similar in definition with the ones used in the one dimensional case. The trade-off parameter a_1, a_2, b and d are derived from the frequency response of a window, which is found using the Hankel transform.

The Hankel transform allows a two dimensional transform to be found from a one dimensional manipulation. Consequently the definitions of the trade-off parameters doesn't change. The graphical interpretation of the parameters are shown in Fig. 4.1.

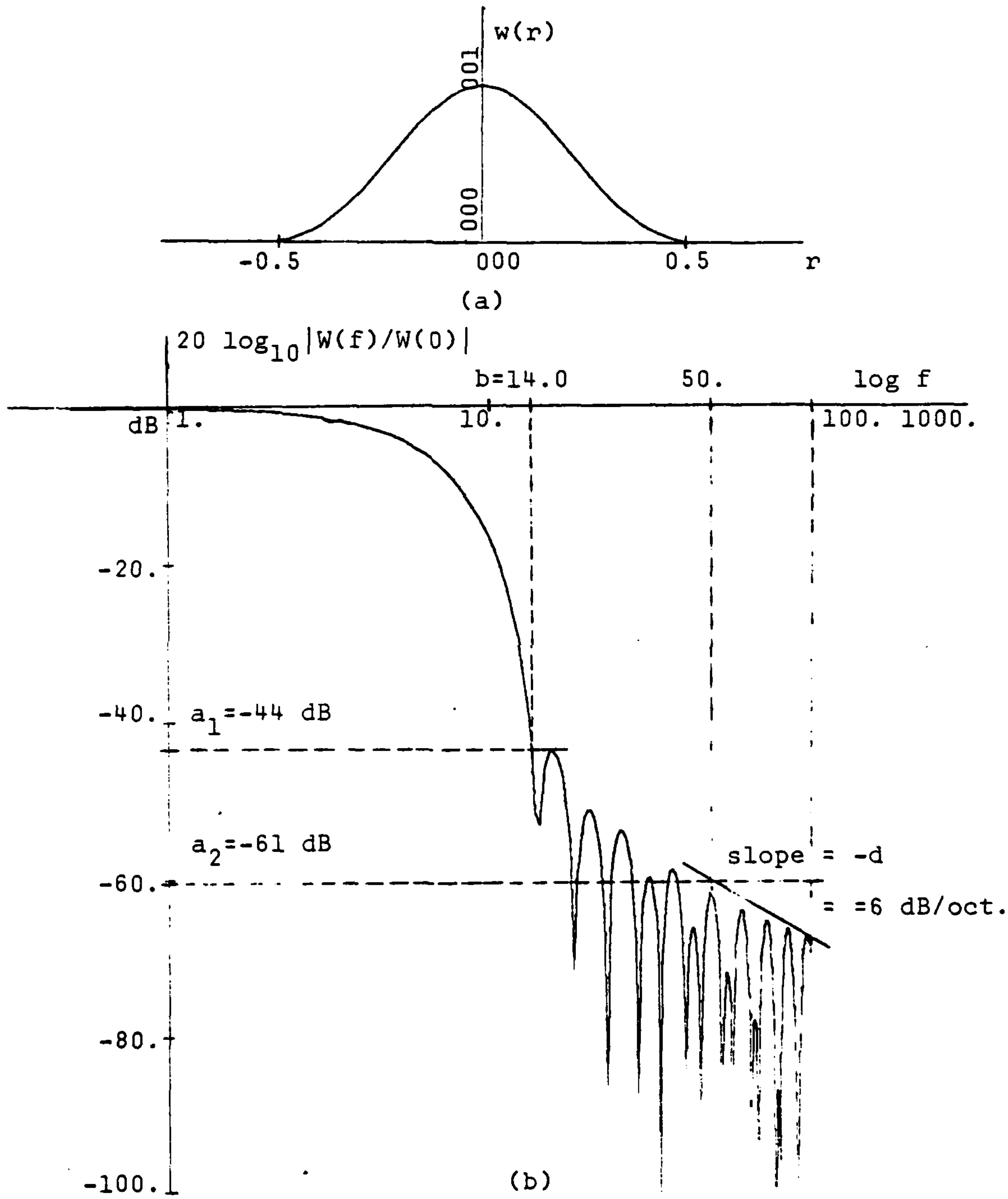


Figure 4.1

Plot Of Kaiser Window, Used To Illustrate The Definition Of Trade-Off Parameters a_1, a_2, b and d

V. TWO DIMENSIONAL WINDOW FUNCTIONS

The two dimensional extension of the previous window functions will be given in this chapter. The window's frequency response is found using Hankel transform. Unfortunately it is hard to present the Hankel transform of a window in a closed form. Only two transforms are presented in closed form, they are Rectangular and Barlett windows, the rest are presented in their original integral form.

Without loss of generality, the window function is given as w(r), a two dimensional circularly symmetric function, restricted to the interval $|r| \leq a$, while the window is set to unity at the origin.

$$w(o) = \int_{-\infty}^{\infty} f\, W(f)\, df = 1 \qquad (5.1)$$

$$w(r) = o \qquad \text{for} \quad |r| > a \qquad (5.2)$$

The frequency response of this window is given as:

$$W(f) = \int_{o}^{\infty} r\, w(r)\, J_o(fr)\, dr \qquad (5.3)$$

The frequency response consists of a main lobe and side lobes which posses a large amplitude ratio in a linear scale. The frequency response plot is given in log-frequency scale versus log-normalized amplitude (W(f)/W(0)).

A. RECTANGULAR WINDOW [2]

The two dimensional version of the circularly symmetric rectangular window is given as follows:

$$w(r) = \begin{cases} 1 & \text{for } |r| \leq a \\ 0 & \text{for } |r| > a \end{cases} \qquad (5.4)$$

This window is cylindrical, with the height equal to one and the diameter of the base equal to 2a. The vertical cross-section of this window is equal to a rectangular window.

The frequency response of this window is given by the Hankel transform as:

$$W(f) = \int_o^a r\, J_o(fr)\, dr \qquad (5.5a)$$

The recurrence formulae is used to solve the integral in (5.5a), where: [10]

$$\frac{d}{dx}\{x^n J_n(x)\} = x^n J_{n-1}(x)$$

$$\frac{d}{dx}\{x\, J_1(x)\} = x\, J_o(x)$$

Then the integral form in (5.5a) reduces to:

$$W(f) = \frac{1}{f_2} \int_o^a (fr) J_o(fr)\, d(fr)$$

$$= \frac{1}{f^2} \int_o^a d(fr) J_1(fr) = \frac{(fr)}{f^2} J_1(fr) \Big|_o^a$$

$$= a^2 \frac{J_1(af)}{af} \quad . \qquad (5.5b)$$

The window frequency response $\{\frac{J_1(af)}{(af)}\}$ has a form similar to $(\frac{\sin x}{x})$ form. The value of this function at the origin, f=0, is one. The envelope of the window frequency response crosses the frequency axis, whenever $J_1(af)=0$. The first zero-crossing occurs at the value of af=3.8. If the value of a is increased (meaning the window impulse response is lengthened), the zero-crossing occurs at smaller value of f (meaning the window frequency response is narrower). If the value of a is decreased (meaning the window impulse response is shortened), the zero-crossing occurs at larger value of f (meaning the window frequency response is wider).

At the value of a=0.5, the first zero-crossing occurs at the value of f=7.6.

