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A GENERAL MIXTURE-TRUNCATION METHOD
FOR GENERATING BIVARIATE RANDOM
VARIABLES FROM UNIVARIATE GENERATORS

Myong Ho Shin

NAVAL POSTGRADUATE SCHOOL

Monterey, California



THESIS

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FOR GENERATING BIVARIATE RANDOM
VARIABLES FROM UNIVARIATE GENERATORS

by

Myong Ho Shin

March 1980

Thesis Advisor:

P.A.W. Lewis

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A simple mixture-truncation method is put forward here to generate bivariate distributions with any marginal distribution from univariate generators.

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A General Mixture-Truncation Method
for Generating Bivariate Random
Variables from Univariate Generators

by

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Submitted in partial fulfillment of the
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ABSTRACT

In this thesis we deal with the generation of bivariate random variables with identical marginal distributions and specified correlation coefficients. Many schemes have been put forward for different cases; they almost always involve computing the inverse probability distribution of the marginal distributions or exploiting special properties of the random variables in question.

A simple mixture-truncation method is put forward here to generate bivariate distributions with any marginal distribution from univariate generators.

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I. INTRODUCTION

In many simulation applications, it is required to generate bivariate random variables which have identical marginal distribution.

For example, in reliability problems, the assumption that two components have independent exponential failure times is very often unrealistic, and much effort has gone into deriving bivariate exponential random variables to handle these situations [Gaver (1972), Olkin & Marshall (1967), Downton (1970)]. Now except in very specific physical situations it may be difficult to specify the complete bivariate distribution of life time of each component. However, it may be realistic to specify the marginal distributions and some measure of dependence (usually the correlation coefficient) between the life time of each component. In this kind of situation, we can use bivariate random vectors having given marginal distribution and dependence to solve the problem in simulation. To generate these vectors, there exists some previous works and existing methods, discussed in Section II. As seen in Section II, most previous work is specific to specified marginal distributions and uses inverse transformation methods as a basic concept.

An example is the recent work by Johnson and Tenenbein (1979), to generate a bivariate random vector (X,Y) which has marginal distribution $F_1(x)$, $F_2(y)$ and correlation ρ , by a weighted linear combination method.

Define

$$X = F_1^{-1}(H_1(U))$$

$$Y = F_2^{-1}(H_2(V))$$

or

$$Y = F_2^{-1}(1 - H_2(V))$$

where H_1 and H_2 are the cumulative distribution functions (c.d.f.) of U and V respectively and

$$U = U'$$

$$V = cU' + (1 - c)V'$$

where U' , V' are i.i.d. random variables with probability density function $g(\cdot)$. In this procedure, F_1^{-1} , F_2^{-1} , and c are specific to the marginal distribution and correlation desired. The functions F_1^{-1} and F_2^{-1} are difficult to compute in most cases and the weighting factor c is also difficult to calculate. Moreover most of the work in univariate random number generation has been aimed at avoiding having to calculate inverse cumulative distribution functions such as $F_1^{-1}(\cdot)$ and $F_2^{-1}(\cdot)$. These are the reasons why many proposed methods are specific to a specified marginal distribution.

Again special properties of certain random variables such as infinite divisibility have been exploited to give easily generated bivariate random variables, often though with limited ranges of dependency. One very clever scheme by Gaver (1972) to generate bivariate exponential random variables uses the fact that the sum of a geometrically distributed number of exponential random variables (Y) is exponentially distributed and that the minimum of this geometrically distributed number of independent logistic random variables (Z) is exponential. Clearly when Y is large, Z is small. This scheme is of course very specific to exponential marginal distributions and, via an exponential transformation, to uniform random variables. To avoid these kinds of limitations and to make the generation of bivariate random variables simpler and more automatic in simulations we develop here a scheme presented in Jacobs and Lewis (1977).

This scheme, the mixture-truncation method, is a very general tool which requires only that a method be available for generating random variables with the desired marginal distribution. The mixture-truncation method scheme for generating bivariate random variables is as follows.

Let $F(x)$ be the common marginal distribution, of the bivariate random variable (Y, Z). Let ρ be the desired correlation between Y and Z (which may or may not be attainable generally or in particular with the mixture-truncation scheme).

Define the transition matrix \underline{P} as

$$\underline{P} = \begin{bmatrix} \alpha_1 & 1 - \alpha_1 \\ 1 - \alpha_2 & \alpha_2 \end{bmatrix}$$

with stationary vector

$$\underline{\pi} \underline{P} = \underline{\pi} = \left(\frac{1 - \alpha_2}{1 - \alpha_1 + 1 - \alpha_2}, \frac{1 - \alpha_1}{1 - \alpha_1 + 1 - \alpha_2} \right)$$

and let the range of the random variable X with distribution function $F(x)$ be χ . Then generate (Y, Z) as follows.

1. (Initialization)

i) Choose an "allowable" x_0 from (x_l, x_u) (the allowable range of x_0 will usually be smaller than χ and depends on ρ and $F(x)$).

ii) Set $\pi_1 = F(x_0)$, $\pi_2 = 1 - \pi_1$ and compute α_1 and α_2 .

iii) Denote by X_1 the random variable X truncated to the left of x_0 and by X_2 truncated to the right of x_0 .

2. (Generate Y). Choose Y from X_1 with probability π_1 or choose Y from X_2 with probability π_2 .

3. (Generate Z)

i) If Y is chosen from X_1 , choose Z from X_1 with probability α_1 or choose Z from X_2 with probability $1 - \alpha_1$.

ii) If Y is chosen from X_2 , choose Z from X_1 with probability $1 - \alpha_2$ or choose Z from X_2 with probability α_2 .

In Section III, we will show that the correlation ρ between Y and Z, if x_0 is properly chosen, is

$$\begin{aligned} \rho &= E\left[\frac{Y - E\{Y\}}{\sigma_Y}\right] \left[\frac{Z - E\{Z\}}{\sigma_Z}\right] \\ &= \beta M, \end{aligned}$$

where

$$\begin{aligned} \beta &= \alpha_1 - (1 - \alpha_2), \\ M &= \frac{(\mu_1 - \mu_2)^2 \pi_1 \pi_2}{\sigma_X^2}, \end{aligned}$$

$$\mu_1 = E\{X_1\}, \quad \mu_2 = E\{X_2\},$$

$$\sigma_X^2 = \text{VAR}[X],$$

Moreover, the generated bivariate random vector (Y, Z) will have marginal distributions $F(x)$ and correlation ρ and the joint distribution function of (Y, Z) will be

$$\begin{aligned}
F(Y, Z) = & \left[\begin{array}{ll} \frac{\alpha_1}{\pi_1} F(z) F(y) & \text{if } z \leq x_0, y \leq x_0 \\ \left\{ \alpha_1 + \frac{(1-\alpha_1)}{\pi_2} [F(z) - F(x_0)] \right\} F(y) & \text{if } z > x_0, y \leq x_0 \\ \left\{ \alpha_1 + \frac{(1-\alpha_1)}{\pi_2} [F(y) - F(x_0)] \right\} F(z) & \text{if } z \leq x_0, y > x_0 \\ \alpha_1 \pi_1 + (1-\alpha_1) [F(z) - F(x_0)] \frac{\pi_1}{\pi_2} & \text{if } z < x_0, y > x_0 \end{array} \right. \\
& + \left\{ (1-\alpha_2) + \frac{\alpha_2}{\pi_2} [F(z) - F(x_0)] \right\} [F(y) - F(x_0)]
\end{aligned}$$

These relationships will be developed in Section III. Note that (Y, Z) may be continuous or discrete random variables or mixtures of both, though in this thesis we concentrate on continuous cases.

The key problem in this very simple algorithm comes at the initialization steps i) and ii). There are two degrees of freedom in the selection matrix \underline{P} but setting $\pi_1 = F(x_0)$ constraints reduce to one degree of freedom. Specifying a desired correlation further constrains the degree of freedom, though not completely. Subject to the constraint that α_1 and α_2 are probabilities there may be

- a. no values x_0 which will give (Y, Z) with correlation ρ
- b. one value x_0
- c. a range \bar{x}_0 .

The main numerical problem of initialization step (i) then is to compute the range of allowable x_0 for a given ρ and $F(x)$.

The main statistical problem is then to choose which of the bivariate random vectors (Y,Z) indexed by $x_0 \in \bar{x}_0$ to use. An alternate solution to the statistical problem, giving another algorithm is then to let x_0 have some distribution in the range \bar{x}_0 , possibly uniform or triangular, this not only alleviates the problem of picking a particular $x_0 \in \bar{x}_0$ but it also smoothes out the distribution of (Y,Z) and possibly makes it continuous. The computation of \bar{x}_0 for given ρ is illustrated in subsequent sections for uniform, exponential and gamma marginals for (Y,Z) .

Before doing this and developing, in Section III, the results already given here, we review a few existing methods for generating bivariate random variables in Section II. These are methods which seem fairly tractable, for use on a computer.

II. REVIEW OF EXISTING METHODS

There are several existing methods for generating bivariate random vectors, but most of them are specific to particular marginal distribution and use inverse transformation method. The inverse transformation method is a very useful univariate procedure which, unfortunately, is not possible to use with many distributions because it is difficult and/or uneconomic to compute the inverse functions. We survey here some of the methods which are germane to this thesis, in particular concentrating on bivariate random variables with uniform, exponential and gamma marginals. First we review the problem of determining the range of correlation coefficient ρ which can be obtained for bivariate distribution with given marginal distributions, again considering only the continuous case.

A. RANGE OF CORRELATION COEFFICIENT ρ

Suppose that Y, Z are random variables with an arbitrary joint distribution $F(y,z)$ with finite second moments. Then in general the correlation coefficient ρ can take all values in the closed interval $[-1,1]$. But with specific marginal distribution $F_1(y)$ and $F_2(z)$, the class of all $F(y,z)$ need not attain the values of $-1, 1$, of ρ . The necessary and sufficient conditions that there exist determination of $F(y,z)$ with ρ equal to 1 and -1 are given by Moran (1967) as follows.

- i) there exist constants α and β such that $\alpha Y + \beta$ has the same distribution as Z ,
- ii) the distribution of Z is symmetrical about its mean.

To see this, rescale Y to have the same mean and variance as Z . If there exists an $F(y,z)$ such that $\rho = 1$ we have $E[Z - Y]^2 = 0$, so that $Y = Z$ with probability one. Then Y must have the same distribution as Z . On the other hand if there exists an $F(y,z)$ such that $\rho = -1$ we shall have $E[Z + Y]^2 = 0$, and $Y = -Z$ with probability one. Thus if both bounds are attainable Z must have a symmetric distribution.

Given general marginal distributions $F_1(y)$ and $F_2(z)$, Mardia (1970) showed Frechet bounds as

$$\max[0, F_1(y) + F_2(z) - 1] \leq F(y, z) \leq \min[F_1(y), F_2(z)] \quad (\text{II-A-1})$$

From this we can find the range of possible values of ρ . For simplicity we now confine our consideration to distributions $F(y,z)$ of positive random variables whose derivatives $F'_1(y)$ and $F'_2(z)$ are strictly positive for $y > 0, z > 0$, respectively. Suppose also that the variates are scaled to have unit variances. Let $G_1(u), G_2(v)$ be the inverse functions of $F_1(y), F_2(z)$, i.e.

$$F_i[G_i(w)] = w$$

where $0 \leq w \leq 1, i = 1, 2$. Then the correlation coefficient

between Y and Z is given by the equation

$$\rho = \int_0^{\infty} \int_0^{\infty} y z dF(y, z) - E[Y] E[Z] \quad (\text{II-A-2a})$$

$$= \int_0^1 \int_0^1 G_1(u) G_2(v) dK(u, v) - E[Y] E[Z] \quad (\text{II-A-2b})$$

where $K(u, v)$ is the joint distribution of the quantities $U = F_1(y)$, $V = F_2(z)$. Then U, V are jointly distributed on the square $0 \leq U \leq 1$, $0 \leq V \leq 1$, in such a way that the marginal distributions are uniform on the unit intervals. From expression (II-A-1) the minimum correlation is attained when the probability is concentrated uniformly on the line $U + V = 1$. The minimum value of ρ then is

$$\rho_{\min} = \int_0^1 G_1(u) G_2(1-u) du - E[Z] E[Y] \quad (\text{II-A-3})$$

The corresponding $F(y, z)$ will be a singular distribution with all the probability concentrated on the line $F_1(y) + F_2(z) = 1$. In fact Y and Z are an antithetic pair. The maximum value of ρ is attained when the probability is concentrated uniformly on the line $U = V$. Then the maximum value of ρ is

$$\rho_{\max} = \int_0^1 G_1(u) G_2(u) du - E[Z] E[Y], \quad (\text{II-A-4})$$

with the corresponding probability concentrated on the line

$$F_1(y) = F_2(z).$$

By using this result we will get lower and upper bounds of the correlation coefficient for uniform, exponential and gamma marginal cases. In the uniform marginal distribution case, we can see $\rho_{\min} = -1$ and $\rho_{\max} = 1$ from Moran's condition ii), i.e., uniform distribution is symmetric about its mean. For the exponential marginal distribution case suppose Y and Z have exponential distributions with unit mean and variance. Then

$$F_1(y) = 1 - e^{-y}, \quad F_2(z) = 1 - e^{-z}$$

and the inverse functions are

$$G_1(u) = -\ln(1-u),$$

$$G_2(v) = -\ln(1-v).$$

From equations (II-A-3) and (II-A-4)

$$\begin{aligned} \rho_{\min} &= \int_0^1 G_1(x) G_2(1-x) dx - E[Y] E[Z] \\ &= \int_0^1 \ln x \ln(1-x) dx - 1 = 1 - \frac{\pi^2}{6} \\ &\approx -0.64493 \end{aligned}$$

$$\begin{aligned} \rho_{\max} &= \int_0^1 G_1(x) G_2(x) dx - E[Y] E[Z] \\ &= \int_0^1 \ln x \ln x dx - 1 = 1 \end{aligned}$$

The ρ_{\min} can be attained when all Y and Z are concentrated on the line $e^{-Y} + e^{-Z} = 1$. Also the ρ_{\max} can be attained when all Y and Z are concentrated on the line $e^{-Y} = e^{-Z}$, i.e., $Y = Z$.

For the gamma marginal distribution case, suppose that Y and Z have gamma type distributions with probability density functions

$$\begin{aligned} f_1(y) &= \frac{1}{\Gamma(\alpha)} e^{-y} y^{\alpha-1} & \alpha > 0, y \geq 0 \\ f_2(z) &= \frac{1}{\Gamma(\beta)} e^{-z} z^{\beta-1} & \beta > 0, z \geq 0. \end{aligned}$$

Then

$$E[Y] = \alpha, \quad E[Z] = \beta, \quad \text{VAR}[Y] = \alpha \quad \text{VAR}[Z] = \beta.$$

The ρ_{\min} will be attained when the probability density is concentrated on the line

$$\frac{1}{\Gamma(\alpha)} \int_0^y e^{-x} x^{\alpha-1} dx + \frac{1}{\Gamma(\beta)} \int_0^z e^{-x} x^{\beta-1} dx = 1.$$

This defines y uniquely as a function of z which can be written $y = A(z)$. The ρ_{\min} is then, on rescaling the covariance

$$\rho_{\min} = \left\{ \int_0^{\infty} u A(u) f_1(u) du - \alpha\beta \right\} / (\alpha\beta)^{1/2}$$

When α, β become large, ρ_{\min} tends to -1 . And the ρ_{\max} will be attained when the probability density is concentrated on the line

$$\frac{1}{\Gamma(\alpha)} \int_0^y e^{-x} x^{\alpha-1} dx = \frac{1}{\Gamma(\beta)} \int_0^z e^{-x} x^{\alpha-1} dx$$

This defines y as a function of z which can be written $y = B(z)$. The ρ_{\max} then is, on rescaling the covariance

$$\rho_{\max} = \left\{ \int_0^{\infty} u B(u) f_1(u) du - \alpha\beta \right\} / (\alpha\beta)^{1/2}$$

when $\alpha = \beta$, ρ_{\max} tends to 1 . Schmeiser and Ram Lal (1979) showed the obtainable correlations between random variables Y and Z having gamma marginal distribution with density function

$$f_i(x) = \left[(x/\beta_i)^{\alpha_i-1} \exp(-x/\beta_i) / (\beta_i \Gamma(\alpha_i)) \right]$$

for $x > 0$, $\alpha_i > 0$, $\beta_i > 0$, $i = 1, 2$.

The Figures (II-a) and (II-b) show the obtainable correlation as a function of α_2 , given $\alpha_1 = 1$ and 5, for $\beta_i = 1$.

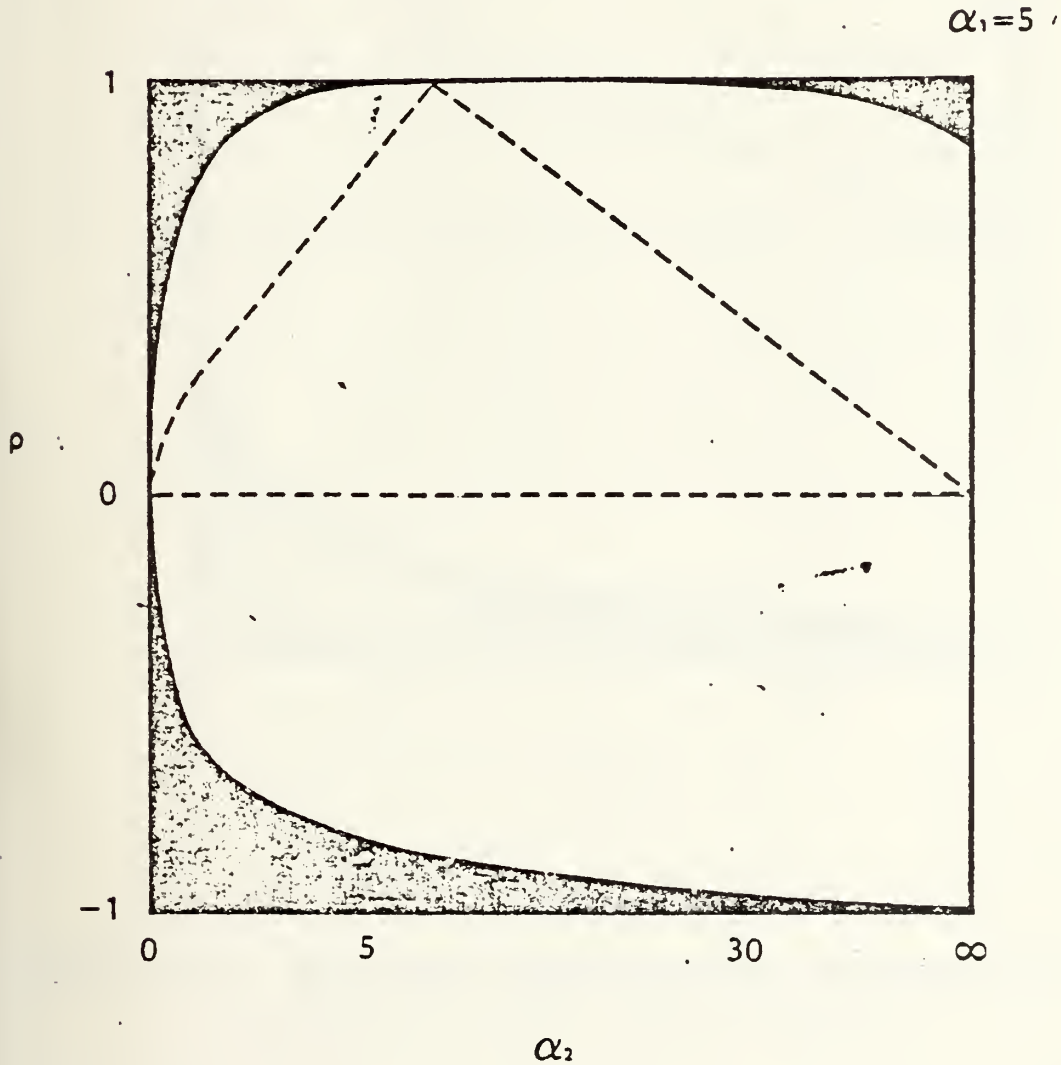


Figure II-a. Obtainable correlation as a function of α_2 for $\alpha_1 = 1$, $\beta_1 = 1$, by Schmeiser

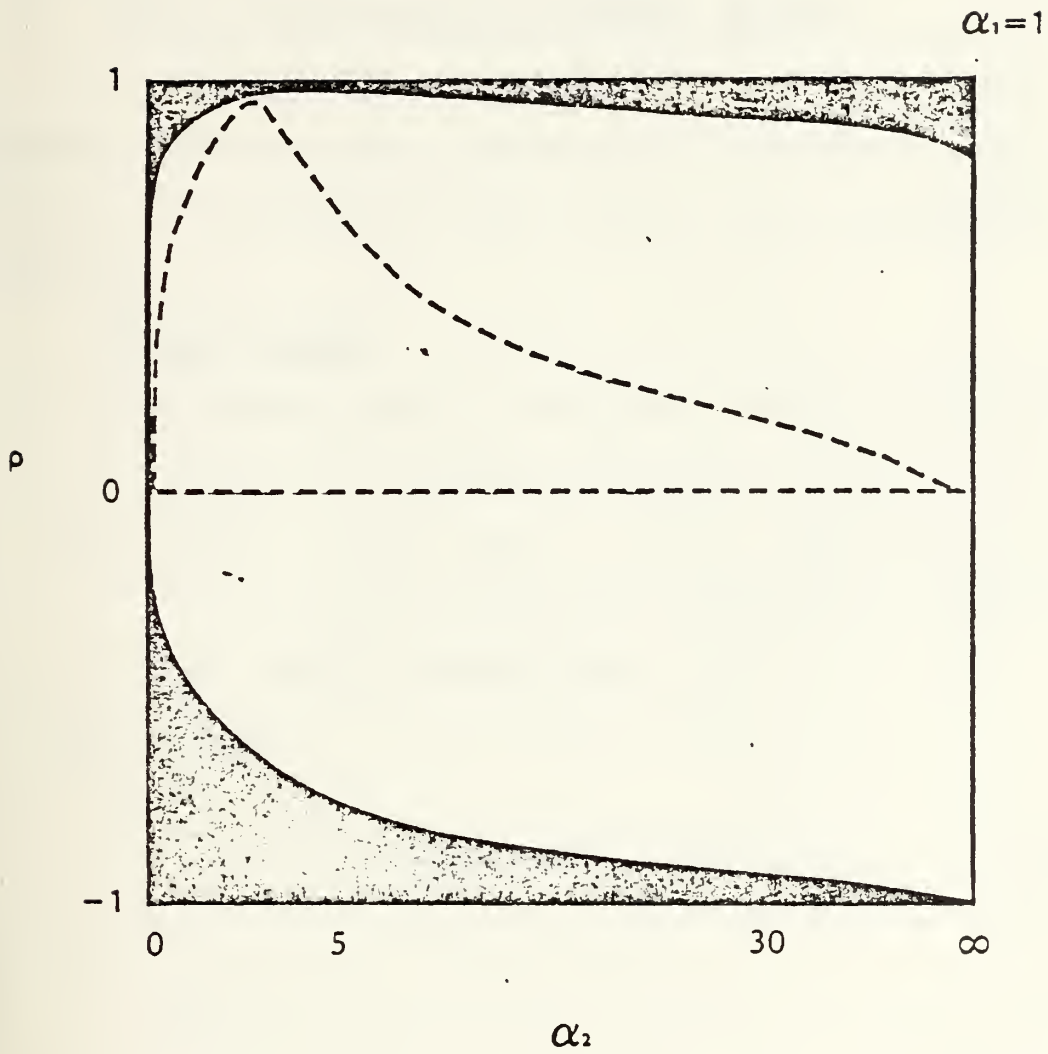


Figure II-b. Obtainable correlation as a function of α_2 for $\alpha_1 = 5$, $\beta_1 = 1$, by Schmeiser

B. GENERAL METHOD REVIEW

1. Johnson and Tenenbein's General Method

A general method of constructing a bivariate distribution, whose marginal distribution functions are $F_1(x)$ and $F_2(y)$, is proposed by Nataf (1962), can be represented as follows.

General Method

- i) Consider any two continuous random variables U and V with probability density function $h(u,v)$.
- ii) Let $X' = H_1(u)$ and $Y' = H_2(v)$, where $H_1(u)$ and $H_2(v)$ are the cumulative distribution functions of U and V , respectively.
- iii) Define

$$X = F_1^{-1}(X') = F_1^{-1}[H_1(u)] \quad (\text{II-B-1})$$

and

$$Y = F_2^{-1}(Y') = F_2^{-1}[H_2(v)] \quad (\text{II-B-2a})$$

or

$$Y = F_2^{-1}(1 - Y') = F_2^{-1}[1 - H_2(v)] \quad (\text{II-B-2b})$$

Since X' , Y' and $1-Y'$ are uniformly distributed over the interval $[0,1]$, X defined by expression (II-B-1) and Y defined either by (II-B-2a) or (II-B-2b) will have a joint distribution whose marginal distribution functions are $F_1(x)$ and $F_2(y)$. The method is probably what one would think of first but again involves inverses. Based on this general method, Johnson and Tenenbein (1979) developed two procedures for generating (and also constructing) bivariate distributions whose marginal distributions and measures of dependence, as given by Kendall's T or the grade correlation coefficient ρ_s , are specified. Note that these Kendall's T and grade correlation coefficient ρ_s are not the same as the ordinary product moment correlation coefficient ρ we have been considering; these measures are discussed by Kendall (1962) and Kruskal (1958) and can be defined as follows. Let X and Y be continuous random variables having some joint probability density function. Let (X_1, Y_1) , (X_2, Y_2) and (X_3, Y_3) be three independent pairs of observations having the same joint density function, then

$$T = 2 P[(X_1 - X_2)(Y_1 - Y_2) > 0] - 1$$

$$\rho_s = 6 P[(X_1 - X_2)(Y_1 - Y_3) > 0] - 3$$

The first procedure for generating bivariate pairs, called the WLC (weighted linear combination), defines

$$U = U' \quad (\text{II-B-3})$$

and

$$V = cU' + (1-c)V' \quad (\text{II-B-4})$$

where $0 \leq c \leq 1$, U' and V' are independent and identically distributed random variables with probability density function $g(\cdot)$. Then X and Y are obtained from the general method using equation (II-B-1) and (II-B-2a) if the dependence measure is positive or equations (II-B-1) and (II-B-2b) if the dependence measure is negative.

In order to apply the WLC procedure, we must obtain expressions for $H_1(u)$, $H_2(v)$, $\sigma_s(u,v)$ and $T(u,v)$, in terms of c and $g(\cdot)$. The values of $H_1(u)$ and $H_2(v)$ allow us to apply the general method for a given choice of c and $g(t)$. The expressions for $T(u,v)$ and $\sigma_s(u,v)$ allow us to specify c for a given choice of $g(\cdot)$ in terms of the required value of either T or σ_s . From equations (II-B-3) and (II-B-4)

$$H_1(u) = \int_{-\infty}^u g(t) dt \quad (\text{II-B-5})$$

$$H_2(v) = \int \int_{R_1} g(u') g(v') du' dv' \quad (\text{II-B-6})$$

where

$$R_1 = \{(u', v') : cu' + (1-c)v' \leq v\}$$

and the joint density function of u and v is

$$h(u, v) = \frac{1}{1-c} g(u) g\left(\frac{v-cu}{1-c}\right) \quad (\text{II-B-7})$$

$$\rho_S(u, v) = 12 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H_1(u) H_2(v) h(u, v) du dv - 3 \quad (\text{II-B-8})$$

$$T(u, v) = 4 \int_{-\infty}^0 G_2\left(\frac{1-c}{c} t\right) g_2(t) dt, \quad (\text{II-B-9})$$

where

$$g_2(t) = \int_{-\infty}^{\infty} g(w+t) g(w) dw$$

$$G_2(t) = \int_{-\infty}^{\infty} g_2(x) dx$$

Using equations (II-B-5), (II-B-6), (II-B-7), (II-B-8) and (II-B-9), we have to evaluate $H_1(u)$, $H_2(v)$, $h(u, v)$, $T(u, v)$, and $\rho_S(u, v)$ for all values of c and for specified $g(\cdot)$.

As Johnson and Tenenbein mentioned in their report, these integrals are quite tedious to perform, so they gave some computation results in their report.

The second procedure, called TVR (trivariate reduction) is discussed by Mardia (1970). In this case U and V are defined as

$$U = U' + \beta Z' \quad (\text{II-B-10})$$

$$V = V' + \beta Z' \quad (\text{II-B-11})$$

for $0 \leq \beta < \infty$, and U' , V' , and Z' are independent and identically distributed random variables with probability density function $g(\cdot)$.

Like the WLC procedure, one needs to define $H_1(u)$, $H_2(v)$, $h(u,v)$, $\rho_s(u,v)$ and $T(u,v)$ as a function of β and $g(\cdot)$. From equations (II-B-10) and (II-B-11) it follows that

$$H_1(u) = H_2(v) = \int_{R_3} \int g(u') g(z') du' dz' \quad (\text{II-B-12})$$

where

$$R_3 = \{(u', z') : u' + \beta z' \leq u\},$$

$$h(u,v) = \int_{-\infty}^{\infty} g(u - \beta z) g(v - \beta z) g(z) dz \quad (\text{II-B-13})$$

and

$$T(u,v) = 4 \int_{-\infty}^{\infty} [G_2(\beta t)]^2 g_2(t) dt - 1 \quad (\text{II-B-14})$$

where

$$g_2(t) = \int_{-\infty}^{\infty} g(w+t) g(w) dw$$

$$G_2(t) = \int_{-\infty}^{\infty} g_2(x) dx$$

using equations (II-B-12), (II-B-13), (II-B-14) and (II-B-8) the same computations are needed.

In this general WLC and TVR methods, there are two problems. One is the need for inverse transformations. The other is the need to generate coupled random variables to create dependence after the inverse transformation is applied. Doing this by linear combinations is not necessarily the simplest and most felicitous with respect to calculation of correlations and range of correlations.

A significant simplification can be achieved by obtaining a "smooth" bivariate uniform pair (U,V) whose correlation can be varied between the limit of correlation which can be obtained for uniform marginals. These limits are (-1,1) since the antithetic pair (U,1-U) has correlation -1.

Note that the problem is essentially one of obtaining a bivariate uniform random variable with positive correlation because

$$\text{corr}(U,V) = \rho_{u,v} = \begin{cases} -\rho_{U,1-V} = \text{corr}(U,1-V) \\ -\rho_{1-U,V} = \text{corr}(1-U,V) \end{cases}$$

Two fairly simple schemes for obtaining bivariate uniform distributions are discussed in the next section.

C. REVIEW OF SOME SPECIFIC METHODS

The mixture-truncation method is a general algorithm for generating bivariate pairs (Y,Z) with given marginal distributions and requires only the availability of a generator for the marginal distribution and calculation of constants. Unless randomization is used, however, it does give a bivariate distribution with a discontinuous joint distribution, as in Section (III-D). This could be a disadvantage. It is interesting therefore to compare it to other generations which are specific to the given marginal distribution and we do this here for the cases for which the mixture-truncation method is illustrated in this thesis, namely uniform, exponential and gamma.

There are many schemes available for uniform and exponential cases though it should be noted that smooth, simply generated pairs with easily calculated correlations have only recently been available. In particular we give here two

very recent schemes for smooth bivariate uniforms which are easily generated and whose correlations can be calculated; similarly two schemes for exponential which are essentially the uniform schemes after transformation. When one comes to gamma marginals one is again in difficulty and we describe a recent scheme by Schmeiser and Ram Lal which involves however, computation of the inverse gamma distribution function and very difficult initialization computations. The mixture-truncation method is quite competitive here. In fact if one can compute the inverse gamma distribution then bivariate gamma's can be computed as $[Y = F^{-1}(U), Z = F^{-1}(V)]$ where U and V are a bivariate uniform generated by the schemes mentioned above; this would be simpler for any marginal distributions than some of the general schemes mentioned in the previous sections. Comparisons of the generating schemes given in this section with the mixture-truncation method are given in Section IV-D and Section V-D.

1. Lawrance and Lewis's Bivariate Uniform

Lawrance and Lewis (1979) show that a simple transformation of the NEAR(1) (New Exponential Autoregressive) process gives a two-parameter family of Markovian random variables with uniform marginal distributions. This method generates a correlated uniform pair as a multiplicative mixture of uniform variables, where the parameters α and β take on values between zero and one, with the condition that they not both be one. Let Y be a uniform $[0,1]$ random variable, and define

$$Z = \begin{cases} \varepsilon Y^\beta & \text{wp } \alpha \\ \varepsilon & \text{wp } 1-\alpha \end{cases}$$

where

$$\varepsilon = \begin{cases} U & \text{wp } \frac{1-\beta}{1-(1-\alpha)\beta} \\ U^{(1-\alpha)\beta} & \text{wp } \frac{\alpha\beta}{1-(1-\alpha)\beta} \end{cases}$$

where U is an independent identically distributed uniform $(0,1)$ random variable. The correlation structure is defined as follows as a function of α and β ;

$$\rho_{Y,Z} = \frac{3}{2+\beta} \left(\frac{\alpha\beta}{1+(1-\alpha)\beta} \right)$$

The model can be reduced to a one-parameter model by suitably relating α and β , e.g., $\alpha = \beta$ and all positive values of ρ can be obtained. Consequently Y and $1-Z$ will have a full range of negative correlations. A generating procedure for this bivariate pair of uniform random variables is as follows.

Generating Procedure

1. (Initialization). For given correlation, find suitable parameter values α and β .
2. Generate U_1 , set $Y \leftarrow U_1$; $r = 1-\alpha$, $P \leftarrow \frac{1-\beta}{1-(1-\alpha)\beta}$.
3. Generate U_2 .

