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The Construction And Asymptotic
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Simple Epidemic Models .

by

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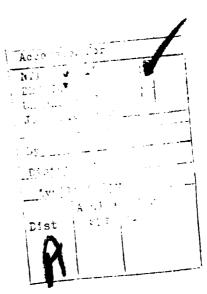
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ABSTRACT

A population of susceptible individuals exposed to m contagious diseases
is considered. The progress of this epidemic among the individuals is modeled
by an m-dimensional stochastic process. The components of this process repre-
sent the number of infective individuals with the respective diseases at time t.

A class of m-dimensional stochastic processes is constructed. These
processes describe the progress of the epidemic models considered in the sequel.
Exact and approximate formulas for the joint and marginal state probabilities
of these models are obtained. It is shown that the approximate formulas are
relatively simple functions of time while, the derivations of the exact for-
mulas involve tedious computations. The results obtained in the paper are
applied to a sample of examples.

Key words: m-dimensional simple epidemics, exponential, negative bino-
mial and Poisson random variables, stochastic processes and convergence in
distribution.

1. Introduction And Summary.

In a simple epidemic situation we assume that a population of suscepti-
ble individuals (susceptibles) is exposed only to one contagious disease
(disease) [Bailey (1975)]. However, frequently susceptibles are exposed
simultaneously to more than one disease, as is the case with different
types of flu. In this paper we consider a population of susceptibles
exposed to m diseases, $m = 1, 2, \ldots$. We say that this population undergoes
an m-dimensional simple epidemic if the following six assumptions hold.

(1.1) At each point in time at most one susceptible contracts a disease.

(1.2) Each susceptible constracts at most one disease.

(1.3) Once a susceptible contracts disease r, $r = 1, \ldots, m$, he remains
contagious for the duration of the epidemic.

(1.4) An infective individual (infective) with disease r, $r = 1, \ldots, m$, can
only transmit that disease.

(1.5) All interactions between a susceptible and an infective with a specified
disease are equally likely to result in an infection.

(1.6) Individuals neither join nor do they depart from the population.

Note that for $m = 1$ the m-dimensional epidemic reduces to the univariate
simple epidemic.

Let T_0 denote the first time we have at least one infective with each
of the various diseases, and n, finite or infinite, the number of suscepti-
bles at time T_0. We describe the progress of m dimensional simple epidemics
among susceptibles by an m-dimensional stochastic process
$\underline{X}_n(t) = [X_{n,1}(t), \ldots, X_{n,m}(t)]$, $t \in [0, \infty)$. The components of $\underline{X}_n(t)$ represent

the number of infectives with the respective diseases at time t measured from T_0. In Section 2 we construct a variety of m-dimensional stochastic processes. These processes describe the progress of the epidemic models considered in the sequel.

The computation of state probabilities in epidemic models is of major interest to researchers. In Section 3 we derive formulas for the joint and marginal state probabilities: $P_{n,\underline{k}}(t) = P\{\underline{X}_n(t)-\underline{X}_n(0)=\underline{k}\}$, and

$P_{n,k,r}(t) = P\{X_{n,r}(t)-X_{n,r}(0)=k\}$, where $\underline{k} = [k_1,\ldots,k_m]$, k, k_1, ..., $k_m \epsilon \{0,1,\ldots\}$, $r = 1, \ldots, m$, and $t \epsilon (0,\infty)$. We calculate these formulas without the traditional use of the differential equations associated with the state probabilities. This is done by utilizing a formula for the distribution function (df) of a sum of independent exponential random variables (rva's) given by Billard, Lacayo and Langberg (BLL) (1980).

The formulas for the state probabilities obtained in Section 3 are rather complicated. To overcome this deficiency we derive relatively simple approximations to the state probabilities when the initial number of susceptibles: n, is sufficiently large. To be more specific some notation and a definition are needed. Let $\underline{X}_\infty(t)$, $\underline{X}_n(t)$, be m-dimensional stochastic processes, $X_\infty(t) = \sum_{r=1}^m X_{\infty,r}(t)$, $X_n(t) = \sum_{r=1}^m X_{n,r}(t)$, the total number of infectives at time t measured from T_0, and let $Y(t)$ be a nonnegative rv, $t \epsilon [0,\infty)$, $n = 1, 2, \ldots$.

Definition 1.1. Let $n = 1, 2, \ldots, \infty$. Then the transition rate of disease r, $r = 1, \ldots, m$, at time t, $t \epsilon [0,\infty)$ is given by $\lim_{h \to 0^+} h^{-1} P\{X_{n,r}(t+h) - X_{n,r}(t) = 1 | \underline{X}_n(t)\}$, and is denoted by $R(\underline{X}_n(t),r)$.

In Section 4 we assume that for all $\underline{k} = [k_1, \ldots k_m]$, and given s, $\beta \epsilon (0, \infty)$

(1.7) $\lim_{n \to \infty} \underline{X}_n(0) = \underline{X}_\infty(0)$,

(1.8) $\lim_{n \to \infty} R(\underline{X}_n(0) + \underline{k}, r) = R(\underline{X}_\infty(0) + \underline{k}, r)$, $r = 1, \ldots, m$,

(1.9) $E\{Y(s)\}^\beta < \infty$, and that

(1.10) for $x \epsilon [X_\infty(0), \infty)$ and almost all $n \epsilon \{1, 2, \ldots\}$, $P\{X_n(s) \geq x\} \leq P\{Y(s) \geq x\}$.

We show that under Conditions (1.7), (1.8), $\underline{X}_n(t)$ converges in distribution as

$n \to \infty$ to $\underline{X}_\infty(t)$ for all $t \epsilon (0, \infty)$. Using this result we can approximate $P_{n,\underline{k}}(t)$,

and $P_{n,k,r}(t)$ by the respective state probabilities of the process $\underline{X}_\infty(t)$. Further,

we show that $\lim_{n \to \infty} E\{X_n(s)\}^\beta = E\{X_\infty(s)\}^\beta$, and that $\lim_{n \to \infty} E\{X_{n,r}(s)\}^\beta = E\{X_{\infty,r}(s)\}^\beta$,

$r = 1, \ldots, m$, provided Conditions (1.7) through (1.10) hold. Note that if

$\beta \epsilon [2, \infty)$, we can approximate $EX_{n,r}(s)$, $EX_n(s)$, $Var\{X_{n,r}(s)\}$, and $Var\{X_n(s)\}$, by

the equivalent counterparts of $\underline{X}_\infty(s)$.

