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Monterey, California

THESIS

SEARCH PRIORITIES FOR A
TARGET PROBABILITY AREA

by

Patricia Ann Tracey

March 1980

Thesis Advisor: W.P. Hughes

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Search Priorities for a
Target Probability Area

by

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Submitted in partial fulfillment of the
requirements for the degree of

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ABSTRACT

The problem of determining whether a line of bearing measured by a local surface-based sensor coincides with a threat whose position has been previously estimated by an ocean surveillance sensor is addressed. Uncertainties in the position estimate of the threat, in the bearing error and in the position estimate of the sensor are considered in measuring the probability that the threat lies on a given bearing from the sensor. A TI-59 calculator program is developed which calculates this likelihood when the threat location density can be assumed to be bivariate normal. Computations required when significant time has elapsed since the original estimate of threat location when the density can no longer be considered bivariate normal are discussed.

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I. INTRODUCTION

As long range surface launched weapons systems continue to be introduced into the fleet, the operational commander is increasingly faced with the problem of being able to launch weapons at targets located beyond the horizon. Successful employment of such weapons is dependent not only on the ability to detect, classify and localize targets at considerable distances, but also on the ability to distinguish the true target from a potentially larger field of false targets. While long range ocean surveillance sensors may be of assistance in the identification and localization of targets, the information provided may not be refined sufficiently to permit effective targeting of long range weapons on that basis alone. The on-scene commander must in general rely on additional data on target location gathered locally and close to the time of weapons launch for accurate targeting. Thus, he must still be able to detect and track the desired target and be able to distinguish it from other targets within range of his sensors.

The procedures developed in this paper are designed to be of assistance in addressing the last of these problems. They are applicable when the information available is an error ellipse around a threat location estimated by an ocean surveillance sensor and bearings only data generated by a local surface-based sensor. The question of whether a target detected by the local sensor is the same as that whose estimated position was provided by an external sensor can only be addressed if information is available on the locations and tracks of all possible targets within range of the local sensor. Since such data is generally not available, this paper does not attempt to answer that question, but rather develops a method by which bearing information from different sensors can be compared as to the likelihood of each bearing being associated with the threat identified previously. It is envisioned that these likelihoods can be then used to induce an ordering among bearing data gathered by different sensors or, conceivably, conflicting data gathered by one sensor. The ordering would be based on the likelihood that each bearing will contribute to refining the original estimate of the location of the target of interest. This information could be applied in a number of ways: as a guide to allocation of more capable sensor resources for purposes of obtaining targeting information; as a guide for allocation of weapons against more than one threat; as a means of pre-processing data before entering it into a target motion model, thereby reducing the chance of introducing unrelated data.

To determine the likelihood that a given line of bearing and the threat coincide, consideration was given to the uncertainties inherent in estimation of target position, in the measurement of bearings by a particular sensor and in estimation of sensor location. It is assumed that at some time t_o, an ocean surveillance sensor detects a threat whose position is estimated to be within an elliptical region with $p_1 \times 100\%$ certainty. The estimated position data are received and converted by the on-scene commander into a probability distribution described by a truncated bivariate normal density function.

It is further assumed that the standard error σ_β characteristic of the local sensor is known. The sensor bearing β with bearing error σ_β is then projected from the sensor position through the threat density function.

Since sensor position relative to the target may itself be subject to navigation error, the uncertainty is introduced as a truncated circular bivariate normal distribution centered at location (u_o, v_o) with standard deviation σ.

A TI-59 calculator program is developed which estimates the likelihood that the threat identified by an external sensor lies along bearing β, given the threat distribution, the bearing error, and sensor position distribution relative to the threat.

The theoretical basis for this calculation is presented in Chapter II. The algorithms used in designing the calculator program are described in Chapter III. A program listing and verbal flow are provided in Appendix A along with instructions for the user. Appendix B contains a verbal flow of a program designed for use when considerable time has elapsed since the initial estimate of the location of a moving target.

II. THEORETICAL BASES

The general approach to determining the likelihood that a measured line of bearing β is the true bearing from the sensor to the threat identified and localized by an external sensor is discussed in this chapter. Calculations required when using threat position information both as initially generated by the external sensor at time zero, t_o, and as distorted to account for an intervening time late, t_L, are discussed. Uncertainties in bearing measurement and sensor position are included.

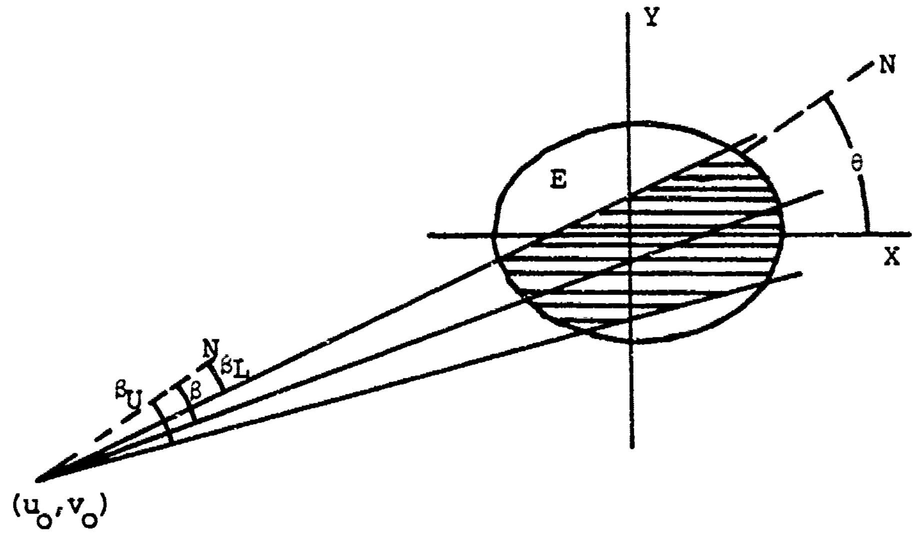
Initially, assume that sensor position is known with certainty. Let β_T be the true bearing of the threat from the sensor. Since threat location is uncertain, β_T is a random quantity with probability density function $f_{\beta_{TRUE}}(\beta_T)$. Let $f_\beta(\beta;\beta_T)d\beta$ be the probability that the errors in bearing measurement are such as to give rise to a bearing on the threat in the interval $d\beta$ about the observed value β when the true bearing is β_T. Then the likelihood of observing a bearing β is:

$$f'(\beta) = \int_{\text{all } \beta_T} f_\beta(\beta;\beta_T)\, f_{\beta_{TRUE}}(\beta_T)\, d\beta_T \, . \qquad (1)$$

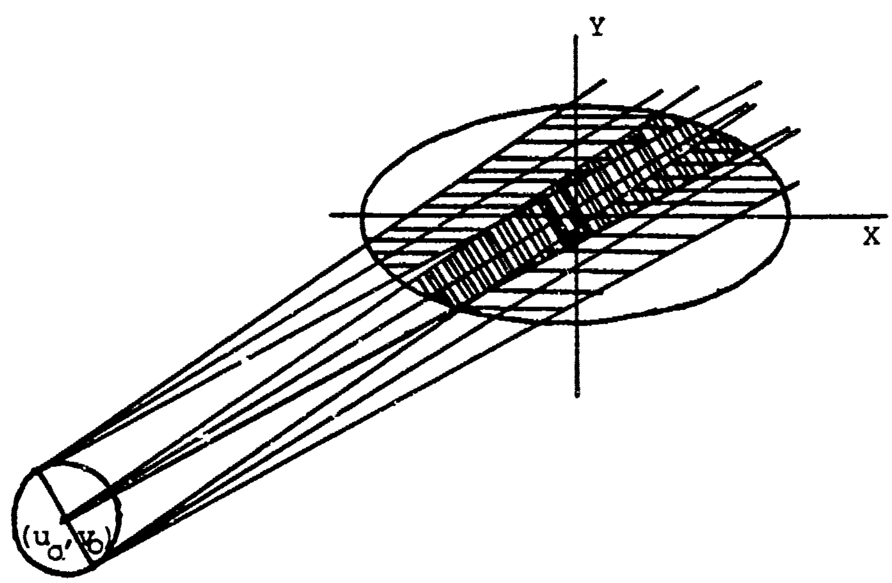
The density $f_{\beta_{TRUE}}(\beta_T)$ is determined by the probability density function of threat location at time t, $f_{X,Y}(x,y;t)$.

Given the probability density function of threat location at time t and assuming that bearing errors are normally distributed with mean zero, consideration is limited to the probability that the observed bearing is the true bearing of the threat, given the threat is contained in a planar region E and the true bearing β_T is in an interval about β with upper and lower bounds β_U and β_L. The region E is selected to be the minimum area planar region which contains the threat with a specified high probability p_1. The interval (β_L, β_U) is selected so that, for all the possible values of β_T contained in the interval, β is within an interval of specified high probability p_2 around β_T. The fan (β_L, β_U) is symmetric about β so that it represents the minimum area region which meets the above criterion. The likelihood of observing β when the threat is in E and β_T is in (β_L, β_U) is determined by integrating expression (1) above, over a region defined in the manner of the shaded area of Figure 1(a).

If sensor position is not certain, assume that it is distributed in accordance with a circular bivariate normal distribution $f_{U,V}(u,v)$. Again, consideration is limited to determining the likelihood of observing a bearing β given that the threat is contained in a planar region, E, β_T is in (β_L, β_U), and the sensor is contained in a planar region C. The region C is selected as the minimum area planar region which contains the sensor with a high probability p_3. The region of E over which $f'(\beta)$ is evaluated expands as the shaded region of Figure 1(b).



(a) Sensor Position Certain



(b) Sensor Position Uncertain

FIGURE 1. REGION OF INTEGRATION

Since the measurement errors involved in estimating the threat position, the sensor position and the bearing angle arise from different measurement procedures, it is reasonable to assume that errors are independent. The probability densities required to perform the above calculations can be estimated as follows.

At time t_o an ocean surveillance sensor estimates the position of the threat to be located within an elliptical area characterized by the parameter set $E = \{X,Y,\theta,A,B\}$ with confidence $p_1 \times 100\%$. The elements of the parameter set E are respectively: X the latitude of the estimated threat position, Y the longitude of estimated threat position, θ the orientation of the major axis from true North, A the length of the semi-major axis, and B the length of the semi-minor axis. Since the measurement errors in determining threat position are generally assumed to be normally distributed, the ellipse characterized by E represents the minimal area $p_1 \times 100\%$ confidence region about the mean (X,Y). Treating this ellipse as the p_1 probability region of a bivariate normal distribution, a density function for the threat position at time t_o can be estimated. For convenience, locate the origin of a rectangular coordinate system at the center of the ellipse, (X,Y), with positive x-axis located along the major-axis of the ellipse at a bearing θ from true North. Assume a flat earth in the region of interest. Let $t_o = 0$. The mean of the threat position density $f_{X,Y}(x,y;0)$ is then the point (0,0). The variances in the X and Y directions can be derived from the fact that the region with minimal area which contains the threat with probability p_1 is a k-sigma ellipse where k is determined from the relationship:

$$P[\text{threat located in k-sigma ellipse}] = 1 - e^{-k^2/2}$$

Thus,

$$\sigma_X^2 = (A/k)^2$$

and

$$\sigma_Y^2 = (B/k)^2$$

where

$$k = \sqrt{-2 \ln(1 - p_1)} \ .$$

X and Y are assumed to be independent.

If the course and speed of the threat are known with certainty to be ψ and s respectively, the probability density of the threat position at time late t_L can be shown to be again a bivariate normal with mean $(st_L\cos(\theta-\psi), st_L\sin(\theta-\psi))$ and variances σ_X^2, σ_Y^2. The p_1 probability region of the density at time t_L would then be an ellipse congruent to that characterized by the set E above but centered at the point $(st_L\cos(\theta-\psi), st_L\sin(\theta-\psi))$ (Figure 2) [Ref. 1].