4. If $U_2 \leq r$, set $U \leftarrow \frac{U_2}{r}$ and go to 8.
5. Otherwise $U \leftarrow \frac{U_2 - r}{1 - r}$.
6. If $U \leq P$, set $Z \leftarrow Y^\beta \frac{U}{1 - P}$ and go to 10.
7. Otherwise, set $Z \leftarrow Y^\beta \left(\frac{U - P}{1 - P}\right)^{r\beta}$ and go to 10.
8. If $U \leq P$, set $Z \leftarrow \frac{U}{P}$ and go to 10.
9. Otherwise, set $Z \leftarrow \left(\frac{U - P}{1 - P}\right)^{r\beta}$.
10. Go to 2 and repeat until the desired number of bivariate uniform variables (Y, Z) are obtained.

The program is listed in the appendix and the scatter plot and bivariate histogram of resulting output are in Section IV-D.

2. Bivariate Uniform By Transformation of Gaver's Bivariate Exponential

In Gaver's bivariate exponential to be discussed in Section II-C-4, we can define a bivariate exponential random vector, (Y, Z) with correlation $\rho = -P/2$ as follows:

$$Y = \ln\left[\frac{1}{U^{1/N}(1-P)} - \frac{P}{1-P}\right] \quad (\text{II-C-1})$$

and Z is, conditional upon $N = n$, a gamma random variable with shape parameter n and mean $n(1-p)$, where N is a geometric random variables with probability density function $f(x) = p(1-p)^{x-1}$, $x = 1, 2, \dots$ and U is a uniform random variable over $[0, 1]$. From this we can generate a pair of

uniform random variables (V,W) by transformation. Since we know that to generate an exponential random variable X from a uniform random variable $U[0,1]$, we use the following transformation:

$$X = -\ln(1 - U) \quad (\text{II-C-2})$$

The resulting X has the exponential distribution with unit mean. From equations (II-C-1) and (II-C-2), we can generate V , a uniform random variable over $[0,1]$:

$$V = \frac{(1-p) U^{1/N}}{1 - p U^{1/N}}$$

Further also we can generate W , a uniform random variable over $[0,1]$:

$$W = \left(\prod_{i=1}^N U_i \right)^{1-p}$$

This follows since we know that Z has, conditionally upon $N = n$, the gamma distribution with shape parameter n and mean is $n(1-p)$.

In general the resulting V and W will be negatively correlated and $(V,1-W)$ will be a positively correlated pair. Because the correlation structure will be changed by transformation, the resulting (V,W) need not have the same

correlation with (Y,Z). The correlation coefficient of (V,W) will be computed as a function of P as follows.

$$\rho = 12 E[VW] - 3$$

where

$$\begin{aligned} E[VW] &= E_N[E[VW|N=n]] \\ &= E_N\left[\int_0^1 \frac{(1-p)U^{1/n}}{1-P U^{1/n}} du - \left[\int_0^1 u^{1-P} du\right]^n\right] \end{aligned}$$

Unfortunately this computation of ρ as a function of P is difficult. As an example we used here the same function for ρ as holds in the exponential case.

Generating Procedure

1. (Initialization)
 - i) Compute P for given correlation ρ
 - ii) Choose N from a geometric distribution with parameter 1-P
2. Generate a uniform [0,1] random variable U_1 and define

$$V = \frac{(1-P) U_1^{1/n}}{1 - P U_1^{1/n}}$$

3. Generate $N = n$ uniform [0,1] random variables U_i , $i = 1, \dots, n$, and define

$$W = \left(\prod_{i=1}^n U_i \right)^{1-p}$$

4. Deliver (V,W) and go to 2 until a sufficient number of bivariate pairs have been generated.

The program is listed in the appendix and the scatter plot and bivariate histogram of resulting random vectors are in Section IV-D.

3. Marshall and Olkin's Positively Correlated Bivariate Exponential

Suppose X is the age, or length of service of the first device at the time of death, governed by two independent Poisson processes $Z_1(t), Z_{12}(t)$ with parameters λ_1, λ_{12} respectively and Y the same of the second device, governed by two independent Poisson processes $Z_2(t), Z_{12}(t)$ with parameters λ_2, λ_{12} , respectively. Further suppose the second device is placed in operating after a time δ later than the first device. Then the joint distribution of (X,Y) is defined as

$$P[X > x, Y > y] \equiv F(x, y)$$

$$= \begin{cases} \exp\{-\lambda_1 x - \lambda_2 y - \lambda_{12} \max[x, y + \min(x, \delta)]\} & \text{if } \delta \geq 0 \\ \exp\{-\lambda_1 x - \lambda_2 y - \lambda_{12} \max[y, x + \min(y, \delta)]\} & \text{if } \delta \leq 0 \end{cases}$$

(II-C-3)

If $\delta = 0$, then equation (II-C-3) reduces to

$$P[X > x, Y > y] \equiv \bar{F}(x, y)$$

$$= \exp\{-\lambda_1 x - \lambda_2 y - \lambda_{12} \max(x, y)\}$$

Note that since the marginal distributions do not depend on δ ,

$$E[X] = \frac{1}{\lambda_1 + \lambda_{12}},$$

$$\text{VAR}[X] = \frac{1}{(\lambda_1 + \lambda_{12})^2},$$

$$E[Y] = \frac{1}{\lambda_2 + \lambda_{12}},$$

$$\text{VAR}[Y] = \frac{1}{(\lambda_2 + \lambda_{12})^2}$$

We also have, for $\delta \geq 0$

$$E[XY] = \frac{1}{(\lambda_1 + \lambda_{12})(\lambda_2 + \lambda_{12})} + \frac{\lambda_{12} e^{-(\lambda_1 + \lambda_{12})\delta}}{(\lambda_1 + \lambda_{12})(\lambda_2 + \lambda_{12})}$$

and the correlation

$$\text{corr}(X, Y) = \rho_{x, y} = \frac{\lambda_{12}}{\lambda} e^{-(\lambda_1 + \lambda_{12})\delta} \quad (\text{II-C-4a})$$

where

$$\lambda = \lambda_1 + \lambda_2 + \lambda_{12}.$$

If $\delta = 0$, then the equation (II-C-4a) reduces to

$$\rho_{xy} = \frac{\lambda_{12}}{\lambda} \quad (\text{II-C-4b})$$

This characterization can be represented in terms of independent random variables. For $\delta \geq 0$, if there exist independent exponential random variables U_1, U_2, U_3 and U_4 with respective parameters $\lambda_1, \lambda_2, \lambda_{12}, \lambda_{12}$, then

$$X = \min(U_1, U_3)$$

$$Y = \begin{cases} \min(U_2, U_3 - \delta) & \text{if } U_3 \geq \delta \\ \min(U_2, U_4) & \text{if } U_3 < \delta \end{cases}$$

It can be verified formally from the relation

$$\begin{aligned} P[X > x, Y > y] &= P[U_1 > x, U_2 > y] \{ P[U_3 > x, U_3 > y + \delta | U_3 > \delta] \\ &\quad \times P[U_3 > \delta] \} + P[U_4 > y] P[U_3 > x | U_3 < \delta] P[U_3 > \delta] \} \end{aligned}$$

In the $\delta = 0$ case, the representation yields

$$X = \min(U_1, U_3)$$

$$Y = \min(U_2, U_3)$$

This bivariate distribution has a line-discontinuity, if $E[X] = E[Y]$, along the line $x = y$, i.e., in the case where $x = y = U_3$. Thus this bivariate exponential is not as smooth as others. In particular the NEAR(1) process of Lawrance and Lewis generates bivariate exponentials with a continuous density function. But if we consider the case $\delta \geq 0$, the line-discontinuity will be removed. In either case, the resulting (X,Y) pairs always have positive correlation as shown in equations (II-C-4a) and (II-C-4b). If we use these methods to generate positively correlated random vectors (X,Y) for exponential marginal distribution with unit mean and given ρ , we have to compute λ_1 , λ_2 and λ_{12} for given correlations to generate independent exponential random variables. For simplicity we will consider the method with $\delta = 0$. From the relationship $\lambda = \lambda_1 + \lambda_2 + \lambda_{12}$,

$$E[X] = \frac{1}{\lambda_1 + \lambda_{12}} = 1$$

$$E[Y] = \frac{1}{\lambda_2 + \lambda_{12}} = 1$$

$$\rho = \frac{\lambda_{12}}{\lambda},$$

we can compute λ_1 , λ_2 and λ_{12} as functions of ρ .

$$\lambda_{12} = 2\rho / (1 + \rho)$$

$$\lambda_1 = \lambda_2 = 1 - \lambda_{12}.$$

Generating Procedure

1. (Initialization). Compute λ_1 , λ_2 and λ_{12} for given $0 \leq \rho \leq 1$.
2. Generate three independent exponential random variables E_1 , E_2 and E_3 with unit mean.
3. Rescale the independent random variables and set as

$$U_1 \leftarrow E_1/\lambda_1$$

$$U_2 \leftarrow E_2/\lambda_2$$

$$U_3 \leftarrow E_3/\lambda_{12}$$

4. Define X and Y as

$$X = \min[U_1, U_3]$$

$$Y = \min[U_2, U_3]$$

5. Deliver (X,Y) and go to 2 until a sufficient number of bivariate pairs have been generated.

The program is listed in the appendix and the scatter plot and bivariate histogram of resulting random vectors are shown in Section V-D.

4. Gaver's Negatively Correlated Bivariate Exponential

This negatively correlated bivariate exponential is generated from the following considerations. Suppose a particular system has N defective elements; here we assume N has geometric distribution with probability density function

$$f(x) = p(1-p)^{x-1}, \quad x = 1, 2, \dots$$

The time to failure of each defective element is T_i with density function $F(t)$. Further suppose repair is carried out after all defective elements are discovered. If we define \underline{R} as repairing time which is distributed as $G^n(x)$, provided $N = n$, and define \underline{T} as minimum of T_i , then the joint distribution function of \underline{T} and \underline{R} is obtainable from

$$P[\underline{T} > t, \underline{R} \leq x] = \sum_{n=1}^{\infty} (1 - F(t))^n P_n G^n(x)$$

where

$$P_n = (1-p) p^{n-1}, \quad 0 < p < 1, \quad n = 1, 2, \dots$$

Then

$$E[\underline{R} | \underline{T} > t] = \frac{E[R]}{1 - P[1 - F(t)]}$$

where $E[R]$ is the expected value of an element repair time.

Moreover,

$$\begin{aligned} E[\underline{R}|\underline{T}=t] &= E[R] \frac{1 + P[1 - F(t)]}{1 - P[1 - F(t)]} \\ &= E[R] \left[1 + \frac{2P}{1-P}(1 - \phi(t)) \right] \end{aligned}$$

where

$$\phi(t) = P[\underline{T} \leq t] = \frac{F(t)}{1 - P[1 - F(t)]}$$

From this regression, we know that if \underline{T} is long, \underline{R} is short and vice versa; the negative relationship is clearly present.

Further we can calculate $\text{Cov}[\underline{R}, \underline{T}]$ as

$$\text{Cov}[\underline{R}, \underline{T}] = P E[R] E[T] \left(1 - \frac{E[T(1)]}{E[T]} \right) < 0,$$

where $T(1)$ has the distribution of the smallest of a sample of two from $\phi(t)$.

It will not be shown that one can select $F(t) = 1 - \bar{F}(t)$ in such a way as to induce exponentially distributed time to system failure, \underline{T} . Since we know that exponential G , geometrically compounded, yields exponential \underline{R} , the outcome will be a $(\underline{T}, \underline{R})$ pair with exponential marginals and negative correlation.

If

$$F(t) = 1 - \bar{F}(t)$$

satisfies

$$\sum_{n=1}^{\infty} [\bar{F}(t)]^n p^{n-1} (1-p) = e^{-\lambda t}$$

then \underline{T} is exponential with mean $1/\lambda$. Solution for F yields,

$$\bar{F}(t) = \frac{1}{p + (1-p) e^{\lambda t}} \quad t \geq 0$$

which is a logistic distribution, left truncated at $t = 0$.

If $\lambda = 1$, then \underline{T} is exponential with unit mean, while

$E[\underline{T}_{(1)}] = 1/2$, so

$$\text{corr}(\underline{R}, \underline{T}) = -\frac{p}{2} E[\underline{R}] E[\underline{T}] .$$

If G is chosen so as to make \underline{R} exponential then it may be shown that

$$\text{corr}(\underline{R}, \underline{T}) = -\frac{p}{2} = -\frac{1}{2} \left[1 - \frac{1}{E[N]} \right]$$

Consequently, a greatest lower bound for the correlation in this model is $-\frac{1}{2}$.

Generating Procedure

1. If $0 < \rho < -0.5$, Set $P \leftarrow -2\rho$
2. Generate a geometric random variable N with parameter $1-P$
3. Generate a uniform $(0,1)$ random variable U and define Y as

$$Y = \ln \left[\frac{1}{U^{1/n}(1-p)} - \frac{P}{1-p} \right]$$

4. Generate a gamma random variable, Z , with shape parameter n , and mean is $n(1-p)$
5. Deliver (Y,Z) and go to 2 until a sufficient number of pairs are obtained.

For obtaining negatively correlated bivariate exponential random vectors, this algorithm is very simple and one of the few available. Moreover the correlation is known explicitly. The program is listed in the appendix and scatter plot and bivariate histogram of resulting random vectors are shown in Section V-D.

5. Arnold's Bivariate Gamma Generator

Arnold (1967) developed a Trivariate reduction method to generate bivariate gamma random vectors having positive correlation with

$$\text{corr } \rho < \min(\alpha_1, \alpha_2) / (\alpha_1 \alpha_2)^{1/2} ,$$

where α_1 and α_2 are the given shape parameters for the marginal distributions. Letting "gamma (α, β)" denote the gamma distribution with shape parameter α and scale parameter β , so that

$$f(x) = \frac{1}{\beta \Gamma(\alpha)} \left(\frac{x}{\beta}\right)^{\alpha-1} \exp(-x/\beta),$$

where $\Gamma(\alpha)$ is the gamma function and

$$E[x] = \alpha\beta, \quad \text{VAR}[x] = \alpha\beta^2,$$

the trivariate reduction algorithm proceeds as follows for given shape parameter values α_1, α_2 and ρ to generate unit scale gamma variate i.e., $\beta = 1$.

Generating Procedure

1. Generate $G_1 \sim \text{gamma}(\alpha_1 - \rho(\alpha_1\alpha_2)^{1/2}, 1)$
2. Generate $G_2 \sim \text{gamma}(\alpha_2 - \rho(\alpha_1\alpha_2)^{1/2}, 1)$
3. Generate $G_3 \sim \text{gamma}(\rho(\alpha_1\alpha_2)^{1/2}, 1)$
4. Define Y and Z

$$Y \leftarrow G_1 + G_3$$

$$Z \leftarrow G_2 + G_3$$

These Y and Z are unit gamma variates with $\beta = 1$, but Y and Z can be multiplied by β_1 and β_2 , respectively, to obtain any desired scaling. In the algorithm, G_3 is a common component of both Y and Z , thus inducing positive correlation.

When a positive, but not extreme, correlation is needed, trivariate reduction is an excellent algorithm, using univariate gamma generators. If the marginal distributions have the same shape parameter value α , then the correlation limitation is $0 \leq \rho < 1$. But in the other case the upper limit is smaller than 1.

Note that this algorithm exploits the infinite divisibility of the marginal gamma variables and is applicable to any infinitely divisible marginal. It is commonly used, for instance, to get bivariate Poisson random variables.

6. Schmeiser's Bivariate Gamma Generator

Schmeiser (1979) developed a family of algorithms, any member of which can generate bivariate gamma random vectors having any shape parameters α_1 , α_2 and allowable correlation ρ . In this section we will discuss his general procedure.

If Z , W_1 and W_2 are independent gamma random variables with shape parameters r , δ_1 , and δ_2 , respectively, and U is an independent uniform $[0,1]$ random variable, and either $V = U$ or $V = 1-U$; then

$$X_1 = F_n^{-1}(U) + Z + W_1 \quad (\text{II-C-5})$$

$$X_2 = F_m^{-1}(V) + Z + W_2$$

where F_n^{-1} and F_m^{-1} are the inverse distribution functions of gamma $(n,1)$ and gamma $(m,1)$, respectively, distribution function. The resulting X_1, X_2 are gamma random variables, with shape parameters

$$\alpha_1 = n + r + \delta_1, \tag{II-C-6}$$

$$\alpha_2 = m + r + \delta_2,$$

respectively. This result follows immediately from the reproducibility property of the gamma distribution and from noting that $F^{-1}(u)$ and $F^{-1}(1-u)$ are each random variables having CDF F . This scheme generalizes the trivariate scheme but brings in the inverse CDF.

The correlation coefficient is defined as

$$\rho_{X_1, X_2} = \frac{E[F_n^{-1}(u) F_m^{-1}(v)] - nm + r}{(\alpha_1 \alpha_2)^{1/2}} \tag{II-C-7}$$

The remaining problem is to select values of the five parameters n, m, r, δ_1 and δ_2 to obtain the desired marginal distributions and correlation. The following conditions must be satisfied.

$$n + r + \delta_1 = \alpha_1$$

$$m + r + \delta_2 = \alpha_2$$

$$\frac{E[F_n^{-1}(u) F_m^{-1}(v)] - (nm) + r}{(\alpha_1 \alpha_2)^{1/2}} = \rho \quad (\text{II-C-8})$$

$$H_1, H_2, r, \delta_1, \delta_2 \geq 0$$

Since we are using five variables to satisfy three equality conditions, finding a set of parameter values corresponds to finding a feasible solution, rather than an optimal solution, to a nonlinear programming problem.

An efficient solution procedure for determining parameter values is important, since substantial computation is required to determine whether or not conditions (II-C-8) are satisfied for given parameter values. Most of the computation is involved in calculating

$$\rho = \frac{E[F_n^{-1}(u) F_m^{-1}(v)] - nm - r}{(\alpha_1 \alpha_2)^{1/2}},$$

since the expected value must be calculated numerically using any one of the following three integrals:

$$\int_0^1 F_n^{-1}(u) F_m^{-1}(u) du \quad (\text{II-C-9a})$$

$$\int_0^\infty F_m^{-1}(F_n(x)) x^n \exp(-x) dx / \Gamma(n) \quad (\text{II-C-9b})$$

$$\int_0^{\infty} F_m^{-1}(F_n(-\ln y)) (-\ln y)^n dy / \Gamma(n) \quad (\text{II-C-9c})$$

if $\rho > 0$. If $\rho < 0$, then replace $F_n^{-1}(u)$ with $F_n^{-1}(1-u)$ in equation (II-C-9a), replace $F_n(x)$ with $1 - F_n(x)$ in equation (II-C-9b), and replace $F_n(-\ln y)$ with $1 - F_n(-\ln y)$ in equation (II-C-9c).

Schmeiser seemed to have best results in terms of a subjective trade off between speed and accuracy, using a 24 point Gaussian method for the integration. He selected the parameter values from the feasible values satisfying conditions (II-C-8) by making the curves of regression $E[X_1|X_2]$ and $E[X_2|X_1]$ behave as desired.

Generating Procedure

1. Generate $X_1 \sim \text{gamma}(n, 1)$
 2. $U \leftarrow F_n(X_1)$
If $\rho < 0$, $U \leftarrow 1 - u$
 3. Generate $Z \sim \text{gamma}(r, 1)$
 4. Generate $W_1 \sim \text{gamma}(\delta_1, 1)$
 5. Generate $W_2 \sim \text{gamma}(\delta_2, 1)$
 6. $X_1 \leftarrow X_1 + Z + W_1$
- $$X_2 \leftarrow F_{H_2}^{-1}(U) + Z + W_2$$

Based on these procedures he developed a family of algorithms which can provide variates having any theoretically possible correlation ρ . Anyway, for gamma marginal distributions, not all correlations are consistent with particular shape parameter values. Schmeiser shows the obtainable correlations as a function of α_2 , given $\alpha_1 = 1$ and 5 as shown in Section II-A. Note that only when $\alpha_1 = \alpha_2$ is it possible to obtain $\rho = 1$. Likewise $\rho = -1$ is not possible except in the limit as α_1 and α_2 tend to infinity. The maximum and minimum possible correlations, given in Moran (1969), occur when

$$X_2 = F_{\alpha_2}^{-1}(F_{\alpha_1}(X_1))$$

and

$$X_2 = F_{\alpha_2}^{-1}(1 - F_{\alpha_1}(X_1)),$$

respectively, where $F_{\alpha}(X)$ and $F_{\alpha}^{-1}(U)$ are the cumulative distribution function and inverse CDF, respectively, of the gamma distribution with shape parameter α .

III. GENERAL MIXTURE-TRUNCATION METHOD

Denote by (Y,Z) the bivariate random pair, where each has identical marginal continuous distribution $F(x)$, and denote a general random variable from this distribution by X . The argument is not specific to continuous random variables; this aspect comes in only in the computation of the correlations and can be developed in a parallel fashion for discrete marginal distributions. Let

$$P = \begin{bmatrix} \alpha_1 & 1 - \alpha_1 \\ 1 - \alpha_2 & \alpha_2 \end{bmatrix},$$

with stationary vector

$$\bar{\pi} = \bar{\pi} P = \left(\frac{1 - \alpha_2}{1 - \alpha_1 + 1 - \alpha_2}, \frac{1 - \alpha_1}{1 - \alpha_1 + 1 - \alpha_2} \right) = (\pi_1, \pi_2),$$

and let X_1 be an X truncated to the left of a fixed point x_0 , X_2 be an X truncated to the right of x_0 , so that

$$F_{X_1}(x) = P[X_1 \leq x] = \begin{cases} \frac{F(x)}{F(x_0)} & \text{if } x \leq x_0 \\ 1 & \text{if } x > x_0 \end{cases}$$

$$F_{X_2}(x) = P[X_2 \leq x] = \begin{cases} 0 & \text{if } x \leq x_0 \\ \frac{F(x) - F(x_0)}{1 - F(x_0)} & \text{if } x > x_0 \end{cases}$$

In addition we set $\pi_1 = F(x_0)$, $\pi_2 = 1 - \pi_1$ and choose Y and Z as follows.

- i) Choose Y from X_1 with probability π_1 and then choose Z from X_1 with probability α_1 , or from X_2 with probability $1 - \alpha_1$.
- ii) Choose Y from X_2 with probability π_2 , where $\pi_1 + \pi_2 = 1$, and then choose Z from X_1 with probability $1 - \alpha_2$, or from X_2 with probability α_2 .

If we choose (Y,Z) as in the above procedure, then we can make the following two theorems.

A. MARGINAL DISTRIBUTIONS

Theorem 1.

The marginal distribution of (Y,Z) becomes $F(x)$ for both Y and Z.

Proof

1. Marginal distribution of Y

By definition Y is the mixture of X_1 and X_2 with probability π_1 , π_2 respectively. That is

$$F_Y(x) = \pi_1 F_{X_1}(x) + \pi_2 F_{X_2}(x)$$

$$F_Y(x) = \begin{cases} \pi_1 \frac{F(x)}{F(x_0)} + \pi_2 \cdot 0 & \text{if } x \leq x_0 \\ \pi_1 \cdot 1 + \pi_2 \frac{F(x) - F(x_0)}{1 - F(x_0)} & \text{if } x > x_0 \end{cases}$$

But since we define $\pi_1 = F(x_0)$; $\pi_2 = 1 - \pi_1 = 1 - F(x_0)$, we have

$$F_Y(x) = \begin{cases} \pi_1 \frac{F(x_0)}{\pi_1} = F(x) & \text{if } x \leq x_0 \\ \pi_1 + \pi_2 \frac{F(x) - \pi_1}{\pi_2} = F(x) & \text{if } x > x_0 \end{cases}$$

= $F(x)$ in all cases.

So Y has the marginal distribution $F(x)$.

2. Marginal distribution of Z

If Y is from X_1 , then

$$F_{Z_1}(x) = \alpha_1 F_{X_1}(x) + (1 - \alpha_1) F_{X_2}(x)$$

$$= \begin{cases} \alpha_1 \frac{F(x)}{F(x_0)} + (1 - \alpha_1) \cdot 0 & \text{if } x \leq x_0 \\ \alpha_1 \cdot 1 + (1 - \alpha_1) \frac{F(x) - F(x_0)}{1 - F(x_0)} & \text{if } x > x_0 \end{cases}$$

If Y is from X_2 , then

$$F_{Z_2}(x) = (1 - \alpha_2) F_{X_1}(x) + \alpha_2 F_{X_2}(x)$$

$$= \begin{cases} (1 - \alpha_2) \frac{F(x)}{F(x_0)} + \alpha_2 \cdot 0 & \text{if } x \leq x_0 \\ (1 - \alpha_2) \cdot 1 + \alpha_2 \frac{F(x) - F(x_0)}{1 - F(x_0)} & \text{if } x > x_0 \end{cases}$$

So,

$$F_Z(x) = \pi_1 F_{Z_1}(x) + \pi_2 F_{Z_2}(x)$$

$$= \begin{cases} \pi_1 \alpha_1 \frac{F(x)}{F(x_0)} + \pi_2 (1 - \alpha_2) \frac{F(x)}{F(x_0)} & \text{if } x \leq x_0 \\ \pi_1 (\alpha_1 + (1 - \alpha_1)) \frac{F(x) - F(x_0)}{1 - F(x_0)} & \text{if } x > x_0 \\ + \pi_2 ((1 - \alpha_2) + \alpha_2) \frac{F(x) - F(x_0)}{1 - F(x_0)} \end{cases}$$

and we defined π_1 and π_2 as follows.

$$\pi_1 = \frac{1 - \alpha_2}{1 - \alpha_1 + 1 - \alpha_2}$$

$$\pi_2 = \frac{1 - \alpha_1}{1 - \alpha_1 + 1 - \alpha_2}$$

From this, we know

$$1 - \alpha_2 = \frac{\pi_1}{\pi_2} (1 - \alpha_1)$$

If we use this relationship, then $F_Z(x) = F(x)$ in both cases. So Z also has the marginal distribution $F(x)$. The result is a consequence of the fact that $\underline{\pi}$ is defined to be the stationary vector associated with \underline{P} .

B. THE PRODUCT-MOMENT CORRELATION

Theorem 2.

The correlation coefficient between Y and Z becomes

$$\rho = \beta M ,$$

where

$$-1 \leq \beta = \alpha_1 - (1 - \alpha_2) \leq 1 ,$$

and

$$M = (\mu_1 - \mu_2)^2 \pi_1 \pi_2 / \sigma_x^2 ,$$

where

$$\mu_1 = \int_0^{x_0} x d \frac{F(x)}{F(x_0)}$$

$$\mu_2 = \int_{x_0}^{\infty} x \, d \frac{F(x) - F(x_0)}{1 - F(x_0)}$$

$$\sigma_1^2 = \int_0^{x_0} x^2 \, d \frac{F(x)}{F(x_0)} - \mu_1^2$$

$$\sigma_2^2 = \int_{x_0}^{\infty} x^2 \, d \frac{F(x) - F(x_0)}{1 - F(x_0)} - \mu_2^2$$

$$\sigma_x^2 = \int_0^{\infty} x^2 \, dF(x) - E[X]^2$$

$$= \sigma_1^2 \pi_1 + \sigma_2^2 \pi_2 + (\mu_1 - \mu_2)^2 \pi_1 \pi_2$$

Proof

$$\rho_{YZ} = \frac{\text{cov}[Y, Z]}{\sigma_Y \sigma_Z}$$

$$= \frac{E[YZ] - E[Y] E[Z]}{\sigma_Y \sigma_Z}$$

Now

$$\begin{aligned}
E[YZ] &= E_S[E[YZ|Y, Z \in S]] \\
&= E[YZ|Y \in X_1, Z \in X_1]P[Y \in X_1, Z \in X_1] \\
&\quad + E[YZ|Y \in X_1, Z \in X_2]P[Y \in X_1, Z \in X_2] \\
&\quad + E[YZ|Y \in X_2, Z \in X_1]P[Y \in X_2, Z \in X_1] \\
&\quad + E[YZ|Y \in X_2, Z \in X_2]P[Y \in X_2, Z \in X_2] \\
&= E[X_1] E[X_1] \pi_1 \alpha_1 + E[X_1] E[X_2] \pi_1 (1 - \alpha_1) \\
&\quad + E[X_2] E[X_1] \pi_2 (1 - \alpha_2) + E[X_2] E[X_2] \pi_2 \alpha_2 \\
&= \mu_1^2 \alpha_1 \pi_1 + \mu_1 \mu_2 (1 - \alpha_1) \pi_1 \\
&\quad + \mu_1 \mu_2 (1 - \alpha_2) \pi_2 + \mu_2^2 \alpha_2 \pi_2
\end{aligned}$$

where

$$\mu_1 = E[X_1], \quad \mu_2 = E[X_2]$$

Further,

$$\begin{aligned}
E[Y] &= E_S[E[Y|Y \in S]] \\
&= E[Y|Y \in X_1]P[Y \in X_1] + E[Y|Y \in X_2]P[Y \in X_2] \\
&= E[X_1] \pi_1 + E[X_2] \pi_2 \\
&= \mu_1 \pi_1 + \mu_2 \pi_2 = E[Z]
\end{aligned}$$

by Theorem 1.

Also by Theorem 1,

$$\begin{aligned}\sigma_Y^2 &= E[Y^2] - (E[Y])^2 \\ &= \sigma_X^2 = \sigma_Z^2\end{aligned}$$

If we put together these formulae into

$$\rho_{Y,Z} = \frac{E[YZ] - E[Y]E[Z]}{\sigma_Y \sigma_Z}$$

we get

$$\rho_{Y,Z} = \frac{(\mu_1 - \mu_2)^2 \pi_1 \pi_2 (\alpha_1 - (1 - \alpha_2))}{\sigma_X^2}$$

and let

$$\beta = \alpha_1 - (1 - \alpha_2)$$

$$M = (\mu_1 - \mu_2)^2 \pi_1 \pi_2 / \sigma_X^2$$

then

$$\rho = \beta M .$$

C. GENERAL ALGORITHM

We give here three algorithms for implementing the bivariate mixture-truncation method, which we call the FXO

method, the UXO method, and the TXO method. All of these methods are exactly the same except in how the algorithm chooses x_0 , the truncation point, from the x_0 range $[x_\ell, x_u]$. The first procedure, called the FXO, choose x_0 as a fixed point of x_ℓ and x_u and uses the same x_0 during the entire routine. The second procedure, called the UXO, chooses x_0 uniformly from $[x_\ell, x_u]$ and repeats this step in every call to the algorithm. The third procedure, called the TXO, is the same as the UXO procedure except in that it uses a triangular distribution instead of uniform. It is necessary to fix these choices of x_0 because in general there is more than one x_0 which will give a bivariate pair (Y, Z) with the given marginal distribution and given correlation. The first procedure, FXO, is defective in terms of their discontinuity of distribution while the second and the third, UXO and TXO, are satisfactory in this respect. The choice of the midpoint of the interval $[x_\ell, x_u]$ for x_0 in FXO is based on experience presented later for various marginal distributions. Note that the algorithm described here is inefficient in that it generates the truncated variables X_1 and X_2 by comparing random variables X to x_0 until one which is respectively greater than or less than x_0 is found. More efficient methods can be found in special cases such as the exponential, but the present algorithm requires only a generation of univariate random variables X without regard to the method used to do this. Of course initialization is required and this is specific to each marginal distribution.

General Mixture-Truncation Method

1. (Initialization)
 - i) For given marginal distribution $F(x)$ and correlation coefficient ρ find x_0 ranges $[x_\ell, x_u]$
2. Define truncation point x_0
 - * FXO method
 - i) $X_0 = \frac{1}{2}(x_\ell + x_u)$
 - * UXO method
 - i) Generate a uniform $[0,1]$ random variable V_1
 - ii) $x_0 = x_\ell + (x_u - x_\ell) * V_1$
 - * TXO method
 - i) Generate two uniform $[0,1]$ random variables V_1, V_2 .
 - ii) $X_0 = x_\ell + x_1 + x_2$

where

$$x_m = \frac{1}{2}(x_\ell + x_u)$$

$$x_1 = (x_m - x_\ell) * V_1$$

$$x_2 = (x_u - x_m) * V_2$$

3. Compute parameters value, $\pi_1, \pi_2, \alpha_1, \alpha_2$
4. Choose type for Y
 - i) Generate a uniform $[0,1]$ random variable U
 - ii) If $U \leq \pi_1$, go to 9

5. Y is an X_2

- i) Generate a random variable X from $F(x)$
- ii) If $X > x_0$, set $Y \leftarrow X$ and go to 6
- iii) Otherwise return to 5.i)

6. Choose type for Z

- i) Set $U \leftarrow ((U - \pi_1)/(1 - \pi_1))$
- ii) If $U \leq 1 - \alpha_2$, go to 8

7. Z is an X_2

- i) Generate a random variable X from $F(x)$
- ii) If $X > x_0$, set $Z \leftarrow X$ and go to 11
- iii) Otherwise return to 7.i)

8. Z is on X_1

- i) Generate a random variable X from $F(x)$
- ii) If $X \leq x_0$, set $Z \leftarrow X$ and go to 11
- iii) Otherwise return to 8.i)

9. Y is an X_1

- i) Generate a random variable X from $F(x)$
- ii) If $X \leq x_0$, set $Y \leftarrow X$ and go to 10
- iii) Otherwise return to 9.i)

10. Choose type for Z

- i) Set $U \leftarrow U/\pi_1$

ii) If $U \leq \alpha_1$, go to 8

iii) Otherwise go to 7

11. Deliver (Y, Z) and go to 4 for the FXO method, or go to 2 for the UXO and TXO method until a sufficient number of random vectors are obtained.

