AD-A085 075 ILLINOIS UNIV AT URBANA-CHAMPAIGN DECISION AND CONTROL LAB F/G 9/3
OUTPUT FEEDBACK POLE-PLACEMENT IN THE DESIGN OF COMPENSATORS FO--ETC(U)
JUN 79 W E HOPKINS N00014-79-C-0424

UNCLASSIFIED DC-847 NL

1 OF 2

AD A
085075

REPORT DC-26 JUNE, 1979

COORDINATED SCIENCE LABORATORY

DECISION AND CONTROL LABORATORY

OUTPUT FEEDBACK POLE-PLACEMENT IN THE DESIGN OF COMPENSATORS FOR SUBOPTIMAL LINEAR QUADRATIC REGULATORS

LEVEL II

WILLIAM EDWARDS HOPKINS, JR.

REPORT R-847 UILU-ENG 78-2240

UNIVERSITY OF ILLINOIS – URBANA, ILLINOIS

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER

2. GOVT ACCESSION NO. AD-A085075

3. RECIPIENT'S CATALOG NUMBER 9

4. TITLE (and Subtitle) 6

OUTPUT FEEDBACK POLE-PLACEMENT IN THE DESIGN OF COMPENSATORS FOR SUBOPTIMAL LINEAR QUADRATIC REGULATORS.

5. TYPE OF REPORT & PERIOD COVERED

Technical Report.

6. PERFORMING ORG. REPORT NUMBER

R-847(DC-26); UILU-78-2240

7. AUTHOR(s) 10

WILLIAM EDWARD HOPKINS, JR

8. CONTRACT OR GRANT NUMBER(s) 15

AFOSR-78-3633

~~DOE EY-76-C-01-2088~~

N00014-79-C-0424

9. PERFORMING ORGANIZATION NAME AND ADDRESS

Coordinated Science Laboratory
University of Illinois at Urbana-Champaign
Urbana, Illinois 61801

10. ~~PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS~~

AFOSR-78-36

11. CONTROLLING OFFICE NAME AND ADDRESS

Joint Services Electronics Program

12. REPORT DATE 11

June [redacted]79

13. NUMBER OF PAGES

132

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)

12 142

15. SECURITY CLASS. (of this report)

UNCLASSIFIED

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

14 DC-847, UILU-ENG-78-2240

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

ℓ (1 < or = ℓ < or = n)

$\ell > r$

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Output feedback pole placement
Linear quadratic regulators
Dynamic compensators

$\ell - r$

4-10775

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Based on a modified output regulator problem, a design oriented methodology is presented for the construction of output feedback compensators retaining $\ell(1 \leq \ell \leq n)$ optimal eigenvectors from a reference state feedback regulator. Viewing ℓ as a design parameter, it is known that in the case $\ell > r$ this requires a dynamic compensator of dimension $\ell - r$ whose parameters are determined in function of the solution of an associated output feedback pole-placement problem. Using an iterative dyadic pole-placement procedure, an algorithm is given which determines the solution of this pole-placement --

DD FORM 1473, 1 JAN 73 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

20. Abstract (continued)

problem without a priori assumptions on the compensator dimension. The methodology is also extended to the class of stabilizable systems and the required compensator is shown to possess a separation property. Finally the design methodology is illustrated by three nontrivial examples.

UILU-ENG 78-2240

OUTPUT FEEDBACK POLE-PLACEMENT IN THE DESIGN OF COMPENSATORS FOR SUBOPTIMAL LINEAR QUADRATIC REGULATORS

by

William Edward Hopkins, Jr.

This work was supported in part by the Air Force Office of Scientific Research under Contract AFOSR 78-3633; in part by the Department of Energy under Grant DOE EX-76-C-01-2088; and in part by the Joint Services Electronics Program (U.S. Army, U.S. Navy, and U.S. Air Force) under Contract N00014-79-C-0424.

OUTPUT FEEDBACK POLE-PLACEMENT IN THE DESIGN OF COMPENSATORS FOR SUBOPTIMAL LINEAR QUADRATIC REGULATORS

BY

WILLIAM EDWARD HOPKINS, JR.

B.S., The College of William and Mary in Virginia, 1975

THESIS

Submitted in partial fulfillment of the requirements
for the degree of Master of Science in Electrical
Engineering in the Graduate College of the
University of Illinois at Urbana-Champaign, 1979

Thesis Advisor: Professor W. R. Perkins

Urbana, Illinois

ABSTRACT

Based on a modified output regulator problem, a design oriented methodology is presented for the construction of output feedback compensators retaining $\ell (1 \leq \ell \leq n)$ optimal eigenvectors from a reference state feedback regulator. Viewing ℓ as a design parameter, it is known that in the case $\ell > r$ this requires a dynamic compensator of dimension $\ell - r$ whose parameters are determined in function of the solution of an associated output feedback pole-placement problem. Using an iterative dyadic pole-placement procedure, an algorithm is given which determines the solution of this pole-placement problem without a priori assumptions on the compensator dimension. The methodology is also extended to the class of stabilizable systems and the required compensator shown to posses- a separation property. Finally the design methodology is illustrated by three nontrivial examples.

ACKNOWLEDGEMENTS

The author wishes to thank his advisors Professors W. R. Perkins and J. V. Medanic for their assistance. Special gratitude is acknowledged for Professor Medanic's suggestion of the research problem and for his encouragement and direction, without which the completion of this work would not have been possible.

The author also wishes to thank the members of the Decision and Control Laboratory for many helpful discussions, and Rose Harris and Wilma McNeal for an excellent preparation of the manuscript.

TABLE OF CONTENTS

LIST OF TABLES

Table Page

LIST OF FIGURES

INTRODUCTION

In the design of linear quadratic regulators for time-invariant systems, the inaccessibility of state variables precludes the implementation of optimal state feedback control laws. Two approaches to the resolution of this difficulty are the reconstruction of unmeasured states by reduced order observers and the reformulation of the optimization problem as an output feedback quadratic regulator problem. The introduction of reduced order observers allows the retention of the optimal state feedback control law, but may often result in the use of a compensator of much higher dimension than is actually needed to satisfactorily control the system. The reformulation of the problem under output feedback suffers a more severe disadvantage. Whereas there is a wealth of literature available on the state feedback regulator and associated state reconstruction problems, or estimation and filtering problems in the presence of noise, very little is known about the existence and properties of solutions to the output feedback regulator problem. It is therefore of interest to develop implementable regulators which retain some measure of optimality provided by the state feedback regulator problem, without requiring the use of compensators of high dimension.

One solution to this problem is to design compensators for output feedback regulators which retain as many optimal eigenvectors of the corresponding state feedback regulators as possible. Such regulators have the properties of achieving the optimal cost in the subspace spanned by the retained eigenvectors, and of providing an easily computed measure of the cost degradation in the remaining state space. It is known that retention of $r+p$ optimal eigenvectors, where r is the rank of the output matrix, requires the construction of a dynamic compensator of dimension p, and that the compensator parameters may be determined in function of the solution of an associated output feedback pole-placement problem [1].

Following a review of a design oriented methodology for the construction of output feedback regulators which retain $r+p$ optimal eigenvectors $(p \geq 0)$, an algorithm will be presented which solves the associated pole-placement problem and determines the dimension p of the required compensator without a priori assumptions. In the event the system is stabilizable by static output feedback, rather than construct a dynamic compensator, it may be preferable to relax the requirement of retention of $r+p$ optimal eigenvectors. The problem of retention of fewer than r optimal eigenvectors will therefore be considered and shown to also give rise to an output feedback pole-placement problem. Finally it will be shown that the design methodology may be extended to the class of stabilizable systems.

In chapter one dyadic solutions to the static output feedback pole-placement problem are reviewed. The second chapter presents the methodology for the design of suboptimal linear quadratic regulators which retain $\ell(1 \leq \ell \leq n)$ optimal eigenvectors from the state feedback regulator. Based on the methodology proposed in [2] for solving the general output feedback pole-placement problem, an algorithm is obtained in chapter three which solves the pole-placement problem associated with the design of suboptimal regulators. In the last chapter three nontrivial examples illustrate the design methodology.

CHAPTER 1

DYADIC SOLUTIONS TO THE OUTPUT FEEDBACK POLE-PLACEMENT PROBLEM

Since the problem of eigenvalue assignment by state feedback was resolved [3] the related problem of eigenvalue assignment by output feedback has been the subject of extensive research. Aside from numerical methods, approaches to the problem may be characterized as giving rise to either dyadic or full rank feedback matrices.

In the dyadic approach the nonlinear equations which describe the complete solution to the problem are rendered linear by arbitrarily fixing certain otherwise free parameters, and the feedback gain matrix is obtained as a sum of dyadic products. Dyadic solutions have been obtained for systems represented in state space [4], [5], [6], [7] and for systems described by transfer functions [8], [9], [10], [11].

Other approaches to the problem attempt to utilize the freedom discarded in the dyadic approach, typically to assign eigenvectors as well as eigenvalues, and while usually resulting in full rank feedback matrices, tend to give rise to solutions which are numerically difficult. Extensive results have been obtained using geometric approaches [12], [13], [14]. Other approaches include the generalized root locus [15], [16], [17], the use of generalized inverses [18], [19] and the use of Kronecker products [20].

Because of the difficulty of obtaining explicit solutions to the output feedback pole placement problem, many numerical procedures have been proposed. In view of the fact that in general only $m+r-1$ eigenvalues may be arbitrarily assigned while nothing can be said about the resulting locations of the remaining eigenvalues, it has been suggested that the problem be reformulated as an optimization problem to minimize the deviations of all n eigenvalues from their desired locations [21], [22]. Other authors have considered the problem of requiring the closed loop eigenvalues to lie in prescribed regions of the complex plane [23], [24]. This leads naturally to recasting the problem as an output feedback linear quadratic regulator problem [24], but theoretical results are lacking. In chapter 2 one approach to designing suboptimal linear quadratic regulators will be seen to lead directly back to the output feedback pole-placement problem.

In this chapter dyadic solutions to the output feedback pole-placement problem are discussed in detail as they will be used in chapter 3 to solve the pole-placement problem associated with the design of suboptimal regulators.

Let the triple (A,B,C) represent the linear time-invariant system

$$\dot{x} = Ax + Bu \ , \quad y = Cx \ , \quad A \in R^{n \times n}, \ B \in R^{n \times n}, \ C \in R^{r \times n} \qquad (1.1)$$

where B and C are assumed to be of full rank. Then except for certain singular systems it is known that $\max(n, m+r-1)$ eigenvalues may be assigned "almost" arbitrarily by output feedback of the form $u = Ky$; $K \in R^{m \times r}$. The proof of this is constructive and relies on the fact that uncontrollable and/or unobservable eigenvalues of a system are invariant under output feedback. The idea is that, after an initial feedback placing some eigenvalues at their desired locations, the system may be collapsed to a single input or output system having $\min(m,r)-1$ of these eigenvalues uncontrollable or unobservable. The remaining $\min(\max(m,r), n-\min(m,r)+1)$ desired eigenvalues may then be assigned by a further feedback. Formulas for the computation of the required gains are given in the proof of Theorem 1.2 below, and Theorem 1.3 gives the final desired gain as a sum of dyadic products.

The following definitions make precise the use of the word "almost".

Definition 1.1 [26]: Let $\{\varphi_i(\bar{x})\}_{i=1}^{N}$ be a finite set of real valued polynomials taking their arguments in a parameter space R^n. Then the set of common zeros $V = \{\bar{x} \in R^n : \varphi_i(\bar{x}) = 0,\ 1 \leq i \leq N\}$ is called a proper variety provided $V \neq R^n$. □

Definition 1.2 [26]: A property is said to hold for almost all points in a parameter space R^n provided the set of points at which the property fails to hold is the union of a finite number of proper varieties. □

Also are needed:

Definition 1.3 [13]: A set Γ_ℓ of ℓ complex numbers is said to be a symmetric set provided $\lambda \in \Gamma_\ell$ if and only if $\lambda^* \in \Gamma_\ell$. □

Definition 1.4: The matrix A is said to have ℓ assignable eigenvalues if for almost all symmetric sets Γ_ℓ there exists an output feedback matrix K such that $\Gamma_\ell \subseteq \sigma(A + BKC)$.

A preliminary result will permit the avoidance of any discussion of the complications which arise if the matrix A has repeated eigenvalues.

Theorem 1.1 [27]: If (A,B,C) is a controllable and observable triple then for almost all matrices K, the eigenvalues of $A + BKC$ will be distinct.

Proof: See [27]. □

The proof of the next theorem provides formulas for computing the factors of dyadic output feedbacks. A result that will be needed in the proof is contained in Lemma 1.1. To simplify the notation, let $|A|$ denote the determinant of A.

Lemma 1.1 [28]: If $x, y \in R^{n \times 1}$ are column vectors then $|I_n + xy^T| = (1 + y^T x)$.

<u>Proof</u>: Let $V \in R^{nx(n-1)}$ have as columns any basis for the null space of y^T. Then $(I_n + xy^T)V = V$ and it follows that 1 is an eigenvalue of $I_n + xy^T$ with algebraic multiplicity $n-1$. Letting λ denote the remaining eigenvalue of $I_n + xy^T$ then

$$|I_n + xy^T| = \lambda \ , \qquad \text{trace}(I_n + xy^T) = \lambda + (n-1) \tag{1.2}$$

and since $\text{trace}(I_n + xy^T) = n + \langle x,y \rangle$ there follows

$$|I_n + xy^T| = \text{trace}(I_n + xy^T) - (n-1) = (1 + y^T x) \tag{1.3}$$ □

<u>Theorem 1.2 [4],[6]</u>: If (A,B,C) is a controllable and observable triple with C having full rank r and A having distinct eigenvalues, then for any symmetric set Γ_r there exists a feedback matrix K such that r eigenvalues of $A + BKC$ are arbitrarily close to the elements of the set Γ_r.

<u>Proof</u>: Let $\Gamma_r = \{\lambda_i\}_{i=1}^{r}$ be a symmetric set of complex numbers to be assigned to the spectrum of $A + BKC$. By Theorem 1.1 it may be assumed without loss of generality that the eigenvalues of A are distinct. The theorem is proved by finding a vector f such that the pair (A,Bf) is controllable and then solving the pole-placement problem for the single input system (A,Bf,C). The final feedback gain will be a dyadic product $K = fg$.

Let $T \in R^{n \times n}$ transform the matrix A to Jordan form so that

$$T^{-1}AT = dg[\sigma_1 \ldots \sigma_n] \ , \quad T^{-1}B = [T^{-1}b_1 \vdots \ldots \vdots T^{-1}b_m] \tag{1.4}$$

and note that by the assumption of controllability, each row of $T^{-1}B$ has at least one nonzero entry. Select $f \in R^{m \times 1}$ such that no entry of $T^{-1}Bf$ is zero. This is always possible, as the set of vectors f such that some entry of $T^{-1}Bf$ is zero is the union of m proper (linear) varieties. With $b = Bf$ the single input system (A,b,C) is then controllable.

Let the characteristic equation of A be

$$p_o(\lambda) = \sum_{i=0}^{n} a_i \lambda^i , \quad a_n = 1 \tag{1.5}$$

and by Leverrier's algorithm write

$$(\lambda I-A)^{-1} = \frac{1}{p_o(\lambda)} \sum_{i=0}^{n-1} F_i \quad \lambda^{n-i-1} , \quad F_i = \sum_{j=0}^{i} a_{n-j} A^{i-j} \tag{1.6}$$

By Lemma 1.1 the closed loop characteristic equation under feedback $u = gy$, $g \in R^{1 \times r}$, may be written

$$\begin{aligned} p_c(\lambda) &= |\lambda I-(A+bgC)| = |\lambda I - bgC(\lambda I-A)^{-1}|\,|\lambda I-A| \\ &= (1 - gC(\lambda I-A)^{-1}b)p_o(\lambda) \qquad (1.7) \\ &= p_o(\lambda) - gC \sum_{i=0}^{n-1} \sum_{j=0}^{i} A^{i-j} b a_{n-j} \lambda^{n-i-1} \end{aligned}$$

Changing the summation indices to $k = i-j$, $\ell = i$ the equation becomes

$$p_c(\lambda) = p_o(\lambda) - gCQRs(\lambda) \tag{1.8}$$

where

$$Q = [b \vdots Ab \vdots \dots \vdots A^{n-1}b], \; R = \begin{bmatrix} 1 & a_{n-1} & \cdots & a_1 \\ & 1 & \ddots & \vdots \\ & & \ddots & a_{n-1} \\ 0 & & & 1 \end{bmatrix}, \; s(\lambda) = \begin{bmatrix} \lambda^{n-1} \\ \vdots \\ \lambda \\ 1 \end{bmatrix} \tag{1.9}$$

Defining $P = [p_o(\lambda_1) \cdots p_o(\lambda_r)] \in R^{r \times 1}$ and $S = [s(\lambda_1) \vdots \cdots \vdots s(\lambda_r)] \in R^{n \times r}$ and constraining $p_c(\lambda_i) = 0$ for $i = 1, \ldots, r$ results in the square linear system of equations

$$gCQRS = P \tag{1.10}$$

which will always have a solution g except for those choices of Γ_r for which $|CQRS| = 0$. If this determinant is zero and the equations are inconsistent then an arbitrarily small perturbation of the λ_i will result in a consistent set of equations, but it should be noted that in practice this will result in arbitrarily large gains in some or all of the entries of g. The theorem is proved on writing the final feedback as $K = fg$. □

By duality there follows immediately from Theorem 1.2:

Corollary 1.2.1 [4]: If (A,B,C) is a controllable and observable triple and B and C are of full rank, then for any symmetric set Γ_p of $p = \max(m,r)$ complex numbers there exists a matrix K such that the eigenvalues of $A + BKC$ are arbitrarily close to the elements of Γ_p. □

It should be noted that the selection of f in the proof of the theorem was completely arbitrary and represents precisely that loss of freedom in the specification of a dyadic feedback K which results in a linear system of equations. Also the requirement of controllability of the pair (A,B) may be relaxed to the requirement that (CQRS) be invertible, but conditions on the matrices A, B, f under which this will be true are not known. That the matrix may fail to be invertible for some uncontrollable systems will lead to the failure of Theorem 1.3 to hold for all triples (A,B,C). Finally it is remarked that the method of the proof of Theorem 1.3

fails in the case Γ_r contains repeated eigenvalues. A straightforward modification of the equations to handle this case is given in [4] and may be of interest, for example, in solving minimum time problems for discrete systems.

In general, for any fixed f the equation $|CQRS| = 0$ defines a proper variety of eigenvalues Γ which are not assignable by any finite output feedback. The following example illustrates this point.

Example 1.1 [12]: Let (A,B,C) be given by

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{1.11}$$

The open loop characteristic equation is $p_o(\lambda) = \lambda^3 + \lambda^2 - 1$. For any given $f = (f_1, f_2)^T$ the gain $K = fg$ is defined by (1.10):

$$(g_1, g_2) \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & f_2 & f_1 - f_2 \\ f_1 & 0 & f_2 \\ f_2 & f_1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^2 & \lambda_2^2 \\ \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} =$$

$$(\lambda_1^3 + \lambda_1^2 - 1, \ \lambda_2^3 + \lambda_2^2 - 1) \tag{1.12}$$

and the equation

$$0 = |CQRS| = (\lambda_2 - \lambda_1)(f_1^2(\lambda_1\lambda_2 + \lambda_1 + \lambda_2 + 1) - f_2^2(\lambda_1 + \lambda_2 + 1) - f_1 f_2) \tag{1.13}$$

defines those symmetric sets (λ_1, λ_2) which are not assignable, though as noted above, the equations may be modified to handle the case $\lambda_1 = \lambda_2$. □

The main theorem may now be given.

Theorem 1.3 [5],[7]: For almost all controllable and observable triples (A,B,C) and for any symmetric set Γ_p of $p = \min(n, m+r-1)$ complex numbers, there exists a matrix K such that the eigenvalues of $A+BKC$ are arbitrarily close to the elements of Γ_p.

This result was obtained in [5], [7] and [12] and provides the best known bound on the number of eigenvalues assignable by output feedback. For a geometric proof which constructs a full rank feedback the interested reader is referred to [12].

Proof: Let $\Gamma_p = \{\lambda_i\}_{i=1}^p$ be a symmetric set of $p = \min(n, m+r-1)$ complex numbers to be assigned to the spectrum of $A+BKC$. By Theorem 1.1 it may be assumed that the eigenvalues of A are distinct. The theorem will be proved by constructing a rank 2 gain $K = f_1 g_1 + f_2 g_2$ where f_1 will be arbitrary, g_1 chosen to place at least r-1 eigenvalues, g_2 to render those r-1 eigenvalues unobservable, and f_2 to place the remaining $\min(n-r+1, m)$ eigenvalues.

With $\hat{A}$, b, c, $\{a_i^1\}_{i=1}^n$, $\{a_i^2\}_{i=1}^n$, $p_o^1(\lambda)$, $p_o^2(\lambda)$ to be defined below let:

$$Q_1 = [b \vdots Ab \vdots \dots \vdots A^{n-1}b] \qquad Q_2 = [c^T \vdots \hat{A}^T c^T \vdots \dots \vdots (\hat{A}^{n-1})^T c^T]^T$$

$$R_1 = \begin{bmatrix} 1 & a_{n-1}^1 & \cdots & a_1^1 \\ & 1 & \ddots & \vdots \\ & 0 & \ddots \quad 1 & a_{n-1}^1 \\ & & & 1 \end{bmatrix} \quad R_2 = \begin{bmatrix} 1 & & & \\ a_{n-1}^2 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \\ \vdots & & \ddots & 1 \\ a_1^2 & \cdots & \cdots & a_{n-1}^2 \end{bmatrix} \quad S_1 = \begin{bmatrix} \lambda_1^{n-1} & \cdots & \lambda_r^{n-1} \\ \vdots & & \vdots \\ \lambda_1 & & \lambda_r \\ 1 & \cdots & 1 \end{bmatrix}$$

$$S_2 = \begin{bmatrix} \lambda_r^{n-1} & \cdots\cdots & \lambda_r & 1 \\ \vdots & & & \\ \lambda_{\min(n,m+r-1)}^{n-1} & \cdots\cdots & \lambda_{\min(n,m+r-1)} & 1 \end{bmatrix} \quad P_1 = [p_o^1(\lambda_1) \cdots p_o^1(\lambda_r)] \tag{1.14}$$

$$P_2 = [p_o^2(\lambda_r) \cdots p_o^2(\lambda_{\min(n,m+r-1)})]^T$$

As in the proof of Theorem 1.2 let $f_1 \in R^{m \times 1}$ be any vector such that the pair (A,b), $b \triangleq Bf_1$, is controllable, and denote the open loop characteristic polynomial of A by

$$p_o^1(\lambda) = \sum_{i=0}^{n} a_i^1 \lambda^i ; \; a_n^1 = 1 \tag{1.15}$$

Then by Theorem 1.2, the solution $g_1 \in R^{1 \times r}$ of the equation

$$g_1[C \; Q_1 R_1 S_1] = P_1 \tag{1.16}$$

will assign the eigenvalues $\{\lambda_i\}_{i=1}^{r}$ to the spectrum of $\hat{A} = A + Bf_1 g_1 C$, subject to a possible perturbation of the numbers λ_i to ensure the consistency of the equations.

Recalling that unobservable eigenvalues are invariant under output feedback let $V_1 \in R^{nx(r-1)}$ and $V_2 \in R^{nx(n-r+1)}$ have as columns the eigenvectors of $\hat{A}$ corresponding respectively to $\{\lambda_i\}_{i=1}^{r-1}$ and the remaining eigenvalues of $\hat{A}$. Then if $g_2 CV_1 = 0$, the single output system $(\hat{A},B,c)$, $c \triangleq g_2 C$, will have r-1 unobservable eigenvalues $\{\lambda_i\}_{i=1}^{r-1}$. The number of nonzero entries of the vector $g_2 CV_2$ will be the number of eigenvalues of $\hat{A}$ subject to influence under further feedback. A solution to $g_2 CV_2 = 0$ always exists as it requires finding an r-vector orthogonal to r-1 other r-vectors, however conditions under which at least min(m,n-r+1) entries of $g_2 CV_2$ will be nonzero are not known.

Let the characteristic polynomial of $\hat{A}$ be

$$p_o^2(\lambda) = \sum_{i=0}^{n} a_i^2 \lambda^i \;\; ; \;\; a_n^2 = 1 \tag{1.17}$$

Applying the dual of Theorem 1.2, the solution f_2 of the equations

$$[S_2 R_2 Q_2 B] f_2 = P_2 \tag{1.18}$$

will assign the remaining eigenvalues $\{\lambda_i\}_{i=r}^{\min(n,m+r-1)}$ to the spectrum of $\hat{A}+Bf_2g_2C$, provided the equations are consistent. The final dyadic feedback will then be $K = f_1g_1 + f_2g_2$. If the equations are inconsistent, then a perturbation of the eigenvalues to be assigned may possibly fail to make $[S_2R_2Q_2B]$ invertible, since the factor R_2Q_2B cannot be guaranteed to be of full rank, the triple (A,B,C) not being observable. □

Since rank-deficient compensators have poor disturbance rejection properties in applications, the procedure of this proof may be iterated to obtain full rank feedback matrices as sums of min(m,r) dyadic products [2]. This modification provides the basis for the algorithm given in section 3.1 and will be discussed in detail there.

The following example illustrates the possibility of a system structurally failing to allow the assignment of min(n,m+r-1) eigenvalues.

Example 1.2: Let $A_1, A_2 \in R^{2 \times 2}$ and $b_1, b_2, c_1, c_2 \in R^{2 \times 1}$ and consider the system

$$\left(\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}, \begin{bmatrix} c_1^T & 0 \\ 0 & c_2^T \end{bmatrix} \right) \tag{1.19}$$

Under output feedback $u = Ky$, $K \in R^{2 \times 2}$ the closed loop system is

$$\begin{bmatrix} A_1 + k_{11}b_1c_1^T & k_{12}b_1c_2^T \\ k_{21}b_2c_1^T & A_2 + k_{22}b_2c_2^T \end{bmatrix} \tag{1.20}$$

The result of Theorem 1.3 would predict that $\min(n, m+r-1) = 3$ eigenvalues may be assigned. However if either $b_1c_2^T = 0$ or $b_2c_1^T = 0$ the closed loop system is block triangular and only $\max(m,r) = 2$ eigenvalues may be assigned, one to each diagonal block by the choice of k_{11}, k_{22}. □

An alternate characterization of the vectors f_i, g_i in the proof of Theorem 1.3 is available. Rather than requiring $p_c(\lambda_i) = 0$, $i = 1,\ldots,\max(m,r)$ in the proof of Theorem 1.2, the desired closed loop characteristic equation may be constrained to be of the form

$$p_c(\lambda) = p_1(\lambda)p_2(\lambda), \quad p_1(\lambda) = \prod_{i=1}^{\max(m,r)} (\lambda-\lambda_i) \tag{1.21}$$

giving rise to a system of equations which determine not only the feedback gain but also the remaining spectrum of the closed loop system as the roots of $p_2(\lambda)$ [8]. Since this will require the computation of the Markov parameters CA^iB, a pole-placement procedure incorporating this approach will be referred to in this thesis as a frequency domain solution, whereas the procedure contained in the proofs of Theorems 1.2 and 1.3 will be referred to as a state space solution. By duality it suffices to define a frequency domain solution by indicating formulas for pole-placement in single-input systems and for rendering poles uncontrollable.

Let the system (A,B,C) have transfer function

$$C(\lambda I-A)^{-1}B = N(\lambda) \;/\; d(\lambda) \;,\; d(\lambda) = \sum_{i=0}^{n} a_i\lambda^i \;,\; a_n = 1 \tag{1.22}$$

and assume that the eigenvalues of A are distinct. Using Leverrier's algorithm the transfer function may be written in terms of the Markov parameters CA^iB:

$$\begin{aligned} N(\lambda) &= C\ \mathrm{adj}(\lambda I-A)B = C(\sum_{i=0}^{n-1}\sum_{j=0}^{i} a_{n-j}A^{i-j}\lambda^{n-i-1})B \\ &= \sum_{k=0}^{n-1}(\sum_{j=0}^{n-k-1} a_{n-j}CA^{(n-k-1)-j}B)\lambda^k \end{aligned} \tag{1.23}$$

Defining $N_k = \sum_{j=0}^{n-k-1} a_{n-j}CA^{n-k-1-j}B$ equation (1.22) becomes

$$C(\lambda I-A)^{-1}B = (\sum_{i=0}^{n-1} N_i\lambda^i) \;/\; d(\lambda) \tag{1.24}$$

The following lemma is needed in obtaining single input systems with prescribed uncontrollable eigenvalues.

<u>Lemma 1.2 [11]</u>: If λ_o is a root of $d(\lambda)$ then rank $N(\lambda_o) \leq 1$.

<u>Proof</u>: Let T transform A to the Jordan form $T^{-1}AT = dg(\lambda_1,\ldots,\lambda_n)$. Then

$$\begin{aligned} N(\lambda) &= C\ \mathrm{adj}(\lambda I-A)B = CT\ \mathrm{adj}(dg(\lambda-\lambda_1,\ldots,\lambda-\lambda_n))T^{-1}B \\ &= CT\ dg(\prod_{i\neq 1}(\lambda-\lambda_i),\ldots,\prod_{i\neq n}(\lambda-\lambda_i))T^{-1}B \end{aligned} \tag{1.25}$$

If $\lambda_o = \lambda_k$ for some k, $1 \leq k \leq n$, then $N(\lambda_o)$ becomes

$$N(\lambda_o) = CT\ dg(0,\ldots,0,\prod_{i\neq k}(\lambda-\lambda_i),0,\ldots,0)T^{-1}B \tag{1.26}$$

and so rank $N(\lambda_o) \leq 1$. □

Recall that a pole of a single input system is uncontrollable if the numerator of the transfer function is zero evaluated at that pole, and let f define the single input system (A,Bf,C) with transfer function

$$H(\lambda) = N(\lambda)f\,/\,d(\lambda)\ , \quad N(\lambda) = \sum_{i=0}^{n-1} N_i\lambda^i\ , \ N_i = \begin{bmatrix} n_{r1} \\ \vdots \\ n_{ri} \end{bmatrix} \in R^{n\times 1} \qquad (1.27)$$

To render m-1 poles $\{\lambda_i\}_{i=1}^{m-1}$ uncontrollable it suffices to satisfy $N(\lambda_i)f = 0$, $i = 1,\ldots,m-1$. If the matrix $M \in R^{m-1\times m}$ is taken to have as its ith row the linearly independent row of $N(\lambda_i)$ then f may be obtained as the solution of $Mf = 0$ [2], [11]. (If $N(\lambda_i) \equiv 0$ then λ_i is already uncontrollable and the ith row may be taken all zeros).

Let $\{\lambda_i\}_{i=1}^{r}$ be the r poles to be assigned by the feedback $g \in R^{1\times r}$ of Figure 1.1. By summing at nodes 1 and 2 in the figure the closed loop transfer function may be written

$$H_c(\lambda) = (I - H(\lambda)g)^{-1}H(\lambda) = H(\lambda)\,/\,(1 - gH(\lambda)) \qquad (1.28)$$

from which the closed loop polynomial is given by

$$p_c(\lambda) = d(\lambda) - gN(\lambda) \qquad (1.29)$$

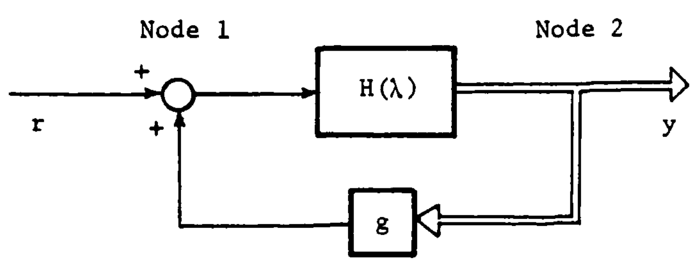


Figure 1.1. Block diagram of transfer function given in (1.28).