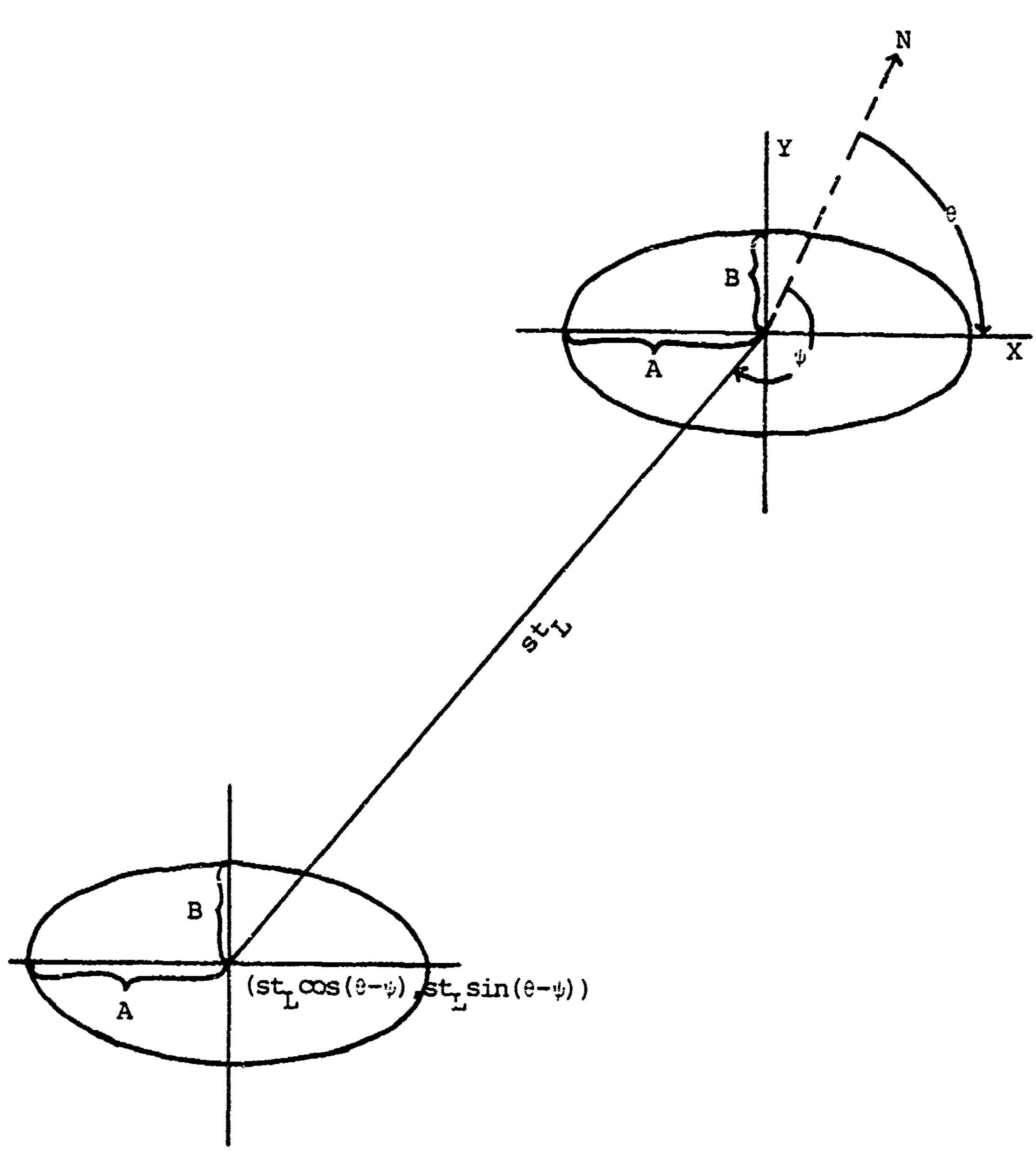


FIGURE 2. TIME LATE ELLIPSE, KNOWN COURSE AND SPEED

In some cases, the motion of a submarine on patrol in a large area can be characterized, when t_L is large, by an expansion of the probability area with time at a rate D. The result in such a case is that $f_{X,Y}(x,y;t_L)$ is still bivariate normal with mean (0,0) but with variances $\sigma_X^2 + Dt_L$ and $\sigma_Y^2 + Dt_L$.

When the motion of the threat cannot be described in the above manner, and the course and speed are not known, but assumed to be distributed according to the densities $f_\psi(\psi)$ and $f_S(s)$, determining the probability density $f_{X,Y}(x,y;t_L)$ is a considerably more complex problem. Let $(x_o(s,\psi),y_o(s,\psi))$ be the coordinates of the point at which the threat would have to be located at time zero in order to reach the point (x_L,y_L) at time t_L if the threat speed were s and course ψ. Thus, $x_o(s,\psi) = x_L - st_L\cos(\theta-\psi)$ and $y_o(s,\psi) = y_L - st_L\sin(\theta-\psi)$ (Figure 3). Then,

$$f_{X,Y}(x,y;t_L) = \int_0^\infty \int_0^{360} f_{X,Y}[x_o(s,\psi),y_o(s,\psi);t_o] \cdot f_\psi(\psi) f_S(s)\, d\psi ds \quad \text{[Ref. 1]} .$$

This density is no longer normal. In the special case when A = B, i.e., $\sigma_X^2 = \sigma_Y^2$, ψ has a uniform distribution over the interval (0°,360°) and S is known with certainty, as shown in Reference 2, the density changes with time as in Figure 4.

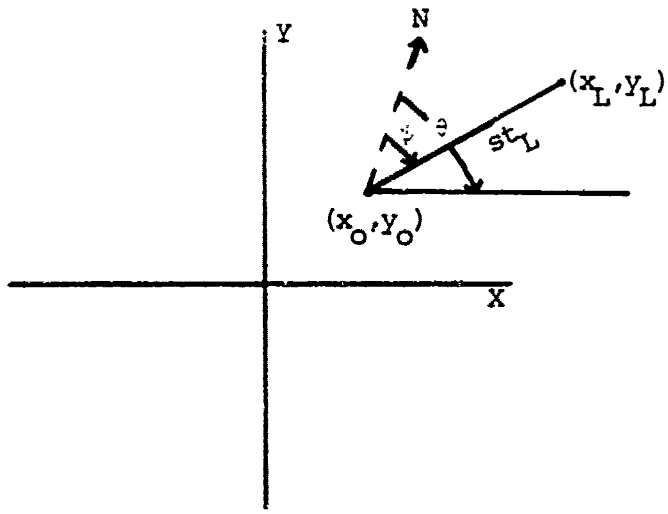


FIGURE 3. TIME LATE POSITION, COURSE AND SPEED UNCERTAIN

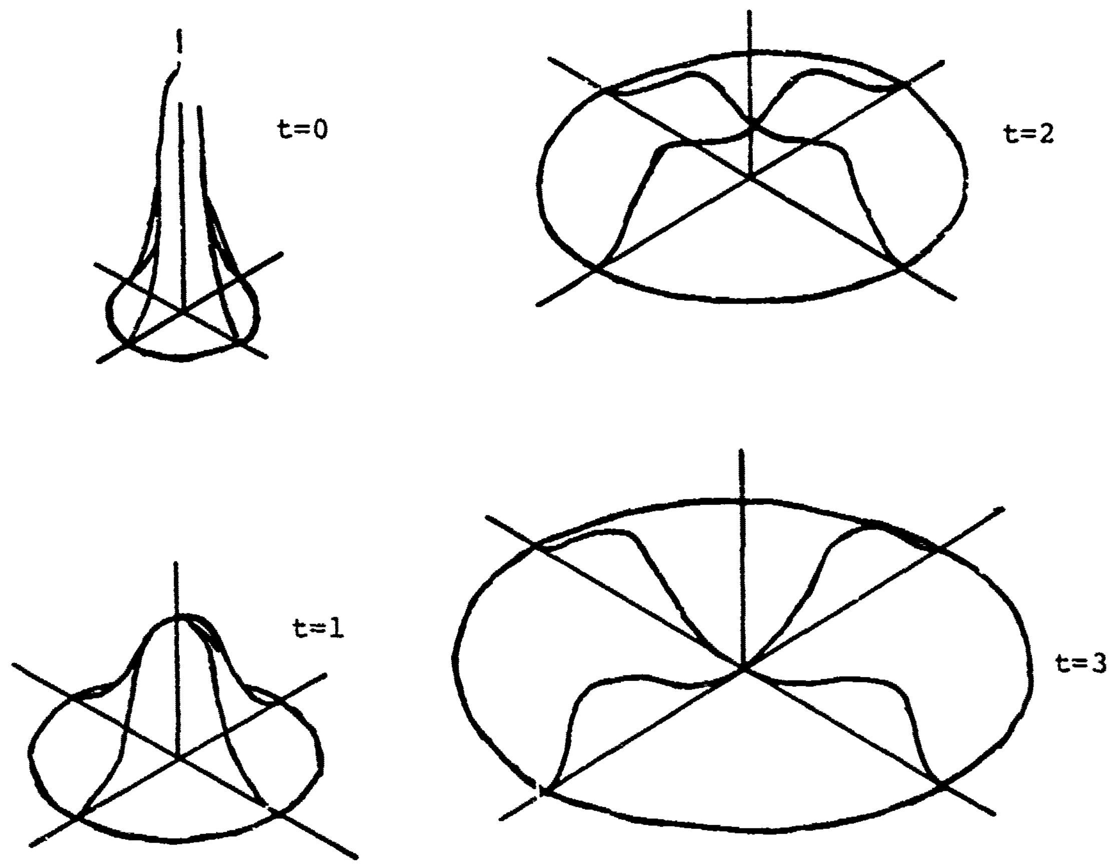


FIGURE 4. TIME LATE THREAT DENSITY, SPEED KNOWN COURSE UNIFORM (0°,360°)

The distribution of the bearing error measured by the local sensor can be estimated if the standard error of the sensor, σ_β, is known. The bearing error is then assumed to be normally distributed with mean zero and variance σ_β^2.

Assume that the estimated sensor location (u_o, v_o) is accurate to within R nautical miles with $p_3 \times 100\%$ confidence. The density $f_{U,V}(u,v)$ of sensor location can be assumed to be a circular bivariate normal with mean (u_o, v_o) and variance $\sigma^2 = R^2/k^2$. The value of k is determined from the relationship:

$$P[(u,v) \text{ contained in k-sigma region}] = 1 - e^{-k^2/2},$$

where the probability on the left is p_3 in this cas .

Since the evaluation of $f'(\beta)$ considering sensor position density and bearing error density does not generalize to a closed form, algorithms are developed in the remainder of this paper for estimation of the probability (likelihood) that the threat identified by an external sensor lies on a line of bearing measured by a local sensor given that the threat is in region E, the sensor is in region C and the true line of bearing lies within (β_L, β_U).

III. ALGORITHMS

As indicated in the previous chapter, the variety of geometrical situations which can arise depending on the location of the sensor relative to the estimated threat location precluded development of a generalized analytical procedure. Rather, algorithms are developed in this paper for numerically evaluating $f'(\beta)$. The procedures applicable to a bivariate normal threat location density have been implemented on a TI-59 calculator. Appendix A contains a listing of that program. The calculations required when the time late threat location density is no longer normal exceeded the available program capacity of the TI-59 and therefore have not been implemented. A detailed verbal flow is provided at Appendix B for future implementation on a larger machine. The algorithms used in both situations are described in this chapter.

A. BIVARIATE NORMAL THREAT LOCATION DENSITY

This case includes situations (1) where the time elapsed since generation of the initial error ellipse by the ocean surveillance sensor is negligible, (2) where the motion of the threat can be assumed to be random in the manner described above, and (3) where course and speed of the threat are assumed to be known with certainty. With appropriate modifications to the input data, all three of these situations can be addressed using the program contained in Appendix A.

In situation (1) the data entered are the parameters of the ellipse as generated by the ocean surveillance sensor. In (2), the location and orientation from North of the ellipse is the same as originally generated, but the size of the ellipse expands at some constant rate of area per unit time which must be estimated by the user. This rate D, times the elapsed time, t_L, yields the factor which must be added to the semi-major and semi-minor axes of the original ellipse. That is, if the original error ellipse is a $p_1 \times 100\%$ confidence ellipse, the semi-major and semi-minor axes of the diffused ellipse will be input as A' and B' respectively:

$$A' = \sqrt{A^2 + (-2 \ln(1-p_1))Dt_L}$$

$$B' = \sqrt{B^2 + (-2 \ln(1-p_1))Dt_L}$$

In situation (3) the dimensions and orientation of the time late ellipse are the same as those of the original error ellipse, but the center of the ellipse is displaced from its original position by the known velocity times elapsed time. The updated position of the error ellipse is treated as the origin of the rectangular coordinate system for this situation and all linear measurements are made relative to this system. All angular measurements are made from true North.

Estimation of the likelihood that the threat lies on a bearing from the local sensor given that the threat is located in a $p_1 \times 100\%$ confidence ellipse and the true bearing lies within the bounds (β_L, β_U) with confidence $p_2 \times 100\%$ proceeds as follows:

1. Estimate parameters of the bivariate normal density $f_{X,Y}(x,y)$ of threat location: $\mu_X = \mu_Y = 0$, σ_X^2, σ_Y^2.

2. Determine β_L and β_U such that an interval of length $2k_\beta \sigma_\beta$ centered on either β_L or β_U would contain β, the measured bearing, with probability p_2: $\beta_L = \beta - k_\beta \sigma_\beta$, $\beta_U = \beta + k_\beta \sigma_\beta$.

3. Determine sensor location coordinates relative to the origin of the threat ellipse.

4. Subdivide the angular interval (β_L, β_U) into 2n subintervals. Each subinterval k intersects the ellipse in a strip with average width W_k which corresponds to $\Delta\beta_t$.