BLL(1979) consider a special class of m-dimensional simple epidemic models

and name them the symmetric m-dimensional simple epidemics. We say that a pop-

ulation exposed to m diseases undergoes a symmetric m-dimensional simple epidemic

if Assumptions (1.1) through (1.6) hold and if the transition rate of disease r,

$r = 1, \ldots, m$, at time t, $t \epsilon [0, \infty)$ is given by:

(1.11) $\quad R(\underline{X}_n(t), r) = \begin{cases} n^{-1} \alpha X_{n,r}(t)(n + X_{n,r}(0) - X_{n,r}(t)), & n = 1, 2, \ldots, \\ \alpha X_{\infty,r}(t) & , \quad n = \infty, \end{cases}$

where $\alpha \epsilon (0, \infty)$. The transition rates given in Equation (1.11) for the case $n = \infty$

depend only on the number of infectives with the respective diseases. This

special structure of the transition rates is shared by a variety of models as

is shown in the last section. In Section 5 we assume that

(1.12) $R(\underline{X}_\infty(t),r) = \Delta(X_{\infty,r}(t),r)$, $r = 1, \ldots, m$, $t\epsilon[0,\infty)$,

where $\underline{\Delta(k,r)}$, $k = X_{\infty,r}(0), X_{\infty,r}(0) + 1, \ldots$, $r = 1, \ldots, m$, are sequences

of positive real numbers. We prove that

(1.13)   for $r = 1, \ldots, m$, $k = 0, 1, \ldots$, and $t\epsilon(0,\infty)$,

$P\{X_{\infty,r}(t)-X_{\infty,r}(0)\geq k\} =$

$=P\{\sum_{q=1}^{k}[\Delta(X_{\infty,r}(0)+q-1,r)]^{-1}U_q\leq t\}$

where $U_1, U_2, \ldots$, are independent indentically distributed (iid) exponential

rva's with mean 1. Further we show, as expected, that

(1.14)   for all $t\epsilon(0,\infty)$, the components of $\underline{X}_\infty(t)$ are independent.

These results are applied in Section 6 to simplify the approximations to the

state probabilities and to related moments. Note that under Condition (1.12)

the processes $X_{\infty,1}(t), \ldots, X_{\infty,m}(t)$, describe the progress of m independent

univariate simple epidemic models among susceptibles in closed populations

of infinite sizes.

In Section 6 we apply the theory developed in the previous sections to

some classes of m-dimensional simple epidemics, and other models in Epidemiology.

2. Model Construction.

In this section we construct m-dimensional stochastic processes. These processes describe the progress of m-dimensional simple epidemics in a population of a finite or infinite number of susceptibles.

First, we introduce some notation. Let $N(n) = n + 1$ for $n < \infty$, and $= \infty$ for $n = \infty$, and $T_{n,k}$, $1 \leq k < N(n)$, the $k\underline{\text{th}}$ interinfection time defined as the time that elapses between the $X_n(0) + k - 1$ and the $X_n(0) + k$ infection. Further, let $\xi_{n,k}$, $1 \leq k < N(n)$, be a rva assuming values in the set $\{1,\ldots,m\}$, and $\xi_{n,0} \equiv 0$. The rva $\xi_{n,k}$, $1 \leq k < N(n)$, designates the disease responsible for the $X_n(0) + k$ infection. Finally, let I be the indicator function, $S_{n,0} = 0$, $S_{n,k} = \sum_{q=1}^{k} T_{n,q}$, and $S_{n,N(n)} = \infty$. The rva $S_{n,k}$, $1 \leq k < N(n)$, describes the time measured from T_0 until the $X_n(0) + k$ infection. In particular $\sum_{q=1}^{N(n)} T_{n,q}$ represents the duration time of the m-dimensional simple epidemic.

Note that for $0 \leq k < N(n)$, $r = 1, \ldots, m$, and $t \epsilon (0,\infty)$, the following event equality holds.

(2.1) $\qquad (X_{n,r}(t) - X_{n,r}(0) = k) =$

$\qquad = \bigcup_{k \leq q < N(n)} (S_{n,q} \leq t < S_{n,q+1}, \sum_{j=1}^{q} I(\xi_{n,j} = r) = k).$

Thus, to construct the process $X_n(t)$ if suffices to determine the df's of the random vectors (rve's) $[T_{n,q}, \xi_{n,q}, q=1,\ldots,k]$, $1 \leq k < N(n)$. Next we determi... the df's of these rve's. We need some more notation. Let $J_{n,k,r}$, be the index set of all infectives with disease r at time $T_0 + S_{n,k-1}$, and $C_{n,k,r} = X_{n,r}(0) + \sum_{q=0}^{k-1} I(\xi_{n,q} = r)$, $1 \leq k < N(n)$, $r = 1, \ldots, m$. The rva $C_{n,k,r}$

represents the number of infectives with disease r at time $T_0 + S_{n,k-1}$. Further, let $\mu(n,k,q,r)$, $1 \leq k,q < N(n)$, $r = 1, \ldots, m$, be positive real numbers and $\tau_{n,i,k}$, $i = 1, \ldots, X_n(0) + k - 1$, $1 \leq k < N(n)$, be rva's defined on a common probability space. The rva $\tau_{n,i,k}$ describes the time measured from the $X_n(0) + k - 1$ infection until the $i\underline{\text{th}}$ contagious individual causes the $X_n(0) + k$ infection. Throughout we assume that

(2.2)   the conditional rva's $\{\tau_{n,i,k} | \xi_{n,0}, \ldots, \xi_{n,k-1}\}$,

     $i = 1, \ldots, X_n(0) + k - 1$, $1 \leq k < N(n)$, are

     independent exponentially distributed, and that

(2.3) $E\{\tau_{n,i,k} | \xi_{n,0}, \ldots, \xi_{n,k-1}\} =$

     $= \mu^{-1}(n,k,C_{n,k,r},r)$, $i \epsilon J_{n,k,r}$, $1 \leq k < N(n)$, $r = 1, \ldots, m$.

We are ready now to determine the df's of the rve's $[T_{n,q}, \xi_{n,q}, q = 1, \ldots, k]$, $1 \leq k < N(n)$. First, we determine the df's of the rve's $[\xi_{n,1}, \ldots, \xi_{n,k}]$, $1 \leq k < N(n)$. We need the following.

