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**ELECTROMAGNETIC ENERGY DEPOSITION
IN A CONCENTRIC SPHERICAL MODEL
OF THE HUMAN OR ANIMAL HEAD**

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USAF SCHOOL OF AEROSPACE MEDICINE
Aerospace Medical Division (AFSC)
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NOTICES

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This technical report has been reviewed and is approved for publication.

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20. ABSTRACT (Continued)

of this users-oriented program covers such details as: structure and sequence of control parameter and data cards, output formats, and subroutine and function subprograms. Sample printouts and plots (linear and contour) of computer results and a listing of the FORTRAN source program are included. ←

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ELECTROMAGNETIC ENERGY DEPOSITION IN A CONCENTRIC SPHERICAL MODEL OF THE HUMAN OR ANIMAL HEAD

INTRODUCTION

The head is modeled by several homogeneous regions of tissue bounded by one or two members of a family of concentric spheres. These tissues include brain, cerebrospinal fluid (CSF), dura, bone, subcutaneous fat, and skin tissue. We assume that this complex of biological material is exposed to nonionizing electromagnetic radiation taking the form of a time-harmonic plane wave of peak amplitude, E_0 . The time variation factor, $\exp(-i\omega t)$, has been suppressed in most of the discussion. Wave propagation is in the positive z -direction, and the electric field, E , is linearly polarized in the x -direction (Fig. 1). A rectangular-spherical coordinate system with origin at the center of an inner core sphere is used. Also, the medium surrounding the concentric spherical model is taken as free space (or vacuum). Thus our embedding medium is a nonconductor, and both the surrounding medium and the model are non-magnetic. Each region ($p = 1, \dots, N-1$) into which the model is partitioned is homogeneous, isotropic, and possesses a unique dielectric constant and conductivity. All magnetic permeabilities are considered to have the value unity. The value " $p = N$ " is reserved for reference to the containing medium.

The need for a multilayer model and the inadequacy of (1) ignoring the relatively thin outer structures, or (2) carrying out a volume average of the electrical properties of the regions can be seen by looking at Figure 2 for the case of the rhesus monkey. There graphically displayed, for comparison purposes, are three superimposed distributions of absorbed-power density along the z -axis. The monkey-head models consist of (1) pure brain tissue, (2) tissue with average volume of electrical properties of the structural components in Figure 1, and (3) unique tissues represented in Figure 1. The predicted distributions are based on 1-volt-per-meter intensity incident wave at 3 GHz.

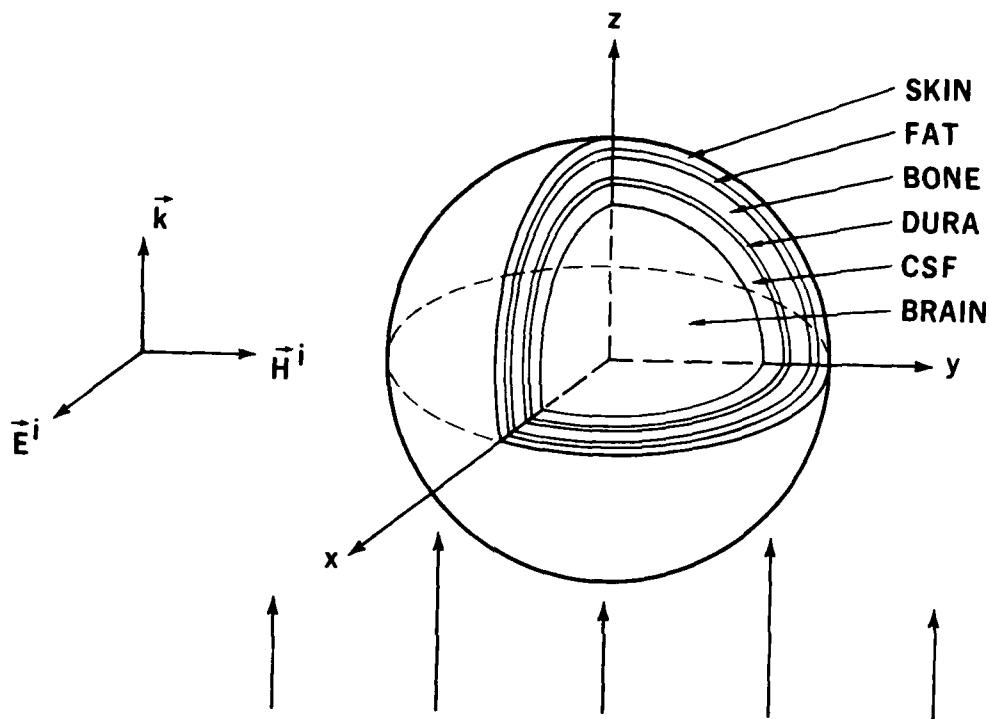


Figure 1. Electromagnetic plane wave impinging on a head model composed of an inner core sphere and five spherical shells.

Dimensions of structural media and electrical parameter values were extracted from a table that was produced by Shapiro et al. (13). Table 1 presents such information. Contour plots--Figures 3 ($\phi = 0$), 4 ($\phi = \pi/2$), and 5--are likewise based on information offered by Table 1.

The linear plot, Figure 6, and the contour plots--Figures 7 ($\phi = 0$), 8 ($\phi = \pi/2$), and 9--are founded on the entries of Table 2, an extraction from a paper by Weil (15). Incident plane-wave characteristics are 1-volt-per-meter intensity and 1-GHz frequency. Other parameter values pertinent to the computations for graphical construction are given in Table 2.