D. BIVARIATE DISTRIBUTION FUNCTIONS

From Theorem 1, in Section III-A, we know that if Y is from X_1 , then

$$F_Z(z|y) = \alpha_1 F_{X_1}(z) + (1 - \alpha_1) F_{X_2}(z)$$
$$= \begin{cases} \alpha_1 \frac{F(z)}{F(x_0)} & \text{if } z \leq x_0 \\ \alpha_1 + (1 - \alpha_1) \frac{F(z) - F(x_0)}{1 - F(x_0)} & \text{if } z > x_0 \end{cases}$$

and if Y is from X_2 , then

$$F_Z(z|y) = (1 - \alpha_2) F_{X_1}(z) + \alpha_2 F_{X_2}(z)$$
$$= \begin{cases} (1 - \alpha_2) \frac{F(z)}{F(x_0)} & \text{if } z \leq x_0 \\ (1 - \alpha_2) + \alpha_2 \frac{F(z) - F(x_0)}{1 - F(x_0)} & \text{if } z > x_0 \end{cases}$$

By using these we can define the bivariate distribution function as follows.

$$\begin{aligned}
 F(y, z) &= P[Y \leq y, Z \leq z] \\
 &= \int_{-\infty}^y P[Z \leq z | Y = u] dP[Y \leq u]
 \end{aligned}$$

where

$$P[Y \leq u] = F(u)$$

$$P[Z \leq z | Y = u] = \begin{cases} \alpha_1 \frac{F(z)}{F(x_0)} & \text{if } u \leq x_0, z \leq x_0 \\ \alpha_1 + (1 - \alpha_1) \frac{F(z) - F(x_0)}{1 - F(x_0)} & \text{if } u \leq x_0, z > x_0 \\ (1 - \alpha_2) \frac{F(z)}{F(x_0)} & \text{if } u > x_0, z \leq x_0 \\ (1 - \alpha_2) + \alpha_2 \frac{F(z) - F(x_0)}{1 - F(x_0)} & \text{if } u > x_0, z > x_0 \end{cases}$$

So, if we put these together, integrating with respect to $dP[Y \leq u]$, we get the final result:

$$\begin{aligned}
 P[Y \leq y, Z \leq z] &= \left\{ \begin{array}{l} \frac{\alpha_1}{\pi_1} F(z)F(y) \quad \text{if } y \leq x_0, z \leq x_0 \quad (\text{III-D-1}) \\ \left\{ \alpha_1 + \frac{(1-\alpha_1)}{\pi_2} [F(z) - F(x_0)] \right\} F(y) \quad \text{if } y \leq x_0, z > x_0 \quad (\text{III-D-2}) \\ \left\{ \alpha_1 + \frac{(1-\alpha_1)}{\pi_2} [F(y) - F(x_0)] \right\} F(z) \quad \text{if } y > x_0, z \leq x_0 \quad (\text{III-D-3}) \\ \alpha_1 \pi_1 + (1-\alpha_1) [F(z) - F(x_0)] \frac{\pi_1}{\pi_2} \quad (\text{III-D-4}) \\ + \left\{ (1-\alpha_2) + \frac{\alpha_2}{\pi_2} [F(z) - F(x_0)] \right\} [F(y) - F(x_0)] \quad \text{if } y > x_0, z > x_0 \end{array} \right.
 \end{aligned}$$

For example, the expression (III-D-4) is obtained as

$$\begin{aligned}
 F(y, z) &= \int_{-\infty}^y P[Z \leq z | Y = u] dF(u) \quad y > x_0, z > x_0 \\
 &= \int_{-\infty}^{x_0} P[Z \leq z | Y = u \leq x_0] dF(u) \\
 &\quad + \int_{x_0}^y P[Z \leq z | Y = u > x_0] dF(u) \\
 &= \left\{ \alpha_1 + \frac{(1-\alpha_1)}{\pi_2} [F(z) - F(x_0)] \right\} F(x_0) \\
 &\quad + \left\{ (1-\alpha_2) + \frac{\alpha_2}{\pi_2} [F(z) - F(x_0)] \right\} [F(y) - F(x_0)]
 \end{aligned}$$

It is easily seen from (III-D-2) that when $z \rightarrow \infty$,

$P[Y \leq y, Z \leq \infty] = F(y)$; from (III-D-3) that as $y \rightarrow \infty$,

$P[Y \leq \infty, Z \leq z] = F(z)$ and from (III-D-4) that as $y \rightarrow \infty$,

$P[Y \leq \infty, Z \leq z] = F(z)$ and that $z \rightarrow \infty$, $P[Y \leq y, Z \leq \infty] = F(y)$.

In particular from (III-D-4) we have that, as $y \rightarrow \infty$,

$$\begin{aligned}
 F(y, z) &\rightarrow \alpha_1 F(x_0) + (1 - \alpha_2) [F(z) - F(x_0)] + (1 - \alpha_2) \pi_2 \\
 &\quad + \alpha_2 [F(z) - F(x_0)] \\
 &= \alpha_1 F(x_0) + (1 - \alpha_2) \pi_2 + [F(z) - F(x_0)] \\
 &= F(z) + (1 - \alpha_2) \pi_2 - (1 - \alpha_1) \pi_1 = F(z)
 \end{aligned}$$

where at the last step we used the facts that $F(x_0) = \pi_1$

and $\pi_1(1 - \alpha_1) = \pi_2(1 - \alpha_2)$. If $F(x)$ is absolutely continuous with probability density function $f(x)$ then the joint p.d.f.

for the bivariate pair (Y, Z) is

$$f(y, z) = \begin{cases} \frac{\alpha_1(1 - \alpha_1)}{\pi_2(1 - \alpha_2)} f(y) f(z) & \text{if } y \leq x_0, z \leq x_0 & \text{(III-D-5)} \\ \frac{1 - \alpha_1}{\pi_2} f(y) f(z) & \text{if } y \leq x_0, z > x_0 & \text{(III-D-6)} \\ \frac{1 - \alpha_1}{\pi_2} f(y) f(z) & \text{if } y > x_0, z \leq x_0 & \text{(III-D-7)} \\ \frac{\alpha_2}{\pi_2} f(y) f(z) & \text{if } y > x_0, z > x_0 & \text{(III-D-8)} \end{cases}$$

Note that there is a discontinuity in the density function as one crosses the boundaries of the four quadrants defined by the lines $y = x_0$; $z = x_0$. The density is the same in the first and third quadrants. The multipliers of $f(y)f(z)/\pi_2$ are the same in all four quadrants iff there is independence. This occurs when $(1 - \alpha_1) = \alpha_2$ so that $\pi_1 = \alpha_2$ and $f(y,z) = f(y)f(z)$ for the whole range of y and z .

IV. BIVARIATE UNIFORM GENERATOR

The uniform random variable is a continuous random variable with probability density function which is constant over the interval (a,b) and zero otherwise; the density function

$$f(x) = \begin{cases} \frac{1}{b-a}; & \text{if } a \leq x \leq b, \\ 0; & \text{otherwise;} \end{cases}$$

$$\text{mean} = E[X] = \frac{b+a}{2};$$

$$\text{variance} = \text{VAR}[X] = \frac{(b-a)^2}{12}.$$

We note that if U has a uniform distribution over [0,1] then $X = a + (b-a)U$ has a uniform distribution over [a,b]. So we will only be concerned with algorithms for generating uniform [0,1] distributions.

Two algorithms for generating uniform bivariate pairs are given in Section II, Gaver's and Lawrance and Lewis's, and since they have relatively smooth distributed functions and are simple to generate, they are probably preferable to the uniform bivariate random variable generated by the mixture-truncation method unless x_0 is taken to be random (of course the correlation for the Gaver bivariate uniform is difficult

to compute). However the development in this section of the uniform mixture-truncation method does help to fix ideas on the problem of initialization and determination of the possible range of correlations attainable in other cases. To use the mixture-truncation method for generating bivariate random vector (Y,Z) whose marginal distribution is uniform over $[0,1]$ identically and has given correlation ρ , we should use uniform $[0,1]$ distribution functions as $F(x)$. Then X_1 , which is X constrained to be on the left side of the truncation point x_0 , becomes uniform over $[0,x_0]$ and X_2 , which is constrained to be on the right side of x_0 becomes uniform over $[x_0,1]$.

A. DETERMINATION OF PARAMETERS IN THE UNIFORM MIXTURE-TRUNCATION METHOD

Because X_1 is uniform $[0,x_0]$ and x_2 is uniform $[x_0,1]$,

$$E[X_1] = \mu_1 = \frac{x_0}{2}$$

$$\text{VAR}[X_1] = \sigma_1^2 = \frac{x_0^2}{12}$$

$$E[X_2] = \mu_2 = \frac{1+x_0}{2}$$

$$\text{VAR}[X_2] = \sigma_2^2 = \frac{(1-x_0)^2}{12}$$

and

$$\pi_1 = F(x_0) = \frac{1 - \alpha_2}{1 - \alpha_1 + 1 - \alpha_2} = x_0, \quad (\text{IV-A-1a})$$

$$\pi_2 = 1 - F(x_0) = \frac{1 - \alpha_1}{1 - \alpha_1 + 1 - \alpha_2} = 1 - x_0 \quad (\text{IV-A-1b})$$

If we use these formulas in Theorem 2 of Section III, we get

$$M = 3 x_0 (1 - x_0)$$

$$\beta = \frac{\alpha_1 - x_0}{1 - x_0}$$

$$\rho = 3 x_0 (\alpha_1 - x_0) \quad (\text{IV-A-2})$$

Then from (IV-A-2), (IV-A-1a) and (IV-A-1b)

$$\alpha_1 = \frac{\rho}{3x_0} + x_0 \quad (\text{IV-A-3a})$$

$$\begin{aligned} \alpha_2 &= 1 - \frac{\pi_1}{\pi_2} (1 - \alpha_1) \\ &= 1 - \frac{x_0^2 + x_0}{1 - x_0} + \frac{\rho}{3(1 - x_0)} \end{aligned} \quad (\text{IV-A-3b})$$

Furthermore, α_1 and α_2 are probabilities so they have to satisfy the following conditions:

$$0 \leq \alpha_1 \leq 1,$$

$$0 \leq \alpha_2 \leq 1 .$$

By solving these two inequality equations, we can get the feasible range of x_0 for given ρ as follows.

x_l = Lower bound of x_0

$$= \begin{cases} 1/2 - \sqrt{1/4 - (1/3)\rho} & \text{if } \rho > 0 \\ \sqrt{-\rho/3} & \text{if } \rho < 0 \end{cases} \quad (\text{IV-A-4})$$

x_u = Upper bound of x_0

$$= \begin{cases} 1/2 + \sqrt{1/4 - (1/3)\rho} & \text{if } \rho > 0 \\ 1 - \sqrt{-\rho/3} & \text{if } \rho < 0 \end{cases} \quad (\text{IV-A-5})$$

After finding x_l and x_u , choose x_0 within the range $[x_l, x_u]$ by way of either the FXO, UXO, or TXO method. And by formulas (IV-A-1a), (IV-A-1b), (IV-A-3a) and (IV-A-3b), we can get π_1 , π_2 , α_1 and α_2 . Also we can expect some limitations in ρ because of x_0 . From

$$M = 3 x_0 (1 - x_0)$$

$$-1 \leq \beta = \alpha_1 - (1 - \alpha_2) \leq 1$$

$$\rho = \beta M$$

We can compute that

$$\rho_{\max} = 3/4 \text{ if } \beta = 1, x_l \leq x_0 = 1/2 \leq x_u$$

$$\rho_{\min} = -3/4 \text{ if } \beta = -1, x_l \leq x_0 = 1/2 \leq x_u$$

In another way, note that if one chooses $x_0 = 1/2$ and $\alpha_2 = \alpha_1 = 1$, from (IV-A-2) we attain the highest allowable correlation $\rho = 3/4$, and when $\alpha_1 = 0$, $\rho = -3/4$, the lowest attainable correlation.

It turns out that for this choice of $x_0 = 1/2$ (in the FXO method), the bivariate uniform distribution is also about the smoothest, as can be seen in a later section (IV-D), and from the bivariate distribution.

B. GENERATING PROCEDURE

We developed here all of the three procedures, the FXO method, the UXO method, the TXO method for generating bivariate random vectors whose marginal distributions are uniform over $[0,1]$ and correlation coefficient is ρ . In this algorithm we used direct generating procedures for X_1 and X_2 instead of comparing random variables X to x_0 until one which is respectively greater than or less than x_0 is found.

Uniform Mixture-Truncation Method

1. (Initialization)

i) For given $-3/4 < \rho < 3/4$, find x_ℓ and x_u

$$x_\ell = \begin{cases} 1/2 - \sqrt{1/4 - 1/3 \rho} & \text{if } \rho > 0 \\ \sqrt{-\rho/3} & \text{if } \rho < 0 \end{cases}$$

$$x_u = \begin{cases} 1/2 + \sqrt{1/4 - 1/3 \rho} & \text{if } \rho > 0 \\ 1 - \sqrt{-\rho/3} & \text{if } \rho < 0 \end{cases}$$

2. Define truncation point x_0 .

* FXO method

$$i) x_0 = (x_\ell + x_u)/2$$

* UXO method

i) Generate a uniform $[0,1]$ random variable U_1

$$ii) x_0 = x_\ell + (x_u - x_\ell) * U_1$$

* TXO method

i) Generate two uniform $[0,1]$ random variables

$$V_1, V_2$$

$$ii) x_0 = x_\ell + x_1 + x_2$$

where

$$x_1 = (x_m - x_\ell) * V_1$$

$$x_2 = (x_u - x_m) * V_2$$

$$x_m = (x_u + x_l)/2$$

3. Compute π_1 , π_2 , α_1 and α_2 as

$$\pi_1 = F(x_0) = x_0$$

$$\pi_2 = 1 - \pi_1$$

$$\alpha_1 = \frac{\rho}{3} x_0 + x_0$$

$$\alpha_2 = 1 - \frac{\pi_1}{\pi_2}(1 - \alpha_1)$$

4. Choose type for Y

i) Generate a uniform [0,1] random variable U

ii) If $U \leq \pi_1$ go to 9

5. Y is on X_2

i) Generate a uniform [0,1] random variable W_1

ii) $Y \leftarrow x_0 + (1.0 - x_0) * W_1$

6. Choose type for Z

i) Set $U \leftarrow ((U - \pi_1)/(1 - \pi_1))$

ii) If $U \leq 1 - \alpha_2$, go to 8

7. Z is an X_2

i) Generate a uniform [0,1] random variable W_2

ii) $Z \leftarrow x_0 + (1.0 - x_0) * W_2$

iii) Go to 11

8. Z is an X_1

i) Generate a uniform [0,1] random variable W_2

ii) $Z \leftarrow x_0 * W_2$

iii) Go to 11

9. Y is an X_1

i) Generate a uniform [0,1] random variable W_1

ii) $Y \leftarrow x_0 * W_1$

10. Choose type for Z

i) Set $U \leftarrow U/\pi_1$

ii) If $U \leq \alpha_1$ go to 8

iii) Otherwise go to 7

11. Deliver (Y,Z) and go to 4, for the FXO method, or go to 2 for the UXO method and the TXO method until a sufficient number of random vectors are obtained.

The programs are listed in the appendix and the scatter plot and bivariate histogram of resulting random vectors are shown in Section IV-D.

C. REGRESSION OF Z ON Y FOR GIVEN ρ

Schmeiser (1979) in developing a bivariate gamma method fixed the free parameter by specifying the regression of Z and Y and vice-versa. We do this now for the uniform case with fixed x_0 and uniformly distributed x_0 . First in fixed x_0 case we have, for $y \leq x_0$

$$\begin{aligned} E[Z|Y=y, y \leq x_0] &= \alpha_1 E[X_1] + (1 - \alpha_1) E[X_2] \\ &= 1/2 - \frac{1}{6x_0} \rho \end{aligned}$$

and for $y > x_0$

$$\begin{aligned} E[Z|Y=y, y > x_0] &= (1 - \alpha_2) E[X_1] + \alpha_2 E[X_2] \\ &= 1/2 + \rho \frac{1}{6(1 - x_0)} \end{aligned}$$

These are the step function and clearly not linear in y.

If x_0 has uniform distribution over $[x_l, x_u]$, we have the following results. These results are dependent on Y values.

If $y \leq x_l$, then we have

$$\begin{aligned} E[Z|Y=y] &= \int_{x_l}^{x_u} E[Z|Y=y, X=x_0, y \leq x_0] f(x_0) dx_0 \\ &= \int_{x_l}^{x_u} \left(\frac{1}{2} - \frac{1}{6x_0} \rho \right) \frac{1}{x_u - x_l} dx_0 \end{aligned}$$

$$E[Z|Y=y] = \frac{1}{x_u - x_l} \left(\frac{1}{2} x_u - \frac{1}{2} x_l - \frac{\rho}{6} \ln \frac{x_u}{x_l} \right)$$

If $x_l \leq Y \leq x_u$, then we have

$$\begin{aligned} E[Z|Y=y] &= \int_{x_l}^y E[Z|Y=y, X=x_0, y > x_0] f(x_0) dx_0 \\ &+ \int_y^{x_u} E[Z|Y=y, X=x_0, y \leq x_0] f(x_0) dx_0 \\ &= \int_{x_l}^y \left(\frac{1}{2} + \frac{\rho}{6(1-x_0)} \right) f(x_0) dx_0 \\ &+ \int_y^{x_u} \left(\frac{1}{2} - \frac{\rho}{6x_0} \right) f(x_0) dx_0 \\ &= \frac{1}{x_u - x_l} \left(\frac{1}{2} x_u - \frac{1}{2} x_l - \frac{\rho}{6} \ln \frac{1-y}{y} \frac{x_u}{1-x_l} \right) \end{aligned}$$

If $y > x_l$, then we

$$\begin{aligned} E[Z|Y=y] &= \int_{x_l}^{x_u} E[Z|Y=y, X=x_0, y > x_0] f(x_0) dx_0 \\ &= \int_{x_l}^{x_u} \left(\frac{1}{2} + \frac{\rho}{6(1-x_0)} \right) f(x_0) dx_0 \\ &= \frac{1}{x_u - x_l} \left(\frac{1}{2} x_u - \frac{1}{2} x_l - \frac{\rho}{6} \ln \frac{1-x_u}{1-x_l} \right) \end{aligned}$$

From the results we can see that if we use uniform distribution for x_0 we can get smoother regression functions if $x_l \leq y \leq x_u$ although we still have step functions for the $y \leq x_l$ and $y \geq x_u$ cases.

D. SIMULATION RESULTS

We will show here the scatter plots and the bivariate histograms of the mixture-truncation bivariate uniform (0,1) random vectors with correlation $\rho = 0.3$ and $\rho = -0.3$ by the FXO, UXO and TXO methods in Figures (IV-a), (IV-b) and (IV-c), respectively.

In addition, we will show the scatter plot and bivariate histogram of sample size 2000 from Gaver's transformation and Lawrance-Lewis method in Figures (IV-e) and (IV-d), respectively. The Gaver method does not have known correlation but the value of p to give ρ in the exponential case is used.

As we mentioned earlier, the FXO method has some discontinuity at the truncation point. But the UXO and TXO methods generate relatively smooth continuous distributions. In this respect, we say that the FXO is defective and the UXO and the TXO are relatively satisfactory.

The computed correlation from subroutine BIVHST (Bivariate histogram) is a little different from given correlation. But this can be assumed as a sampling error. To check this error, we simulated 10 times with given correlation $\rho = 0.1$. From this simulation we get the following results:

$\bar{\rho}$ = mean of computed correlation = 0.105

$\text{VAR}[\rho]$ = variance of computed correlation = 0.0004

$\sigma[\bar{\rho}]$ = standard deviation of mean = 0.0064

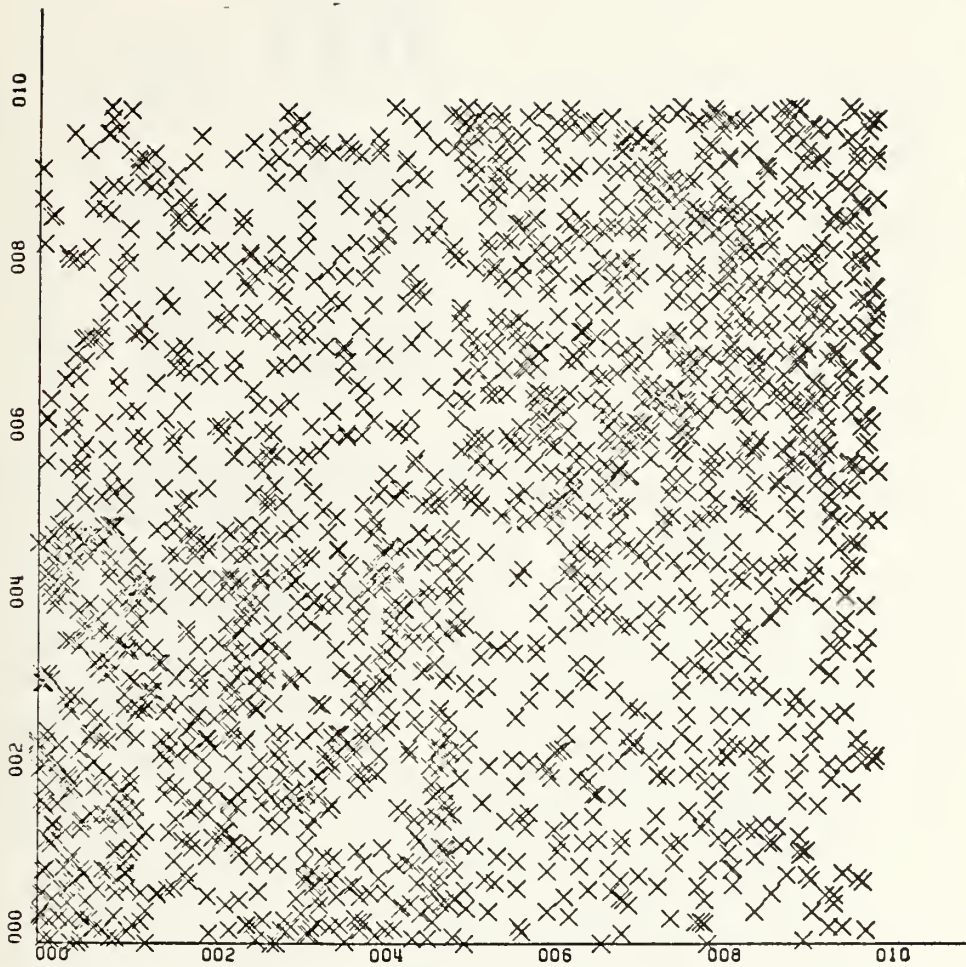
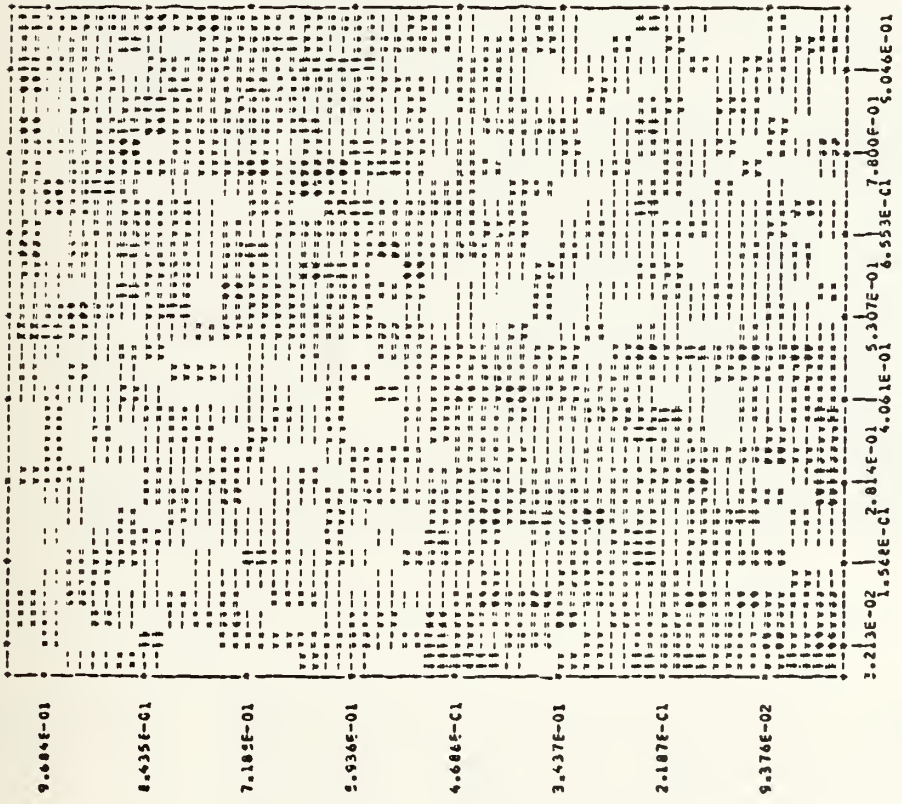


Figure IV-a1. Scatter plot for sample of size 2000 from mixture-truncation bivariate uniform with $\rho = 0.3$. Here x_0 is fixed at the midpoint between the lower and upper bounds.



KEY

SYMBOL PRINTED 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15

MEASURES OF ASSOCIATION
 COVARIANCE 2.5197E-02
 CORRELATION COEFFICIENT 0.301639
 SPEARMAN'S RANK CORRELATION COEF. 0.301639

TESTS FOR EQUIDISTRIBUTION
 KOLMOGOROV-SMIRNOV TEST 0.072720
 WALD-WILFENY TEST 1902142
 PEARSON TEST 3993142
 SIEGEL-TUKEY TEST 3913187

UNIVARIATE STATISTICS
 X SAMPLE Y SAMPLE
 MEAN 3.0512E-01 3.0297E-01
 MEDIAN 3.1594E-01 3.15917E-01
 VARIANCE 0.66894E-02 8.12119E-02
 STD DEV 2.94437E-01 2.84977E-01
 RANGE 9.57130E-01 9.99629E-01
 SKWNESS -5.48591E-02 -4.27645E-02
 KURTOSIS -1.23524E-00 -1.14810E-00
 MAXIMUM 9.98096E-01 9.99741E-01
 MINIMUM 9.86540E-01 4.43756E-01

NORMALIZED STATISTIC
 0.60001
 -0.21817
 -2.38832

Figure IV-a2. Bivariate histogram for sample of size 2000 from mixture-truncation bivariate uniform with $\rho = 0.3$. Here x_0 is fixed at the midpoint between the lower and the upper bounds.

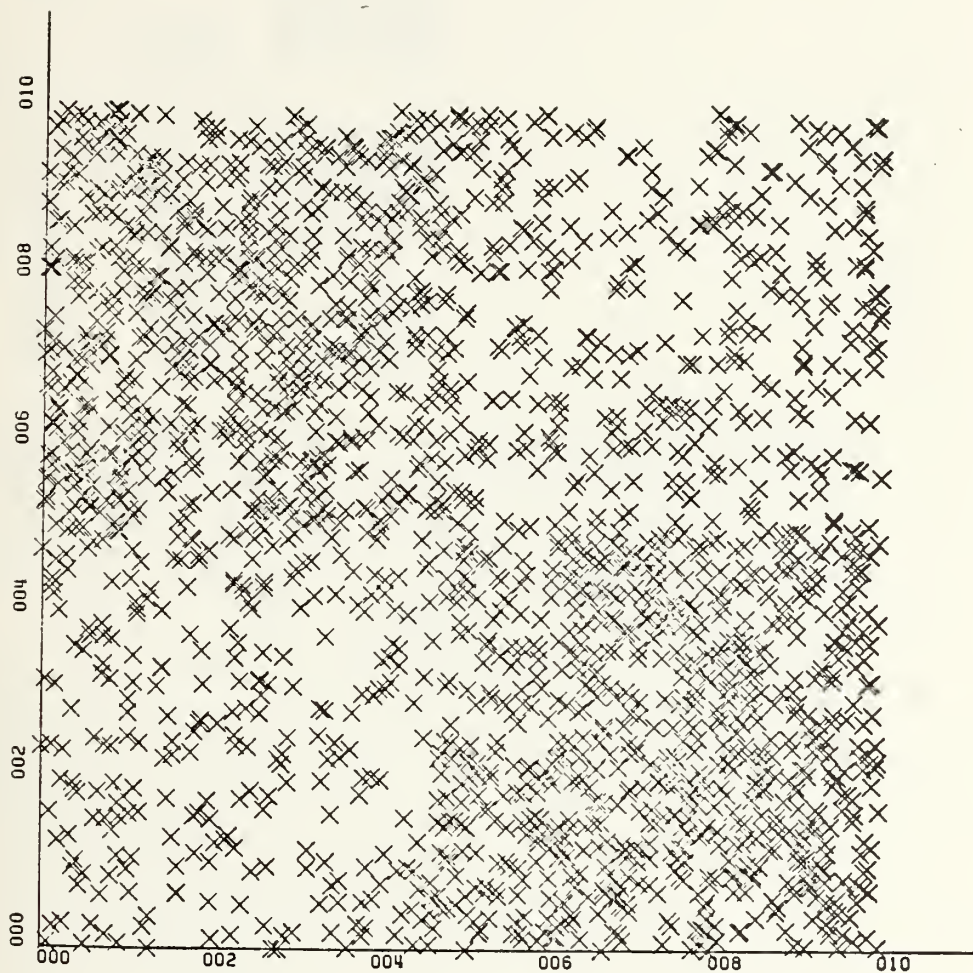


Figure IV-a3. Scatter plot for sample of size 2000 from mixture-truncation bivariate uniform with $\rho = 0.3$. Here x_0 is fixed at the midpoint between the lower^o and the upper bounds.

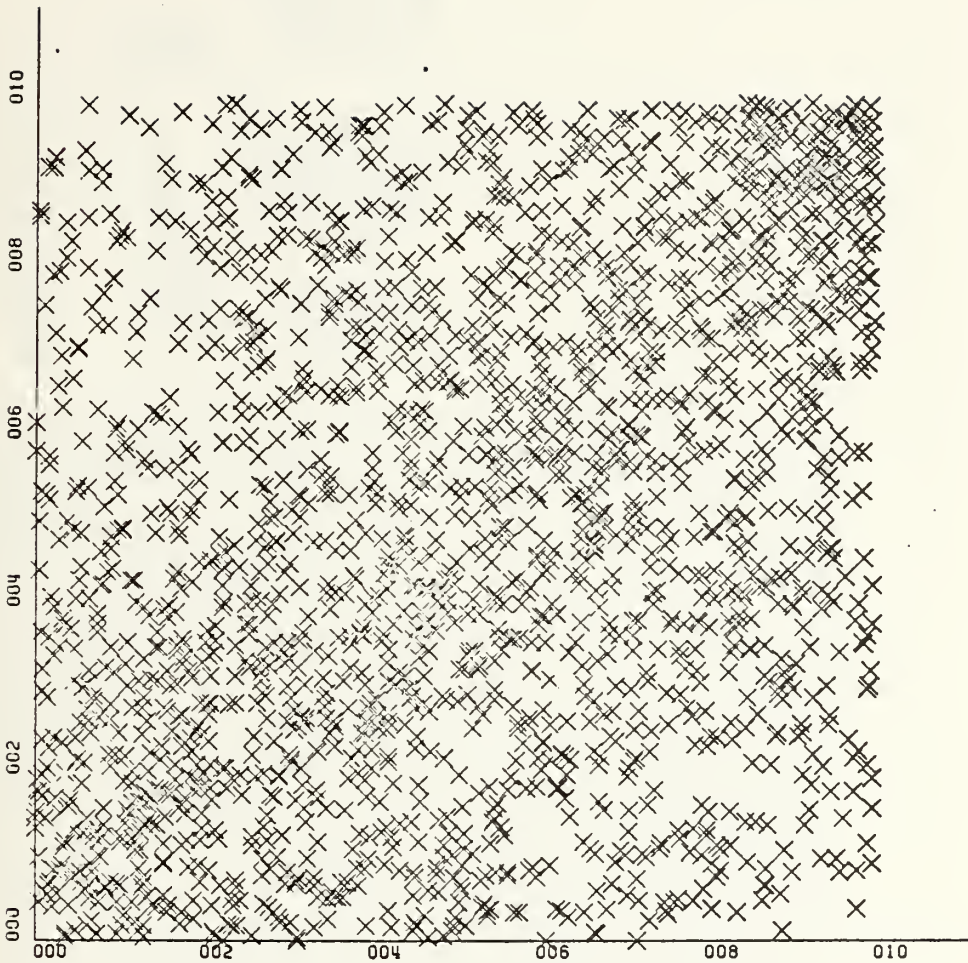
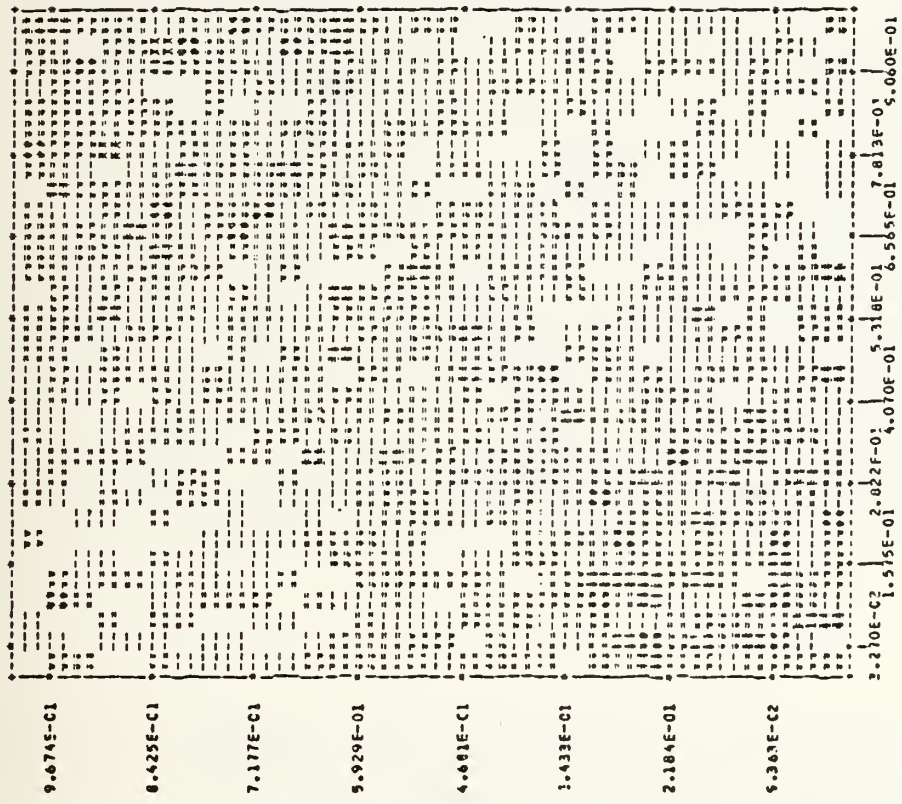


Figure IV-b1. Scatter plot for sample of size 2000 from mixture-truncation bivariate uniform with $\rho = 0.3$. Here x_0 is taken to be uniformly distributed between the lower and the upper bounds.