5. At the midpoint of each subinterval k determine the equation of the line through the sensor position at the true bearing β_k from North.

6. Let the equation of the line of bearing β_k be $X = \frac{Y-c}{m}$. Then the plane perpendicular to the xy-plane which contains this line intersects the bivariate normal threat density in a curve whose equation is

$$g(y) = \frac{1}{2\pi\sigma_X\sigma_X} e^{-\frac{1}{2}\left(\frac{(y-c)^2}{m^2\sigma_X^2} + \frac{y^2}{\sigma_Y^2}\right)} \tag{2}$$

found by making the substitution $X = \frac{Y-c}{m}$ in the density function $f_{X,Y}(x,y)$. It will prove convenient to expand the right side of (2) as follows [Ref. 3]:

$$g(y) = \frac{m}{\sqrt{2\pi}\,\sqrt{\sigma_Y^2 + m^2\sigma_X^2}}\, e^{-\frac{1}{2}\frac{c^2}{\sigma_Y^2 + m^2\sigma_X^2}} \tag{3}$$

$$\cdot\left[\frac{\sqrt{\sigma_Y^2 + m^2\sigma_X^2}}{\sqrt{2\pi}\,\sigma_X\sigma_Y m}\, e^{-\frac{1}{2}\left(\frac{\sigma_Y^2 + m^2\sigma_X^2}{m^2\sigma_X^2\sigma_Y^2}\right)\left(y - \frac{c\sigma_Y^2}{\sigma_Y^2 + m^2\sigma_X^2}\right)^2}\right]$$

For computational purposes, assume that the width W_k of the region of the ellipse cut out by the angular subinterval around β_k will be nearly constant through the ellipse. Let the points of intersection of the line of bearing β_k with the threat ellipse be (X_{k1}, Y_{k1}) and (X_{k2}, Y_{k2}). Then approximate the volume of the normal density over this region by the absolute value of the product of the area under the curve g between Y_{k1} and Y_{k2} and W_k (Figure 5). Observe that the term in brackets in equation (3) above is the density function of a univariate normal random variable with mean

$$\frac{c\sigma_Y^2}{\sigma_Y^2 + m^2\sigma_X^2}$$

and variance

$$\frac{m^2\sigma_X^2\sigma_Y^2}{\sigma_Y^2 + m^2\sigma_X^2}\,.$$

The term preceding the brackets in (3) is the slope of the line of bearing, m, times the density of a univariate normal random variable with mean zero and variance $\sigma_Y^2 + m^2\sigma_X^2$

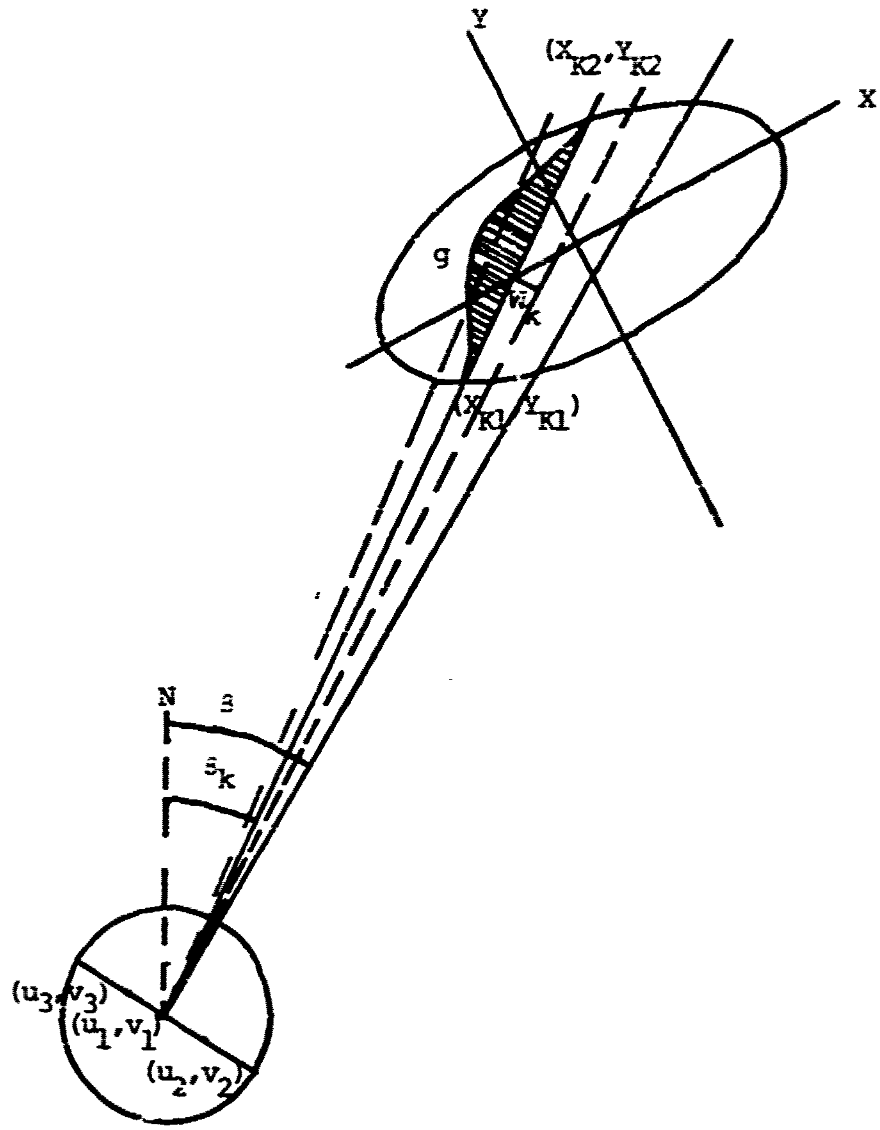


FIGURE 5

evaluated at c. Thus, evaluating the expression

$$\left| W_k \int_{Y_{k1}}^{Y_{k2}} g(y)\, dy \right|$$

is equivalent to the computation

$$\left| W_k m \left(\frac{1}{\sqrt{\sigma_Y^2 + m^2 \sigma_X^2}} \phi \left(\frac{c}{\sqrt{\sigma_Y^2 + m^2 \sigma_X^2}} \right) \right) \left(\Phi(Z_2) - \Phi(Z_1) \right) \right| ,$$

where

$$Z_i = \left(Y_{ki} - \frac{c \sigma_Y^2}{\sigma_Y^2 + m^2 \sigma_X^2} \right) \left(\frac{\sqrt{\sigma_Y^2 + m^2 \sigma_X^2}}{m \sigma_X \sigma_Y} \right)$$

for $i = 1,2$, Φ is the $P[Z \leq z]$ when Z is a standard normal random variable, and ϕ is the density function of a standard normal random variable.

If the substitution $Y = mX + c$ were made in the density $f_{X,Y}(x,y)$, an analogous situation would arise with the limits of the integration being X_{k1} and X_{k2}.

7. If the value of the density under the bivariate normal curve over the region of the ellipse defined by the subinterval $\Delta\beta_k$ is weighted by the probability of observing a bearing error $(\beta_k - \beta)$ the result is the probability of observing the bearing of the threat as β when the true threat location is in the segment of the ellipse defined by $\Delta\beta_k$. This probability is approximated as follows:

$$\left|W_k \int_{Y_{k1}}^{Y_{k2}} f_{X,Y}\left(\frac{y-c}{m}, y\right) dy\right| f_\beta(\beta;\beta_k).$$

Since the bearing error is assumed normally distributed with mean zero and variance σ_β^2, the value of $f_\beta(\beta;\beta_k)$ can be determined by the expression

$$\frac{1}{\sigma_\beta}\,\phi\left(\frac{\beta_k - \beta}{\sigma_\beta}\right),$$

where ϕ is as above the density function of a standard normal random variable.

8.

$$f'(\beta) = \int_{\text{all } \beta_t} f_\beta(\beta;\beta_t)\, f_{\beta_{TRUE}}(\beta_t)\, d\beta_t$$

is approximated by

$$\sum_{k=1}^{2n} \left|W_k\left[\int_{Y_{k1}}^{Y_{k2}} f_{X,Y}\left(\frac{y-c}{m}, y\right) dy\right]\right| f_\beta(\beta;\beta_k)$$

The value of the sum is determined by repeating steps A.6 and A.7 above at the midpoints of each of the 2n sub-intervals defined in step A.4 and summing the results of each of these calculations. Obviously, the finer the sub-division of (β_L, β_U), the more accurate will be the estimate of the likelihood, but also the longer the calculation will take.

B. SENSOR POSITION UNCERTAINTY

The result of the above calculation will be the likelihood $f'(\beta)$ that the threat lies along the bearing measured by the sensor given that the threat is located within the threat ellipse and the true bearing is within the interval (β_L, β_U) and the sensor is at the position used to perform the calculation. We next will introduce additional calculations that are required to account for the fact that the sensor position is not known with certainty.

1. If the position of the sensor is estimated as being within R nautical miles of (u_o, v_o), its assumed coordinates in the xy-system, with $p_3 \times 100\%$ confidence, estimate the parameters of the sensor location density $f_{U,V}(u,v)$ with mean zero and variance σ^2: mean $= (u_o, v_o)$, variance, $\sigma^2 = R^2/(-2\ \ln(1-p_3))$ in threat centered coordinates.

2. The bearing measured by the sensor is β regardless of the sensor location. Assume that the area of intersection of the angular wedge (β_L, β_U) and the threat ellipse does not change significantly as the sensor position is moved along the line of bearing β. Then the effect of the bivariate normal distribution of sensor location can be approximated by considering only the univariate normal density along a line through (u_o, v_o) perpendicular to β. Repeat the calculations in steps A.3 through A.8 above with the sensor located at each of the three points (u_o, v_o), $(u_o+.97\sigma\cos(\theta-\beta-90),$ $v_o+.97\sigma\sin(\theta-\beta-90))$, and $(u_o-.97\sigma\cos(\theta-\beta-90), v_o-97\sigma\sin(\theta-\beta-90))$. If the line through (u_o, v_o) perpendicular to β is subdivided

symmetrically about (u_o, v_o) such that 1/3 of the univariate normal density lies above each subinterval, the three points chosen above represent the "center of gravity" of each third of the density (Figure 6).

3. If R is chosen to include a significant proportion of the sensor density, i.e., on the order of 2σ or greater, the probability of the sensor being located in each of the three regions is approximately 1/3. Thus, if p_3 is on the order of .86, multiply each result in step B.2 by 1/3.

4. Summing the results of steps B.2 and B.3 yields an estimate of the likelihood that the threat lies at bearing β given the threat is in the $p_1 \times 100\%$ confidence ellipse, β_T is in (β_L, β_U) and the sensor is in a $p_3 \times 100\%$ confidence region. That is, the likelihood is estimated by

$$f'(\beta) = \sum_{j=1}^{3} \frac{1}{3} f_j'(\beta) ,$$

where j is the index of sensor position in figures 5 and 6.

Instructions for application of the TI-59 program to calculate the above are included in Appendix A.

C. THREAT DISTRIBUTION NOT BIVARIATE NORMAL AFTER TIME LATE ELAPSED

The basic approach to evaluating $f'(\beta)$ when the time late distribution of the threat is no longer bivariate normal is the same as that just discussed. The principal difference

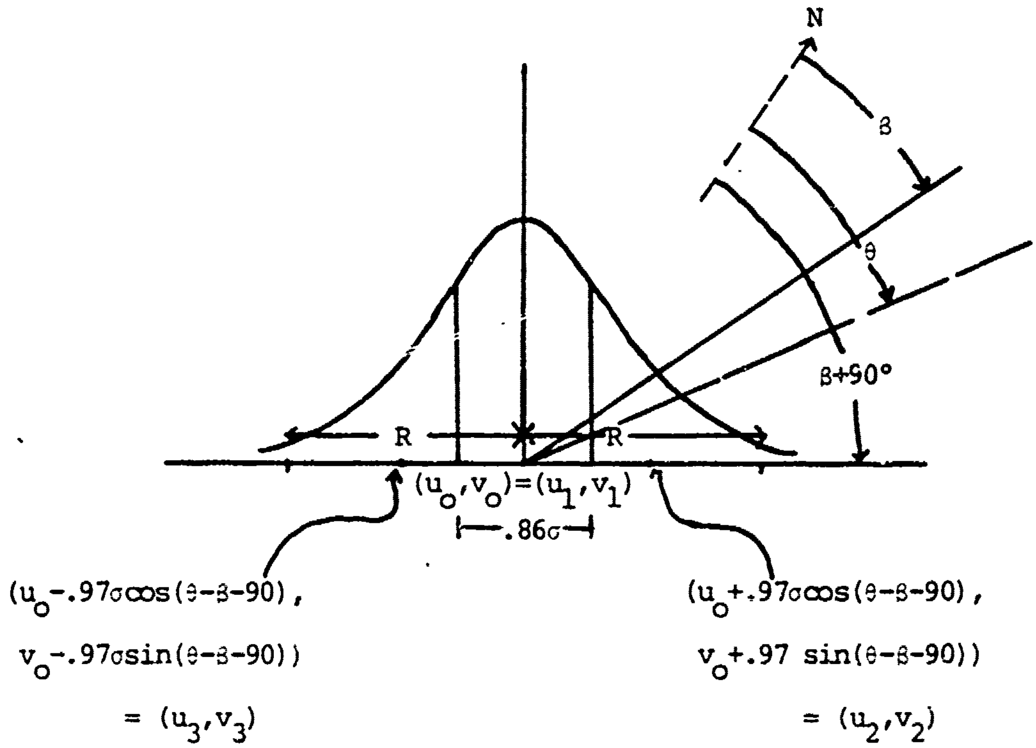


FIGURE 6. ESTIMATE OF SENSOR POSITION DENSITY

arises from the fact that the ime late density is significantly more complex in this case.