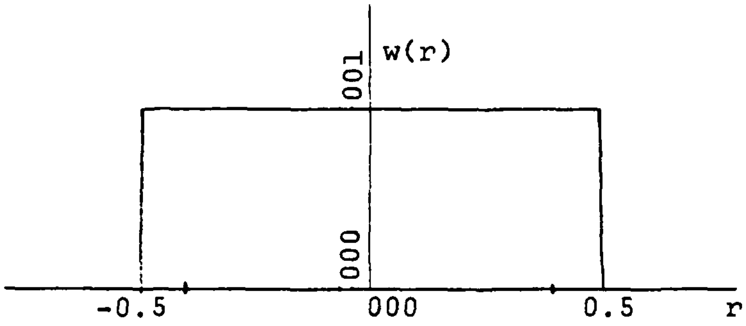
The ENBW of this window with the value of a is given as a=0.5:

$$\text{ENBW} = \pi(1) = 3.14$$

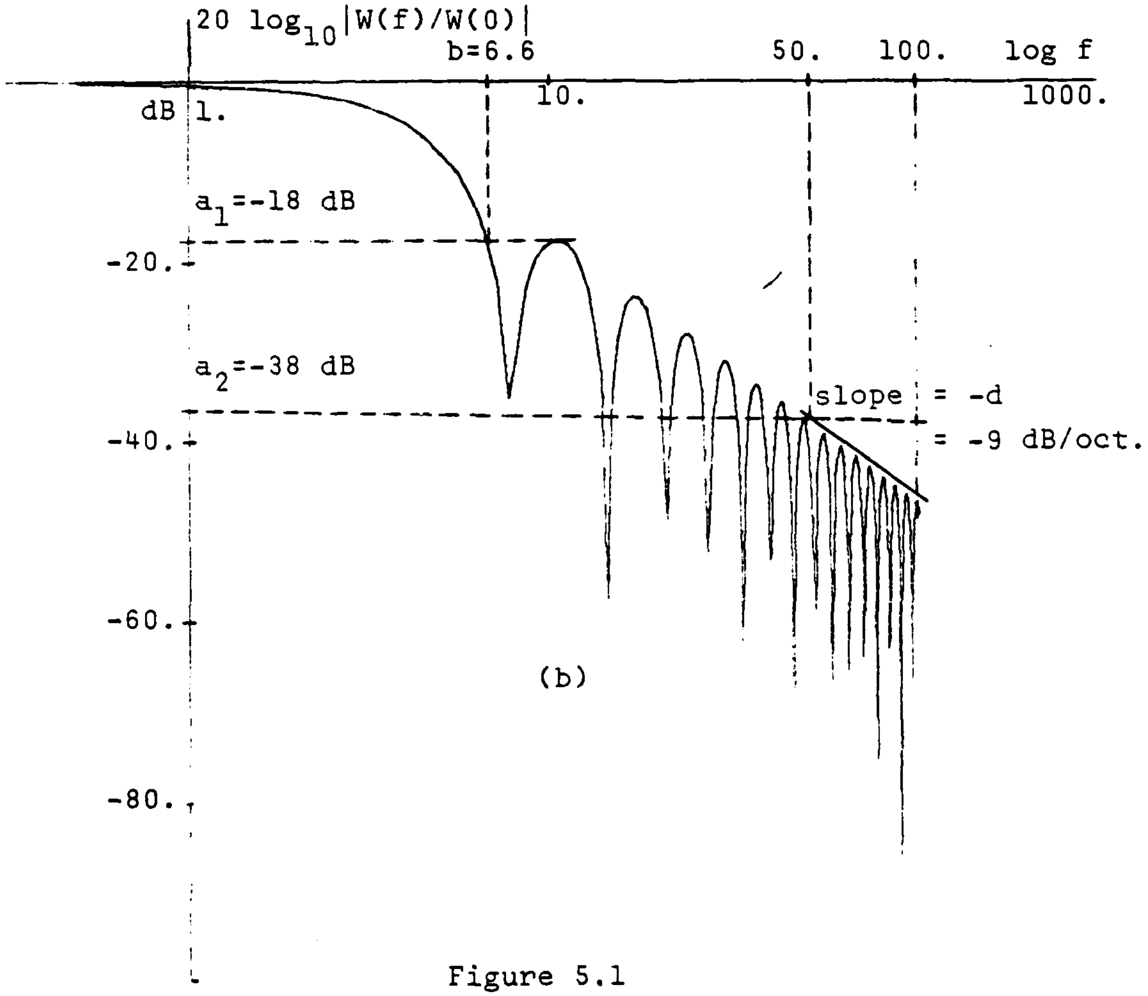
The COH.GAIN is given as:

$$\text{COH.GAIN} = \frac{2\pi}{\pi(0.5)^2} \int_o^{0.5} r\, dr = \frac{0.78}{0.78} = 1.0$$

The window and its frequency response are given in Figure 5.1(a) and (b) respectively. The main lobe width (the width between the origin and the first zero-crossing) is 7.6. The first side lobe level in 18 dB down from the main lobe peak and the side lobe fall-off rate is 9 dB/octabe.



(a)



(b)

Figure 5.1

(a) 2-D Rectangular Window, (b) 2-D Spectral Window

B. BARTLETT WINDOW

The two dimensional circularly zymetric Bartlett window is given as:

$$w(r) = \begin{cases} 1 - \frac{|r|}{a} & |r| \leq \\ 0 & \text{elsewhere} \end{cases} \tag{5.7}$$

The shape of this window function is similar to a cone, with the height equal to one, and the base in a circle with a radius r=a. The vertical cross-section of this window is equal to a one-dimensional Bartlett window.

The frequency response of this window is given by the Hankel transform as follows:

$$W(f) = \int_o^a r(1 - \frac{|r|}{a})\, J_o(fr)\, dr$$

$$= \int_o^a r\, J_o(fr) dr - \int_o^a \frac{r^2}{a}\, J_o(fr)\, dr$$

$$= a\, \frac{J_1(af)}{f} - \frac{1}{a} \int_o^a r\, \frac{d\ (fr)J_1(fr)}{f^2}$$

$$= a\, \frac{J_1(af)}{f} - a\, \frac{J_1(af)}{f} + \frac{1}{a} \int_o^a \frac{(fr)J_1(fr)}{f^2}\, dr$$

$$= \frac{1}{a} \int_o^a \frac{(fr)^2 J_1(fr)}{f^4 r}\, d(fr)$$

$$= \frac{1}{a} \int_o^a \frac{d(fr)^2 J_2(fr)}{f^4 r}$$

$$= \frac{1}{fr} J_2(af) - \frac{1}{a} \int_o^a (fr)^2 J_2(fr)\, d\left(\frac{1}{f^4 r}\right)$$

$$= \frac{J_2(af)}{f^2} + \frac{1}{af^3} \int_o^a J_2(fr)\, d(fr)$$

$$= \frac{J_2(af)}{f^2} + \frac{1}{af^3} \left\{ \int_o^a J_o(fr)\, d(fr) - 2J_1(af)\right\}$$

$$= \frac{1}{af^3} \{af\, J_2(af) - 2\, J_1(af)\} + \frac{1}{af^3} \int_o^a J_o(fr)\, d(fr)$$

$$= \frac{1}{af^3} \{-af\, J_o(af)\} + \frac{a^2}{(af)^3} \int_o^a J_o(fr)\, d(fr)$$

$$W(f) = a^2\left\{\frac{-1}{(af)^2} J_o(af) + \frac{1}{(af)^3} \int_o^a J_o(fr)\, d(fr)\right\} \quad (5.8)$$

If ρ=af is substituted into (5.8) then the end result is:

$$W(\rho_a) = a^2\{\rho^{-3} \int_o^\rho J_o(t)dt - \rho^{-2} J_o(\rho)\} \quad (5.9)$$

where t is a dummy variable of integration.