Lemma 2.1. Let us assume that Conditions (2.2), (2.3) hold, let $r = 1, \ldots, m$, and $1 \leq k < N(n)$. Then

(2.4) $P\{\xi_{n,k} = r | \xi_{n,0}, \ldots, \xi_{n,k-1}\} =$

     $= C_{n,k,r} \mu(n,k,C_{n,k,r},r) [\sum_{\ell=1}^{m} C_{n,k,\ell} \mu(n,k,C_{n,k,\ell},\ell)]^{-1}$.

Proof. Note that the event $(\xi_{n,k} = r)$ is equal to $(\min\{\tau_{n,i,k}: i \epsilon J_{n,k,r}\} < \min\{\tau_{n,i,k}: i \epsilon \bigcup_{\substack{\ell=1 \\ \ell \neq r}}^{m} J_{n,k,\ell}\})$. Consequently Statement (2.4) follows by Conditions (2.2), (2.3) and simple integral evaluations. ||

Note that Equation (2.4) determines the df's of the rve's $[\xi_{n,1}, \ldots, \xi_{n,k}]$, $1 \leq k < N(n)$. Next, we determine the df's of the conditional rve's $[\{T_{n,q} | \xi_{n,0}, \ldots \xi_{n,q-1}\}, q = 1, \ldots, k]$, $1 \leq k < N(n)$.

Lemma 2.2. Let us assume that Conditions (2.2), (2.3) hold. Then the conditional rva's $\{T_{n,k}|\xi_{n,0},\ldots,\xi_{n,k-1}\}$, $1 \leq k < N(n)$, are independent exponentially distributed with means equal respectively to $[\sum_{r=1}^{m} C_{n,k,r} \mu(n,k,C_{n,k,r},r)]^{-1}$.

Proof. Note that $T_{n,k} = \min\{\tau_{n,i,k}: i = 1,\ldots,X_n(0)+k-1\}$, $1 \leq k < N(n)$. Consequently the result of the lemma follows by Conditions (2.2), (2.3) and some simple properties of exponential rva's. ||

Clearly Lemmas 2.1 and 2.2 together determine the df's of the rve's $[T_{n,q},\xi_{n,q},q=1,\ldots,k]$, $1 \leq k < N(n)$.

Finally, we show that the transition rates of an m-dimensional simple epidemic, as expected, determine uniquely the epidemic. We need the following lemma.

Lemma 2.3. Let us assume that Conditions (2.2), (2.3) hold. Then $R(\underline{X}_n(t),r) = X_{n,r}(t)\mu(n,X_n(t),X_{n,r}(t),r)$, $r = 1, \ldots, m$, $t\in[0,\infty)$.

Proof. The result of the lemma follows by the memoryless property of exponential rva's [Barlow, Proschan (1975), p. 56], Equation (2.1) and Conditions (2.2), (2.3). ||

Consequently by Lemma 2.3 and Condition (2.3) we obtain for $i\in J_{n,k,r}$, $1 \leq k < N(n)$, $r = 1, \ldots, m$, that

$$(2.5) \qquad E\{\tau_{n,i,k}|\xi_{n,0},\ldots,\xi_{n,k-1}\} =$$

$$= C_{n,k,r}[R(\{C_{n,k,1},\ldots,C_{n,k,m}\},r)]^{-1}$$

Thus, clearly m-dimensional simple epidemic models are determined by their transition rates.

3. Formulas for the State Probabilities.

Let k, k_1, ..., $k_m \epsilon \{0,...,n\}$, $n = 1, 2, ..., \underline{k} = [k_1,...,k_m]$, $s(\underline{k}) = \sum_{r=1}^m k_r$, and θ_1, θ_2, ..., be positive real numbers.

This section contains formulas for the joint and marginal state probabilities: $P_{n,\underline{k}}(t)$, and $P_{n,k,r}(t)$, $r = 1, ..., m$, $t \epsilon (0,\infty)$. These formulas are calculated without the traditional use of the differential equations associated with the state probabilities. Rather, we utilize an available formula for the distribution function of a sum of independent exponential rva's. For the sake of completeness we present this formula.

Theorem 3.1. [BLL (1980), Theorem 2]. Let M be a positive integer, and $\ell_M(j) = \sum_{j_1+...+j_M=j} \prod_{q=1}^m \theta^{j_q+1}$, $j = M, M+1, ...$. Then for $t \epsilon (0,\infty)$

(3.1)
$$P\{\sum_{q=1}^M \theta_q^{-1} U_q \le t\} = \sum_{j=M}^\infty \ell_M(j)(-t)^j/j!.$$

To aid in computing the joint and marginal state probabilities we introduce the following notation. Let $\underline{\ell}(k) = [\ell_0,...,\ell_k]$, $\ell_0=0$, $\ell_1,...,\ell_k \epsilon \{1,..,m\}$, $B(\underline{k}) = \{\underline{\ell}(s(\underline{k})): \sum_{q=0}^{s(\underline{k})} I(\ell_q=r)=k_r, r=1,...,m\}$, and $A(r,j,k) = \{\underline{\ell}(j): \sum_{e=0}^j I(\ell_e=r)=k\}$, $j = k, k+1, ...$.

By Equation (2.1) we obtain that for $t \epsilon (0,\infty)$

(3.2) $P_{n,\underline{k}}(t) = \sum_{\underline{\ell}(s(\underline{k}))} [P\{S_{n,s(\underline{k})} \le t < S_{n,s(\underline{k})+1} | \xi_{n,q}=\ell_q, q=0,...,s(\underline{k})\} \cdot$

$\cdot P\{\xi_{n,q}=\ell_q, q=0,...,s(\underline{k})\} I(\underline{\ell}(s(\underline{k})) \epsilon B(\underline{k}))]$, and that for $r = 1, ..., m$

(3.3) $P_{n,k,r}(t) = \sum_{j=k}^n \sum_{\underline{\ell}(j)} [P\{S_{n,j} \le t < S_{n,j+1} | \xi_{n,e}=\ell_e, e=0,...,j\} \cdot$

$\cdot P\{\xi_{n,e}=\ell_e, e=0,...,j\} I(\underline{\ell}(j) \epsilon A(r,j,k))]$.