From this equation there arise two possibilities for solving for g. Constraining $p_c(\lambda) = 0$, $i = 1,\ldots,r$, gives the system of equations (1.8), (1.9) in the proof of Theorem 1.2:

$$[p_o(\lambda_1),\ldots,p_o(\lambda_r)] = g[N(\lambda_1)f \,\vdots\, \cdots \,\vdots\, N(\lambda_r)f]$$

$$= g\begin{bmatrix} n_{1(n-1)} & \cdots & n_{11} \\ \vdots & & \vdots \\ n_{r(n-1)} & \cdots & n_{r1} \end{bmatrix}\begin{bmatrix} \lambda_1^{n-1} & \cdots & \lambda_r^{n-1} \\ \vdots & & \vdots \\ \lambda_1 & \cdots & \lambda_r \\ 1 & \cdots & 1 \end{bmatrix} \tag{1.30}$$

The second method solves a higher order system of equations but also obtains the coefficients of the polynomial whose roots are the remaining n-r poles of the closed loop system. Let

$$\prod_{i=1}^{r}(\lambda - \lambda_i) = \sum_{i=0}^{r} d_i\lambda^i \;,\; d_r = 1 \tag{1.31}$$

and factor

$$p_c(\lambda) = p_1(\lambda)p_2(\lambda) = (\sum_{i=0}^{r} d_i\lambda^i)(\sum_{i=0}^{n-r} h_i\lambda^i),\; h_{n-r} = 1 \tag{1.32}$$

Then equation (1.29) becomes

$$(\sum_{i=0}^{r} d_i\lambda^i)(\sum_{i=0}^{n-r} h_i\lambda^i) = (\sum_{i=0}^{n} a_i\lambda^i) - (\sum_{i=0}^{n-1} gN_i\lambda^i) \tag{1.33}$$

This may be rewritten

$$\sum_{i=0}^{n-1} gN_i\lambda^i + (\sum_{i=0}^{r} d_i\lambda^i)(\sum_{i=0}^{n-r-1} h_i\lambda^i) = \sum_{i=0}^{n} a_i\lambda^i - \sum_{i=0}^{r} d_i\lambda^{n-r+i} \tag{1.34}$$

Equating coefficients of λ gives the nth order linear system of equations [8]:

$$[g_1,\ldots,g_r \vdots h_0,\ldots,h_{n-r-1}] \times \begin{bmatrix} n_{11} & \cdots & & & & & n_{1r} \\ \vdots & & & & & & \vdots \\ n_{r1} & \cdots & & & & & n_{rr} \\ \hline d_0 & d_1 & \cdots & d_{r-1} & 1 & \cdots & 0 \\ 0 & d_0 & \ddots & & d_{r-1} & 1 \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \ddots\, 1 & \vdots \\ 0 & \cdots & d_0 & d_1 & \cdots & d_{r-1} & 1 \end{bmatrix} \tag{1.35}$$

$$= (a_0,\ldots,a_{n-1}) - (0,\ldots,0,d_0,d_1,\ldots,d_{r-1})$$

These equations determine the gain g, and the remaining spectrum of $A + BfgC$ as the roots of $p_2(\lambda)$. As in Theorem 1.2 it may arise that the coefficient matrix in (1.35) is singular. If the system of equations is inconsistent then an arbitrarily small perturbation of the λ_i will render the coefficient matrix invertible by altering the lower (n-r)xn block.

To illustrate frequency domain and state space solutions of the pole-placement problem reconsider Example 1.1.

Example 1.1 (continued) [12]: Let A,B,C be given by (1.11), and assume the desired closed loop spectrum is $(-2,-1\pm 1j)$. Since $m+r-1=3$, arbitrary pole-placement is possible.

A dyadic feedback $K = f_1g_1 + f_2g_2$ is computed in two stages. At the first stage two poles are placed and at the second stage one of these poles is rendered invariant to further feedback and two additional poles are assigned. There are several possibilities for computing f_1,g_1,f_2,g_2. At the first stage f_1 may be chosen arbitrarily and g_1 computed from (1.16) to place $r=2$ poles, or g_1 may be selected arbitrarily and f_1 computed from

(1.18) to place $m=2$ poles. At the second stage there are also two choices. Either f_2 may be chosen to define a single-input system with m-1 uncontrollable poles and g_2 computed from (1.16) to place r additional poles, or g_2 may be chosen to define a single-output system with r-1 unobservable poles and f_2 computed from (1.18) to place m additional poles. In this example, for both the state space and the frequency domain solutions, r poles will be assigned at the first stage. At the second stage g_2 will be chosen to render r-1 poles unobservable and then f_2 will be computed to place the remaining m poles at their desired locations. Because the complex pair $-1 \pm 1j$ may not be split, $\lambda = -2$ must be assigned at the first stage.

The state space solution is as follows. Arbitrarily select $f_1 = (1,0)^T$ and let the spectrum to be assigned by g_1 be (-2,0). With $p_0^1(\lambda) = \lambda^3 + \lambda^2 - 1$, $\lambda_1 = -2$, $\lambda_2 = 0$ equation (1.16) becomes

$$g_1 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ -2 & 0 \\ 1 & 1 \end{bmatrix} = (-5,-1) \tag{1.36}$$

which has solution $g_1 = (-1,-3)$. The resultant spectrum is $\sigma(\hat{A}) = (0,-2,-2)$ and

$$\hat{A} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -3 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad p_0^2(\lambda) = \lambda^3 + 4\lambda^2 + 4\lambda \tag{1.37}$$

The repeated eigenvalue $\lambda = -2$ has only one eigenvector $V_1 = (-1,-2,1)^T$. To render one eigenvalue at -2 unobservable the equation $g_2 C V_1 = 0$ is solved for $g_2 = (2,1)$. Using $p_o^2(-1 \pm 1j) = -2 \mp 2j$ equation (1.18) becomes

$$\begin{bmatrix} -2j & -1+j & 1 \\ 2j & -1-j & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 4 & 4 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 1 & -1 & -1 \\ -2 & 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} f_2$$

$$= \begin{bmatrix} -2-2j \\ -2+2j \end{bmatrix} \qquad (1.38)$$

Premultipling by $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2}j & \frac{1}{2}j \end{bmatrix}$ to obtain a set of real equations, the solution is $f_2 = (4,-2)^T$. The final feedback is then

$$K = f_1 g_1 + f_2 g_2 = \begin{bmatrix} 7 & 1 \\ -4 & -2 \end{bmatrix} \qquad (1.39)$$

and the spectrum of $A + BKC$ is $(-2, -1 \pm 1j)$.

The frequency domain solution is as follows. Again selecting $f_1 = (1,0)^T$ the transfer function for the single input system (A, Bf_1, C) is

$$H(\lambda) = \frac{1}{\lambda^3 + \lambda^2 - 1} \begin{bmatrix} \lambda + 1 \\ \lambda^2 + \lambda \end{bmatrix} \qquad (1.40)$$

Placing poles at -2,0, (1.31) becomes $p_1(\lambda) = \lambda^2 + 2\lambda$ and (1.35) gives

$$[g_1 \vdots h] \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ \hline 0 & 2 & 1 \end{bmatrix} = (-1,0,1) - (0 \vdots 0,2) \qquad (1.41)$$

The solution is $g_1 = (-1,-3)$ and $h = 2$. The spectrum of $\hat{A}$ is therefore $(0,-2)$ together with the root -2 of $p_2(\lambda) = \lambda + h$. Since the spectrum of $\hat{A}$ is known, its characteristic equation is also known and the transfer function for the system $(\hat{A},B,C)$ may be computed from the Markov parameters (1.23):

$$C(\lambda I-\hat{A})B = N(\lambda)/d(\lambda) = \frac{1}{\lambda^3 + 4\lambda^2 + 4\lambda}\begin{bmatrix} \lambda+1 & \lambda^2+4\lambda+3 \\ \lambda^2+\lambda & -\lambda \end{bmatrix} \tag{1.42}$$

To render the pole at -2 unobservable, g_2 is computed as the solution $g_2 = (2,1)$ of $g_2 N(-2) = 0$:

$$g_2 N(-2) = g_2 \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} = 0 \tag{1.43}$$

The transfer function of the single-output system $(\hat{A},B,g_2C)$ is then

$$H(\lambda) = \frac{1}{\lambda^3+4\lambda^2+4\lambda}(\lambda^2+3\lambda+2, 2\lambda^2+7\lambda+6) \tag{1.44}$$

To place two poles at $-1 \pm 1j$, (1.31) becomes $p_2(\lambda) = \lambda^2+2\lambda+2$, and the dual of (1.35) gives

$$\left[\begin{array}{cc|c} 2 & 2 & 2 \\ 3 & 7 & 2 \\ 1 & 6 & 1 \end{array}\right] \begin{bmatrix} f_2 \\ \cdots \\ h \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \tag{1.45}$$

The solution is $f_2 = (4,-2)^T$, $h = 2$. The final feedback gain K is given by (1.39), and the spectrum of $A + BKC$ is $-1 \pm 1j$ together with the root -2 of $p_2(\lambda) = \lambda + h$. □

CHAPTER 2

DESIGN OF SUBOPTIMAL LINEAR QUADRATIC REGULATORS

This chapter considers the problem of designing suboptimal linear quadratic regulators having the property of retaining any number ℓ $(1 \leq \ell \leq n)$ of eigenvectors at their optimal locations as defined by a reference state feedback regulator problem. The system under consideration will be taken to be

$$\dot{x} = Ax + Bu \ , \quad y = Cx \ , \quad A \in R^{nxn}, \ B \in R^{nxm}, \ C \in R^{rxn} \qquad (2.1)$$

and will be assumed to be controllable and observable.

In the case $\ell \leq r$ the solution will be seen to be given by a static output feedback compensator, while in the case $\ell > r$ a dynamic compensator will be required. In the complete methodology ℓ is viewed as a free parameter to be specified during the design procedure and not fixed a priori. When $\ell \neq r$ there exists freedom in the design, which is translated into the choice of feedback gains if $\ell < r$ and into the choice of parameters of the dynamic compensator if $\ell > r$. In both cases this freedom is used to shape the complementary spectral characteristics of the closed loop system by solving an associated output feedback pole-placement problem, a solution to which will be given in section 3.1.

2.1. Review of Necessary Conditions

In this section solutions to three related linear quadratic regulator problems are presented and their properties reviewed.

Let $Q \in R^{nxn}$, $R \in R^{mxm}$ where $Q = Q^T \geq 0$ and $R = R^T > 0$.

For reference purposes the state feedback linear quadratic regulator problem is defined as:

$$\underset{\substack{u=-Kx,K\in R^{mxn}\\ \dot{x} = Ax+Bu}}{\text{minimize}} \left\{ \frac{1}{2}\int_{o}^{\infty}(x^TQx+u^TRu)dt \right\} \tag{2.1.1}$$

Defining the state feedback Ricatti equation as

$$A^TM+MA-MBR^{-1}B^TM+Q = 0 \tag{2.1.2}$$

it is well known that if $(A,\sqrt{Q})$ is a detectable pair then the minimizing control law is given by $u=-Kx$ where $K=R^{-1}B^TM$ and M is the unique symmetric positive definite solution of (2.1.2). If $(A,\sqrt{Q})$ is not a detectable pair and none of the eigenvalues of the matrix

$$E = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \tag{2.1.3}$$

lie on the imaginary axis, then (2.1.2) has at least two positive semi-definite solutions M. More generally, if E has p unobservable eigenvalues with positive real parts then there are at least 2p such solutions, including the optimal solution M_o and the unique stabilizing solution M_s, and they satisfy [29]

$$0\le M_o \le M \le M_s \tag{2.1.4}$$

In particular, for the minimum energy problem $(Q=0,\ R=I)$ the minimizing solution is $M_0=0$, and if A is unstable (2.1.2) has a unique stabilizing solution M_s with the property of "reflecting" unstable eigenvalues of A about the imaginary axis.

The output feedback regulator problem is formulated under the assumption that the initial state x_0 is a zero mean random variable with covariance matrix Q_0 in order to eliminate the dependence of the cost functional upon the initial state. The problem is to

$$\min_{\substack{u=-Ky,\,K\in R^{mxr}\\ \dot{x}=Ax+Bu\\ y=Cx}} E\left\{\frac{1}{2}\int_o^\infty (x^TQx+u^TRu)dt\right\} \tag{2.1.5}$$

Introducing the symmetric positive definite matrices $L\in R^{nxn}$, $M\in R^{nxn}$ defined by

$$L=\int_o^\infty E(xx^T)dt\ ,\quad \frac{1}{2}x_0^TMx_0=\frac{1}{2}\int_o^\infty (x^TQx+u^TRu)dt \tag{2.1.6}$$

the necessary conditions for a solution to the problem are the coupled Ricatti equations:

$$F^TM+MF+Q+C^TK^TRKC=0 \tag{2.1.7a}$$

$$FL+LF^T+Q_0=0 \tag{2.1.7b}$$

$$K=R^{-1}B^TMLC^T(CLC^T)^{-1} \tag{2.1.7c}$$

$$F=A-BKC \tag{2.1.7d}$$

Little is known regarding the existence and properties of solutions of these equations beyond the following sufficiency condition.

Theorem 2.1.1 [30]: If there exists an output feedback matrix $K\in R^{mxr}$ such that $A-BKC$ is a stable matrix, then there exists a solution to equations (2.1.7) for all $Q_0>0$, $R>0$, and $Q\geq 0$ provided the pair $(A,\sqrt{Q})$ is observable. □

Thus if the triple (A,B,C) may be stabilized by output feedback, then the output feedback regulator problem has a solution. Furthermore, using the stabilizing matrix K_0 as an initial guess for the solution of (2.1.7), the following numerical iteration scheme has been proposed [25]:

$$F_i^T M_{i+1} + M_{i+1} F_i + Q + C^T K_i^T R K_i C = 0, \quad F_i = A - BK_i C \tag{2.1.8a}$$

$$K_{i+1} = R^{-1} B^T M_{i+1} L_{i+1} C^T (C L_{i+1} C^T)^{-1} \tag{2.1.8b}$$

$$F_{i+1} L_{i+1} + L_{i+1} F_{i+1}^T + Q_0 = 0 \tag{2.1.8c}$$

Given K_i, (2.1.8a) is solved for M_{i+1} which determines $K_{i+1} = K_{i+1}(L_{i+1})$ by (2.1.8b). Solving equation (2.1.8c) for L_{i+1} gives K_{i+1} numerically and completes the ith iteration. In practice this scheme frequently converges but there is no general convergence proof.

For later purposes it is noted here that if an arbitrary output feedback K is applied to the triple (A,B,C) the associated cost is $\frac{1}{2}\mathrm{trace}(MQ_0)$ where $M = M(K)$ is the solution of (2.1.7a). If M, L is a solution of (2.1.7a-d) then of course this is the optimal cost.

In order to obtain further insight into the properties of the output feedback regulator problem, the problem has recently been reformulated so as to eliminate the dependence of M and L on the covariance matrix Q_0 [31]. Noting that equation (2.1.7a) may be rewritten as

$$A^T M + MA + MBR^{-1} B^T M + Q + W(M,K) = 0 \tag{2.1.9a}$$

$$W(M,K) = (R^{-1} B^T M - KC)^T R (R^{-1} B^T M - KC) \tag{2.1.9b}$$

it may be shown that if K^* minimizes $W(M,K)$ then the solution $M^* = M^*(K^*)$

of (2.1.9a) is the optimal solution of the output regulator problem, independent of the distribution of initial states. However, a K^* that makes $W(M,K)$ minimal in the positive semi-definite sense exists only if either C is invertible, or if the pair (A,C) is completely aggregable ($CA = A_0C$ for some A_0) and the weighting matrix Q may be decomposed as $Q = C^TQ_0C$. Both of these cases are equivalent to the state feedback regulator problem , under a transformation of basis in the first case, and under a reduction of state in the later. It was therefore suggested in [31] that K be chosen to minimize the term $R^{-1}B^TM-KC$ in $W(M,K)$ with respect to the matrix norm induced by the inner product (x,Ly) in order to make the contribution of $W(M,K)$ in (2.1.9a) small. For a given L the minimizing K is given by

$$K = R^{-1}B^TMLC^T(CLC^T)^{-1} \tag{2.1.10}$$

and substitution in equation (2.1.9a) yields the equations:

$$A^TM + MA - MBR^{-1}B^TM + Q + (I-P)^TMR^{-1}B^TM(I-P) = 0 \tag{2.1.11a}$$

$$P = LC^T(CLC^T)^{-1}C \tag{2.1.11b}$$

Thus for a given L the necessary conditions for this modified output feedback regulator problem are the existence of a positive definite matrix M satisfying equations (2.1.11). The corresponding feedback gain is given by (2.1.10).

Relating this modified problem to the output feedback regulator problem defined above it is known that if for some (Q,R) for which $(A,\sqrt{Q})$ is observable there exists an $L > 0$ such that (2.1.11) has a positive definite solution M, then for any $Q \geq 0$, $R > 0$, $Q_0 > 0$, $(A, \sqrt{Q})$ observable, the necessary conditions (2.1.7) for the output feedback problem have a solution $L > 0$, $M > 0$ [31].

The properties of solutions of the modified regulator problem will be discussed in the next section.

2.2. Retention of Optimal Invariant Subspaces by Static Output Feedback Compensation

Since the solution of the output feedback regulator problem does not have an analytic characterization it is of interest to obtain sub-optimal output regulators associated with the state regulator. This section gives a solution to the problem of determining output feedback gains which assign ℓ dimensional invariant subspaces $(1 \leq \ell \leq r)$ of the optimal state regulator. Consideration is restricted to those output feedbacks which may be obtained as "generalized projections" of state feedbacks:

$$K_0 = K_s P_1 + P_2 \tag{2.2.1}$$

Let $K_s = R^{-1} B^T M_c$ be the solution to the state feedback regulator problem for a given Q,R,where M_c is the solution of the Ricatti equation, and define the two problems:

a) For $1 \leq \ell < r$ determine an output feedback gain K_0 such that a prescribed ℓ dimensional invariant subspace of $(A-BK_s)$ is also an invariant subspace of $(A-BK_0C)$.

b) For $\ell = r$ determine an output feedback gain K_0 such that an r dimensional invariant subspace of $A-BK_s$ is also an invariant subspace of $A-BK_0C$ and K_0 is optimal with respect to the modified output feedback regulator problem define above.

The first problem has a non-unique solution and may be solved using generalized inverses. This gives rise to an output feedback pole-placement problem which may be used to shape the complementary spectrum of $A-BK_0C$. Thus K_0 may be chosen to retain an ℓ dimensional subspace which is optimal with respect to a state feedback regulator and the remaining freedom in K_0 used to shape the spectrum of the closed loop system.

Let $\{u_i\}_{i=1}^{n}$, $u_k \in R^{nx1}$ and $\{\lambda_i\}_{i=1}^{n}$, $\lambda_i \in \mathcal{C}^1$ be the eigenvectors and eigenvalues of the optimal closed closed loop system $F = A - BK_s$, where the first ℓ eigenvectors span the ℓ dimensional invariant subspace of F to be assigned to $A - BK_0C$. It is assumed that $(\lambda_\ell, \lambda_{\ell+1})$ is not a complex pair. In order to work over the reals define a transformation T_r by

$$[\mathrm{row}_i(T_r)] = (0,\ldots,0,\underset{\uparrow \text{ ith position}}{1},0,\ldots,0) \qquad \text{if } \lambda_i \text{ is real}$$

$$\begin{bmatrix} \mathrm{row}_i(T_r) \\ \mathrm{row}_{i+1}(T_r) \end{bmatrix} = \begin{bmatrix} 0,\ldots,0,\frac{1}{2},-\frac{1}{2}j,0,\ldots,0 \\ 0,\ldots,0,\underset{\uparrow \text{ ith position}}{\frac{1}{2}},\frac{1}{2}j,\ 0,\ldots,0 \end{bmatrix} \quad \text{if } (\lambda_i,\lambda_{i+1}) \text{ are a complex pair} \tag{2.2.2}$$

Then under T_r the complex pairs $\begin{pmatrix} \sigma + j\omega & 0 \\ 0 & \sigma - j\omega \end{pmatrix}$ and $[u + jv \vdots u - jv]$ are mapped to $\begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$ and $[u \vdots v]$ respectively.

Defining

$$U_\ell = (u_1 \vdots \ldots \vdots u_\ell)T_r\ , \quad \Lambda_\ell = T_r^{-1}\, dg(\lambda_1,\ldots,\lambda_\ell)T_r \tag{2.2.3}$$

the problem may be stated as that of finding a gain K_o and a matrix P such that

$$\begin{aligned} (A - BK_oC)U_\ell &= U_\ell \Lambda_\ell \\ (A - BK_sP)U_\ell &= U_\ell \Lambda_\ell \end{aligned} \tag{2.2.4}$$

or equivalently, assuming B is of full rank, such that

$$
\begin{aligned}
K_o C U_\ell &= K_s U_\ell \\
K_s P &= K_o C
\end{aligned}
\tag{2.2.5}
$$

The general solution of equations (2.2.5) is

$$
\begin{aligned}
K_o &= K_s U_\ell [CU_\ell]^1 + X[I_r - CU_\ell (CU_\ell)^1], \quad X \in R^{mxr} \\
P &= K_s^1 K_o C
\end{aligned}
\tag{2.2.6}
$$

where A^1 denotes any 1-inverse of A^* [32]. By assumption (A,C) is observable and if CU_ℓ has rank $\ell < r$ one choice for $(CU_\ell)^1$ is $[(CU_\ell)^T (CU_\ell)]^{-1} (CU_\ell)^T$. It should be noted that $(CU_\ell)^1 (CU_\ell) = I_\ell$. If also $(A, \sqrt{Q})$ is observable then $K_s = R^{-1} B^T M_c$ is of full rank since $M_c > 0$, and the 1-inverse of K_s may be taken to be $K_s^1 = K_s^T (K_s \; K_s^T)^{-1}$, which satisfies $K_s K_s^1 = I_m$. However, if $(A, \sqrt{Q})$ is not observable, and in particular in the case of the minimum energy problem, M_c and K_s need not be of full rank and the formula for K_s^1 is replaced by

$$
K_s^1 = Y \begin{bmatrix} K_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} Z \quad , \quad ZK_s Y = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \tag{2.2.7}
$$

where $K_{11} \in R^{pxp}$, $Y \in R^{mxm}$, $Z \in R^{nxn}$, and rank $(K_s) = p$.

Since rank $(I_r - CU_\ell (CU_\ell)^1) = r - \ell$ there are only $m(r-\ell)$ degrees of freedom in the matrix X in (2.2.6). To eliminate the redundancy let

---

*X is a 1-inverse of A if $AXA = A$.

$S \in R^{(r-\ell)xr}$ be of full rank and define $Y \in R^{mx(r-\ell)}$ by $X = YS$. Also note that since P is to be eventually multiplied on the left by K_s, the factor $K_s^1 K_s$ in (2.2.6) may be omitted. The expression for P then becomes

$$P = U_\ell (CU_\ell)^1 C + K_s^1 YS(I_r - CU_\ell (CU_\ell)^1)C \qquad (2.2.8)$$

Defining the matrices:

$$\begin{aligned} A_0 &= A - BK_s U_\ell (CU_\ell)^1 C \\ B_0 &= -BK_s K_s^1 \ (= -B \text{ if } K_s \text{ is of full rank}) \\ C_0 &= S(I_r - CU_\ell (CU_\ell)^1)C \end{aligned} \qquad (2.2.9)$$

and substituting (2.2.8) into $(A-BK_sP)$ gives the output feedback pole-placement problem of finding $Y \in R^{mx(r-\ell)}$ such that the spectrum of $A_0 + B_0YC_0$ is satisfactory.

Since by construction $A_0 + B_0YC_0$ contains an ℓ dimensional invariant subspace it is possible to exploit this fact to reduce the dimensionality of the problem. Let $T = [T_1 \vdots T_2]$ be any invertible matrix with $T_1 = U_\ell$ and $T_2 \in R^{(n-\ell)xn}$. Applying the transformation T to the triple (A_0, B_0, C_0) gives:

$$T^{-1}A_0 T = \left[\begin{array}{c|c} A_\ell & X \\ \hline 0 & A_1 \end{array}\right], \quad T^{-1}B_0 = \left[\begin{array}{c} X \\ \hline B_1 \end{array}\right], \quad C_0 T = [0 \vdots C_1] \qquad (2.2.10)$$

where $A_1 \in R^{(n-\ell)x(n-\ell)}$, $B_1 \in R^{(n-\ell)xm}$, $C_1 \in R^{rx(n-\ell)}$. The pole-placement problem to be solved is then that of satisfactorily shaping the spectrum of $A_1 + B_1YC_1$ by an appropriate $Y \in R^{mx(r-\ell)}$. The final output feedback gain will be given by

$$K_0 = K_s U_\ell (CU_\ell)^{1} + YS[I_r - CU_\ell (CU_\ell)^{1}] \tag{2.2.11}$$

and the corresponding projection matrix P by (2.2.8).

In summary equation (2.2.11) characterizes the $m(r-\ell)$ degrees of freedom available in solving the problem of retaining a prescribed ℓ dimension invariant subspace of the optimal state regulator. The associated pole-placement problem may be solved by any pole-placement procedure, but in particular the dyadic solution for which software support has been developed is appropriate.

If it is desired to retain an r dimensional invariant subspace of the state feedback regulator, it is possible to choose K_0,P so as to additionally solve the modified output feedback regulator problem defined in the previous section. Whereas in the case $\ell < r$ it is only guaranteed that using output feedback achieves the optimal state regulator cost in an ℓ dimensional subspace, in the case $\ell = r$ the output feedback control law will also be optimal in the sense of the modified regulator problem.

In order to simplify the calculations it is assumed that a transformation has been applied to the triple (A,B,C) such that $C = [I_r \ \ 0]$. Denote by M_c the solution of the state feedback Ricatti equation and by M(L) the solution of the modified regulator problem. Let $\{u_i\}_{i=1}^{r}$ and $\{\lambda_i\}_{i=1}^{r}$ be r eigenvectors and eigenvalues of $(A-BR^{-1}B^TM_c)$ and introduce the partitions:

$$A = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix}, \quad M_c B R^{-1} B^T M_c = \begin{vmatrix} \Gamma_{11} & \Gamma_{21}^T \\ \Gamma_{21} & \Gamma_{22} \end{vmatrix}$$

$$L = \begin{vmatrix} L_{11} & L_{21}^T \\ L_{21} & L_{22} \end{vmatrix}, \quad K = R^{-1} B^T M_c = [K_1 \vdots K_2]$$

$$(u_1 \vdots \ldots \vdots u_r) T_r = U_r = \begin{vmatrix} Y \\ Z \end{vmatrix}, \quad N = ZY^{-1}, \quad P = \begin{vmatrix} I_r & 0 \\ N & 0 \end{vmatrix} \qquad (2.2.12)$$

$A_{11}, \Gamma_{11}, L_{11}, Y \in R^{r x r}, K_1 \in R^{m x r}, T_r$ defined in (2.2.2)

Then the following holds [31].

<u>Theorem 2.2.1</u>: If the matrix $A_r = (A_{22} - ZY^{-1} A_{12})$ has all its eigenvalues in the left half plane then:

1) The modified output feedback regulator problem has a solution $M(L) = M(N)$ for each L satisfying $L_{21} L_{11}^{-1} = ZY^{-1} = N$, and the associated feedback gain is given by

$$\hat{K} = R^{-1} B^T M(N) P = [K_1 + K_2 ZY^{-1} \vdots 0] \qquad (2.2.13)$$

2) The spectrum of the closed loop output feedback system $A - B\hat{K}$ is given by $\{\lambda_i\}_{i=1}^{r} \cup \sigma(A_r)$.

3) The cost matrix $M(N)$ may be decomposed as $M(N) = M_c + D(N)$, $D(N) \geq 0$, where $D(N)$ represents the cost increase over the optimal state feedback solution, and the null space of D is spanned by $\{u_i\}_{i=1}^{r}$ so that in

this r dimensional retained subspace there is no increase in cost associated with using the output feedback control. D is given by

$$D = \begin{bmatrix} N^T D_{22} N & -N^T D_{22} \\ -D_{22} N & D_{22} \end{bmatrix}, \; D_{22} \in R^{(n-r)x(n-r)} \qquad (2.2.14)$$

where $D_{22} = D_{22}^T > 0$ is the solution of the Lyapunov equation

$$(A_{22} - NA_{12})^T D_{22} + D_{22}(A_{22} - NA_{12}) + \Gamma_{22} = 0 \qquad (2.2.15) \square$$

Furthermore the matrix D(N) provides a bound on the optimal cost for the output feedback regulator problem. Defining J_s as the cost associated with the optimal state feedback regulator and J_o as the cost associated with the optimal output feedback regulator then [31]

$$J_s \le J_o \le J_s + \frac{1}{2}\text{trace}(DQ_o) \qquad (2.2.16)$$

In summary the problem of determining a gain to retain r optimal eigenvectors has the unique solution $K_o = K_1 + K_2 N$. It is noted that this is precisely equation (2.2.11) under the constraints $\ell = r$ and $C = [I_r \;\; 0]$.

2.3. Retention of Optimal Invariant Subspaces by Dynamic Output Feedback Compensation

In the event that it is not possible to stabilize the system by static output feedback while retaining a desired ℓ dimensional subspace of the state feedback regulator, or that the spectrum of the resultant system is not acceptable, a dynamic compensator of dimension p may be designed which will retain an $r+p$ dimensional subspace of the optimal state feedback regulator.

In this section the derivation given in [1] of a design oriented approach to the construction of such a compensator of prespecified dimension p will be presented. This approach reduces the specification of the parameters of the compensator to the solution of an output feedback pole-placement problem similiar to that encountered in the previous section in the design of static output feedback compensators. In the next chapter an algorithm will be given which simultaneously solves this pole-placement problem and determines without apriori assumptions the dimension of the desired compensator.

Again for simplicity the output matrix is assumed to be in the form $C=[I_r \quad 0]$. Introducing the compensator $\dot{z}=Hz+Dy$, $H\in R^{pxp}$, $D\in R^{pxr}$ and momentarily assuming that the matrices H and D and the dimension p are known, the compensator design problem may be treated as a pole placement problem to which Theorem 2.2.1 may be applied. Defining the matrices:

$$\hat{A}=\left[\begin{array}{c|c} H & DC \\ \hline 0 & A \end{array}\right]=\left[\begin{array}{c|c|c} H & D & 0 \\ \hline 0 & A_{11} & A_{21} \\ \cline{2-3} & A_{12} & A_{22} \end{array}\right]\in R^{(n+p)x(n+p)}$$

$$\hat{B}=\begin{bmatrix} 0 \\ B \end{bmatrix}\in R^{(n+p)xm},\quad \hat{C}=\begin{bmatrix} I_p & 0 \\ 0 & C \end{bmatrix}=\begin{bmatrix} I_p & 0 & 0 \\ 0 & I_r & 0 \end{bmatrix}\in R^{(r+p)x(n+p)} \tag{2.3.1}$$

$$\hat{Q}=\begin{bmatrix} 0 & 0 \\ 0 & Q \end{bmatrix},\quad \hat{Q}_0=\begin{bmatrix} 0 & 0 \\ 0 & Q_0 \end{bmatrix},\quad \hat{R}=R$$

and the augmented state $\hat{x}=\begin{bmatrix} z \\ x \end{bmatrix}$, the objective is to find an output feedback $u=-\hat{K}\hat{C}\hat{x}$ which retains an $r+p$ dimensional subspace of the optimal state feedback regulator associated with $(\hat{A},\hat{B},\hat{Q},\hat{R})$ and which is optimal in the sense of the modified regulator problem.

Since the compensator state is not observable through $\sqrt{\hat{Q}}$ the solution of the state feedback regulator problem is

$$\hat{M}_c = \begin{bmatrix} 0 & 0 \\ 0 & M_c \end{bmatrix}, \; K_s = R^{-1}\hat{B}^T\hat{M}_c = [0 \;\vdots\; R^{-1}B^TM_c] \tag{2.3.2}$$

where M_c is the solution of the Ricatti equation associated with (A,B,Q,R). Note however that if H is not stable, then $(\hat{A},\sqrt{\hat{Q}})$ is not a detectable pair and though optimal, the solution (2.3.2) will not be stabilizing. (See remarks preceding equation (2.1.4).) This poses no difficulty in the design methodology as the optimal closed loop system will then consist of the system $\dot{x} = (A-BR^{-1}B^TM_c)x$ driving an unstable open loop compensator. Even though the compensator states will diverge, the plant response will be stable. The implemented output feedback control law will stabilize the total closed loop system.

Let $\{u_i\}_{i=1}^n$ and $\{\lambda_i\}_{i=1}^n$ be the eigenvectors and eigenvalues of the optimal state feedback regulator, where the first $r+p$ eigenvectors span the $r+p$ dimensional subspace to be retained and $(\lambda_r,\lambda_{r+1})$ and $(\lambda_{r+p},\lambda_{r+p+1})$ are not complex pairs. Denote:

$$(u_1 \;\vdots \cdots \vdots\; u_r)T_r = \begin{bmatrix} Y \\ Z \end{bmatrix}, \; Y \in R^{rxr}, \; Z \in R^{(n-r)xr}, \; T_r \text{ defined in (2.3.2)}$$

$$(u_{r+1} \;\vdots \cdots \vdots\; u_{r+p})T_r = \begin{bmatrix} U \\ V \end{bmatrix}, \; U \in R^{rxp}, \; V \in R^{(n-r)xp} \tag{2.3.3}$$

$$\Lambda_r = T_r^{-1}\, dg(\lambda_1 \ldots \lambda_r)T_r, \; \Lambda_p = T_r^{-1}\, dg(\lambda_{r+1} \cdots \lambda_{r+p})T_r$$

and define $W_p \in R^{pxp}$ and $W_r \in R^{pxr}$ by the eigenvector equation

$$\left[\begin{array}{c|cc} H & D & 0 \\ \hline 0 & & F \end{array}\right]\begin{bmatrix} W_r & W_p \\ Y & U \\ Z & V \end{bmatrix} = \begin{bmatrix} W_r & W_p \\ Y & U \\ Z & V \end{bmatrix}\begin{bmatrix} \Lambda_r & 0 \\ 0 & \Lambda_p \end{bmatrix}$$

$$F = A - BR^{-1}B^TM_c \qquad (2.3.4)$$

By Theorem 2.2.1, stabilization of the system by output feedback of the measured variables and the compensator states requires the selection of H,D and therefore W_p and W_r such that the matrix A_r has a satisfactory spectrum where:

$$\hat{N} = [Z \quad V]\begin{bmatrix} W_r & W_p \\ Y & U \end{bmatrix}^{-1} \triangleq [N_p \quad N_r], \quad N_p \in R^{(n-r)xp}, \; N_r \in R^{(n-r)xr}$$

$$A_r = A_{22} - \hat{N}\begin{bmatrix} 0 \\ A_{12} \end{bmatrix} = A_{22} - N_rA_{12} \qquad (2.3.5)$$

Using the formula for the inverse of a partitioned matrix the expression for A_r may be written $A_1 + B_0PA_{12}$ where:

$$A_1 = A_{22} - ZY^{-1}A_{12} \in R^{(n-r)x(n-r)}$$

$$B_0 = V - ZY^{-1}U \in R^{(n-r)xp} \qquad (2.3.6)$$

$$P = L(Y-UL)^{-1} \in R^{pxr}, \quad L = W_p^{-1}W_r \in R^{pxr}$$

Thus the satisfaction of the condition of Theorem 2.2.1 that A_r have an acceptable spectrum is reduced to the solution of an output feedback pole-placement problem. In the next chapter this problem will be solved by an algorithm which computes P as the sum of a sequence of dyadic feedbacks and determines the compensator dimension p. It is noted here that the number of columns of B_0 is equal to the compensator dimension and that rank (B_0) is always maximal since the columns of B_0 are eigenvectors of a particular matrix arising in the block triangularization of F [1].