The window and its frequency response is given in Figure 5.2(a) and (b) respectively, where a=0.5. The ENBW and COH.GAIN parameters are given as:

$$\text{ENBW} = \pi(1.33) = 4.18$$

$$\text{COH.GAIN} = \frac{2\pi}{M(0.5)^2} \int_o^{0.5} r\left(1 - \frac{|r|}{0.5}\right) dr = \frac{0.26}{0.78} = 0.34$$

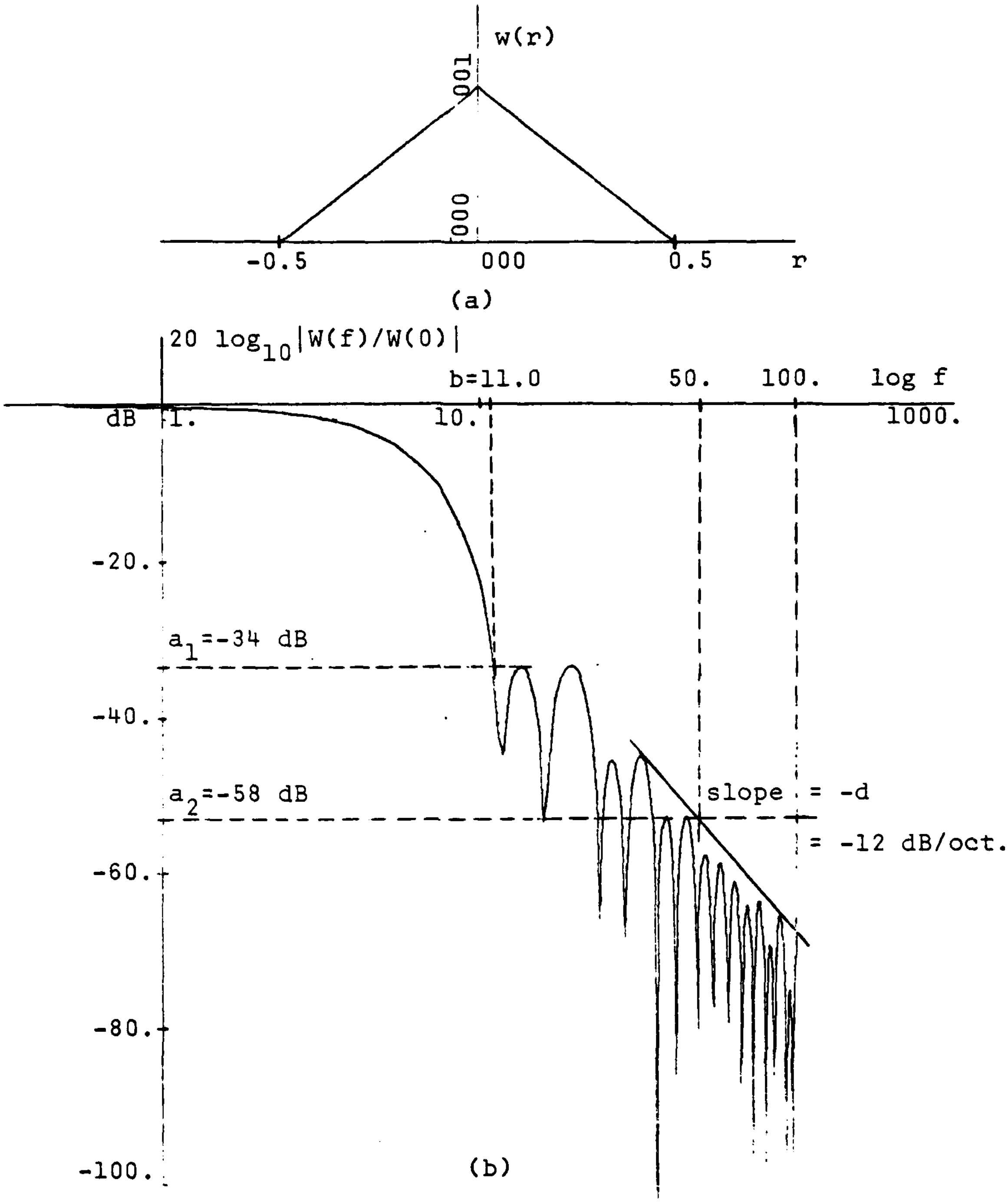


Figure 5.2

(a) 2-D Barlett Window, (b) 2-D Spectral Window

The main lobe width is about 11.5. The first side lobe level is 34 dB down from the main lobe level, and the side lobe fall off rate is 12 dB/octave.

The term $\int_o^{\rho} J_o(t)dt$ in (5.9) is hardly a closed form, but it is tabulated. Then the rectangular and relatively the Bartlett windows are two-dimensional windows that have a closed-form frequency response.

C. COSINE WINDOW

The third two-dimensional window to be discussed is a cosine window. The two dimensional circularly symmetric window is given as:

$$w(r) = \begin{cases} \cos \frac{\pi}{2a} r & \text{for } |r| \leq a \\ o & \text{elsewhere} \end{cases} \qquad (5.10)$$

The shape of this window is similar to a half sphere and the height at r=o is one. The base of the window is a circle with radius r=a.

The frequency response of the window is given as a Hankel transform with no closed form.

$$w(f) = \int_o^a r \cos \frac{\pi}{2a} r \, J_o(fr) \, dr \qquad (5.11)$$

The effort to derive a closed form is given in the Appendix.

If a=0.5, then the window and its frequency response are given in Figure 5.3(a) and (b) respectively. The parameters

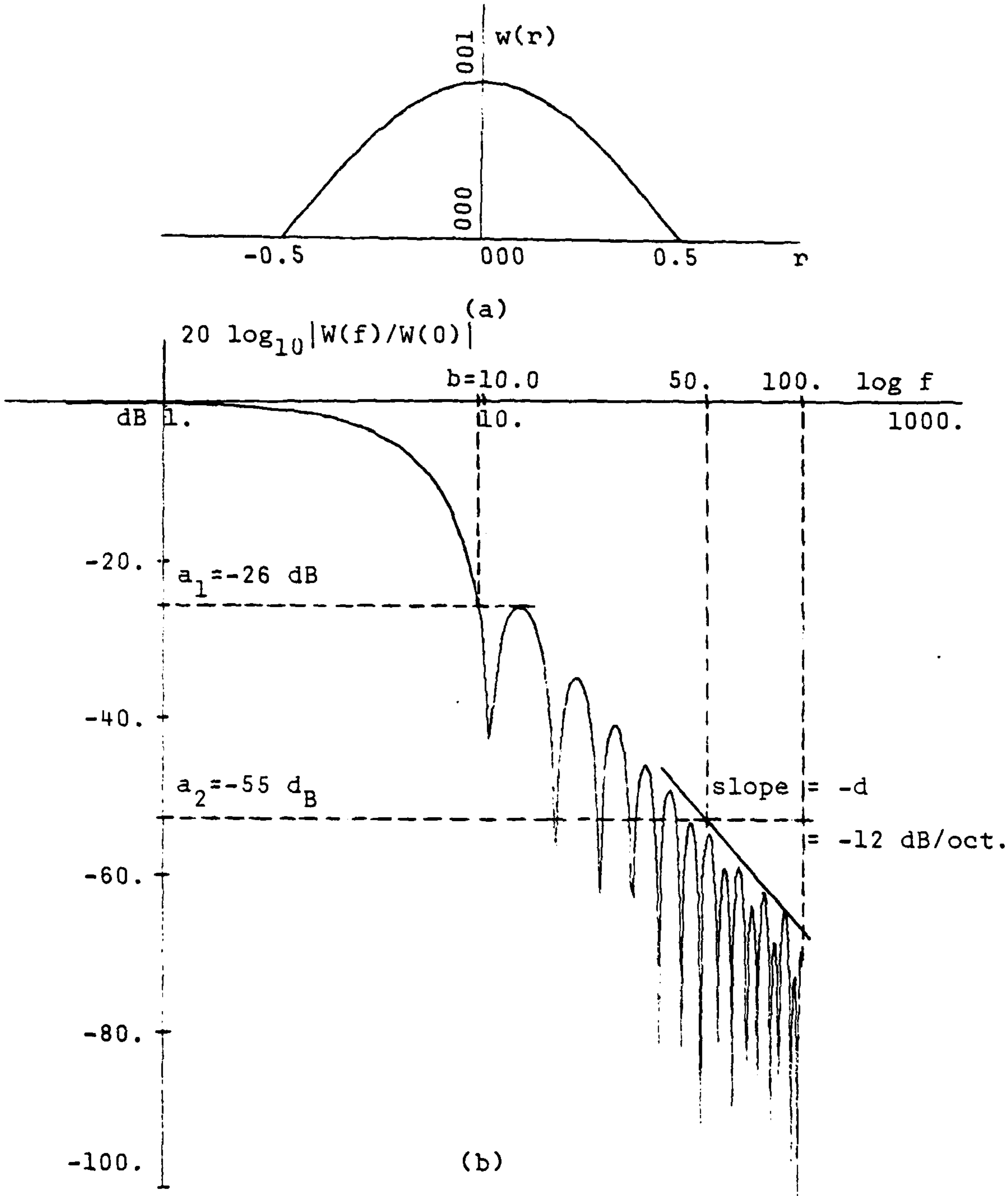


Figure 5.3

(a) 2-D Version Of Cosine Window (cross-section),
(b) 2-D Spectral Window

ENBW and COH.GAIN are given as:

$$\text{ENBW} = \pi(1.33) = 3.86$$

$$\text{COH.GAIN} = \frac{2\pi}{\pi(0.5)^2} \int_{o}^{0.5} r \cos \pi r \, dr = \frac{0.36}{0.78} = 0.46$$

The main lobe width is 10.5 while the first side lobe level is 26 dB down from the main lobe and the side lobe fall off rate is 12 dB/octave.