BIVARIATE SAMPLE SUMMARY SAMPLE SIZE = 2000



SYMBOL PRINTED NO. OBSERVATIONS 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15

MEASURES OF ASSOCIATION
 COVARIANCE 3.639132E-C2
 CORRELATION COEFFICIENT 3.312961
 SPEARMAN'S RANK CORRELATION COEFF. 3.314104

TESTS FOR EQUIDISTRIBUTION
 MULTICOMPLIMENTATION TEST 8.021577
 MANN-WHITNEY U TEST 202418
 PEARSON'S CHI-SQUARE TEST 509519
 SPECTRUM TEST 3916810
 MODALIZED STATISTIC 1.65463
 0.66002
 -0.66224

UNIVARIATE STATISTICS
 X SAMPLE 4.547595E-01
 Y SAMPLE 5.506757E-01
 MEAN 5.027518E-01
 MEDIAN 5.000000E-01
 VARIANCE 9.518925E-C2
 STD. DEV. 9.756519E-01
 RANGE 5.980980E-01
 SKEWNESS 4.685491E-02
 KURTOSIS -1.205842E-02
 MAXIMUM 9.984115E-01
 MINIMUM 1.519133E-01

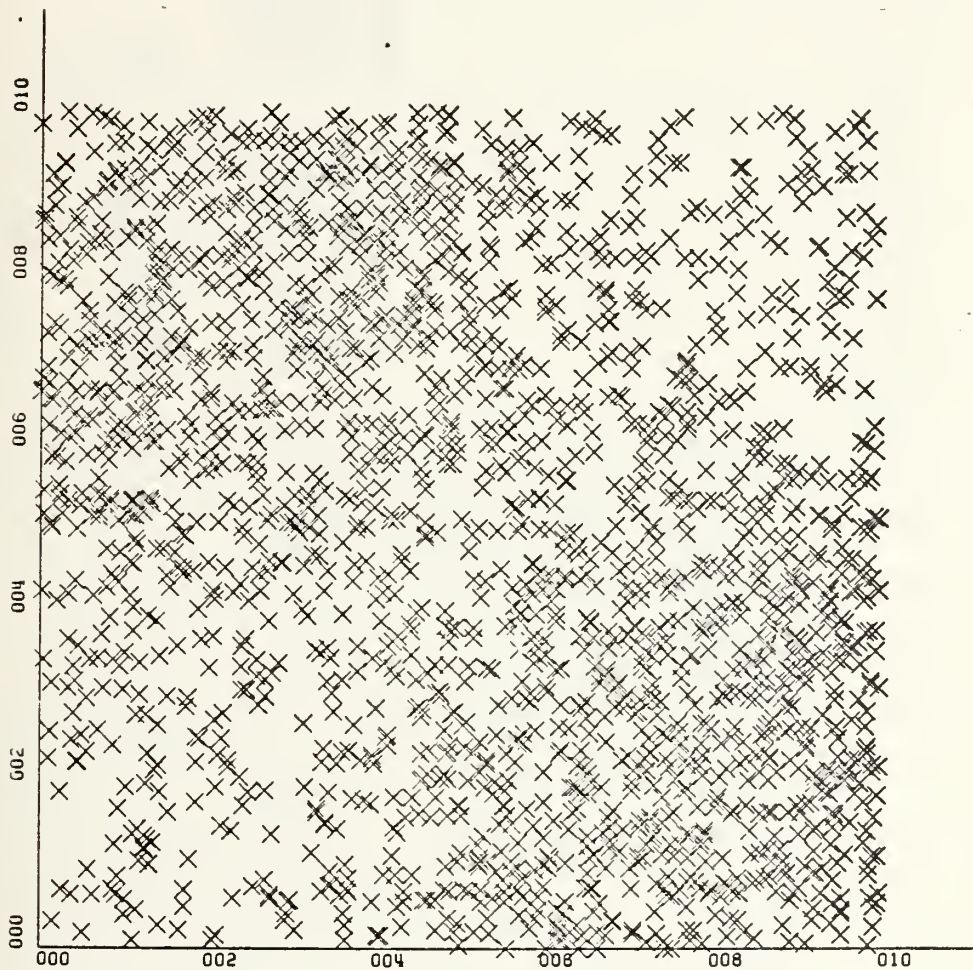
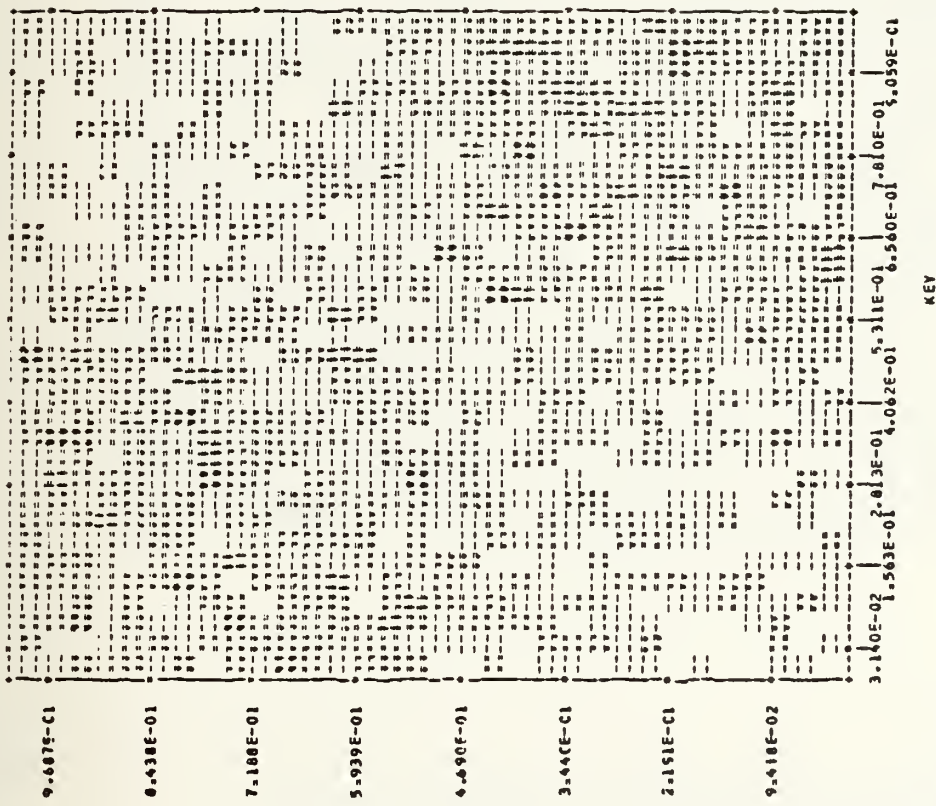


Figure IV-b3. Scatter plot for sample of size 2000 from mixture-truncation bivariate uniform with $\rho = -0.3$. Here x_0 is taken to be uniformly distributed between the lower and the upper bounds

BIVARIATE SAMPLE SUMMARY SAMPLE SIZE = 2000



3.140E-02 1.563E-01 2.813E-01 4.062E-01 5.311E-01 6.560E-01 7.810E-01 9.059E-01
 SYMBOL PRINTED NO. OBSERVATIONS 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15

MEASURES OF ASSOCIATION
 COVARIANCE 0.421544E-02
 CORRELATION COEFFICIENT -0.293129
 SPADMAN CORRELATION COEFF. -0.30137

TESTS FOR EMPIRICALITY
 TEST STATISTIC NORMALIZED STATISTIC
 KOLMOGOROV-SMIRNOV TEST 0.076570
 RASCH-KOLMOGOROV TEST 207.074
 MILKOSEN TEST 407.621
 SIEGEL-TUKEY TEST 2992.69

UNIVARIATE STATISTICS
 X SAMPLE Y SAMPLE
 MEAN 5.176325E-01 4.990785E-01
 MEDIAN 5.244598E-01 5.09562E-01
 VARIANCE 8.187401E-07 8.059071E-07
 STD DEV 2.861679E-01 2.83674E-01
 RANGE 9.994072E-01 9.594176E-01
 SKENNESS -0.426173E-02 9.50959E-03
 KURTOSIS -1.815927E-00 -1.178793E-01
 MAXIMUM 9.555749E-01 9.99133E-01
 MINIMUM 1.677235E-04 4.826549E-04

9.607E-01
 0.438E-01
 7.100E-01
 5.939E-01
 6.690E-01
 3.44E-01
 2.151E-01
 9.410E-02

Figure IV-b4. Bivariate histogram for sample of size 2000 from mixture-truncation bivariate uniform with $\rho = -0.3$. Here x_0 is taken to be uniformly distributed between the lower and the upper bounds.

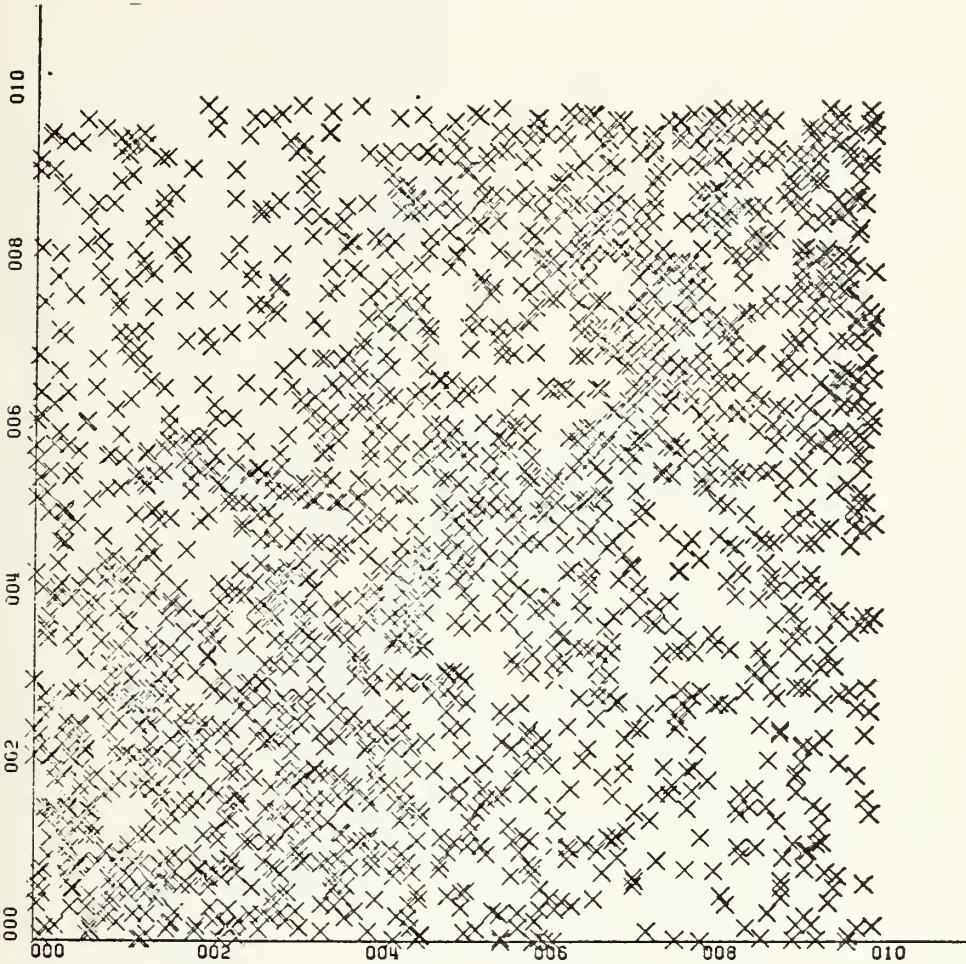


Figure IV-cl. Scatter plot for sample of size 2000 from mixture-truncation bivariate uniform with $\rho = 0.3$. Here x_0 has triangular distribution between the lower and the upper bounds.

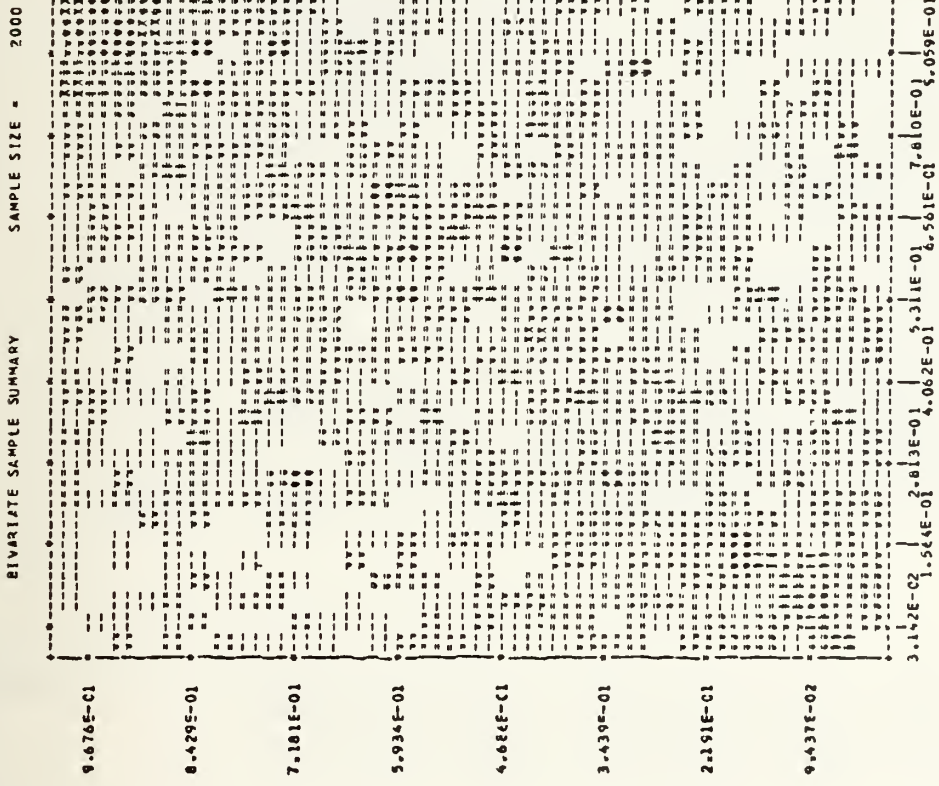


Figure IV-c2. Bivariate histogram for sample of size 2000 from mixture-truncation bivariate uniform with $\rho = 0.3$. Here x_0 has triangular distribution between the lower and the upper permissible bounds.

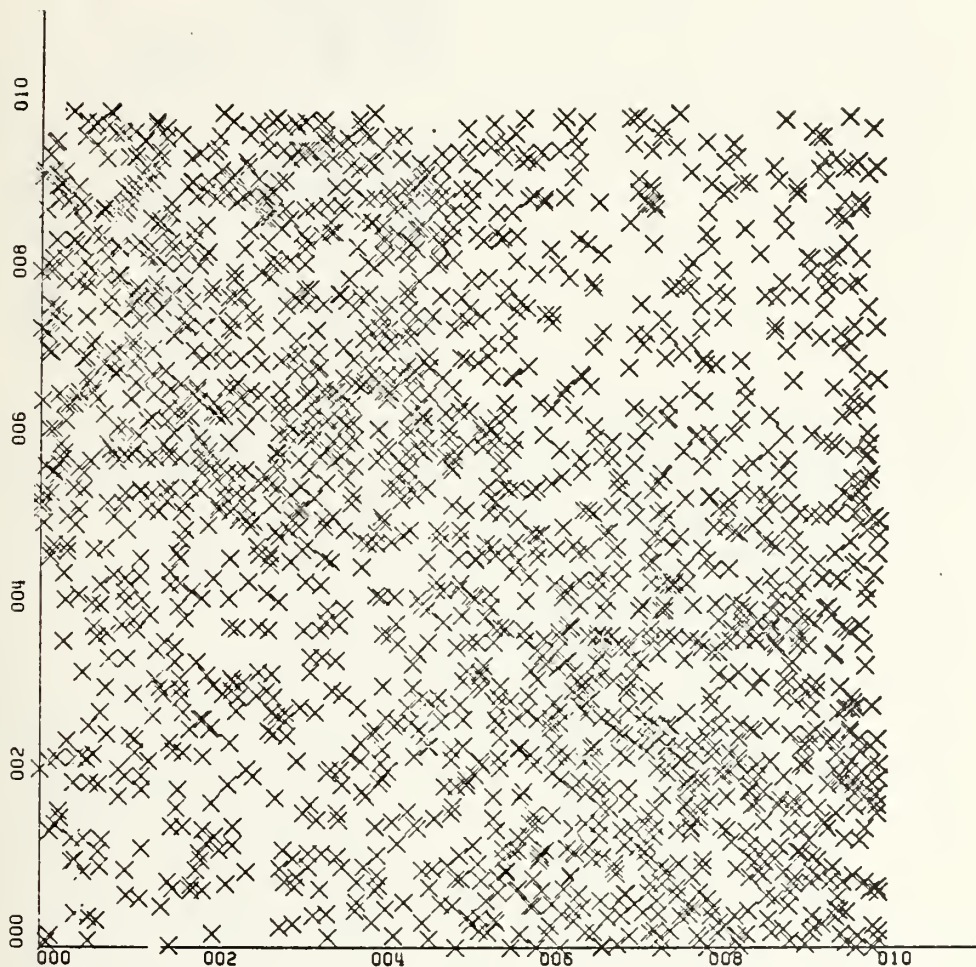


Figure IV-c3. Scatter plot for sample of size 2000 from mixture-truncation bivariate uniform with $\rho = -0.3$. Here x_0 has triangular distribution between the lower and the upper bounds.

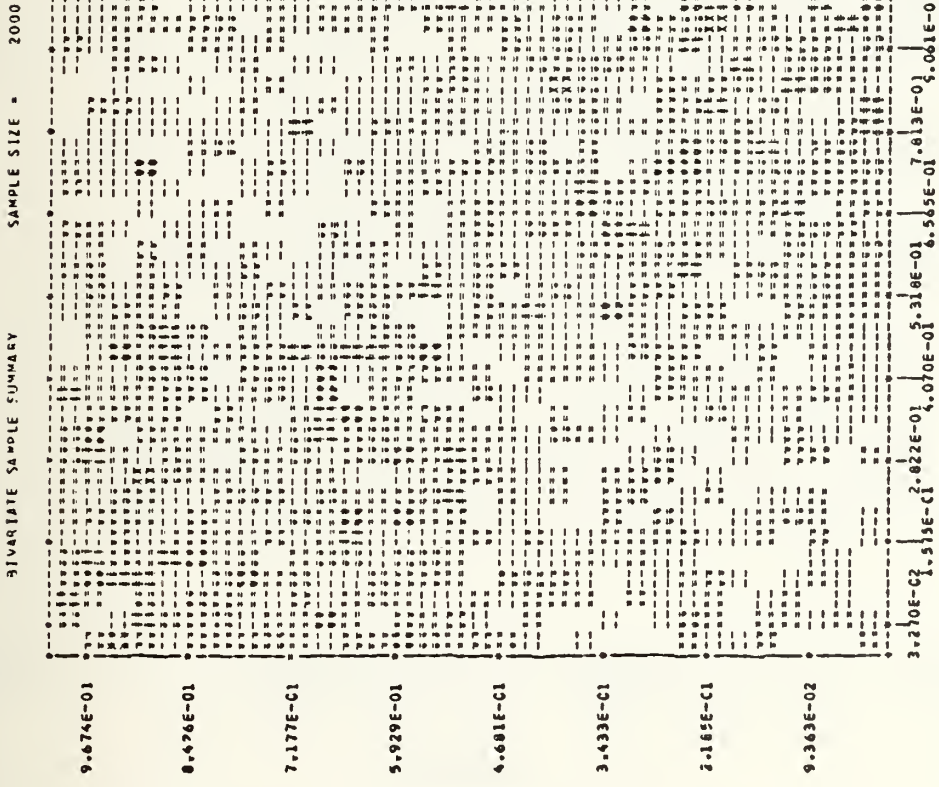


Figure IV-c4. Bivariate histogram for sample of size 2000 from mixture-truncation bivariate uniform with $\rho = -0.3$. Here x_0 has triangular distribution between the lower and the upper bounds.

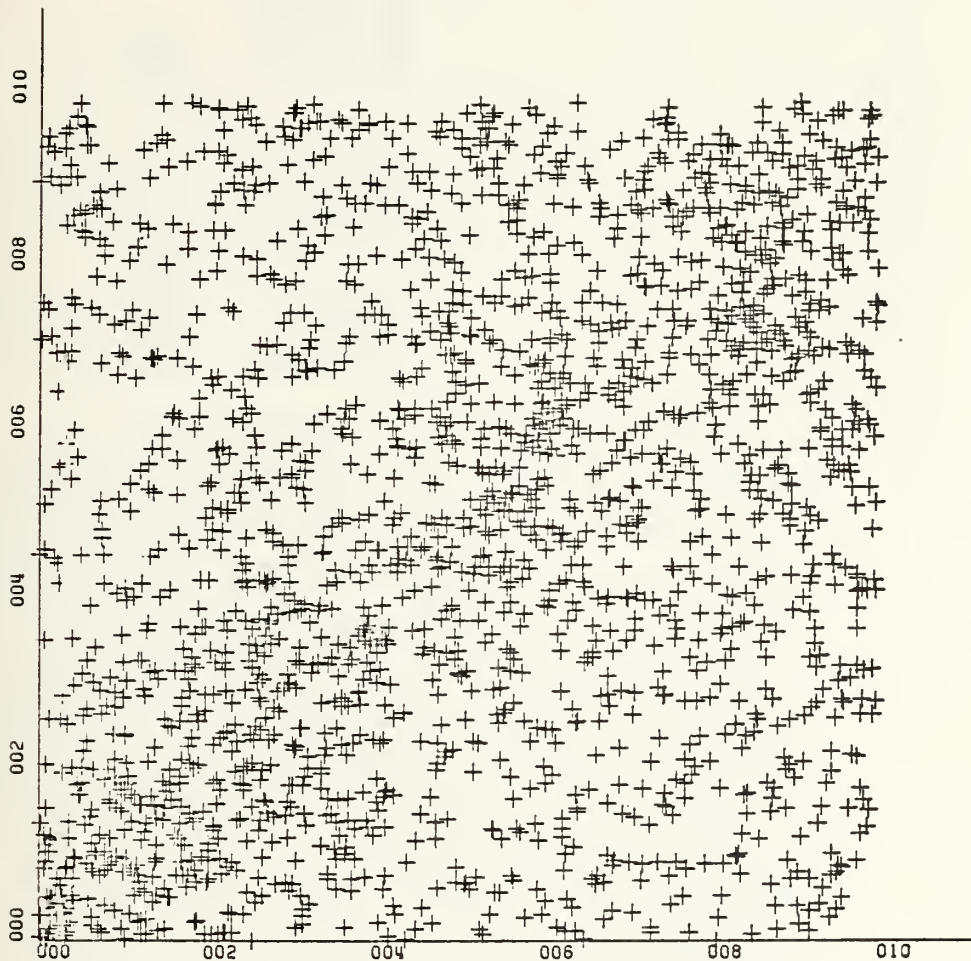


Figure IV-d1. Scatter plot for sample of size 2000 from Lawrance-Lewis's method with $\rho = 0.3$ and

$$\alpha = \frac{(2+\beta)}{3+(2+\beta)\rho} \left(\frac{1}{\beta} + 1 \right)$$

and β is uniform over $[\frac{2\rho}{3-\rho}, 1]$.

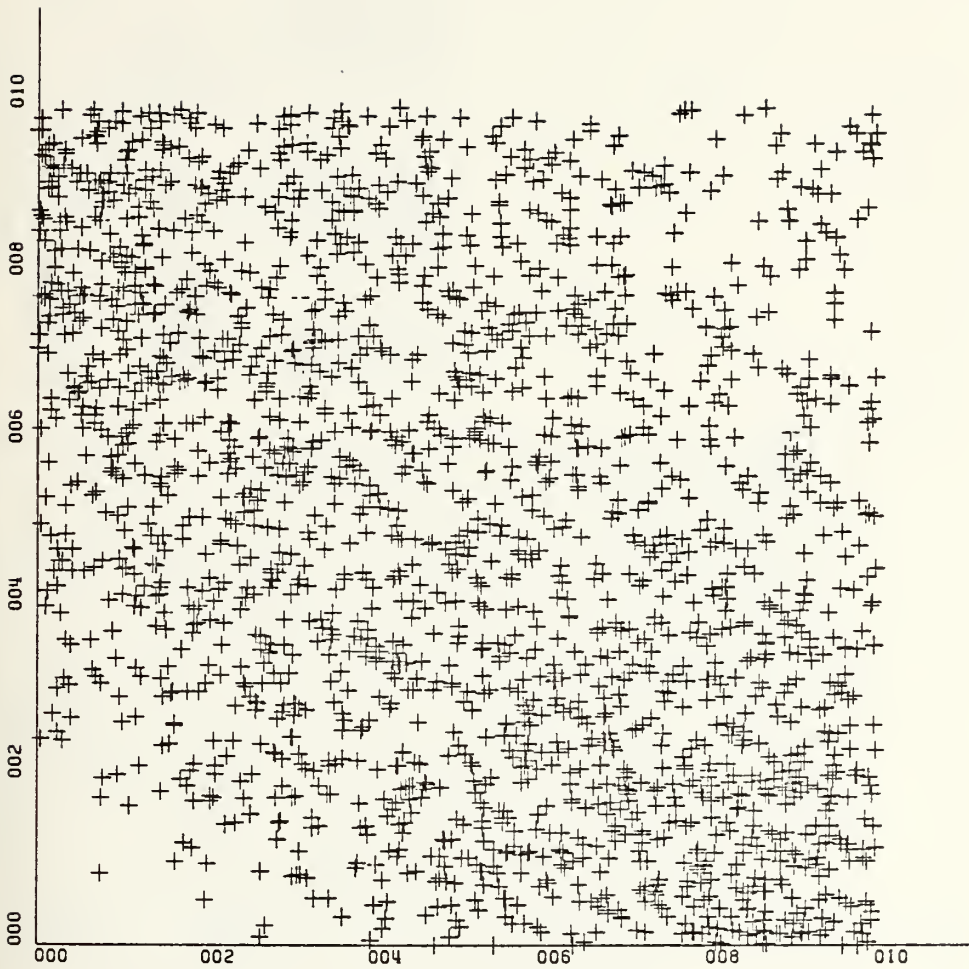


Figure IV-el. Scatter plot for sample of size 2000 from Gaver's transformation with $p = 0.6$.



Figure IV-e2. Bivariate histogram for sample of size 2000 from Gaver's transformation with $p = 0.6$.

V. BIVARIATE EXPONENTIAL GENERATOR

Another probability distribution of major interest in simulations is the exponential. The cumulative distribution function and probability density function for the exponential are, respectively,

$$F(x) = 1 - e^{-\lambda x} \quad x \geq 0$$

and

$$f(x) = \lambda e^{-\lambda x} \quad x \geq 0$$

The expected value of the exponential distribution is

$$E[X] = 1/\lambda$$

and the variance is

$$\text{VAR}[X] = 1/\lambda^2.$$

The problem of generating exponential deviates reduces to one of generating "unit" exponentials, i.e., those with $\lambda = 1.0$, and then multiplying the result by whichever $1/\lambda$ is necessary to give the desired distribution. That is, if the random variable E has the exponential ($\lambda = 1$) distribution then,

$$X = E * \frac{1}{n}$$

has the exponential ($\lambda = n$) distribution. In this section, we considered only unit exponentials as marginal distributions for bivariate pairs.

A. DETERMINATION OF PARAMETERS IN THE EXPONENTIAL MIXTURE-TRUNCATION METHOD

Because X_1 is an exponential ($\lambda = 1$) truncated to the left of x_0 and X_2 is also exponential ($\lambda = 1$) truncated to the right of x_0 , we have

$$\begin{aligned} E[X_1] = \mu_1 &= \int_0^{x_0} x \, d \frac{F(x)}{F(x_0)} \\ &= 1 - \frac{\pi_2}{\pi_1} x_0; \end{aligned}$$

$$\begin{aligned} \text{VAR}[X_1] = \sigma_1^2 &= \int_0^{x_0} x^2 \, d \frac{F(x)}{F(x_0)} - \mu_1^2 \\ &= 1 - \frac{\pi_2}{\pi_1} x_0^2; \end{aligned}$$

$$\begin{aligned} E[X_2] = \mu_2 &= \int_{x_0}^{\infty} x \, d \frac{F(x) - F(x_0)}{1 - F(x_0)} \\ &= 1 + x_0; \end{aligned}$$

$$\begin{aligned} \text{VAR}[X_2] &= \int_{x_0}^{\infty} x^2 d \frac{F(x) - F(x_0)}{1 - F(x_0)} - \mu_2^2 \\ &= 1; \end{aligned}$$

and from definition,

$$\begin{aligned} \pi_1 &= \frac{1 - \alpha_2}{1 - \alpha_1 + 1 - \alpha_2} \\ &= F(x_0) \\ &= 1 - e^{-x_0}, \end{aligned} \tag{V-A-1a}$$

$$\begin{aligned} \pi_2 &= \frac{1 - \alpha_1}{1 - \alpha_1 + 1 - \alpha_2} \\ &= 1 - F(x_0) \\ &= e^{-x_0}. \end{aligned} \tag{V-A-1b}$$

If we use these formulas in Theorem 2 in Section III, we get

$$M = \frac{\pi_2}{\pi_1} x_0^2 \tag{V-A-2}$$

$$\beta = \frac{1}{\pi_2} (\alpha_1 - \pi_1) \tag{V-A-3}$$

and

$$\rho = \frac{\alpha_1 - \pi_1}{\pi_1} x_0^2 \quad (\text{V-A-4})$$

For given correlation coefficient ρ , we can compute α_1 as a function of x_0 from the formula (V-A-4) as

$$\begin{aligned} \alpha_1 &= \frac{\rho \pi_1}{x_0^2} + \pi_1 \\ &= \left(1 + \frac{\rho}{x_0^2}\right) (1 - e^{-x_0}) \end{aligned}$$

and we know that $\alpha_2 = 1 - \frac{\pi_1}{\pi_2}(1 - \alpha_1)$ from the formulas (V-A-1a) and (V-A-1b). From this,

$$\begin{aligned} \alpha_2 &= 1 - \frac{\pi_1}{\pi_2}(1 - \alpha_1) \\ &= e^{-x_0} + e^{x_0}(1 - e^{-x_0})^2 \frac{\rho}{x_0^2} \end{aligned}$$

These α_1 and α_2 are probabilities, so they have to be greater than or equal to zero and less than or equal to one;

$$0 \leq (1 - e^{-x_0}) \left(1 + \frac{\rho}{x_0^2}\right) \leq 1 \quad (\text{V-A-5a})$$

$$0 \leq e^{-x_0} + e^{x_0}(1 - e^{-x_0})^2 \frac{\rho}{x_0^2} \leq 1 \quad (\text{V-A-5b})$$

From these two inequality equations, (V-A-5a) and (V-A-5b), we can find the x_0 ranges for given correlation coefficient ρ . To solve these equations, we can divide into two cases,

one for positive correlation case and the other for negative correlation case. If correlation coefficient ρ is positive, then both equations are always positive. Thus we only need to find the x_0 ranges which makes (V-A-5a) and (V-A-5b) less than 1. From the equation (V-A-5a), for the α_1 case,

$$\alpha_1 = (1 - e^{-x_0}) \left(1 + \frac{\rho}{x_0^2}\right) \leq 1$$

becomes

$$\rho(e^{x_0} - 1) \leq x_0^2$$

and let

$$y_1 = x_0^2, \quad y_2 = \rho(e^{x_0} - 1).$$

Because of the first derivatives of y_1 and y_2 are always positive, we know that these two functions are monotone increasing functions. Thus we can find x_0 ranges which satisfy $y_1 \geq y_2$ by the Newton Raphson method. When using the Newton Raphson method, let $y = y_1 - y_2 = 0$ and find an approximate solution, by approximating exponential series, which we can use as a starting point. That is,

$$\begin{aligned} y &= y_1 - y_2 = x_0^2 - \rho \left(1 + x_0 + \frac{x_0^2}{2!} + \frac{x_0^3}{3!} - 1\right) \\ &= \frac{\rho}{6} x_0^2 - \left(1 - \frac{\rho}{2}\right) x_0 + \rho = 0 \end{aligned}$$

Then

$$x_0 = \frac{(1 - \frac{\rho}{2}) \pm (1 - \rho - \frac{5}{12} \rho^2)^{1/2}}{1/3 \rho}$$

Starting with this approximate value in the Newton Raphson method, we can find x_0 range, say $[x_{l_1}, x_{u_1}]$, which satisfies $0 \leq \alpha_1 \leq 1$. And for the α_2 case,

$$\alpha_2 = e^{-x_0} + e^{-x_0} (1 - e^{-x_0})^2 \frac{\rho}{x_0^2} \leq 1$$

becomes

$$\rho (e^{x_0} - 1) \leq x_0^2$$

This result is exactly the same as the α_1 case, that is, at the same range α_1 and α_2 satisfy constraints $\alpha_1 \leq 1$ and $\alpha_2 \leq 1$. Thus we can use x_{l_1} and x_{u_1} as the lower bound of x_0, x_l , and the upper bound of x_0, x_u . If correlation coefficient ρ is negative, then the equations (V-A-5a) and (V-A-5b) are always less than 1. Therefore we need to consider only one constraint which makes $\alpha_1 \geq 0, \alpha_2 \geq 0$. From the α_1 equation (V-A-5a), solve the inequality equation

$$0 \leq \alpha_1 = (1 - e^{-x_0}) (1 + \frac{\rho}{x_0^2})$$

Since $1 - e^{-x_0}$ is always positive, we see that to satisfy the inequality $1 + \frac{\rho}{x_0^2}$ should be positive, i.e.,

$$\frac{\rho}{x_0^2} \geq -1$$

equivalently, we have

$$x_0 \geq \sqrt{-\rho}$$

In the α_2 case, from equation (V-A-5b),

$$0 \leq \alpha_2 = e^{-x_0} + e^{x_0}(1 - e^{-x_0})^2 \frac{\rho}{x_0^2}$$

or, equivalently, we have

$$x_0^2 \geq -\rho(e^{x_0} - 1)^2$$

As in the positive correlation case, we can find a starting point by approximation to solve this equation by Newton Raphson. The result comes out as

$$x_s = (\sqrt{-\rho} + \rho) / (-\frac{\rho}{2})$$

with this starting point we find another bound of x_0 which satisfies $0 \leq \alpha_2$. This becomes the upper bound of x_0 , x_u , and from the α_1 case, we have a constraint $x_0 \geq \sqrt{-\rho}$ which

becomes the lower bound of x_0 , i.e., $x_l = \sqrt{-\rho}$. The lowest and highest correlations available for bivariate exponential pairs in mixture-truncation method are approximately -0.480 and 0.647 respectively. By comparison note that the most negative correlation available for bivariate exponential pairs with identical fixed marginals is $1 - \frac{\pi^2}{6}$ [Moran (1967)]. Gaver's (1972) negatively correlated pair has correlations in the range [-0.5,0]. The table (V-1) shows the lower and upper bound of x_0 in the mixture-truncation method with identical marginal exponential and given correlation.