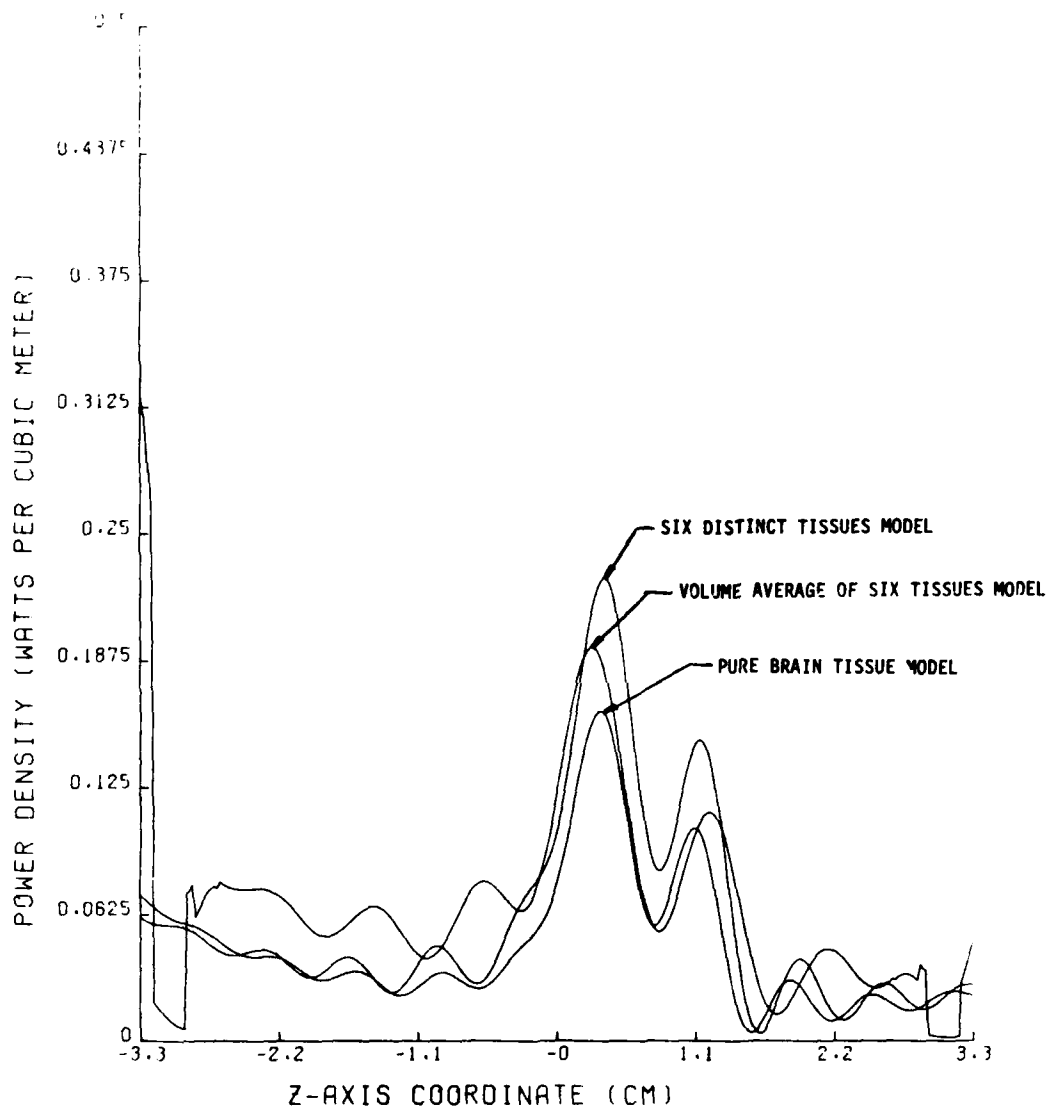


Figure 2. Distribution of power density along the z-axis for three different head models of the rhesus monkey. Spheres are of 3.3-cm radius and frequency is at 3 GHz.

TABLE 1. RHESUS-MONKEY-HEAD DATA

Region (p)	Tissue modeled	Thickness (cm)	Radius of surface boundary, r_p (cm)	Relative dielectric constant, ^a ϵ_p	Conductivity, ^a σ_p (mho/m)
1	Brain	sphere	2.68	42.0	2.0
2	CSF	0.20	2.88	77.0	1.9
3	Dura	0.05	2.93	45.0	2.5
4	Bone	0.20	3.13	5.0	0.2
5	Fat	0.07	3.20	5.0	0.2
6	Skin	0.10	3.30	45.0	2.5

^aAt $T = 37^{\circ}\text{C}$ and $f = 3 \text{ GHz}$.

TABLE 2. IDEALIZED HUMAN-HEAD DATA

Region (p)	Tissue modeled	Thickness (cm)	Radius of surface boundary, r_p (cm)	Relative dielectric constant, ^a ϵ_p	Conductivity, ^a σ_p (mho/m)
1	Brain	sphere	9.10	60.00	0.90
2	CSF	0.20	9.30	76.00	1.70
3	Dura	0.05	9.35	45.00	1.00
4	Bone	0.40	9.75	8.50	0.11
5	Fat	0.15	9.90	5.50	0.08
6	Skin	0.10	10.00	45.00	1.00

^aAt $f = 1 \text{ GHz}$.