D. HANN WINDOW

The two dimensional circularly symmetric Hann window is given as:

$$w(r) = \begin{cases} \cos^2 \frac{\pi}{2a} r = 0.5+0.5 \cos \frac{\pi}{a} r & |r| \leq a \\ o & \text{elsewhere} \end{cases} \quad (5.12)$$

The vertical cross-section of the window is equal to a one dimensional Hann window.

The frequency response is given by the Hankel transform as:

$$w(f) = \int_{a}^{o} r(0.5+0.5 \cos \frac{\pi}{a} r) \, J_o(fr) \, dr$$

$$= 0.5 \, a \frac{J_1(af)}{f} + 0.5 \int_{o}^{a} \cos \frac{\pi}{a} r \, J_o(fr) \, dr \quad (5.13)$$

The first term in (5.13) has a closed form as in the case of a Rectangular window. The second term has no closed form as in the case of the cosine window.

For a window length of a=0.5, the window and its frequency response are given in Figure 5.4(a) and (b) respectively. The ENBW and COH.GAIN parameters are given as:

$$\text{ENBW} = \pi(1.5) = 4.7$$

$$\text{COH.GAIN} = \frac{2\pi}{\pi(0.5)^2} \int_0^{0.5} r \cos^2 \pi r \, dr = \frac{0.23}{0.78} = 0.29$$

The main lobe width is 13.7 and the first side lobe level is 34 dB down from the main lobe level. The fall-off rate of the side lobes is 18 dB/octave.

E. HAMMING WINDOW

The two dimensional circularly symmetric Hamming window is given as:

$$w(r) = \begin{cases} 0.54 + 0.46 \cos \frac{\pi}{a} r & |r| \leq a \\ 0 & \text{elsewhere} \end{cases} \qquad (5.14)$$

The shape of this window is a smooth cone sitting on a cylindrical base. The vertical cross-section of the window is equal to the one dimensional case.

The frequency spectrum of the window is given by the Hankel transform as:

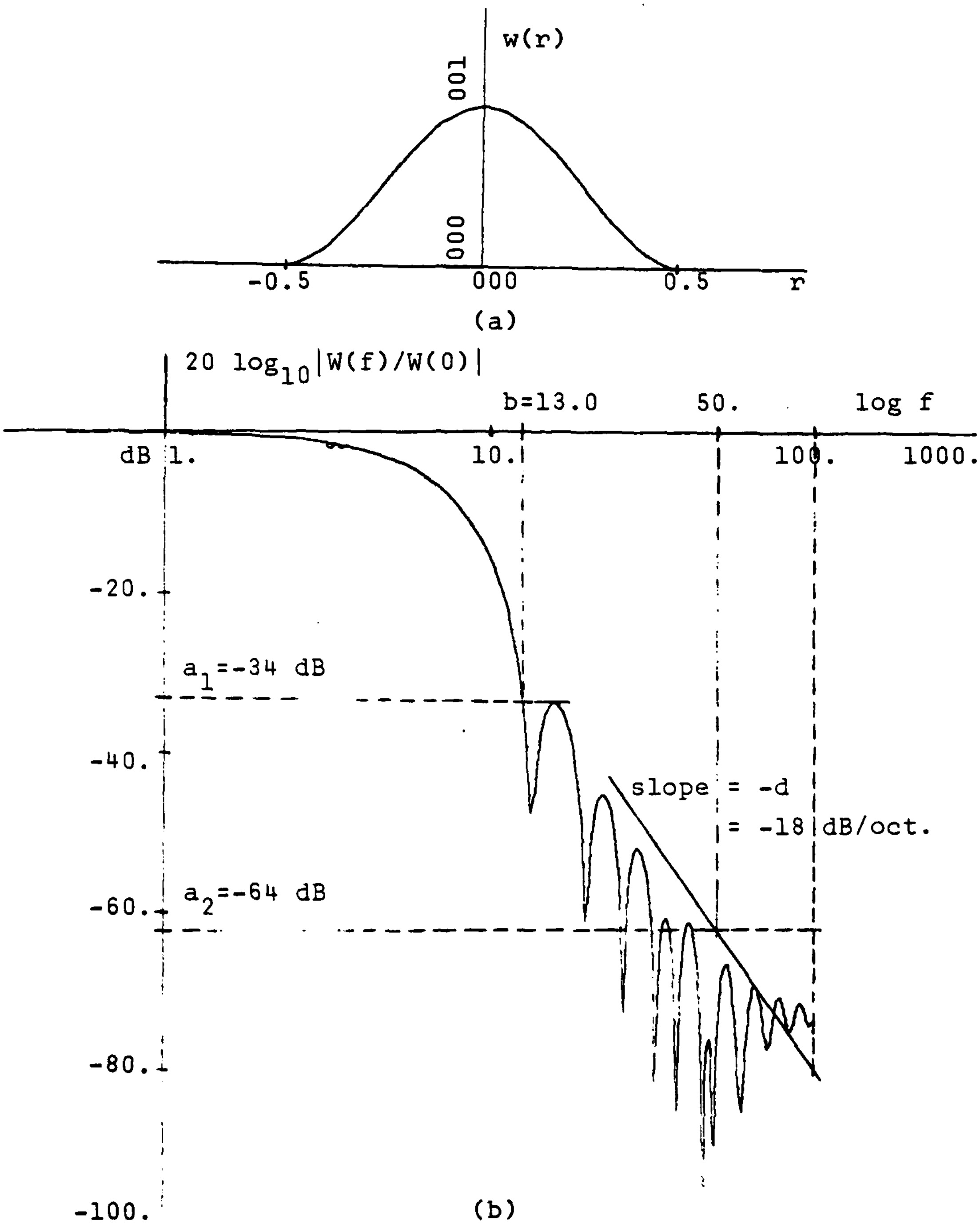


Figure 5.4

(a) Von Hann Window in 2-D, (b) Spectral Window

$$w(f) = \int_o^a r(0.54 + 0.46 \cos \frac{\pi}{a} r)\, J_o(fr)\, dr$$

$$= 0.54\, a \frac{J_1(af)}{f} + 0.46 \int_o^a r \cos \frac{\pi}{a} r\, J_o(fr)\, dr \qquad (5.15)$$

The first term in (5.15) is the familiar frequency response of the rectangular window and the second term is the response of a cosine window.

If the length of the window is a=0.5, then the window and its frequency response are given in Figure 5.5(a) and (b) respectively. The ENBW and COH.GAIN parameters are given as:

$$\text{ENBW} = \pi(1.36) = 4.3$$

$$\text{COH.GAIN} = \frac{2\pi}{\pi(0.5)^2} \int_o^{0.5} r(0.54 + 0.46 \cos \frac{\pi r}{0.5})dr = \frac{0.28}{0.78} = 0.36$$

The main lobe width is equal to 13.5 and the first side lobe level is 47 dB down from the main lobe level. The side lobe fall off rate is 6 dB/octave.