Assuming that this pole-placement problem has been solved and that the dimension p and the gain P are known, the parameters of the compensator are given by:

$$
\begin{aligned}
& H = W_p H_o W_p^{-1}, \quad H_o = [\Lambda_p - L\Lambda_r Y^{-1} U][I_p + PU] \\
& D = W_p D_o, \quad D_o = [L\Lambda_r - \Lambda_p L]Y^{-1}[I_r + UP] \\
& L = (I + PU)^{-1} PY \\
& N_p = N_{po} W_p^{-1}, \quad N_{po} = (V - ZY^{-1}U)(I_p + PU) \\
& N_r = ZY^{-1} - (V - ZY^{-1}U)P \\
& \hat{K}_s = [K_z \;\vdots\; K_y \;\vdots\; 0] = R^{-1}\hat{B}^T\hat{M}_c\hat{P}, \quad \hat{P} = \begin{bmatrix} I_{r+p} & 0 \\ \hat{N} & 0 \end{bmatrix} = \begin{bmatrix} I_p & 0 & 0 \\ 0 & I_r & 0 \\ N_p & N_r & 0 \end{bmatrix}
\end{aligned}
\tag{2.3.7}
$$

Thus the matrix P determines the compensator up to a similarity transformation W_p which may be used to obtain a favorable representation of the pair (H,D). The final closed loop spectrum consists of the $r+p$ retained optimal eigenvalues together with the spectrum of A_r.

Before considering the solution of the pole-placement problem in the next chapter some remarks regarding the resultant compensator dimension p are appropriate.

By Theorem 1.3, if $p \geq n-2r+1$ then the spectrum of A_r may almost always be assigned arbitrarily. Thus a bound on the dimension of compensator required to satisfactorily control the system is $n-2r+1$. In the case this bound is achieved, the resultant closed loop system will retain $r+p=n-r+1$ optimal eigenvectors and will admit an arbitrary complementary spectrum through A_r. As will be seen in numerical examples in Chapter 4, acceptable designs may be obtained with compensators of dimension well below this bound, particularly when the goal is to place all the eigenvalues in a prescribed region of the complex plane rather than at prescribed locations.

In the case $p=n-r$ all n optimal eigenvectors will be retained and the resultant compensator may be identified with the reduced order Luenberger observer. To make the correspondence explicit consider an observer given by [33]:

$$\dot{w} = Ew + Gy + Ru \quad , \quad E \in R^{(n-r)x(n-r)},\ G \in R^{(n-r)xr},\ R \in R^{(n-r)xm}$$

$$\hat{x}_2 = w + Sy \quad , \quad S \in R^{(n-r)xr} \tag{2.3.8}$$

where $\hat{x}_2$ is an estimate of the unmeasured state variables x_2. (Recall $C = [I\ 0]$). Defining an error $e = \hat{x} - x$ and introducing the partitions:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} , \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} , \quad A_{11} \in R^{rxr},\ B_1 \in R^{rxm}$$

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} , \quad F = A - BR^{-1}B^TM_c \tag{2.3.9}$$

it may be shown that

$$\dot{e} = Ee + (A_{22} - SA_{12} - E)x_2 + (A_{21} - SA_{11} - G + ES)x_1$$
$$+ (B_2 - SB_1 - R)u \qquad (2.3.10)$$

Thus the error will converge asymtopically to zero independent of x and u provided

$$\begin{aligned} E &= A_{22} - SA_{12} \\ G &= A_{21} - SA_{11} + A_{22}S - SA_{12}S \\ R &= B_2 - SB_1 \end{aligned} \qquad (2.3.11)$$

and E is a stable matrix. Observability of (A,C) implies that the pair (A_{22},A_{12}) is observable and so the pole-placement problem for E has a solution.

Implementing the control law as

$$u = r - (K_1 \; K_2)\begin{pmatrix} x_1 \\ \hat{x}_2 \end{pmatrix} = r - (K_2 : K_1 + K_2 S)\begin{pmatrix} w \\ y \end{pmatrix}$$
$$(K_1 \; K_2) = R^{-1}B^T M_c \qquad (2.3.12)$$

and using (2.3.11) the total closed loop system becomes

$$\frac{d}{dt}\begin{bmatrix} w \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} F_{22}-SF_{12} & F_{21}-SF_{11}+F_{22}S-SF_{12}S & 0 \\ -B_1K_2 & A_{11} - B_1(K_1+K_2S) & A_{12} \\ -B_2K_2 & A_{21} - B_2(K_1+K_2S) & A_{22} \end{bmatrix}\begin{bmatrix} w \\ x_1 \\ x_2 \end{bmatrix} +$$

$$\begin{bmatrix} B_2 - SB_1 \\ B_1 \\ B_2 \end{bmatrix} r \tag{2.3.13}$$

By comparison, the compensator of this section is of the form $\dot{v} = Hv + Dy$ and may be viewed as producing an estimate $\hat{x}_2 = N_p v + N_r y$. Under the control law

$$u = r - (0\ K_1\ K_2)\ \hat{P}\begin{pmatrix} v \\ x_1 \\ x_2 \end{pmatrix} = r - (K_2 N_p \ \vdots\ K_1 + K_2 N_r \ \vdots\ 0)\begin{pmatrix} v \\ x_1 \\ x_2 \end{pmatrix} \tag{2.3.14}$$

the closed loop system is

$$\frac{d}{dt}\begin{bmatrix} v \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} H & D & 0 \\ -B_1K_2N_p & A_{11}-B_1(K_1+K_2N_r) & A_{12} \\ -B_2K_2N_p & A_{21}-B_2(K_1+K_2N_r) & A_{22} \end{bmatrix}\begin{bmatrix} v \\ x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ B_1 \\ B_2 \end{bmatrix} r \tag{2.3.15}$$

Provided N_p^{-1} exists, then under the transformation $(\tilde{v}, x_1, x_2) = (N_p^{-1} v, x_1, x_2)$ (2.3.15) becomes

$$\frac{d}{dt}\begin{bmatrix} \tilde{v} \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} N_pHN_p^{-1} & N_pD & 0 \\ -B_1K_2 & A_{11}-B_1(K_1+K_2N_r) & A_{12} \\ -B_2K_2 & A_{21}-B_2(K_1+K_2N_r) & A_{22} \end{bmatrix}\begin{bmatrix} \tilde{v} \\ x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ B_1 \\ B_2 \end{bmatrix} r \tag{2.3.16}$$

Now in [1] it is shown that the transformation $T = \begin{bmatrix} I & 0 \\ -N & I \end{bmatrix}$ upper block triangularizes the optimal closed loop state feedback matrix provided $N = ZY^{-1}$ exists and $\begin{bmatrix} Y \\ Z \end{bmatrix}$ is a matrix of optimal eigenvectors of the closed loop system. Applying this result to the compensated system of this section gives

$$T\begin{bmatrix} H & D & 0 \\ 0 & F_{11} & F_{12} \\ 0 & F_{21} & F_{22} \end{bmatrix} T^{-1} = \begin{bmatrix} H & D & 0 \\ F_{12}N_p & F_{11}+F_{12}N_r & F_{12} \\ f(N_p,N_r) & g(N_p,N_r) & F_{22}-N_rF_{12} \end{bmatrix} \tag{2.3.17a}$$

where

$$T = \begin{bmatrix} I_p & 0 & 0 \\ 0 & I_r & 0 \\ -N_p & -N_r & I_{n-r} \end{bmatrix} \tag{2.3.17b}$$

$$\begin{aligned} f(N_p,N_r) &= (-N_pHN_p^{-1}+F_{22}-N_rF_{12})N_p \equiv 0 \\ g(N_p,N_r) &= (-N_pD+F_{21}-N_rF_{11}+F_{22}N_r-N_rF_{12}N_r) \equiv 0 \end{aligned} \tag{2.3.17c}$$

Identifying $S \leftrightarrow N_r$ and comparing (2.3.13) and (2.3.16) it follows that the compensator of this section is a Luenberger reduced order observer in an appropriate basis provided the reference input is zero. Furthermore the dynamics of the error equation are governed by the matrix

$$A_r = A_{22} - N_rA_{12} = A_1 + B_0PA_{12} \tag{2.3.18}$$

whose eigenvalues may be arbitrarily assigned by state feedback pole-placement since B_0 is square and of full rank n-r.

CHAPTER 3

FURTHER ASPECTS OF THE DESIGN METHODOLOGY

3.1. An Algorithm Solving the Associated Pole-Placement Problem

This section presents an algorithm which solves the output feedback pole-placement problem of section 2.3, associated with the design of compensators for suboptimal regulators. The problem is to determine the compensator dimension p and a matrix $P \in R^{p\times r}$ such that the spectrum of $A_r = A_1 + B_0PA_{12}$ is satisfactory (see equation (2.3.6). A solution is obtained here by iterating the proof of Theorem 1.3 in the manner suggested in [2]. Recalling that the number of columns of B_0 is the dimension of the compensator, the idea is to start with B_0 a column vector and successively increase the column span of B_0, solving a partial pole-placement at each stage, until satisfactory spectral characteristics are obtained for A_r. The degree of the required compensator will then be the number of columns of B_0 and the parameters H,D,K_z,K_y of the design will be given in function of P by (2.3.7).

The algorithm consists of the solution of a sequence of pole-placement problems defined as follows. Given from the first part of the design are the observable pair (A_{22},A_{12}) and the optimal closed loop eigenvalues and eigenvectors $\{\lambda_i\}_{i=1}^{n}$, $\{v_i\}_{i=1}^{n}$ where the λ_i are distinct. It is assumed that the λ_i and v_i have been ordered such that $(\lambda_r,\lambda_{r+1})$ is not a complex pair and such that the matrix Y defined in (2.3.3) is invertible. The ordering may also have taken into account the relative importance which the designer may attach to the optimal eigenvalues. Define:

$$\begin{bmatrix} Y \\ Z \end{bmatrix} = (v_1 \vdots \cdots \vdots v_r)T_r \ , \quad Y \in R^{r\times r},\ Z \in R^{(n-r)\times r},\ T_r \text{ defined in (2.2.2)}$$

$$\left[\begin{array}{c|c|c} u_1 & \text{---------} & u_{n-r} \\ \hline v_1 & \text{---------} & v_{n-r} \end{array}\right] = (v_{r+1} \vdots \cdots \vdots v_n)T_r,\ u_i \in R^{r\times 1},\ v_i \in R^{(n-r)\times 1} \tag{3.1.1}$$

$$b_i = v_i - N_0 u_i,\ N_0 = ZY^{-1}, \qquad B_i = (b_1 \vdots \cdots \vdots b_i),\ i = 1,\ldots,n-r$$

Since A_{12} has the role of an output matrix and since it will be required for the output matrix to have full rank in the pole-placement procedures to be described, denote $\ell = \text{rank}(A_{12})$, let $T \in R^{\ell\times r}$ be any matrix of full rank, and define a new output matrix $C = TA_{12}$. Also since the pole-placement problem for the triple (A_1, B_o, A_{12}) may be solved as a state feedback pole-placement problem if $\ell \geq n-r$, this case will be discussed at the end of the section. Presently it will be assumed that $\ell + r < n$.

The algorithm may be stated in terms of two pole-placement procedures, to be defined below, as follows.

Algorithm

0. Initialize $i = 0$.

1. Let $i = i+1$.

2. Using either procedure 1 or 2 below, compute a dyadic feedback

$$f_i g_i,\ f_i \in R^{i\times 1},\ g_i \in R^{1\times \ell} \tag{3.1.2}$$

to assign $i + \ell - 1$ desired eigenvalues to the matrix:

$$A_{i+1} = A_i + B_i f_i g_i C \tag{3.1.3}$$

3. If the resultant spectrum of A_{i+1} is unsatisfactory or if the algorithm

may not terminate because λ_i, λ_{i+1} is a complex pair, go to 1.

4. Let the compensator dimension be $p = i$ and let

$$P = \sum_{j=1}^{i} \begin{bmatrix} f_j \\ \hdashline 0_{(i-j)\times 1} \end{bmatrix} g_j\, T\ , \quad A_r = A_{i+1} \qquad (3.1.4)\ \square$$

Since the spectrum of A_r may be assigned arbitrarily if $p + \ell - 1 = n-r$ this algorithm will terminate after at most $n-r-\ell+1$ stages.

Procedures 1 and 2 apply Theorem 1.3 to the systems (A_i, B_i, C) defined by the algorithm. At the ith stage the system (A_i, B_i, C) has i inputs and ℓ outputs. Thus, with possible exceptions due to the system lying on a hypersurface on which the results of the theorem fail to hold, $i + \ell - 1$ eigenvalues of A_{i+1} may be assigned arbitrarily closely to desired values. The procedure then is to choose an input (or output) space projection which reduces the system to a single-input (or single-output) system and renders $i - 1$ eigenvalues uncontrollable (or ℓ-1 eigenvalues unobservable). These eigenvalues correspond to eigenvalues already assigned during the (i-1)st stage. Next feedback gains are computed to place at desired locations as many additional eigenvalues as there are outputs (or inputs). Because uncontrollable and/or unobservable eigenvalues are invariant under output feedback this will result in the assignment of $i + \ell - 1$ eigenvalues of the matrix A_{i+1}. Both state space and frequency domain based procedures will be given.

It will be assumed that (A_1, b_1) is a controllable pair, and therefore that each pair (A_i, B_i) is controllable. The case when (A_1, b_1) is uncontrollable will be discussed at the end of the section.

Procedure 1

The system (A_i,B_i,C) is transformed to the single-input system (A_i,B_if_i,C) where $f_i \in R^{i\times 1}$ is chosen to render i-1 eigenvalues of A_i uncontrollable. A vector $g_i \in R^{1\times \ell}$ is then found to place ℓ eigenvalues of $A_{i+1} = A_i + B_if_ig_iC$. If $i=1$, then $f_1 = 1$, and the first part of the procedure is omitted.

State Space Solution

Let $\{\epsilon_k\}_{k=1}^{n-r}$ be the eigenvalues of A_i, the first i-1 of which are to be retained, and let $\{v_k\}_{k=1}^{n-r}$, $v_k \in R^{1\times(n-r)}$, be the corresponding left eigenvectors. With T_r given by (2.2.2) define

$$V_1 = T_r^T \begin{bmatrix} v_1 \\ \vdots \\ v_{i-1} \end{bmatrix} \in R^{(i-1)\times(n-r)}, \quad V_2 = T_r^T \begin{bmatrix} v_i \\ \vdots \\ v_{n-r} \end{bmatrix} \in R^{(n-r-i+1)\times(n-r)} \tag{3.1.5}$$

Recalling that an eigenvalue is uncontrollable if the corresponding left eigenvector is in the null space of the input matrix, select a vector $f_i \in R^{i\times 1}$ satisfying $V_1B_if_i = 0$. Then $\{\epsilon_k\}_{k=1}^{i-1}$ will be invariant under further feedback and for each k such that the kth entry of $V_2B_if_i$ is nonzero, the eigenvalue ϵ_{k+i-1} will be controllable. A solution of $V_1B_if_i = 0$ always exists since this corresponds to finding an i-vector orthogonal to (i-1) i-vectors. However conditions under which the n-r-i+1 remaining eigenvalues will be controllable are not available.

Equations (1.5) and (1.6) determine the gain g_i to place ℓ eigenvalues of the triple (A_i,B_if_i,C) at desired locations $\{\sigma_k\}_{k=1}^{\ell}$. (If ξ entries of $V_2B_if_i$ are nonzero with $\xi < \ell$ then ξ should be used in place of ℓ in the following formulas.) Let the characteristic polynomial of A_i be

$$p_0(\lambda) = \sum_{k=0}^{n-r} a_k\lambda^i \;, \quad a_{n-r} = 1 \tag{3.1.6}$$

and define the matrices:

$$Q = [B_i f_i \;\vdots\; A_i B_i f_i \;\vdots \cdots \vdots\; A_i^{n-r-1} B_i f_i] \in R^{(n-r)x(n-r)}$$

$$R = \begin{bmatrix} 1 & a_{n-r-1} & \cdots & a_2 & a_1 \\ & 1 & \ddots & & a_2 \\ & & \ddots & \ddots & \vdots \\ & 0 & & 1 & a_{n-r-1} \\ & & & & 1 \end{bmatrix} \in R^{(n-r)x(n-r)} \tag{3.1.7}$$

$$S = \begin{bmatrix} \sigma_1^{n-r-1} & \cdots & \sigma_\ell^{n-r-1} \\ \vdots & & \vdots \\ \sigma_1 & \cdots & \sigma_\ell \\ 1 & \cdots & 1 \end{bmatrix} T_r \in R^{(n-r)x\ell}, \quad \hat{S} = \begin{bmatrix} \sigma_1^{n-r} & \cdots & \sigma_\ell^{n-r} \\ \vdots & & \vdots \\ \sigma_1 & \cdots & \sigma_\ell \\ 1 & \cdots & 1 \end{bmatrix} T_r \in R^{(n-r+1)x\ell}$$

$$\hat{P} = (p_0(\sigma_1), \ldots, p_0(\sigma_\ell))T_r = (1, a_{n-r-1}, \ldots, a_1, a_0)\hat{S} \in R^{1x\ell}$$

where T_r is given in (2.2.2). Then the solution g_i of

$$g_i(CQRS) = \hat{P} \tag{3.1.8}$$

will assign to the spectrum of $A_{i+1} = A_i + B_i f_i g_i C$ the $\ell + i - 1$ eigenvalues $\{\sigma_k\}_{k=1}^{\ell}$ and $\{\epsilon_k\}_{k=1}^{i-1}$.

<u>Frequency Domain Solution</u>

The transfer function of the system (A_i, B_i, C) may be written

$$C(\lambda I - A_i)^{-1} B_i = N(\lambda) / d(\lambda)$$
$$d(\lambda) = \sum_{k=0}^{n-r} a_k \lambda^k, \quad a_{n-r} = 1 \tag{3.1.9}$$

where the roots of $d(\lambda)$ are $\{\epsilon_k\}_{k=1}^{n-r}$ of which the first i-1 are to be retained. The numerator polynomial matrix may be written

$$N(\lambda) = \begin{bmatrix} n_{11}(\lambda) & \cdots & n_{1i}(\lambda) \\ \vdots & & \vdots \\ n_{\ell 1}(\lambda) & \cdots & n_{\ell i}(\lambda) \end{bmatrix} = \sum_{k=0}^{n-r-1} N_k \lambda^k \tag{3.1.10}$$

$$n_{ij}(\lambda) = \sum_{k=0}^{n-r-1} n_{ijk}\lambda^k, \quad N_k = \begin{bmatrix} n_{11k} & \cdots & n_{1ik} \\ \vdots & & \vdots \\ n_{\ell 1k} & \cdots & n_{\ell ik} \end{bmatrix}$$

(In n_{ijk} the index i indicates the row of $N(\lambda)$, j the column of $N(\lambda)$, and k the degree of the term of $n_{ij}(\lambda)$ in which n_{ijk} appears.)

Recall that the goal is to render i-1 eigenvalues uncontrollable by a choice of f_i and to choose g_i to place ℓ eigenvalues. Since an eigenvalue of a single-input system is uncontrollable if the numerator of the transfer function is zero evaluated at that eigenvalue, it follows that for $\{\epsilon_k\}_{k=1}^{i-1}$ to be uncontrollable eigenvalues of the single-input system $(A_i, B_i f_i, C)$ it must hold that $N(\epsilon_k) f_i = 0$ for $k = 1,\ldots,i-1$. By Lemma 1.2 rank $N(\epsilon_k) = 1$. Let $\{j_k\}_{k=1}^{i-1}$ be any sequence such that the j_k^{th} row of $N(\epsilon_k)$ is nonzero, and define

$$M = \begin{bmatrix} e_{j_1} N(\epsilon_1) \\ \hdashline \vdots \\ \hdashline e_{j_{(i-1)}} N(\epsilon_{i-1}) \end{bmatrix} R^{(i-1)xi}, \quad e_k = (0,\ldots,0,\underset{\uparrow\, k^{th}\text{ position}}{1},0,\ldots,0) \tag{3.1.11}$$

Then f_i may be computed as the solution of

$$T_r^T M f_i = 0 \tag{3.1.12}$$

where T_r is given in (2.2.2). For purposes of numerical evaluation the rows of the matrix $T_r^T M$ may be obtained as

$$\begin{array}{ll} (1 \;\; \epsilon_k, \ldots, \epsilon_k^{n-r-1})\hat{N}_{j_k} & \text{if } \epsilon_k \text{ is real} \\ \begin{bmatrix} 1 & Re(\epsilon_k), \ldots, Re(\epsilon_k)^{n-r-1} \\ 0 & Im(\epsilon_k), \ldots, Im(\epsilon_k)^{n-r-1} \end{bmatrix} \hat{N}_{j_k} & \text{if } (\epsilon_k, \epsilon_{k+1}) \text{ is a complex pair} \end{array} \tag{3.1.13}$$

where

$$\hat{N}_j = \begin{bmatrix} n_{j10} & & n_{ji0} \\ \vdots & & \vdots \\ n_{j1(n-r-1)} & \cdots & n_{ji(n-r-1)} \end{bmatrix} \in R^{(n-r)xi} \tag{3.1.14}$$

It should be noted that the same difficulties regarding the choice of f_i are present here as in the state space solution. That is, M may be rank deficient in which case there will be a multiplicity of choices for f_i, and not all the remaining eigenvalues $\{\epsilon_k\}_{k=i}^{n-r}$ need be controllable with respect to the pair $(A_i, B_i f_i)$. Their controllability may be verified by computing the vectors $N(\epsilon_k) f_i$.

To assign the eigenvalues $\{\sigma_k\}_{k=1}^{\ell}$ the gain g_i is computed from equation (1.31)-(1.35). Since the transfer function for the single input system $(A_i, B_i f_i, C)$ is

$$C(\lambda I - A_i)^{-1} B_i f_i = \sum_{k=0}^{n-r-1} N_k f_i \lambda^k \; / \; d(\lambda) \tag{3.1.15}$$

define $x_k = N_k f_i$, $k = 0,\ldots,n-r-1$. Also let $p_c(\lambda) = p_1(\lambda)p_2(\lambda)$ where

$$p_1(\lambda) = \sum_{k=1}^{\ell} (\lambda - \sigma_k) = \sum_{k=0}^{\ell} d_k \lambda^k , \quad d_\ell = 1 \tag{3.1.16}$$

$$p_2(\lambda) = \sum_{k=0}^{n-r-\ell} h_k \lambda^k , \quad h_{n-r-\ell} = 1$$

Then g_i is determined from the (n-r)th order system of equations

$$(g \,\vdots\, h)\, X = Y , \quad h = (h_0 \; h_1 \; \ldots \; h_{n-r-\ell-1})$$

$$X = \left[\begin{array}{cccccccc} x_0 & \vdots & \cdots & \cdots & \cdots & \cdots & \vdots & x_{n-r-1} \\ \hline d_0 & d_1 & \cdots & d_{\ell-1} & 1 & 0 & \cdots & 0 \\ 0 & d_0 & d_1 & & d_{\ell-1} & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & & \ddots & \ddots & \vdots \\ 0 & \cdots & d_0 & d_1 & \cdots & \cdots & d_{\ell-1} & 1 \end{array} \right] \in R^{(n-r)x(n-r)} \tag{3.1.17}$$

$$Y = (a_0 \; a_1 \; \ldots \; a_{n-r-1}) - (0 \; \ldots \; 0 \; d_0 \;\; d_1 \; \ldots \; d_{r-1})$$

The remaining spectrum of A_{i+1} is determined as the roots of the polynomial $p_2(\lambda)$.

If ξ eigenvalues of $(A_i, B_i f_i)$ are controllable with $\xi < \ell$ then in general (3.1.17) will be inconsistent and must be modified by replacing ℓ with ξ.

To illustrate the notation consider an example.

<u>Example 3.a.1</u>: Suppose the system (A_i, B_i, C) at the ith stage of the algorithm is given by

$$A_i = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad B_i = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$n-r = 4, \ i = 2, \ \ell = 3 \tag{3.1.18}$$

The transfer function is

$$C(\lambda I-A_2)^{-1} B_2 = \frac{1}{\lambda^2(\lambda-1)(\lambda+2)} \begin{bmatrix} \lambda^3+\lambda^2-2\lambda+1 & \lambda-1 \\ \lambda & \lambda^2-\lambda \\ \lambda^2+2\lambda & \lambda^3+\lambda^2-2\lambda \end{bmatrix}$$

$$= \sum_{k=0}^{3} N_k\lambda^k / d(\lambda) \tag{3.1.19}$$

$$N_0 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad N_1 = \begin{bmatrix} -2 & 1 \\ 1 & -1 \\ 2 & -2 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad N_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

To render the pole at -2 uncontrollable an $f_2 \in R^{2x1}$ must be found satisfying

$$(1 \quad -2 \quad 4 \quad -8)\hat{N}_1 f_2 = 0, \quad \hat{N}_1 = \begin{bmatrix} 1 & -1 \\ -2 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \tag{3.1.20}$$

This has solution $f_2 = (3,1)^T$ and the resulting transfer function for the single input system $(A_2, B_2 f_2, C)$ is

$$C(\lambda I - A_2)^{-1}B_2 f_2 = \frac{1}{d(\lambda)} \left[\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \lambda^3 + \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \lambda^2 + \begin{pmatrix} -5 \\ 2 \\ 4 \end{pmatrix} \lambda + \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \right] \tag{3.1.21}$$

If the remaining poles are to be placed at $-1 \pm j1$, -3, the desired partial characteristic equation is $p_1(\lambda) = \lambda^3 + 5\lambda^2 + 8\lambda + 6$. Equation (3.1.15) becomes

$$[g_2 \,\vdots\, h] \begin{bmatrix} 2 & -5 & 3 & 3 \\ 0 & 2 & 1 & 0 \\ 0 & 4 & 4 & 1 \\ \hline 6 & 8 & 5 & 1 \end{bmatrix} = (0 \quad 0 \quad -2 \quad 1) - (0 \,\vdots\, 6 \quad 8 \quad 5) \qquad (3.1.22)$$

and has solution $g_2 = (-6, -50, 12)$, $h_o = 2$. The eigenvalues of A_3 may be verified to be $-1 \pm 1j$, -3 together with the root -2 of $\lambda + h_o$. □

Procedure 2

The procedure is the dual of Procedure 1. The idea is to transform the system (A_i, B_i, C) to the single-output system $(A_i, B_i\, g_i, C)$ where $g_i \in R^{1x\ell}$ is chosen to render $\ell - 1$ eigenvalues of A_i unobservable. A vector $f_i \in R^{ix1}$ is then found to place i eigenvalues of $A_{i+1} = A_i + B_i f_i g_i C$. If $\ell = 1$ then the first part of the procedure is omitted.

State Space Solution

Let $\{\varepsilon_k\}_{k=1}^{n-r}$ be the eigenvalues of A_i, the first $\ell - 1$ of which are to be retained, and let $\{v_k\}_{k=1}^{n-r}$, $v_k \in R^{n-rx1}$ be the corresponding eigenvectors. Define

$$\begin{aligned} V_1 &= (v_1 \,\vdots \ldots \vdots\, v_{\ell-1}) T_r \in R^{(n-r)x(\ell-1)} \\ V_2 &= (v_\ell \,\vdots \ldots \vdots\, v_{n-r}) T_r \in R^{(n-r)x(n-r-\ell+1)} \end{aligned} \qquad (3.1.23)$$

To render $\{\varepsilon_k\}_{k=1}^{\ell-1}$ unobservable select g_i such that $g_i C V_1 = 0$. The observability of the remaining eigenvalues may be checked by computing the vector

$g_i CV_2$.

Let $\{\sigma_k\}_{k=1}^{i}$ be the i desired eigenvalues to be assigned to A_{i+1} by f_i. By the dual of equations (1.5), (1.6) f_i is given by $(SRQB_i)f_i = \hat{P}$ where:

$$P_o(\lambda) = |\lambda I - A_i| = \sum_{k=0}^{n-r} a_k \lambda^k, \quad a_{n-r} = 1$$

$$S = T_r^T \begin{bmatrix} \sigma_1^{n-r-1} & \cdots\cdots & \sigma_1 & 1 \\ \vdots & & \vdots & \vdots \\ \sigma_i^{n-r-1} & \cdots\cdots & \sigma_i & 1 \end{bmatrix} \in R^{ixn-r}, \quad \hat{S} = T_r^T \begin{bmatrix} \sigma_1^{n-r} & \cdots\cdots & \sigma_1 & 1 \\ \vdots & & \vdots & \vdots \\ \sigma_i^{n-r} & \cdots\cdots & \sigma_i & 1 \end{bmatrix} \in R^{ixn-r+1} \tag{3.1.24}$$

$$\hat{P} = T_r^T \begin{bmatrix} P_0(\sigma_1) \\ \vdots \\ P_0(\sigma_i) \end{bmatrix} = \hat{S} \begin{bmatrix} 1 \\ a_{n-r-1} \\ \vdots \\ a_1 \\ a_0 \end{bmatrix} \in R^{ix1}, \quad Q = \begin{bmatrix} g_i C \\ g_i CA_i \\ \hline \\ \hline g_i CA_i^{n-r-1} \end{bmatrix} \in R^{n-rxn-r}$$

$$R = \begin{bmatrix} 1 & & & & 0 \\ a_{n-r-1} & 1 & & & \\ \vdots & \ddots & \ddots & & \\ a_2 & & \ddots & 1 & \\ a_1 & a_2 & \cdots & a_{n-r-1} & 1 \end{bmatrix} \in R^{n-rxn-r}$$

If ξ eigenvalues of A_i are observable with $\xi < i$ then ξ should be used in place of i in (3.1.24).

<u>Frequency Domain Solution</u>

Let the transfer function of (A_i, B_i, C) be given by (3.1.9), (3.1.10), and let the eigenvalues of A_i be $\{\varepsilon_k\}_{k=1}^{n-r}$, the first $\ell - 1$ of which are to be rendered unobservable by the selection of g_i. Then g_i must satisfy

$$g_i N(\epsilon_k) = 0 \ , \quad k = 1, \ldots, \ell-1 \tag{3.1.25}$$

and may be obtained as the solution of $g_i M T_r = 0$ where the kth column of M is a linearly independent column of $N(\epsilon_k)$, $k = 1, \ldots, \ell-1$. If $\{j_k\}_{k=1}^{\ell-1}$ is a sequence such that the j_k^{th} column of $N(\epsilon_k)$ is nonzero then the columns of MT_r may be computed as

$$\hat{N}_{j_k} \begin{bmatrix} 1 \\ \epsilon_k \\ \vdots \\ \epsilon_k^{n-r-1} \end{bmatrix} \quad \text{if } \epsilon_k \text{ is real}$$

$$\hat{N}_{j_k} \begin{bmatrix} 1 & 0 \\ \mathrm{Re}(\epsilon_k) & \mathrm{Im}(\epsilon_k) \\ \vdots & \vdots \\ \mathrm{Re}(\epsilon_k^{n-r-1}) & \mathrm{Im}(\epsilon_k^{n-r-1}) \end{bmatrix} \text{if } (\epsilon_k, \epsilon_{k+1}) \text{ are a complex pair} \tag{3.1.26}$$

where

$$\hat{N}_j = \begin{bmatrix} n_{1j0} & \cdots\cdots\cdots & n_{1j(n-r-1)} \\ \vdots & & \vdots \\ n_{\ell j0} & \cdots\cdots\cdots & n_{\ell j(n-r-1)} \end{bmatrix} \tag{3.1.27}$$

To assign the eigenvalues $\{\sigma_k\}_{k=1}^{i}$ the gain f_i is computed from the dual of equations (1.31)-(1.35). The transfer function of the single-output system $(A_i, B_i, g_i C)$ is

$$g_i C(\lambda I - A_i) B_i = \sum_{k=0}^{n-r-1} g_i N_k \lambda^k / d(\lambda) \tag{3.1.28}$$

Define $x_k = g_i N_k$, $k = 0,\dots,n-r-1$ and let $p_c(\lambda) = p_1(\lambda)p_2(\lambda)$ where

$$\begin{aligned} p_1(\lambda) &= \sum_{k=1}^{i}(\lambda-\sigma_k) = \sum_{k=0}^{i} d_k\,\lambda^k, \quad d_i = 1 \\ p_2(\lambda) &= \sum_{k=0}^{n-r-i} h_k\,\lambda^k, \quad h_{n-r-i} = 1 \end{aligned} \tag{3.1.29}$$

Then f_i is the solution of

$$X\begin{bmatrix} f_i \\ \hline h \end{bmatrix} = Y, \quad h = (h_0 \;\; h_1 \; \dots \; h_{n-r-i-1})^T$$

$$X = \left[\begin{array}{c|cccc} x_0 & d_0 & 0 & \cdots & 0 \\ \hline & d_1 & d_0 & \ddots & 0 \\ & \vdots & d_1 & \ddots & d_0 \\ & d_{i-1} & \vdots & \ddots & d_1 \\ & 1 & d_{i-1} & & \vdots \\ & \vdots & 1 & \ddots & d_{i-1} \\ & \vdots & \vdots & \ddots & \\ \hline x_{n-r-1} & 0 & 0 & \cdots & 1 \end{array}\right], \quad Y = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ \vdots \\ a_{n-r-1} \end{bmatrix} - \begin{bmatrix} 0 \\ \vdots \\ 0 \\ d_0 \\ d_1 \\ \vdots \\ d_{i-1} \end{bmatrix} \tag{3.1.30}$$

If ξ eigenvalues of A_i are observable with $\xi < i$ then ξ should be used instead of i in equation (3.1.30).