The threat distribution becomes distorted from the normal after some time late when the course and speed of the threat are constant but not known with certainty. Application of the method described herein requires that the user assume a discrete distribution of the speed of the threat with upper and lower bounds S_M and S_1, respectively. In addition, the threat course is assumed to be uniformly between 0° and 360°.

The density of threat location after some time late t_L when the threat speed is s_i then becomes [Ref. 1]:

$$f_{X,Y}(x,y;t_L,s_i) = \frac{1}{2\pi\sigma_X\sigma_Y}\int_0^{360} \exp\left[-\frac{1}{2}\left(\frac{(x-s_i t_L\cos(\theta-\psi))^2}{\sigma_X^2} + \frac{(y-s_i t_L\sin(\theta-\psi))^2}{\sigma_Y^2}\right)\right]\frac{d\psi}{360}$$

where θ is the bearing of the major axis from North. Note that the new threat density is still centered at the same position as the time zero ellipse but its shape changes as in Figure 4 of Chapter II. If the threat speed is s_i, and the course is uniformly distributed over (0°,360°), the outer limit of the new planar region containing the threat after time t_L has elapsed, given that it was originally located in the $p_1 \times 100\%$ confidence ellipse with semi-axes A and B, can be represented by an ellipse with semi-major axis $A+s_i t_L$ and semi-minor axis $B+s_i t_L$ (Figure 7). Thus the

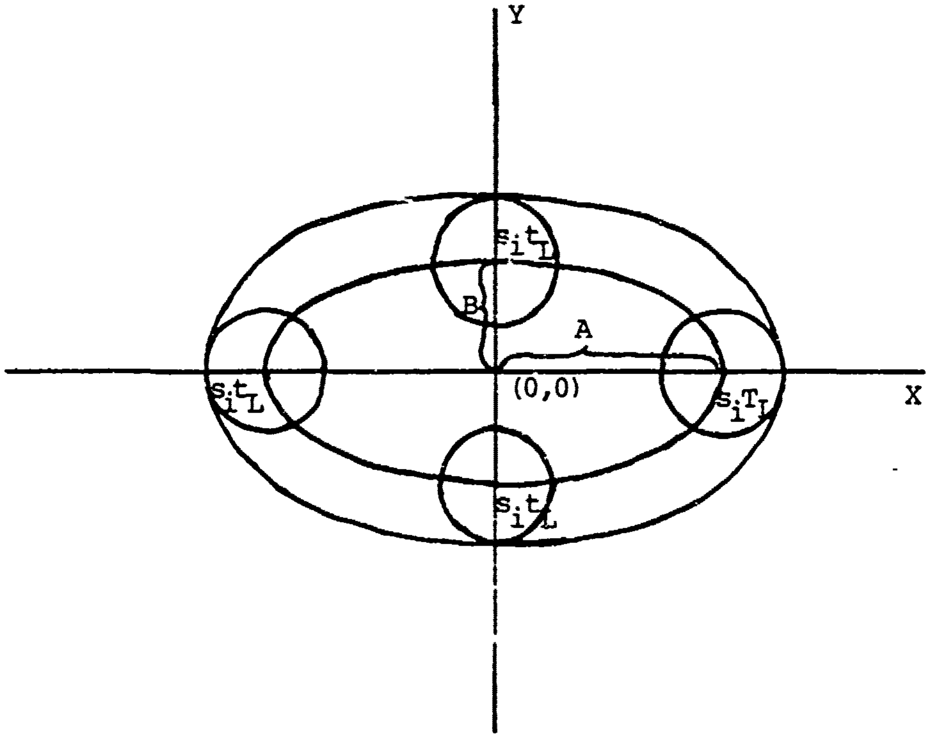


FIGURE 7. TIME LATE PLANAR REGION

region over which the density will be evaluated is still the intersection of an ellipse with an angular wedge, recalling however that the density is no longer normal. Further, the new elliptical region does not represent a $p_1 \times 100\%$ confidence region of the time late density.

Calculation of the likelihood that the threat lies along bearing β given that the threat lies in the $p_1 \times 100\%$ ellipse at time zero, that the true bearing lies in (β_L, β_U), that the sensor lies in the $p_3 \times 100\%$ circle and that the speed is s_i proceeds as follows:

1. Estimate the parameters of the original normal distribution: mean = (0,0), ${\sigma_X}^2 = A^2/(-2 \ln(1-p_1))$, ${\sigma_Y}^2 = B^2/(-2 \ln(1-p_1))$.

2. Determine the upper and lower bounds on the true bearing wedge, $\beta + k_\beta \sigma_\beta$ and $\beta - k_\beta \sigma_\beta$.

3. Determine the time late planar region as the ellipse with semi-major axis equal to $A + s_i t_L$ and semi-minor axis equal to $B + s_i t_L$.

4. Determine the position of the sensor relative to the ellipse center.

5. Subdivide the bearing fan $(\beta - k_\beta \sigma_\beta, \beta + k_\beta \sigma_\beta)$ into 2n subintervals.

6. At the midpoint of each subinterval, determine the equation of the line through the ship position at that bearing β_k.

7. Let the equation of the line of bearing β_k be $X = \frac{Y - c}{m}$. Then, the plane perpendicular to the xy-plane which

contains this line intersects the time late threat density in a curve whose equation is

$$g_L(y) = \left(\frac{m}{\sqrt{2\pi}\,\sqrt{\sigma_Y^2 + m^2\sigma_X^2}}\, e^{-\frac{1}{2}\frac{c^2}{\sigma_Y^2 + m^2\sigma_X^2}}\right)\left(\frac{\sqrt{\sigma_Y^2 + m^2\sigma_X^2}}{\sqrt{2\pi}\; m\; \sigma_X\; \sigma_Y}\right.$$

$$\left. \cdot\, e^{-\frac{1}{2}\left[\left(\frac{\sigma_Y^2 + m^2\sigma_X^2}{m^2\sigma_X^2\sigma_Y^2}\right)\left(y - \frac{c\,\sigma_Y^2}{\sigma_Y^2 + m^2\sigma_X^2}\right)^2\right]}\right)$$

$$\cdot \int_0^{360} \exp\left[-\frac{1}{2}\left(\frac{(s_i t_L)^2\cos^2(\theta-\psi) - 2 s_i t_L y \cos(\theta-\psi)}{m^2\sigma_X^2}\right.\right.$$

$$\left.\left. + \frac{(s_i t_L)^2\sin^2(\theta-\psi) - 2 s_i t_L y \sin(\theta-\psi)}{\sigma_Y^2}\right)\right] \frac{d\psi}{360} \qquad (4)$$

found by making the substitution $x = \frac{y-c}{m}$ in the density function $f_{X,Y}(x,y;t_L)$ and expanding.

For computational purposes, assume that the width W_k of the region of the time late threat ellipse cut out by the angular subinterval around β_k will be nearly constant through the ellipse. Let the points of intersection of the line of bearing β_k with the time late threat ellipse be (X_{k1}, Y_{k1}), (X_{k2}, Y_{k2}). Then approximate the time late density at speed s_i over this region by the absolute value of the product of W_k and the area under the curve g_L between Y_{k1} and Y_{k2}. The area under curve g_L can be approximated as follows. Subdivide the interval (Y_{k1}, Y_{k2}) into n_1 segments of length h. Evaluate g_L at the midpoint of each segment, y_J. Note that the first term in parenthesis in (4) is a constant equal to m times the value of the density of a normal random variable with mean zero and variance $\sigma_Y^2 + m^2\sigma_X^2$ evaluated at c. The second term in parenthesis is the density function of a normal random variable with mean

$$\frac{c\sigma_Y^2}{\sigma_Y^2 + m^2\sigma_X^2}$$

and variance

$$\frac{m^2\sigma_X^2\sigma_Y^2}{\sigma_Y^2 + m^2\sigma_X^2}$$

evaluated at y. The variable y also appears in the integral term in equation (4). Numerically evaluate this term of (4) with $y = y_J$. Let $g_I(y_J)$ be the result of this computation. Then, evaluating the area under the curve g_L between Y_{k1} and Y_{k2} is equivalent to the calculation:

$$\sum_{J=1}^{n_1} \frac{m}{\sqrt{\sigma_Y^2 + m^2\sigma_X^2}} \phi\left(\frac{c}{\sqrt{\sigma_Y^2 + m^2\sigma_X^2}}\right) \frac{\sqrt{\sigma_Y^2 + m^2\sigma_X^2}}{m\sigma_X\sigma_Y} \phi(z_J)\; g_I(y_J)$$

where $\phi(\cdot)$ is the density function of a standard normal random variable, and

$$z_J = \left(y_J - \frac{c\sigma_Y^2}{\sigma_Y^2 + m^2\sigma_X^2}\right)\left(\frac{\sqrt{\sigma_Y^2 + m^2\sigma_X^2}}{m\sigma_X\sigma_Y}\right) .$$

An analagous situation arises if the substitution $Y = mX + c$ is made for y in the time late density $f_{X,Y}(x,y;t_L)$.

8. The value of the density over the subinterval containing β_k is then weighted by the instantaneous probability that the bearing error is $\beta_k - \beta$:

$$\left| W_k \int_{Y_{k1}}^{Y_{k2}} f_{X,Y}(x_L,y_L;t_L)\,dy_L \right| f_\beta(\beta;\beta_k)$$

9. Steps C.7 and C.8 are then repeated for each subinterval of (β_L,β_U) and the results of each calculation are summed.

10. The uncertainty in sensor location is accounted for by repeating steps C.4 through C.9 with the assumption the sensor is located at each of the three points in Figure 5, multiplying by the probability the sensor lies in that interval and summing each result.

11. The result of calculations in steps C.1 through C.10 is the likelihood that the threat lies on bearing β given the threat was originally located in the $p_1 \times 100\%$ confidence ellipse, $\beta_T \in (\beta_L,\beta_U)$, the sensor is located in the $p_3 \times 100\%$ circle and the threat speed is s_i. The condition that the speed is s_i is removed by repeating the

calculations C.1 through C.10 for each of the speeds s_i, $i = 1,\ldots,M$ multiplying the result by the probability that the speed equals s_i and summing all M results. The final result is the likelihood that the threat lies on β given it was originally located in the $p_1 \times 100\%$ ellipse, $\beta_T \in (\beta_L, \beta_U)$, and the sensor is in the $p_3 \times 100\%$ circle estimated as

$$\sum_{i=1}^{M} P[S = s_i] \sum_{j=1}^{3} \frac{1}{3} f_j'(\beta) .$$

IV. CONCLUSIONS

Possible applications of and extensions to the algorithms developed in Chapter III are discussed in this chapter.

As indicated in Chapter I, the objective of this paper has been to develop a means of assessing the likelihood a threat whose position has previously been estimated lies at a given measured bearing from a local sensor. The procedures were developed with a view towards permitting the user to make comparisons among lines of bearing measured by different sensors or among conflicting bearing information generated by one sensor. The approach chosen has been to estimate the likelihood that a threat lies on bearing $\hat{\beta}$ given that the threat is located in an ellipse of specified confidence $p_1 \times 100\%$, that $\hat{\beta}$ is measured with $p_2 \times 100\%$ accuracy, and that the sensor position is measured with $p_3 \times 100\%$ accuracy. The algorithm to calculate the likelihood in the cases where the probability distribution of the target can be assumed to be bivariate normal at the time of the bearing measurement has been implemented on the TI-59 calculator. The cases in which this program applies are the following: (1) when the time elapsed since generation of the threat error ellipse is small enough to justify using the original estimate of the ellipse; (2) when threat course and speed are known, in which case the ellipse center is translated from the original position according to the course, speed and elapsed time information;

(3) when the threat can be assumed to be moving about in a random manner over a significant region in such a way that the ellipse center remains unchanged, but the x and y variances have increased. When none of these cases hold, but the course is assumed uniformly distributed over (0°,360°) and speed has a discrete distribution over a finite interval, the threat density at the time late t_L is not a bivariate normal density. The algorithm applicable in this case has not been implemented, but is described in some detail in Chapter III and Appendix B.