F. BLACKMAN WINDOW

The two dimensional circularly symmetric Blackman window is given as:

$$w(r) = \begin{cases} 0.42+0.5 \cos\frac{\pi r}{a} + 0.08 \cos\frac{2\pi}{a} r & |r| \leq a \\ o & \text{elsewhere} \end{cases} \qquad (5.16)$$

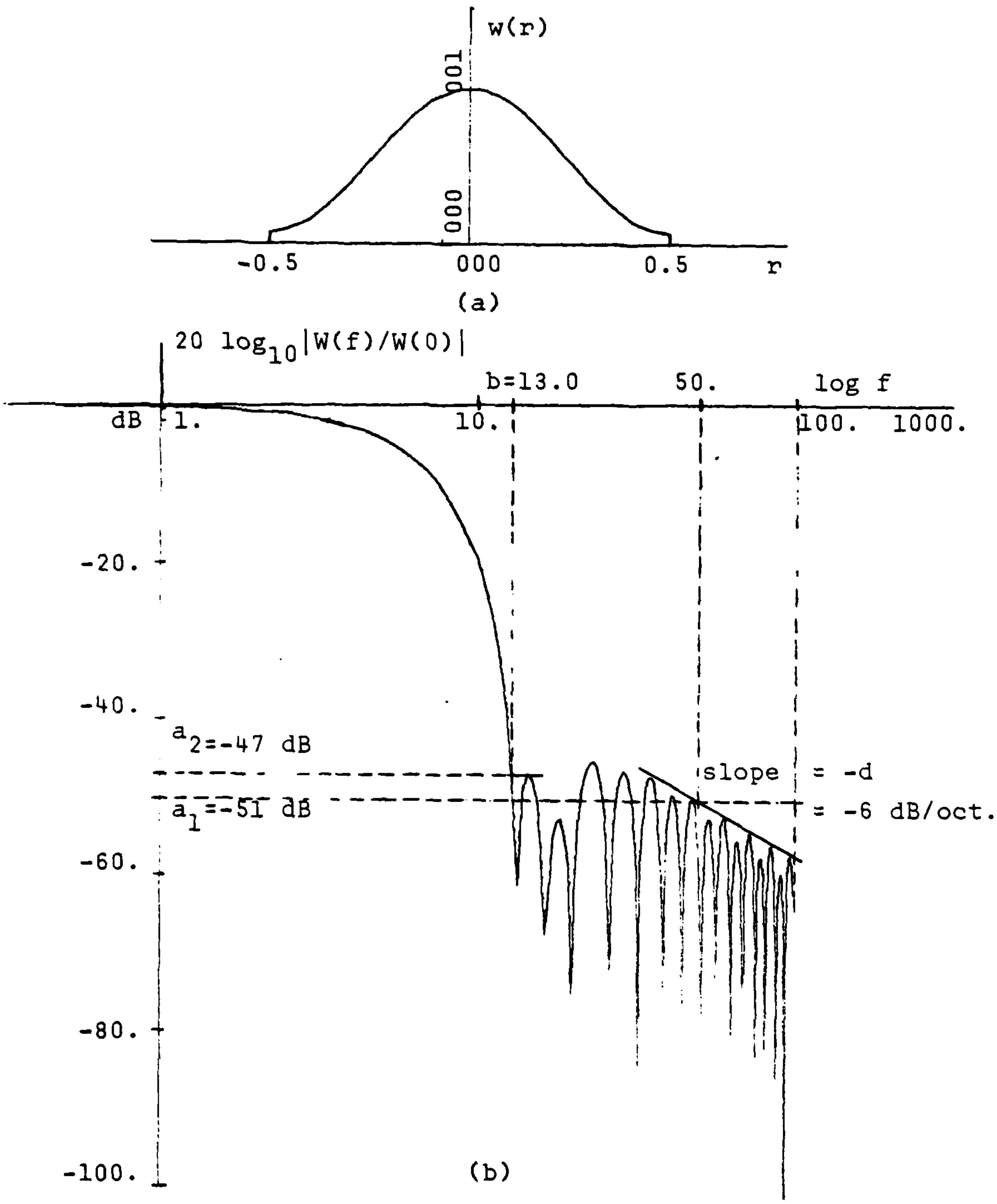


Figure 5.5

(a) Hamming Window in 2-D, (b) Spectral Window

The vertical cross-section of this window is equal to the one-dimensional Blackman window.

The window frequency response is given by the Hankel transform:

$$W(f) = \int_0^a r\{0.42+0.5\cos\frac{\pi}{a}r + 0.08\cos\frac{2\pi}{a}r\}J_o(fr)dr$$

$$= 0.42\, a\frac{J_1(af)}{f} + 0.5\int_0^a r\cos\frac{\pi r}{a}J_o(fr)dr$$

$$+ 0.08\int_0^a r\cos\frac{2\pi}{a}r\, J_o(fr)dr \qquad (5.17)$$

If the extent of the window given as a=0.5, the window and its frequency response are given in Figure 5.6(a) and (b) respectively. The ENBW and COH.GAIN parameters are given as:

$$ENBW = \pi(1.57) = 4.9$$

$$COH.GAIN = \frac{2\pi}{(0.5)^2\pi}\int_0^{0.5} r\,(0.42+0.5\cos\frac{\pi r}{0.5}+0.08\cos\frac{2\pi}{0.5}r)dr$$

$$= \frac{17}{0.78} = 0.22$$

The main lobe width is equal 18.5, which is almost triple of that of the rectangular window. The first side lobe level is 60 dB down from the main lobe and the side lobe fall off rate is 9 dB/octave.