In the discussion of Procedures 1 and 2 it has been assumed that $\ell < n-r$. In the case that $\ell \geq n-r$ the spectrum of A_1 may be assigned by state feedback and the algorithm will end at the first stage. Let $P' \in R^{1\times n-r}$ be a vector assigning a desired spectrum to $A_r = A_1 + b_1 P'$. Then P may be taken to be any solution of

$$PTA_{12} = P' \tag{3.1.31}$$

and the degree of the required compensator will be one. Equation (3.1.31) always has a solution, being a set of n-r equations in ℓ unknowns.

If (A_1,b_1) is not controllable the algorithm may still be applied only with the distinction that at the ith stage any uncontrollable eigenvalues of (A_i,B_i) will be contained in the spectrum of A_{i+1} in addition to those placed by the designer. This may be used to advantage if the uncontrollable eigenvalues are "acceptable". If they are not then it may be advantageous to choose a different ordering of the eigenvectors $\{v_k\}_{k=1}^{n}$ of the orginal system. It should be noted that the system (A_{n-r},B_{n-r},C) is controllable since the columns of B_{n-r} span R^{n-r} [1]. The algorithm will therefore eventually reach a stage where (A_i,B_i,C) is controllable regardless of the ordering of the eigenvectors.

In summary, this algorithm, when incorporated into the methodology for designing low order dynamic regulators, determines the degree and implicitly the parameters of the required compensator that shapes the entire spectrum of the resulting closed loop system. This is achieved by computing a feedback matrix P as a sum of dyadic products such that the spectrum of $A_r = A_1 + B_0PA_{12}$ is satisfactory. The spectrum of the total closed loop system is then determined as the spectrum of A_r together with those eigenvalues corresponding to the selected N_0 and $\{b_k\}_{k=1}^{p}$, and the parameters of the compensator are fixed as functions of P.

3.2. Review of the Design Methodology

Before considering several numerical examples in the next chapter, it may be useful to summarize the design methodology that has been presented.

Because the information structure of most linear systems prohibits the implementation of optimal control laws based on state feedback, a design criterion has been defined to be the construction of suboptimal control laws that retain as large an invariant subspace of an optimal state regulator as possible in the resulting closed loop system.

A preliminary study of the system should, in addition to an identification of the controllability and observability structure, include an analysis of the possibility of satisfactorily shaping the closed loop spectrum by static output feedback. As will be seen in the examples, insight helpful in the selection of Λ_r and Λ_p may be gleamed from such an analysis.

Having defined a state feedback regulator problem through the selection of weighting matrices Q,R, the Ricatti equation must be solved and the resultant closed loop eigenvectors and eigenvalues computed. The possible choices for Λ_r will be no more than C(n,r), and by inspection of the optimal eigenvectors may be easily identified in accordance with the requirement that complex pairs not be split and that the matrix Y be invertible. On the basis of the spectra of the resultant matrices A_1, r eigenvalues must be selected for retention. If none of the spectra are acceptable, and a compensator is to be designed, then this choice may be guided by an "identification" of those eigenvalues which have contributed most to the unacceptability of the spectrum of A_1, in their departure under output feedback from their optimal locations. The selection of Λ_r may also be based on information obtained from the preliminary pole-placement analysis, or on the retention of dominant eigenvalues.

When this first part of the design is completed, a decision must be made whether to improve the dynamics of the system by retaining p additional optimal eigenvectors through the introduction of a dynamic compensator or by reducing the number of retained eigenvectors from r to $\ell < r$. Because the preliminary static output feedback pole-placement problem corresponds to the case $\ell = 0$, unless this problem had a satisfactory solution the design of a dynamic compensator should be undertaken.

If a compensator is to be designed, the remaining n-r eigenvalues must be ordered, the vectors b_i computed, and the pole-placement problem for the triple (A_1, B_0, A_{12}) solved. This ordering may again be based on the desire to retain at their optimal locations those eigenvalues most contributing to the unacceptability of the spectrum of the matrix A_l corresponding to the selected Λ_r. It would be desirable to also take into consideration the controllability properties of the pairs (A_i, B_i), but a convenient criterion for ordering the vectors b_i to enhance the solvability of the pole-placement problem is unfortunately not available. The pole-placement problem is solved by computing a sequence of dyadic feedbacks, at each stage increasing by one the number of optimal eigenvectors retained (as well as the number of assignable eigenvalues of A_r and the number of columns of B_o), until a satisfactory tradeoff is achieved between the spectrum of A_r and the dimension p of the compensator. At each stage of the algorithm certain previously assigned eigenvalues are chosen to be retained and a number of additional eigenvalues of A_r are specified. This provides the designer with considerable freedom to meet design specifications for the n-(r+p) remaining eigenvalues of the eventual closed loop system. In particular though arbitrary pole-placement for the matrix will not be possible if $p < n-2r+1$, this freedom may be used to place the eigenvalues of A_r in desired regions of the complex plane, as may be determined for example by minimum damping ratio requirements, or other considerations.

Once the pole-placement problem has been solved satisfactorily, the dimension of the desired compensator has been determined and the parameters H, D, K_z, K_y may be computed in any convenient basis, completing the design procedure.

The computational aspects of the design procedure are straight forward, involving only the solution of eigenvector equations, the solution of linear systems of equations, and associated algebraic manipulations. Following a transformation of basis to bring the output matrix to the form $[I_r \quad 0]$, the solution of a Ricatti equation, and the determination of the optimal eigenvectors, at most C(n,r) eigenvalue calculations are required to compute the spectra of the matrices A_1. The solution of the pole-placement problem by state space procedures requires at each stage the computation of the left (or right) eigenvectors and the characteristic equation of A_i, the solution of a homogeneous system of equations of order r (or i) to determine f_i (or g_i) and finally the solution of an inhomogenous system of equations of order i (or r) to find g_i (or f_i). As the pole-placement problem is solved interactively, allowing for the repeated execution of each stage of the algorithm until the designer is satisfied with the spectrum of A_{i+1}, the solution may be costly if the dimension n-r of A_i is large and many repetitions are employed.

The final computation of the parameters of the compensator involves only the algebraic manipulation of matrices.

The design methodology will be illustrated by several nontrivial examples in the next chapter.

3.3. Extension to Stabilizable Systems

Although it has been assumed that the triple (A,B,C) is controllable and observable, this restriction may be relaxed. In this section the application of the design methodology to the class of observable, stabilizable systems $(\hat{A},\hat{B},\hat{C})$ is considered and it is shown that for such systems the dynamic compensator of section 2.3 possesses a separation property.

Let the system be represented in the canonic form

$$\frac{d}{dt}\begin{bmatrix}\hat{x}_1\\ \hat{x}_2\end{bmatrix}=\begin{bmatrix}\hat{A}_{11} & 0\\ \hat{A}_{21} & \hat{A}_{22}\end{bmatrix}\begin{bmatrix}\hat{x}_1\\ \hat{x}_2\end{bmatrix}+\begin{bmatrix}0\\ \hat{B}_2\end{bmatrix}u,\quad \hat{x}_1\in R^{n_1\times 1},\quad \hat{x}_2\in R^{n_2\times 1}$$

$$\hat{y}=(\hat{C}_1\quad \hat{C}_2)\begin{pmatrix}\hat{x}_1\\ \hat{x}_2\end{pmatrix} \tag{3.3.1}$$

where $\hat{A}_{11}$ is a stable matrix, the triple $(\hat{A}_{22},\hat{B}_2,\hat{C}_2)$ is controllable and observable, and the pair $(\hat{A}_{11},\hat{C}_1)$ is observable.

Consider first the solution of the state feedback regulator problem for this system. It may be assumed without loss of generality that $R=I$ (under a transformation $\hat{u}=\sqrt{R}\,u$). Compatibly partitioning the solution M of the Ricatti equation (2.1.2) gives the three equations:

$$\hat{A}_{22}^T M_{22} + M_{22}\hat{A}_{22} - M_{22}\hat{B}_2\hat{B}_2^T M_{22} + Q_{22} = 0$$

$$(\hat{A}_{22}^T - M_{22}\hat{B}_2\hat{B}_2^T)M_{21} + M_{21}\hat{A}_{11} + (Q_{21} + M_{22}\hat{A}_{21}) = 0 \tag{3.3.2}$$

$$\hat{A}_{11}^T M_{11} + M_{11}\hat{A}_{11} + (\hat{A}_{21}^T M_{21} + M_{21}^T \hat{A}_{21} - M_{21}^T \hat{B}_2 \hat{B}_2^T M_{21} + Q_{11}) = 0$$

the first of which is the Ricatti equation for the subsystem $(\hat{A}_{22}, \hat{B}_2)$ with penalty Q_{22}. The optimal control is then

$$u = -(\hat{K}_1 \hat{K}_2) \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}, \quad (\hat{K}_1 \ \hat{K}_2) = (\hat{B}_2^T M_{21} \ \vdots \ \hat{B}_2^T M_{22}) \tag{3.3.3}$$

and the closed loop matrix is

$$F = \begin{bmatrix} \hat{A}_{11} & 0 \\ F_{21} & F_{22} \end{bmatrix}, \quad \begin{aligned} F_{21} &= \hat{A}_{21} - \hat{B}_2\hat{K}_1 \\ F_{22} &= \hat{A}_{22} - \hat{B}_2\hat{K}_2 \end{aligned} \tag{3.3.4}$$

Thus the optimal control has the property that the controllable subsystem F_{22} is the optimal closed loop system matrix for the pair $(\hat{A}_{22}, \hat{B}_2)$ with penalty Q_{22}. The optimal control shapes the dynamics of $\hat{A}_{22}$ exactly the same as if there were no driving uncontrollable subsystem $(\hat{A}_{21} = 0)$, but also expends energy in shaping the eigenvectors (and therefore the response) of the uncontrollable state variables. This is true even if the uncontrollable states are not penalized $(Q = dg(0\ Q_{22}))$.

Let the controllable and uncontrollable eigenvalues of the optimal closed loop system be

$$\left\{\sigma_i\right\}_{i=1}^{n_1} = \sigma(A_{11}), \quad \left\{\lambda_i\right\}_{i=1}^{n_2} = \sigma(F_{22}) \tag{3.3.5}$$

The optimal controllable eigenvectors may be written explicitly as

$$\begin{bmatrix} 0 \\ v_i^{c} \end{bmatrix}, \quad F_{22} v_i^{c} = v_i^{c} \lambda_i, \quad i = 1,\ldots,n_2 \tag{3.3.6}$$

and the uncontrollable eigenvectors have the form

$$\begin{bmatrix} u_i^{\bar{c}} \\ v_i^{\bar{c}} \end{bmatrix}, \quad \begin{matrix} \hat{A}_{11} u_i^{\bar{c}} = u_i^{\bar{c}} \sigma_i \\ F_{12} u_i^{\bar{c}} + F_{22} v_i^{\bar{c}} = v_i^{\bar{c}} \sigma_i \end{matrix} \quad i = 1,\ldots,n_1 \tag{3.3.7}$$

In order to apply the methodology for the design of dynamic compensators it is necessary that the matrix Y in equation (2.3.3) be invertible. In an arbitrary basis this requires the selection of r eigenvectors such that the matrix $\hat{C}U$ appearing in equation (2.2.11) is invertible. Assuming that a design criterion is the retention of as many *optimal controllable eigenvectors* as possible, the maximum number of controllable eigenvectors which may be retained in Λ_r is therefore r_2 where

$$r_2 = \operatorname{rank}(\hat{C}_2 [v_1^{c} \vdots \ldots \vdots v_{n_2}^{c}]) \tag{3.3.8}$$

Since the pair $(\hat{A}_{22}, \hat{C}_2)$ is observable and the vectors v_i^{c} span R^{n_2}, (3.3.8) simplyfies to $r_2 = \operatorname{rank}(C_2)$. Thus r_2 controllable eigenvectors may be retained in Λ_r, and the remaining $r_1 = r - r_2$ eigenvectors must be selected from the uncontrollable subsystem.

To obtain the separation property for the compensator, the output matrix is first transformed to $[I \vdots 0]$. Let rank $(\hat{C}_2) = r_2$ and let S be any full rank output-space transformation such that

$$S\hat{C}_2 = \begin{bmatrix} 0 \\ \hat{C}_{22} \end{bmatrix}, \quad \text{rank } (\hat{C}_{22}) = r_2 \tag{3.3.9}$$

Then under the transformation $y = S\hat{y}$ the output matrix takes the form

$$\hat{C} = \begin{bmatrix} \hat{C}_{11} & 0 \\ \hat{C}_{21} & \hat{C}_{22} \end{bmatrix}, \quad \hat{C}_{11} \in R^{r_1 x r_1}, \quad C_{22} \in R^{r_2 x r_2}, \quad r_1 + r_2 = r \tag{3.3.10}$$

Now let T_1 and T_2 be any matrices such that the transformation $x' = T\hat{x}$ is invertible where

$$T = \left[\begin{array}{c|c} \hat{c}_{11} & \\ \cdots\cdots & 0 \\ T_1 & \\ \cdots\cdots & \cdots\cdots \\ \hat{C}_{21} & \hat{C}_{22} \\ \cdots\cdots & \cdots\cdots \\ 0 & T_2 \end{array}\right] \tag{3.3.11}$$

Under this transformation the system (3.3.1) becomes

$$\frac{d}{dt} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} \begin{bmatrix} A_{11}' & 0 \\ A_{21}' & A_{22}' \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} + \begin{bmatrix} 0 \\ B_2' \end{bmatrix} u \tag{3.3.12}$$

$$y = \begin{bmatrix} I_{r_1} & 0 & 0 & 0 \\ 0 & 0 & I_{r_2} & 0 \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}$$

Introduce the partitions:

$$A'_{11} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad A'_{21} = \begin{bmatrix} A_{31} & A_{32} \\ A_{41} & A_{42} \end{bmatrix} \quad A'_{22} = \begin{bmatrix} A_{33} & A_{34} \\ A_{43} & A_{44} \end{bmatrix}$$

$$B'_2 = \begin{bmatrix} B_3 \\ B_4 \end{bmatrix} \quad x'_1 = \begin{bmatrix} x_1^{\bar{c}} \\ x_2^{\bar{c}} \end{bmatrix} \quad x'_2 = \begin{bmatrix} x_1^{c} \\ x_2^{c} \end{bmatrix} \tag{3.3.13}$$

$$x_1^{\bar{c}} \in R^{r_1 \times 1}, \quad x_2^{\bar{c}} \in R^{n_1 - r_1 \times 1}, \quad x_1^{c} \in R^{r_2 \times 1}, \quad x_2^{c} \in R^{n_2 - r_2 \times 1}$$

Then under the permutation of states:

$$x = \begin{bmatrix} x_1^{\bar{c}} \\ x_1^{c} \\ x_2^{\bar{c}} \\ x_2^{c} \end{bmatrix} = \begin{bmatrix} I_{r_1} & 0 & 0 & 0 \\ 0 & 0 & I_{r_2} & 0 \\ 0 & I_{n_1 - r_1} & 0 & 0 \\ 0 & 0 & 0 & I_{n_2 - r_2} \end{bmatrix} \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} \tag{3.3.14}$$

the system is represented as:

$$\frac{d}{dt} x = \begin{bmatrix} A_{11} & 0 & A_{12} & 0 \\ A_{31} & A_{33} & A_{32} & A_{34} \\ A_{21} & 0 & A_{22} & 0 \\ A_{41} & A_{43} & A_{42} & A_{44} \end{bmatrix} x + \begin{bmatrix} 0 \\ B_1 \\ 0 \\ B_2 \end{bmatrix} u \tag{3.3.15}$$

$$y = \begin{bmatrix} I_{r_1} & 0 & 0 & 0 \\ 0 & I_{r_2} & 0 & 0 \end{bmatrix} x$$

In this basis the controllable and uncontrollable optimal closed loop eigenvectors have the form:

$$\begin{bmatrix} 0 \\ \phi_3^{\,i} \\ 0 \\ \phi_4^{\,i} \end{bmatrix}, \quad \begin{bmatrix} \phi_3^{\,i} \\ \phi_4^{\,i} \end{bmatrix} \triangleq v_i^{\,c}, \; i = 1,\ldots,n_2, \quad \begin{bmatrix} \omega_1^{\,i} \\ \omega_3^{\,i} \\ \omega_2^{\,i} \\ \omega_4^{\,i} \end{bmatrix}, \quad \begin{bmatrix} \omega_1^{\,i} \\ \omega_2^{\,i} \end{bmatrix} \triangleq u_i^{\,\bar{c}},$$

$$\begin{bmatrix} \omega_3^{\,i} \\ \omega_4^{\,i} \end{bmatrix} \triangleq v_i^{\,\bar{c}}, \; i = 1,\ldots,n_1 \tag{3.3.16}$$

At the first stage of the design it is necessary to select Y,Z and to compute $\sigma(A_1)$. From (3.3.8) and the structure of the output matrix in (3.3.15) it follows that at most r_2 controllable eigenvectors may be retained in Y. In accordance with the design criterion of retaining as many optimal controllable eigenvectors as possible, let:

$$Y = \begin{bmatrix} Y_1^{\,\bar{c}} & 0 \\ Y_2^{\,\bar{c}} & Y_2^{\,c} \end{bmatrix}, \quad \begin{matrix} Y_1^{\,\bar{c}} = [\omega_1^1 \ldots \omega_1^{r_1}], \quad Y_2^{\,c} = [\phi_3^1 \ldots \phi_3^{r_2}] \\ Y_2^{\,\bar{c}} = [\omega_3^1 \ldots \omega_3^{r_1}] \end{matrix} \tag{3.3.17}$$

$$Z = \begin{bmatrix} Z_1^{\,\bar{c}} & 0 \\ Z_2^{\,\bar{c}} & Z_2^{\,c} \end{bmatrix}, \quad \begin{matrix} Z_1^{\,\bar{c}} = [\omega_2^1 \ldots \omega_2^{r_1}], \quad Z_2^{\,c} = [\phi_4^1 \ldots \phi_4^{r_2}] \\ Z_2^{\,\bar{c}} = [\omega_4^1 \ldots \omega_4^{r_1}] \end{matrix}$$

$$u_i = \begin{bmatrix} 0 \\ \phi_3^{\,i+r_2} \end{bmatrix}, \quad v_i = \begin{bmatrix} 0 \\ \phi_4^{\,i+r_2} \end{bmatrix}, \quad i = 1,\ldots,n_2-r_2$$

$$u_{i+(n_2-r_2)} = \begin{bmatrix} \omega_1^{\,i+r_1} \\ \omega_3^{\,i+r_1} \end{bmatrix}, \quad v_{i+(n_2-r_2)} = \begin{bmatrix} \omega_2^{\,i+r_1} \\ \omega_4^{\,i+r_1} \end{bmatrix}, \; i = 1,\ldots,n_1-r_1$$

Then from equation (3.1.1):

$$N_o = ZY^{-1} = \begin{bmatrix} N_o^{\bar{c}} & 0 \\ N_{21} & N_o^{c} \end{bmatrix}, \quad \begin{aligned} N_o^{\bar{c}} &= Z_1^{\bar{c}}(Y_1^{\bar{c}})^{-1} \\ N_{21} &= Z_2^{\bar{c}}(Y_1^{\bar{c}})^{-1} - Z_2^{c}(Y_2^{c})^{-1}Y_2^{\bar{c}}(Y_1^{\bar{c}})^{-1} \\ N_o^{c} &= Z_2^{c}(Y_2^{c})^{-1} \end{aligned}$$

$$b_i = \left\{ \begin{array}{ll} \begin{bmatrix} 0 \\ b_i^{c} \end{bmatrix} & i = 1,\ldots,n_2-r_2 \\ \begin{bmatrix} b_i^{\bar{c}} \\ b_i' \end{bmatrix} & i = n_2-r_2+1,\ldots,n-r \end{array} \right\}, \quad \begin{aligned} b_i^{c} &= \phi_3^{\,i+r_2} - N_o^{c}\,\phi_4^{\,i+r_2} \\ b_i^{\bar{c}} &= \omega_2^{\,i+r-n_2} - N_o^{\bar{c}}\,\omega_1^{\,i+r-n_2} \\ b_i' &= \omega_4^{\,i+r-n_2} - N_{21}\,\omega_1^{\,i+r-n_2} \\ &\quad + N_o^{c}\,\omega_3^{\,i+r-n_2} \end{aligned} \tag{3.3.18}$$

$$A_1 = \begin{bmatrix} A_{22} - N_o^{\bar{c}} A_{12} & 0 \\ A_{42} - (N_{21}A_{12} + N_o^{c}A_{32}) & A_{44} - N_o^{c} A_{34} \end{bmatrix}$$

Note that the vectors $b_i^{\bar{c}}$ and b_i^{c} span $R^{n_1-r_1}$ and $R^{n_2-r_2}$ respectively (see remarks following (2.3.6)).

Assuming that only controllable eigenvalues are to be retained in Λ_p, define $B_i = [b_1^{c} \vdots \ldots \vdots\, b_i^{c}]$. Then the pole-placement problem is that of finding a feedback $P = [P_1 \; P_2]$ to satisfactorily assign the spectrum of

$$A_r = A_1 + \begin{bmatrix} 0 \\ B_i \end{bmatrix} (P_1 \quad P_2) \begin{bmatrix} A_{12} & 0 \\ A_{32} & A_{34} \end{bmatrix} \tag{3.3.19}$$

It should be noted that A_r contains an uncontrollable subsystem $A_{22} - N_o^{\bar{c}} A_{12}$ whose spectrum is the $n_1 - r_1$ uncontrollable eigenvalues not retained in Λ_r. The remaining spectrum of A_r may be shaped by solving a pole-placement problem for the triple $(A_{44} - N_o{}^c A_{34},\ B_i,\ A_{34})$.

Assuming the solution of this reduced pole-placement problem is P_2, the required compensator parameters may be computed from equation (2.3.7) with $P = [0\ \ P_2]$. Let:

$$U_2 = [\phi_3{}^{1+r_2} \ldots \phi_3{}^{r_2+p}],\ V_2 = [\phi_4{}^{1+r_2} \ldots \phi_4{}^{p+r_2}]$$
$$B_o = [b_1 \ldots b_p],\ \Lambda_r = dg(\Lambda_{r_1},\ \Lambda_{r_2}) \tag{3.3.20}$$

where Λ_{r_1}, Λ_{r_2} correspond to the r_1 uncontrollable and r_2 controllable eigenvalues retained in the first stage. Then it may be shown that:

$$H_o = H_o{}^c,\qquad H_o{}^c = [\Lambda_p - L^c\Lambda_{r_2}(Y_2{}^c)^{-1}U_2][I + P_2U_2]$$
$$D_o = [D_o{}^{\bar{c}}\ D_o{}^c],\ D_o{}^c = (L^c\Lambda_{r_2} - \Lambda_p L^c)(Y_2{}^c)^{-1}(I + U_2P_2)$$
$$D_o{}^{\bar{c}} = [L^{\bar{c}}\Lambda_{r_1} - L^c\Lambda_{r_2}(Y_2{}^c)^{-1}Y_2{}^{\bar{c}}](Y_1{}^{\bar{c}})^{-1}$$
$$L^c = [I + P_2U_2]^{-1}P_2Y_2{}^c,\ L^{\bar{c}} = [I + P_2U_2]^{-1}P_2Y_2{}^{\bar{c}} \tag{3.3.21}$$
$$N_p = \begin{bmatrix} 0 \\ N_p{}^c \end{bmatrix},\qquad N_p{}^c = [V_2 - N_o{}^cU_2][I + P_2U_2] = B_o'[I + P_2U_2]$$
$$N_r = \begin{bmatrix} N_o{}^{\bar{c}} & 0 \\ N_{21} & N_r{}^c \end{bmatrix},\qquad N_r{}^c = N_o{}^c - (V_2 - N_o{}^cU_2)P_2 = N_o{}^c - B_o'P_2$$

Partitioning the optimal state feedback gains in this basis as $K = [K_1\ K_2]$, $K_1 = [K_1{}^{\bar{c}}\ \ K_1{}^c]$, $K_2 = [K_2{}^{\bar{c}}\ \ K_2{}^c]$, the suboptimal feedback gains are given by:

$$K_y = K_1 + K_2 N_r = [K_y^{\bar{c}} \;\; K_y^{c}]$$

$$K_z = K_2 N_p = K_z^{c}$$

$$K_y^{\bar{c}} = K_1^{\bar{c}} + K_2^{\bar{c}} N_o^{\bar{c}} + K_2^{c} N_{21} \qquad (3.3.22)$$

$$K_y^{c} = K_1^{c} + K_2^{c} N_r^{c}$$

$$K_z^{c} = K_2^{c} N_p^{c}$$

It is noted that $K_y^{\bar{c}}$ is independent of the solution P of the pole-placement problem.

By the superposition principle for linear systems, the controller may be represented as two compensators in parallel:

$$u = u^{c} + u^{\bar{c}} \qquad (3.3.21a)$$

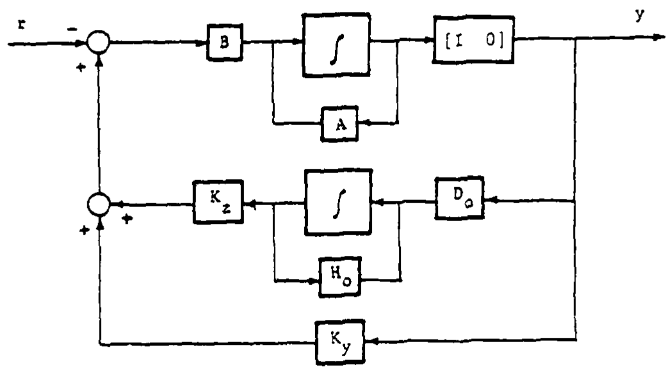
$$u^{c} = K_z^{c} v^{c} + K_y^{c} y^{c}, \quad \dot{v}^{c} = H_o^{c} v^{c} + D_o^{c} y^{c} \qquad (3.3.21b)$$

$$u^{\bar{c}} = K_z^{c} v^{\bar{c}} + K_y^{\bar{c}} y^{\bar{c}}, \quad \dot{v}^{\bar{c}} = H_o^{c} v^{\bar{c}} + D_o^{\bar{c}} y^{\bar{c}} \qquad (3.3.21c)$$

The compensator given in (3.3.21b) is precisely that compensator which would have been designed if the controllable and observable subsystem $(A_{22}', B_2', (I_{r_2} \vdots 0))$ were operating in isolation. The second compensator (3.3.21c) represents a modification in the control scheme due to the presence of the driving uncontrollable subsystem A_{11}'. A block diagram of this controller is given in Figure 3.3.1.

In summary it has been shown that the design methodology may be applied to stabilizable systems, and that the resultant compensator satisfies a separation property. It is noted that the restriction that the system be

For controllable systems the compensator of (2.3.7) has the block diagram:



For stabilizable systems the compensator of (3.3.21), (3.3.22) has the block diagram:

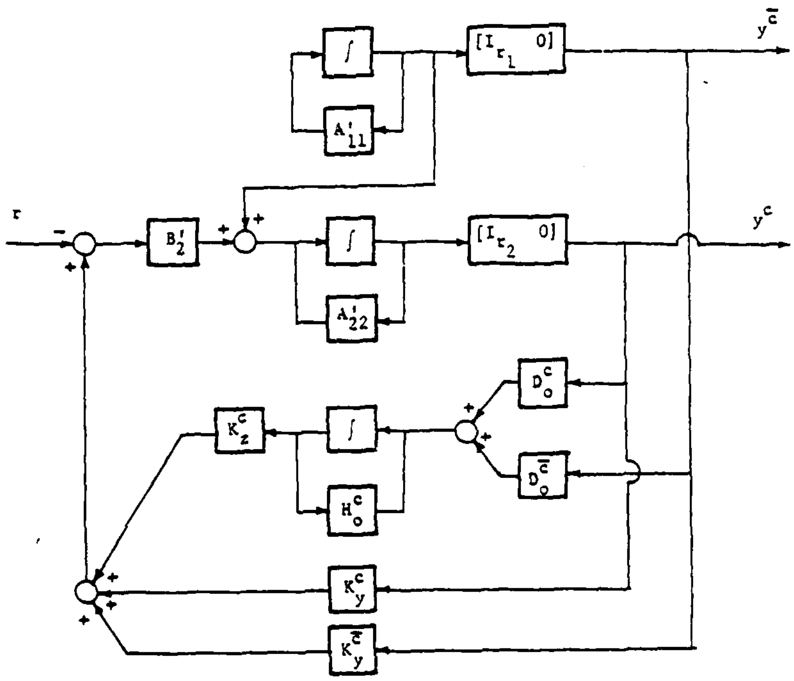


Figure 3.3.1. Compensator structure for controllable and stabilizable systems

observable may be relaxed provided the designer ensures that the matrix Y in (2.3.3) is invertible.

3.4. Software Support

The examples in the next chapter were solved using an interactive Fortran program DSGN·FOR written to implement the methodology described in this thesis. The algorithm solving the pole-placement problem was implemented using the state space formulations of Procedures 1 and 2. A flow chart for the program is given in Figure 3.4.1. The user is assumed to have previously obtained the solution M_c to the Ricatti equation and to have stored on disc in column major order the matrices A, B, R, M_c in a basis representation in which $C = [I_r \quad 0]$. The matrix Q is not needed.

All eigenvalue and eigenvector calculations are performed using the IMSL subroutine EIGRF, which returns an estimate (on the user's console) of the accuracy of the computed eigenvectors. The eigenvector computation is satisfactory if this estimate is less than one, fair if between one and 100, and poor if over 100. The eigenvalues are then ordered by increasing real part, and their corresponding eigenvectors are normalized to unit length. In the case of a complex eigenvalue, the corresponding eigenvector is represented in its real and imaginary parts and is only determined to within a complex multiplicative constant $\cos\theta + j\sin\theta$. All matrix inversions are performed using the IMSL subroutine LINV2F which also returns an error code which should be zero.