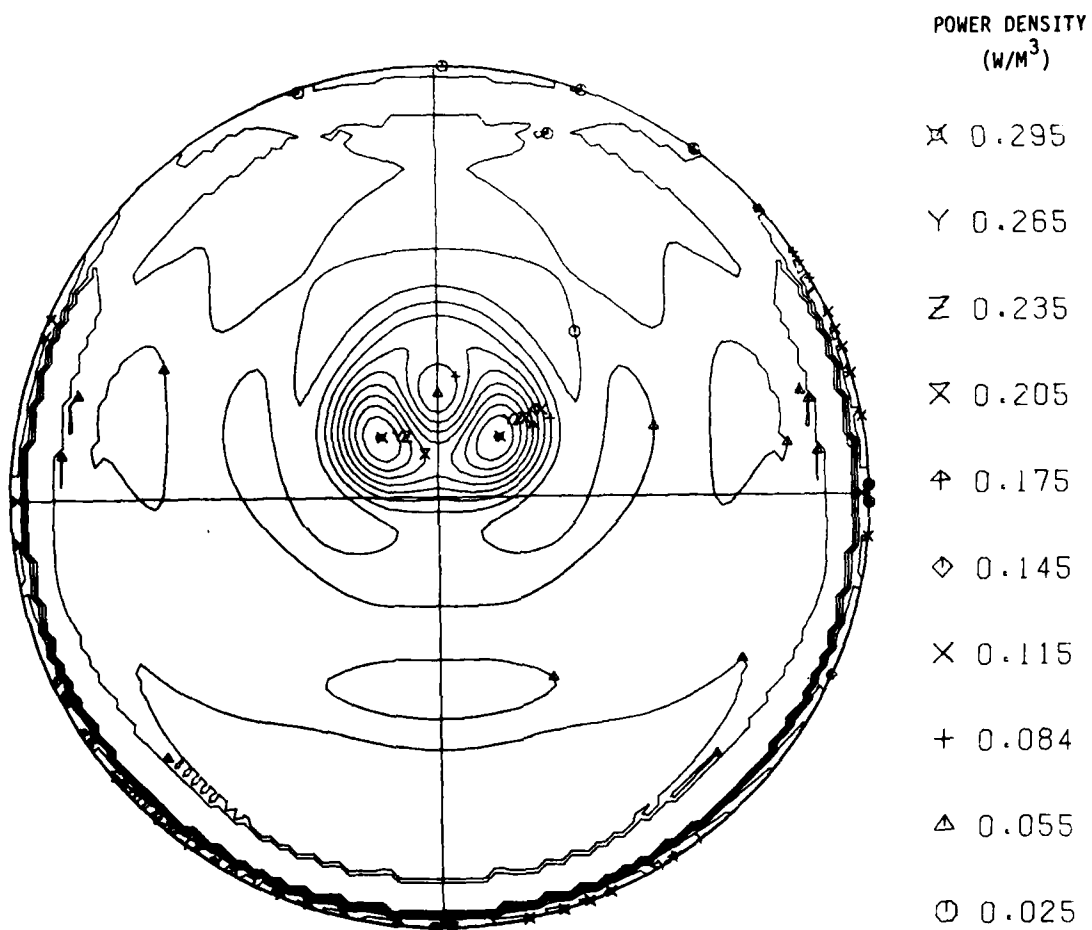


Figure 3. Distribution of power density in a 3.3-cm-radius sphere, a model of the rhesus-monkey head, at 3 GHz. (E-plane)

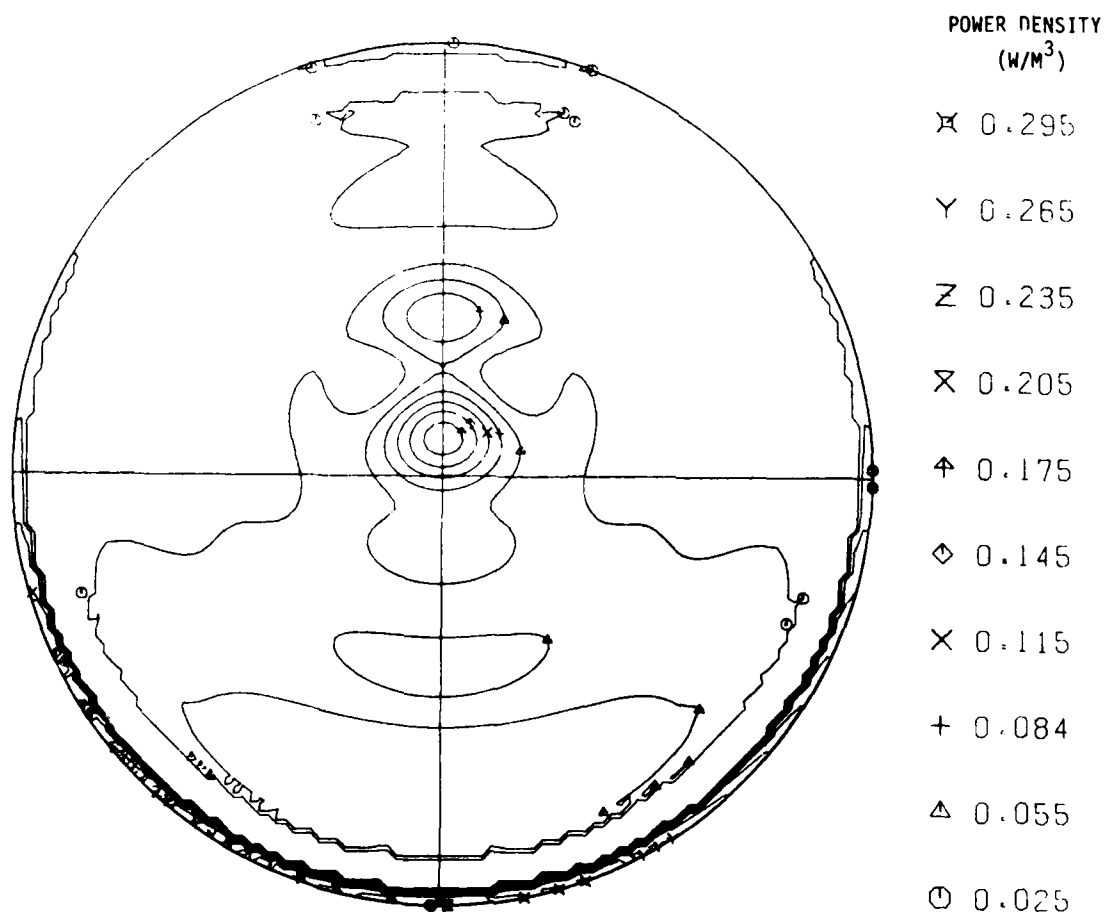


Figure 4. Distribution of power density in a 3.3-cm-radius sphere, a model of the rhesus-monkey head, at 3 GHz. (H-plane)