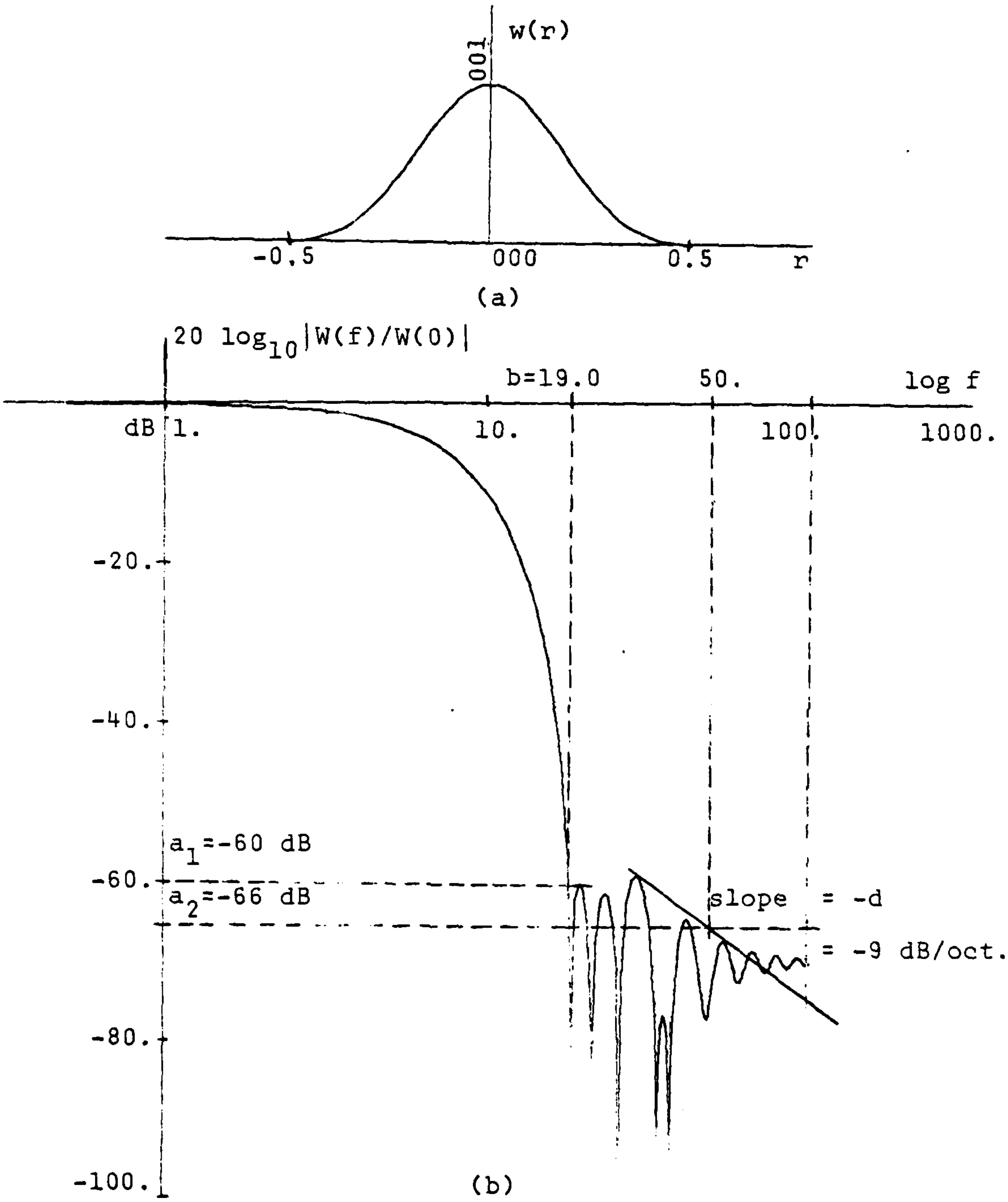


Figure 5.6

(a) 2-D Blackman Window, (b) Spectral Window

G. KAISER WINDOW

Two dimensional circularly symmetric Kaiser window is given as:

$$w(r) = \begin{cases} \dfrac{I_o[\beta\sqrt{1-(\frac{|r|}{a})^2}]}{I_o[\beta]} & |r| \leq a \\ o & \text{elsewhere} \end{cases} \qquad (5.18)$$

The vertical cross-section of the window is equal to a one dimensional Kaiser window.

The frequency spectrum of the window is given by the Hankel transform:

$$W(f) = \int_o^a r \frac{I_o[\beta\sqrt{1-(\frac{|r|}{a})^2}}{I_o[\beta]} J_o(fr)dr \qquad (5.19)$$

which cannot be reduced to a closed form.

If a=0.5 and β=6 then the window and its frequency response are given in Figure 5.7(a) and (b). The ENBW and COH.GAIN parameter are given as:

$$\text{ENBW} = \pi(1.5) = 4.71$$

$$\text{COH.GAIN} = \frac{2\pi}{\pi(0.5)^2} \int_o^{0.5} r \frac{I_o[\beta\sqrt{1+(\frac{r}{0.5})^2}]}{I_o[\beta]} dr = \frac{0.48}{0.78} = 0.6$$

The main lobe width is 14.0 and the first side lobe leve is 44 dB down from the main lobe peak. The side lobe fall o rate is 6 dB/octave.

In this case, the window's characteristics depend strong on the values of a and β. With β=6, the window is similar t a Hamming window.

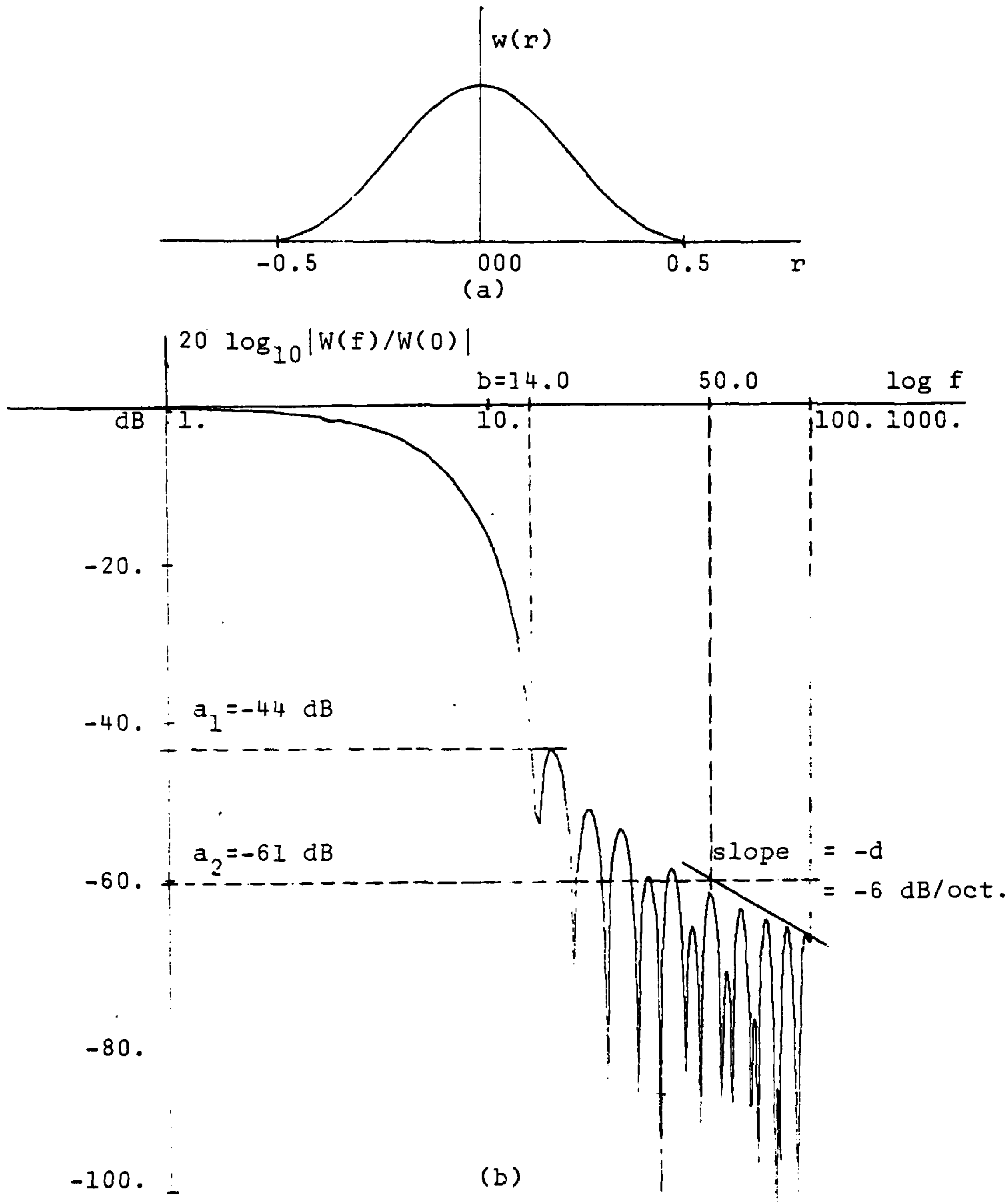


Figure 5.7

(a) 2-D Kaiser Window, (b) Spectral Window

TABLE VI.1

Two-Dimensional Window Functions and Figures of Merit

	WINDOW	ENBW	COH.GAIN	a_1	a_2	b	d
				dB	dB	Hz	dB/octave
1.	Retangular	3.14	1.0	-18	-38	6.6	-9
2.	Bartlett	4.18	.34	-34	-58	11.0	-12
3.	Cosine	3.86	.46	-26	-55	10.0	-12
4.	Hann	4.71	.29	-34	-64	13.0	-18
5.	Hamming	4.3	.36	-47	-51	13.0	-6
6.	Blackman	4.9	.22	-60	-66	19.0	-9
7.	Kaiser	4.7	.61	-44	-61	14.0	-6

VI. CONCLUSION

One of the most straightforward applications of the rectangular window is to truncate an infinite length impulse response. This can be thought as multiplying the impulse response by a window function that is unity in some finite interval between ±a, and zero elsewhere. This multiplication is equivalent to convolving the Fourier transform of the impulse response with a function of the form $\frac{\sin x}{x}$. This causes ringing in the original frequency response. Increasing the length of the finite impulse response (by increasing the length of the rectangular window) decreases the duration of the ringing, but has no effect on the peak overshoot. This effect is called the Gibbs phenomenon.