Once the appropriate computation has been completed for each of the bearings considered, the results can be used to weight the value of several bearings in refining the threat location estimate provided by the external sensor. Note that, although unlikely, $f'(\beta)$ may correctly be greater than one. Comparisons using these likelihoods should be made only when the upper and lower bounds on the true bearing fan for each bearing are chosen at the same probability level p_2 and the uncertainty areas for all sensors include the same probability level p_3. Further, these likelihood levels should be selected so as not to exclude a significant portion of the appropriate density. If p_2 or p_3 are not the same in all cases to be compared, $f'(\beta)$ must be divided by the applicable value of p_2 or p_3 for each measured bearing β to be considered.

Having established the relative value of available bearing information, the user can allocate weapons or further search effort accordingly. However, the probabilities calculated are strictly ordinal data and do not define a redistribution of target location probability based on additional information. Further, the threat ellipse does not contain the target with certainty. The power to predict the probability of success of a search or weapons allocation plan based on the priorities established by these procedures is limited by these constraints. In this area in particular further research would be useful.

In situations such as that for which the procedures in this paper have been developed, where a track has not been developed on the target, introduction of unrelated bearing data to a target motion model could impact significantly on the reliability of future position predictions. If there is high confidence in the reliability of the estimate of the threat ellipse provided by the ocean surveillance sensor, the prioritization established herein could be used to process bearing data prior to input to a target motion model. Using a pre-established threshold, only those bearings which coincide with the threat with an acceptable level of likelihood could be used to refine or update a track on the threat.

Desirable enhancements to the algorithms include providing for the instances in which the interval of uncertainty of the target course is known to be less than (0°,360°). Further, if the circular region of radius R contains the sensor with significantly less than 86% confidence or the assumption of a bivariate normal distribution of sensor location is unsatisfactory, it is left to the user to modify the calculations accordingly.

The utility of these algorithms would also be improved by implementation on a larger and faster system than the TI-59 calculator.

APPENDIX A. TI-59 PROGRAM VERBAL FLOW AND USER'S INSTRUCTIONS

Part I Step Number	Verbal Flow
000 - 029	Enter the confidence level p_1 for the threat ellipse. Calculate the value of k for the given p_1: $$k = \sqrt{-2 \ln(1-p_1)} .$$
020 - 029	Enter the length of the semi-major axis, A. Calculate $\sigma_X = A/k$.
030 - 052	Enter the length of the semi-major axis, B. Calculate $\sigma_Y = B/k$. Calculate B/A, $\sigma_X\sigma_Y$, σ_X^2 and σ_Y^2.
053 - 058	Enter orientation of semi-major axis, θ.
059-063	Enter bearing from sensor to center of threat ellipse, α. Calculate $\theta-\alpha$.
064 - 073	Enter distance r from sensor position to center of threat ellipse. Determine rectangular coordinates of sensor position (U,V) from polar coordinates $(-r,\theta-\alpha)$.
074 - 089	Store the constants 360, $\sqrt{2\pi}$. Initialize register 35 to 0.

090 - 096	Enter number of standard deviations desired for bearing fan, k_β.
097 - 104	Enter standard deviation of bearing error, σ_β. Calculate $k_\beta \sigma_\beta$.
105 - 143	Enter the angular stepsize desired, $\Delta\beta$, for incrementally stepping through (β_L, β_U). Calculate $\frac{1}{2}\Delta\beta$. Calculate the largest number n of increments of size $\Delta\beta$ contained in $k_\beta \sigma_\beta - \frac{1}{2}\Delta\beta$ degrees. Initialize counter 00 to n+1. Save n+1 in register 20. Determine $\delta\Delta\beta = k_\beta \sigma_\beta - \frac{1}{2}\Delta\beta - n\Delta\beta$, the residual increment. Calculate $\frac{1}{2}\delta\Delta\beta$.
144 - 146	Initialize counter 01 to 2. Calculations will be made at the midpoint of each interval from $\beta + \frac{1}{2}\Delta\beta$ to $\beta + k_\beta \sigma_\beta$, then at the midpoint of each interval from $\beta - \frac{1}{2}\Delta\beta$ to $\beta - k_\beta \sigma_\beta$, and finally at β. Counter 01 indicates whether calculations are complete on both sides of β.
147 - 151	Enter bearing measured by sensor, β.
152 - 154	Enter standard deviation of sensor position σ.

155	Enter index of the sensor position to be used for this run.
156 - 157	Coordinates of the sensor position are selected in accordance with run number entered above in Subroutine sin.
158 - 238	Determine whether a bearing β' parallel to either axis is included in the fan (β_L, β_U). If (β_L, β_U) includes a bearing parallel to the y-axis, use program 1 for Part II. The appropriate program number is displayed in calculator display register. If (β_L, β_U) does not contain a bearing parallel to either axis, use program 1.
239 - 259	Subroutine P/R.
260 - 340	Subroutine sin.

Part II, Programs 1 and 2

Step Number	Verbal Flow
000 - 003	Initiate β'.
004 - 007	If the last angular increment on this side of β has been considered, go to step 513. Otherwise continue.
008 - 012	Decrement counter for angular increment. If counter = 0, go to 021. Otherwise continue.
013 - 020	Remove flag to indicate this is not the last angular increment. Recall $\Delta\beta$, the input angular stepsize. Go to 029.
021 - 028	Set flag to indicate this is the last angular increment on this side. Add one-half the input angular stepsize $\Delta\beta$ and one-half the residual stepsize $\delta\Delta\beta$: $\frac{1}{2}\Delta\beta + \frac{1}{2}\delta\Delta\beta$.
029 - 030	Increment β' by the appropriate stepsize.
031 - 045	Convert β' to an angle between 0° and 360°.
046 - 051	Calculate $\theta-\beta'$. Print $\theta-\beta'$.

052 - 064	If $\lvert \theta - \beta' \rvert = 90°$ or $= 270°$, go to 236.
065 - 074	If $\lvert \theta - \beta' \rvert = 0°$ or $= 180°$, go to 075. Otherwise go to 133.
075 - 086	If the absolute value of the y coordinate of sensor position is greater than the length of the semi-minor axis, β, go to 004. In this case β' does not intersect the error ellipse. Otherwise continue.
087 - 088	Set flag 2 to indicate that the bearing β' is parallel to the x-axis.
089 - 111	Calculate the coordinates of the points of intersection of β' with the threat ellipse. The y-coordinates are equal to V, the y-coordinate of sensor position. x-coordinates are determined in Subroutine y^x. The points of intersection are symmetric about the y-axis. Thus, $X_{k1} = -X_{k2}$. Store the smaller x value in register 27, the larger in register 28.
112 - 132	Save locations of X_{k1}, X_{k2}, Y_{k1}, σ_X, and σ_Y. Go to 291.
133 - 138	Remove flags 2 and 3 to indicate that β' is not parallel to either the x- or y- axes.

139 - 145	Calculate the slope of β': $m = \tan(\theta - \beta')$.
146 - 154	Calculate the y-intercept of β': $c = V - mU$.
155 - 176	If $c^2 > A^2m^2 + B^2$, go to 004. In this case, β' does not intersect the error ellipse. Otherwise continue.
177 - 235	Calculate X_{k1} and X_{k2}, the x-coordinates of the points of intersection of β' with the ellipse:

$$X_{k1} = \frac{-mc + \frac{B}{A}\sqrt{A^2m^2 + B^2 - c^2}}{\frac{A^2m^2 + B^2}{A^2}}$$

$$X_{k2} = \frac{-mc - \frac{B}{A}\sqrt{A^2m^2 + B^2 - c^2}}{\frac{A^2m^2 + B^2}{A^2}} .$$

Calculate the y-coordinates of the points of intersection of β' with the ellipse, Y_{k1}, Y_{k2}:

$$Y_{k1} = mX_{k1} + c$$

$$Y_{k2} = mX_{k2} + c .$$

Go to 299.

236 - 247	If $\lvert U \rvert$, the absolute value of the x-coordinate of sensor position, is greater than the length of the semi-major axis, A, go to 004. β'

does not intersect the error ellipse in this case. Otherwise continue.

248 - 249 Set flag 3 to indicate that β' is parallel to the y-axis.

250 - 272 Calculate the coordinates of the points of intersection of β' with the error ellipse. The x-coordinates are equal to U, the x-coordinate of sensor position. The y-coordinates are determined in Subroutine y^x. The points of intersection are symmetric about the x-axis. Thus, $Y_{k1} = -Y_{k2}$. Store the smaller y value in register 29, the larger in 30.

273 - 298 Save the locations of Y_{k2}, Y_{k1}, X_{k1}, σ_Y, σ_X.

299 - 328 Determine the width of the strip around β', W:

If flag 0 set,

$$W = (d_1+d_2) \tan(\tfrac{1}{2}\delta\Delta B),$$

Otherwise

$$W = (d_1+d_2) \tan \tfrac{1}{2}\Delta B).$$

d_1 and d_2 are the distances from (U,V), the sensor position, to the intersection points (X_{k1},Y_{k1}) and (X_{k2},Y_{k2}), respectively. Distances are calculated in Subroutine log.

Multiply W by $\Delta\beta$ if flag 0 not set. Otherwise, multiply by $\delta\Delta\beta$. Save result in register 37.

329 - 334 — If β' is parallel to either the x- or the y-axis, go to 468. Otherwise continue.

<u>When using program 1:</u>

335 - 380 — Express the y-coordinates of the intersection points as standard normal random variables, Z_1 and Z_2:

$$Z_i = \frac{Y_{ki} - \dfrac{c\sigma_Y^2}{\sigma_Y^2 + m^2\sigma_X^2}}{\dfrac{|m|\,\sigma_X\sigma_Y}{\sqrt{\sigma_Y^2 + m^2\sigma_X^2}}},$$

$i = 1,2.$

381 - 391 — Sort the values Z_i in descending order. Let Z_2' be the larger value. Z_1' is the smaller.

392 - 401 — Calculate $\Phi(Z_2') - \Phi(Z_1')$, the probability that a standard normal random variate lies between Z_1' and Z_2'.

402 - 424 Multiply the results of steps 392-401 by:

$$\frac{1}{\sqrt{\sigma_Y^2 + m^2\sigma_X^2}} \phi\left(\frac{c}{\sqrt{\sigma_Y^2 + m^2\sigma_X^2}}\right) \times |m| \times W,$$

where ϕ is the standard normal density function.

The result of the calculations in steps 335-424 is

$$\left| W \int_{Y_{k1}}^{Y_{k2}} f_{X,Y}\left(\frac{y-c}{m}, y\right) dy \right| .$$

<u>When using program 2:</u>

335 - 372 Express the x-coordinates of the intersection points as standard normal random variables Z_1 and Z_2:

$$Z_i = \frac{X_{k2} + \dfrac{mc\sigma_X^2}{\sqrt{\sigma_Y^2 + m^2\sigma_X^2}}}{\dfrac{\sigma_X\sigma_Y}{\sqrt{\sigma_X^2 + m^2\sigma_Y^2}}},$$

$i = 1,2.$

373 - 383 Sort the values of Z_i in descending order. Let Z_2' = larger value. Z_1' = smaller.

384 - 393 Calculate $\Phi(Z_2') - \Phi(Z_1')$, the probability that a standard normal random variate lies between Z_1' and Z_2'.

394 - 413 Multiply the results of steps 384-393 by:

$$\frac{1}{\sqrt{\sigma_Y^2 + m^2\sigma_X^2}}\,\phi\left(\frac{c}{\sqrt{\sigma_Y^2 + m^2\sigma_X^2}}\right) \times W,$$

where ϕ is the standard normal density function.

The result of the calculations in steps 335-413 is

$$\left| W \int_{Y_{k1}}^{Y_{k2}} f_{X,Y}(x,mx+c)\ dx \right| .$$

Regardless of which program is in use:

425 - 465 Multiply results of previous calculation by

$$\frac{1}{\sigma_\beta}\,\phi\left(\frac{\beta''-\beta}{\sigma_\beta}\right)$$

where $\beta'' = \begin{cases} \beta' & , \beta' < 180° \\ \beta'-360° & , \beta' \geq 180° \end{cases}$

and ϕ is the standard normal density.

The result of this calculation is

$$f_\beta(\beta;\beta')\ \left| W \int f_{X,Y}(x,y)\ dy \right| .$$

466 - 467 Go to 508

468 - 507 Calculate the area under the curve formed by the intersection of β' with $f_{X,Y}(x,y)$. When β' parallels the x-axis this calculation becomes:

$$\frac{1}{\sigma_Y}\,\phi\left(\frac{V}{\sigma_Y}\right)\left[\Phi(Z_2) - \Phi(Z_1)\right] ,$$

where $Z_2 = X_{k2}/\sigma_X$ and $Z_1 = X_{k1}/\sigma_X$, V is the y-coordinate of the sensor position.