Thus, to compute $P_{n,\underline{k}}(t)$ and $P_{n,k,r}(t)$ it suffices to evaluate $P\{\xi_{n,q}=\ell_q, q=0,...,k\}$, and $P\{S_{n,k} \le t < S_{n,k+1} | \xi_{n,q}=\ell_q, q=0,...,k\}$.

Now, we present formulas for these probabilities. Let

$$\underline{D}_{n,k,r}(\underline{\ell}(k),q) = X_{n,r}(0) + \sum_{j=0}^{q-1} I(\ell_j = r), \text{ and}$$

$$n(n,k,\underline{\ell}(k),q) = \sum_{r=1}^{m} \mu(n,q,D_{n,k,r}(\underline{\ell}(k),q),r), \quad q = 1, \ldots, k+1. \text{ First, by}$$

Statement (2.4)

(3.4) $P\{\xi_{n,q} = \ell_q, q = 0, \ldots, k\} =$

$$= \prod_{q=1}^{k} [\{n(n,k,\underline{\ell}(k),q)\}^{-1} \prod_{r=1}^{m} \{\mu(n,q,D_{n,k,r}(\underline{\ell}(k),q,r)\}^{I(\ell_q=r)}].$$

Next, let $\underline{f_k(\theta_1, \ldots, \theta_k, t)}$ be the density function of the rva $\sum_{q=1}^{k} \theta_q^{-1} U_q$. Then for $k = 1, 2, \ldots, t \in (0, \infty)$

(3.5) $P\{\sum_{q=1}^{k} \theta_q^{-1} U_q \le t < \sum_{q=1}^{k+1} \theta_q^{-1} U_q\} =$

$$= \int_0^t e^{-\theta_{k+1}(t-u)} f_k(\theta_1, \ldots, \theta_k, u) du = \theta_{k+1}^{-1} f(\theta_1, \ldots, \theta_{k+1}, t).$$

Thus, by Lemma 2.2 and Theorem 3.1

(3.6)

$P\{S_{n,k} \le t < S_{n,k+1} | \xi_{n,q} = \ell_q = 0, \ldots, k\} =$

$$= \begin{cases} e^{-n(n,k,\underline{\ell}(k),1)t} & k = 0 \\ \\ \sum_{j=n}^{\infty} \frac{(-1)^{n+j-1} t^j}{j!} \sum_{j_1 + \ldots + j_n = j} \prod_{q=1}^{n} \{n(n,k,\underline{\ell}(k),q)\}^{j_q+1} & k = n \\ \\ \{n(n,k,\underline{\ell}(k),k+1)\}^{-1} \sum_{j=k+1}^{\infty} \frac{(-1)^{k+j} t^{j-1}}{(j-1)!} \sum_{j_1 + \ldots + j_{k+1} = j} \\ \prod_{q=1}^{k+1} \{n(n,k,\underline{\ell}(k),q)\}^{j_q+1} & 1 \le k < n. \end{cases}$$

Consequently, the formulas for the joint and marginal state probabilities can be obtained from (3.2) and (3.3) by substitution. Note that, as expected, the exact formulas for the joint and marginal state probabilities are rather complicated.

4. The Asymptotic Approximations.

In this section we present the asymptotic approximations of the joint and marginal state probabilities and of some related moments. All limits are calculated as $n \to \infty$. First, we show that under Conditions (1.7), (1.8), $\underline{X}_n(t)$ converges in distribution to $\underline{X}_\infty(t)$ for all $t \in (0,\infty)$. Then we prove that $\lim E\{X_n(s)\}^\beta = E\{X_\infty(s)\}$, and that $\lim E\{X_{n,r}(s)\}^\beta = E\{X_{\infty,r}(s)\}^\beta$ for $r = 1, \ldots, m$, provided Conditions (1.7) through (1.10) hold.

To establish the convergence of $\underline{X}_n(t)$ to $\underline{X}_\infty(t)$ we need the following two lemmas.

Lemma 4.1. Let us assume that Conditions (1.7), (1.8) hold, let $k = 1, 2, \ldots$, and $\ell_1, \ldots, \ell_k \in \{1,\ldots,m\}$. Then

$$(4.1) \qquad \lim P\{\xi_{n,q}=\ell_q, q=1,\ldots,k\} = P\{\xi_{\infty,q}=\ell_q, q=1,\ldots,k\}.$$

Proof. Let $\ell_0 = 0$. Then $P\{\xi_{n,q}=\ell_q, q=1,\ldots,k\} = $
$$\prod_{q=1}^{k} P\{\xi_{n,q}=\ell_q | \xi_{n,j}=\ell_j, j=0,\ldots,q-1\}.$$ Consequently Statement (4.1) follows by Conditions (1.7), (1.8), and Statements (2.4), (2.5). ||

Lemma 4.2. Let us assume that Conditions (1.7), (1.8) hold, and let $k = 1, 2, \ldots$. Then the conditional rva's $\{S_{n,k} | \xi_{n,q}, q=0,\ldots,k-1\}$ converge in distribution to the conditional rva $\{S_{\infty,k} | \xi_{\infty,q}, q=0,\ldots,k-1\}$.

Proof. To prove the result of the lemma it suffices, by the Cramer-Wold device [Billingsley (1968), p. 49], to show that the conditional rve's $\{[T_{n,1},\ldots,T_{n,k}] | \xi_{n,q}, q=0,\ldots,k-1\}$ converge in distribution to the conditional rve $\{[T_{\infty,1},\ldots,T_{\infty,k}] | \xi_{\infty,q}=\ell_q, q=0,\ldots,k-1\}$. The preceeding statement follows by Conditions (1.7), (1.8), Lemma 2.2, and Statement (2.5). ||

We are ready now to show that $\underline{X}_n(t)$ converges to $\underline{X}_\infty(t)$.

Theorem 4.3. Let us assume that Conditions (1.7), (1.8) hold and let $t \in (0,\infty)$. Then the rve's $\underline{X}_n(t)$ converge in distribution to the rve $\underline{X}_\infty(t)$.