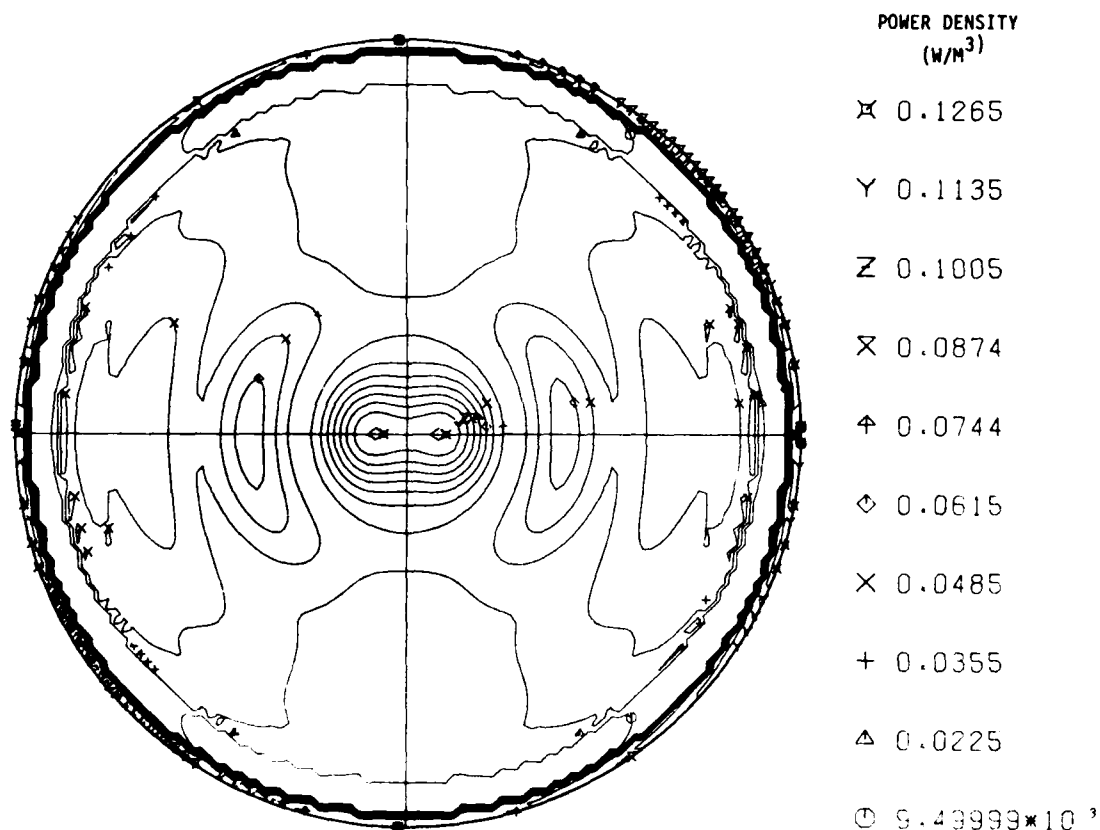


Figure 5. Distribution of power density in a 3.3-cm-radius sphere, a model of the rhesus-monkey head, at 3 GHz. (X,Y-plane)