In the solution of the pole-placement problem for $A_r = A_1 + B_o P A_{12}$ the vectors f_i are computed as zero eigenvectors of the matrices obtained by augmenting the homogeneous systems $V_1 B_i f_i = 0$ with a zero row. The vectors

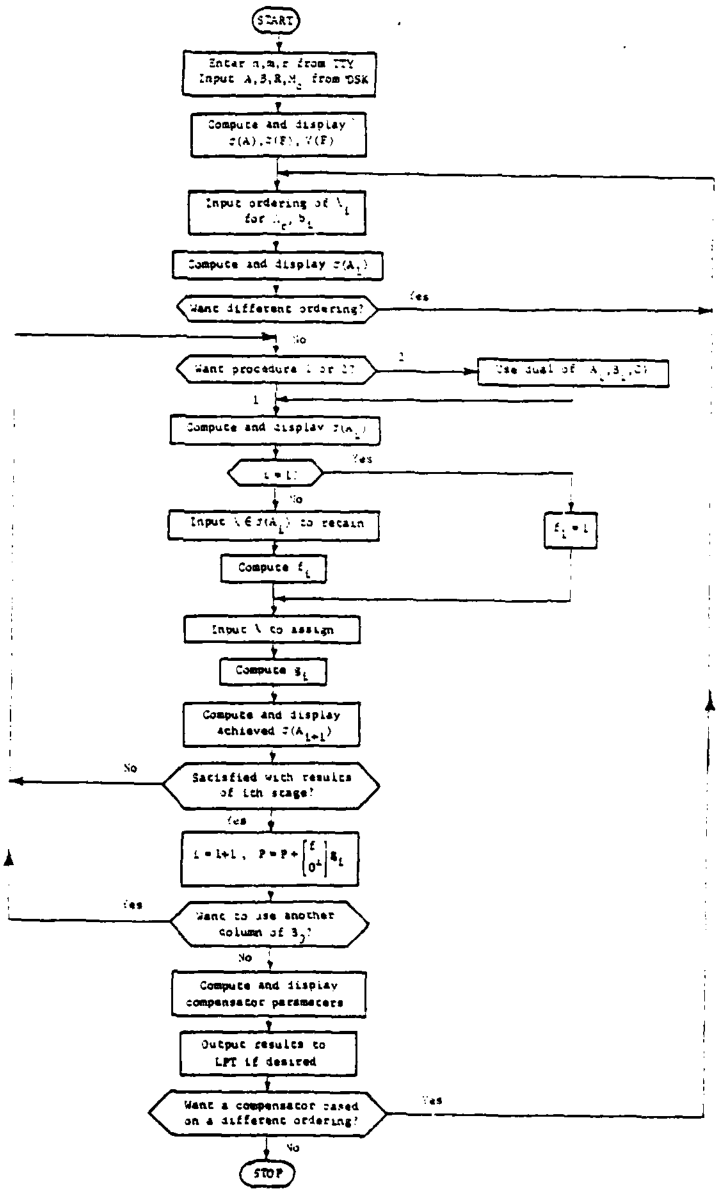


Figure 3.4.1. Flow chart for DSGN·FOR

g_i are computed as the solutions of the square inhomogeneous systems $(CQRS)^T\ g^T = \hat{P}^T$ using the subroutine LINEQ available from [34]. The routine returns a condition number for the coefficient matrix which is typed on the user's console.

All output is to the lineprinter only, though the user has the possibility of suppressing the listing of any undesired data.

CHAPTER 4

EXAMPLES

4.1. Saturn V Booster Model

In this example a second order compensator is designed for a seventh order, single-input, two-output model of a Saturn V booster. The model has appeared in three articles [23],[24],[35] on output feedback pole-placement in which numerical algorithms were employed to stabilize the system. In [24] and [35] the real part of the least stable eigenvalue of the closed loop system was minimized by two different methods and in [23] the eigenvalues of the closed loop system were constrained to lie in a prescribed region of the complex plane.

The model is given by $\dot{x} = Ax + Bu$, $y = Cx$ where A,B,C are given in Table 4.1.1.

As a preliminary analysis the possibility of stabilizing the system by output feedback was considered. In both [23] and [24] the real part of the least stable eigenvalue was required to be less than -0.07, and this resulted in a damping ratio* of less than 0.02 ($\eta = 89^o$). Relaxing the requirement that all the eigenvalues of the closed loop system lie to the left of $\sigma = -0.07$, and attempting instead to increase the damping ratio, the results given in Table 4.1.2 were obtained using the pole-placement subroutine of the compensator design software. The damping ratio has been increased to 0.1 ($\eta = 84^o$) while the least stable eigenvalue has been shifted

---

*The damping ratio of a stable matrix A is here taken to mean the smallest damping ratio of all the complex eigenvalues of $A : \xi = \min_{\sigma+j\omega\in\sigma(A)} \{-\sigma/\sqrt{\sigma^2+\omega^2}\}$. The associated angle $\eta = \cos^{-1}\xi$ is measured from the negative real axis.

Table 4.1.1. System matrices for the Saturn V booster model

The model is $\dot{x} = Ax+Bu, \quad y = Cx$

$$A = \left[\begin{array}{cc|ccccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & -0.65 & -0.002 & 2.6 & 0 \\ \hline -0.014 & 1 & -0.041 & 0.0002 & -0.015 & -0.033 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -45.0 & -0.13 & 255.0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -50.0 & -10.0 \end{array}\right] \qquad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C = \left[\begin{array}{cc|ccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Table 4.1.2. Comparison of static output feedback compensator designs for the Saturn V boosters Solution using

Here $F = A-BKC$

		Sirensa and Choi	Miller, et. al.	Solution using PPL subroutine
K		(-20.31, -16.56)	(-26.68,-16.27)	(-152.541,-42.623)
σ(F)	1.	(-4.841, 5.433)	(-4.823, 5.401)	(-4.340, 6.018)
	2.	(-4.841,-5.433)	(-4.823,-5.401)	(-4.340,-6.018)
	3.	(-0.125, 0.497)	(-0.118, 0.642)	(-0.471, 4.683)
	4.	(-0.125,-0.497)	(-0.118,-0.642)	(-0.471,-4.683)
	5.	(-0.098, 0)	(-0.105, 6.204)	(-0.250, 2.400)
	6.	(-0.070, 6.204)	(-0.105,-6.204)	(-0.250,-2.400)
	7.	(-0.070,-6.204)	(-0.078, 0)	(-0.050, 0)
ξ		0.0113	0.0169	0.1001
η		89°	89°	84°

to -0.05. The gains required to achieve this solution are large given the slight improvement in the pattern of the closed loop spectrum over the solutions of [23] and [24].

In view of these results a linear quadratic regulator problem was formulated with the expectation that a low order compensator would be required to satisfactorily shape the dynamics of the final closed loop system. Solving the state feedback regulator problem with $Q = \alpha C^T C$ and R=1 for a few values of α indicated that unless α were of the order of 10^3 or 10^4, the optimal solution would possess a complex pair with a small damping ratio, and this led to the selection of

$$R = 0.01 \qquad Q = C^T \begin{bmatrix} 500 & 0 \\ 0 & 100 \end{bmatrix} C . \tag{4.1.1}$$

The solution of the state feedback regulator problem for this choice of Q and R is given in Tables 4.1.3 and 4.1.4. In comparison to the minimum energy solution (Q=0, R=I) which simply reflects the unstable eigenvalues about the imaginary axis, this optimal regulator has perturbed the eigenvalues at $-5 \pm j5$ and -0.014 only slightly to λ_1, λ_2, and λ_7, whereas the pair $-0.065 \pm j6.708$ has been moved to the pair λ_5, λ_6 and the two real eigenvalues -0.475 and -0.420 have formed a complex pair and moved to λ_3, λ_4 (see Figure 4.1.1). It is noted that the least stable eigenvalue is very near an invariant zero of the system.*

Since r=2 and the optimal spectrum of F contains three complex pairs of eigenvalues there are only three choices for Λ_r. Computing the

---

*A survey of the literature on zeros of linear time-invariant multivariable systems is available in [36].

Table 4.1.3. Open loop eigenvalues, optimal closed loop eigenvalues, and invariant zeros of the Saturn V booster

	Open Loop Spectrum	Closed Loop Spectrum	Invariant Zeros
1.	(-5.000, 5.000)	(-5.106, 4.483)	(-4.327, 0)
2.	(-5.000,-5.000)	(-5.106,-4.483)	(-0.0462, 0)
3.	(-0.475, 0.000)	(-2.305, 7.648)	(4.401, 0)
4.	(-0.065, 6.708)	(-2.305,-7.648)	
5.	(-0.065,-6.708)	(-1.757, 0.820)	
6.	(0.014, 0.000)	(-1.757,-0.820)	
7.	(0.420, 0.000)	(-0.046, 0.000)	

Table 4.1.4. Optimal state feedback regulator solution

The Riccati solution is M_c =

$$\begin{bmatrix} 620.881 & 326.193 & 11.062 & -11.812 & -4.778 & -40.707 & -2.235 \\ 326.188 & 305.860 & 23.048 & -9.677 & -4.775 & -48.934 & -2.826 \\ 11.061 & 23.047 & 10.431 & -0.461 & -0.384 & -4.713 & -0.289 \\ -11.812 & -9.677 & -0.461 & 0.510 & 0.140 & 0.336 & -0.013 \\ -4.778 & -4.775 & -0.384 & 0.140 & 0.082 & 0.988 & 0.054 \\ -40.705 & -48.933 & -4.713 & 0.335 & 0.988 & 18.839 & 1.158 \\ -2.235 & -2.826 & -0.289 & -0.013 & 0.054 & 1.158 & 0.082 \end{bmatrix}$$

$K = R^{-1}B^TM_c$ =

$$[-223.486 \quad -282.557 \quad -28.919 \quad -1.343 \quad 5.370 \quad 115.817 \quad 8.211]$$

The closed loop matrix is $F = A-BK$ =

$$\begin{bmatrix} 0.000 & 1.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.200 & -0.650 & -0.002 & 2.600 & 0.000 \\ -0.014 & 1.000 & -0.041 & 0.0002 & -0.015 & -0.033 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 1.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & -45.000 & -0.130 & 255.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 1.000 \\ 223.486 & 282.557 & 28.919 & 1.343 & -5.370 & -165.817 & -18.211 \end{bmatrix}$$

The optimal eigenvectors are:

$$v_{\lambda_1,\lambda_2} = \begin{bmatrix} 0.001 \\ -0.010 \\ 0.000 \\ 0.026 \\ -0.755 \\ 0.030 \\ -0.253 \end{bmatrix} \pm j \begin{bmatrix} 0.001 \\ -0.003 \\ -0.001 \\ 0.138 \\ -0.587 \\ 0.023 \\ 0.018 \end{bmatrix} \quad v_{\lambda_3,\lambda_4} = \begin{bmatrix} -0.002 \\ 0.004 \\ 0.000 \\ -0.122 \\ 0.388 \\ 0.002 \\ -0.134 \end{bmatrix} \pm j \begin{bmatrix} 0.000 \\ -0.012 \\ 0.000 \\ -0.014 \\ -0.903 \\ 0.017 \\ -0.022 \end{bmatrix} \quad v_{\lambda_5,\lambda_6} = \begin{bmatrix} -0.021 \\ 0.037 \\ -0.024 \\ 0.204 \\ -0.030 \\ 0.033 \\ 0.004 \end{bmatrix} \pm j \begin{bmatrix} 0.000 \\ -0.018 \\ 0.005 \\ -0.401 \\ 0.872 \\ -0.076 \\ 0.162 \end{bmatrix}$$

$$v_{\lambda_7} = \begin{bmatrix} -0.000 \\ 0.000 \\ 0.685 \\ 0.717 \\ -0.033 \\ 0.126 \\ -0.006 \end{bmatrix}$$

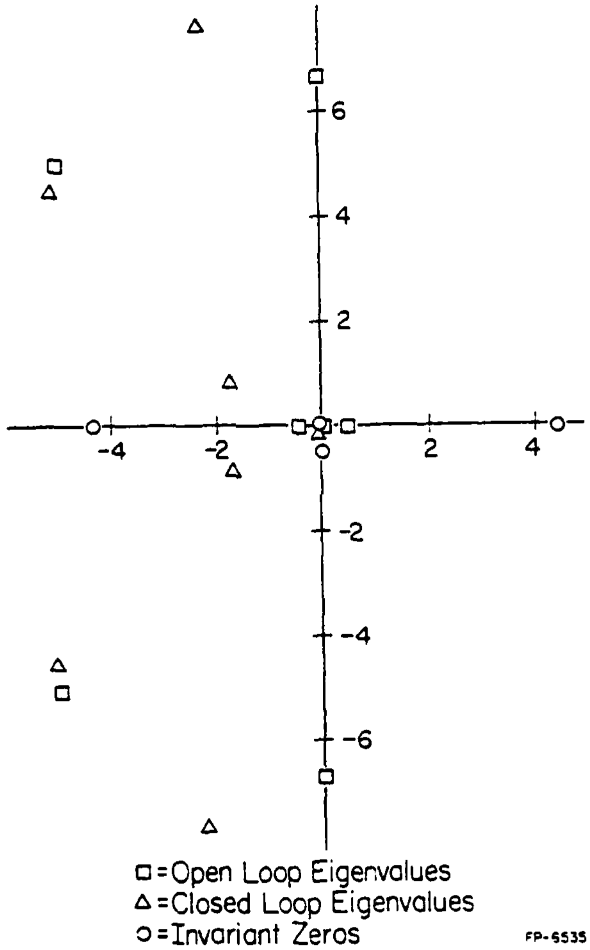


Figure 4.1.1. Open loop and optimal closed loop spectra and invariant zeros of the Saturn V booster

matrices A_1 and their spectra shows that in each case A_1 is unstable and so as expected a compensator will be required to stabilize the system. The data for this first stage of the design are given in Table 4.1.5, and Figure 4.1.2. In all three cases it should be noted that there is an eigenvalue very near the invariant zero at -0.046. It is therefore expected that in designing a dynamic compensator the spectrum of A_r will "likely" contain an eigenvalue near this location. Hence Λ_p *should not contain* $\lambda_7 = -0.046$ but rather eigenvalues which depart more from their optimal locations under static output feedback (see Figure 4.1.2).

This was confirmed as it was not possible to design a first order compensator by stabilizing A_r for any of the orderings $(\lambda_k, \lambda_{k+1}, \lambda_7)$, $k = 1,3,5$. Using two columns of B_o however, A_r was easily stabilized for several choices of Λ_r and Λ_p. Two compensators based on the orderings $\lambda_1\lambda_2\lambda_5\lambda_6$ and $\lambda_3\lambda_4\lambda_5\lambda_6$ will be discussed here. In both designs the spectrum of A_r was shaped in accordance with two criteria: first that as many eigenvalues as possible be placed at the locations of the n-r-p unretained optimal eigenvalues, and second that the damping ratio of A_o (as defined above) be no less than that of the optimal solution. In view of the latter requirement it is noted that the damping ratio of the optimal closed loop system is $\xi = 0.289$ ($\eta \simeq 73^o$).

Retaining $\lambda_1\lambda_2$ and ordering the remaining eigenvalues $\lambda_5\lambda_6\lambda_7\lambda_3\lambda_4$, with the real eigenvalue λ_7 placed third in anticipation of designing either a second or a third order compensator, the pole-placement subroutine was used to place the spectrum of A_r. Since the first row of A_{12} is all zeros it was necessary to introduce $P = \hat{P}T$, $T = (0,1)$ and to solve the pole-placement problem

Table 4.1.5. Data for various choices of Λ_r

Based on retention of λ_1, λ_2:

The spectrum of A_1 is:

1. (-0.194, 7.095)
2. (-0.194, -7.095)
3. (-0.065, 0.000)
4. (0.247, 0.729)
5. (0.247, -0.729)

$$N_o = \begin{bmatrix} -0.771 & -0.113 \\ 128.864 & 8.119 \\ -374.840 & 45.949 \\ 14.231 & -1.854 \\ 85.580 & 33.161 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} -0.018 & -0.073 & -0.015 & 0.260 & 0.000 \\ -1.624 & 5.278 & 1.016 & -21.111 & 0.000 \\ -9.190 & -15.133 & -0.038 & 135.534 & 0.000 \\ 0.371 & -1.205 & -0.004 & 4.820 & 1.000 \\ -6.632 & 21.555 & 0.066 & -136.220 & -10.000 \end{bmatrix}$$

Based on retention of λ_3, λ_4:

The spectrum of A_1 is:

1. (-4.767, 3.087)
2. (-4.767, -3.087)
3. (-0.047, 0.000)
4. (2.010, 2.973)
5. (2.010, -2.973)

$$N_o = \begin{bmatrix} -0.168 & -0.021 \\ 80.281 & 1.251 \\ -79.796 & 74.516 \\ -4.519 & -1.401 \\ 89.397 & 1.939 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} -0.037 & -0.014 & -0.015 & 0.022 & 0.000 \\ -0.250 & 0.813 & 1.003 & -3.252 & 0.000 \\ -14.903 & 3.436 & 0.019 & 61.257 & 0.000 \\ 0.280 & -0.911 & -0.003 & 3.643 & 1.000 \\ -0.388 & 1.260 & 0.004 & -55.041 & -10.000 \end{bmatrix}$$

Based on retention of λ_5, λ_6:

The spectrum of A_1 is:

1. (-5.565, 8.109)
2. (-5.565, -8.109)
3. (-0.050, 0.000)
4. (2.261, 4.187)
5. (2.261, -4.187)

$$N_o = \begin{bmatrix} 0.630 & -0.284 \\ 30.743 & 23.080 \\ -86.803 & -50.379 \\ 6.124 & 4.401 \\ -16.553 & -9.346 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 0.016 & -0.184 & -0.016 & 0.704 & 0.000 \\ -4.616 & 15.002 & 1.046 & -60.009 & 0.000 \\ 10.076 & -77.746 & -0.231 & 385.985 & 0.000 \\ -0.880 & 2.861 & 0.009 & -11.443 & 1.000 \\ 1.869 & -6.075 & -0.019 & -25.702 & -10.000 \end{bmatrix}$$

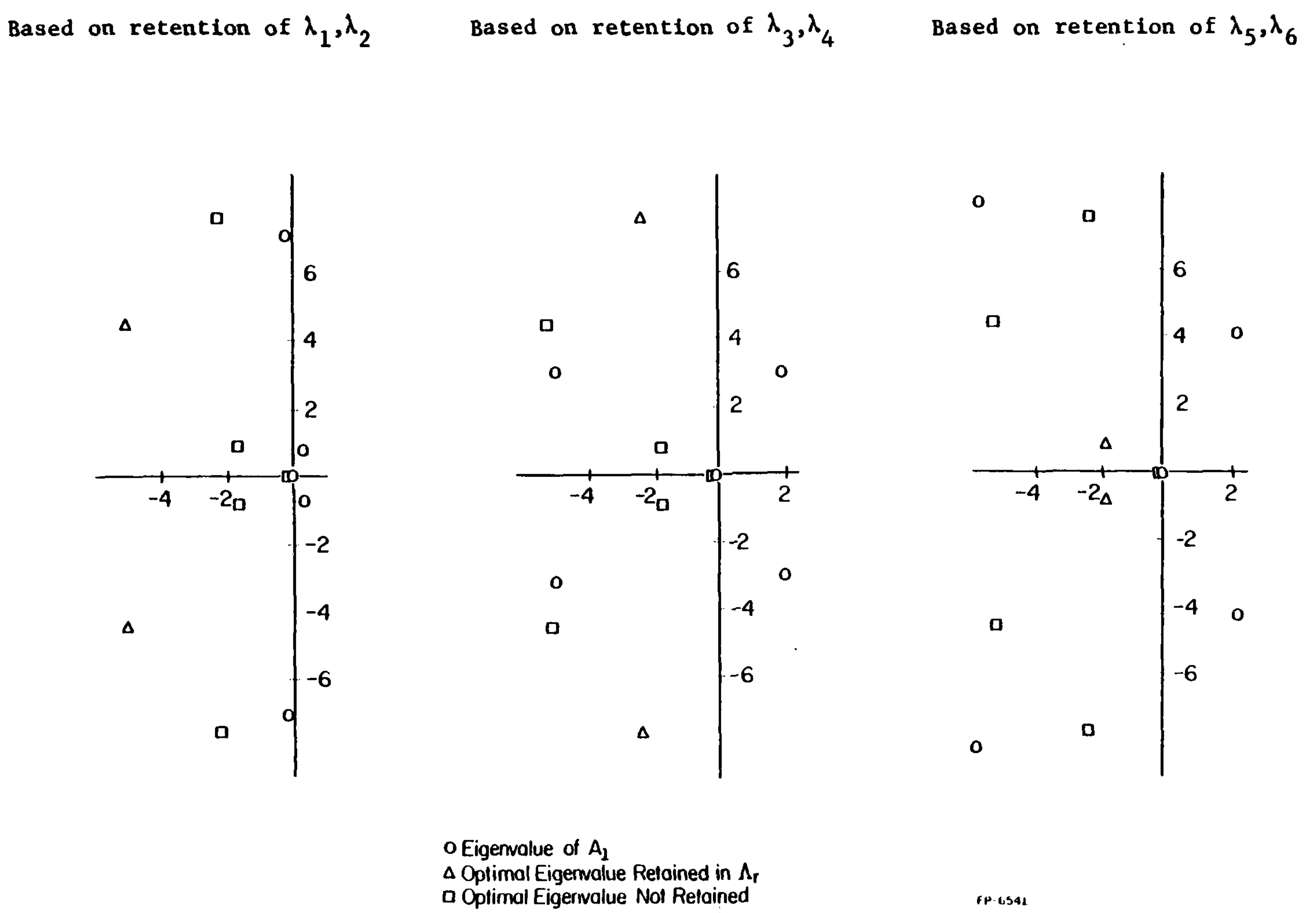


Figure 4.1.2. Spectra of suboptimal closed loop systems under static compensation

$$A_r = A_1 + B_o\hat{P}(TA_{12}) \in R^{5\times 5}. \tag{4.1.2}$$

Thus at the ith stage of the algorithm only i poles could be arbitrarily assigned. Since at least two columns of B_o were to be used, at the first stage a real pole was arbitrarily placed at -2 using Procedure 1. Then using Procedure 2 a complex pair was placed at $-1 \pm j3.5$ ($\xi = 0.275$, $\eta = 74^o$) resulting in another complex pair at $-1.625 \pm j5.41$ ($\xi = 0.288$, $\eta = 73^o$) and a real pole very near λ_7 (and the invariant zero at -0.0462). Attempts at placing poles near λ_3, λ_4 resulted in an unstable A_r. In view of the criteria above, this solution of the pole-placement problem was satisfactory and the degree of the required compensator taken to be $p = 2$. Data for the pole-placement problem is given in Table 4.1.6.

The parameters of the compensator are:

$$\begin{aligned} H_o &= \begin{bmatrix} -6.258 & 0.315 \\ -24.226 & -2.602 \end{bmatrix} & K_z &= [-1.221 \quad 1.959] \\ D_o &= \begin{bmatrix} -267.925 & -31.216 \\ -1212.909 & -59.339 \end{bmatrix} & K_y &= [-36.437 \quad -30.255] \end{aligned} \tag{4.1.3}$$

and the total closed loop matrix is

$$A_c = \left[\begin{array}{rr|rr|rrrrr} -6.258 & 0.315 & -267.925 & -31.216 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ -24.226 & -2.602 & -1212.909 & -59.339 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ \hline 0.000 & 0.000 & 0.000 & 1.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.200 & -0.650 & -0.002 & 2.600 & 0.000 \\ 0.000 & 0.000 & -0.014 & 1.000 & -0.041 & 0.000 & -0.015 & -0.033 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 1.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & -45.000 & -0.130 & 255.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 1.000 \\ 1.221 & -1.959 & 36.437 & 30.255 & 0.000 & 0.000 & 0.000 & -50.000 & -10.000 \end{array}\right] \tag{4.1.4}$$

Table 4.1.6. Data for pole-placement problem based on ordering $\lambda_1,\lambda_2,\lambda_5,\lambda_6$

$$f_1 = [1.000] \qquad f_2 = \begin{bmatrix} 3.029 \\ 21.055 \end{bmatrix} \qquad p = \begin{bmatrix} 0.000 & 4.651 \\ 0.000 & 21.055 \end{bmatrix}$$

$$g_1 = [1.621] \qquad g_2 = [1.000]$$

The spectrum of A_2 is:

1. (-2.000, 0.000)
2. (-0.039, 0.000)
3. (-0.038, 7.130)
4. (-0.038, -7.130)
5. (1.107, 0.000)

The spectrum of $A_3 = A_r$ is:

1. (-1.625, 5.407)
2. (-1.625, 5.407)
3. (-1.000, 3.500)
4. (-1.000, -3.500)
5. (-0.055, 0.000)

$$B_o = \begin{bmatrix} -0.036 & 0.003 \\ 2.624 & -0.279 \\ -9.639 & 1.742 \\ 0.402 & -0.111 \\ 0.585 & 0.731 \end{bmatrix}$$

$$A_2 = A_1 + b_1 f_1 g_1 (TA_{12}) =$$

$$\begin{bmatrix} -0.030 & -0.035 & -0.015 & 0.108 & 0.000 \\ -0.773 & 2.512 & 1.008 & -10.049 & 0.000 \\ -12.316 & -4.974 & -0.007 & 94.898 & 0.000 \\ 0.501 & -1.629 & -0.005 & 6.516 & 1.000 \\ -6.443 & 20.939 & 0.064 & -133.755 & -10.000 \end{bmatrix}$$

$$A_r = A_3 = A_2 + [b_1 b_2] f_2 g_2 (TA_{12}) =$$

$$\begin{bmatrix} -0.038 & -0.009 & -0.015 & 0.003 & 0.000 \\ -0.360 & 1.170 & 1.004 & -4.679 & 0.000 \\ -10.818 & -9.841 & -0.022 & 114.363 & 0.000 \\ 0.276 & -0.896 & -0.003 & 3.586 & 1.000 \\ -3.009 & 9.781 & 0.030 & -89.123 & -10.000 \end{bmatrix}$$

Referring to Table 4.1.7 and Figure 4.1.3 it is verified that the compensated system has retained from the optimal state regulator a four dimensional subspace spanned by the eigenvectors corresponding to $\lambda_1, \lambda_2, \lambda_5, \lambda_6$. Also the compensator is open loop stable.

Recalling that H, D, and K_z are determined only up to a similarity transformation under W_p, the pair (D, K_z) may be balanced by introducing

$$\hat{H}_z = W_p H_o W_p^{-1}, \quad \hat{D} = W_p D_o, \quad \hat{K}_z = K_z W_p^{-1}. \tag{4.1.5}$$

For example, if $W_p = \frac{1}{25} I_{2\times 2}$ then H_o and K_y are unchanged and

$$\hat{D} = \begin{bmatrix} -10.717 & -1.249 \\ -48.516 & -2.374 \end{bmatrix} \qquad \hat{K}_z = [-30.525 \quad 48.975] \tag{4.1.6}$$

For purposes of comparison a design is given based on the ordering $\lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_1, \lambda_2$. The pole-placement problem was solved by first placing a pole arbitrarily at -1.0, and then in the second stage of the algorithm, placing a complex pair at $-1.0 \pm j3.0$. This resulted in a spectrum for A_r which met the criteria given before, and again placed a real pole near λ_7. The data for the pole-placement problem are given in Table 4.1.8. The parameters of the resultant compensator are:

$$\begin{aligned} H_o &= \begin{bmatrix} -30.530 & 4.925 \\ -68.884 & 8.858 \end{bmatrix} \quad K_z = [\,-15.881 \quad 3.919] \\ D_o &= \begin{bmatrix} -970.743 & 224.189 \\ -2204.778 & 582.655 \end{bmatrix} \quad K_y = [-544.318 \quad 76.387]. \end{aligned} \tag{4.1.7}$$

Again the matrices D_o and K_z may be scaled to give ($W_p = \frac{1}{12} I_{2x2}$):

$$\hat{D}_o = \begin{bmatrix} -80.895 & 18.682 \\ -183.732 & 48.555 \end{bmatrix} \quad \hat{K}_z = [-190.572 \quad 47.028]. \tag{4.1 .8}$$

Comparing the two compensator designs it is seen that that based on retention of $\lambda_1 \lambda_2 \lambda_5 \lambda_6$ resulted in feedback gains more than an order of magnitude smaller than those obtained in the design based on retention of

Table 4.1.7. Spectra of closed loop compensated Saturn V booster and of open loop compensator

		Design based on retention of $\lambda_1\lambda_2\lambda_5\lambda_6$	Design based on retention of $\lambda_3\lambda_4\lambda_5\lambda_6$
$\sigma(A_c)$	1.	(-5.106, 4.483) : λ_1	(-18.750, 0.000)
	2.	(-5.106, -4.483) : λ_2	(-2.908, 0.000)
	3.	(-1.757, 0.820) : λ_5	(-2.305, 7.648) : λ_3
	4.	(-1.757, -0.820) : λ_6	(-2.305, -7.648) : λ_4
	5.	(-1.625, 5.407)	(-1.757, 0.820) : λ_5
	6.	(-1.625, -5.407)	(-1.757, -0.820) : λ_6
	7.	(-1.000, 3.500)	(-1.000, 3.000)
	8.	(-1.000, -3.500)	(-1.000, -3.000)
	9.	(-0.055, 0.000)	(-0.061, 0.000)
$\sigma(H_o)$	1.	(-4.430, 2.070)	(-17.808, 0.000)
	2.	(-4.430, -2.070)	(-3.863, 0.000)

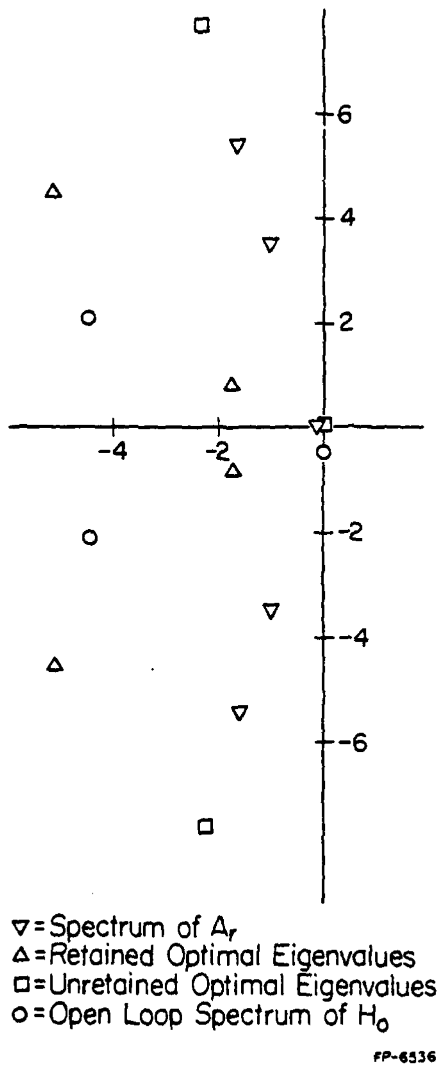


Figure 4.1.3. Spectrum of compensated Saturn V booster under retention of $\lambda_1, \lambda_2, \lambda_5, \lambda_6$

Table 4.1.8. Data for pole-placement problem based on ordering $\lambda_3, \lambda_4, \lambda_5, \lambda_6$

$$f_1 = [\ 1.000] \qquad g_1 = [4.766] \qquad p = \begin{bmatrix} 0.000 & 14.548 \\ 0.000 & 33.041 \end{bmatrix}$$

$$f_2 = \begin{bmatrix} 9.781 \\ 33.041 \end{bmatrix} \qquad g_2 = [1.000]$$

The spectrum of A_2 is:

1. (-9.739, 0.000)
2. (-1.000, 0.000)
3. (-0.305, 3.496)
4. (-0.305, -3.496)
5. (-0.060, 0.000)

The spectrum of $A_3 = A_r$ is:

1. (-18.750, 0.000)
2. (-2.908, 0.000)
3. (-1.000, 3.000)
4. (-1.000, -3.000)
5. (-0.061, 0.000)

$$B_o = \begin{bmatrix} -0.027 & 0.005 \\ 1.852 & -0.392 \\ -4.468 & 2.197 \\ -0.010 & -0.100 \\ 1.819 & 0.181 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -0.062 & 0.069 & -0.015 & -0.307 & 0.000 \\ 1.515 & -4.925 & 0.985 & 19.701 & 0.000 \\ -19.162 & 17.276 & 0.062 & 5.894 & 0.000 \\ 0.271 & -0.879 & -0.003 & 3.517 & 1.000 \\ 1.346 & -4.375 & -0.013 & -32.499 & -10.000 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} -0.083 & 0.135 & -0.015 & -0.574 & 0.000 \\ 2.547 & -8.277 & 0.975 & 33.107 & 0.000 \\ -13.386 & -1.494 & 0.004 & 80.976 & 0.000 \\ -0.413 & 1.341 & 0.004 & -5.363 & 1.000 \\ 6.102 & -19.831 & -0.061 & 29.324 & -10.000 \end{bmatrix}$$

AD-A085 075   ILLINOIS UNIV AT URBANA-CHAMPAIGN DECISION AND CONTROL LAB   F/G 9/3
OUTPUT FEEDBACK POLE-PLACEMENT IN THE DESIGN OF COMPENSATORS FO--ETC(U)
JUN 79   W E HOPKINS   N00014-79-C-0424

UNCLASSIFIED   DC-847   NL

2 of 2
AD A
085075

END
DATE
FILMED
7-80
DTIC

$\lambda_3\lambda_4\lambda_5\lambda_6$. Recall that in comparing the optimal state feedback regulator with the minimum energy solution, it was noted that the pair $\lambda_1\lambda_2$ was slightly perturbed from $-5 \pm j5$ whereas the pair $\lambda_3\lambda_4$ arose from two real poles near -0.45 forming a complex pair and leaving the real axis. It may be inferred that this has been reflected in the second design by a large expenditure of energy to retain the pair $\lambda_3\lambda_4$. Apparently the first design is preferable.

It should also be noted that in all of the static and dynamic designs, including the unstable designs based on a zero order compensator, there is a real pole near the invariant zero at -0.0462. This is consistent with the fact that under high gain output feedback a number of the poles of the closed loop system tend to the finite invariant zeros [36]. It is interesting to note that the system appears to have two widely separated time scales in the sense that the gains of interest are "high gain" relative to the zero at -0.0462, but not relative to the other two zeros, which do not have any easily discernable influence on the system.