When β' is parallel to the y-axis, the calculation is:

$$\frac{1}{\sigma_X}\,\phi\left(\frac{U}{\sigma_X}\right)\left[\Phi(Z_2) - \Phi(Z_1)\right] ,$$

where $Z_2 = Y_{k2}/\sigma_Y$ and $Z_1 = Y_{k1}/\sigma_Y$, U is the x-coordinate of the sensor position.

Go to 422 to complete calculation of $f_\beta(\beta;\beta')[W \int f_{X,Y}(x,y)]$.
(Go to 410 in Program 2).

508-512 Accumulate the probability at each angular interval. Display result.

513 - 515 If flag 1 is set go to 558. Otherwise continue.

516 - 520 If flag 0 is not set, that is if the angular increment just considered was not the last on this side of β, go to 004 and continue calculation on same side of β. Otherwise continue.

521 - 524	Decrement the counter 01. If counter is now 0 go to 543. In this case, the probabilities have been calcu...ed on both sides of $\hat{s}$. The calculation at $\hat{s}$ remains to be done. Otherwise continue.
525 - 542	Remove flag 0. Multiply $\Delta\hat{s}$, $\frac{1}{2}\Delta\hat{s}$, and $\frac{1}{2}\delta\Delta\hat{s}$ by -1. Reinitialize counter 00 to n+1. Go to 000 to begin calculation on second side of $\hat{s}$.
543 - 567	Set flag 01 indicating that calculations on both sides of $\hat{s}$ have been completed. The next iteration will do the calculation at $\hat{s}' = \hat{s}$. Remove flag 00. Initialize $\hat{s}'$ to 0. Recall $\hat{s}$. Go to 029.
558 - 566	Remove flag 01. Display the accumulated likelihood. STOP

The result of this calculation is approximately $f_j'(\hat{s})$ with sensor at (U_j, V_j) where j = run number entered in Part I.

567 - 587	Subroutine y^x calculates the points of intersection when $\hat{s}'$ is parallel to either the x or y axis:

$$Y_{k1} = Y_{k2} = V$$

$$X_{k1} = -A\sqrt{1 - (\frac{U}{B})^2}$$

$$X_{k2} = A\sqrt{1 - (\frac{U}{B})^2}$$

Parallel to y-axis:

$$X_{k1} = X_{k2} = U$$

$$Y_{k1} = -B \sqrt{1 - \left(\frac{V}{A}\right)^2}$$

$$Y_{k2} = B \sqrt{1 - \left(\frac{V}{A}\right)^2} \quad .$$

589 - 610 Subroutine log calculates the distance from sensor position to point of intersection of s' with the ellipse:

$$d_i = \sqrt{(U - X_{ki})^2 + (V - Y_{ki})^2}$$

Part III Step Number	Verbal Flow
000 - 012	Enter result of run 1. Multiply by 1/3.
013 - 021	Enter result of run 2. Multiply by 1/3.
022 - 029	Enter result of run 3. Multiply by 1/3.
030 - 032	Display likelihood.
033 - 041	If p_2 is the same for all bearings to be compared, enter 1. Go to 042. Otherwise, enter p_2 for this bearing. Divide likelihood by p_2. Display result.
042 - 050	If p_3 is the same for all bearings to be compared, enter 1 and STOP. Otherwise, enter p_3 for this bearing. Divide likelihood by p_3. Display result. STOP.

USER'S INSTRUCTIONS

The program to determine the likelihood that the threat lies along bearing β given the threat is in the confidence ellipse, $\beta_T \varepsilon (\beta_L, \beta_U)$ and the sensor is within the $p_3 \times 100\%$ confidence circle is in three parts. All parts require the use of a printer and the use of the Applied Statistics Library Module. Prior to running, the calculator must be repartitioned:

1. Enter 4
2. Press 2nd OP 17

Part I

1. Read sides 1 and 2 of Part I
2. Read side 4 of Part II either program 1 or program 2. Since program 2 is used more often, unless it is known that the bearing fan (β_L, β_U) contains a bearing parallel to the major axis, recommend using program 2 of Part II.
3. Press R.
4. Enter p_1, the confidence level of the threat ellipse. Press A
5. Enter A, length of the semi-major axis. If data provided is length of the entire major axis, divide by 2 before entering.

Press B.

6. Enter B, length of the semi-minor axis. If data provided is length of entire minor axis, divide by 2 before entering. Press C.

7. Enter θ, the bearing of major axis from North. $0° \le \theta \le 180°$. θ is entered in degrees. Press D.

8. Enter α, the bearing of the threat ellipse center from the estimated sensor position. $0° \le \alpha \le 360°$, in degrees. Press E.

9. Enter r, the range from the estimated sensor position to the center of the threat ellipse. Press 2nd A'.

10. Enter k_β, the number of standard deviations desired in one direction from β. The program will construct a fan of equal size on the other side of β. Press 2nd B'.

11. Enter σ_β, the standard deviation of bearing error. Press 2nd C'.

12. Enter $\Delta\beta$, the desired angular stepsize. Press 2nd D'.

13. Enter β, the measured bearing. Press 2nd E'.

14. Enter σ, the standard deviation of sensor position. Press R/S.

15. Enter the number of this run, 1, 2 or 3. When run number = 1, sensor is located at its estimated position (u_o, v_o). When run number = 2, the location will be $(u_o + .97\sigma\cos(\theta-\beta-90), v_o + .97\sigma\sin(\theta-\beta-90))$. When run number = 3, the location will be $(u_o - .97\sigma\cos(\theta-\beta-90), v_o - .97\sigma\sin(\theta-\beta-90))$. Press R/S. If program number displayed matches that of the Part II side 4 read in, continue to 16. Otherwise, press 2nd CMS, RST. Read in side 4 of the Part II program which corresponds to the number displayed. Repeat 3 through 15.

16. Press 4 2nd WRITE. Rerecord side 4 of the Part II program read in. This enables data entered in Part I to be transferred to Part II.

Part II

1. Read sides 1, 2, 3 and 4 of Part II Program 1 or 2 as selected by the Part I program.

2. Press RST

3. Press R/S. Values of β_t and $W\int f_\beta(\beta;\beta_t) f_{\beta_{TRUE}}(\beta_t)$ will print alternately. Final result also prints out at end of calculation.

4. Record final result: Likelihood threat lies along bearing β given threat is in $p_1 \times 100\%$ ellipse, true bearing is in $p_2 \times 100\%$ fan and sensor is at location used in this run. Parts I and II must be completed 3 times (Run numbers 1, 2 and 3) before proceeding to Part III if uncertainty in sensor position is being considered.

Part III

1. Read side 1 of Part I.I.

2. Enter result of run 1. Press A.

3. Enter result of run 2. Press B.

4. Enter result of run 3. Press C.

5. Enter p_2, confidence level of (β_L, β_U), if necessary. Otherwise enter 1. Press D.

6. Enter p_3 confidence level of sensor position, if necessary. Otherwise, enter 1. Press E.

APPENDIX B. TIME LATE VERBAL FLOW

VERBAL FLOW

1. Enter p_1. Determine the value of k: $k = \sqrt{-2\ln(1-p_1)}$
2. Enter A. Determine $\sigma_X{}^2 = (A/k)^2$
3. Enter B. Determine $\sigma_Y{}^2 = (B/k)^2$
4. Enter θ, orientation of major axis
5. Enter α, bearing from sensor to ellipse center
6. Enter range r from sensor to ellipse center
7. Calculate (u_o, v_o) the coordinates of mean sensor position. $u_o = -r\cos(\theta-\alpha)$, $v_o = -r\sin(\theta-\alpha)$
8. Enter σ_β, the standard deviation of bearing error
9. Enter k_β, the number of standard deviations to be included in the bearing fan on each side of β.
10. Enter desired angular stepsize, $\Delta\beta$
11. Calculate the number of iterations of size $\Delta\beta$ required on each side of β: $I = [(k_\beta\sigma_\beta - \frac{1}{2}\Delta\beta)/\Delta\beta]$ where $[\cdot]$ means the greatest integer less than or equal to the value within.
12. In general $(k_\beta\sigma_\beta - \frac{1}{2}\Delta\beta)/\Delta\beta$ is not an integer. Determine the size of the fractional increment:

$$\delta\Delta\beta = ((k_\beta\sigma_\beta - \tfrac{1}{2}\Delta\beta)/\Delta\beta - I)\Delta\beta.$$

13. Enter β, the measured bearing. Let PROB2=0, PROB1=0
14. Enter the time late t_L
15. For each of the discrete threat speeds to be considered, repeat steps 16 to 43. Then go to 44.
16. Enter the target speed s. Let PROB = 0

17. Expand the outer limit of the threat ellipse:

$$\text{Let } A' = A + st_L$$
$$B' = B + st_L$$

18. Let $\beta' = \beta$

19. For each of the I increments of size $\Delta\beta$ repeat steps 20 to 38.

20. Let $\beta' = \beta' + \Delta\beta$

21. Calculate $(\theta-\beta')$. If $|\theta-\beta'| = 0°$ or $180°$, go to 27. If $|\theta-\beta'| = 90°$ or $270°$, go to 30

22. Calculate $m = \tan(\theta-\beta')$, the slope of the line of bearing.

23. Calculate $c = mU+V$, where (U,V) is the sensor location for this iteration. c is the y-intercept of the line of bearing.

24. Calculate $m^2A^2 + B^2 - c^2$. If this quantity is less than zero, the line of bearing β' does not intersect the threat ellipse. Go to step 20 and process next increment of size $\Delta\beta$, if any remain. If all I intervals of size $\Delta\beta$ on this side of β have been considered go to step 40.

25. Calculate the points of intersection of the line of bearing with the expanded ellipse:

$$X_1 = \frac{(-mc + (B'/A')\sqrt{A'^2m^2 + B'^2 - c^2})(A'^2)}{A'^2m^2 + B'^2}$$

$$X_2 = \frac{(-mc - (B'/A')\sqrt{A'^2m^2 + B'^2 - c^2})(A'^2)}{A'^2m^2 + B'^2}$$

$$Y_1 = mX_1 + c, \quad Y_2 = mX_2 + c$$

26. Go to 32

27. If $|V| > B'$, the bearing β' parallel to the x-axis does not intersect the threat ellipse. Go to step 20 and process next increment of size $\Delta\beta$ if any remain. If all I intervals if size $\Delta\beta$ on this side of β have been considered, go to 40.

28. Calculate the points of intersection

$$Y_1 = Y_2 = V$$

$$X_1 = -B'\sqrt{1 - (V^2/B'^2)}$$

$$X_2 = B'\sqrt{1 - (V^2/B'^2)}$$

29. Go to 32

30. If $|U| > A'$, the bearing β' parallel to the y-axis does not intersect the threat ellipse. Go to step 20 and process the next increment of size $\Delta\beta$ if any remain. If all I intervals of size $\Delta\beta$ on this side of β have been considered, go to 40.

31. Calculate the points of intersection:

$$X_1 = X_2 = U$$

$$Y_1 = -A'\sqrt{1 - (U^2/A'^2)}$$

$$Y_2 = A'\sqrt{1 - (U^2/A'^2)}$$

32. Calculate the median width of the strip of the ellipse defined by the angular subinterval under consideration. If the subinterval is of size $\Delta\beta$:

$$W = \frac{d_1 + d_2}{2}(2 \tan \tfrac{1}{2}\Delta\beta)$$

If the subinterval is of size $\delta\Delta\beta$:

$$W = \frac{d_1 + d_2}{2}(2 \tan \tfrac{1}{2}\delta\Delta\beta)$$

In these expressions d_i is the distance from the sensor to the ith point of intersection, i = 1, 2: $d_i = \sqrt{(U-X_i)^2 + (V-Y_i)^2}$

33. If $|\theta-\beta'| = 0°$ or $180°$, go to 36. If $|\theta-\beta'| = 90°$ or $270°$, go to 35.

34. Evaluate the target density from Y_1' to Y_2' along the line $y = mx + c$:

Let $Y_1' = \min(Y_1,Y_2)$

$Y_2' = \max(Y_1,Y_2)$

$\phi(z)$ = density function of a standard normal random variable evaluated at z

$$\lambda = (m/\sqrt{\sigma_Y^2 + m^2\sigma_X^2})\ \phi(c/\sqrt{\sigma_Y^2 + m^2\sigma_X^2})$$

Subdivide the interval (Y_1',Y_2') into n segments of length h. At the midpoint of each segement, y_j, compute:

$$hK\left(\sqrt{\sigma_Y^2+m^2\sigma_X^2}\,/\,m\sigma_X\sigma_Y\right)\ \phi\left(\left(y_j-\frac{c\sigma_Y^2}{\sigma_Y^2+m^2\sigma_X^2}\right)\left(\sqrt{\sigma_Y^2+m^2\sigma_Y^2}\,/\,m\sigma_X\sigma_Y\right)\right)$$

$$\int_0^{360} \exp\left[-\frac{1}{2}\left(\frac{(st_L)^2\cos^2(\theta-\psi)-2st_Ly_j\cos(\theta-\psi)}{m^2\sigma_X^2} + \frac{(st_L)^2\sin^2(\theta-\psi)-2st_Ly_j\sin(\theta-\psi)}{\sigma_Y^2}\right)\right]\frac{d\psi}{360}$$

when the integral must be numerically evaluated.