Proof. Note that $\underline{X}_\infty(t)$, $\underline{X}_n(t)$, $n = 1, 2, \ldots$, are discrete rve's. Thus, [Billingsley (1968), p. 16], to prove the result of the theorem it suffices to show that for all $\underline{k} = [k_1, \ldots k_m]$, $k_1, \ldots, k_m \in \{0,1,\ldots\}$

(4.2) $\qquad \underset{}{\ell\text{im}} \, P\{\underline{X}_n(t) - \underline{X}_n(0) = \underline{k}\} = P\{\underline{X}_\infty(t) - \underline{X}_\infty(0) = \underline{k}\}.$

Statement (4.2) follows by Equation (3.2), Conditions (1.7), (1.8) and Lemmas 4.1, 4.2. ||

By Cramer-Wold device and Theorem 4.3 we obtain that

Corollary 4.4. Let us assume that Conditions (1.7), (1.8) hold, let $t \in (0,\infty)$, and $r = 1, \ldots, m$. Then (i) $X_{n,r}(t)$ converges in distribution to $X_{\infty,r}(t)$, and (ii) $X_n(t)$ converges in distribution to $X_\infty(t)$.

Clearly Theorem 4.3 and Corollary 4.4(i) insure that the joint and marginal state probabilities: $P_{n,\underline{k}}(t)$, $P_{n,k,r}(t)$, can be approximate by $P_{\infty,\underline{k}}(t)$, $P_{\infty,k,r}(t)$, respectively.

Now, we establish the moments convergence. First, we show that $\ell\text{im} \, E\{X_n(s)\}^\beta = E\{X_\infty(s)\}^\beta$.

Theorem 4.5. Let us assume that Conditions (1.7) through (1.10) hold. Then $\ell\text{im} \, E\{X_n(s)\}^\beta = E\{X_\infty(s)\}^\beta$.

Proof. Note that $E\{X_n(s)\}^\beta = \beta \int_0^\infty y^{\beta-1} P\{X_n(s) > y\} dy$. Thus, the result of the theorem follows by Corollary 4.4(ii), Conditions (1.9), (1.10) and the dominated convergence theorem [Loève (1963), p. 125]. ||

Next, we show the convergence of $E\{X_{n,r}(s)\}^\beta$.

Theorem 4.6. Let us assume that Conditions (1.7) through (1.10) hold, and let $r = 1, \ldots, m$. Then $\ell im\{X_{n,r}(s)\}^\beta = E\{X_{\infty,r}(s)\}^\beta$.

Proof. Note that $E\{X_{n,r}(s)\}^\beta = \beta\int_0^\infty y^{\beta-1} P\{X_{n,r}(s)>\}dy$, and that for $y\epsilon(-\infty,\infty)$, $n = 1, 2, \ldots$, $P\{X_{n,r}(s)>y\} \le P\{X_n(s)>y\}$. Thus, the result of the theorem follows by Corollary 4.4(i), Conditions (1.9), (1.10), and the dominated convergence theorem. ||

Note that if Condition (1.9) holds then $E\{Y(s)\}^\delta<\infty$ for $\delta\epsilon(-\infty,\beta]$. Now, assume that Conditions (1.9), (1.10) hold for all $s\epsilon(0,\infty)$ and some $\delta\epsilon[2,\infty)$. Then we can approximate $EX_n(t)$, $EX_{n,r}(t)$, $Var\{X_n(t)\}$, and $Var\{X_{n,r}(t)\}$ by the equivalent counterparts of $\underline{X}_\infty(t)$ for all $t\epsilon(0,\infty)$, and $r = 1, \ldots, m$.

5. Some Special m-Dimensional Simple Epidemic Models.

Let $\underline{X}_\infty(t)$, $t \in [0,\infty)$, be an m-dimensional stochastic process describing the progress of an m-dimensional simple epidemic among infinitely many susceptibles. We assume that the transition rates of $\underline{X}_\infty(t)$ depend only on the number of infectives with the respective diseases. Under this assumption we identify the distribution functions of $X_{\infty,1}(t)$, ..., $X_{\infty,m}(t)$, and show, as expected, that $\underline{X}_\infty(t)$ has independent components for $t \in (0,\infty)$. These results are then used in Section 6 to obtain relatively simple approximations to the state probabilities and to some related moments of certain classes of m-dimensional simple epidemics.

We need some notation. Let $Z_r(t)$, $t \in [0,\infty)$, $r = 1$, ..., m, be independent stochastic processes assuming values in the sets $\{X_{\infty,r}(0), X_{\infty,r}(0)+1, \ldots\}$, respectively and determined by the following two equations.

(5.1)
$$Z_r(0) = X_{\infty,r}(0), \quad r = 1, \ldots, m.$$

(5.2)
$$P\{Z_r(t) - Z_{\ldots}(\ldots)^{\ldots}\} =$$

$$= P\{\sum_{q=1}^{k} [\Delta(Z_r(0)+q-1,r)]^{-1} U_q \leq t\}$$

$$t \in (0,\infty), \quad r = 1, \ldots, m, \quad k = 1, 2, \ldots .$$

Further, let $\underline{Z(t)} = \sum_{r=1}^{m} Z_r(t)$, $W_0(t) = 0$, $\underline{W_k(t)} = \text{Inf}\{t: Z(t+\sum_{q=0}^{k-1} W_q)-Z(0)=k\}$, $t \in (0,\infty)$, and $\underline{D_k} = \sum_{q=0}^{k} W_q$, $k = 0, 1, \ldots .$ Finally, let λ_k, $k = 1, 2, \ldots$, be rva's assuming values in the set $\{1,\ldots,m\}$, determined by the following set equality

(5.3)
$$(\lambda_k = r) = (Z_\ell(D_k) - Z_\ell(D_{k-1}) = 0,$$

$$Z_r(D_k) - Z_r(D_{k-1}) = 1, \quad r \neq \ell = 1,\ldots,m), \quad r = 1, \ldots, m.$$

The processes $Z_1(t), \ldots, Z_m(t)$, describe the progress of m independent simple epidemics among susceptibles in closed populations of infinite sizes. Assume we observe the progress of these m independent simple epidemics. Then $Z(t)$ represents the total number of infectives at time t, $W_k(t)$ models the time that elapses between the $Z(0) + k - 1$ and the $Z(0) + k$ infection, D_k describes the time until the $Z(0) + k$ infection, and λ_k indicated the simple epidemic responsible for the $Z(0) + k$ infection.

In this section we assume that Condition (1.12) holds, and prove that the processes $\underline{X}_\infty(t)$ and $\underline{Z}(t) = [Z_1(t), \ldots, Z_m(t)]$, $t \in [0, \infty)$, are stochastically equal. In particular we obtain Statements (1.13), (1.14).