In summary the second order compensator design obtained for the ordering $\lambda_1\lambda_2\lambda_5\lambda_6$ compares favorably in dimension with the order of the reduced order observer $(n-r=5)$ and with the bound on the dimension required for arbitrary placement of the spectrum of A_r $(n-r-\ell+1=5)$. The compensator is open loop stable, retains a four dimensional invariant subspace of the optimal state feedback regulator in which there is no cost degradation, requires modest gains, and achieves a damping ratio of 0.275 ($\eta \cong 74^{\circ}$).

4.2. Fifth Order Example

This is a trivial example illustrating the application of the design methodology to stabilize systems. Let the system be given by

$$\dot{\hat{x}} = \left[\begin{array}{cc|ccc} 0 & 1 & 0 & 0 & 0 \\ -6 & -5 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 \end{array}\right] \hat{x} + \left[\begin{array}{c} 0 \\ 0 \\ \hline 0 \\ 0 \\ 1 \end{array}\right] u, \qquad y = \left[\begin{array}{cc|ccc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right] \hat{x} \tag{4.2.1}$$

and introduce the permutation of states $x = T\hat{x}$

$$T = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.2.2}$$

to obtain the system $\dot{x} = Ax + Bu$, $y = Cx$ where

$$A = \begin{bmatrix} -5 & 0 & -6 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}. \tag{4.2.3}$$

In this basis x_1, x_3 are the state variables of the uncontrollable subsystem with eigenvalues at -2,-3, while x_2, x_4, x_5 are the state variables of the controllable subsystem with eigenvalues at -1,0,0.

The solution of the state feedback regulator problem with R=1 and $Q = dg(1,5,0,2,0)$ is given in Tables 4.2.1 and 4.2.2. It is noted that the optimal controllable eigenvectors have the form given in (3.3.16). To insure the invertibility of Y at most one controllable eigenvector may be retained in Λ_r, and the possible choices are $(\sigma_1,\sigma_2),(\sigma_1,\lambda_1),(\sigma_2,\lambda_1)$. The spectra of the resulting matrices A_1 are given in Table 4.2.2. It is noted that the controllable subspectrum of A_1 is the same in both the cases $\Lambda_1 = dg(\sigma_1\lambda_1)$ and $\Lambda_1 = dg(\sigma_2\lambda_1)$ as predicted by the form of the block diagonal entries of A_1 in (3.3.18).

Table 4.2.1. Solution of state feedback regulator for fifth order example

$$M_c = \begin{bmatrix} 0.1146 & 0.0518 & 0.0817 & 0.1826 & 0.1309 \\ 0.0518 & 8.7407 & 0.5519 & 6.6400 & 2.2361 \\ 0.0817 & 0.5519 & 1.0782 & 1.3728 & 0.8610 \\ 0.1826 & 6.6400 & 1.3728 & 9.3716 & 3.9090 \\ 0.1309 & 2.2361 & 0.8610 & 3.9090 & 1.9695 \end{bmatrix}$$

$$K = [\ 0.1309 \quad 2.2361 \quad 0.8610 \quad 3.9090 \quad 1.9695\ ]$$

$$F = \begin{bmatrix} -5.0000 & 0.0000 & -6.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 1.0000 & 0.0000 \\ 1.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 \\ -0.1309 & -2.2361 & 0.1390 & -3.9090 & -2.9695 \end{bmatrix}$$

The optimal closed loop eigenvectors are (see Table 4.2.2)

$$v_{\sigma_1} = \begin{bmatrix} 0.9361 \\ \underline{0.0170} \\ -0.3120 \\ -0.0510 \\ 0.1530 \end{bmatrix} \qquad v_{\sigma_2} = \begin{bmatrix} 0.8057 \\ \underline{0.0948} \\ -0.4028 \\ -0.1896 \\ 0.3791 \end{bmatrix}$$

$$v_{\lambda_1} = \begin{bmatrix} 0.0000 \\ \underline{0.4332} \\ 0.0000 \\ -0.5547 \\ 0.7104 \end{bmatrix} \qquad v_{\lambda_2\lambda_3} = \begin{bmatrix} 0.0000 \\ \underline{0.3711} \\ 0.0000 \\ -0.5031 \\ 0.2017 \end{bmatrix} \pm j \begin{bmatrix} 0.0000 \\ \underline{0.1866} \\ 0.0000 \\ 0.2196 \\ -0.6968 \end{bmatrix}$$

Table 4.2.2. Eigenvalues of various matrices of the fifth order example

Spectrum of A	Spectrum of F	
1. (-3.000, 0.000)	1. (-3.000, 0.000)	: σ_1
2. (-2.000, 0.000)	2. (-2.000, 0.000)	: σ_2
3. (-1.000, 0.000)	3. (-1.281, 0.000)	: λ_1
4. (0.000, 0.000)	4. (-0.844, 1.061)	: λ_2
5. (0.000, 0.000)	5. (-0.844, -1.016)	: λ_3

Spectrum of A_c	Spectrum of H_o
1. (-3.000, 0.000)	1. (-2.485, 2.485)
2. (-2.000, 0.000)	2. (-2.485, -2.485)
3. (-1.500, 1.500)	
4. (-1.500, -1.500)	
5. (-1.281, 0.000)	
6. (-0.844, 1.016)	
7. (-0.844, -1.016)	

Spectra of possible matrices A_1

Eigenvalues retained in Λ_r	Spectrum of A_1
$(\sigma_1\sigma_2)$:	1. (-0.702, 0.274)
	2. (-0.702, -0.274)
	3. (0.404, 0.000)
$(\sigma_1\lambda_1)$:	1. (-2.000, 0.000)
	2. (0.140, 0.583)
	3. (0.140, -0.583)
$(\sigma_2\lambda_1)$:	1. (-3.000, 0.000)
	2. (0.140, 0.583)
	3. (0.140, -0.583)

Since all three static compensators are unstable, a dynamic compensator is designed. Consistent with the desire to retain as many controllable optimal eigenvectors as possible the design is based on the ordering $(\sigma_2\lambda_1\lambda_2\lambda_3\sigma_1)$. (The ordering $(\sigma_1\lambda_1\lambda_2\lambda_3\sigma_2)$ would serve equally well.) A second order compensator will be required since (λ_2,λ_3) is a complex pair.

The reduced pole-placement problem of (3.3.19) is solved in two stages. Since the final closed loop spectrum will retain the entire optimal spectrum of F, the choice of poles to assign is arbitrary and taken to be $-1.5\pm j1.5$. At the first stage a pole is arbitrarily placed at 0.0 using Procedure 1, and at the second stage the complex pair is assigned using Procedure 2. The data for this pole-placement problem are given in Table 4.2.3. In the notation of (3.3.21),(3.3.22) the parameters of the resultant compensator are:

$$H_o^c = \begin{bmatrix} -0.3698 & 1.2551 \\ -8.4828 & -4.5997 \end{bmatrix} \quad D_o^{\bar{c}} = \begin{bmatrix} -0.2332 \\ 0.0830 \end{bmatrix} \quad D_o^c = \begin{bmatrix} -1.2791 \\ 20.1197 \end{bmatrix}$$

$$N_p = \begin{bmatrix} 0 \\ N_p^c \end{bmatrix} = \begin{bmatrix} 0.0000 & 0.0000 \\ \hline -1.2453 & -0.1537 \\ -0.7261 & -1.1634 \end{bmatrix} \quad N_r = \begin{bmatrix} N_o^{\bar{c}} & 0 \\ N_{21} & N_r^c \end{bmatrix} = \left[\begin{array}{c|c} -0.5000 & 0.0000 \\ \hline -0.0846 & 2.0000 \\ 0.2777 & 2.5000 \end{array}\right]$$

$$K_z^c = [-6.2978 \quad -2.8921]$$

$$K_y^{\bar{c}} = [-0.0835] \qquad K_y^c = [14.9777]. \tag{4.2.4}$$

In the original basis the total closed loop matrix is

$$A_c = \left[\begin{array}{cc|c:cc:cc} -0.3698 & 1.2551 & 0 & -0.2332 & -1.2791 & 0 & 0 \\ -8.4828 & -4.5997 & 0 & 0.0830 & 20.1197 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -6 & -5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 6.2978 & 2.8921 & 1 & 0.0835 & -14.9777 & 0 & -1 \end{array}\right] \tag{4.2.5}$$

Table 4.2.3. Data for pole-placement problem for fifth order example

Data for full problem (see equation (3.3.19)):

$$N_o = \left[\begin{array}{c|c} -0.5000 & -0.0000 \\ \hline -0.0846 & -1.2805 \\ 0.2777 & 1.6398 \end{array}\right] \qquad B_o = \left[\begin{array}{cc} 0.0000 & 0.0000 \\ \hline -0.0279 & 0.4586 \\ -0.4069 & -1.0028 \end{array}\right]$$

$$A_1 = \left[\begin{array}{c|cc} -3.0000 & 0.0000 & 0.0000 \\ \hline -0.5078 & 1.2805 & 1.0000 \\ 2.6660 & -1.6398 & -1.0000 \end{array}\right] \qquad P = \left[\begin{array}{c|c} 0.0000 & 17.1739 \\ 0.0000 & -6.1098 \end{array}\right]$$

$$\begin{bmatrix} A_{12} & 0 \\ A_{32} & A_{34} \end{bmatrix} = \left[\begin{array}{c|cc} -6 & 0 & 0 \\ \hline 0 & 1 & 0 \end{array}\right]$$

Data for reduced problem:

$$f_1 = [\ 1.0000] \qquad f_2 = \begin{bmatrix} 18.0003 \\ -6.1098 \end{bmatrix}$$

$$g_1 = [-0.826] \qquad g_2 = [1.000]$$

$$A_2 = \begin{bmatrix} 1.3036 & 1.0000 \\ -1.3036 & -1.0000 \end{bmatrix} \qquad A_3 = \begin{bmatrix} -2.0000 & 1.0000 \\ -2.5000 & -1.0000 \end{bmatrix}$$

The spectrum of A_2 is:

1. (0.000, 0.000)
2. (0.304, 0.000)

The spectrum of A_3 is:

1. (-1.500, 1.500)
2. (-1.500, -1.500)

This compensator retains all the optimal eigenvalues of the state regulator and is open loop stable.

4.3. Twelfth Order Nuclear Reactor Model

In this example a single input, three output, twelfth order model of a nuclear reactor is considered. In a preliminary analysis the system structure of the model will be discussed and the output feedback pole-placement problem solved, and then a linear quadratic regulator problem will be defined on the basis of which both a static and a first order dynamic compensator design will be given.

The model is taken from [24] and referring to Table 4.3.1 is given by

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ B_2 \end{bmatrix}u, \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{4.3.1}$$

The system consists of a seventh order uncontrollable subsystem A_{11} driving a fifth order controllable subsystem A_{22}. The eigenvalues of these two systems will be denoted $\{\sigma_i\}_{i=1}^{7}$ and $\{\lambda_i\}_{i=1}^{5}$ respectively and are given in Table 4.3.2 and Figure 4.3.1. The observability structure of the system may be obtained by inspection of the eigenvectors given in Table 4.3.3:

a) The uncontrollable eigenvalues (σ_1, σ_6) are also unobservable.

b) The first row of C observes a fifth order uncontrollable subsystem with eigenvalues $(\sigma_{2,3}, \sigma_{4,5}, \sigma_7)$.

c) The second row of C observes a first order controllable subsystem with a zero eigenvalue.

Table 4.3.1. System matrices for the nuclear reactor model

The model is $\dot{x} = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix} x + \begin{pmatrix} 0 \\ B_2 \end{pmatrix} u, \quad y = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} x$ where:

$$A_{11} = \begin{bmatrix} -0.4044 & 0.000 & 0.000 & 0.4044 & 0.000 & 0.000 & 0.000 \\ 0.000 & -0.4044 & 0.000 & 0.000 & 0.4044 & 0.000 & 0.000 \\ 0.000 & 0.000 & -0.4044 & 0.000 & 0.000 & 0.4044 & 0.000 \\ 0.01818 & 0.000 & 0.000 & -0.5363 & 0.000 & 0.000 & 0.4545 \\ 0.000 & 0.0818 & 0.000 & 0.4545 & -0.5363 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.0818 & 0.000 & 0.4545 & -0.5363 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.150 & 0.000 & -0.150 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & -7.500 & 0.000 & 0.000 & 75.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \end{bmatrix}$$

$$A_{22} = \begin{bmatrix} 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 600.000 & -74.995 & 0.033 & 0.346 & 0.621 \\ 0.000 & 2.475 & -0.033 & 0.000 & 0.000 \\ 0.000 & 25.950 & 0.000 & -0.346 & 0.000 \\ 0.000 & 46.570 & 0.000 & 0.000 & -0.621 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad c_1 = [0000001] \qquad c_2 = \begin{bmatrix} 10000 \\ 01000 \end{bmatrix}$$

Table 4.3.2. Open and closed loop eigenvalues of the nuclear reactor model

The uncontrollable and controllable eigenvalues of A are the eigenvalues of A_{11} and A_{22} respectively. The eigenvalues of the optimal state feedback regulator are those of A_{11} and F_{22}. The eigenvalues of the final compensated system are those of A_{11} together with the controllable eigenvalues of A_c.

Spectrum of A_{11} (σ_i)	Spectrum of A_{22} (λ_i)	Spectrum of F_{22} (λ_i)
1. (-0.664, 0.000)	1. (-75.502, 0.000)	1. (-75.197, 0.000)
2. (-0.631, 0.195)	2. (-0.446, 0.000)	2. (-13.051, 12.119)
3. (-0.631, -0.195)	3. (-0.0476, 0.000)	3. (-13.051, -12.119)
4. (-0.379, 0.034)	4. (0.000 , 0.000)	4. (-0.399, 0.000)
5. (-0.379, -0.034)	5. (0.000 , 0.000)	5. (-0.034, 0.000)
6. (-0.277, 0.000)		
7. (-0.0112, 0.000)		

Controllable eigenvalues of A_c	Open loop spectrum of H_o
1. (-75.197, 0.000)	1. (-76.132, 0)
2. (-13.051, 12.119)	
3. (-13.051,-12.119)	
4. (-7.000, 0.000)	
5. (-0.408, 0.000)	
6. (-0.034, 0.000)	

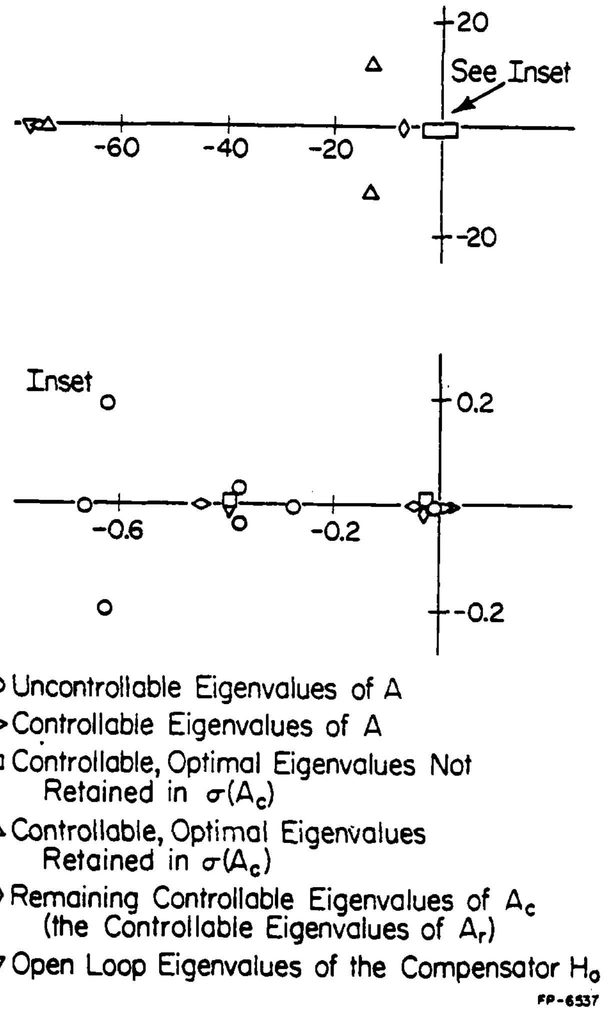


Figure 4.3.1. Eigenvalues of the nuclear reactor model

Table 4.3.3. Open loop eigenvectors of the nuclear reactor model

The eigenvectors corresponding to controllable eigenvalues of A_{22} are (see Table 4.3.2 for the definition of λ_i, σ_i):

$$v_{\lambda_1} = \left[\begin{array}{r} 0.0000 \\ 0.0000 \\ 0.0000 \\ 0.0000 \\ 0.0000 \\ 0.0000 \\ 0.0000 \\ \hline 0.0000 \\ -0.8146 \\ 0.0267 \\ 0.2813 \\ 0.5066 \end{array}\right] \quad v_{\lambda_2} = \left[\begin{array}{r} 0.0000 \\ 0.0000 \\ 0.0000 \\ 0.0000 \\ 0.0000 \\ 0.0000 \\ 0.0000 \\ \hline 0.0000 \\ 0.0027 \\ -0.0161 \\ -0.7003 \\ 0.7137 \end{array}\right] \quad v_{\lambda_3} = \left[\begin{array}{r} 0.0000 \\ 0.0000 \\ 0.0000 \\ 0.0000 \\ 0.0000 \\ 0.0000 \\ 0.0000 \\ \hline 0.0000 \\ 0.0048 \\ -0.8179 \\ 0.4205 \\ 0.3927 \end{array}\right] \quad v_{\lambda_4} = \left[\begin{array}{r} 0.0000 \\ 0.0000 \\ 0.0000 \\ 0.0000 \\ 0.0000 \\ 0.0000 \\ 0.0000 \\ \hline 0.0000 \\ -0.0077 \\ -0.5774 \\ -0.5774 \\ -0.5773 \end{array}\right]$$

The eigenvectors corresponding to uncontrollable eigenvalues of A_{11} are:

$$v_{\sigma_1} = \left[\begin{array}{r} -0.0000 \\ 0.0000 \\ -0.8417 \\ 0.0000 \\ -0.0000 \\ 0.5399 \\ -0.0000 \\ \hline 0.0000 \\ -0.0000 \\ 0.0000 \\ -0.0000 \\ 0.0000 \end{array}\right] \quad v_{\sigma_2}, v_{\sigma_3} = \left[\begin{array}{r} 0.0071 \\ 0.0029 \\ -0.0210 \\ -0.0049 \\ 0.0043 \\ 0.0137 \\ -0.0003 \\ \hline 0.0000 \\ 0.0039 \\ -0.0139 \\ -0.2167 \\ 0.1087 \end{array}\right] \pm j \left[\begin{array}{r} 0.0020 \\ -0.0123 \\ -0.0041 \\ 0.0023 \\ 0.0082 \\ -0.0078 \\ -0.0027 \\ \hline 0.0000 \\ 0.0007 \\ -0.0072 \\ -0.2075 \\ -0.9470 \end{array}\right]$$

$$v_{\sigma_4}, v_{\sigma_5} = \left[\begin{array}{r} 0.0550 \\ -0.0398 \\ -0.0001 \\ 0.0072 \\ -0.0009 \\ -0.0023 \\ 0.0001 \\ \hline 0.0000 \\ -0.0003 \\ 0.0008 \\ -0.5548 \\ -0.0966 \end{array}\right] \pm j \left[\begin{array}{r} -0.0453 \\ -0.0189 \\ 0.0272 \\ 0.0018 \\ -0.0045 \\ 0.0017 \\ 0.0030 \\ \hline 0.0000 \\ -0.0017 \\ 0.0122 \\ 0.7594 \\ -0.3133 \end{array}\right] \quad v_{\sigma_6} = \left[\begin{array}{r} -0.0000 \\ -0.0000 \\ 0.9537 \\ 0.0000 \\ 0.0000 \\ 0.3007 \\ 0.0000 \\ \hline 0.0000 \\ 0.0000 \\ -0.0000 \\ 0.0000 \\ -0.0000 \end{array}\right] \quad v_{\sigma_7} = \left[\begin{array}{r} 0.0003 \\ 0.0003 \\ 0.0003 \\ 0.0003 \\ 0.0003 \\ 0.0003 \\ 0.0003 \\ \hline 0.0000 \\ -0.0064 \\ -0.7223 \\ -0.4926 \\ -0.4853 \end{array}\right]$$

d) The third row of C observes a tenth order system consisting of two fifth order controllable and uncontrollable subsystems with eigenvalues $(\lambda_1,\lambda_{2,3},\lambda_4,\lambda_5)$ and $(\sigma_{2,3},\sigma_{4,5},\sigma_7)$ respectively.

Since output feedback only affects the eigenvalues of the controllable and observable subsystem, a preliminary analysis of the pole-placement problem under static output feedback compensation may be carried out on the controllable and observable triple (A_{22},B_2,C_2) under feedback $u=-(k_2k_3)C_2$. (Here $K=(k_1,k_2,k_3)\in R^{3\times 1}$.) When the linear quadratic regulator problem is defined, however, it will not be possible to exploit the uncontrollability of A_{11} to reduce the dimensionality of the system since uncontrollable eigenvectors are not invariant under feedback, and are shaped in accordance with the selected cost structure (Q,R). It will then be advantageous to retain the feedback gain k_1 through C_1.

The matrices of the subsystem (A_{22},B_2,C_2) are given by:

$$A_{22}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \gamma & -(\beta_1+\beta_2+\beta_3) & \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & \beta_1 & -\alpha_1 & 0 & 0 \\ 0 & \beta_2 & 0 & -\alpha_2 & 0 \\ 0 & \beta_3 & 0 & 0 & -\alpha_3 \end{bmatrix} \quad B_2=\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad C_2=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} \alpha_1 &= 0.033 & \beta_1 &= 2.475 \\ \alpha_2 &= 0.346 & \beta_2 &= 25.95 \\ \alpha_3 &= 0.621 & \beta_3 &= 46.57 \\ \alpha_1+\alpha_2+\alpha_3 &= 1 & \gamma &= 600.0 \end{aligned} \tag{4.3.2}$$

and have the controllability canonic form :

$$\hat{A}_{22} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -a_2 & -a_3 & -a_4 \end{bmatrix} \quad \hat{B}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \hat{C}_2 = \begin{bmatrix} 0 & a_2 & a_3 & a_4 & 1 \\ \delta & \varepsilon & \gamma & \gamma & 0 \end{bmatrix}$$

$$\begin{aligned} \delta &= 4.2543 \\ \varepsilon &= 148.066 \\ a_4 &= 75.995 \\ a_3 &= 37.2614 \\ a_2 &= 1.60241 \end{aligned} \tag{4.3.3}$$

from which the open loop characteristic equation is

$$p_o(\lambda) = \lambda^5 + a_4\lambda^4 + a_3\lambda^3 + a_2\lambda^2. \tag{4.3.4}$$

Under output feedback $u = -(k_2 \; k_3)C_2$ the closed loop characteristic equation is

$$\begin{aligned} p_c(\lambda) = \lambda^5 + (k_2+a_4)\lambda^4 + (k_2a_4+\gamma k_3+a_3)\lambda^3 + (k_2a_3+k_3\gamma+a_2)\lambda^2 \\ + (k_2a_2+k_3\varepsilon)\lambda + k_3\delta. \end{aligned} \tag{4.3.5}$$

Before investigating the root locus of this equation it is of interest to determine if the system has any invariant zeros. Since this is the controllable and observable subsystem, the invariant zeros may be determined as the roots of the greatest common divisor of all the minors of full order of the system matrix

$$\left[\begin{array}{c|c} A_{22}-\lambda I_5 & -B_2 \\ \hline C_2 & 0 \end{array}\right]_{7\times 6} \tag{4.3.6}$$

or equivalently in this case as the intersection of the invariant zeros of the two square systems [35]

$$\left[\begin{array}{c|c} A_{22}-\lambda I_5 & -B_2 \\ \hline 10000 & 0 \end{array}\right], \qquad \left[\begin{array}{c|c} A_{22}-\lambda I & -B_2 \\ \hline 01000 & 0 \end{array}\right] \tag{4.3.7}$$

The zeros of the first system are the distinct eigenvalues of A_{22}: $(\lambda_1\lambda_2\lambda_3\lambda_4)$ and those of the second system are the parameters: $(\alpha_1\alpha_2\alpha_3)$. Since these two sets are disjoint it follows that the controllable and observable subsystem has no invariant zeros.

As the closed loop characteristic equation is a function of the two parameters k_2,k_3 a conventional root locus plot may be replaced with a graph of those values of gain (k_2k_3) for which the eigenvalues of $A_{22}-B_2(k_2k_3)C_2$ lie in prescribed regions of the complex plane. This will be done here for the stability region (the left half plane) and for the region to the left of the hyperbola $\sigma = \sqrt{\omega^2+\alpha^2}$ having as assymptopes the constant damping lines $\sigma = \pm\omega$.

As the number of assignable eigenvalues is two, these may momentarily be denoted $-s$ and $-t$ and the closed loop characteristic equation written

$$p_c(\lambda) = (\lambda+s)(\lambda+t)(\lambda^3+h_2\lambda^2+h_1\lambda+h_0). \tag{4.3.8}$$

Equating (4.3.5) and (4.3.8) gives the linear system of equations $X\begin{bmatrix}k\\h\end{bmatrix}=Y$

$$\left[\begin{array}{cc|ccc} 0 & -\delta & st & 0 & 0 \\ -a_2 & -\epsilon & s+t & st & 0 \\ -a_3 & -\gamma & 1 & s+t & st \\ -a_4 & -\gamma & 0 & 1 & s+t \\ -1 & 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c} k_2 \\ k_3 \\ \hline h_0 \\ h_1 \\ h_2 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ \hline a_2 \\ a_3-st \\ a_4-(s+t) \end{array}\right] \tag{4.3.9}$$

which has a solution for all values of s,t for which $|X| \neq 0$. (Compare equation (1.35).) The boundary of the region of stabilizing gains may be found as the parameterized solution $(k_2,k_3) = (k_2(s,t),k_3(s,t))$ under the constraints

a) a real pole crosses the imaginary axis: $s=0$, $t>0$

b) a complex pair crosses the imaginary axis: $s+t=0$, $st=\omega^2$, $\omega \geq 0$.

Similarly the boundary of the region to the left of the hyperbola $\sigma = \sqrt{\omega^2+\alpha^2}$ ($\alpha = .030$, $\lambda = -\sigma \pm j\omega$) is the solution under the constraints

a) a real pole crosses the line $\sigma = 0.03$: $s = 0.03$, $t > 0.03$

b) a complex pair crosses the hyperbola: $s+t = 2\sigma$, $st = \sigma^2+\omega^2$, $\sigma \geq .03$.

A graph of these two regions is shown in Figure 4.3.2, and suggests that a static design for the output feedback regulator problem may be found satisfactory. It should be noted that placing the eigenvalues of A_{22} to the left of the hyperbola has required the gain k_2 to be roughly an order of magnitude larger than the gain k_3. Recalling that the third row of C observed the entire subsystem A_{22} whereas the second row only observed a one dimensional generalized eigenvector of $\lambda_{4,5}=0$, this suggests that in designing a static compensator for the linear quadratic regulator the observability properties of the system may be reflected in disparate values for the gains k_2 and k_3.

A linear quadratic regulator problem is now defined with

$$Q = \begin{bmatrix} 0 & 0 \\ 0 & Q_{22} \end{bmatrix} \qquad R = 1 \qquad Q_{22} = dg[1,1,\alpha_1,\alpha_2,\alpha_3]. \qquad (4.3.10)$$

Only the states of the controllable subsystem have been penalized, but since $A_{12} \neq 0$, the optimal solution shapes the eigenvectors of the uncontrollable eigenvalues and it is not possible to exploit the uncontrollability of A_{11} to reduce the regulator problem to one for pair (A_{22},B_2).*

*It is true that there is a two dimensional subsystem with eigenvalues (σ_1,σ_6) which could be removed to reduce the dimensionality of the problem from twelve to ten, but this does not seem worthwhile.

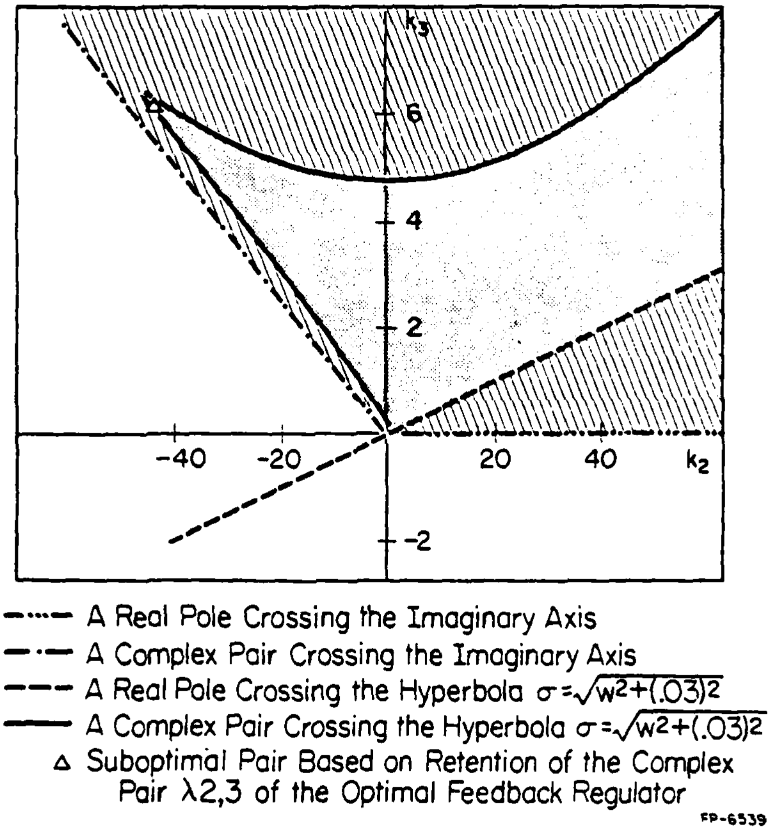


Figure 4.3.2. Stability regions for the nuclear reactor model

This uncontrollability will prove useful however, in analyzing the pole-placement problem for the triple (A_1,B_0,A_{12}).

The solution to the regulator problem is given in Table 4.3.4, and the resultant optimal eigenvectors are given in Table 4.3.5. Qualitatively, the optimal solution has moved the double pole at the origin to the complex pair $-13\pm j12$ while only slightly perturbing the remaining three real poles, suggesting that an output feedback design should concentrate on retaining this complex pair. Examination of the eigenvectors of F in Table 4.3.5 shows that there are severe restrictions on the permissible choices of Λ_r. Since Λ_r must satisfy $\text{rank}(CV_{\Lambda_r}) = 3$ the choices may include at most two eigenvalues of F_{22} and are: $(\lambda_1\sigma_2\sigma_3),(\lambda_1\sigma_4\sigma_5),(\lambda_2\lambda_3\sigma_7)$ $(\lambda_4\sigma_2\sigma_3),(\lambda_4\sigma_4\sigma_5),(\lambda_1\ \lambda_4\ \sigma_7)$. (Those involving λ_5 are omitted as the eigenvector corresponding to λ_5 is nearly in the null space of C.) Since it is desired to retain as many optimal eigenvectors as possible,a design is based on retention of $(\lambda_2\lambda_3\sigma_7)$ with the expectation that if a compensator is used, additionally one or more of $\lambda_1,\lambda_4,\lambda_5$ may be retained.

Based on this choice of Λ_r and under the permutation of states

$$T = \begin{bmatrix} 0 & I_3 & 0 \\ I_6 & 0 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \tag{4.3.11}$$

bringing C to the form [I 0], the resulting matrices A_1,N_o,B_o,A_{12} are given in Table 4.3.6. The spectrum of A_1 consists of the uncontrollable eigenvalues $(\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_5,\sigma_6)$ together with three controllable eigenvalues located at -6.066, -0.407, -0.034. Thus the system is stable as may have been anticipated from the preliminary output feedback analysis and the corresponding gain $K = (-4.502,-43.385,6.249)$ lies within the region of Figure 4.3.2.