Sum the results over all n segments.

Go to 37.

35. Evaluate the target density from Y_1' to Y_2' along the line $x = U$:

Let $Y_1' = \min(Y_1, Y_2)$

$Y_2' = \max(Y_1, Y_2)$

$\phi(z)$ = density function of standard normal random variable

$$K = \frac{1}{\sigma_X}\,\phi\left(\frac{U}{\sigma_X}\right)$$

Subdivide the interval (Y_1', Y_2') into n segments of length h. At the midpoint of each segment, y_j, compute:

$$hK\,\frac{1}{\sigma_X}\left(\phi\left(\frac{y_j}{\sigma_Y}\right)\right)\int_0^{360} \exp\left[-\frac{1}{2}\left(\frac{(st_L)^2\cos(\theta-\psi)-2st_LU\cos(\theta-\psi)}{\sigma_X^2} + \frac{(st_L)^2\sin^2(\theta-\psi)-2st_Ly_j\sin(\theta-\psi)}{\sigma_Y^2}\right)\right]\frac{d\psi}{360}$$

where the integral must be numerically evaluated.

Sum the results over all n segments. Go to 37.

36. Evaluate the target density from X_1' to X_2' along the line y = V:

$$\text{Let } X_1' = \min(X_1, X_2)$$

$$X_2' = \max(X_1, X_2)$$

$\phi(z)$ = density of standard normal random variable

$$K = \frac{1}{\sigma_Y} \phi\left(\frac{V}{\sigma_Y}\right)$$

Subdivide the interval (X_1', X_2') into n segments of length h. At the midpoint of each segment, x_j, compute:

$$hK \frac{1}{\sigma_X} \left(\phi\left(\frac{x_j}{\sigma_X}\right)\right) \int_0^{360} \exp\left[-\frac{1}{2}\left(\frac{(st_L)^2\cos(\theta-\psi) - 2st_L x_j(\cos(\theta-\psi)}{\sigma_X^2} + \frac{(st_L)^2\sin^2(\theta-\psi) - 2st_L V\sin(\theta-\psi)}{\sigma_Y^2}\right)\right] \frac{d\psi}{360}$$

where the integral must be numerically evaluated. Sum results over all n segments.

37. Multiply the value of the target density just computed by W.

38. Multiply this result by

$$\frac{1}{\sigma_\beta} \phi\left(\frac{\beta' - \beta}{\sigma_\beta}\right)$$

where ϕ is the standard normal density function.

39. Let PROB = PROB+ (the results of the calculations in steps 20 through 38 for each if the I subintervals of size $\Delta\beta$).

40. If the fractional interval of size $\delta\Delta\beta$ on this side has been considered go to 41. Otherwise, let $\beta' = \frac{1}{2}\delta\Delta\beta + \frac{1}{2}\Delta\beta$. Repeat calculations 21 through 38 once. Let PROB = PROB+ (the result calculated at this step).

41. If the computations on both sides of β have been computed, go to 42. Otherwise repeat the computations from 18 to 40 on the other side of β by letting $\Delta\beta = -\Delta\beta$, $\delta\Delta\beta = -\delta\Delta\beta$.

42. Let $\beta' = \beta$. Repeat steps 21 through 38 once. Let PROB = PROB+ (the results of this calculation).

43. Let j = the number of discrete threat speeds to be considered. Let PROB1 = PROB1 + P[S=s]PROB. Go to 15.

44. Repeat steps 15 through 43 once with the sensor located at each of three points:

(1) $(U,V) = (u_o,v_o)$, the mean of the sensor density;

(2) $(U,V) = (u_o + .97\sigma\cos(\theta-\beta-90),$
$v_o + .97\sigma\sin(\theta-\beta-90))$;

(3) $(U,V) = (u_o - .97 \cos(\theta-\beta-90),$
$v_o - .97 \sin(\theta-\beta-90))$;

where σ is the standard deviation of the sensor density. The value of PROB1 calculated at each iteration will be weighted by the approximate probability that the sensor is located in the region of the sensor error circle represented by the applicable value (U,V). If p_3 is of the order .86 or greater, multiply by 1/3.

45. The value of PROB2 calculated after completion of step 44 is the relative probability that the threat lies along bearing β at time t_L, given the threat was in $p_1 \times 100\%$ ellipse at time t_o, β_t is in (β_L, β_U) and the sensor is in a $p_3 \times 100\%$ confidence circle.

CALCULATOR PROGRAM

PART I

Step	Code	Key	Step	Code	Key
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003	[illegible]	[illegible]	053	[illegible]	LBL
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007	55	[illegible]	057	[illegible]	[illegible]
008	02	[illegible]	058	[illegible]	[illegible]
009	[illegible]	[illegible]	059	[illegible]	[illegible]
010	[illegible]	[illegible]	060	[illegible]	[illegible]
011	[illegible]	[illegible]	061	[illegible]	[illegible]
012	[illegible]	[illegible]	062	[illegible]	[illegible]
013	[illegible]	[illegible]	063	[illegible]	[illegible]
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019	[illegible]	[illegible]	069	[illegible]	[illegible]
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043	[illegible]	[illegible]	093	[illegible]	[illegible]
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045	[illegible]	[illegible]	095	[illegible]	[illegible]
046	[illegible]	[illegible]	096	[illegible]	[illegible]
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048	[illegible]	[illegible]	098	[illegible]	[illegible]
049	03	[illegible]	099	[illegible]	[illegible]

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203	77	GE
204	01	01
205	93	93
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207	43	RCL
208	30	30
209	77	GE
210	01	01
211	95	95
212	61	GTO
213	01	01
214	93	93
215	42	STO
216	30	30
217	43	RCL
218	11	11
219	32	X:T
220	77	GE
221	01	01
222	95	95
223	32	X:T
224	85	+
225	01	1
226	02	2
227	00	0
228	95	=
229	32	X:T
230	43	RCL
231	30	30
232	22	INV
233	77	GE
234	01	01
235	95	95
236	61	GTO
237	01	01
238	93	93
239	76	LBL
240	37	P/R
241	53	(
242	53	(
243	53	(
244	24	CE
245	85	+
246	43	RCL
247	14	14
248	54	)
249	55	÷

250	43	RCL
251	14	14
252	54	)
253	22	INV
254	34	INT
255	65	×
256	43	RCL
257	14	14
258	54	)
259	92	RTN
260	76	LBL
261	38	SIN
262	01	1
263	67	EQ
264	03	03
265	40	40
266	02	2
267	22	INV
268	67	EQ
269	03	03
270	73	73
271	86	STF
272	02	02
273	03	3
274	67	EQ
275	03	03
276	94	94
277	87	IFF
278	03	03
279	03	03
280	96	96
281	94	+/-
282	34	√x
283	92	RTN
284	86	STF
285	03	03
286	53	(
287	93	.
288	09	9
289	07	7
290	65	×
291	43	RCL
292	31	31
293	65	×
294	53	(
295	43	RCL
296	11	11
297	75	-
298	43	RCL
299	22	22

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300  75  -
301  09  9
302  00  0
303  54  )
304  39  COS
305  54  )
306  67  IFF
307  02  02
308  03  03
309  11  11
310  94  +/-
311  44  SUM
312  12  12
313  53  (
314  93  .
315  09  9
316  07  7
317  65  ×
318  43  RCL
319  31  31
320  65  ×
321  53  (
322  43  RCL
323  11  11
324  75  -
325  43  RCL
326  22  22
327  75  -
328  09  9
329  00  0
330  54  )
331  38  SIN
332  54  )
333  67  IFF
334  02  02
335  03  03
336  38  38
337  94  +/-
338  44  SUM
339  13  13
340  92  RTN
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PART II - PROGRAM 1

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012	[illegible]	[illegible]	062	[illegible]	[illegible]
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039	43	[illegible]	089	[illegible]	[illegible]
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```
42 STO
32 32
03 3
42 STO
33 33
71 SBR
45 YX
42 STO
28 28
34 FX
42 STO
27 27
02 2
08 8
42 STO
38 38
02 2
07 7
42 STO
39 39
02 2
09 9
42 STO
34 34
05 5
42 STO
36 36
06 6
42 STO
24 24
61 GTO
02 02
91 91
22 INV
86 STF
03 03
22 INV
86 STF
02 02
43 RCL
24 24
30 TAN
42 STO
24 24
42 STO
36 36
43 RCL
12 12
94 +/-
49 PRD
```