Table V-1: The lower bound and upper bound of x_0 in the mixture-truncation method with identical exponential marginal distributions

x_0 range for each correlation

| ρ | x_L | x_u | ρ | x_L | x_u |
|--------|-------|-------|--------|-------|-------|
| 0.1 | 0.106 | 5.832 | -0.1 | 0.317 | 1.984 |
| 0.2 | 0.225 | 4.723 | -0.2 | 0.448 | 1.439 |
| 0.3 | 0.362 | 3.990 | -0.3 | 0.548 | 1.103 |
| 0.4 | 0.527 | 3.395 | -0.4 | 0.633 | 0.855 |
| 0.5 | 0.741 | 2.842 | -0.45 | 0.671 | 0.751 |
| 0.6 | 1.082 | 2.223 | | | |

B. GENERATING PROCEDURE

The generating procedure of bivariate exponential random vector is almost the same as in the uniform case. As in the uniform case, we develop three procedures which are the FXO method, the UXO method and the TXO method. As mentioned

in Section III, all of these methods are exactly the same except in how x_0 is chosen from the x_0 range $[x_\ell, x_u]$. The FXO method chooses x_0 as a fixed point between x_ℓ and x_u . This procedure makes some discontinuity at the truncation point. In this respect, this method is defective.

The UXO and TXO methods are the same except we use different distributions for x_0 . In the UXO method we use the uniform distribution and the triangular distribution for the TXO method. These methods have more smooth distribution than the FXO method.

Exponential Mixture-Truncation Method

1. (Initialization)

i) For given $-0.48 < \rho < 0.64$, find x_ℓ and x_u

2. Define truncation point x_0

* FXO method

$$i) \quad x_0 = \frac{1}{2}(x_\ell + x_u)$$

* UXO method

i) Generate a uniform $[0,1]$ random variable U_1

$$ii) \quad x_0 = x_\ell + (x_u - x_\ell) * U_1$$

* TXO method

i) Generate two uniform $[0,1]$ random variables

$$V_1, V_2$$

$$ii) \quad x_0 = x_\ell + x_1 + x_2$$

where

$$x_m = (x_l + x_u)/2$$

$$x_1 = (x_m - x_l) * V_1$$

$$x_2 = (x_u - x_m) * V_2$$

3. Compute parameter values

$$\pi_1 = F(x_0) = 1 - e^{-x_0}$$

$$\pi_2 = 1 - \pi_1$$

$$\alpha_1 = \pi_1 \left(1 + \frac{\rho}{x_0^2}\right)$$

$$\alpha_2 = 1 - \frac{\pi_1}{\pi_2} (1 - \alpha_1)$$

4. Choose type for Y

- i) Generate a uniform [0,1] random variable U
- ii) If $U \leq \pi_1$, go to 9

5. Y is an X_2

- i) Generate an exponential random variable E_1
- ii) If $E_1 > x_0$, set $Y \leftarrow E_1$ and go to 6
- iii) Otherwise, return to 5.i)

6. Choose type for Z

- i) Set $U \leftarrow ((U - \pi_1)/(1 - \pi_1))$

- ii) If $U \leq 1 - \alpha_2$, go to 8
7. Z is an X_2
- i) Generate an exponential random variable E_2
 - ii) If $E_2 > x_0$, set $Z \leftarrow E_2$ and go to 11
 - iii) Otherwise return to 7.i)
8. Z is an X_1
- i) Generate an exponential random variable E_2
 - ii) If $E_2 \leq x_0$, set $Z \leftarrow E_2$ and go to 11
 - iii) Otherwise return to 8.i)
9. Y is an X_1
- i) Generate an exponential random variable E_1
 - ii) If $E_1 \leq x_0$, set $Y \leftarrow E_1$ and go to 10
 - iii) Otherwise return to 9.i)
10. Choose type for Z
- i) Set $U \leftarrow U/\pi_1$
 - ii) If $U \leq \alpha_1$, go to 8
 - iii) Otherwise go to 7
11. Deliver (Y, Z) and go to 4 for the FXO method, or go to 2 for the UXO and TXO methods until a sufficient number of random vectors are obtained.

For the exponential case it is possible to give a more efficient algorithm in which X_1 and X_2 are generated exactly. The algorithm is as follows.

Efficient Exponential Mixture-Truncation Method

1. (Initialization)

i) For given $-0.48 < \rho < 0.64$, find x_ℓ and x_u

2. Define the truncation point x_0

* FXO method

$$x_0 = \frac{1}{2}(x_\ell + x_u)$$

* UXO method

i) Generate a uniform $[0,1]$ random variable U_1

$$\text{ii) } x_0 = x_\ell + (x_u - x_\ell) * U_1$$

* TXO method

i) Generate two uniform $[0,1]$ random variables

$$V_1, V_2$$

$$\text{ii) } x_0 = x_\ell + x_1 + x_2$$

where

$$x_m = (x_\ell + x_u)/2$$

$$x_1 = (x_m - x_\ell) * V_1$$

$$x_2 = (x_u - x_m) * V_2$$

3. Compute parameter values

$$\pi_1 = F(x_0) = 1 - e^{-x_0}$$

$$\pi_2 = 1 - \pi_1$$

$$\alpha_1 = \pi_1 \left(1 + \frac{\rho}{x_0}\right)$$

$$\alpha_2 = 1 - \frac{\pi_1}{\pi_2} (1 - \alpha_1)$$

4. Choose type for Y

- i) Generate a uniform [0,1] random variable U
- ii) If $U \leq \pi_1$, go to 9

5. Y is an X_2

- i) Generate an exponential random variable E_1
- ii) Set $Y \leftarrow x_0 + E_1$

6. Choose type for Z

- i) Set $U \leftarrow ((U - \pi_1) / (1 - \pi_1))$
- ii) If $U \leq 1 - \alpha_2$, go to 8

7. Z is an X_2

- i) Generate an exponential random variable E_2
- ii) Set $Z \leftarrow x_0 + E_2$ and go to 11

8. Z is an X_1

- i) Generate a uniform [0,1] random variable W_2
- ii) Set $Z \leftarrow -\ln(1.0 - W_2 * \pi_1)$ and go to 11

9. Y is an X_1

- i) Generate a uniform [0,1] random variable W_1

ii) Set $Z \leftarrow -\ln(1.0 - W_1 * \pi_1)$

10. Choose type for Z

i) Set $U \leftarrow U/\pi_1$

ii) If $U \leq \alpha_1$, go to 8

iii) Otherwise, go to 7

11. Deliver (Y, Z) and go to 4 for the FXO method, or go to 2 for the UXO and TXO methods until a sufficient number of random vectors are obtained.

Note that to compute $x_{\underline{y}}$ and $x_{\underline{u}}$ in step 1 of both algorithms we use subroutine BOUND which is listed in the appendix and the program with effective algorithm also. The scatter plot and bivariate histogram of resulting random vectors are shown in Section V-D.

C. REGRESSION OF Z ON Y FOR GIVEN ρ

Schmeiser (1979) has used the regression of Z on $Y = y$ to fix the parameters in his bivariate gamma distribution. Consequently we investigate this for the mixture-truncation method case. The regression is different depending on whether $Y \leq x_0$ or $Y > x_0$. We consider two cases here, one for fixed x_0 and the other for x_0 having uniform distribution. For fixed x_0 , we have

$$\begin{aligned} E[Z|Y=y, Y \leq x_0] &= \alpha_1 E[x_1] + (1 - \alpha_1) E[x_2] \\ &= 1 + x_0 - \alpha_1 x_0 \left(1 + \frac{\pi_2}{\pi_1}\right) \end{aligned}$$

Substituting the value for α_1 ,

$$\alpha_1 = \pi_1 \left(1 + \frac{\rho}{x_0^2}\right)$$

then we have

$$\begin{aligned} E[Z | Y = y, Y \leq x_0] &= 1 + x_0 - x_0 \left(1 + \frac{\rho}{x_0^2}\right) \\ &= 1 - \frac{\rho}{x_0} \end{aligned}$$

And if $y > x_0$, then

$$\begin{aligned} E[Z | Y = y, Y > x_0] &= (1 - \alpha_2)E[x_1] + \alpha_2 E[x_2] \\ &= 1 - \frac{\pi_2}{\pi_1} x_0 + \frac{x_0}{\pi_1} \alpha_2 \end{aligned}$$

Substituting the value for α_2 ,

$$\begin{aligned} \alpha_2 &= 1 - \frac{\pi_1}{\pi_2} (1 - \alpha_1) \\ &= 1 - \pi_1 \left(1 - \frac{\pi_1}{\pi_2} \frac{\rho}{x_0^2}\right) \end{aligned}$$

then we have

$$E[Z | Y = y, Y > x_0] = 1 + \frac{\pi_1}{\pi_2} \frac{\rho}{x_0}$$

Thus the regression is constant over $(0, x_0)$ and changes for $y \geq x_0$. This is not surprising in light of the joint distribution given in Section IV-D. For uniformly distributed x_0 , the computation is different for different ranges of Y . If $y \leq x_\ell$, then we have

$$\begin{aligned}
 E[Z|Y=y] &= \int_{x_\ell}^{x_u} E[Z|Y=y, X=x_0, y \leq x_0] f(x_0) dx_0 \\
 &= \int_{x_\ell}^{x_u} \left(1 - \frac{\rho}{2} \frac{1}{x_0}\right) e^{-x_0} dx_0 \\
 &= -e^{-x_u} + e^{-x_\ell} - \rho \int_{x_\ell}^{x_u} \frac{e^{-x_0}}{x_0} dx_0 \\
 &= e^{-x_\ell} - e^{-x_u} + \rho \left(\frac{1}{x_u} e^{-x_u} - \frac{1}{x_\ell} e^{-x_\ell} \right. \\
 &\quad \left. + \ln \frac{x_u}{x_\ell} - x_u + x_\ell \right)
 \end{aligned}$$

If

$$x_\ell \leq y \leq x_u$$

then we have

$$\begin{aligned}
E[Z|Y=y] &= \int_{x_L}^y E[Z|Y=y, X=x_0, y \geq x_0] f(x_0) dx_0 \\
&+ \int_y^{x_u} E[Z|Y=y, X=x_0, y \leq x_0] f(x_0) dx_0 \\
&= 2e^{-y} + 2\rho y - \rho(x_l + x_u) + \rho\left(\frac{1}{x_u} e^{-x_u} - \frac{1}{y} e^{-y}\right) \\
&\quad + \rho \ln \frac{x_u}{y}
\end{aligned}$$

If $y > x_u$, then we have

$$\begin{aligned}
E[Z|Y=y] &= \int_{x_l}^{x_u} E[Z|Y=y, X=x_0, y > x_0] f(x_0) dx_0 \\
&= \int_{x_l}^{x_u} \left(1 + \frac{\pi}{2} \frac{\rho}{x_0}\right) e^{-x_0} dx_0 \\
&= e^{-x_l} - e^{-x_u} + \rho(y - x_l)
\end{aligned}$$

By making x_0 uniformly distributed over the available range of x_0 for given correlation, we can get smoother behavior for the regression function.

D. SIMULATION RESULTS

We will show here the scatter plots and bivariate histogram of the mixture-truncation bivariate exponential

random vectors with correlation $\rho = 0.3$ and $\rho = -0.3$ by the FXO, UXO and TXO methods in Figures (V-a), (V-b) and (IV-c), respectively. Also we show the results from Gaver's bivariate exponential and Marshall and Olkin's in Figures (V-d) and (V-e), respectively.

In the mixture-truncation bivariate exponential distribution, the generated random vectors by the FXO method also has discontinuity at truncation point but the other cases have relatively smooth distributions.

The computed correlation printed in the left side of the bivariate histogram is a little different from the given correlation. But this difference can be assumed as a sampling error.

To check this error we simulated 10 times with given correlation $\rho = 0.1$. From these simulations we get:

$$\bar{\rho} = \text{mean of computed correlation} = 0.092$$

$$\text{VAR}[\rho] = \text{variance of computed correlation} = 0.0008$$

$$\sigma(\bar{\rho}) = \text{standard deviation of mean} = 0.009.$$

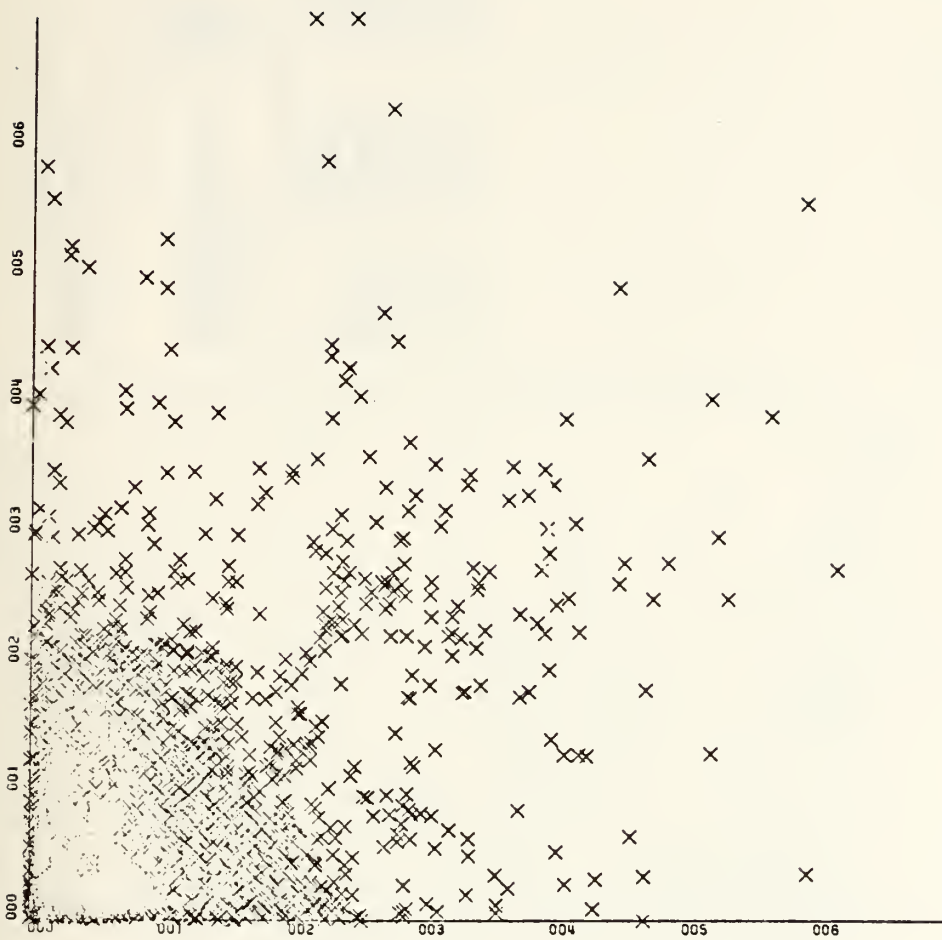
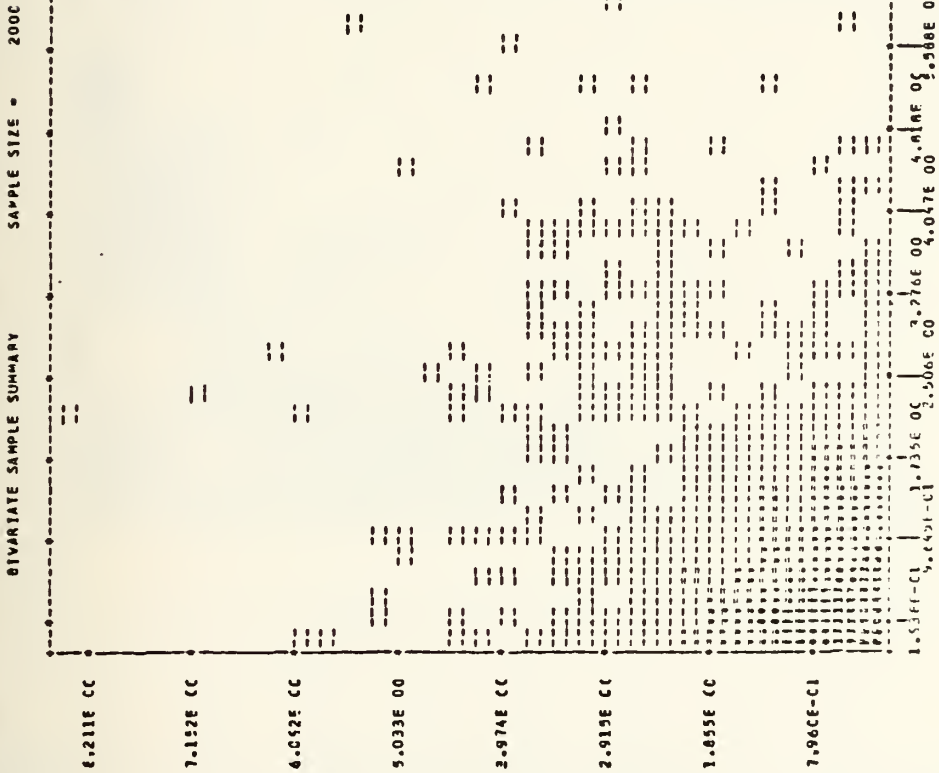


Figure V-a1. Scatter plot for sample of size 2000 from mixture-truncation bivariate exponential with $\rho = 0.3$. Here x_0 is fixed at midpoint between the lower and the upper bounds.



KEY

| | | | | | | | | | | | | | | | | |
|-----------------|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| SYMBOL PRINTED | 0 | 11 | 17 | 23 | 29 | 35 | 41 | 47 | 52 | 59 | 65 | 71 | 77 | 83 | 85 | 95 |
| AC. COORDINATES | | | | | | | | | | | | | | | | |

MEASURES OF ASSOCIATION
 COVARIANCE COEFFICIENT 2.414210E-01
 SPEARMAN'S RANK CORRELATION COEF. 0.710371
 0.136156

TESTS FOR ECUIDISTRIBUTION
 KOLMOGOROV-SMIRNOV TEST 2.7275E-01
 WILCOXON TEST 0.2115E-01
 PEARSON-K-S TEST 1.9848E-01
 PEARSON-K-S TEST 1.9848E-01
 WILCOXON TEST 3.0271E-01
 STEIGEL-TUKHY TEST 4.0223E-01
 NORMALIZED STATISTIC
 -1.45493
 -0.35973
 -0.25573
 1.26017

UNIVARIATE STATISTICS
 X SAMPLE 9.5260E-01
 Y SAMPLE 7.0950E-01
 MEAN 2.9524E-01
 VARIANCE 8.7442E-01
 STD. DEV 9.3510E-01
 RANGE 2.1849E-01
 SKEWNESS 1.7224E 00
 KURTOSIS 3.7624E 00
 MAXIMUM 6.1561E-01
 MINIMUM 1.1198E-01

Figure V-a2. Bivariate histogram for sample of size 2000 from mixture-truncation bivariate exponential with $\rho = 0.3$. Here x_0 is fixed at the midpoint between the lower and the upper bounds.

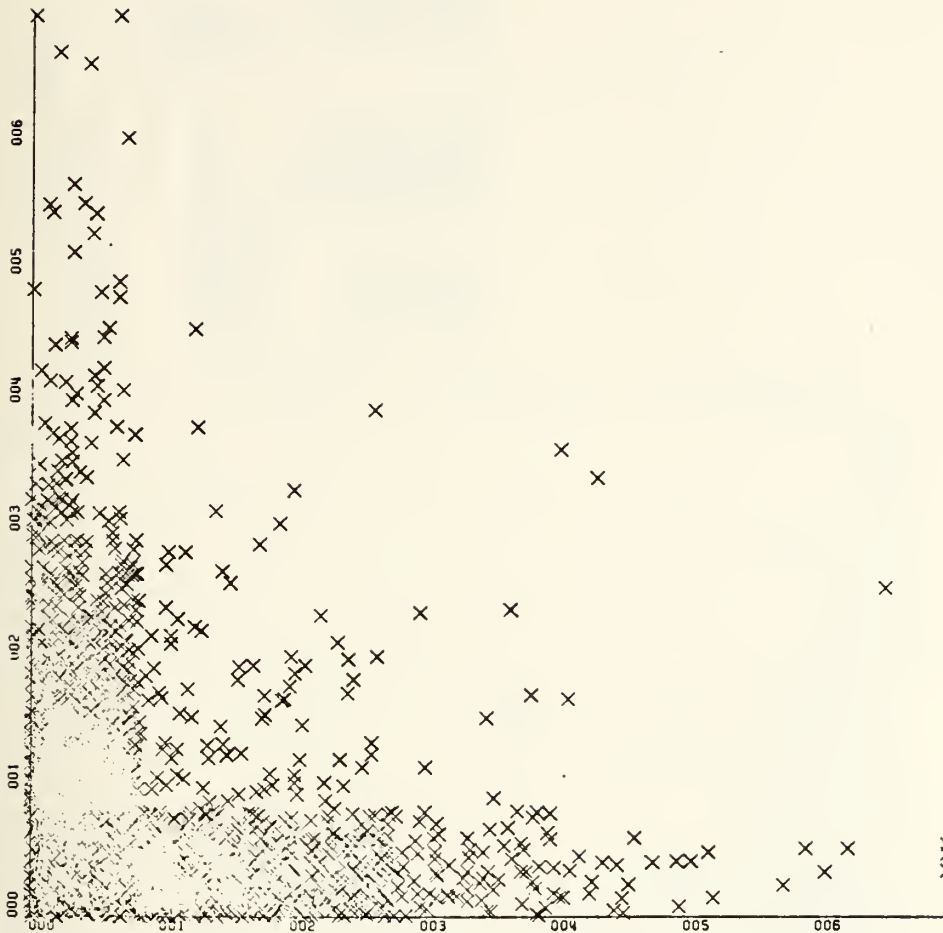
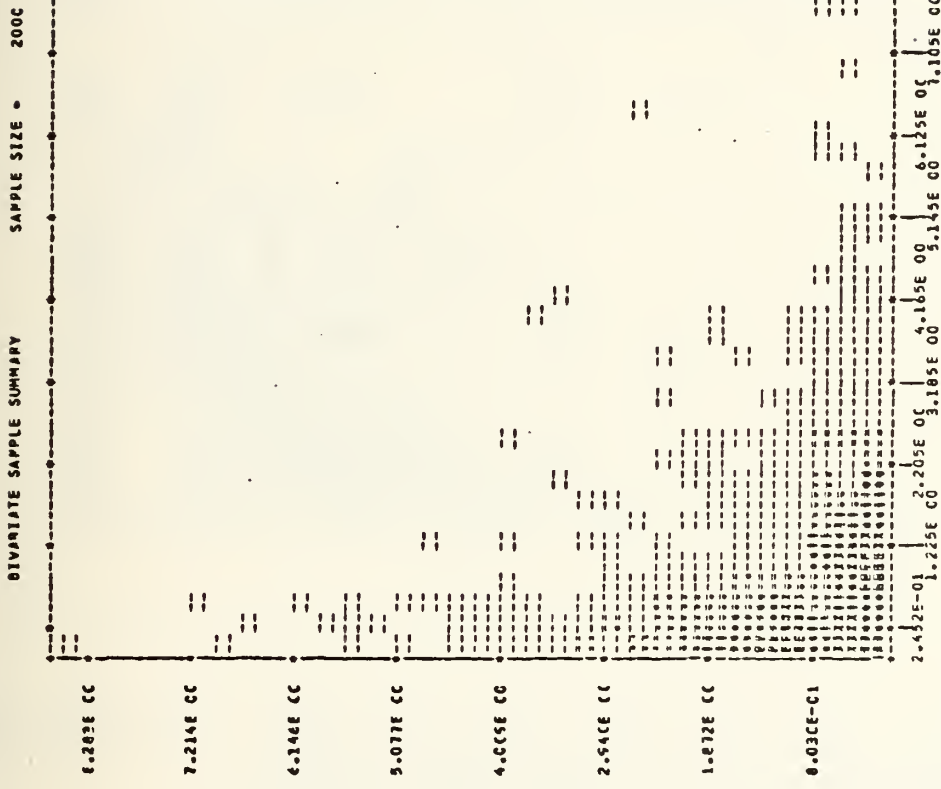


Figure V-a3. Scatter plot for sample of size 2000 from mixture-truncation bivariate exponential with $\rho = -0.3$. Here x_0 is fixed at the midpoint between the lower and the upper bounds.



SYMBOL PRINTED KEY

AC. OBSERVATIONS 0 5 14 19 24 29 34 39 44 49 54 55 64 69 74 79

MEASURES OF ASSOCIATION

COVARIANCE -2.89974E-01

CORRELATION COEFFICIENT -0.29472E-01

SPEARMAN'S RANK CORRELATION COEFF. -0.431071

TESTS FOR EQUIDISTRIBUTION

KOLMOGOROV-SMIRNOV TEST 0.0013072

MILGRAM-ROFFMAN TEST 0.72741

MANN-WHITNEY TEST 15459.0

WILCOXSON TEST 35846.1

STEELE-TUKEY TEST 39375.01

NORMALIZED STATISTIC -0.31323

UNIVARIATE STATISTICS

MEAN 9.75000E-01

MEDIAN 4.50000E-01

VARIANCE 9.52387E-01

STD DEV 9.76158E-01

RANGE 7.82564E-01

SKEWNESS 2.04643E-01

KURTOSIS 6.21374E-01

MAXIMUM 7.82564E-01

MINIMUM 1.86881E-01

Y SAMPLE SIZE 2000

X SAMPLE SIZE 2000

Y STATISTIC 9.27476E-01

X STATISTIC 5.63057E-01

Y CORRELATION COEFF. 2.54837E-01

X CORRELATION COEFF. 2.54837E-01

Y MEAN 9.75000E-01

X MEAN 4.50000E-01

Y VARIANCE 9.52387E-01

X VARIANCE 9.52387E-01

Y STD DEV 9.76158E-01

X STD DEV 9.76158E-01

Y RANGE 7.82564E-01

X RANGE 7.82564E-01

Y SKEWNESS 2.04643E-01

X SKEWNESS 2.04643E-01

Y KURTOSIS 6.21374E-01

X KURTOSIS 6.21374E-01

Y MAXIMUM 7.82564E-01

X MAXIMUM 7.82564E-01

Y MINIMUM 1.86881E-01

X MINIMUM 1.86881E-01

Figure V-a4. Bivariate histogram for sample of size 2000 from mixture-truncation bivariate exponential with $\rho = -0.3$. Here x_0 is fixed at the midpoint between the lower and the upper bounds.

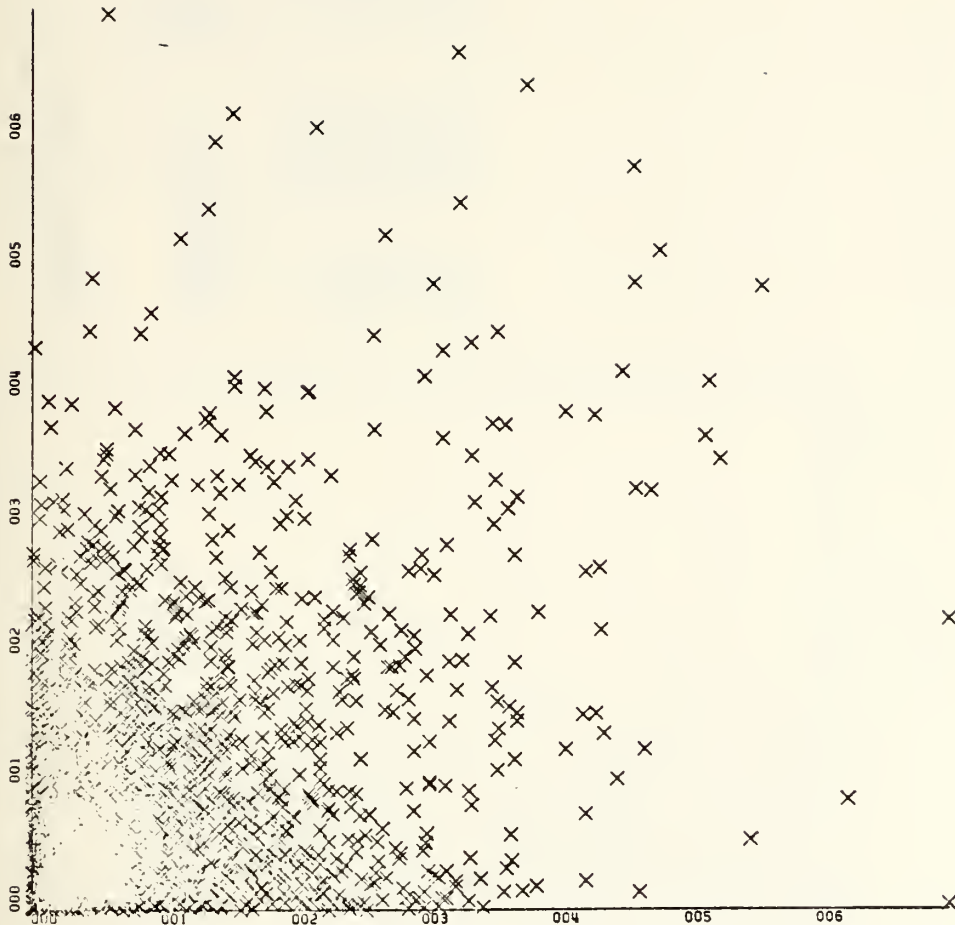


Figure V-b1. Scatter plot for sample of size 2000 from mixture-truncation bivariate exponential with $\rho = 0.3$. Here x_0 is taken to be uniformly distributed between the lower and the upper bounds.

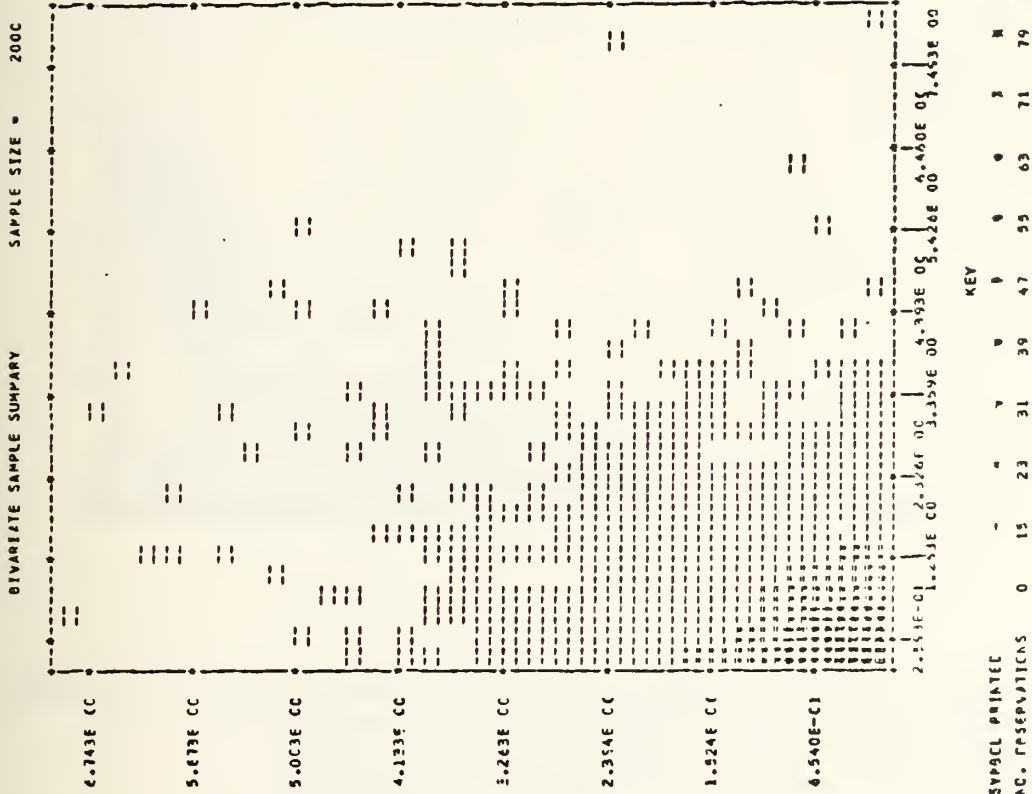


Figure V-b2. Bivariate histogram for sample of size 2000 from mixture-truncation bivariate exponential with $\rho = 0.3$. Here x_0 is taken to be uniformly distributed between the lower and the upper bounds.