A considerable amount of effort has been put into finding a good window function, i.e., window functions which reduce the peak overshoot of the Gibbs oscillation. It is generally agreed that the frequency response of a good window should have a narrow central lobe (with a large amplitude) and side lobes with comparatively small amplitude. There are many window functions readily available in the literature, however most window functions are given only in one dimension. The one dimensional window functions and their Fourier transform are given in detail in Chapter III of this thesis. Figures of merit for one dimensional window are powerful measure and are used to compare the characteristic of the various windows.

It has been the objective of this thesis to examine two dimensional window functions. The two dimensional window functions presented here have been a direct extension of corresponding one dimensional window functions. The two dimensional window function is formed from a one dimensional window function, using the relation $w_2(x,y)=w(\sqrt{x^2+y^2})$. This obviously rotates the one dimensional function about the origin in the x-y plane. However it does not simply rotate the frequency response of the one dimensional about the origin in the u,v plane. The two dimensional frequency response of the window can be calculated using the Hankel transform. The principal advantage of this approach is that the frequency transform of two dimensional window can be found using a one dimensional manipulation.

The two dimensional window figures of merit are a direct extension of the one dimensional case. However the definition of the ENBW is changed from the width of a rectangular filter in the one dimensional case to the circumference of a circular filter in the two dimensional case. Table VI.1 shows the comparison between one dimensional and two dimensional window figure of merit. From the table it is clear that the two ENBW parameters are related by the equation (4.42).

The coherent gain decreases slightly except for the Kaiser window where it is higher, and the rectangular window where it is constant. The first side lobe level, which is shown by the parameter a_1, is lower for the two dimensional windows, however the width of the main lobe is much greater in each case. The side lobe fall-off rate stays the same except for the Blackman window where it is slower and the rectangular window where it is faster.

In conclusion, while one dimensional and two dimensional windows can be simply related to one another in the time domain, no direct, general parallels exist to relate their transforms in the frequency domain.

TABLE VI.1

Comparison Between Two Dimensional And One Dimensional Window Figures of Merit

WINDOW	ENBW		COH.GAIN		a_1		b		d	
	2D	1D	2D	1D	2D	1D	2D	1D	2D	1D
Rectangular	3.14	1.0	1.0	1.0	-18	-13	6.6	0.81	-9	-6
Bartlett	4.18	1.33	.34	.5	.34	-26	11.0	1.63	-12	-12
Cosine	3.86	1.23	.46	0.64	-26	-23	10.0	1.35	-12	-12
Hann	4.71	1.5	.29	0.5	-34	-32	13.0	1.87	-18	-18
Hamming	4.3	1.36	.36	0.54	-47	-43	13.0	1.91	-6	-6
Blackman	4.9	1.57	.22	.46	-60	-58	19.0	2.82	-9	-18
Kaiser	4.7	1.5	.61	.49	-44	-43	14.0	2.2	-6	-6

APPENDIX

The effort to put a Hankel transform of a cosine function in a closed form is given below. The cosine function arises from Hankel transform of a cosine window.

The Hankel transform of a cosine function is given below:

$$w(f) = \int_o^a r \cos \frac{\pi}{2a} r \, J_o(rf) \, dr \qquad |r| \leq a \qquad (A.1)$$

The above integral form has no known closed form, as proved by the following discussion.

Three different methods are used to bring the integral into a closed form, they are: straight integration, partial differential and table of integration.

A. STRAIGHT INTEGRATION

To use straight integration method to solve the integration, the variable needs to be changed into trigonometric variables.

Let: $\theta = \frac{\pi r}{2a}$, $r = \frac{2a}{\pi} \theta$.

The interval of the integral becomes, θ=o and $=\frac{\pi}{2}$ for r=o and r=a respectively.

Substitution of the new variables into Hankel transform gives:

$$w(f) = \left(\frac{\lambda}{f}\right)^2 \int_o^{\pi/2} \theta \cos \theta \, J_o(\lambda\theta) \, d\theta \qquad (A.2)$$

where $\lambda = 2(\frac{af}{\pi})$. This new integral form gives more pleasing appearance than the previous one.

Substitute the Bessel function $J_o(\lambda\theta)$ inside the integral with Bessel equation, given as follows:

$$J_o''(x) + \frac{1}{x} J_o'(x) + J_o(x) = 0 \qquad (A.3)$$

Let $x = \lambda\theta$, and do some manipulation. The Bessel equation becomes:

$$J_o(\lambda\theta) = -\frac{1}{\lambda^2}\{J_o''(\lambda\theta) + \frac{1}{\theta} J_o'(\lambda\theta)\} \qquad (A.4)$$

The integral form in A.2 becomes:

$$w(\lambda) = -\frac{1}{f^2}\int_o^{\pi/2} \theta \cos\theta \, \{J_o''(\lambda\theta) + \frac{1}{\theta} J_o'(\lambda\theta)\}d\theta$$

$$= -\frac{1}{f^2}\{\theta \cos\theta \, J_o'(\lambda\theta)\Big|_o^{\pi/2} \int_o^{\pi/2} \{(\cos\theta - \theta\sin\theta)J_o'(\lambda\theta)$$

$$- \cos\theta \, J_o'(\lambda\theta)\}d\theta\}$$

$$= -\frac{1}{f^2}\{\int_o^{\pi/2} \theta \sin\theta \, J_o'(\lambda\theta)\, d\theta\}$$

$$= -\frac{1}{f^2}\{\theta \sin\theta \, J_o(\lambda\theta)\Big|_o^{\pi/2} - \int_o^{\pi/2} (\sin\theta + \theta\cos\theta)J_o(\lambda\theta)d\theta\}$$

$$= -\frac{1}{f^2}\{\frac{\pi}{2} J_o(\frac{\pi}{2}\lambda)\} + \frac{1}{f^2}\int_o^{\pi/2} \sin\theta \, J_o(\lambda\theta)\, d\theta + \frac{1}{\lambda^2} w(\lambda)$$

$$(1 - \frac{1}{\lambda^2})w(\lambda) = -\frac{\pi}{2f^2} J_o(\frac{\pi}{2}\lambda) + \frac{1}{f^2} \int_o^{\pi/2} \sin\theta \, J_o(\lambda\theta) \, d\theta$$

The end result is:

$$w(\lambda) = \frac{\lambda^2-1}{\lambda^2 f^2} \int_o^{\pi/2} \sin\theta \, J_o(\lambda\theta) \, d\theta - \frac{\pi(\lambda^2-1)}{2\lambda^2 f^2} J_o(\frac{\pi}{2}\lambda) \quad (A.5)$$

By replacing $J_o(\lambda\theta)$ inside the integral as before, it may be continued at will, but the term with the integral form cannot be eliminated. The conclusion is, the integral form in A.2 cannot be put in a closed form by this method.

B. PARTIAL DIFFERENTIAL METHOD

Return to the integral form given in A.2:

$$W(\lambda) = (\frac{\lambda}{f})^2 \int_o^{\pi/2} \theta \cos\theta \, J_o(\lambda\theta) \, d\theta$$

Define new equations: $C(\lambda)$ and $S(\lambda)$

$$C(\lambda) = \int_o^{\pi/2} \theta \cos\theta \, J_o(\lambda\theta) \, d\theta \quad (A.6)$$

$$S(\lambda) = \int_o^{\pi/2} \theta \sin\theta \, J_o(\lambda\theta) \, d\theta \quad (A.7)$$

Differentiate to λ as $C'(\lambda)$ and $S'(\lambda)$

$$C'(\lambda) = \frac{d}{d\lambda} C(\lambda) = \int_o^{\pi/2} \theta^2 \cos\theta \, J_o'(\lambda\theta) \, d\theta \quad (A.8)$$

$$S'(\lambda) = \frac{d}{d\lambda} S(\lambda) = \int_0^{\pi/2} \theta^2 \sin\theta \, J_o'(\lambda\theta)\, d\theta \qquad (A.9)$$