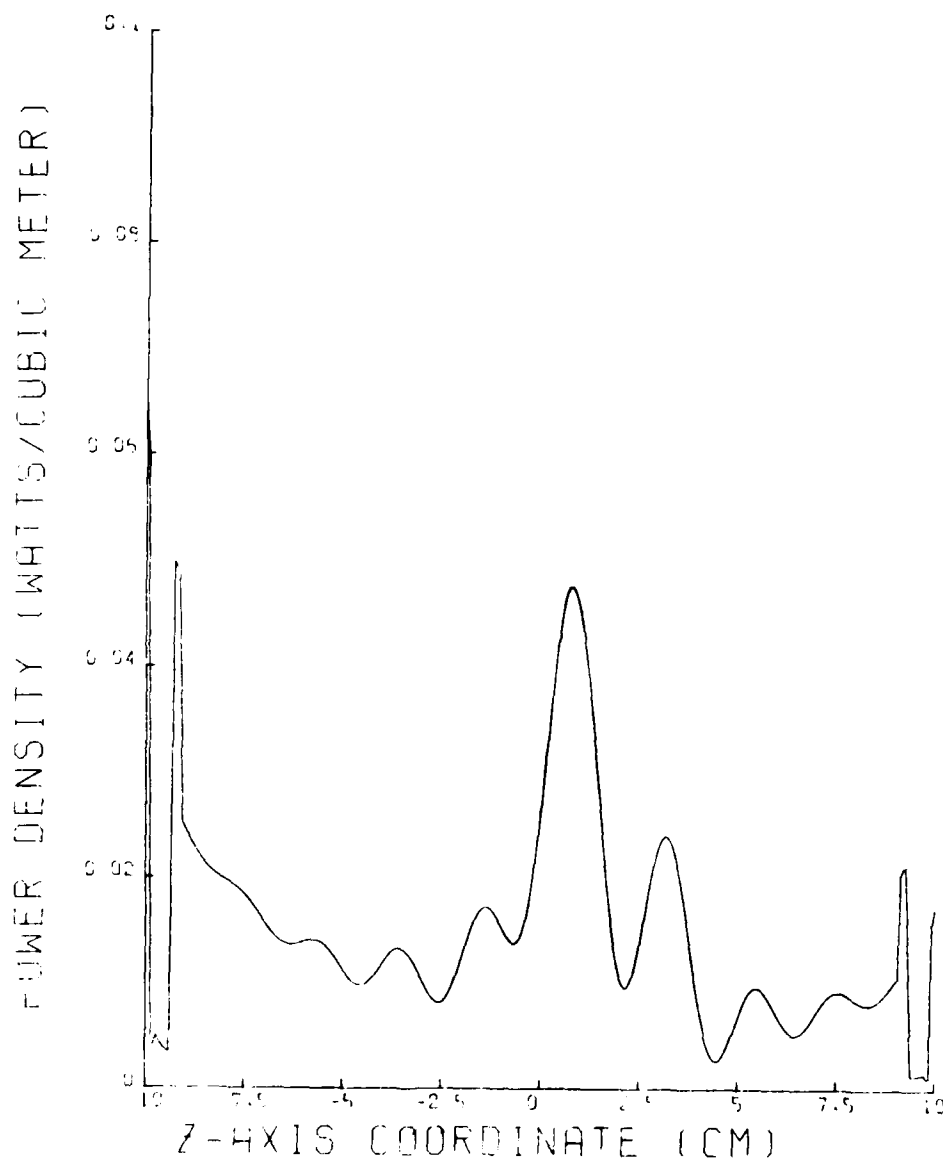


Figure 6. Distribution of power density in a 10-cm-radius sphere, a model of the human head, at 1 GHz along the z-axis.

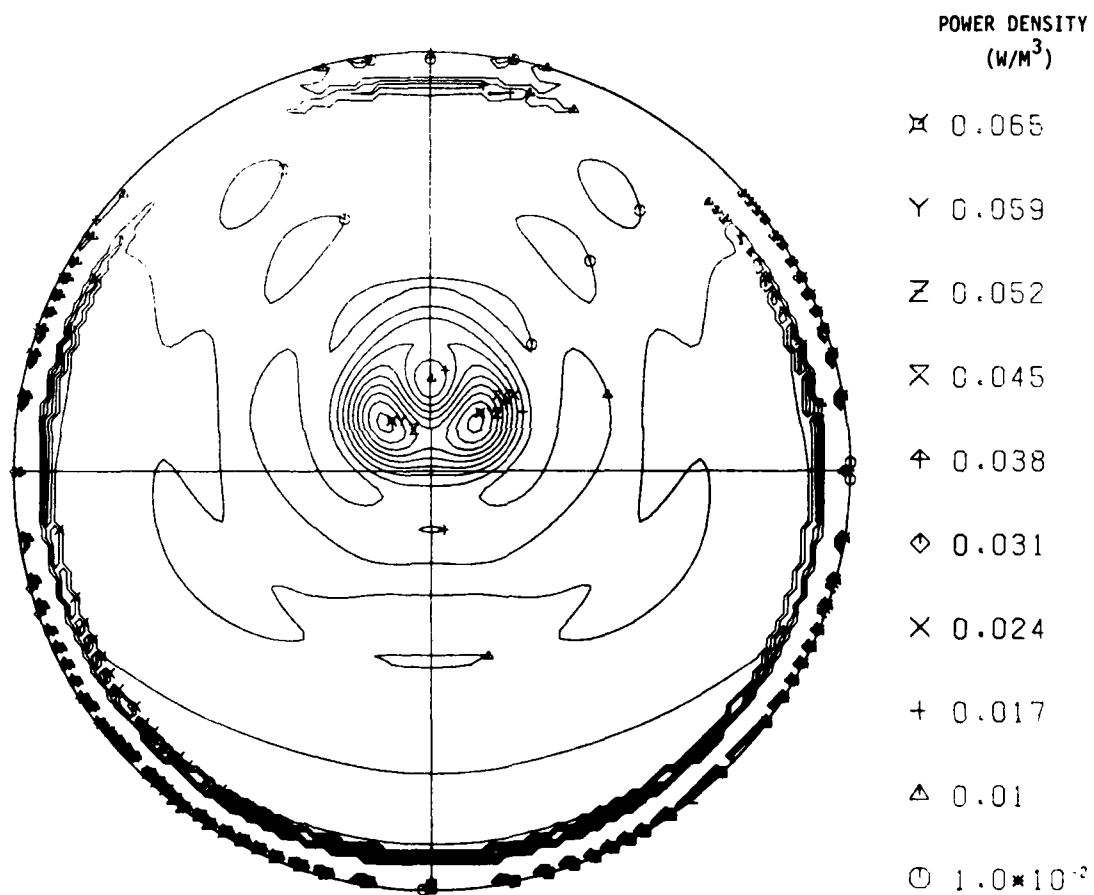


Figure 7. Distribution of power density in a 10-cm-radius sphere, a model of the human head, at 1 GHz. (E-plane)