Table 4.3.4. Optimal state feedback regulator solution

$$
M_c = \left[\begin{array}{ccccccc|ccccc}
0.0000 & 0.0002 & 0 & 0.0008 & 0.0009 & 0 & 0.0024 & 0.0000 & -0.0000 & 0.0000 & -0.0000 & -0.0000 \\
0.0002 & 0.0047 & 0 & 0.0034 & -0.0348 & 0 & 0.0115 & -0.3064 & -0.0069 & -0.0002 & -0.0037 & -0.0084 \\
0.0000 & -0.0000 & 0 & -0.0000 & 0.0000 & 0 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
0.0008 & 0.0034 & 0 & 0.0214 & 0.0281 & 0 & 0.0521 & 0.0568 & 0.0001 & 0.0000 & 0.0001 & 0.0001 \\
0.0009 & -0.0348 & 0 & 0.0281 & 0.4209 & 0 & 0.0552 & 3.1448 & 0.0690 & 0.0024 & 0.0373 & 0.0846 \\
0.0000 & -0.0000 & 0 & -0.0000 & 0.0000 & 0 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
0.0024 & 0.0115 & 0 & 0.0521 & 0.0552 & 0 & 0.1580 & 0.0001 & -0.0000 & 0.0001 & -0.0000 & -0.0001 \\
\hline
0.0000 & -0.3064 & 0 & 0.0568 & 3.1448 & 0 & 0.0001 & 25.7364 & 0.5511 & 0.0191 & 0.2980 & 0.6757 \\
-0.0000 & -0.0069 & 0 & 0.0001 & 0.0690 & 0 & -0.0000 & 0.5511 & 0.0271 & 0.0008 & 0.0126 & 0.0291 \\
0.0000 & -0.0002 & 0 & 0.0000 & 0.0024 & 0 & 0.0001 & 0.0191 & 0.0008 & 0.4953 & -0.0132 & -0.0175 \\
-0.0000 & -0.0037 & 0 & 0.0001 & 0.0373 & 0 & -0.0000 & 0.2980 & 0.0126 & -0.0132 & 0.3843 & -0.1897 \\
-0.0000 & -0.0084 & 0 & 0.0001 & 0.0846 & 0 & -0.0001 & 0.6757 & 0.0291 & -0.0175 & -0.1897 & 0.1615
\end{array}\right]
$$

$$
K = [0 \quad -0.3064 \quad 0 \quad 0.0568 \quad 3.1448 \quad 0 \quad 0.0001 \,|\, 25.7364 \quad 0.5511 \quad 0.0191 \quad 0.2980 \quad 0.6757]
$$

Table 4.3.5. Optimal eigenvectors of state feedback regulator solution

The eigenvectors corresponding to controllable eigenvalues of F_{22} are:

$$v_{\lambda_1} = \left[\begin{array}{r} 0.0000\\ 0.0000\\ 0.0000\\ 0.0000\\ 0.0000\\ 0.0000\\ 0.0000\\ \hline 0.0004\\ 0.8135\\ -0.0268\\ -0.2820\\ -0.5080 \end{array}\right] \quad v_{\lambda_2}, v_{\lambda_3} = \left[\begin{array}{r} 0.0000\\ 0.0000\\ 0.0000\\ 0.0000\\ 0.0000\\ 0.0000\\ 0.0000\\ \hline 0.0305\\ 0.2477\\ -0.0429\\ -0.4549\\ -0.8246 \end{array}\right] \pm j \left[\begin{array}{r} 0.0000\\ 0.0000\\ 0.0000\\ 0.0000\\ 0.0000\\ 0.0000\\ 0.0000\\ \hline -0.0141\\ -0.1862\\ -0.0045\\ -0.0535\\ -0.1062 \end{array}\right] \quad v_{\lambda_4} = \left[\begin{array}{r} 0.0000\\ 0.0000\\ 0.0000\\ 0.0000\\ 0.0000\\ 0.0000\\ 0.0000\\ \hline -0.0004\\ -0.0019\\ 0.0126\\ 0.9202\\ -0.3912 \end{array}\right] \quad v_{\lambda_5} = \left[\begin{array}{r} 0.0000\\ 0.0000\\ 0.0000\\ 0.0000\\ 0.0000\\ 0.0000\\ 0.0000\\ \hline -0.0001\\ -0.0002\\ 0.9997\\ -0.0186\\ -0.0178 \end{array}\right]$$

The eigenvectors corresponding to uncontrollable eigenvalues of A_{11} are:

$$v_{\sigma_1} = \left[\begin{array}{r} 0.0000\\ -0.0000\\ 0.4817\\ -0.0000\\ 0.0000\\ -0.5399\\ 0.0000\\ \hline -0.0000\\ 0.0000\\ 0.0000\\ 0.0000\\ -0.0000 \end{array}\right] \quad v_{\sigma_2}, v_{\sigma_3} = \left[\begin{array}{r} -0.2172\\ 0.1347\\ 0.6117\\ 0.0876\\ -0.2509\\ -0.2205\\ 0.0521\\ \hline 0.0330\\ 0.0000\\ -0.0000\\ -0.0000\\ -0.0001 \end{array}\right] \pm j \left[\begin{array}{r} 0.0707\\ 0.3648\\ -0.2534\\ -0.1441\\ -0.1394\\ 0.4361\\ 0.0646\\ \hline 0.0220\\ -0.0000\\ 0.0000\\ -0.0000\\ -0.0001 \end{array}\right]$$

$$v_{\sigma_4}, v_{\sigma_5} = \left[\begin{array}{r} 0.3313\\ 0.3228\\ -0.2965\\ -0.0405\\ 0.0516\\ -0.0116\\ -0.0327\\ \hline -0.0024\\ 0.0000\\ 0.0000\\ 0.0000\\ -0.0000 \end{array}\right] \pm j \left[\begin{array}{r} 0.7326\\ -0.3781\\ -0.0812\\ 0.0731\\ 0.0035\\ -0.0298\\ -0.0071\\ \hline -0.0052\\ 0.0000\\ -0.0000\\ -0.0000\\ 0.0000 \end{array}\right] \quad v_{\sigma_6} = \left[\begin{array}{r} 0.0000\\ 0.0000\\ 0.9537\\ -0.0000\\ 0.0000\\ 0.3007\\ -0.0000\\ \hline -0.0000\\ -0.0000\\ 0.0000\\ 0.0000\\ -0.0000 \end{array}\right] \quad v_{\sigma_7} = \left[\begin{array}{r} 0.3678\\ 0.3791\\ 0.3907\\ 0.3576\\ 0.3685\\ 0.3799\\ 0.3984\\ \hline -0.0413\\ 0.0000\\ 0.0000\\ 0.0000\\ 0.0000 \end{array}\right]$$

Table 4.3.6. Data for pole-placement problem for the nuclear reactor model based on retention of $\lambda_2, \lambda_3, \sigma_7, \lambda_1$

$$
A_{12} = \left[\begin{array}{cccccc|ccc}
0 & 0 & 0 & 0 & 0.15 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & -7.5 & 0 & 0 & 75.0 & 0 & 0.033 & 0.346 & 0.632
\end{array}\right]
$$

$$
N_o = \left[\begin{array}{c|cc}
0.9232 & 0.0000 & 0.0000 \\
0.9516 & 0.0000 & 0.0000 \\
0.9808 & 0.0000 & 0.0000 \\
0.8976 & 0.0000 & 0.0000 \\
0.9252 & 0.0000 & 0.0000 \\
0.9536 & 0.0000 & 0.0000 \\
\hline
-0.4328 & -4.1715 & 0.3397 \\
-4.6563 & -44.8802 & 3.6811 \\
-8.5479 & -82.3894 & 6.8002
\end{array}\right]
\qquad
B_o = \left[\begin{array}{c}
0.0000 \\
0.0000 \\
0.0000 \\
0.0000 \\
0.0000 \\
0.0000 \\
\hline
0.3014 \\
3.2578 \\
6.0054
\end{array}\right]
$$

$$
A_1 = \left[\begin{array}{cccccc|ccc}
-0.4044 & 0.0000 & 0.0000 & 0.4044 & -0.1385 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
0.0000 & -0.4044 & 0.0000 & 0.0000 & 0.2617 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
0.0000 & 0.0000 & -0.4044 & 0.0000 & -0.1471 & 0.4044 & 0.0000 & 0.0000 & 0.0000 \\
0.0182 & -0.0000 & 0.0000 & -0.5363 & -0.1346 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
0.0000 & 0.0818 & 0.0000 & 0.4545 & -0.6751 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
0.0000 & -0.0000 & 0.0818 & 0.0000 & 0.3115 & -0.5363 & 0.0000 & 0.0000 & 0.0000 \\
\hline
0.0000 & 2.5478 & 0.0000 & 0.0000 & -25.4130 & 0.0000 & -0.0442 & -0.1175 & -0.2110 \\
0.0000 & 27.6085 & 0.0000 & 0.0000 & -275.3962 & 0.0000 & -0.1215 & -1.6197 & -2.2860 \\
0.0000 & 51.0014 & 0.0000 & 0.0000 & -508.7316 & 0.0000 & -0.2244 & -2.3529 & -4.8439
\end{array}\right]
$$

Qualitatively this output feedback solution retains the eigenvectors corresponding to the optimal complex pair λ_2,λ_3 and places two eigenvalues (though not their eigenvectors) very near their optimal locations λ_4,λ_5. However it has also shifted the fast eigenvalue of the system from -75.2 to only -6.066. This suggests trying to design a first order compensator to additionally retain λ_1 with the expectation that two eigenvalues may still remain near λ_4,λ_5. The data for the pole-placement problem are given in Table 4.3.6.

Because of the block structure of (A_1,B_0,A_{12}) which clearly satisfies the requirement that the spectrum of A_r contain the unretained uncontrollable eigenvalues $(\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_5,\sigma_6)$, and the fact that only the third row of A_{12} observes the controllable subsystem of A_1,the pole-placement problem may be replaced with the single-input single-output pole-placement problem for the triple $(A_1^{22},B_o^2,A_{12}^{22})$. This problem may then be solved analytically by transforming to the controllability canonic form:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad [c_0 \ c_1 \ c_2]$$

$$\begin{aligned} a_0 &= 0.083374 & c_0 &= 0.067455 \\ a_1 &= 2.6912 & c_1 &= 2.1602 \\ a_2 &= 6.5080 & c_2 &= 4.8663 \end{aligned} \tag{4.3.12}$$

The closed loop characteristic equation is

$$p_c(\lambda) = \lambda^3 + (a_2-c_3p_3)\lambda^2 + (a_1-c_1p_3)\lambda + (a_o-c_op_3). \tag{4.3.13}$$

This gives rise to the root locus

$$\frac{\lambda^3+a_2\lambda^2+a_1\lambda+a_0}{c_2\lambda^2+c_1\lambda+c_0} = p_3 \tag{4.3.14}$$

where the roots of the numerator are the controllable eigenvalues of A_1 and the roots of the denominator are -0.0338, -0.4101. This near pole-zero cancellation confirms the expectation that a first order compensator would allow retention of λ_1 and yet still keep two eigenvalues very near their optimal locations λ_4, λ_5 without requiring large gains. Analysis of the root locus shows that the system is stable for $p_3 < 1.235$, with two poles very nearly at -0.41, -0.034 for $-\infty < p_3 < 1.0$. Arbitrarily placing a pole at -7.0 gives $p_3 = -0.192$ and the controllable subspectrum for A_r becomes -7.0, -0.408, -0.034.

The parameters of the compensator are then:

$$
\begin{aligned}
H_0^c &= [-76.132] \\
D_0^{\bar{c}} &= [\,-9.186] \qquad D_0^c = [-88.540 \quad -1.103] \\
K_z^c &= [\quad 5.820] \\
K_y^{\bar{c}} &= [\,-4.502] \qquad K_y^c = [-43.385 \quad 7.216]
\end{aligned}
\tag{4.3.15}
$$

and in the original basis the final closed loop system is $\frac{d}{dt}\begin{pmatrix} v \\ x \end{pmatrix} = A_c \begin{pmatrix} v \\ x \end{pmatrix}$

where $A_c =$

$$
\left[\begin{array}{c|cccccc}
-76.132 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
\hline
0.000 & -0.4044 & 0.000 & 0.000 & 0.4044 & 0.000 & 0.000 \\
0.000 & 0.000 & -0.4044 & 0.000 & 0.000 & 0.4044 & 0.000 \\
0.000 & 0.000 & 0.000 & -0.4044 & 0.000 & 0.000 & 0.4044 \\
0.000 & 0.01818 & 0.000 & 0.000 & -0.5363 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.0818 & 0.000 & 0.4545 & -0.5363 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.0818 & 0.000 & 0.4545 & -0.5363 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.150 & 0.000 \\
\hline
-5.820 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & -7.500 & 0.000 & 0.000 & 75.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000
\end{array}\right.
$$

$$
\begin{array}{rrrrrr}
-9.186 & -88.540 & -1.103 & 0.000 & 0.000 & 0.000 \\
\hline
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.4545 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
-0.150 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
\hline
4.502 & 43.385 & -7.216 & 0.000 & 0.000 & 0.000 \\
0.000 & 600.000 & -74.995 & 0.033 & 0.346 & 0.621 \\
0.000 & 0.000 & 2.475 & -0.033 & 0.000 & 0.000 \\
0.000 & 0.000 & 25.950 & 0.000 & -0.346 & 0.000 \\
0.000 & 0.000 & 46.570 & 0.000 & 0.000 & -0.621
\end{array}
\qquad (4.3.16)
$$

Thus by taking into account the eigenvector structure of the optimal state feedback regulator solution, the methodology for designing regulators has been applied to a nontrivial system having uncontrollable and unobservable modes. A first order compensator has been constructed having the property of retaining from the optimal regulator solution a four dimensional invariant subspace spanned by eigenvectors corresponding to three controllable and one uncontrollable optimal eigenvalues, at the same time placing eigenvalues near all the remaining optimal locations.

4.4. Two Interconnected Power System Model

This example considers a model of a power system consisting of two interconnected steam generators. The model, derived in [37], is given by $\dot{x} = Ax + Bu$, $y = Cx$, A,B,C defined in Table 4.4.1, and represents a linearization of the system about an operating point, describing the system behavior under real power and frequency variations. The state vector has the physical correspondence:

x_1, x_7 - valve displacements in areas one and two

x_2, x_8 - power displacements of high pressure turbines in areas one and two

Table 4.4.1. System matrices for the two interconnected power system model

The system is represented as:

$$\dot{x} = \begin{bmatrix} A_s & -a_1 & 0 \\ a_2 & 0 & -a_2 \\ 0 & a_1 & A_s \end{bmatrix} x + \begin{bmatrix} b & 0 \\ 0 & 0 \\ 0 & b \end{bmatrix} u, \qquad y = \begin{bmatrix} c & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c \end{bmatrix} x$$

where:

$$A_s = \begin{bmatrix} -2.0 & 0 & 0 & 0 & -4.0 \\ 4.75 & -5.0 & 0 & 0 & 0 \\ 0 & 0.16667 & -0.16667 & 0 & 0 \\ 0 & 0 & 2.0 & -2.0 & 0 \\ 0 & 0.025 & 0.02333 & 0.035 & -0.1125 \end{bmatrix}$$

$$a_1 = [\,0 \quad 0 \quad 0 \quad 0 \quad 0.08333\,]^T$$

$$a_2 = [\,0 \quad 0 \quad 0 \quad 0 \quad 22.21439\,]$$

$$b = [\,4.0 \quad 0 \quad 0 \quad 0 \quad 0\,]^T$$

$$c = [\,0 \quad 0 \quad 0 \quad 0 \quad 1\,]$$

x_3, x_9 - power displacements of intermediate pressure turbines in areas one and two

x_4, x_{10} - power displacements of low pressure turbines in areas one and two

x_5, x_{11} - frequency deviations in areas one the two

x_6 - tie-line power flow deviation from area one into area two

and the controls are the set point adjustments in the two areas. It is assumed that the tie-line power flow and the two frequency deviations are available for measurement.

Since the system is open loop stable a preliminary analysis of the static output feedback pole-placement problem may be replaced by an investigation of the eigenstructure of both the open and closed loop systems. A clear understanding of this structure will be essential in applying the design methodology.

The eigenvalues of the open loop matrix may be classified according to whether the tie-line power flow variable x_6 is zero in the corresponding eigenspace. The eigenvector equation for the open loop matrix is:

$$\begin{aligned} A_s v - \alpha a_1 &= \lambda v \\ a_2 v - a_2 w &= \alpha\lambda , \\ \alpha a_1 + A_s w &= \lambda w \end{aligned} \qquad \mathcal{V} = \begin{bmatrix} v \\ \alpha \\ w \end{bmatrix}, \quad v,w \in R^{5\times 1}, \quad \alpha \in R^{1\times 1}. \tag{4.4.1}$$

These equations may be rewritten:

$$\alpha(2a_2 \operatorname{adj}(\lambda I - A_s)a_1 + \lambda\det(\lambda I - A_s)) = 0 \tag{4.4.2a}$$

$$(A_s - \lambda I)v = \alpha a_1 \tag{4.4.2b}$$

$$(A_s - \lambda I)w = -\alpha a_1 . \tag{4.4.2c}$$

If $\alpha = 0$ then there are five solutions:

$$\{\lambda_i^d\}_{i=1}^5, \quad \{\mathcal{V}_i^d\}_{i=1}^5, \quad \mathcal{V}_i^d = \begin{bmatrix} v_i^d \\ o_d \\ v_i^d \end{bmatrix}, \quad A_s v_i^d = \lambda_i^d v_i^d. \tag{4.4.3}$$

If $\alpha \neq 0$ then (4.4.2a) is a sixth degree polynomial in λ and the solution of (4.4.2) is

$$\{\lambda_i^c\}_{i=1}^6, \quad \{\mathcal{V}_i^c\}_{i=1}^6, \quad \mathcal{V}_i^c = \begin{bmatrix} v_i^c \\ \alpha_{i_c} \\ -v_i \end{bmatrix}, \quad [\lambda_i^c I - A_s \ \vdots \ a_1] \begin{bmatrix} v_i^c \\ \\ \alpha_i \end{bmatrix} \tag{4.4.4}$$

The numerical values of $\lambda_i^d, \lambda_i^c, \mathcal{V}_i^d, \mathcal{V}_i^c, \alpha_i$ are given in Tables 4.4.2, 4.4.3.

Thus,given any initial condition, the transient response of the system may be represented as the sum of a coupled and a decoupled response

$$x(t) = x_c(t) + x_d(t)$$

$$x_c(t) = \sum_{i=1}^{6} \begin{bmatrix} v_i^c \\ \alpha_i \\ -v_i^c \end{bmatrix} \gamma_i^c e^{\lambda_i^c t}, \qquad x_d(t) = \sum_{i=1}^{5} \begin{bmatrix} v_i^d \\ 0 \\ v_i^d \end{bmatrix} \gamma_i^d e^{\lambda_i^d t} \tag{4.4.5}$$

$$x_0 = x(0) = \sum_{i=1}^{6} \gamma_i^c \begin{bmatrix} v_i^c \\ \alpha_i \\ -v_i^c \end{bmatrix} + \sum_{i=1}^{5} \gamma_i^d \begin{bmatrix} v_i^d \\ 0 \\ v_i^d \end{bmatrix}$$

The first term $x_c(t)$ represents a projection of the system response into a subspace in which the tie-line power flow variable is nonzero ($\alpha_i \neq 0$, $x_6 \neq 0$) and in which the response of the second generator is in exact opposition to that of the first generator. The second term $x_d(t)$ represents the projection of the response into a subspace in which the tie-line power flow variable is zero ($x_6 = 0$) and the two generators operate synchronously. For example, consider the case that an initial displacement from the nominal operating point occurs symmetrically in the two areas. If $x_i(0) = x_{i+6}(0)$, $i=1,\ldots,5$, $x_6(0)=0$, then the two coupled steam generators evolve with the

Table 4.4.2. Open loop and optimal closed loop spectra of power system model

	Open loop eigenvalues	Optimal closed loop eigenvalues
λ_i^c	1. (-5.028, 0.000)	1. (-9.171, 0.000)
	2. (-1.982, 0.101)	2. (-4.994, 0.000)
	3. (-1.982, -0.101)	3. (-1.994, 0.000)
	4. (-0.166, 0.000)	4. (-0.241, 1.943)
	5. (-0.061, 1.938)	5. (-0.241, -1.943)
	6. (-0.061, -1.938)	6. (-0.220, 0.000)
λ_i^d	1. (-5.032, 0.000)	1. (-9.171, 0.000)
	2. (-1.970, 0.143)	2. (-4.988, 0.000)
	3. (-1.970, -0.143)	3. (-2.001, 0.000)
	4. (-0.154, 0.149)	4. (-0.171, 0.093)
	5. (-0.154, -0.149)	5. (-0.171, -0.093)

Table 4.4.3. Open loop eigenvectors of the two interconnected power system model

See Table 4.4.2 for the definition of the eigenvalues λ_i^c, λ_i^d. The eigenvectors have the form $\mathcal{V}_i^c = \begin{bmatrix} v_i^c \\ \alpha_i \\ -v_i^c \end{bmatrix}$, $\mathcal{V}_i^d = \begin{bmatrix} v_i^d \\ 0 \\ v_i^d \end{bmatrix}$.

$$\mathcal{V}_1^c = \left[\begin{array}{r} 0.0041\\ -0.7062\\ 0.0242\\ -0.0160\\ 0.0031\\ \hline -0.0276\\ \hline -0.0041\\ 0.7062\\ -0.0242\\ 0.0160\\ -0.0031\end{array}\right] \quad \mathcal{V}_2^c, \mathcal{V}_3^c = \left[\begin{array}{r} 0.2034\\ 0.3222\\ -0.0292\\ -0.2381\\ 0.0003\\ \hline -0.0121\\ \hline -0.2034\\ -0.3222\\ 0.0292\\ 0.2380\\ -0.0003\end{array}\right] \pm j \left[\begin{array}{r} 0.0463\\ 0.0620\\ -0.0073\\ 0.5329\\ -0.0054\\ \hline 0.1197\\ \hline -0.0463\\ -0.0620\\ 0.0073\\ -0.5329\\ 0.0054\end{array}\right] \quad \mathcal{V}_4^c = \left[\begin{array}{r} -0.0027\\ -0.0027\\ -0.4507\\ -0.4914\\ 0.0012\\ \hline -0.3326\\ \hline 0.0027\\ 0.0027\\ 0.4507\\ 0.4914\\ -0.0012\end{array}\right]$$

$$\mathcal{V}_5^c, \mathcal{V}_6^c = \left[\begin{array}{r} -0.0389\\ -0.0162\\ 0.0046\\ 0.0032\\ 0.0429\\ \hline -0.1502\\ \hline 0.0389\\ 0.0162\\ -0.0046\\ -0.0032\\ -0.0429\end{array}\right] \pm j \left[\begin{array}{r} 0.0497\\ 0.0541\\ 0.0016\\ -0.0015\\ -0.0052\\ \hline -0.9794\\ \hline -0.0497\\ -0.0541\\ -0.0016\\ 0.0015\\ 0.0052\end{array}\right] \quad \mathcal{V}_1^d = \left[\begin{array}{r} -0.0047\\ 0.7065\\ -0.0242\\ 0.0160\\ -0.0036\\ \hline 0.0000\\ \hline -0.0047\\ 0.7065\\ -0.0242\\ 0.0160\\ -0.0036\end{array}\right] \quad \mathcal{V}_2^d, \mathcal{V}_3^d = \left[\begin{array}{r} 0.0975\\ 0.1702\\ -0.0129\\ -0.5093\\ 0.0079\\ \hline 0.0000\\ \hline 0.0975\\ 0.1702\\ -0.0129\\ -0.5092\\ 0.0079\end{array}\right] \pm j \left[\begin{array}{r} 0.2414\\ 0.3703\\ -0.0353\\ 0.0730\\ -0.0053\\ \hline 0.0000\\ \hline 0.2414\\ 0.3703\\ -0.0353\\ 0.0730\\ -0.0053\end{array}\right]$$

$$\mathcal{V}_4^d, \mathcal{V}_5^d = \left[\begin{array}{r} 0.2434\\ 0.2319\\ -0.2193\\ -0.2603\\ -0.1203\\ \hline 0.0000\\ \hline 0.2434\\ -0.2319\\ -0.2193\\ -0.2603\\ -0.1203\end{array}\right] \pm j \left[\begin{array}{r} -0.2146\\ -0.2175\\ -0.2775\\ -0.2796\\ 0.0900\\ \hline 0.0000\\ \hline -0.2146\\ -0.2175\\ -0.2775\\ -0.2796\\ 0.0900\end{array}\right]$$

dynamics of a single fifth order system. Note that the total system is not observable from the tie-line power flow variable. If $x_i(0) = -x_{i+6}(0)$, $i=1,\ldots,5$, then the system evolves in a six dimensional invariant subspace, the two steam generators operating in exact opposition to each other.

Inspection of the numerical values of λ_i^d, λ_i^c in Table 4.4.2 shows that the eigenvalues of A may also be identified in pairs:

$\{\lambda_1^c\} - \{\lambda_1^d\}$	two real eigenvalues at ~ -5
$\{\lambda_2^c, \lambda_3^c\} - \{\lambda_2^d, \lambda_3^d\}$	two complex pairs at $\sim -2 \pm j0.1$
$\{\lambda_5^c, \lambda_6^c\} - \{\lambda_4^d, \lambda_5^d\}$	two different dominant complex pairs.

The eigenstructure of the system indicates that rather than representing a pairing of distinct modes, one from each of the two steam generators, this symmetry represents the pairing of coupled and decoupled modes. For example, the frequency pair $-0.15 \pm j0.15$ may be associated with decoupled synchronous mechanical rotation of the two generator shafts, while the pair $-0.06 \pm j2.0$ may be associated with inversely coupled mechanical rotation of the two shafts. If the initial frequency deviations in the two areas were equal then the response would be well damped, while if the initial frequency deviation in one area were the negative of that in the other area, the response would be slow and highly oscillatory.

It may be shown that the closed loop system possesses the same eigenstructure symmetry as the open loop system provided the linear quadratic regulator problem is defined such that the states of the generators are weighted equally: $Q = dg(Q', q, Q')$. The resultant closed loop matrix has the symmetric form

$$F = \begin{bmatrix} F_1 & -f_1 & F_2 \\ a_2 & 0 & -a_2 \\ F_2 & f_1 & F_1 \end{bmatrix} \tag{4.4.6}$$

and the closed loop eigenvectors and eigenvalues are given by:

$$\begin{aligned} &\{\lambda_i^d\}_{i=1}^5, \quad \{\mathcal{V}_i^d\}_{i=1}^5, \quad \mathcal{V}_i^d = \begin{bmatrix} v_i^d \\ \mathbf{0}_d \\ v_i^d \end{bmatrix}, \quad (F_1+F_2)v_i^d = \lambda_i^d v_i^d \\ &\{\lambda_i^c\}_{i=1}^6, \quad \{\mathcal{V}_i^c\}_{i=1}^6, \quad \mathcal{V}_i^c = \begin{bmatrix} v_i^c \\ \alpha_i \\ -v_i^c \end{bmatrix}, \quad (\lambda_i^c I - F_1 + F_2 \vdots f_1) \begin{bmatrix} v_i^c \\ \hline \alpha_i \end{bmatrix} = 0 \end{aligned} \tag{4.4.7}$$

Therefore all the remarks made regarding the open loop transient response apply also to the closed loop response.

A linear quadratic regulator problem may now be defined by letting $Q = dg(5,0,0,0,30,10,5,0,0,0,30)$, $R = I_{2\times 2}$. The solution to the Riccati equation is given in Table 4.4.4, and the optimal closed loop eigenvectors are given in Table 4.4.5. Referring to Figure 4.4.1, the optimal solution has increased the damping of the two complex pairs associated with coupled and decoupled mechanical rotation of the generator shafts. The real eigenvalue associated with the tie-line interaction has increased in magnitude: $\lambda_6^d = -0.220$. The remaining spectrum consists of three pairs of real eigenvalues at -2,-5,-9 corresponding to coupled and uncoupled modes. It is noted that the closed loop eigenvectors $\mathcal{V}_2^c, \mathcal{V}_2^d$ (Table 4.4.5) corresponding to the real eigenvalues at -5 are nearly equal to the eigenvectors $\mathcal{V}_1^c, \mathcal{V}_1^d$ (Table 4.4.3) corresponding to the two open loop eigenvalues near -5. Since the optimal control has expanded no energy shaping these two eigenvectors it is concluded that these two modes might be neglected in designing an output feedback compensator.

Table 4.4.4. Optimal state feedback regulator solution

With $Q = dg(5,0,0,0,30,10,5,0,0,0,30)$ and $R = I_{2\times 2}$ the solution of the Riccati equation and the associated feedback matrix are:

$$M_c = \begin{bmatrix} M_1 & -m_1 & M_2 \\ m_2 & 16.9456 & -m_2 \\ M_2 & m_1 & M_1 \end{bmatrix} \qquad K = \begin{bmatrix} k_1 & -0.8761 & k_2 \\ k_2 & 0.8761 & k_1 \end{bmatrix}$$

where:

$$M_1 = \begin{bmatrix} 0.4627 & 0.0292 & 0.1811 & 0.0719 & 4.3802 \\ 0.0292 & 0.0600 & 0.3147 & 0.1426 & 10.3402 \\ 0.1811 & 0.3147 & 6.2726 & 0.5895 & 6.1776 \\ 0.0719 & 0.1426 & 0.5895 & 0.3795 & 23.7609 \\ 4.3802 & 10.3403 & 6.1773 & 23.7608 & 2478.8810 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} -0.0112 & -0.0222 & -0.0439 & -0.0626 & -4.0168 \\ -0.0222 & -0.0458 & -0.0492 & -0.1233 & -9.2460 \\ -0.0439 & -0.0492 & -0.9310 & -0.2711 & 11.1443 \\ -0.0626 & -0.1233 & -0.2711 & -0.3484 & -21.9648 \\ -4.0170 & -9.2462 & 11.1440 & -21.9648 & -2367.8620 \end{bmatrix}$$

$$m_2 = [-0.2190 \quad -0.2945 \quad -3.9340 \quad -1.2252 \quad 25.3958]$$

$$m_1 = \begin{bmatrix} 0.2190 \\ 0.2945 \\ 3.9342 \\ 1.2252 \\ -25.3955 \end{bmatrix}$$

$$k_1 = [\ 1.8507 \quad 0.1169 \quad 0.7244 \quad 0.2875 \quad 17.5206]$$

$$k_2 = [-0.0449 \quad -0.0889 \quad -0.1755 \quad -0.2502 \quad -16.0673]$$

Table 4.4.5. Optimal eigenvectors of state feedback regulator solution

See Table 4.4.2 for the definition of the eigenvalues λ_i^c, λ_i^d. The eigenvectors have the form: $\mathcal{V}_i^c = \begin{bmatrix} v_i^c \\ \alpha_i \\ -v_i^c \end{bmatrix}$, $\mathcal{V}_i^d = \begin{bmatrix} v_i^d \\ 0 \\ v_i^d \end{bmatrix}$.

$$\mathcal{V}_1^c = \left[\begin{array}{r} 0.4676 \\ -0.5326 \\ 0.0099 \\ -0.0027 \\ 0.0014 \\ \hline -0.0067 \\ \hline -0.4653 \\ 0.5300 \\ -0.0098 \\ 0.0027 \\ -0.0014 \end{array}\right] \quad \mathcal{V}_2^c = \left[\begin{array}{r} -0.0008 \\ -0.7063 \\ 0.0244 \\ -0.0163 \\ 0.0031 \\ \hline -0.0279 \\ \hline 0.0008 \\ 0.7062 \\ -0.0244 \\ 0.0163 \\ -0.0031 \end{array}\right] \quad \mathcal{V}_3^c = \left[\begin{array}{r} 0.0139 \\ 0.0219 \\ -0.0020 \\ -0.6993 \\ 0.0064 \\ \hline -0.1429 \\ \hline -0.0139 \\ -0.0219 \\ 0.0020 \\ 0.6993 \\ -0.0064 \end{array}\right] \quad \mathcal{V}_4^c, \mathcal{V}_5^c = \left[\begin{array}{r} 0.4175 \\ 0.3858 \\ -0.0077 \\ -0.0225 \\ -0.0248 \\ \hline 0.1580 \\ \hline -0.4175 \\ -0.3858 \\ 0.0077 \\ 0.0225 \\ 0.0248 \end{array}\right] \pm j \left[\begin{array}{r} 0.0820 \\ -0.0757 \\ -0.0328 \\ -0.0124 \\ 0.0039 \\ \hline 0.5468 \\ \hline -0.0820 \\ 0.0757 \\ 0.0328 \\ 0.0124 \\ -0.0039 \end{array}\right]$$

$$\mathcal{V}_6^c = \left[\begin{array}{r} -0.1394 \\ -0.1385 \\ 0.4318 \\ 0.4852 \\ -0.0014 \\ \hline 0.2813 \\ \hline 0.1394 \\ 0.1385 \\ -0.4318 \\ -0.4852 \\ 0.0014 \end{array}\right] \quad \mathcal{V}_1^d = \left[\begin{array}{r} -0.4642 \\ 0.5285 \\ -0.0098 \\ 0.0027 \\ 0.0014 \\ \hline 0.0000 \\ \hline -0.4689 \\ 0.5340 \\ -0.0099 \\ 0.0028 \\ -0.0015 \end{array}\right] \quad \mathcal{V}_2^d = \left[\begin{array}{r} -0.0018 \\ -0.7066 \\ 0.0244 \\ -0.0164 \\ 0.0036 \\ \hline 0.0000 \\ \hline -0.0018 \\ -0.7064 \\ 0.0244 \\ -0.0163 \\ 0.0036 \end{array}\right] \quad \mathcal{V}_3^d = \left[\begin{array}{r} -0.0034 \\ -0.0053 \\ 0.0005 \\ -0.7070 \\ 0.0132 \\ \hline 0.0000 \\ \hline -0.0034 \\ -0.0053 \\ 0.0005 \\ -0.7070 \\ 0.0132 \end{array}\right]$$

$$\mathcal{V}_4^d, \mathcal{V}_3^d = \left[\begin{array}{r} 0.1885 \\ 0.1877 \\ 0.1995 \\ 0.1985 \\ -0.2225 \\ \hline 0.0000 \\ \hline 0.1885 \\ 0.1877 \\ 0.1995 \\ 0.1985 \\ -0.2225 \end{array}\right] \pm j \left[\begin{array}{r} 0.1255 \\ 0.1198 \\ -0.3451 \\ -0.3874 \\ -0.0358 \\ \hline 0.0000 \\ \hline 0.1255 \\ 0.1198 \\ -0.3451 \\ -0.3874 \\ -0.0358 \end{array}\right]$$

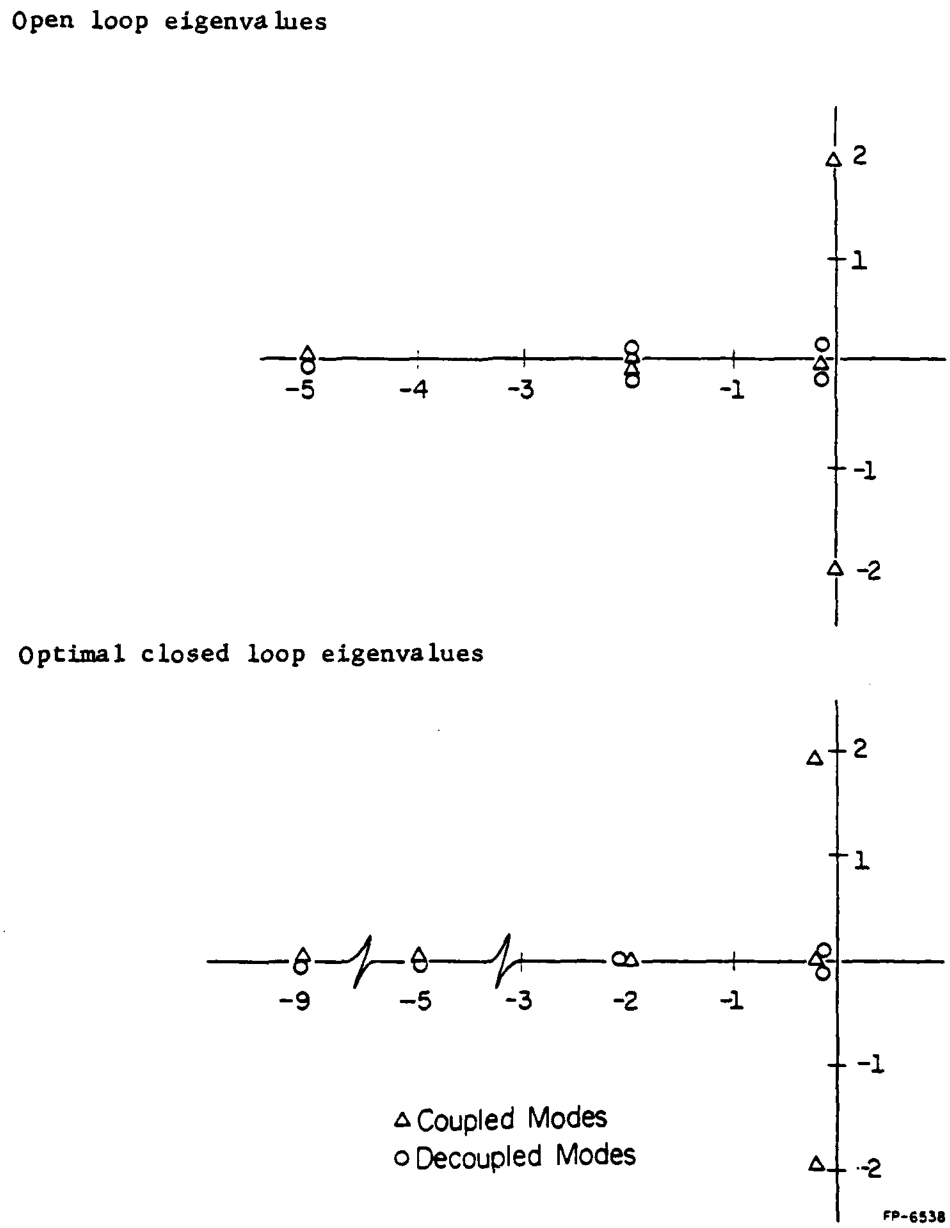


Figure 4.4.1. Open and closed loop spectra of the power system model

To apply the design methodology a permutation of states $\hat{x} = Tx$ is introduced where

$$T = \begin{bmatrix} 0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1 \\ 1&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0 \end{bmatrix}. \qquad (4.4.8)$$

In this basis the observation matrix has the form $(I_{3\times3} \vdots 0)$ and the state vector is $(x_5,x_6,x_{11},x_1,x_2,x_3,x_4,x_7,x_8,x_9,x_{10})$.