Multiply both A.8 and A.9 with λ:

$$\lambda C'(\lambda) = \int_0^{\pi/2} \theta^2 \cos\theta \, J_o'(\lambda\theta)\, d(\lambda\theta) \qquad (A.10)$$

$$\lambda S'(\lambda) = \int_0^{\pi/2} \theta^2 \sin\theta \, J_o'(\lambda\theta)\, d(\lambda\theta) \qquad (A.11)$$

Using Bessel identity $J_o'(\lambda\theta)d(\lambda\theta)=dJ_o(\lambda\theta)$, substitute it into A.10 and A.11:

$$\lambda C'(\lambda) = \int_0^{\pi/2} \theta^2 \cos\theta \, dJ_o(\lambda\theta) \qquad (A.12)$$

$$\lambda S'(\lambda) = \int_0^{\pi/2} \theta^2 \cos\theta \, dJ_o(\lambda\theta) \qquad (A.13)$$

Expand both integral in A.12 and A.13 into partial integration:

$$\lambda C'(\lambda) = \theta^2 \cos\theta \, J_o(\lambda\theta) \Big|_0^{\pi/2} - \int_0^{\pi/2} (2\theta\cos\theta - \theta^2\sin\theta) J_o(\lambda\theta) d\theta$$

$$= - \int_0^{\pi/2} (2\theta \cos\theta - \theta^2 \sin\theta)\, J_o(\lambda\theta)\, d\theta \qquad (A.14)$$

$$\lambda S'(\lambda) = \theta^2 \sin\theta \, J_o(\lambda\theta) \Big|_0^{\pi/2} (2\theta \sin\theta + \theta^2 \cos\theta) J_o(\lambda\theta) d\theta$$

$$= \frac{\pi}{2} J_o(\frac{\pi}{2}\lambda) - \int_o^{\pi/2} (2\theta \sin\theta + \theta^2 \cos\theta) J_o(\lambda\theta) d\theta \qquad (A.15)$$

Rewrite both equation in A.14 and A.15, using definition of C and S. It will end up with the following pair of differential equations:

$$\lambda C'(\lambda) = -2C(\lambda) + S'(\lambda) \qquad (A.16)$$

$$\lambda S'(\lambda) = \frac{\pi}{2} J_o(\frac{\pi}{2}\lambda) - 2S - C' \qquad (A.17)$$

Rearrange the last two equations:

$$\lambda C' + 2C + S' = 0 \qquad (A.18)$$

$$\lambda S' + 2S + C' = \frac{\pi}{2} J_o(\frac{\pi}{2}\lambda) \qquad (A.19)$$

The last two pairs of equations show that it may give a single equation in terms of S or C. And that means it will end up in integral form of C or S. The conclusion is, the integral form in A.2 cannot be put in a closed form by this method.

C. TABLE OF INTEGRATION

Taylor series expands any analytic function f(x) into a power series of x. Consequently cos x may be expanded into a power series of x^2:

$$\cos x = f(x^2) \tag{A.20}$$

Substitute A.20 into A.2:

$$w(x) = \int_o^f x\, f(x^2)\, J_o(fx)dx \tag{A.21}$$

where we change the integration variables with the dummy variables x and f.

From Table of Integration [Ref. 10], the integral form shown below:

$$I_{2k}(x) = \int_o^k t^{2k+1}\, J_o(t)dt \tag{A.22}$$

can be written in a closed form. The term $t(j_o(t)$ in the above equation is replaced using Bessel equation: $tJ_o(t) = -tJ_o''(t)-J_o'(t)$.

The integral form in A.22 becomes:

$$I_{2k}(x) = \int_o^x t^{2k}\{-tJ_o''(t)-J_o'(t)\}\, dt$$

$$= -\int_o^k t^{2k+1}J_o'(t)dt - \int_o^x t^{2k}J_o'(t)dt$$

$$= -x^{2k+1}J_o'(x)+(2k+1)\int_o^x t^{2k}J_o'(t)dt - \int_o^k t^{2k}J_o'(t)dt$$

$$= -x^{2k+1}J_o'(x)+2k\{x^{2k}J_o(x)-2k \int_o^x t^{2k-1}J_o(t)dt\}$$

$$I_{2k}(x) = 2kx^{2k}J_o(x)-x^{2k+1}J_o'(x)-4k^2I_{2k-1}(x) \qquad (A.23)$$

The last equation shows that $I_{2k}(x)$ reduces to lower order $I_{2k-1}(x)$. This form is called reduction form. For k=o, A.22 and A.23 become:

$$I_o(x) = \int_o^x t\ J_o(t)dt = =xJ_o'(x) \qquad (A.24)$$

The equation in A.24 agrees with the property of Bessel function.

If the reduction form applied repeatedly and written in a general form, A.23 may be written as:

$$\int_o^x t^{2k}tJ_o(t)dt = 2k.Poly(2k).J_o(x)+x.Poly(2k).J_o'(x) \qquad (A.25)$$

Lets return to equation A.20:

$$\cos x = f(x^2) = \sum_{k=o}^{\infty} f_k\ x^{2k} \qquad (A.26)$$

Substitute A.26 into A.25:

$$\int_o^x f(t^2)t\ J_o(t)dt = \sum_{k=o}^{\infty} a_k x^{2k}\ .\ J_o(x)+ \sum_{k=o}^{\infty} b_k x^{2k+1}.(J_o'(x)$$

(A.27)

Apply differentiation to both left and right hand side of A

$$\sum_{k=o}^{\infty} f_k x^{2k+1} J_o(x) = J_o(x) \sum_{k=o}^{\infty} 2ka_k x^{2k-1} + J_o'(x) \sum_{k=o}^{\infty} (2k+$$

$$+ J_o'(x) \Sigma a_k x^{2k} + J_o''(x) \Sigma b_k x^{2k+}$$

Replace $J_o''(x)$, using Bessel equation:

$$J_o''(x) = - x^{-1} J_o'(k) - J_o(x) \qquad (A.2$$

$$\sum_{k=o}^{\infty} f_x x^{2k+1} J_o(x) = J_o(x) \{ \sum_{1}^{\infty} 2ka_k x^{2k-1} - \Sigma b_k x^{2k+1} \}$$

$$+ J_o'(x) \{ \Sigma a_k x^{2k} + \Sigma (2k+1) b_k x^{2k} - \Sigma b_k$$

$$= J_o(x) \{ \sum_{o}^{\infty} (2k+1) a_{kt} x^{2k+1} - \sum_{o}^{\infty} b_k x^{2k+1} \}$$

$$+ J_o'(x) \{ \sum_{o}^{\infty} (a_k + 2kb_k) x^{2k} \}$$

Equate the coeficient of the equal term J_o and J_o' from th left hand side and right hand side:

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$$\sum_{o}^{\infty} f_k x^{2k+1} = \sum_{1}^{\infty} \{(2k+1)a_{k+1}-b_k\}\ x^{2k+1} \tag{A.29}$$

$$0 = \Sigma\ (a_k+2kb_k)x^{2k} \tag{A.30}$$

Therefore:

$$f_k = (2k+1)a_{k+1}-b_k \rightarrow a_k = \frac{f_{k-1}+b_{k-1}}{2k} \tag{A.31}$$

$$0 = a_k+2kb_k \qquad \rightarrow b_k = -\frac{a_k}{2k} \tag{A.32}$$

These last two equations give the solution to the integral of the cos x form. Lets expand cos x into power series:

$$\cos x = f_o + f_1x^2 + f_2x^4 + f_3x^6 + \ldots .$$

then using relations given in (A.30) and (A.31):

$a_o = 0$ $\qquad$ $b_o = 0$

$a_1 = \frac{f_o}{2}$ $\qquad$ $b_1 = -\frac{f_o}{4}$

$a_2 = \frac{f_1-f_o/4}{4}$ $\qquad$ $b_2 = -\frac{f_1-f_o/4}{16}$

$a_3 = \frac{f_1-f_o/4}{32}$ $\qquad$ $b_3 = -\frac{f_1-f_o/4}{192}$

etc.

The conclusion is, we succeeded in solving Hankel transform of cos x function, but we failed to express it in a closed form.

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