Let $t \in (0, \infty)$, $k = 1, 2, \ldots$, and $r = 1, \ldots, m$. Then the following set equality holds.

(5.4)
$$(Z_r(t) - Z_r(0) \geq k) =$$

$$= \bigcup_{q=k}^{\infty} (D_q \leq t < D_{q+1}, \sum_{j=1}^{q} I(\lambda_j = r) \geq k).$$

By the set Equalities (2.1), (5.4), the processes $\underline{X}_\infty(t)$, $\underline{Z}(t)$ are determined by $[T_{\infty,q}, \xi_{\infty,q}, q=1,2,\ldots]$, and $[W_q, \lambda_q, q=1,2,\ldots]$, respectively. Thus, to prove that $\underline{X}_\infty(t)$, $\underline{Z}(t)$ are stochastically equal it suffices to show that $[T_{\infty,q}, \xi_{\infty,q}, q=1,2,\ldots]$, $[W_q, \lambda_q, q=1,2,\ldots]$, are stochastically equal.

Next, we prove the preceeding statement. We need the following.

Lemma 5.1. Let us assume that Condition (1.12) holds. Then $[T_{\infty,1}, \xi_{\infty,1}]$ and $[W_1, \lambda_1]$ are equal in distribution.

Proof. Let $t \in (0, \infty)$, and $r = 1, \ldots, m$. Then $P\{W_1 > t, \lambda_1 = r\} =$

$= P\{t < [\Delta(X_{\infty,r}(0), r)]^{-1} U_1 < [\Delta(X_{\infty,\ell}(0), \ell)]^{-1} U_\ell, r \neq \ell = 1, \ldots, m\} = P\{T_{\infty,1} > t, \xi_{\infty,1} = r\}.$ ||

We are ready now to prove the main result of this section.

Theorem 5.2. Let us assume that Condition (1.12) holds. Then the two sequences $[T_{\infty,q},\xi_{\infty,q},q=1,2,\ldots]$ and $[W_q,\lambda_q,q=1,2,\ldots]$, are stochastically equal.

Proof. We have to show that for all $k = 1, 2, \ldots$, the rve's $[T_{\infty,q},\xi_{\infty,q},q=1,\ldots,k]$, $[W_q,\lambda_q,q=1\ldots,k]$ are equal in distribution. We prove this statement by an induction arguement on k.

First, we present some notation. Let $\ell = 1, \ldots, m$, $\underline{Z}(t,\ell) = \underline{Z}(t+W_1)$, $t\epsilon(0,\infty)$, $Z_r(0,\ell) = Z_r(0)$, $\ell\neq r = 1, \ldots, m$, and $Z_\ell(0,\ell) = Z_\ell(0) + 1$. Further, let $W_k(\ell)$, $\lambda_k(\ell)$, be defined as W_k, λ_k, respectively, $k = 1, 2, \ldots$, where $\underline{Z}(t,\ell)$ replaces $\underline{Z}(t)$. Finally, let $\underline{X}_\infty(t,\ell)$, $t\epsilon(0,\infty)$, be a m-dimensional stochastic process as defined in Section 2 such that $X_{\infty,r}(0,\ell) = X_{\infty,r}(0)$, $\ell\neq r = 1, \ldots, m$, $X_{\infty,\ell}(0,\ell) = X_{\infty,\ell}(0) + 1$, and let $T_{\infty,k}(\ell)$, $\xi_{\infty,k}(\ell)$, $k = 1, 2, \ldots$, be the interinfection times and infection causes associated with $X_\infty(t,\ell)$.

We now return to the proof of the theorem. Let $k = 2, 3, \ldots$, and $t\epsilon(0,\infty)$. By the memoryless property of exponential rva's the two conditional rve's $\{[T_{\infty,q},\xi_{\infty,q},q=2,\ldots,k]|T_{\infty,1}=t,\xi_{\infty,1}=\ell\}$ and $\{[W_q,\lambda_q,q=2,\ldots,k]|W_1=t,\lambda_1=\ell\}$, are equal in distribution to the two rve's $[T_{\infty,q}(\ell),\xi_{\infty,q}(\ell),q=1,\ldots,k-1]$ and $[W_q(\ell),\lambda_q(\ell),q=1,\ldots,k-1]$ respectively. Consequently the result of the theorem follows by Lemma 5.1, Bayes formula, and an induction arguement on k. ‖

For reference purposes we summarize the main result of this section in the following theorem.

Theorem 5.3. Let us assume that the transition rates of the m-dimensional process $\underline{X}_\infty(t)$, $t\epsilon[0,\infty)$, are given by Equation (1.12). Then Statements (1.13), (1.14) hold.

6. Examples.

Now we apply the theory developed in the previous sections to some specific m-dimensional simple epidemic classes. Throughout we assume that Condition (1.7) holds.

Following the classical approach to the univariate simple epidemic [Bailey (1975)] one can assume that the transition rates of $\underline{X}_n(t)$ are proportional to the number of infectives with the respective diseases and the total number of susceptibles. Let $\alpha_1, \ldots, \alpha_m \in (0, \infty)$. Then the transition rates of the classical m-dimensional simple epidemic are given by

$$(6.1) \qquad R(\underline{X}_n(t), r) = \alpha_r X_{n,r}(t)(n - X_n(t) + X_n(0)), \quad r = 1, \ldots, m, \quad t \in [0, \infty).$$

From Lemmas 2.2, 2.3 the duration time of a classical m-dimensional simple epidemic is less than or equal to $[\min\{\alpha_r : r = 1, \ldots, m\}]^{-1} \sum_{q=1}^{n} [(X_n(0) + q - 1)(n - q + 1)]^{-1}$. Thus, the duration time tends to zero as $n \to \infty$. Consequently the classical models are not describing properly the progress of the epidemic when n is large. BLL(1979) adjust the classical transition rates and use the symmetric m-dimensional simple epidemic to discribe the progress of the epidemic when n is large. There is no apparent reason for the various transition rates to have the same proportionally coefficient, as is assumed by BLL(1979). We modify those transition rates, given by Equation (1.11), and assume that for $t \in [0, \infty)$, and $r = 1, \ldots, m$

$$(6.2) \qquad R(\underline{X}_n(t), r) = \begin{cases} \alpha_r n^{-1} X_{n,r}(t)(n - X_n(t) + X_n(0)), & n = 1, 2, \ldots \\ \alpha_r X_{\infty,r}(t) & , \quad n = \infty. \end{cases}$$

Next, we derive for the models determined by Equation (6.2) the desired approximations. Let $\underline{V(b, \gamma, t)} = \max\{k : k = 1, 2, \ldots, \sum_{q=1}^{k} (b + q - 1)^{-1} U_q \le \gamma t\}$,

$b = 1, 2, \ldots, t, \gamma\epsilon(0,\infty)$. For the sake of completeness we present the following definition.