```
36 36
43 RCL
13 13
44 SUM
36 36
43 RCL
36 36
33 X2
32 X:T
43 RCL
03 03
33 X2
65 X
43 RCL
24 24
33 X2
85 +
43 RCL
04 04
33 X2
95 =
42 STO
34 34
22 INV
77 GE
00 00
04 04
75 -
32 X:T
95 =
34 FX
65 X
43 RCL
10 10
95 =
42 STO
27 27
94 +/-
75 -
53 (
43 RCL
24 24
65 X
43 RCL
36 36
54 )
22 INV
44 SUM
27 27
95 =
```

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200	42	STO
201	19	19
202	43	RCL
203	03	03
204	32	X:T
205	55	÷
206	43	RCL
207	34	34
208	95	=
209	49	PRD
210	27	27
211	49	PRD
212	28	28
213	43	RCL
214	27	27
215	42	STO
216	29	29
217	43	RCL
218	28	28
219	42	STO
220	30	30
221	43	RCL
222	24	24
223	49	PRD
224	29	29
225	49	PRD
226	30	30
227	43	RCL
228	36	36
229	44	SUM
230	29	29
231	44	SUM
232	30	30
233	61	GTO
234	02	02
235	91	R/S
236	76	LBL
237	35	1/X
238	43	RCL
239	12	12
240	50	I×I
241	32	X:T
242	43	RCL
243	02	02
244	22	INV
245	77	GE
246	00	00
247	04	04
248	86	STF
249	03	03
250	43	RCL
251	12	12
252	42	STO
253	27	27
254	42	STO
255	28	28
256	01	1
257	07	7
258	42	STO
259	31	31
260	03	3
261	42	STO
262	32	32
263	04	4
264	42	STO
265	33	33
266	71	SBR
267	45	Y×
268	42	STO
269	30	30
270	94	+/-
271	42	STO
272	29	29
273	03	3
274	00	0
275	42	STO
276	38	38
277	02	2
278	09	9
279	42	STO
280	34	34
281	02	2
282	[illegible]	[illegible]
283	42	STO
284	34	[illegible]
285	06	6
286	42	STO
287	36	36
288	05	5
289	42	STO
290	24	24
291	02	2
292	07	7
293	42	STO
294	31	31
295	02	2
296	09	9
297	42	STO
298	13	13
299	71	SBR

300	28	LOG
301	42	STO
302	37	37
303	02	2
304	08	8
305	42	STO
306	31	31
307	03	3
308	00	0
309	42	STO
310	32	32
311	71	SBR
312	28	LOG
313	44	SUM
314	37	37
315	43	RCL
316	13	13
317	22	INV
318	87	IFF
319	00	00
320	03	03
321	24	24
322	43	RCL
323	31	31
324	50	I×I
325	30	TAN
326	50	I×I
327	49	PRD
328	37	37
329	87	IFF
330	02	02
331	67	EQ
332	87	IFF
333	03	03
334	67	EQ
335	43	RCL
336	36	36
337	65	×
338	43	RCL
339	08	08
340	55	÷
341	53	(
342	43	RCL
343	08	08
344	85	+
345	43	RCL
346	09	09
347	55	
348	43	RCL
349	24	24

350	33	X²
351	54	)
352	42	STO
353	38	38
354	95	=
355	22	INV
356	44	SUM
357	29	29
358	22	INV
359	44	SUM
360	30	30
361	43	RCL
362	38	38
363	34	√X
364	49	PRD
365	29	29
366	49	PRD
367	30	30
368	43	RCL
369	24	24
370	50	I×I
371	65	×
372	43	RCL
373	07	07
374	95	=
375	22	INV
376	49	PRD
377	29	29
378	22	INV
379	49	PRD
380	30	30
381	43	RCL
382	[illegible]	[illegible]
383	[illegible]	[illegible]
384	43	RCL
385	30	30
386	77	GE
387	03	03
388	92	92
389	42	STO
390	29	29
391	32	X:T
392	36	PGM
393	19	19
394	11	A
395	75	-
396	43	RCL
397	29	29
398	36	PGM
399	19	19

400	11	E	450	16	[illegible]
401	55	=	451	55	-
402	45		452	43	RCL
403	53		453	[illegible]	[illegible]
404	43	RCL	454	54	
405	36	[illegible]	455	36	PGM
406	55	-	456	[illegible]	[illegible]
407	43	RCL	457	11	A
408	38	38	458	54	
409	34	[illegible]	459	65	
410	54		460	43	RCL
411	36	PGM	461	16	16
412	[illegible]	[illegible]	462	55	-
413	11	A	463	43	RCL
414	55	-	464	[illegible]	[illegible]
415	43	RCL	465	95	=
416	32	[illegible]	466	[illegible]	[illegible]
417	34	[illegible]	467	[illegible]	[illegible]
418	65		468	[illegible]	[illegible]
419	43	RCL	469	[illegible]	[illegible]
420	24	24	470	[illegible]	[illegible]
421	50	[illegible]	471	[illegible]	[illegible]
422	45		472	[illegible]	[illegible]
423	43	RCL	473	[illegible]	[illegible]
424	[illegible]	[illegible]	474	[illegible]	[illegible]
425	[illegible]		475	[illegible]	[illegible]
426	[illegible]		476	[illegible]	[illegible]
427	[illegible]		477	[illegible]	[illegible]
428	[illegible]		478	[illegible]	[illegible]
429	[illegible]		479	[illegible]	[illegible]
430	[illegible]		480	[illegible]	[illegible]
431	[illegible]		481	[illegible]	[illegible]
432	[illegible]		482	[illegible]	[illegible]
433	[illegible]		483	[illegible]	[illegible]
434	43	RCL	484	[illegible]	[illegible]
435	33	[illegible]	485	[illegible]	[illegible]
436	75	-	486	[illegible]	[illegible]
437	43	RCL	487	[illegible]	[illegible]
438	22	22	488	[illegible]	[illegible]
439	54		489	[illegible]	=
440	22	INV	490	[illegible]	[illegible]
441	77	GE	491	[illegible]	[illegible]
442	04	04	492	[illegible]	[illegible]
443	48	48	493	[illegible]	[illegible]
444	75	-	494	55	-
445	43	RCL	495	[illegible]	[illegible]
446	14	14	496	[illegible]	[illegible]
447	54		497	[illegible]	
448	65		498	[illegible]	PGM
449	43	RCL	499	[illegible]	[illegible]

500	11	A
501	55	÷
502	73	RC*
503	34	34
504	95	=
505	61	GTO
506	04	04
507	22	22
508	76	LBL
509	77	GE
510	44	SUM
511	35	35
512	99	PRT
513	87	IFF
514	01	01
515	78	Σ+
516	22	INV
517	87	IFF
518	00	00
519	00	00
520	04	04
521	22	INV
522	97	DSZ
523	01	01
524	89	π
525	22	INV
526	86	STF
527	00	00
528	01	1
529	94	+/-
530	49	PRD
531	13	13
532	49	PRD
533	19	19
534	49	PRD
535	21	21
536	43	RCL
537	20	20
538	42	STO
539	00	00
540	61	GTO
541	00	00
542	00	00
543	76	LBL
544	89	π
545	86	STF
546	01	01
547	22	INV
548	86	STF
549	00	00
550	25	CLR
551	42	STO
552	33	33
553	43	RCL
554	22	22
555	61	GTO
556	00	00
557	29	29
558	76	LBL
559	78	Σ+
560	22	INV
561	86	STF
562	01	01
563	43	RCL
564	35	35
565	99	PRT
566	91	R/S
567	76	LBL
568	45	YX
569	53	(
570	53	(
571	01	1
572	75	-
573	53	(
574	73	RC*
575	31	31
576	55	÷
577	73	RC*
578	32	32
579	54	)
580	33	X²
581	54	)
582	34	√X
583	45	YX
584	73	RC*
585	33	33
586	54	)
587	92	RTN
588	76	LBL
589	28	LOG
590	53	(
591	53	(
592	73	RC*
593	31	31
594	75	-
595	43	RCL
596	12	12
597	54	)
598	33	X²
599	85	+

```
600  53
601  73 RC-
602  32   32
603  75   -
604  43 RCL
605  13   13
606  54
607  33 MA
608  54
609  34 CM
610  92 RTN
```

PART II PROGRAM 2

000	43	RCL
001	21	21
002	42	STO
003	23	23
004	87	IFF
005	00	00
006	05	05
007	13	13
008	22	INV
009	97	DSZ
010	00	00
011	00	00
012	21	21
013	22	INV
014	86	STF
015	00	00
016	43	RCL
017	16	16
018	61	GTO
019	00	00
020	29	29
021	86	STF
022	00	00
023	43	RCL
024	13	13
025	85	+
026	43	RCL
027	21	21
028	95	=
029	44	SUM
030	23	23
031	43	RCL
032	14	14
033	44	SUM
034	23	23
035	22	INV
036	49	PRD
037	23	23
038	65	×
039	48	EXC
040	23	23
041	22	INV
042	59	INT
043	95	=
044	42	STO
045	23	23
046	[illegible]	[illegible]
047	95	=
048	43	RCL
049	11	11
050	95	=
051	99	PRT
052	42	STO
053	24	24
054	50	I×I
055	32	X:T
056	09	9
057	00	0
058	67	EQ
059	35	1/X
060	02	02
061	07	07
062	00	0
063	67	EQ
064	25	CLR
065	00	0
066	67	EQ
067	78	Σ+
068	01	1
069	08	8
070	00	0
071	22	INV
072	67	EQ
073	01	01
074	33	33
075	76	LBL
076	78	Σ+
077	43	RCL
078	13	13
079	50	I×I
080	32	X:T
081	43	RCL
082	24	24
083	22	INV
084	77	GE
085	00	00
086	04	04
087	86	STF
088	02	02
089	43	RCL
090	13	13
091	42	STO
092	29	29
093	42	STO
094	30	30
095	02	2
096	09	9
097	42	STO
098	31	31
099	04	4

100	42	STO
101	32	32
102	03	3
103	42	STO
104	33	33
105	71	SBR
106	45	YX
107	42	STO
108	29	29
109	94	+/-
110	42	STO
111	27	27
112	02	2
113	03	3
114	42	STO
115	38	38
116	02	2
117	07	7
118	42	STO
119	39	39
120	02	2
121	09	9
122	42	STO
123	34	34
124	05	5
125	42	STO
126	36	36
127	06	6
128	42	STO
129	24	24
130	42	STO
131	02	02
132	91	R/S
133	22	INV
134	86	STF
135	03	03
136	22	INV
137	86	STF
138	02	02
139	43	RCL
140	24	24
141	30	TAN
142	42	STO
143	24	24
144	42	STO
145	36	36
146	43	RCL
147	12	12
148	94	+/-
149	49	PRD

150	36	36
151	43	RCL
152	11	11
153	44	SUM
154	36	36
155	43	RCL
156	36	36
157	33	X²
158	85	+
159	43	RCL
160	03	03
161	33	X²
162	95	=
163	43	RCL
164	12	12
165	33	X²
166	85	+
167	43	RCL
168	04	04
169	33	X²
170	95	=
171	42	STO
172	34	34
173	22	INV
174	77	GE
175	00	00
176	04	04
177	75	-
178	32	X:T
179	45	=
180	34	FX
181	55	÷
182	43	RCL
183	10	10
184	45	=
185	42	STO
186	21	21
187	54	)
188	75	-
189	53	(
190	43	RCL
191	24	24
192	65	×
193	43	RCL
194	34	34
195	54	)
196	22	INV
197	44	SUM
198	21	21
199	95	=

```
200  42  STO
201  28   28
202  43  RCL
203  03   03
204  30  [illegible]
205  55   -
206  43  RCL
207  34   34
208  95   =
209  49  PRD
210  27   27
211  49  PRD
212  28   28
213  43  RCL
214  27   27
215  42  STO
216  29   29
217  43  RCL
218  28   28
219  42  STO
220  30   30
221  43  RCL
222  24   24
223  49  PRD
224  29   29
225  49  PRD
226  30   30
227  43  RCL
228  36   36
229  44  SUM
230  29   29
231  44  SUM
232  30   30
233  61  GTO
234  02   02
235  91   91
236  76  LBL
237  35  1/X
238  43  RCL
239  12   B
240  50  I×I
241  32  X:T
242  43  RCL
243  03   03
244  22  INV
245  77   GE
246  00   00
247  04   04
248  86  STF
249  03   03
250  43  RCL
251  12   12
252  42  STO
253  27   27
254  42  STO
255  28   28
256  02   2
257  07   7
258  42  STO
259  31   31
260  03   3
261  42  STO
262  32   32
263  04   4
264  42  STO
265  33   33
266  71  SBR
267  45  [illegible]
268  42  STO
269  30   30
270  94  +/-
271  42  STO
272  29   29
273  03   3
274  00   0
275  42  STO
276  33   33
277  02   2
278  09   9
279  42  STO
280  34   34
281  [illegible]
282  [illegible]
283  42  STO
284  34   34
285  06   6
286  42  STO
287  36   36
288  05   5
289  42  STO
290  24   24
291  02   2
292  07   7
293  42  STO
294  31  [illegible]
295  03  [illegible]
296  03  [illegible]
297  42  STO
298  31   31
299  71  SBR
```

500	11	A	550	25	CLR
501	55	−	551	42	STO
502	73	RC*	552	23	23
503	34	√X	553	43	RCL
504	95	=	554	22	22
505	61	GTO	555	61	GTO
506	04	04	556	00	00
507	10	E'	557	29	29
508	76	LBL	558	76	LBL
509	77	GE	559	79	X̄
510	44	SUM	560	22	INV
511	35	35	561	86	STF
512	99	PRT	562	01	01
513	87	IFF	563	43	RCL
514	01	01	564	35	35
515	79	X̄	565	99	PRT
516	22	INV	566	91	R/S
517	87	IFF	567	76	LBL
518	00	00	568	45	YX
519	00	00	569	53	(
520	04	04	570	53	(
521	22	INV	571	01	1
522	97	DSZ	572	75	−
523	01	01	573	53	(
524	89	π	574	73	RC*
525	22	INV	575	31	31
526	86	STF	576	55	÷
527	00	00	577	73	RC*
528	01	1	578	32	32
529	94	+/-	579	54	)
530	49	PRD	580	33	X²
531	18	18	581	54	)
532	49	PRD	582	34	√X
533	19	19	583	65	×
534	49	PRD	584	73	RC*
535	21	21	585	33	33
536	43	RCL	586	54	)
537	20	20	587	92	RTN
538	42	STO	588	76	LBL
539	00	00	589	28	LOG
540	61	GTO	590	53	(
541	00	00	591	53	(
542	00	00	592	73	RC*
543	76	LBL	593	31	31
544	89	π	594	75	−
545	86	STF	595	43	RCL
546	01	01	596	12	12
547	22	INV	597	54	)
548	86	STF	598	33	X²
549	00	00	599	85	+

```
600  53
601  73 [illegible]
602  32 [illegible]
603  75 -
604  43 RCL
605  13 [illegible]
606  54
607  33 [illegible]
608  54 .
609  34 [illegible]
610  92 RTN
```

PART III

000	76	LBL
001	11	A
002	65	×
003	53	(
004	01	1
005	35	1/X
006	54	)
007	42	STO
008	00	00
009	95	=
010	42	STO
011	01	01
012	91	R/S
013	76	LBL
014	12	B
015	65	×
016	43	RCL
017	00	00
018	95	=
019	44	SUM
020	01	01
021	91	R/S
022	76	LBL
023	13	C
024	55	÷
025	43	RCL
026	00	00
027	95	=
028	44	SUM
029	01	01
030	43	RCL
031	01	01
032	99	PRT
033	76	LBL
034	14	D
035	22	INV
036	49	PRD
037	01	01
038	43	RCL
039	01	01
040	99	PRT
041	91	R/S
042	76	LBL
043	15	E
044	22	INV
045	49	PRD
046	01	01
047	43	RCL
048	01	01
049	99	PRT
050	91	R/S

LIST OF REFERENCES

1. Naval Postgraduate School Technical Report NPS55Fo-75041, Some Notes on Search, Detection and Localization Modeling, by R. N. Forrest, p. 28-30, April 1975.

2. Operations Evaluation Group Report 51, Antisubmarine Warfare in World War II, by C. M. Sternhell and A. M. Thorndike, p. 146, 10 April 1946.

3. Barr, D. R. and Zehna, P. W., Probability, p. 254, Wadsworth, 1971.

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