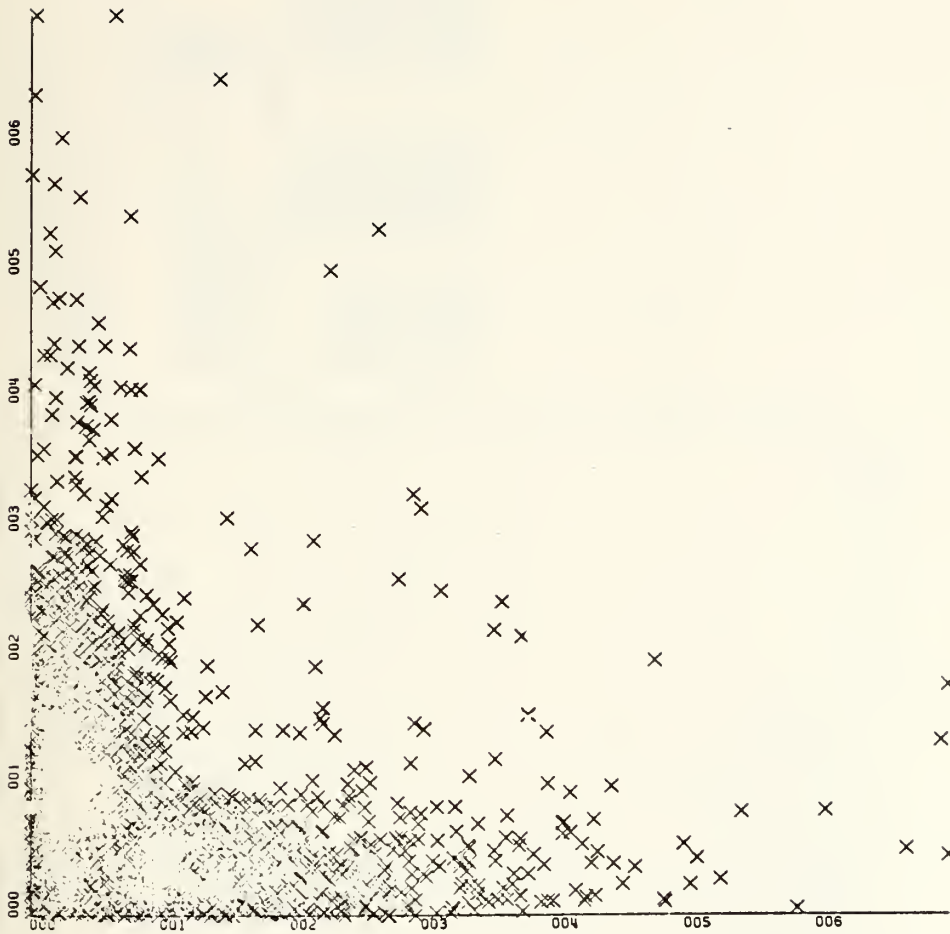
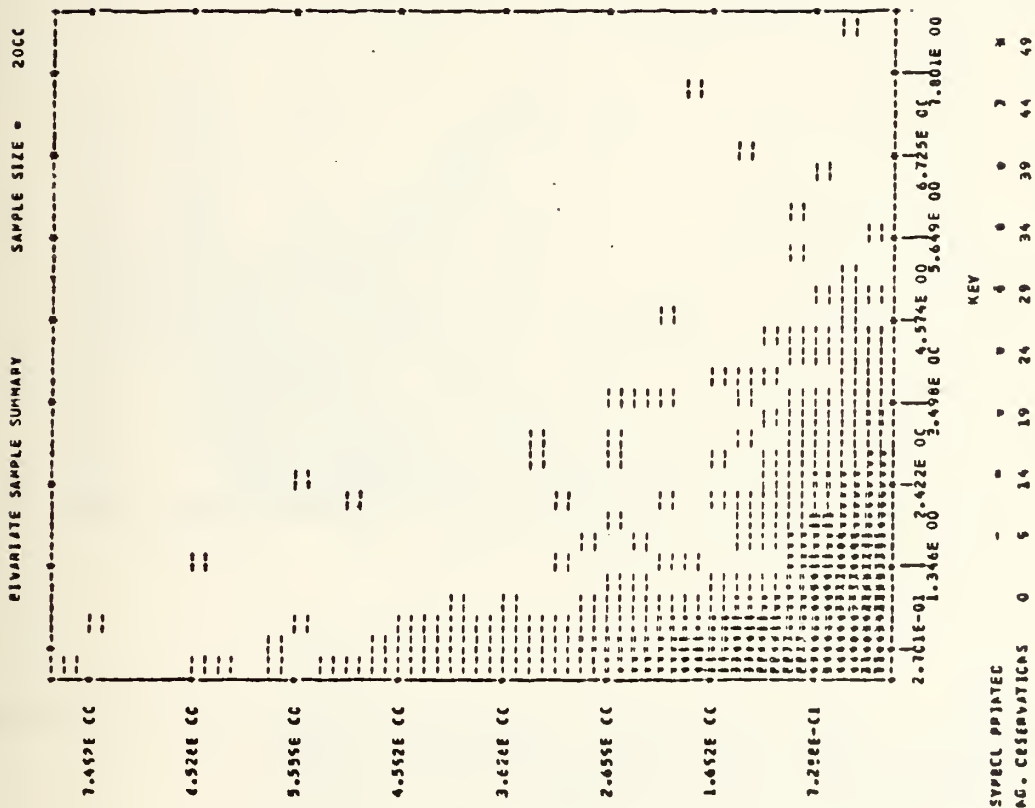


Figure V-b3. Scatter plot for sample of size 2000 from mixture-truncation bivariate exponential with $\rho = -0.3$. Here x_0 is taken to be uniformly distributed between the lower and the upper bounds.



MEASURES OF ASSOCIATION
 COVARIANCE -2.461510E-01
 CORRELATION COEFFICIENT -0.292006
 SPEARMAN'S RANK CORRELATION COEF. -0.417139

TESTS FOR EQUIDISTRIBUTION

| | | |
|--------------------------|-----------|----------------------|
| KOLMOGOROV-SMIRNOV TEST | STATISTIC | NORMALIZED STATISTIC |
| KAL-MILKOMITZ PLUMS TEST | 0.034030 | 1.23344 |
| WILCOXON TEST | 204180 | 1.13283 |
| STEGEL-TUNNEY TEST | 388918 | -3.08383 |

UNIVARIATE STATISTICS

| | | |
|-------------|-------------|-------------|
| MEAN | X SAMPLE | Y SAMPLE |
| 1.00552E CC | 7.00472E-01 | 9.50335E-01 |
| VARIANCE | 9.74284E-01 | 8.34525E-01 |
| STD DEV | 3.12087E-01 | 2.88885E-01 |
| RANGE | 8.66835E CC | 7.73328E CC |
| SKEWNESS | 1.52885E CC | 2.14527E CC |
| KURTOSIS | 5.76141E CC | 6.97512E CC |
| MINIMUM | 1.13843E-01 | 7.72404E-04 |

Figure V-b4. Bivariate histogram for sample of size 2000 from mixture-truncation bivariate exponential with $\rho = -0.3$. Here x_0 is taken to be uniformly distributed between the lower and the upper bounds.

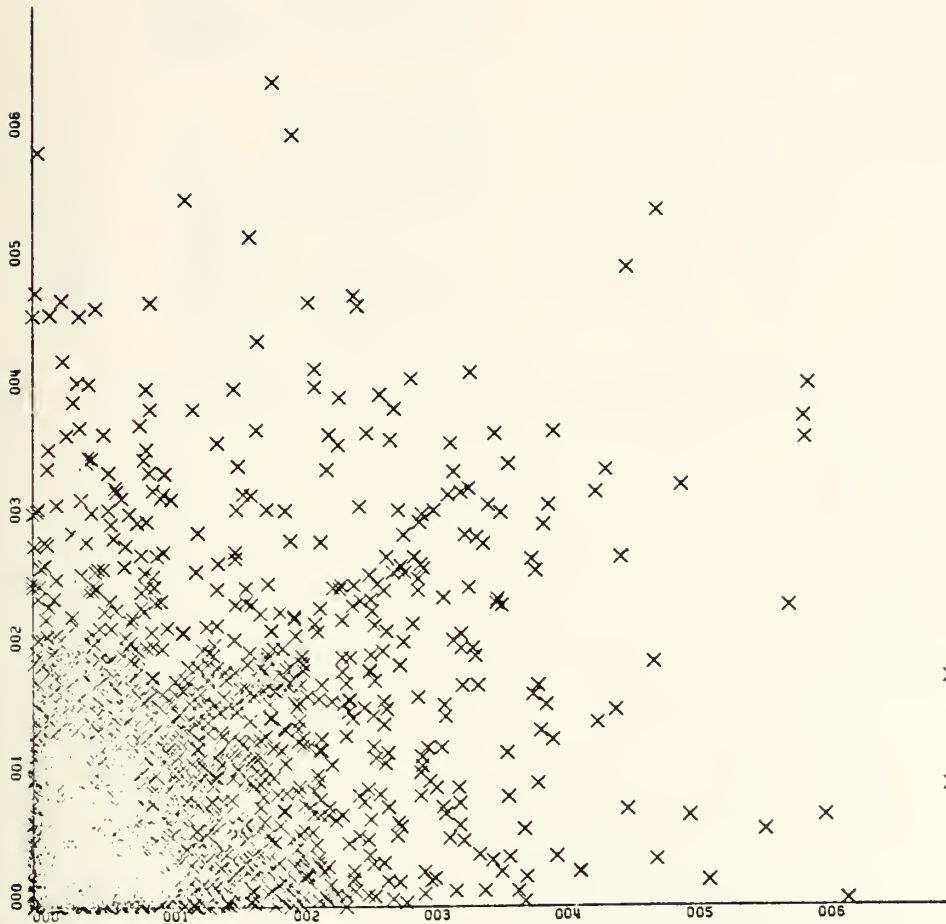


Figure V-cl. Scatter plot for sample of size 2000 from mixture-truncation bivariate exponential with $\rho = 0.3$. Here x_0 has triangular distribution between the lower and the upper bounds.

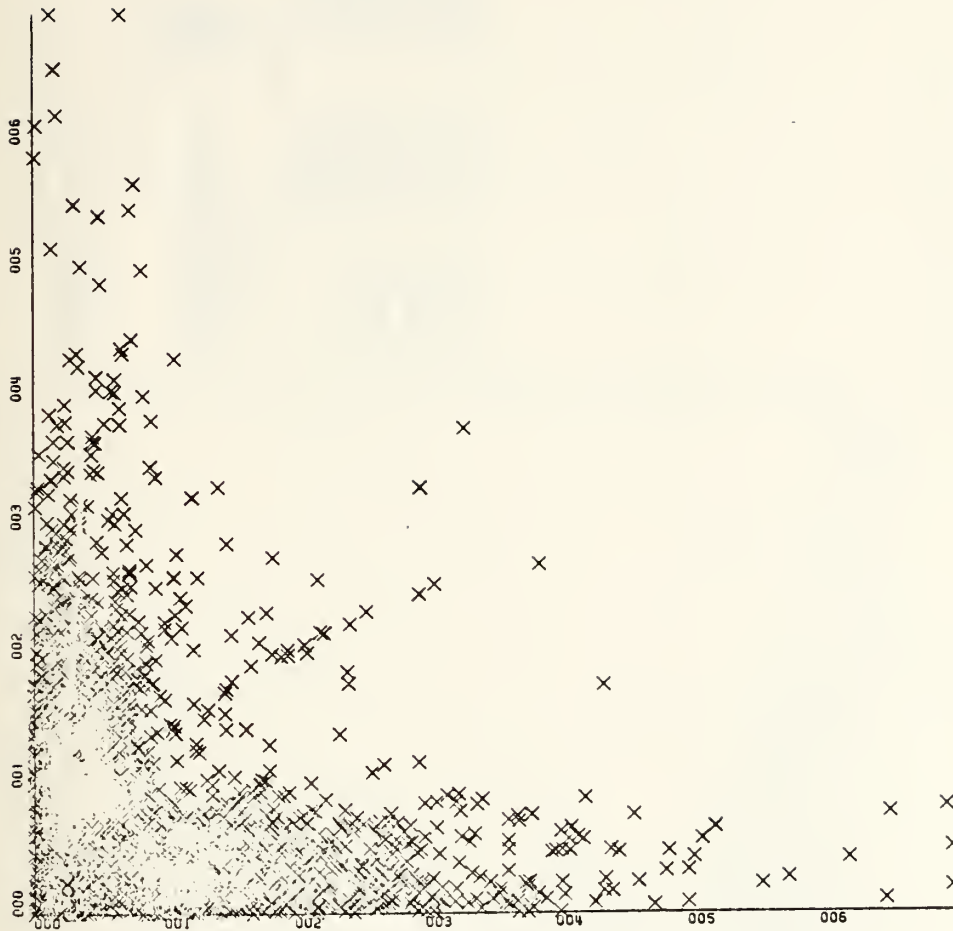
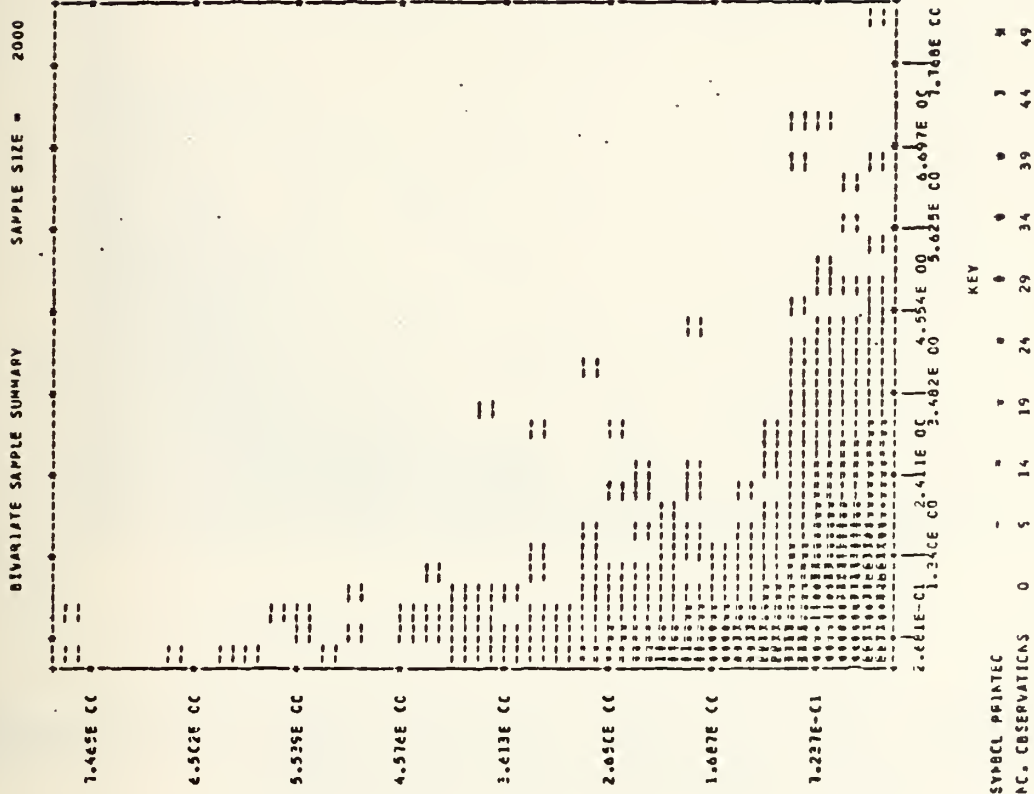


Figure V-c3. Scatter plot for sample of size 2000 from mixture-truncation bivariate exponential with $\rho = -0.3$. Here x_0 has triangular distribution between the lower and the upper bounds.



MEASURES OF ASSOCIATION

COVARIANCE COEFFICIENT -2.91923E-01
 CORRELATION COEFFICIENT -0.313390
 SERRAN'S RANK CORRELATION COEFF. -0.313390

TESTS FOR ECUIDISTRIBUTION

MCLENDON-SAMPSON TEST STATISTIC 0.043209 NORMALIZED STATISTIC 0.55975
 MAIR-BRIDGMAN TEST STATISTIC 204.7602 STATISTIC 1.58116
 SILVERMAN TEST STATISTIC 4048.004 STATISTIC 1.58116
 SIEGEL-TUKEY TEST STATISTIC 3529478 STATISTIC -1.56685

UNIVARIATE STATISTICS

X SAMPLE Y SAMPLE
 MEAN 1.017091E 00 5.57134E-01
 MEDIAN 7.31467E-01 6.73439E-01
 VARIANCE 1.00276E 00 8.74554E-01
 STD. DEV. 1.00137E 00 9.35157E-01
 RANGE 8.51144E 00 7.76437E 00
 SKEWNESS 1.55121E 00 2.04443E 00
 KURTOSIS 5.62822E 00 6.54320E 00
 MAXIMUM 8.51144E 00 7.76577E 00
 MINIMUM 2.81015E-04 1.436161E-03

Figure V-c4. Bivariate histogram for sample of size 2000 from mixture-truncation bivariate with $\rho = -0.3$. Here x_0 has triangular distribution between the lower and the upper bounds.

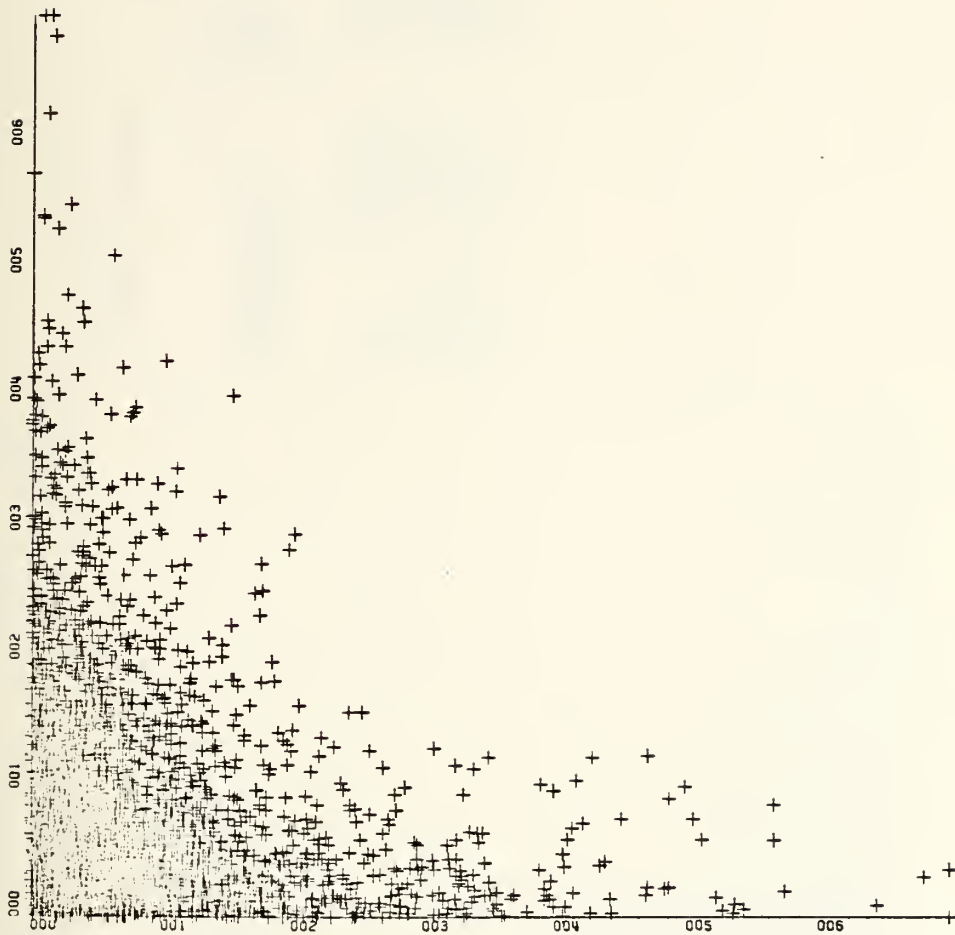


Figure V-d1. Scatter plot for sample of size 2000 from Gaver's bivariate exponential with $\rho = -0.3$.

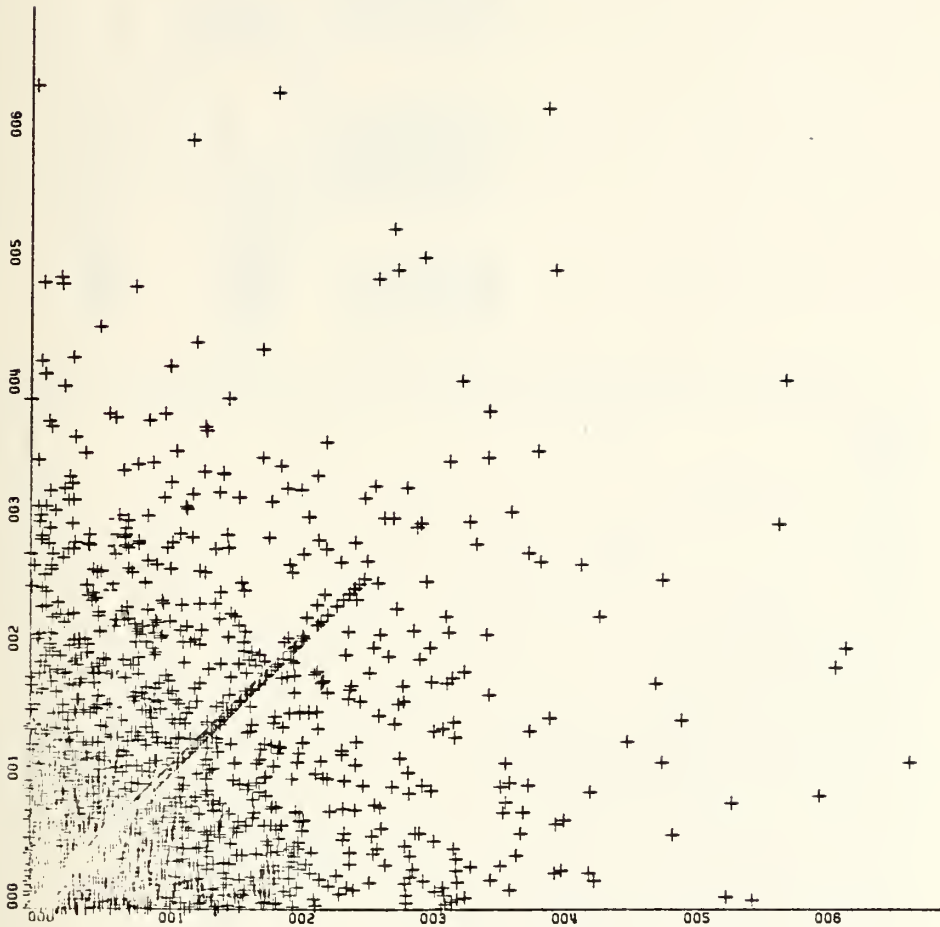


Figure V-e1. Scatter plot for sample of size 2000 from Marshall and Olkin's bivariate exponential with $\rho = 0.3$.

VI. BIVARIATE GAMMA GENERATOR

The gamma distribution with shape parameter r and scale parameter λ has the density function

$$f(x) = \begin{cases} \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \quad (\text{VI-0-1})$$

and cumulative distribution function

$$F(y) = \int_0^y \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x} dx . \quad (\text{VI-0-2a})$$

In particular, if the shape parameter r is a positive integer then

$$F(y) = 1 - \sum_{j=0}^{r-1} \frac{e^{-\lambda y} (\lambda y)^j}{j!} , \quad (\text{VI-0-2b})$$

where $\Gamma(r)$ is Euler's gamma function

$$\Gamma(r) = \int_0^{\infty} x^{r-1} e^{-x} dx .$$

If the random variable X has Gamma (r, λ) distribution then

$$E[x] = \frac{r}{\lambda}$$

$$\text{VAR}[x] = \frac{r}{\lambda^2} ;$$

when $r = 1$, X has the exponential distribution while X , suitably scaled, has an asymptotically normal distribution as $r \rightarrow \infty$. We note that if X has a Gamma($r, 1$) distribution then $\frac{X}{\lambda}$ has a Gamma(r, λ) distribution. So we may set $\lambda = 1$ as far as the generating algorithm is concerned. The output from the generator may then be appropriately scaled. Two algorithms for generating bivariate gamma random vectors are given in Section II. Since they have complicated computations to define the parameter values and use inverse transformation functions, the mixture-truncation method is probably preferable to those methods. To generate bivariate random vectors with identical Gamma ($r, 1$) marginal distributions and having given correlation ρ by mixture-truncation method, we only need univariate gamma ($r, 1$) random variate generator. Under the assumption that the univariate gamma generator is available we developed this procedure. In fact we used Lewis and Robinson's gamma generator as a univariate gamma generator.

A. DETERMINATION OF PARAMETERS IN THE GAMMA MIXTURE-TRUNCATION METHOD

If the marginal distribution is gamma with positive integer shape parameter r and scale parameter $\lambda = 1$, then

we know that $F(x)$ becomes cumulative distribution function of gamma $(r,1)$. Then the X_1 is gamma $(r,1)$ truncated to the left of x_0 and X_2 is also gamma $(r,1)$ truncated to the right of x_0 . Thus we have

$$\begin{aligned}
 E[X_1] &= \mu_1 = \int_0^{x_0} x \, d \frac{F(x)}{F(x_0)} \, dx \\
 &= \frac{1}{F(x_0)\Gamma(r)} \int_0^{x_0} x^r e^{-x} \, dx \\
 &= \frac{1}{\pi_1 \Gamma(r)} \{r! - e^{-x_0} \left[\sum_{i=0}^r \frac{r!}{(r-i)!} x_0^{r-i} \right]\}
 \end{aligned}$$

$$\begin{aligned}
 \text{VAR}[X_1] &= \sigma_1^2 = \int_0^{x_0} x^2 \, d \frac{F(x)}{F(x_0)} - \mu_1^2 \\
 &= \frac{1}{\pi_1 \Gamma(r)} \int_0^{x_0} x^{r+1} e^{-x} \, dx - \mu_1^2 \\
 &= \frac{1}{\pi_1 \Gamma(r)} \{ (r+1)! - e^{-x_0} \left[\sum_{i=0}^{r+1} \frac{(r+1)!}{(r+1-i)!} x_0^{r+1-i} \right] \} \\
 &\quad - \mu_1^2
 \end{aligned}$$

$$\begin{aligned}
E[X_2] = \mu_2 &= \int_{x_0}^{\infty} x \, d \frac{F(x) - F(x_0)}{1 - F(x_0)} \\
&= \frac{1}{\pi_2 \Gamma(r)} \int_{x_0}^{\infty} x^r e^{-x} \, dx \\
&= \frac{1}{\pi_2 \Gamma(r)} \left[e^{-x_0} \sum_{i=0}^r \frac{r!}{(r-i)!} x_0^{r-i} \right]
\end{aligned}$$

$$\begin{aligned}
\text{VAR}[X_2] = \sigma_2^2 &= \int_{x_0}^{\infty} x^2 \, d \frac{F(x) - F(x_0)}{1 - F(x_0)} \\
&= \frac{1}{\pi_2 \Gamma(r)} \int_{x_0}^{\infty} x^{r+1} e^{-x} \, dx \\
&= \frac{1}{\pi_2 \Gamma(r)} \left[e^{-x_0} \sum_{i=0}^{r+1} \frac{(r+1)!}{(r+1-i)!} x_0^{r+1-i} \right] - \mu_2^2
\end{aligned}$$

Let

$$A = e^{-x_0} \sum_{i=0}^r \frac{r!}{(r-i)!} x_0^{r-i}$$

$$B = e^{-x_0} \sum_{i=0}^{r+1} \frac{(r+1)!}{(r+1-i)!} x_0^{r+1-i}$$

and substitute, then we have

$$E[X_1] = \mu_1 = \frac{r! - A}{\pi_1 \Gamma(r)},$$

$$\text{VAR}[X_1] = \sigma_1^2 = \frac{(r+1)! - B}{\pi_1 \Gamma(r)} - \frac{(r! - A)^2}{\pi_1^2 \Gamma(r)^2},$$

$$E[X_2] = \mu_2 = \frac{A}{\pi_2 \Gamma(r)},$$

$$\text{VAR}[X_2] = \sigma_2^2 = \frac{B}{\pi_2 \Gamma(r)} - \frac{A^2}{\pi_2^2 \Gamma(r)^2}.$$

If we use these formulae in Theorem 2 in Section III then we have

$$M = \frac{(\pi_2 r! - A)^2}{\pi_1 \pi_2 (\Gamma(r))^2 r}$$

and

$$\beta = \frac{\alpha_1 - \pi_1}{\pi_2},$$

Thus

$$\rho = \frac{(\alpha_1 - \pi_1)(\pi_2 r! - A)^2}{r \pi_1 \pi_2^2 (\Gamma(r))^2}. \quad (\text{VI-A-1})$$

For given ρ , x_0 and r ,

$$\alpha_1 = \frac{\rho r! (r-1)! \pi_1 \pi_2^2}{(\pi_2 r! - A)^2} + \pi_1 \quad (\text{VI-A-2a})$$

and

$$\alpha_2 = \frac{\pi_2 - \pi_1(1 - \alpha_1)}{\pi_2} . \quad (\text{VI-A-2b})$$

The value of (VI-A-2a) and (VI-A-2b) are determined with given ρ , x_0 and r . Thus for given marginal distribution and correlation coefficient ρ we can find x_0 ranges as before. For simplicity, we will consider them with marginal distribution gamma (2,1). From the formula of (IV-0-2b),

$$\pi_1 = F(x_0) = 1 - e^{-x_0} - x_0 e^{-x_0}$$

$$\pi_2 = 1 - \pi_1 = e^{-x_0} + x_0 e^{-x_0}$$

and

$$A = e^{-x_0} (x_0^2 + 2x_0 + 2) .$$

Then (VI-A-2a) and (VI-A-2b) become

$$\alpha_1 = \frac{2 \rho \pi_1 \pi_2^2}{x_0^4 e^{-2x_0}} + \pi_1 \quad (\text{VI-A-3a})$$

$$\alpha_2 = \pi_2 + \frac{2 \rho \pi_1^2 \pi_2}{x_0^4 e^{-2x_0}} . \quad (\text{VI-A-3b})$$

These α_1 and α_2 are probabilities, so they have to be greater than or equal to zero and less than or equal to one;

$$0 \leq \alpha_1 = \frac{2 \rho \pi_1 \pi_2^2}{x_0^4 e^{-2x_0}} + \pi_1 \leq 1$$

$$0 \leq \alpha_2 = \pi_2 + \frac{2 \rho \pi_1^2 \pi_2}{x_0^4 e^{-2x_0}} \leq 1$$

From these two inequality equations we can find the x_0 ranges for given correlation coefficient ρ . If $\rho \geq 0$, then α_1 and α_2 are always positive. Thus we only need to find the x_0 ranges which makes (VI-A-3a) and (VI-A-3b) less than one. From equation (VI-A-3a), the α_1 case, and from equation (VI-A-3b), the α_2 case, we have exactly the same inequality equation;

$$2\rho e^{x_0}(1+x_0) \leq x_0^4 + 2\rho(1+x_0)^2 \quad (\text{VI-A-4})$$

If $\rho \leq 0$, then α_1 and α_2 are always less than one. Thus we only need to find the x_0 ranges which makes (VI-A-3a) and (VI-A-3b) greater than zero. From the α_1 equation, we have

$$\sqrt{2\rho^*}(1+x_0) \leq x_0^2 \quad (\text{VI-A-5a})$$

where

$$\rho^* = |\rho|$$

and from the α_2 equation, we have

$$\sqrt{2\rho^*} e^{x_0} \leq x_0^2 + \sqrt{2\rho^*}(1 + x_0) \quad (\text{VI-A-5b})$$

where

$$\rho^* = |\rho| .$$

Note that if we let y_1 for the left side equation of inequality sign and y_2 for the right side equation of inequality equations (VI-A-4), (VI-A-5a) and (VI-A-5b), then we know that all y_1 and y_2 are monotone increasing functions. Thus as in the exponential case we can find the range of x_0 by the Newton Raphson method. For the positive correlation case, equation (VI-A-4) will give the x_ℓ and x_u and for the negative correlation case, equation (VI-A-5a) gives x_ℓ and equation (VI-A-5b) gives x_u . By using subroutine GBOUND which is listed in the appendix, we can find x_ℓ and x_u . The tables (VI-a) and (VI-b) show the results from subroutine GBOUND. Unfortunately we have limitations on the possible range of correlations; we can generate bivariate random vectors with correlations in approximately the range $-.5 \leq \rho \leq .6$ for the Gamma (2,1) marginal distribution. In the same way as the above, the values x_ℓ and x_u can be computed with increasing complexity for integer values of r greater than 2. In principle it is also possible, using numerical integration, to compute allowable values for non-integer values of r . The computation is not as complicated as those required for Schmeiser's bivariate gamma. Moreover the mixture-truncation method does not require that the user be able to compute the inverse gamma distribution function.

Table VI-a: x_0 ranges with Gamma (2,1)
marginal distribution
and given ρ

| ρ | x_L | x_u | ρ | x_L | x_u |
|--------|-------|-------|--------|-------|-------|
| 0.1 | 0.41 | 7.38 | -0.1 | 0.93 | 3.41 |
| 0.2 | 0.65 | 6.23 | -0.2 | 1.18 | 2.75 |
| 0.3 | 0.89 | 5.42 | -0.3 | 1.35 | 2.34 |
| 0.4 | 1.15 | 4.74 | -0.4 | 1.50 | 2.03 |
| 0.5 | 1.47 | 4.09 | -0.5 | 1.62 | 1.79 |
| 0.6 | 1.95 | 3.32 | | | |

Table VI-b: x_0 ranges with Gamma (3,1)
marginal distribution
and given ρ

| ρ | x_L | x_u | ρ | x_L | x_u |
|--------|-------|-------|--------|-------|-------|
| 0.1 | 0.82 | 8.99 | -0.1 | 1.64 | 4.73 |
| 0.2 | 1.17 | 7.72 | -0.2 | 1.37 | 3.98 |
| 0.3 | 1.49 | 6.82 | -0.3 | 2.21 | 3.5 |
| 0.4 | 1.83 | 6.05 | -0.4 | 2.4 | 3.15 |
| 0.5 | 2.24 | 5.31 | -0.5 | 2.56 | 2.89 |
| 0.6 | 2.84 | 4.42 | | | |

B. GENERATING PROCEDURE

The generating procedure of bivariate gamma random vectors is almost the same as the uniform and exponential cases. We also developed here three methods which are the FXO method, the UXO method and the TXO method. As in the uniform and exponential cases, these three methods are the same except in the choice of x_0 from the x_0 range $[x_\ell, x_u]$. The FXO method chooses x_0 as a fixed point from the x_0 range and uses it during the entire routine while the UXO and TXO methods choose another x_0 in every routine. From experience, the midpoint of the x_0 range gives the best result of the FXO method. In the UXO method and the TXO method, we assume that x_0 has the uniform distribution and triangular distribution, respectively, over $[x_\ell, x_u]$. This assumption removes some discontinuity which occurs at the truncation point in the FXO method.