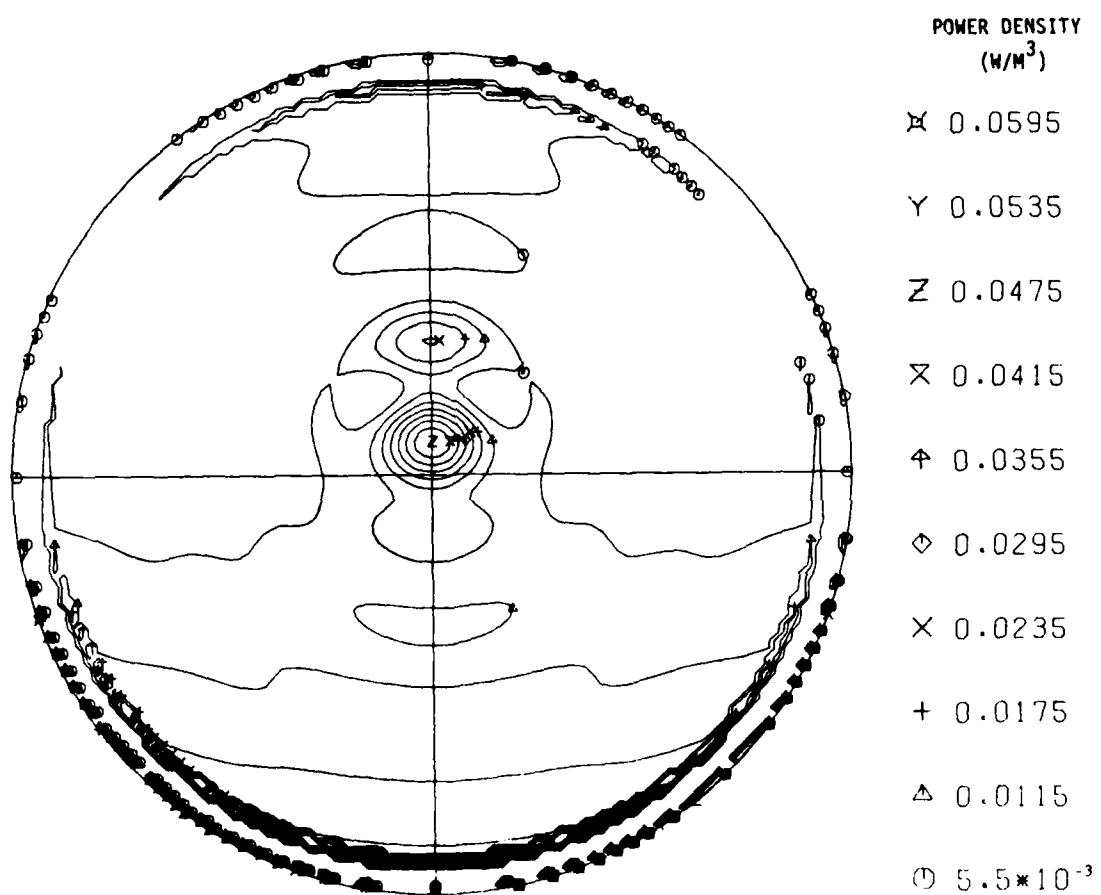


Figure 8. Distribution of power density in a 10-cm-radius sphere, a model of the human head, at 1 GHz. (H-plane)

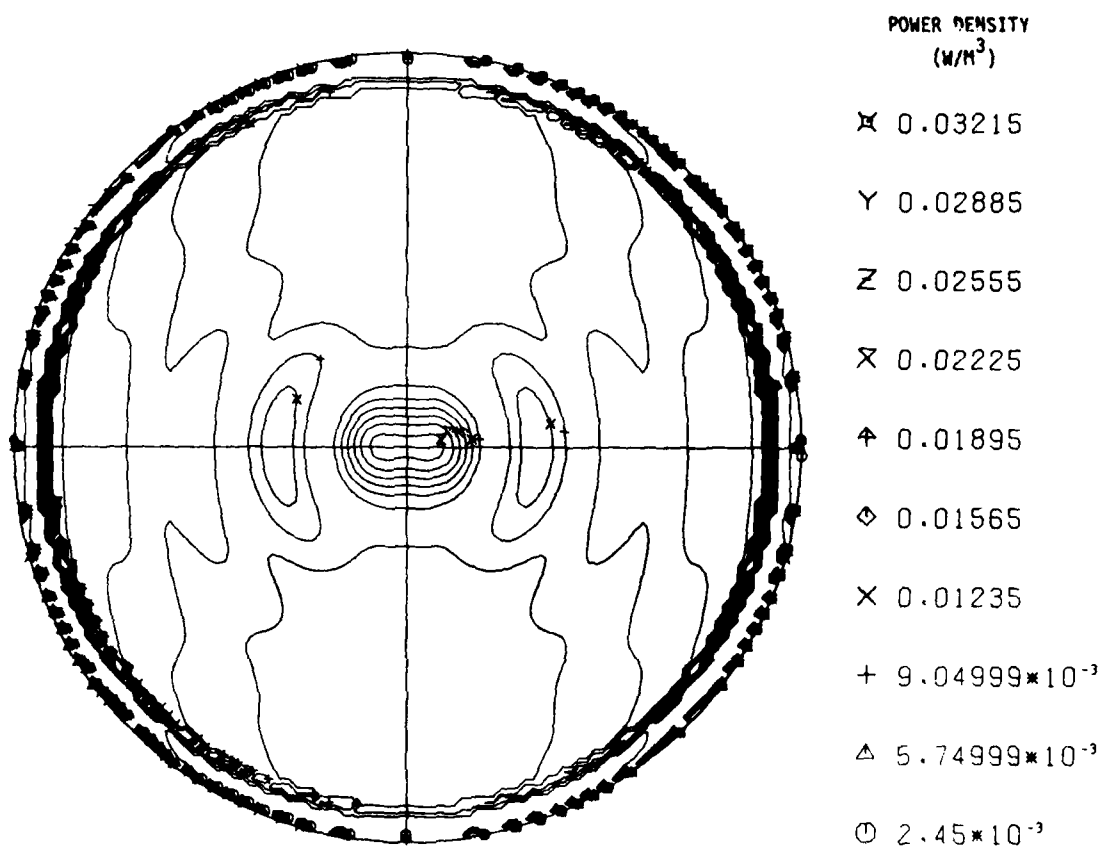


Figure 9. Distribution of power density in a 10-cm-radius sphere, a model of the human head, at 1 GHz. (X,Y-plane)

Nonuniform distribution of the absorbed energy inside of spheres gives rise to the internal appearance of "hot spots." Kritikos and Schwan (5) showed that "hot spots" appear in lossy, homogeneous, brain-like material spheres for the range of values of the radius and frequency: $0.1 < R < 8$ cm and $300 < f < 1200$ MHz. Shapiro et al. (13) and Weil (15) have shown the existence of hot spots inside multilayered spheres. Each has taken into account the importance of the frequency dependence of the sphere electrical properties. What precise conditions will precipitate the phenomenon are still not well known. We do know that the radius of the sphere and frequency, among others, do play a significant role. Occurrence takes place at the front surface or inside the head. It is a phenomenon that is importantly connected to small animals and infants.