At the first stage of the design it is necessary to select three eigenvectors such that Y^{-1} exists, where the ith column of Y consists of the 5th, 6th, and 11th entries of the corresponding selected eigenvector. In view of the above discussion of the closed loop eigenstructure, it follows that the columns of Y have the form $(\beta,0,\beta)^T$ if a decoupled eigenvalue is selected, and $(\beta,\gamma,-\beta)^T$ if a coupled eigenvalue is selected. Thus, for Y to be invertible, Λ_r must contain exactly one decoupled eigenvalue and two coupled eigenvalues. In particular, the selection of the three dominant coupled eigenvalues $\lambda_4^c,\lambda_5^c,\lambda_6^c$ is not possible.

Subject to complex pairing there are 21 possible choices for Λ_r, of which four give a stable spectrum for A_1: $(\lambda_2^c,\lambda_3^c,\lambda_2^d),(\lambda_2^c,\lambda_3^c,\lambda_3^d)$, $(\lambda_4^c,\lambda_5^c,\lambda_2^d),(\lambda_4^c,\lambda_5^c,\lambda_3^d)$. For each of these choices however, the spectrum of A_1 contains at least one eigenvalue with real part to the right of $\sigma = -0.060$. Since the spectrum of the open loop system lies to the lect of the line $\sigma = -0.060$, a static output feedback design is not acceptable and a dynamic compensator will be designed.

To solve the pole-placement problem of the second stage of the design, Λ_r must be selected and the remaining optimal eigenvalues ordered. Because of the nature of the tie-line equation the second row of A_{12} is all zeros and it is necessary to introduce

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad TA_{12} = \begin{bmatrix} c_1 & 0 \\ 0 & c_1 \end{bmatrix}, \quad c_1 = [0.035 \; 0.02333 \; 0.025 \; 0]. \tag{4.49}$$

Then $\ell = 2$ and at the ith stage of the algorithm only $i+\ell-1 = i+1$ eigenvalues may be assigned to A_{i+1}. It is assumed that the dimension of the desired compensator may not exceed four. At first designs based on stable matrices A_1 are discussed and then, as these will prove unsatisfactory, designs based on unstable matrices A_1 will be investigated.

Consider first the four choices of Λ_r which give rise to a stable matrix A_1. As the optimal control expended little energy shaping the eigenvectors $\mathcal{V}_2^c, \mathcal{V}_2^d$ the two choices $(\lambda_2^c, \lambda_3^c, \lambda_2^d), (\lambda_2^c, \lambda_3^c, \lambda_3^d)$ will not be pursued here. The other two choices $(\lambda_4^c, \lambda_5^c, \lambda_3^d), (\lambda_4^c, \lambda_5^c, \lambda_2^d)$ retain the dominant coupled frequency pair and since compensator designs based on the first choice were unsuccessful, only the latter will be considered here.

Let $\Lambda_r = dg(\lambda_2^d, \lambda_4^c, \lambda_5^c)$. Based on a desire to retain the dominant eigenvalues, the pole-placement problem is solved for the ordering $\lambda_6^c, \lambda_3^c, \lambda_4^d, \lambda_5^d$. All calculations are performed using Procedure 1, and the data are given in Table 4.4.6. In the following discussion eigenvalues without superscripts refer to spectra in Table 4.4.6. At the first stage of the algorithm a complex pair is placed at $-0.1 \pm j0.1$ in the expectation of removing the two small eigenvalues λ_7, λ_8. Unfortunately the resultant spectrum of A_2 still contains a small eigenvalue $\lambda_8 = -0.060$. The compensator based on this solution is (D_o, K_z have been scaled)

Table 4.4.6. Data for pole-placement problem based on the ordering: $\lambda_2^d, \lambda_4^c, \lambda_5^c, \lambda_6^c, \lambda_3^c$

$$A_1^1 = \begin{bmatrix} -2.0000 & 0.1961 & 0.1830 & 0.2746 \\ 4.7500 & -2.3665 & 2.4576 & 3.6869 \\ 0.0000 & 0.0833 & -0.2445 & -0.1168 \\ 0.0000 & 0.0473 & 2.0441 & -1.9338 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} A_1^1 & A_1^2 \\ A_1^2 & A_1^1 \end{bmatrix}$$

$$A_1^2 = \begin{bmatrix} 0.0000 & -0.1838 & -0.1715 & -0.2573 \\ 0.0000 & 2.2413 & 2.0916 & 3.1378 \\ 0.0000 & -0.0851 & -0.0794 & -0.1192 \\ 0.0000 & 0.0655 & 0.0612 & 0.0918 \end{bmatrix}$$

$$N_o \begin{bmatrix} -7.8452 & 0.2596 & 7.3487 \\ -105.3412 & -0.0249 & -89.6134 \\ 3.3363 & -0.0595 & 3.4031 \\ -1.8903 & -0.0280 & -2.6205 \\ 7.3517 & -0.2596 & -7.8421 \\ -89.6515 & 0.0249 & -105.3800 \\ 3.4045 & 0.0595 & 3.3377 \\ -2.6216 & 0.0280 & -1.8914 \end{bmatrix} \quad B_o = \begin{bmatrix} -0.2336 & 0.1484 \\ -0.1529 & 0.1192 \\ 0.4484 & -0.0101 \\ 0.4941 & -0.7080 \\ 0.2336 & -0.1484 \\ 0.1540 & -0.1192 \\ -0.4484 & 0.0101 \\ -0.4940 & 0.7080 \end{bmatrix}$$

$$f_1 = [1.0000] \qquad f_2 = \begin{bmatrix} 0.7072 \\ 0.7070 \end{bmatrix} \qquad g_1 = [3914.940 \quad 3918.477]$$

$$g_2 = [1014.327 \quad 1017.972]$$

The spectrum of A_1 is:	The spectrum of A_2 is:	The spectrum of A_3 is:
1. (-5.174, 0.000)	1. (-5.193, 0.000)	1. (-5.197, 0.000)
2. (-2.101, 0.000)	2. (-2.102, 0.000)	2. (-2.102, 0.000)
3. (-1.922, 0.000)	3. (-1.921, 0.000)	3. (-1.921, 0.000)
4. (-1.799, 0.306)	4. (-1.780, 0.315)	4. (-1.741, 0.297)
5. (-1.799, -0.306)	5. (-1.780, -0.315)	5. (-1.741, -0.297)
6. (-0.230, 0.000)	6. (-0.112, 0.104)	6. (-0.152, 0.153)
7. (-0.039, 0.000)	7. (-0.112, -0.104)	7. (-0.152, -0.153)
8. (-0.024, 0.000)	8. (-0.060, 0.000)	8. (-0.037, 0.000)

$$H_0 = [-0.1904] \qquad D_0 = [-182.623,\ 0.00297,\ 182.634]$$

$$K_y = \begin{bmatrix} -762.3500 & -0.4577 & -764.9227 \\ 760.9804 & 0.4577 & 763.5529 \end{bmatrix} \quad K_z = \begin{bmatrix} 20.0423 \\ -20.0423 \end{bmatrix}. \tag{4.4.10}$$

Continuing the solution of the pole-placement problem because of the large gains in K_y and the small eigenvalue λ_8, f_2 is chosen at the second stage to render λ_3 uncontrollable and a complex pair is placed at $0.13 \pm j0.13$. The compensator based on this solution is (D_0, K_z have been scaled)

$$H_0 = \begin{bmatrix} -0.2322 & 0.0632 \\ -0.0567 & -1.9364 \end{bmatrix} \quad D_0 = \begin{bmatrix} -214.281 & 0.005176 & -214.298 \\ -18.645 & 0.002220 & -18.605 \end{bmatrix}$$

$$K_y = \begin{bmatrix} -841.9190 & -0.4577 & -844.7776 \\ 840.5499 & 0.4577 & 843.4084 \end{bmatrix} \quad K_z = \begin{bmatrix} 20.131 & -9.015 \\ -20.131 & 9.015 \end{bmatrix} \tag{4.4.11}$$

$$\sigma(H_0) = (-1.934,\ -0.234).$$

The degree of stability of A_r has decreased: $\lambda_8 = -0.037$, and the gains K_y have increased.

Attempts at continuing the solution of the pole-placement problem to increase the degree of stability of the closed loop system and reduce the feedback gains were unsuccessful. Therefore, a different ordering for Λ_r, Λ_p was considered. Since the major objection to the design above is the poor stability margin, and based on the suspicion that the small real pole in $\sigma(A_1)$ is due to the tie-line interaction mode λ_6^c departing from its optimal location, Λ_r is selected to retain λ_6^c. The remaining two eigenvalues are chosen based on the expectation that if the spectrum of A_1 contains a complex pair near $\pm j2$ then this will correspond to a departure of the frequency pair λ_4^c, λ_5^c from their optimal locations. Based on physical considerations it should then be relatively easy to shape the real part of this complex pair, though the complex part should be insensitive to feedback.

From an inspection of the spectra of the 21 possible matrices A_1 (this data is not reported here), the choice $\Lambda_r = dg(\lambda_3^d, \lambda_3^c, \lambda_6^c)$ was made. The ordering for the pole-placement problem is taken to be $\lambda_1^d, \lambda_1^c, \lambda_4^d, \lambda_5^d$. The pole-placement problem is solved using Procedure 1 at each stage and the data are given in Table 4.4.7. In the following discussion eigenvalues without superscripts refer to spectra in Table 4.4.7. At the first stage a complex pair is placed at the location of λ_4^c, λ_5^c. At the second stage f_2 is chosen to render λ_1 uncontrollable and a complex pair is again placed at λ_4^c, λ_5^c. At the third stage f_3 is chosen to render uncontrollable the complex pair just assigned and then eigenvalues are placed at -2,-5. At the last stage f_4 is chosen to render $\lambda_4, \lambda_7, \lambda_8$ uncontrollable and a complex pair is placed at $-3 \pm j2$.

Three compensators are defined by this solution to the pole-placement problem. Based on the first stage of the solution (D_0, k_z have been scaled)

$$H_0 = [-6.960] \qquad D_0 = (50.941 \quad -0.239 \quad -63.913)$$
$$K_y = \begin{bmatrix} -608.8379 & 0.2478 & 755.8289 \\ -638.6288 & -0.2479 & 791.7451 \end{bmatrix} \qquad K_z = \begin{bmatrix} 56.57 \\ 58.91 \end{bmatrix}. \tag{4.4.12}$$

Based on two stages of the solution (D_0, K_z have been scaled)

$$H_0 = \begin{bmatrix} -6.960 & 16.116 \\ 0 & -9.171 \end{bmatrix} \qquad D_0 = \begin{bmatrix} 50.933 & -0.239 & -63.905 \\ 0 & 0 & 0 \end{bmatrix}$$
$$K_y = \begin{bmatrix} -608.7381 & 0.2478 & 755.7395 \\ -638.5250 & -0.2479 & 791.6521 \end{bmatrix} \qquad K_z = \begin{bmatrix} 56.57 & -273.27 \\ 58.91 & -115.18 \end{bmatrix}. \tag{4.4.13}$$

Based on the complete solution (D_0, K_z have been scaled)

Table 4.4.7. Data for pole-placement problem based on the ordering: $\lambda_3^d, \lambda_3^c, \lambda_6^c, \lambda_1^d, \lambda_1^c, \lambda_4^d, \lambda_5^d$

$$A_1 = \begin{bmatrix} A_1^1 & A_1^2 \\ A_1^2 & A_1^1 \end{bmatrix} \qquad A_1^1 = \begin{bmatrix} -2.0000 & 0.1279 & 0.1194 & 0.1791 \\ 4.7500 & -4.8888 & 0.1037 & 0.1556 \\ 0.0000 & -0.3099 & -0.6114 & -0.6672 \\ 0.0000 & 1.6633 & 3.5522 & 0.3287 \end{bmatrix}$$

$$A_1^2 = \begin{bmatrix} 0.0000 & -0.1215 & -0.1134 & -0.1701 \\ 0.0000 & -0.1010 & -0.0943 & -0.1415 \\ 0.0000 & 0.4756 & 0.4439 & 0.6659 \\ 0.0000 & -0.3211 & -0.2997 & -0.4496 \end{bmatrix}$$

$$N_0 = \begin{bmatrix} -5.1168 & -0.5450 & 4.8607 \\ -4.4470 & -0.5345 & 4.0414 \\ 19.0616 & 1.7236 & -19.0248 \\ -66.5339 & 1.3320 & 12.8315 \\ 4.8610 & 0.5450 & -5.1170 \\ 4.0418 & 0.5345 & -4.4473 \\ -19.0253 & -1.7236 & 19.0622 \\ 12.8455 & -1.3308 & -66.5481 \end{bmatrix} \qquad B_0 = \begin{bmatrix} 0.4571 & -0.4775 & -0.1525 & 0.0870 \\ -0.5199 & 0.5240 & -0.1311 & 0.0583 \\ 0.0104 & 0.0315 & 0.2584 & 0.3075 \\ 0.0728 & -0.1166 & 5.9765 & -10.3782 \\ 0.4765 & 0.4759 & -0.1525 & 0.0870 \\ -0.5412 & -0.5222 & -0.1311 & 0.0583 \\ 0.0091 & -0.0315 & 0.2584 & 0.3075 \\ 0.0776 & 0.1164 & 5.9765 & -10.3783 \end{bmatrix}$$

$$f_1 = [1.0000] \qquad f_2 = \begin{bmatrix} -1.0000 \\ 0.0004 \end{bmatrix} \qquad f_3 = \begin{bmatrix} -0.9954 \\ 0.0628 \\ 0.0728 \end{bmatrix}$$

$$g_1 = [745.263 \quad -925.893] \qquad g_2 = [0.122 \quad -0.109] \qquad g_3 = [748.068 \quad -929.488]$$

$$f_4 = \begin{bmatrix} 0.2210 \\ -0.1526 \\ 0.7809 \\ 0.5640 \end{bmatrix} \qquad p = \begin{bmatrix} 8.9730 & 0.0000 & -11.2749 \\ 41.1179 & 0.0000 & -50.9564 \\ 84.3020 & 0.0000 & -105.4287 \\ 21.5255 & 0.0000 & -27.2383 \end{bmatrix}$$

$$g_4 = [38.169 \quad -48.298]$$

Table 4.4.7. continued

The spectrum of A_1 is:

1. (-5.090, 0.000)
2. (-5.000, 0.000)
3. (-1.999, 0.000)
4. (-1.978, 0.000)
5. (-0.167, 0.000)
6. (-0.112, 0.000)
7. (0.002, 2.026)
8. (0.002, -2.026)

The spectrum of A_2 is:

1. (-5.079, 0.000)
2. (-1.979, 0.000)
3. (-1.203, 2.501)
4. (-1.203, -2.501)
5. (-1.093, 0.196)
6. (-1.093, -0.196)
7. (-0.241, 1.943)
8. (-0.241, -1.943)

The spectrum of A_3 is:

1. (-5.079, 0.000)
2. (-1.979, 0.000)
3. (-1.203, 2.501)
4. (-1.203, -2.501)
5. (-1.093, 0.196)
6. (-1.093, -0.196)
7. (-0.241, 1.943)
8. (-0.241, -1.943)

The spectrum of A_4 is:

1. (-5.000, 0.000)
2. (-3.224, 2.336)
3. (-3.224, -2.336)
4. (-2.001, 0.000)
5. (-1.884, 0.000)
6. (-0.304, 0.000)
7. (-0.241, 1.943)
8. (-0.241, -1.943)

The spectrum of A_5 is:

1. (-4.999, 0.000)
2. (-3.000, 2.000)
3. (-3.000, -2.000)
4. (-2.001, 0.000)
5. (-1.842, 0.000)
6. (-0.546, 0.000)
7. (-0.214, 1.943)
8. (-0.241, -1.943)

$$H_0 = \begin{bmatrix} -9.1432 & 0.2209 & 1.7208 & -3.2424 \\ 0.1214 & -8.1721 & 7.3545 & -13.8580 \\ -0.0662 & -0.3129 & -4.5239 & 8.2957 \\ -0.0142 & -0.0517 & -1.0476 & 1.6279 \end{bmatrix} \quad D_0 = \begin{bmatrix} 7.3655 & 0.0949 & -8.994 \\ 33.439 & 0.4032 & -40.400 \\ -7.968 & 0.8692 & 11.217 \\ -0.837 & 0.236 & 1.741 \end{bmatrix}$$

$$K_y = \begin{bmatrix} 22.6798 & 0.2478 & -28.5104 \\ -50.4229 & -0.2479 & 61.0083 \end{bmatrix} \quad K_z = \begin{bmatrix} 8.276 & -7.668 & 5.957 & -10.206 \\ 8.334 & 6.804 & -11.387 & 22.475 \end{bmatrix} \tag{4.4.14}$$

The first order compensator gives a nice total closed loop spectrum but has the same disadvantage encountered before of large feedback gains. The second order compensator has no apparent advantage over the first beyond the retention of the optimal pair (λ_1^c, ν_1^c). The fourth order compensator however has nice properties. It requires gains no larger than 60 as compared to 20 for the full state feedback solution. Furthermore it retains a seven dimensional optimal invariant subspace corresponding to $(\lambda_1^c, \lambda_3^c, \lambda_6^c, \lambda_1^d, \lambda_3^d, \lambda_4^d, \lambda_5^d)$ and the spectrum of A_r contains eigenvalues near $\lambda_2^c, \lambda_4^c, \lambda_5^c$. The spectrum of the resultant closed loop system is given in Table 4.4.8 and Figure 4.4.2. The compensator is also open loop stable.

In summary if large feedback gains are tolerable then the first order compensator defined in (4.4.12) is satisfactory, however, if it is desired to reduce the magnitude of the feedback gains then the dimension of the compensator must be increased. The fourth order controller defined by (4.4.14) is one possibility and its degree compares favorably with that of the Luenberger reduced order observer $(n-r = 8)$. It is noted that there are many other possibilities for designing satisfactory controllers. The analysis presented here has only served to illustrate the design methodology.

Table 4.4.8. Spectra of closed loop system under dynamic compensation

For the compensator given in (4.4.12)	For the compensator given in (4.4.14)
The spectrum of A_c is:	The spectrum of A_c is:
1. (-9.171, 0.000)	1. (-9.171, 0.000)
2. (-5.079, 0.000)	2. (-9.171, 0.000)
3. (-2.001, 0.000)	3. (-4.999, 0.000)
4. (-1.994, 0.000)	4. (-3.000, 2.000)
5. (-1.979, 0.000)	5. (-3.000, -2.000)
6. (-1.203, 2.501)	6. (-2.001, 0.000)
7. (-1.203, -2.501)	7. (-2.001, -0.000)
8. (-1.093, 0.196)	8. (-1.994, 0.000)
9. (-1.093, -0.196)	9. (-1.842, 0.000)
10. (-0.241, 1.943)	10. (-0.546, 0.000)
11. (-0.241, -1.943)	11. (-0.241, 1.943)
12. (-0.220, 0.000)	12. (-0.241, -1.943)
	13. (-0.220, 0.000)
	14. (-0.171, 0.093)
	15. (-0.171, -0.093)
The spectrum of H_0 is:	The spectrum of H_0 is:
1. (-6.960, 0.000)	1. (-9.170, 0.000)
	2. (-7.841, 0.000)
	3. (-2.644, 0.000)
	4. (-0.557, 0.000)

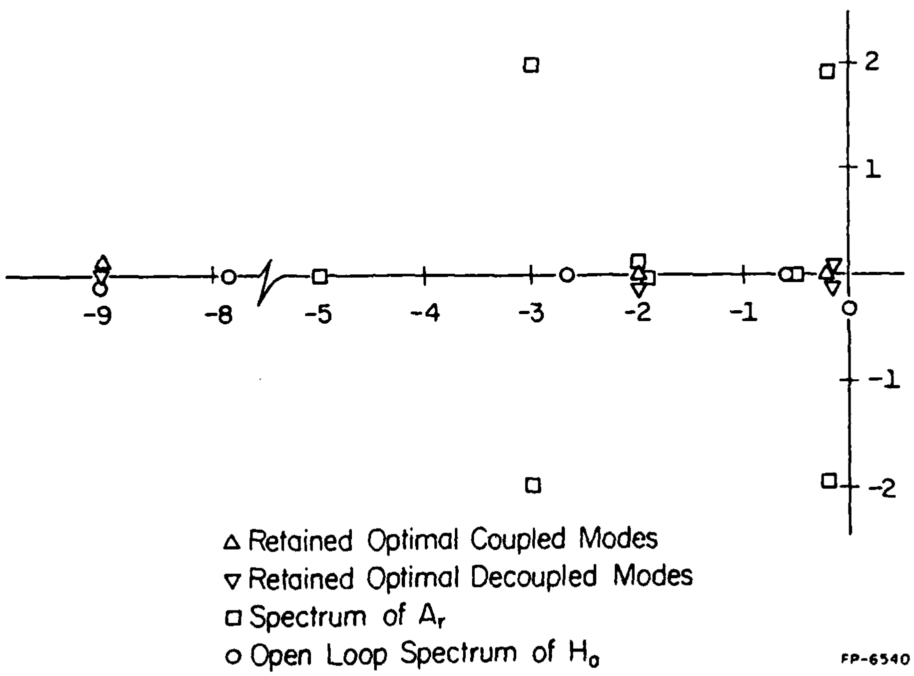


Figure 4.4.2. Closed loop spectrum using the compensator given in (4.4.14)

CONCLUSIONS

A design oriented methodology for the construction of suboptimal linear quadratic regulators has been presented. A design criterion has been taken to be the retention of as many optimal eigenvectors as possible from a reference state feedback regulator. This gives rise to an associated output feedback pole-placement problem both in designs using static and dynamic compensation. For the latter case an algorithm has been given which solves this pole-placement problem, implicitly fixing the parameters of the controller, and determining its dimension without a priori assumptions. It has also been shown that the methodology may be extended to the class of stabilizable systems. Finally the design methodology has been illustrated with three nontrivial examples.

REFERENCES

1. J. Medanic, "Design of Low Order Optimal Dynamic Regulators for Linear Time-Invariant Systems," 1979 Conference on Information Theory and Systems, Johns Hopkins University, March 1979 (Submitted for publication).

2. N. Munro and S. Novin-Hirbod, "Pole Assignment Using Full-Rank Output-Feedback Compensators," Int. J. Systems Sci., Vol. 10, No. 3, March 1979, pp. 285-306.

3. W. M. Wonham, "On Pole Assignment in Multi-Input Controllable Linear Systems," IEEE Trans. on AC, Vol. AC-12, No. 6, Dec. 1967, pp. 660-665.

4. E. J. Davison, "On Pole Assignment in Linear Systems with Incomplete State Feedback," IEEE Trans. on AC, Vol. AC-15, No. 3, June 1970, pp. 348-351.

5. E. J. Davison and S. H. Wang, "On Pole Assignment in Linear Multivariable Systems Using Output Feedback," IEEE Trans. on AC, Vol. AC-20, No. 4, Aug. 1975, pp. 516-518.

6. A. Jameson, "Design of a Single-Input System for Specified Roots Using Output Feedback," IEEE Trans. on AC, Vol. AC-15, No. 3, June 1970, pp. 345-348.

7. T. Topaloglu and D. E. Seborg, "A Design Procedure for Pole Assignment Using Output Feedback," Int. J. Control, Vol. 22, No. 6, 1975, pp. 741-748.

8. C. T. Chen and C. H. Hsu, "Design of Dynamic Compensators for Multivariable Systems," 1971 JACC, Paper no. 8-E1 (Unpublished).

9. F. Fallside and H. Seraji, "Design of Multivariable Systems Using Unity-Rank Feedback," Int. J. Control, Vol. 17, No. 2, 1973, pp. 351-364.

10. H. Seraji, "Restrictions on Attainable Poles and Methods for Pole Assignment with Output Feedback," Proc. IEE , Vol. 121, No. 3, March 1974, pp. 205-212.

11. H. Seraji, "A New Method for Pole Assignment Using Output Feedback," Int. J. Control, Vol. 28, No. 1, 1978, pp. 147-155.

12. H. Kimura, "Pole Assignment by Gain Output Feedback," IEEE Trans. on AC, Vol. AC-20, No. 4, Aug. 1975, pp. 509-516.

13. H. Kimura, "A Further Result on the Problem of Pole Assignment by Output Feedback," IEEE Trans. on AC, Vol. AC-22, No. 3, June 1977, pp. 458-463.

14. A. Bradshaw and B. Porter, "Design of Linear Multivariable Discrete-Time Output-Feedback Regulators," Int. J. Systems Sci., Vol. 9, No. 8, 1978, pp. 857-863.

15. A. I. Vardulakis, "Generalized Root Locus Assignment of all Poles of a Multivariable System by Output Feedback," Int. J. Control, Vol. 23, No. 1, 1976, pp. 39-47.

16. A. G. J. MacFarlane, B. Kouvaritakis, and J. M. Edmunds, "Complex Variable Methods for Multivariable Feedback System Analysis and Design," in Alternates for Linear Multivariable Control, edited by M. Sain, J. Pechzkoraski, and J. Melsa, National Engineering Consortium, Inc., Chicago, 1978.

17. B. Kouvartakis, "Optimal Root Loci of Linear Multivariable Systems," Int. J. Control, Vol. 28, No. 1, 1978, pp. 33-62.

18. N. Munro and A. Vardulakis, "Pole Shifting Using Output Feedback," Int. J. Control, Vol. 18, No. 6, 1973, pp. 1267-1273.

19. N. Munro, "Further Results on Pole-Shifting Using Output Feedback," Int. J. Control, Vol. 20, No. 5, 1974, pp. 775-786.

20. H. M. Power, "Dyadic Output Feedback Laws Generated as Kronecker Products: Optimal and Suboptimal Solutions," Int. J. Control, Vol. 23, No. 6, 1976, pp. 785-798.

21. Y. Bar-Ness, "Optimal Closed-Loop Poles Assignment," Int. J. Control, Vol. 27, No. 3, 1978, pp. 421-430.

22. B. Sridhar and D. P. Lindorff, "Pole-Placement with Constant Gain Output Feedback," Int. J. Control, Vol. 18, No. 5, 1973, pp. 993-1003.

23. H. R. Sirisena and S. S. Choi, "Pole-Placement in Prescribed Regions of the Complex Plane Using Output Feedback," IEEE Trans. on AC, Vol. AC-20, No. 6, Dec. 1975, pp. 810-812.

24. L. F. Miller, R. G. Cochran, and J. W. Howze, "Output Feedback Stabilization by Minimization of a Spectral Radius Function," Int. J. Control, Vol. 27, No. 3, 1978, pp. 455-462.

25. W. S. Levine and M. Athans, "On the Determination of the Optimal Constant Output Feedback Gains for Linear Multivariable Systems," IEEE Trans. on AC, Vol. AC-15, No. 1, Feb. 1970, pp. 44-48.

26. W. M. Wonham, Linear Multivariable Control, A Geometric Approach, Berlin, Springer Verlag, 1974.

27. E. J. Davison and S. H. Wang, "Properties of Linear Time-Invariant Multivariable System Subject to Arbitrary Output and State Feedback," IEEE Trans. on AC, Vol. AC-18, No. 1, Feb. 1973, pp. 24-32.

28. D. Q. Mayne and P. Murcoch, "Modal Control of Linear Time Invariant Systems," Int. J. Control, Vol. 11, No. 2, 1970, pp. 223-227.

29. K. Martensson, "On the Matrix Ricatti Equation," Information Sciences, Vol. 3, Jan. 1971, pp. 17-49.

30. H. P. Horisberger and P. R. Belanger, "Solution of the Optimal Constant Output Feedback Problem by Conjugate Gradients," IEEE Trans. on AC, Vol. AC-19, No. 4, Aug. 1974, pp. 434-435.

31. J. Medanic, "On Stabilization and Optimization by Output Feedback," Twelfth Asilomar Conference on Circuits, Systems, and Computers, Nov. 1978 (Submitted for publication).

32. A. Ben-Israel and T. N. E. Greville, Generalized Inverses, New York, John Wiley & Sons, 1974.

33. D. G. Luenberger, "An Introduction to Observers," IEEE Trans. on AC, Vol. AC-16, No. 6, Dec. 1971, pp. 596-602.

34. G. E. Forsythe, M. A. Malcolm and C. B. Moler, Computer Methods for Mathematical Computations, Prentice-Hall, 1977.

35. D. E. McBrinn and R. J. Roy, "Stabilization of Linear Multivariable Systems by Output Feedback," IEEE Trans. on AC, Vol. 17, No. 2, April 1972, pp. 243-245.

36. D. K. Lindner, "Zeros of Multivariable Systems: Definitions and Algorithms," Report DC-23, Coordinated Science Laboratory, University of Illinois, Urbana, Illinois, May 1979.

37. M. Calovic, "Dynamic State-Space Models of Electric Power Systems," Dept. of Electrical Engr., University of Illinois, Urbana, 1971.