Gamma Mixture-Truncation Method (for integer shape parameters)

1. (Initialization)

- i) For given allowable correlation coefficient ρ ,
find x_ℓ and x_u

2. Define truncation point x_0

* FXO method

- i) $x_0 = \frac{1}{2}(x_\ell + x_u)$

* UXO method

- i) Generate a uniform [0,1] random variable U_1
- ii) $x_o = x_l + (x_u - x_l) * U_1$

* TXO method

- i) Generate two uniform [0,1] random variables,
 V_1, V_2
- ii) $x_o = x_l + x_1 + x_2$

where

$$x_m = (x_l + x_u) / 2.0 ,$$

$$x_1 = (x_m - x_l) * V_1 ,$$

$$x_2 = (x_u - x_m) * V_2 .$$

3. Compute parameter values

$$\pi_1 = F(x_o) = 1 - \sum_{j=0}^{r-1} \frac{e^{-x_o} x_o^j}{j!} ,$$

$$\pi_2 = 1 - \pi_1 ,$$

$$\alpha_1 = \frac{\rho r! (r-1)! \pi_1 \pi_2^2}{(\pi_2 r! - A)^2} + \pi_1 ,$$

$$\alpha_2 = 1 - \frac{\pi_1}{\pi_2} (1 - \alpha_1) ,$$

where

$$A = e^{-x_0} \sum_{i=0}^r \frac{r}{(r-i)!} x_0^{r-i}$$

4. Choose type for Y

- i) Generate a uniform [0,1] random variable U
- ii) If $U \leq \pi_1$, go to 9

5. Y is an X_2

- i) Generate a gamma(r,1) random variable G_1
- ii) If $G_1 > x_0$, set $Y \leftarrow G_1$ and go to 6
- iii) Otherwise return to 5.i)

6. Choose type for Z

- i) Set $U \leftarrow ((U - \pi_1)/(1 - \pi_1))$
- ii) If $U \leq 1 - \alpha_2$, go to 8

7. Z is an X_2

- i) Generate a gamma (r,1) random variable G_2
- ii) If $G_2 > x_0$, set $Z \leftarrow G_2$ and go to 11
- iii) Otherwise return to 7.i)

8. Z is an X_1

- i) Generate a gamma (r,1) random variable G_2
- ii) If $G_2 \leq x_0$, set $Z \leftarrow G_2$ and go to 11
- iii) Otherwise return to 8.i)

9. Y is an X_1
 - i) Generate a gamma $(r,1)$ random variable G_1
 - ii) If $G_1 \leq x_0$, set $Y \leftarrow G_1$ and go to 10
 - iii) Otherwise return to 9.i)

10. Choose type for Z
 - i) Set $U \leftarrow U/\pi_1$
 - ii) If $U \leq \alpha_1$, go to 8
 - iii) Otherwise go to 7

11. Deliver (Y,Z) and go to 4 for the FXO method, or go to 2 for the UXO and TXO methods until a sufficient number of random vectors are obtained.

For the non-integer shape parameter case, step 1 and step 3 need more complicated computation procedures but here we are only concerned for integer shape parameter cases. In step 1 we used the subroutine GBOUND which is listed in the appendix. The scatter plots and bivariate histograms of the resulting random vectors from this algorithm are shown in the next section.

C. SIMULATION RESULTS

The scatter plot and bivariate histograms for random vectors of the mixture-truncation bivariate gamma variables with marginal gamma $(2,1)$ distribution and given correlation (0.3 and -0.3) are given here. As in the uniform and

exponential cases, the UXO, TXO and FXO methods are used and are shown in Figures (VI-a), (VI-b) and (VI-c), respectively.

The FXO method has some discontinuity at the truncation point, but the UXO and TXO methods appear to give a relatively smooth continuous distribution. The computed correlation in subroutine BIVHST (Bivariate histogram) is a little different from the given correlation. But we can assume this difference is a result of sampling error.

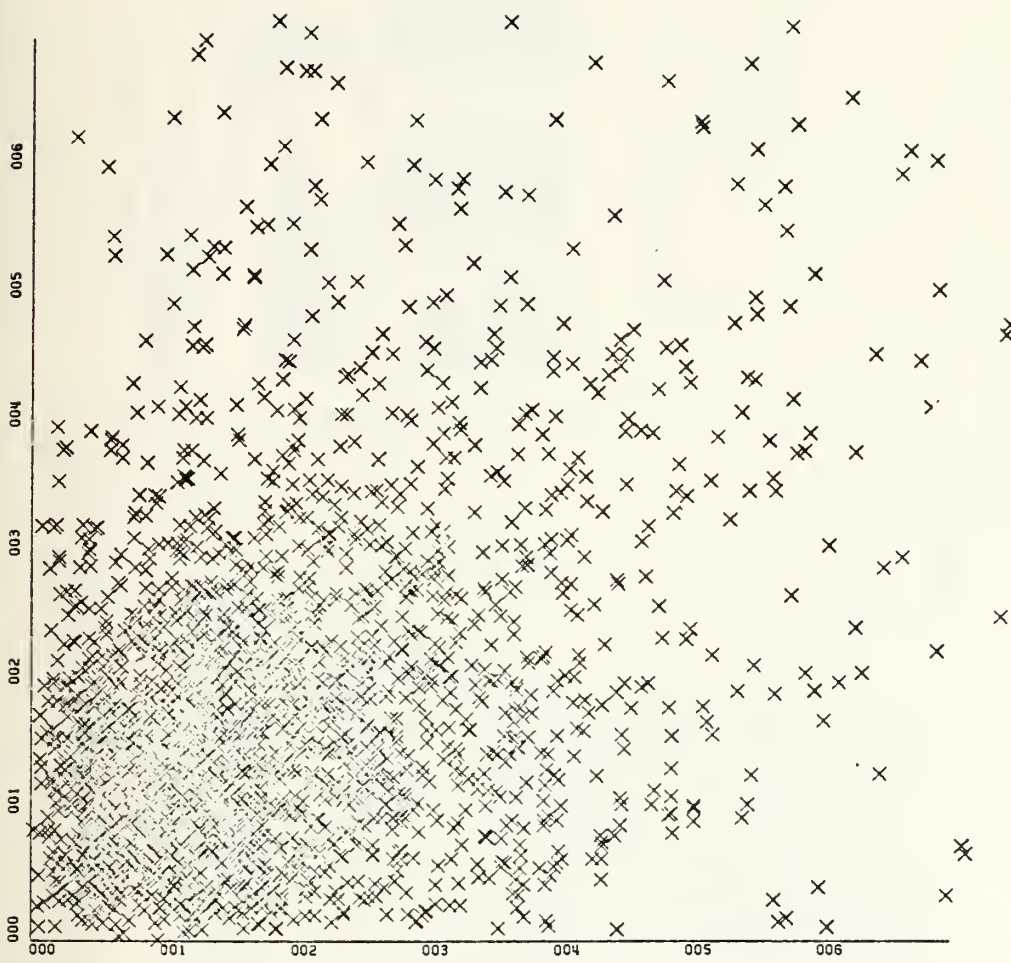


Figure VI-a1. Scatter plot for sample of size 2000 from mixture-truncation bivariate Gamma distribution with $\rho = 0.3$. Here x_0 has uniform distribution over (x_l, x_u) .

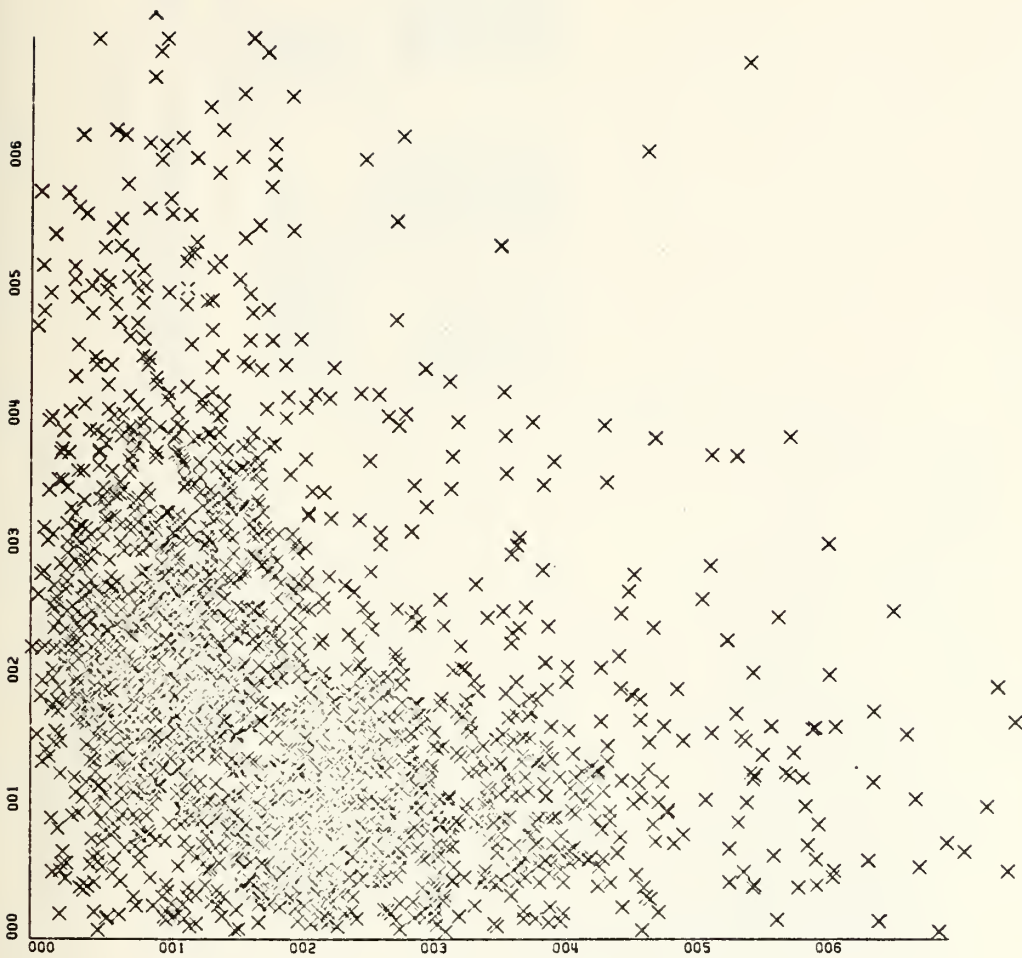
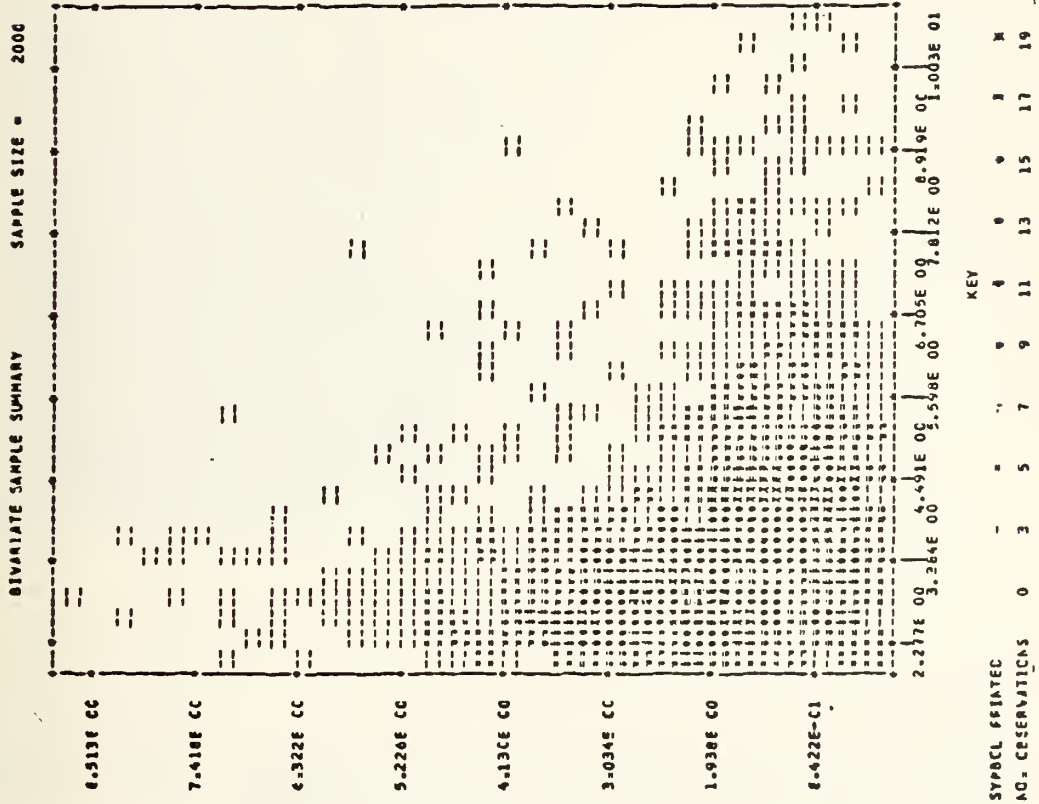


Figure VI-a3. Scatter plot for sample of size 2000 from mixture-truncation bivariate Gamma distribution with $\rho = -0.3$. Here x_0 has uniform distribution over (x_l, x_u) .



MEASURES OF ASSOCIATION
 COVARIANCE COEFFICIENT -0.339569E-01
 CORRELATION COEFFICIENT -0.3371824
 SPEARMAN'S RANK CORRELATION COEFF. -0.371001

TESTS FOR ECUIDISTRIBUTION
 TEST STATISTIC NORMALIZED
 KOLMOGOROV-SMIRNOV TEST 0.559500
 BILD-COLECTOR RUMS TEST -63.22182
 WILCOXON TEST 37.78343
 SIEGEL-TUKEY TEST 400200
 400100
 400100
 0.0

UNIVARIATE STATISTICS

| | X SAMPLE | Y SAMPLE |
|----------|--------------|--------------|
| MEAN | 1.980721E CC | 1.588055E 00 |
| MECHAN | 1.073000E CC | 1.701942E 00 |
| VARIANCE | 2.146494E CC | 1.864730E 00 |
| STD DEV | 1.464717E CC | 1.366331E 00 |
| RANGE | 8.856415E CC | 8.767168E 00 |
| SKWNESS | 3.422014E CC | 1.227572E 00 |
| KURTOSIS | 2.609993E CC | 1.947337E 00 |
| MAXIMUM | 8.885908E CC | 9.787447E 00 |
| MINIMUM | 2.000000E CC | 2.024871E-02 |

Figure VI-a4. Bivariate histogram for sample of size 2000 from mixture-truncation bivariate Gamma distribution with $\rho = -0.3$. Here x_0 has uniform distribution over (x_l, x_u) .

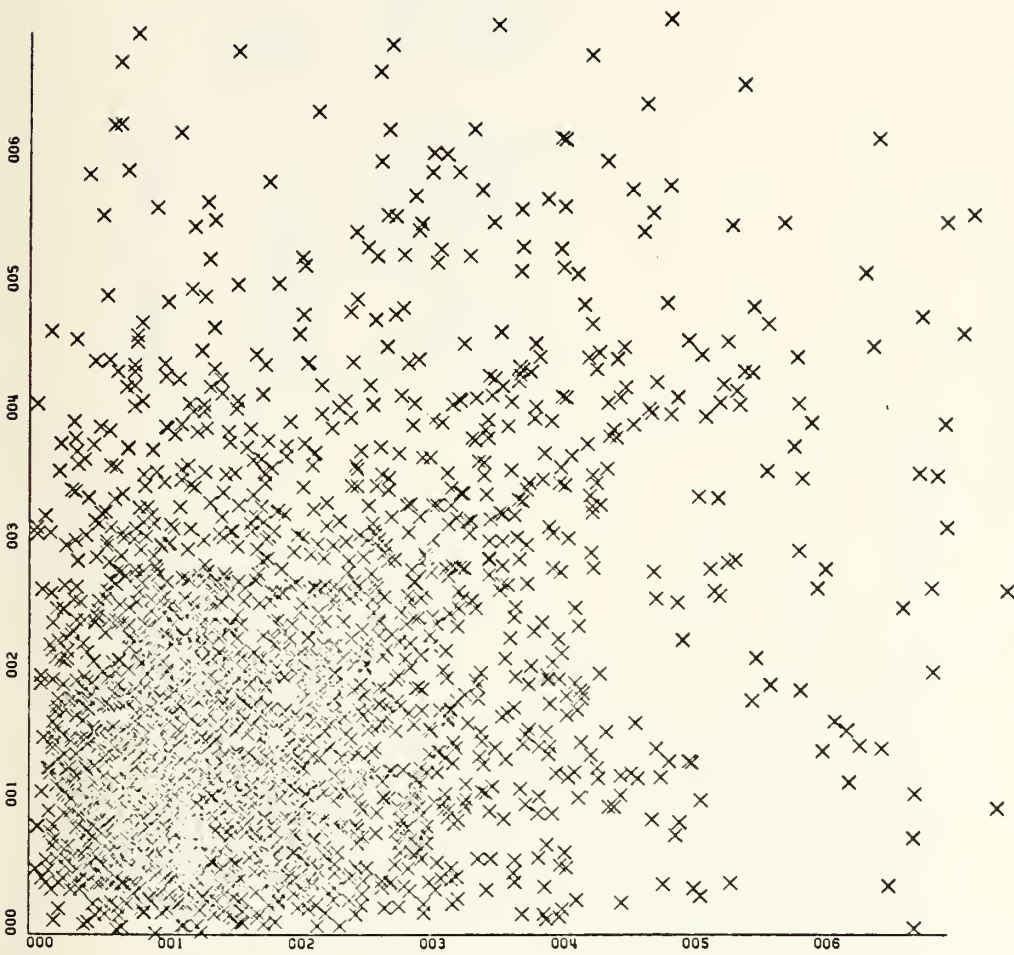
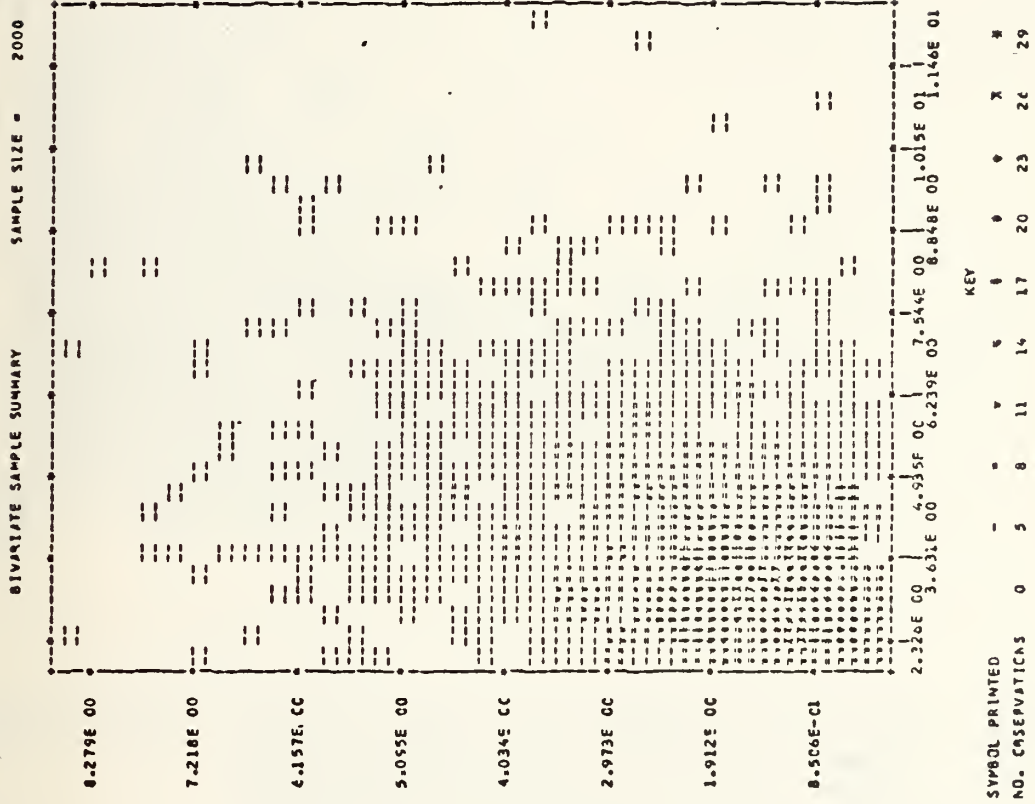


Figure VI-b1. Scatter plot for sample of size 2000 from mixture truncation bivariate Gamma distribution with $\rho = 0.3$. Here x_0 has triangular distribution over (x_l, x_u^0) .



MEASURES OF ASSOCIATION
 COVARIANCE 5.801844E-01
 CORRELATION COEFFICIENT 0.287443
 SPEARMAN'S RANK CORRELATION COEF. 0.238552

TESTS FOR EQUIDISTRIBUTION
 KOLMOGOROV-SMIRNOV TEST 0.955500
 WILCOXON TEST 400000
 MANN-WHITNEY TEST 400000
 SIEGEL-TUKEY TEST 400000
 NORMALIZED STATISTIC -63.22182
 54.76340
 54.76340
 0.0

UNIVARIATE STATISTICS
 MEAN 2.05265E 00
 MEDIAN 1.068200E 00
 VARIANCE 2.064559E 00
 STANDARD DEV 1.436854E 00
 RANGE 1.063549E 01
 SKEWNESS 1.397695E 00
 KURTOSIS 2.668313E 00
 MAXIMUM 2.000000E 00
 MINIMUM 2.000000E 00

Figure VI-b2. Bivariate histogram for sample of size 2000 from mixture-truncation bivariate Gamma distribution with $\rho = 0.3$. Here x_0 has triangular distribution over (x_l, x_u) .

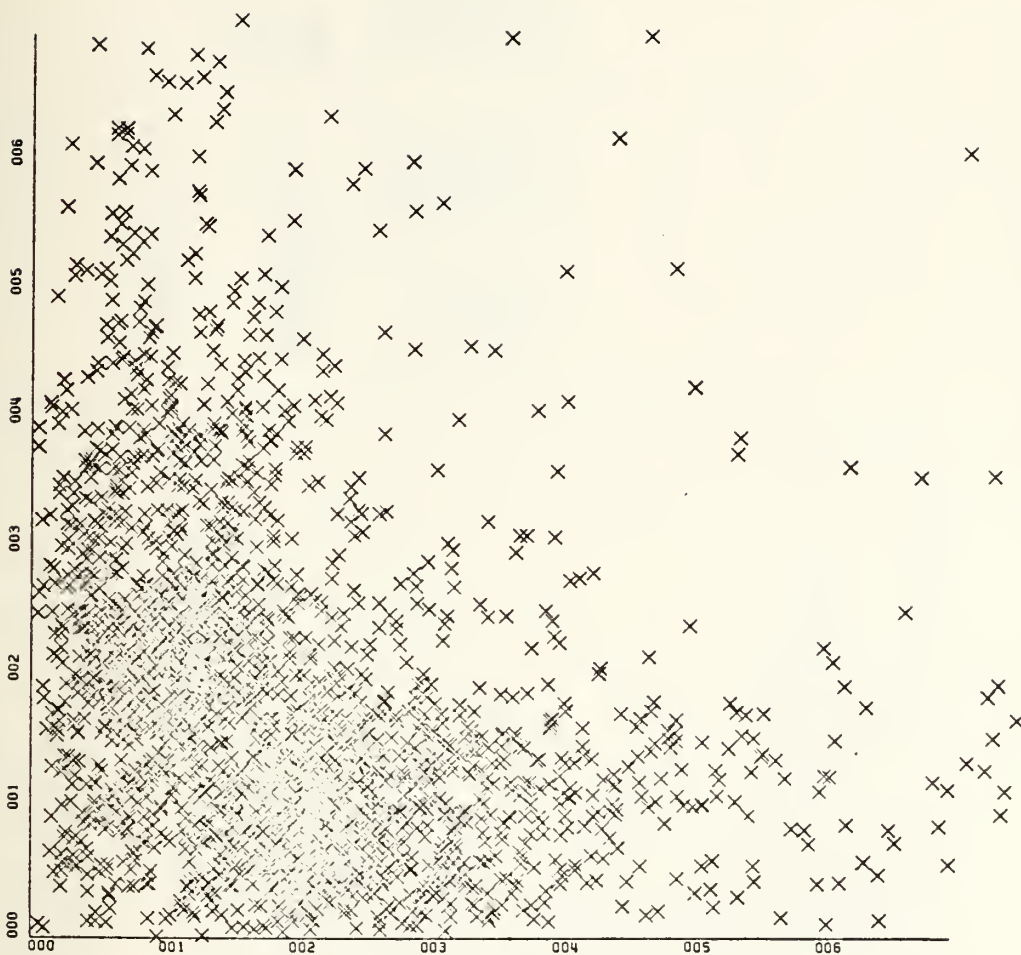


Figure VI-b3. Scatter plot for sample of size 2000 from mixture-truncation bivariate Gamma distribution with $\rho = -0.3$. Here x_u has triangular distribution over (x_l, x_u) .

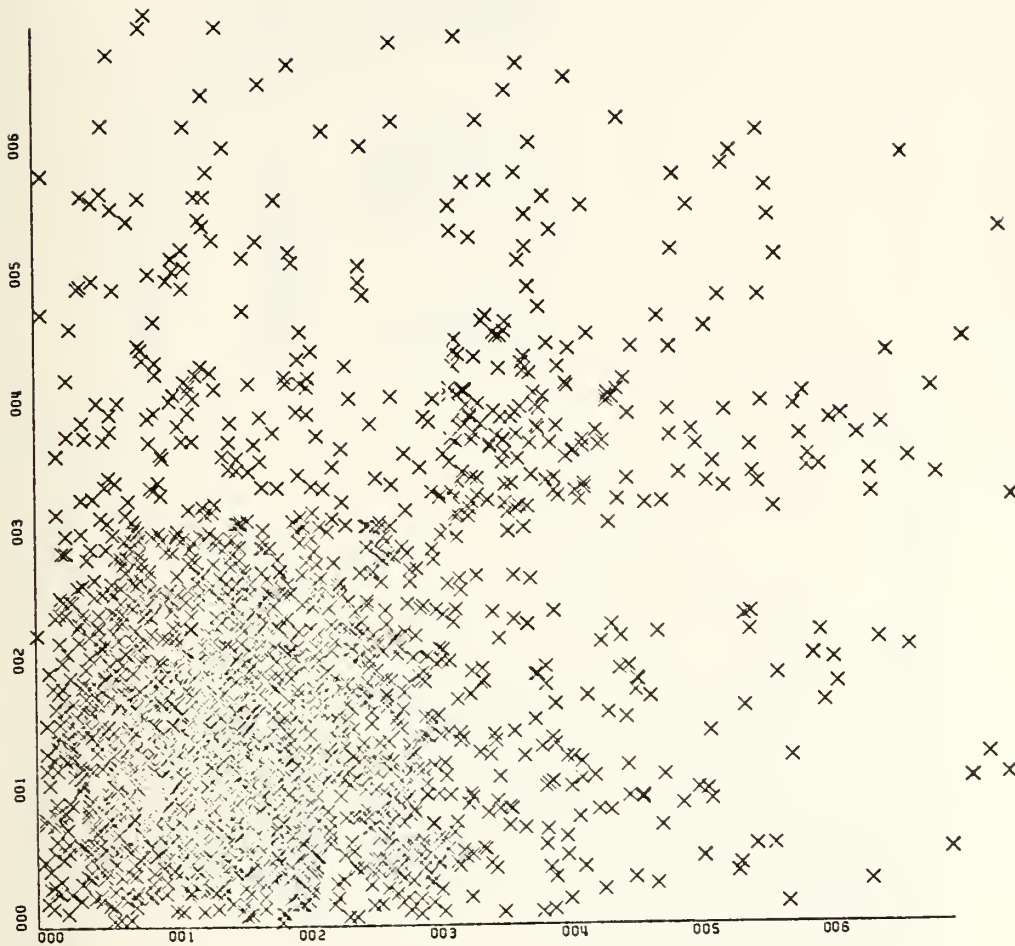
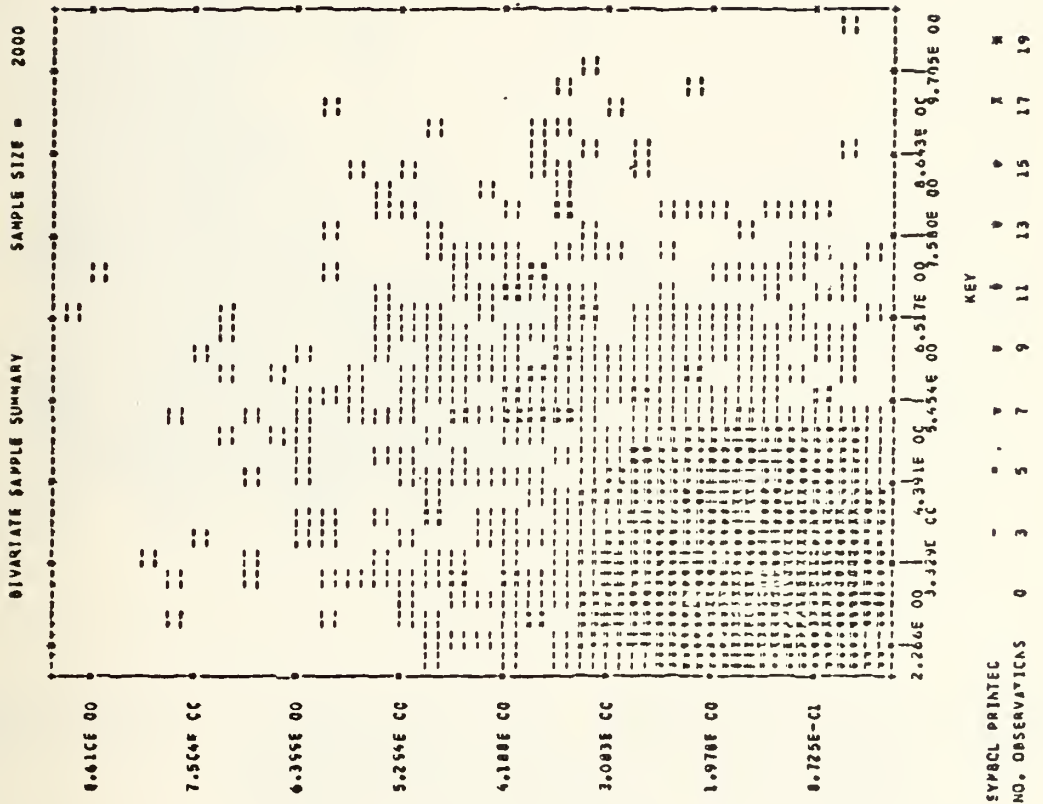


Figure VI-cl. Scatter plot for sample of size 2000 from mixture-truncation bivariate Gamma distribution with $\rho = 0.3$. Here x_0 is fixed at the midpoint of x_ℓ and x_u .



MEASURES OF ASSOCIATION

COVARIANCE 5.789708E-01
CORRELATION COEFFICIENT 0.262123
SPEARMAN'S RANK CORRELATION COEF. 0.262123

TESTS FOR EQUIDISTRIBUTION

TEST NORMALIZED STATISTIC

KOLMOGOROV-SMIRNOV TEST 0.999777
WALD-COLEMAN TEST -63.72182
MANN-WHITNEY TEST 400000
WILCOXSON TEST 600100
SIEGEL-TUKY TEST 400100

UNIVARIATE STATISTICS

X SAMPLE Y SAMPLE

MEAN 1.94784E 00 1.06342E 00
MEDIAN 1.57000E 00 1.66360E 00

VARIANCE 1.85752E 00 1.95505E 00
STD DEV 1.36107E 00 1.39853E 00
RANGE 8.50241E 00 8.84236E 00

SKEWNESS 1.22531E 00 1.30203E 00
KURTOSIS 1.65940E 00 2.09170E 00

MAXIMUM 9.52489E 00 8.99260E 00
MINIMUM 2.00000E 00 2.33489E 00

SYMBOL PRINTED 0 3 5 7 9 11 13 15 17 19 21 23 25 27 29 31

NO. OBSERVATIONS

KEY

Figure VI-c2. Bivariate histogram for sample of size 2000 from mixture-truncation bivariate Gamma distribution with $\rho = 0.3$. Here x_0 is fixed at the midpoint of x_l and x_u .