The concentric spherical model represents one step forward in approximating reality as compared to the single, homogeneous sphere. Even so, the shortcomings of this model are to be found in (1) shape, (2) electrical properties, (3) thicknesses of tissue media, (4) assumption of tissue media being homogeneous and isotropic, and (5) inoperative conduction, convection, and radiation-heat-transfer mechanisms. The present computer program will be updated in the near future by an attempt to incorporate the mechanism of blood flow (along with other features). This may result in an appreciable reduction in the temperature rise now calculated.

The knowledge to be gleaned from this current research is directly related to the research effort of the Radiation Sciences Division at the USAF School of Aerospace Medicine. Briefly, here studies are being conducted to (1) determine the radiofrequency radiation-induced effects in biological specimens, (2) seek out possible hazards to personnel in a radiofrequency environment, (3) accurately measure and determine the distribution of energy in the whole biological body or just in a particular organ, (4) extrapolate response to radiation from the test animal

to man in a meaningful manner, and (5) contribute in the design of realistic safety standards with a solid basis.

The division of this paper entitled "Mathematical Description" consists of four sections. Since spherical harmonics (Stratton [14], pp. 399-423) are used to expand the incident, induced, and scattered fields, we include in the section "Mathematical Preliminaries" details of the exact evaluation of inner products entering into the computation of expansion coefficients. In the section "Expansion of Induced Fields in Terms of Vector Wave Functions," the expansions are used to solve Maxwell's equations (Stratton [14], p. 26), subject to the condition that the tangential components of electric field $\vec{E}$ and magnetic field $\vec{H}$ are continuous across the spheres delimiting different regions of the head model. The section "Determination of Total Absorbed Power" considers the integrals that appear in Poynting's theorem. Such integrals are evaluated in closed form, thereby yielding a formula for the total power absorbed. The section "Summary of Key Equations and Formulas" contains a detailed summary of the formulas implemented by the computer program, Concentric Spherical Model (CSM), for automatically calculating the radiofrequency electromagnetic energy deposited in a concentric spherical model of the human or animal head.

The succeeding division, entitled "Program Description," contains pertinent information about the computer program that is beneficial to the user. The appendixes consist of a sample problem with computer results and a source listing of the program CSM.

To benefit users of this report, program CSM is described in sufficient depth to permit job setups and implementation on any modern computer. The mathematical theory and formulas basic in accomplishing the computations are discussed in an extensive manner. Discussion of this users-oriented computer program covers such details as structure and sequence of control parameter and data cards, output formats, and subroutine and function subprograms. Sample printouts and plots (linear and contour) of computer results and a listing of the FORTRAN IV source program are included.

MATHEMATICAL DESCRIPTION

Mathematical Preliminaries

This part of the mathematical discussion introduces an inner product on doubly periodic vector valued functions, presents a study of some of the properties of Legendre polynomials used in evaluating inner products, introduces vector wave functions, and verifies some of their properties.

Definition 1. Let $S = \{(\theta, \phi) : 0 \leq \theta < \pi \text{ and } 0 \leq \phi < \pi\}$. Then let $H(S, \mathbb{C})$ denote the continuous complex valued functions A defined on S that satisfy the inequality

$$||A||^2 = \int_0^{2\pi} \left[\int_0^\pi |A(\theta, \phi)|^2 \sin \theta d\theta \right] d\phi < \infty. \quad (1)$$

For any functions, A and B in $H(S, \mathbb{C})$ define the inner product $\langle, \rangle$ by the rule

$$\langle A, B \rangle = \int_0^{2\pi} \left[\int_0^\pi A(\theta, \phi) \overline{B(\theta, \phi)} \sin \theta d\theta \right] d\phi. \quad (2)$$

Proposition 1. The space $H(S, \mathbb{C})$ with the inner product $\langle, \rangle$ is a pre-Hilbert space.

This follows immediately from the definition.

Now we need some properties of the associated Legendre functions.

Definition 2. For all nonnegative integers, m and n define

$$P_n^m(x) = \frac{(1-x^2)^{m/2}}{2^n n!} D^{n+m} (x^2 - 1)^n. \quad (3)$$

Proposition 2. If $m+n$ is even (odd), then $D^{n+m}(x^2-1)^n$ is a linear combination of even (odd) powers of x .

Proof. Observe that

$$(x^2-1)^n = \sum_{k=0}^n \binom{n}{k} x^{2k} (-1)^{n-k}. \quad (4)$$

Since an even (odd) number of derivatives of an even power of x is an even (odd) power of x , the proposition is true.

Corollary 1. If $n+m$ is even (odd), then $p_n^m(x)$ is an even (odd) function of x .

Proof of Corollary 1. Since $(1-x^2)^{m/2}$ is an even function of x , the corollary follows immediately from Proposition 2.