Definition 6.1. We say that a rva W is <u>negative binomial</u> with parameters $a = 1, 2, \ldots$, and $p\epsilon(0,1)$, and write $W{\sim}NB(a,p)$ if $P\{W=k\}= \begin{pmatrix} a+k-1 \\ a-1 \end{pmatrix}p^a(1-p)^k$, $k = 1, 2, \ldots$.

Note that by BLL(1979) p. 198 we obtain that

(6.3) $V(b,\gamma,t){\sim}NB(b,e^{-\gamma t})$, $b = 1, 2, \ldots, t, \gamma\epsilon(0,\infty)$.

We show next that for all $t\epsilon(0,\infty)$, and $r = 1, \ldots, m$,

(6.4) $\lim_{n\to\infty} P_{n,\underline{k}}(t) = \prod_{r=1}^{m} P\{NB(X_{\infty,r}(0),e^{-\alpha_r t})=k_r\}$, and

(6.5) $\lim_{n\to\infty} P_{n,k,r}(t) = P\{NB(X_{\infty,r}(0),e^{-\alpha_r t})=k\}$.

Note that Conditions (1.8) and (1.12), with $\Delta(k,r) = \alpha_r k$, hold. Consequently Statements (6.4), (6.5) follow by Theorems 4.3, 5.3, and Statement (6.3). Finally, we show that for $t\epsilon(0,\infty)$, $r = 1, \ldots, m$

(6.6) $\lim_{n\to\infty} EX_{n,r} = X_{\infty,r}(0)(e^{\alpha_r t}-1)$, and that

(6.7) $\lim_{n\to\infty} Var\{X_{n,r}(t)\} = X_{\infty,r}(0)(e^{2\alpha_r t}-e^{\alpha_r t})$.

In particular we obtain from Statements (6.6), (6.7), that for all $t\epsilon(0,\infty)$

(6.8) $\lim_{n\to\infty} EX_n(t) = \sum_{r=1}^{m}X_{\infty,r}(0)(e^{\alpha_r t}-1)$, and

(6.9) $\lim_{n\to\infty} Var\{X_n(t)\} = \sum_{r=1}^{m}X_{\infty,r}(0)(e^{2\alpha_r t}-e^{\alpha_r t})$.

To prove Statements (6.6), (6.7), let $\bar{\alpha} = \sum_{r=1}^{m}\alpha_r$. Then for all $t\epsilon(0,\infty)$ and almost all $n\epsilon\{1,2,\ldots\}P\{X_n(t)\geq k+X_n(0)\} \leq P\{\sum_{q=1}^{k}[n^{-1}(n-q+1)(X_n(0)+q-1)]^{-1}U_q\leq\bar{\alpha}t\} \leq$

$\leq P\{V(X_\infty(0),\bar{\alpha},t)\geq k\}$, $k = 0, 1, \ldots$. Thus, Conditions (1.9), (1.10), hold for

all t, $\beta \in (0,\infty)$, with $Y(t) = V(X_\infty(0),\bar{\alpha},t)$. Consequently Statements (6.6),

(6.7), follow from Theorems (4.6), (5.3), Statement (6.3), and some simple

moments evaluations of negative binomial rv 's.

Frequently infectives are denied the freedom of movement, as is the case

when they are hospitalized immediately after they contract a disease. We

describe the progress of this epidemic by an m-dimensional stochastic process

determined by the following transition rates:

$$(6.10) \quad R(\underline{X}_n(t),r) = \begin{cases} \alpha_r n^{-1} X_{n,r}(t)(n-X_n(t)+X_n(0))[\sum_{\ell=1}^m X_{n,\ell}(t)]^{-1}, & n=1,2,\ldots, \\ \alpha_r X_{\infty,r}(t)[\sum_{\ell=1}^m X_{\infty,r}(t)]^{-1} & , n=\infty, r=1,\ldots,m. \end{cases}$$

Now, we derive for the models determined by Equation (6.10) the desired approx-

imations. Let $\underline{B(a_1,a_2)} = \int_0^1 z^{a_1-1}(1-z)^{a_2-1} dz$, and $\underline{(k_1,\ldots,k_m)} = k![\prod_{r=1}^m k_r]^{-1}$,

$k_1, \ldots, k_m \in \{0,1,\ldots\}$, $k = \sum_{r=1}^m k_r$. For the sake of simplicity we obtain these

approximations only for the symmetric case: when $\underline{\alpha_1 = \alpha_2 \ldots = \alpha_m = \alpha}$. By Lemma

(2.1) it follows that for $k = 1, 2, \ldots$

$$(6.11) \quad P\{\xi_{\infty,q}=\ell_q, q=1,\ldots,k\} = \left[\prod_{r=1}^m \left(\frac{X_{\infty,r}(0)+d_r(k)}{d_r(k)}\right)\right] \cdot$$

$$\cdot \left[\left(d_1(k),\ldots,d_m(k)\right)\left(\frac{X_\infty(0)+k-1}{k}\right)\right]^{-1}, \text{ where } \ell_1, \ldots, \ell_k \in \{1,\ldots,m\}, \text{ and}$$

$$d_r(k) = \sum_{q=1}^k I(\ell_q=r), \quad r = 1, \ldots, m.$$

By Lemma 2.2 we conclude that the interinfection times: $T_{\infty,1}$, $T_{\infty,2}$, ..., are

iid exponential rva's with a mean equal to α^{-1} independent of the infection

causes: $\xi_{\infty,1}$, $\xi_{\infty,2}$, ... In particular note that $X_\infty(t) - X_\infty(0)$ is a Poisson

rva with parameter αt, $t \in (0,\infty)$. Thus, for $t \in (0,\infty)$, and $r = 1, \ldots, m$