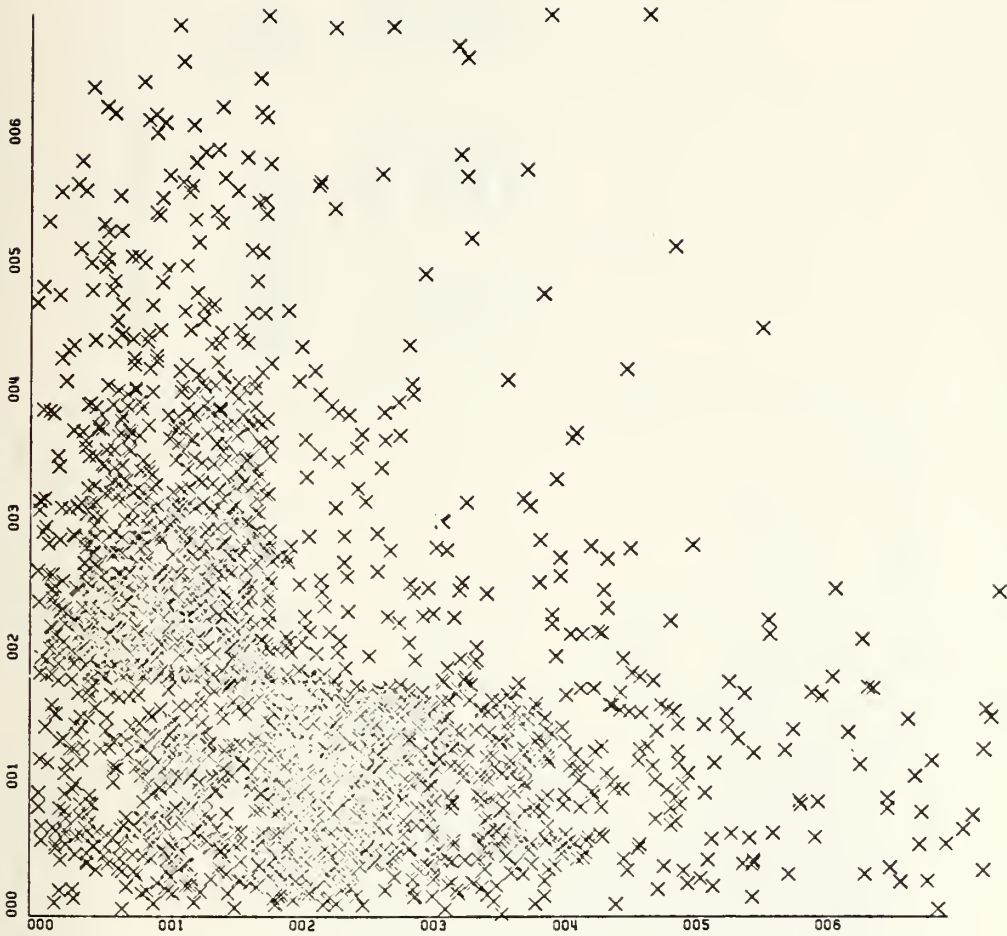


Figure VI-c3. Scatter plot for sample of size 2000 from mixture truncation bivariate Gamma distribution with $\rho = -0.3$. Here x_0 is fixed at the midpoint of x_l and x_u .

VII. CONCLUSION

The mixture-truncation method is a general method which can generate bivariate random vectors having any theoretical marginal distribution and allowable correlation. The generating procedure is very simple and doesn't need much computation for defining parameter values. In this respect, the mixture-truncation method is a very attractive method for generating bivariate random vectors. A price is paid for this simplicity and generality in that the Frechet bounds of correlation for the bivariate distributions specified by the marginal distribution given by Moran (1967) are not always attained. Also there is some discontinuity in the bivariate distribution. However this discontinuity can be decreased by giving some distribution to the truncation point over its range for given ρ . Thus the mixture-truncation method is very attractive for simulation studies involving only partly specified dependency structures. The mixture-truncation method may be extended to generate bivariate random vectors having negative values. Another extension may be made to use grade correlation or rank correlation which are invariant under transformation instead of using the product moment correlation.

CCORRELATION IN THIS PROGRAM FOR CR=' ,F7.3)

STOP
ENC

SUBROUTINE UPJOB(CR,XO,AP1,AP2,PIE1,PIE2,BETA)

C---UNIFORM

C---FIND

INITIAL VALUE
AP1=(CR+3.0*XC**2)/(3.0*XO)
AP2=1.0-(1.0-AP1)*(XO/(1.0-XO))
PIE1=(1.0-AP2)/(1.0-AP1+1.0-AP2)
PIE2=(1.0-AP1)/(1.0-AP1+1.0-AP2)
BETA=AP1-(1.0-AP2)
RETURN
ENC

SLBRROUTINE BEXP(CR,NV,Y,Z,IS)

DIMENSION Y(NV),Z(NV)

IX=IS*0.8
IU=IS*0.3+123456
IF(CR.GT.0.64 .OR. CR.LT.-0.48) GO TO 70
CALL BOUND(CR,XS1,XS2,XL,XU)

DO 100 I=1,NV
CALL RANDOM(IU,U,1)
XO=XL+(XU-XL)*U
CALL EPJOB(CR,XO,AP1,AP2,PIE1,PIE2,BETA)

C---C+

COSE TYPE FOR Y
CALL RANDOM(IU,U,1)
IF(U.LE.PIE1) GO TO 50
L=(U-PIE1)/(1.0-PIE1)
AP=1.0-AP2

C---Y

FROM X2
20 CALL EXPON(IX,X,1)
Y(I)=X+XO

C---C+

COSE TYPE FOR Z
IF(U.LE.AP) GO TO 40

C---Z

FROM X2
30 CALL EXPON(IX,X,1)
Z(I)=X+XO

C---Z

FROM X1
40 CALL RANDOM(IL,U,1)
Z(I)=-1.0*ALOG(1.0-U*PIE1)
GO TO 100

C---Y

FROM X1
50 U=U/PIE1
60 CALL RANDCM(IL,U,1)
Y(I)=-1.0*ALOG(1.0-U*PIE1)
IF(U.LE.AP1) GO TO 40

100

CONTINUE
RETURN

70 WRITE(6,80)CR
80 FORMAT(5X,'CHECK CORRELATION LIMIT,CC NOT ALLCW GIVEN,

CCORRELATION IN THIS PROGRAM FOR CR=' ,F7.3)

STCF
END

SUBROUTINE EPJOB(CR,XO,AP1,AP2,PIE1,PIE2,BETA)

EXPONENTIAL

PIE1=1.0-(1.0/EXP(XC))
PIE2=1.0-PIE1
AP1=PIE1*(1.0+(CR/XC**2))
AP2=1.0-((PIE1/PIE2)*(1.0-AP1))

```

BETA=AP1-(1.0-AP2)
RETURN
END

```

```

SUBROUTINE BOUND(RO,XS1,XS2,XL,XU)
IF(RC.LT.0.0) GO TO 10
AA=(1.0-RC-((RO**2)*(5.0/12.0)))*0.5
XS1=((1.0-0.5*RO)-AA)/(RO/3.0)
XS2=((1.0-0.5*RO)+AA)/(RO/3.0)
CALL RPSON(XS1,XX,RC)
XL=XX
CALL RPSON(XS2,XX,RC)
XU=XX
RETURN
10 AR=ABS(RO)
XL=AR**0.5
XS1=C.0
XS2=((AR**0.5)-AR)/(AR*0.5)
CALL RPSON(XS2,XX,RC)
XU=XX
RETURN
END

```

```

SUBROUTINE RPSON(XS,XX,RO)
EE=C.0001
Y1=XS
N=1
10 IF(RC.LT.0.0) GO TO 20
CALL FUNP(Y1,FV,DV,RC)
GO TO 25
20 CALL FUNN(Y1,FV,DV,RO)
25 Y2=Y1-(FV/DV)
IF((ABS(Y1-Y2)).LE.EE) GO TO 30
Y1=Y2
GO TO 10
30 XX=Y2
RETURN
END

```

```

SUBROUTINE FUNP(Y1,FV,DV,RO)
FV=(Y1**2)-(RC*(EXP(Y1)-1.0))
DV=(2.0*Y1)-(RO*EXP(Y1))
RETURN
END

```

```

SUBROUTINE FUNN(Y1,FV,DV,RO)
RR=ABS(RO)
FV=(Y1**2)-RR*((EXP(Y1)-1.0)**2)
DV=2.0*Y1-(2.0*RR*EXP(2.0*Y1))+(2.0*RR*EXP(Y1))
RETURN
END

```

```

SUBROUTINE BGAM2(CR,NV,Y,Z,IS)

```

```

DIMENSION Y(NV),Z(NV)

```

```

IX=IS*0.8
IU=IS*0.3+123456
IF(CR.GT.0.64 .OR. CR.LT.-0.55) GO TO 70
CALL GROUND(CR,XS1,XS2,XL,XU)
DO 100 I=1,NV
CALL RANDOM(IL,U,1)
XO=XL+(XU-XL)*U

```

```

CALL GPJOB(CR,X0,AP1,AP2,PIE1,PIE2,BETA)
C---C+ COSE TYPE FOR Y
CALL RANDCM(IL,U,1)
IF(L.LE.PIE1) GO TO 50
U=(L-PIE1)/(1.0-PIE1)
AP=1.0-AP2
C---Y FRCM X2
20 CALL GAMA(2.0,IX,X,1)
IF(X.LE.XC) GC TO 20
Y(I)=X
C---C+ COSE TYPE FOR Z
IF(U.LE.AP) GC TO 40
C---Z FROM X2
30 CALL GAMA(2.0,IX,X,1)
IF(X.LE.XC) GC TO 30
Z(I)=X
GO TO 100
C---Z FRCM X1
40 CALL GAMA(2.0,IX,X,1)
IF(X.GT.XC) GC TO 40
Z(I)=X
GO TO 100
C---Y FROM X1
50 L=U/PIE1
60 CALL GAMA(2.0,IX,X,1)
IF(X.GT.XC) GC TO 60
Y(I)=X
IF(L.LE.AP1) GO TO 40
GO TO 30
100 CONTINUE
RETURN
70 WRITE(6,80)CR
80 FORMAT(5X,'CHECK CORRELATION LIMIT,DO NOT ALLOW GIVEN,
C CORRELATION IN THIS PROGRAM FOR CR=',F7.3)
STOP
END

```

```

C
C
C
SUBROUTINE GPJOB(CR,X0,AP1,AP2,PIE1,PIE2,BETA)
GAMMA(2,1)
EX=1.0/EXP(X0)
PIE1=1.0-EX-XC*EX
PIE2=1.0-PIE1
AD=(X0**4)*(1.0/EXP(2.0*X0))
AP1=PIE1+((2.0*CR*PIE1*PIE2**2)/AD)
AP2=1.0-((PIE1/PIE2)*(1.0-AP1))
BETA=AP1-(1.0-AP2)
RETURN
END

```

```

C
C
SUBROUTINE GBCUND(RO,XS1,XS2,XL,XU)
IF(RO.LT.0.0) GO TO 10
A1=((4.0*RO)/3.0)+SQRT(((4.0*RO**2)/9.0)+(4.0*RO))
A2=2.0*(1.0-(RO/3.0))
XS1=A1/A2
XS1=XS1+1.0
XS2=XS1+10.0
CALL GRPSON(XS1,XX,RO)
XL=XX
CALL GRPSON(XS2,XX,RO)
XU=XX
RETURN
10 AR=2.0*ABS(RO)
XL=(SQRT(AR)+SQRT(AR+SQRT(AR)*4.0))/2.0
XS1=0.0
A1=SQRT((SQRT(AR)/6.0)+(RO/9.0))-(SQRT(AR)/6.0)
A2=SQRT(AR)/12.0
XS2=A1/A2
CALL GRPSON(XS2,XX,RO)
XU=XX
RETURN

```

END

```
C
C
SUBROUTINE GRFSON(XS,XX,RO)
EE=C.001
Y1=XS
10 IF(RC.LT.0.0) GO TO 20
CALL GFUNP(Y1,FV,DV,RO)
GC TO 25
20 CALL GFUNN(Y1,FV,DV,RC)
25 Y2=Y1-(FV/DV)
IF((ABS(Y1-Y2)).LE.EE) GO TO 30
Y1=Y2
GO TO 10
30 XX=Y2
RETURN
END
```

```
C
C
SUBROUTINE GFLNP(Y1,FV,DV,RO)
F1=(Y1**4)+(2.0*RO*(1.0+Y1)**2.0)
F2=2.0*RO*EXP(Y1)*(1.0+Y1)
FV=F1-F2
CF1=(4.0*Y1**3)+(4.0*RO*(1.0+Y1))
CF2=2.0*RO*EXP(Y1)*(2.0+Y1)
DV=CF1-DF2
RETURN
END
```

```
C
C
SUBROUTINE GFUNN(Y1,FV,DV,RO)
TR=SQRT(2.0*ABS(RO))
F1=(Y1**2.0)+(TR*(1.0+Y1))
F2=TR*EXP(Y1)
FV=F1-F2
DF1=(2.0*Y1)+TR
CF2=TR*EXP(Y1)
DV=CF1-DF2
RETURN
END
```

```
C
C
SUBROUTINE BIVHST (X, Y, N, WORK)
```

```
IMPLICIT REAL*8 (D)
INTEGER*2 CELL, MAX, SCALE, ONE, ONE5
INTEGER RUNS, WN, SN, WEIGHT, U, FLAG
REAL D, DELTA, DELTAX
```

```
C
C
DIMENSION X(N), Y(N), WORK(N)
DIMENSION XMOM(6), YMOM(6), DWORK(8), CELL(32,32)
DIMENSION KEY(16), XLABEL(8)
DATA NOUT/6/, ONE/1/, ONE5/15/
```

```
C
C
IF(N.GT.15) GO TO 20
WRITE(NOUT,10) N
10 FORMAT('1***BIVHST*** TOO FEW SAMPLE POINTS. N =',I9)
RETURN
```

```
C
C
20 AN = N
AN2 = AN * AN
NM = N - 1
C-----ORDER X-SAMPLE AND FIND RANGES
CALL SORTON (X,1,N,Y)
XRANGE = X(N) - X(1)
YMAX = Y(1)
YMIN = Y(1)
DO 30 I=2,N
IF(Y(I) .GT. YMAX) YMAX = Y(I)
```

```

IF(Y(I) .LT. YMIN) YMIN = Y(I)
30 CONTINUE
YRANGE = YMAX - YMIN
IF(XRANGE * YRANGE .GT. 0.) GO TO 50
WRITE(NOUT,40)
40 FORMAT('1***BIVHST*** X OR Y SAMPLE HAS ZERO RANGE')
RETURN

C-----ZERC OUT DATA ARRAY
50 DO 70 I=1,32
   DO 60 J=1,32
     CELL(I,J) = 0
   60 CONTINUE
   70 CONTINUE

C-----FIND SCALE FACTORS
AX = 31.999 / XRANGE
BX = 1.0001 - AX * X(1)
DELTA X = XRANGE / 32.
AY = 31.999 / YRANGE
BY = 1.0001 - AY * YMIN
DELTA Y = YRANGE / 32.
MAX = 0

C-----ACCLMULATE CELL COUNTS
DO 100 I=1,N
  IX = AX * X(I) + BX
  IY = AY * Y(I) + BY
  IY = 33 - IY
  CELL(IX, IY) = CELL(IX, IY) + ONE
  IF(CELL(IX, IY) .GT. MAX) MAX = CELL(IX, IY)
100 CONTINUE

C-----SCALE FACTOR FOR COUNTS
SCALE = (MAX - ONE) / ONE5 + ONE

C-----FIND SAMPLE MOMENTS
DWORK(1) = 0.0
CALL ACCUM(X,N,DWORK)
CALL MOMENT(DWORK, XMOM)
DWORK(1) = 0.0
CALL ACCUM(Y,N,DWORK)
CALL MOMENT(DWORK, YMOM)

C-----FIND COVARIANCE
DSUM = 0.0
DO 110 I=1,N
  DSUM = DSUM + (X(I) - XMOM(1)) * (Y(I) - YMOM(1))
110 CONTINUE
COVAR = DSUM / AN
XDEV = SQRT(XMOM(2))
YDEV = SQRT(YMOM(2))
RHO = COVAR / (XDEV * YDEV)

C-----KENDALL'S TAU COEFFICIENT
T = 0.0
IF(N .GT. 500) GO TO 119
DO 118 I=1,NM
  DO 114 J=I,N
    Z = Y(J) - Y(I)
    IF(Z .GT. 0.) T = T + 2.0
    IF(Z .LT. 0.) T = T - 2.0
  114 CONTINUE
118 CONTINUE
T = T / (AN2 - AN)

C-----SET UP WORK AREA AND ORDER Y SAMPLE
119 CONTINUE
WI = 1.0
DO 120 I=1,N
  WORK(I) = WI
  WI = WI + 1.0
120 CONTINUE
CALL SORTON(Y, 1, N, WORK)

C-----FIND RANK-ORDER STATISTICS
RN = 0
SN = C
RUNS = 0
IF(Y(1) .LT. X(1)) RUNS = 1

```

```

FLAG = 0
C = 0.0
Y = 1
IX = 1
IY = 1
130 Z = ABS(FLOAT(IX-IY)) / AN
IF(C.LT.Z) C = Z
IF(X(IX) - Y(IY)) 150, 160, 140
C-----CURRENT VALUE IN POOLED SAMPLE IS A Y
140 RUNS = RUNS + FLAG
FLAG = 0
IY = IY + 1
I = I + 1
IF(IY.GT.N) GO TO 200
GO TO 130
C-----CURRENT VALUE IN POOLED SAMPLE IS AN X
150 RUNS = RUNS + 1 - FLAG
FLAG = 1
WN = WN + I
SN = SN + WEIGHT(I,N)
IX = IX + 1
IF(IX.GT.N) GO TO 220
I = I + 1
GO TO 130
C-----X AND Y VALUES ARE TIED. USE MIDRANK CORRECTION.
160 BASE = X(IX)
M = 1
J1 = I + I + 1
J2 = WEIGHT(I,N) + WEIGHT(I+1,N)
K = I
I = I + 1
C-----GET TIED X VALUES
170 IX = IX + 1
IF(IX.GT.N) GO TO 180
IF(X(IX).NE.BASE) GO TO 180
M = M + 1
I = I + 1
J1 = J1 + 1
J2 = J2 + WEIGHT(I,N)
GO TO 170
C-----FIND TIED Y VALUES
180 IY = IY + 1
IF(IY.GT.N) GO TO 190
IF(Y(IY).NE.BASE) GO TO 190
I = I + 1
J1 = J1 + 1
J2 = J2 + WEIGHT(I,N)
GO TO 180
C-----WEIGHTED AVERAGE OF TIED VALUES
190 L = I - K + 1
Z = FLOAT(M * J1) / FLOAT(L) + 0.5
WN = WN + Z
Z = FLOAT(M * J2) / FLOAT(L) + 0.5
SN = SN + Z
C-----AD FCC CORRECTION FOR RUNS
RUNS = RUNS + FLOAT(2 * M * (L-M)) / FLOAT(L)
I = I + 1
IF(IX.GT.N) GO TO 220
IF(IY.LE.N) GO TO 130
C-----Y OBSERVATIONS EXHAUSTED
200 RUNS = RUNS + 1
N2 = N + N
DO 210 I=I,N2
WN = WN + I
SN = SN + WEIGHT(I,N)
210 CONTINUE
C-----FIND MANN-WHITNEY STATISTIC
220 U = WN - (N * (N + 1)) / 2
C-----SPEARMAN'S RANK CORRELATION COEFFICIENT
DSUM = 0.
WI = 1.0
DO 230 I=1,N

```

```

DSUM = DSUM + (WI - WORK(I)) ** 2
WI = WI + 1.0
230 CONTINUE
R = 1. - 6. * DSUM / (AN * (AN2 - 1.))
C-----FIND SAMPLE MEDIANS
IF (MOD(N,2) .EQ. 0) GO TO 240
M = 1 + N/2
XMEC = X(M)
YMEC = Y(M)
GO TO 250
240 M = N/2
XMEC = 0.5 * (X(M) + X(M+1))
YMEC = 0.5 * (Y(M) + Y(M+1))
C-----FIND NORMALIZED STATISTICS
250 RNCRM = (RUNS - N - 1) * SQRT((AN + AN - 1.) / (AN2 -
CAN))
FAC = SQRT(12. / (AN2 * (AN + AN + 1.)))
UNORM = (U - C.5 * AN2) * FAC
WNORM = (WN - AN2 - 0.5 * AN) * FAC
SNORM = (SN - AN2 - 0.5 * AN) * FAC
C-----REORDER Y SAMPLE
CALL SORTON (WORK, 1, N, Y)
C
C
C WRITE OUTPUT
C
C-----HEADING
WRITE(NOUT, 300) N
300 FORMAT('I',20X,'BIVARIATE SAMPLE SUMMARY',10X,'SAMPLE
C SIZE =',I9//)
WRITE(NOUT,31C)
310 FORMAT(13X,'+---',7('*'),7('-')), '*-----+')
C
C-----BODY OF PLOT
CALL OUTPUT(CELL(1, 1), SCALE, 1, 0.)
CALL REPEAT
CALL OUTPUT(CELL(1, 2), SCALE, 2, YMAX - DELTA)
WRITE(NOUT,32C)
320 FORMAT('+',93X,'MEASURES OF ASSOCIATION')
CALL REPEAT
CALL OUTPUT(CELL(1, 3), SCALE, 3, 0.)
WRITE(NOUT,33C) COVAR
330 FORMAT('+',83X,'COVARIANCE',25X,1PE14.6)
CALL REPEAT
WRITE(NOUT,34C) RHC
340 FORMAT('+',83X,'CORRELATION COEFFICIENT',13XF9.6)
CALL OUTPUT(CELL(1, 4), SCALE, 4, 0.)
WRITE(NOUT,35C) R
350 FORMAT('+',83X,'SPEARMAN'S RANK CORRELATION CCEF. '
C,F9.6)
CALL REPEAT
IF ( N .LE. 500 ) WRITE(NOUT,360) T
360 FORMAT('+',83X,'KENDALL'S TAU CCEFFICIENT',11XF9.6)
CALL OUTPUT(CELL(1, 5), SCALE, 5, 0.)
CALL REPEAT
CALL OUTPUT(CELL(1, 6), SCALE, 6, YMAX - 5. * DELTA)
WRITE(NOUT,37C)
370 FORMAT('+',95X,'TESTS FOR EQUIDISTRIBUTION')
CALL REPEAT
CALL OUTPUT(CELL(1, 7), SCALE, 7, 0.)
WRITE(NOUT,38C)
380 FORMAT('+',111X,'TEST',7X,'NORMALIZED')
CALL REPEAT
WRITE(NOUT,39C)
390 FORMAT('+',109X,'STATISTIC',5X,'STATISTIC')
CALL OUTPUT(CELL(1, 8), SCALE, 8, 0.)
WRITE(NOUT,40C) D
400 FORMAT('+',83X,'KOLMOGOROV-SMIRNOV TEST ',F10.6)
CALL REPEAT
WRITE(NOUT,41C) RUNS, RNORM
410 FORMAT('+',83X,'WALD-WOLFOWITZ RUNS TEST',I10,5XF9.5)
CALL OUTPUT(CELL(1, 9), SCALE, 9, 0.)
WRITE(NOUT,42C) U, UNORM

```

```

420 FORMAT('+',83X,'MANN-WHITNEY TEST',7X110,5XF9.5)
CALL REPEAT
WRITE(NOUT,43C) WN, WNCRM
430 FORMAT('+',83X,'WILCOXSON TEST',10X110,5XF9.5)
CALL OUTPUT(CELL(1,10), SCALE, 10, YMAX - 9. * DELTA)
WRITE(NOUT,44C) SN, SNCRM
440 FORMAT('+',83X,'STIEGEL-TUKEY TEST',7X110,5XF9.5)
CALL REPEAT
CALL OUTPUT(CELL(1,11), SCALE,11, 0.)
CALL REPEAT
WRITE(NOUT,45C)
450 FORMAT('+',92X,'UNIVARIATE STATISTICS')
CALL OUTPUT(CELL(1,12), SCALE,12, C.)
CALL REPEAT
WRITE(NOUT,46C)
460 FCPMAT('+',95X,'X SAMPLE',9X,'Y SAMPLE')
CALL OUTPUT(CELL(1,13), SCALE,13, 0.)
WRITE(NOUT,47C) XMOM(1), YMOM(1)
470 FORMAT('+',83X,'MEAN',5X1PE14.6,3XE14.6)
CALL REPEAT
WRITE(NOUT,48C) XMEC, YMED
480 FORMAT('+',83X,'MEDIAN',3X1PE14.6,3XE14.6)
CALL OUTPUT(CELL(1,14), SCALE, 14, YMAX -13. * DELTA)
CALL REPEAT
WRITE(NOUT,49C) XMOM(2), YMOM(2)
490 FORMAT('+',83X,'VARIANCE',1X1PE14.6,3XE14.6)
CALL OUTPUT(CELL(1,15), SCALE,15, C.)
WRITE(NOUT,50C) XDEV, YDEV
500 FORMAT('+',83X,'STD DEV',2X1PE14.6,3XE14.6)
CALL REPEAT
WRITE(NOUT,51C) XRANGE, YRANGE
510 FCPMAT('+',83X,'RANGE',4X1PE14.6,3XE14.6)
CALL OUTPUT(CELL(1,16), SCALE,16, 0.)
CALL REPEAT
WRITE(NOUT,52C) XMOM(4), YMOM(4)
520 FORMAT('+',83X,'SKEWNESS',1X1PE14.6,3XE14.6)
CALL OUTPUT(CELL(1,17), SCALE,17, 0.)
WRITE(NOUT,53C) XMOM(6), YMOM(6)
530 FORMAT('+',83X,'KURTOSIS',1X1PE14.6,3XE14.6)
CALL REPEAT
CALL OUTPUT(CELL(1,18), SCALE, 18, YMAX -17. * DELTA)
WRITE(NOUT,54C) X(N), YMAX
540 FORMAT('+',83X,'MAXIMUM',2X1PE14.6,3XE14.6)
CALL REPEAT
WRITE(NOUT,55C) X(1), YMIN
550 FORMAT('+',83X,'MINIMUM',2X1PE14.6,3XE14.6)
Z = YMAX - 17. * DELTA
DC 560 J = 19,32
Z = Z - DELTA
CALL OUTPUT(CELL(1,J), SCALE, J, Z)
CALL REPEAT
560 CONTINUE
WRITE(NOUT,31C)
XLABEL(1) = X(1) + DELTAX
FOURD = 4. * DELTAX
CO 570 I=2,8
XLABEL(I) = XLABEL(I-1) + FCURD
570 CONTINUE
WRITE(NOUT,58C)(XLABEL(I),I=1,7,2),(XLABEL(J),J=2,8,2)
580 FORMAT(16X,8(' ',7X)/ 12X,4(1PE10.3,' ')/ 20X,4(E
C10.3,6X))
KEY(1) = 0
CO 590 I=2,16
KEY(I) = SCALE * I - 1
590 CONTINUE
WRITE(NOUT,60C)(KEY(I),I=1,16)
600 FORMAT(/50X,'KEY',/' SYMBOL PRINTED',12X,'-
1 - = T > A =
2 F H F F' / '+' ,38X,' E E T E * E'
3 + X < X V E - E T E = E'
4/ '+' ,56X,' X' / '+' ,110X,'T'//
5/

```



```

IF ( ON(MIDDLE) .LT. ON(LOKEEP) ) IL = MIDDLE
IF ( ON(HIKEEP) .LT. ON(IL) ) IL = HIKEEP
IF ( ON(LOKEEP) .GT. ON(IH) ) IH = LOKEEP
ISSUE = LOKEEP + MIDDLE + HIKEEP - IH - IL
TEST = ON(ISSUE)
GO TO 50

```

C

```

30 IF (LO .EQ. HI) GO TO 50
TEMP = CN(HI)
ON(FI) = CN(LC)
CN(LO) = TEMP
ITEMP = WITH(HI)
WITH(HI) = WITH(LO)
WITH(LO) = ITEMP
50 HI = HI - 1
IF ( ON(HI) .GT. TEST) GO TO 50
70 LC = LO + 1
IF ( CN(LO) .LT. TEST) GO TO 70
IF ( LO .LE. HI ) GO TO 30
IF (FI - LCKEEP .LE. HIKEEP - LO) GO TO 90
LOSTK(STKPT) = LOKEEP
HISTK(STKPT) = HI
STKPT = STKPT + 1
LOKEEP = LO
GO TO 110
90 LOSTK(STKPT) = LO
HISTK(STKPT) = HIKEEP
STKPT = STKPT + 1
HIKEEP = FI
GO TO 110
100 STKPT = STKPT - 1
IF (STKPT .EQ. 0) RETURN
LOKEEP = LOSTK(STKPT)
HIKEEP = HISTK(STKPT)
110 IF ( HIKEEP - LOKEEP .GE. 11) GO TO 20
IF ( LOKEEP .EQ. 11) GO TO 10
LOKEEP = LOKEEP - 1
120 LOKEEP = LOKEEP + 1
IF ( LOKEEP .EQ. HIKEEP ) GO TO 100
IF ( ON(LOKEEP) .LE. CN(LOKEEP+1) ) GO TO 120
TEMP = CN(LOKEEP + 1)
LO = LCKEEP
130 CN(LO+1) = CN(LO)
LC = LO - 1
IF ( TEMP .LT. CN(LC)) GO TO 130
CN(LO+1) = TEMP
IW = LCKEEP + 1
N = LCKEEP - LO
ITEMP = WITH(IW)
CO 140 IMOVE = 1, N
WITH(IW) = WITH(IW-1)
IW = IW - 1
140 CONTINUE
WITH(IW) = ITEMP
GO TO 120
END

```

```

FOR GENERATING BIVARIATE RANDOM VECTORS(Y,Z),CALL
FOLLOWING SUBROUTINE.
1. UNIFORM(0,1) MARGINAL DISTRIBUTION.
   CALL LEWUFM(CR,NV,Y,Z,IS); POSITIVE CORRELATION BY
   LAWRENCE AND LEWIS METHOD.
   CALL GAVUFM(CR,NV,Y,Z,IS); NEGATIVE CORRELATION BY
   TRANSFORMATION OF GAVER'S EXPONENTIAL.
2. EXPONENTIAL(MEAN=1.0) MARGINAL DISTRIBUTION.
   CALL MAREXF(CR,NV,Y,Z,IS); POSITIVE CORRELATION BY
   MARSHAL AND OLKINS METHOD.
   CALL GAVEXF(CR,NV,Y,Z,IS); NEGATIVE CORRELATION BY
   GAVER'S METHOD.

```

```

WHERE
CR; GIVEN CORRELATION.
NV; NUMBER OF RANDOM VECTORS BE GENERATING.
Y; THE FIRST ELEMENT OF VECTOR WITH DIMENSION NV.
Z; THE SECOND ELEMENT OF VECTOR WITH DIMENSION NV.
IS; INITIAL RANDOM SEED.

```

```

SUBROUTINE LEWUFM(CR,NV,Y,Z,IU)

```

```

DIMENSION Y(NV),Z(NV),U(3)
IF (CR.LT.0.0) GO TO 20
A=SQRT(36.0+(12.0*CR)+(16.0*(CR**2.0)))-6.0
BETA=A/(2.0*CR)
ALPA=2.0-(1.0/BETA)
P=(1.0-BETA)/(1.0-BETA+(ALPA*BETA))
R=1.0-ALPA
DO 100 I=1,NV
CALL RANDOM(IU,U,2)
Y(I)=U(1)
IF(U(2).LE.R) GO TO 50
W=(U(2)-R)/(1.0-R)
IF(W.LE.P) GO TO 10
Z(I)=(Y(I)**BETA)*((W-P)/(1.0-P))**(R*BETA)
GO TO 100
10 Z(I)=(Y(I)**BETA)*(W/P)
GO TO 100
50 W=U(2)/R
IF(W.LE.P) GO TO 60
Z(I)=((W-P)/(1.0-P))**(R*BETA)
GO TO 100
60 Z(I)=W/P
100 CONTINUE
RETURN
20 WRITE(6,30)CR
30 FORMAT(5X,'THIS METHOD NOT ALLOW CR=',F7.3)
RETURN
END

```

```

SUBROUTINE GAVUFM(CR,NV,Y,Z,IS)

```

```

DIMENSION Y(NV),Z(NV),WK(1),G(100)
COUPELE PRECISION DXS
DXS=IS*0.8
IU=IS*0.5+12345
IF (CR.GT.0.0) GO TO 20
P=-2.0*CR
Q=1.0-P
DO 100 I=1,NV
CALL GGEOI(DXS,1,0,WK1,IG)
CALL RANDOM(IU,U,1)
RN=1.0/IG
Y(I)=(Q*(U**RN))/(1.0-(P*(U**RN)))

```

```

CALL RANDOM(IU,G,IG)
W=1.0
DO 10 K=1,IG
W=W*G(K)
10 CONTINUE
Z(I)=W**((1.0-P)
100 CONTINUE
RETURN
20 WRITE(6,30)CR
30 FORMAT(5X,'THIS METHOD NOT ALLOW CR=',F7.3)
RETURN
END

```

```

SUBROUTINE MAREXP(CR,NV,Y,Z,IX)

```

```

DIMENSION Y(NV),Z(NV),E(3)
IF(CR.LT.0.0) GO TO 20
R12=(2.0*CR)/(CR+1.0)
R1=1.0-R12
R2=R1
DO 50 I=1,NV
CALL EXPON(IX,E,3)
E(1)=E(1)/R1
E(2)=E(2)/R2
E(3)=E(3)/R12
Y(I)=AMIN1(E(1),E(3))
Z(I)=AMIN1(E(2),E(3))
50 CONTINUE
RETURN
20 WRITE(6,30)CR
30 FORMAT(5X,'THIS METHOD NOT ALLOW CR=',F7.3)
RETURN
END

```

```

SUBROUTINE GAVEXP(CR,NV,Y,Z,IS)

```

```

DIMENSION Y(NV),Z(NV),WK1(1),WK2(100)
DOUBLE PRECISION DSEED1,DSEED2,DSEED3
DSEED1=IS*0.2+123456
DSEED2=IS*0.5+1234
DSEED3=IS*0.9+123
IF(CR.LT.-0.5 .OR. CR.GT.0.0) GO TO 20
G=-2.0*CR
P=1.0-Q
DO 100 I=1,NV
CALL GGEOI(DSEED1,1,P,WK1,IG)
A=IG
B=P
CALL GGAMS(DSEED2,A,B,1,WK2,GA)
Z(I)=GA
CALL GGUBS(DSEED3,1,U)
RN=1.0/IG
Y(I)=ALOG((1.0/(P*U**RN))-(G/P))
100 CONTINUE
RETURN
20 WRITE(6,30)CR
30 FORMAT(5X,'THIS METHOD NOT ALLOW CR=',F7.3)
RETURN
END

```

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