$$(6.12) \qquad P_{\infty,\underline{k}}(t) = \left[e^{-\alpha t}(\alpha t)^{s(\underline{k})} \prod_{r=1}^{m} \begin{pmatrix} X_{\infty,r}(0)+k_r-1 \\ k_r \end{pmatrix} \right] \cdot$$

$$\cdot \left[(s(\underline{k}))! \begin{pmatrix} X_{\infty}(0)+s(\underline{k})-1 \\ s(\underline{k}) \end{pmatrix} \right]^{-1} , \text{ and}$$

$$(6.13) \qquad P_{\infty,k,r}(t) = e^{-\alpha t}(\alpha t)^{k} \mathcal{B}(X_{\infty,r}(0),X_{\infty}(0)-X_{\infty,r}(0)) \cdot$$

$$\cdot \int_{q=0}^{\infty} (\alpha t)^{q} \mathcal{B}(X_{\infty,r}(0)+k,X_{\infty}(0)-X_{\infty,r}(0)+q)(q!)^{-1} =$$

$$= \mathcal{B}^{-1}(X_{\infty,r}(0),X_{\infty}(0)-X_{\infty,r}(0)) \int_{0}^{1} \left[z^{X_{\infty,r}(0)-1}(1-z)^{X_{\infty}(0)-X_{\infty,r}(0)-1} \cdot \right.$$

$$\left. \cdot (\alpha t z)^{k} e^{-\alpha t z}(k!)^{-1} \right] dz.$$

Since, Condition (1.8) holds we obtain by Theorem 4.3 and Corollary 4.4 that $P_{n,\underline{k}}(t)$, $P_{n,k,r}(t)$ can be approximate respectively by $P_{\infty,\underline{k}}(t)$, $P_{\infty,k,r}(t)$, given by Equations (6.12), (6.13).

Next, note that for $t \in (0,\infty)$ and all $n \in \{1,2,...\}P\{X_n(t) \geq k + X_n(0)\} \leq P\{\sum_{q=1}^{k} T_{\infty,q} \leq \alpha t\}$, $k = 0, 1, ...$. Thus, Conditions (1.9), (1.10) hold for all t, $t \in (0,\infty)$, with $Y(t)$ being a Poisson rva with mean αt, $t \in (0,\infty)$. Consequently by Theorem 4.6 and Statement (6.12) we obtain for $t \in (0,\infty)$ and $r = 1, ..., m$, that

$$(6.14) \qquad \lim_{n \to \infty} EX_{n,r}(t) = \alpha t X_{\infty,r}(0) X_{\infty}^{-1}(0), \text{ and that}$$

$$(6.15) \qquad \lim_{n \to \infty} Var\{X_{n,r}(t)\} = \alpha t X_{\infty,r}(0) X_{\infty}^{-1}(0) +$$

$$+ \alpha^2 t^2 X_{\infty,r}(0)(X_{\infty}(0)-X_{\infty,r}(0)) X_{\infty}^{-1}(0).$$

Further, by Theorem 4.5 we obtain that for $t \in (0,\infty)$

$$(6.16) \qquad \lim_{n \to \infty} EX_n(t) = \alpha t \qquad \text{and } \lim_{n \to \infty} Var\{X_n(t)\} = \alpha t.$$

Next, we consider m-dimensional simple epidemic models given by the
following transition rates

$$
(6.17) \qquad R(\underline{X}_n(t), r) = \begin{cases} \alpha_r n^{-\beta} X_{n,r}^{\delta}(t)(n - X_n(t) + X_n(0))^{\beta}, & n = 1, 2, \ldots, \\ \alpha_r X_{\infty,r}^{\delta}(t) & , \quad n = \infty, \end{cases}
$$

where $\beta, \delta \in (0, \infty)$, $r = 1, \ldots, m$.

The m-dimensional simple epidemic models determined by Equation (6.17)
generalize univariate simple epidemic models used and motivated by Severo
(1969). Note that Conditions (1.8) and (1.12), with $\Delta(k,r) = \alpha_r k$, hold.
Consequently by Theorems 4.3, 5.3, we obtain for $t \in (0, \infty)$, $r = 1, \ldots, m$, that

$$
(6.18) \qquad \lim_{n \to \infty} P_{n,k,r}(t) = P\{\sum_{q=1}^{k} (X_{\infty,r}(0) + q - 1)^{-\delta} U_q \leq \alpha_r t\},
$$

and that

$$
(6.19) \qquad \lim_{n \to \infty} P_{n,\underline{k}}(t) = \prod_{r=1}^{m} \lim_{n \to \infty} P_{n,k_r,r}(t).
$$

Let $\underline{W(t)} = \max\{k : \sum_{q=1}^{k} (X_{\infty}(0) + q - 1)^{-\delta} U_q \leq \bar{\alpha} t\}$. Then for all $t \in (0, \infty)$ and almost
all $n \in \{1, 2, \ldots\}$ $P\{X_n(t) \geq k + X_n(0)\} \leq P\{W(t) \geq k\}$, $k = 0, 1, \ldots$. Thus, Conditions
(1.9), (1.10), hold for all t, $\beta \in (0, \infty)$, with $Y(t) = W(t)$. Consequently the
moments approximations hold for the models determined by Equation (6.17).

Finally following Gart (1968), (1972), consider a population of sus-
ceptibles exposed to a disease and partitioned to m subpopulations. Gart
assumes that the susceptibility level of an individual varies according to
his membership in the various subpopulations. With small adjustments the
process $\underline{X}_n(t)$ can be used to describe the progress of this univariate simple
epidemic. In this case the components of $\underline{X}_n(t)$ represent the number of in-
fectives in the respective subpopulations at time t measured from T_0. Further,
all the results obtained in this paper carry through for these univariate models.

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20. ABSTRACT

A population of susceptible individuals exposed to m contagious diseases is considered. The progress of this epidemic among the individuals is modeled by an m-dimensional stochastic process. The components of this process represent the number of infective individuals with the respective diseases at time t.

A class of m-dimensional stochastic processes is constructed. These processes describe the progress of the epidemic models considered in the sequel. Exact and approximate formulas for the joint and marginal state probabilities of these models are obtained. It is shown that the approximate formulas are relatively simple functions of time while, the derivations of the exact formulas involve tedious computations. The results obtained in the paper are applied to a sample of examples.

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