Proposition 3. For all nonnegative integers m and n

$$I = \int_0^\pi p_n^m(\cos\theta) \left(\frac{d}{d\theta}\right) (p_n^m(\cos\theta)) \sin\theta d\theta = 0. \quad (5)$$

Proof. Since

$$\frac{1}{2} \left(\frac{d}{d\theta}\right) (p_n^m(\cos\theta)^2) = p_n^m(\cos\theta) \left(\frac{d}{d\theta}\right) (p_n^m(\cos\theta)), \quad (6)$$

an integration by parts implies that

$$I = \frac{1}{2} [p_n^m(\cos\theta)^2 \sin\theta]_0^\pi - \int_0^\pi p_n^m(\cos\theta)^2 \cos\theta d\theta. \quad (7)$$

Substituting $x = \cos\theta$ and using the fact that

$$d\theta = - (1/\sqrt{1-x^2}) dx, \quad (8)$$

it follows that

$$I = - \frac{1}{2} \int_{-1}^1 P_n^m(x)^2 (x/\sqrt{1-x^2}) dx, \quad (9)$$

which in view of Corollary 1, implies that $I = 0$.

Proposition 4. For all nonnegative integers m and n , we have that

$$\int_0^\pi P_n^m(\cos\theta) P_r^m(\cos\theta) \sin\theta d\theta = \begin{cases} 0, r \neq n \\ \frac{2(n+m)!}{(2n+1)(n-m)!}, r=n. \end{cases} \quad (10)$$

Proof. The proof is carried out completely in Whittaker and Watson (16, pp. 324-325).

Proposition 5. For all nonnegative integers m , n , and r , we have

$$\begin{aligned} \int_0^\pi \frac{d}{d\theta}(P_n^m(\cos\theta)) \frac{d}{d\theta}(P_r^m(\cos\theta)) \sin\theta d\theta &= A_{(n,r)}^m \\ &= \delta_{(n,r)} \frac{2(n+m)!}{2n+1(n-m)!} n(n+1) \\ &+ \int_0^\pi \left(\frac{-m^2}{\sin^2\theta} \right) P_n^m(\cos\theta) P_r^m(\cos\theta) \sin\theta d\theta. \end{aligned} \quad (11)$$

Proof. Observe that

$$\begin{aligned} \frac{d}{dx} \left[\frac{(1-x^2)^{m/2}}{2^n n!} \frac{d^{n+m}}{dx^{n+m}} (x^2-1)^n \right] &= \frac{\frac{m}{2}(1-x^2)^{m/2-1}(-2x)}{2^n n!} \frac{d^{n+m}}{dx^{n+m}} (x^2-1)^n \\ &\quad + \frac{(1-x^2)^{m/2}}{2^n n!} \frac{d^{n+m+1}}{dx^{n+m+1}} (x^2-1)^n. \end{aligned} \quad (12)$$

Let $(d/d\theta)F(\cos\theta) = F'(\cos\theta)(-\sin\theta)$. Thus, $x = \cos\theta$ implies that

$$\frac{dx}{d\theta} \frac{d}{dx} = \frac{d}{d\theta}. \quad (13)$$

Hence

$$\begin{aligned} A_{(n,r)}^m &= \int_0^\pi (-\sin\theta) P_n^{m'}(x) (-\sin\theta) P_r^{m'}(x) \sin\theta d\theta \\ &= \int_1^{-1} (1-x^2) P_n^{m'}(x) P_r^{m'}(x) (-dx) \\ &= \int_{-1}^1 (1-x^2) P_n^{m'}(x) P_r^{m'}(x) dx. \end{aligned} \quad (14)$$

Integrating by parts in the above integral, we find that

$$\begin{aligned}
 A_{(n,r)}^m &= - \int_{-1}^1 \frac{d}{dx}((1-x^2) \frac{d}{dx} P_n^m(x)) P_r^m(x) dx \\
 &= \int_{-1}^1 [n(n+1) - \frac{m^2}{1-x^2}] P_n^m(x) P_r^m(x) dx \\
 &= \int_{-1}^1 [\frac{-m^2}{1-x^2} + r(r+1)] P_n^m(x) P_r^m(x) dx. \tag{15}
 \end{aligned}$$

Since $A_{(n,r)}^m = A_{(r,n)}^m$ the above relation shows that if $r \neq n$,

$$\int_{-1}^1 P_n^m(x) P_r^m(x) dx = 0. \tag{16}$$

Substituting back $x = \cos\theta$, we find that

$$\begin{aligned}
 A_{(n,r)}^m &= \delta_{(n,r)} \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!} n(n+1) \\
 &+ \int_0^\pi \left(\frac{-m^2}{\sin^2\theta} \right) P_n^m(\cos\theta) P_r^m(\cos\theta) \sin\theta d\theta. \tag{17}
 \end{aligned}$$

Proposition 6. For all nonnegative integers m, n, and r, we have

$$\int_0^\pi \left[\frac{d}{d\theta}(P_n^m(\cos\theta)) \frac{d}{d\theta}(P_r^m(\cos\theta)) + (m^2/\sin^2\theta) P_n^m(\cos\theta) P_r^m(\cos\theta) \right] \sin\theta d\theta$$

$$= \delta_{(n,r)} \left(\frac{2}{2n+1} \right) \frac{(n+m)!}{(n-m)!} n(n+1) . \quad (18)$$

Proof of Proposition 6. From Proposition 5 we deduce equation 18.

Definition 3. Let $\vec{i}$, $\vec{j}$, and $\vec{k}$ denote the unit vectors in the Cartesian coordinate system. Define

$$\vec{e}_r = \sin\theta\cos\phi\vec{i} + \sin\theta\sin\phi\vec{j} + \cos\theta\vec{k}, \quad (19)$$

$$\vec{e}_\theta = \cos\theta\cos\phi\vec{i} + \cos\theta\sin\phi\vec{j} - \sin\theta\vec{k}, \quad (20)$$

and

$$\vec{e}_\phi = -\sin\phi\vec{i} + \cos\phi\vec{j} . \quad (21)$$

Definition 4. If S is a surface in R^3 bounded by a simple closed-curve C , and $\vec{A}$ is a C^1 vector field defined in a neighborhood of S , then $\text{curl}(\vec{A})$ is a vector field such that

$$\int_S \text{curl}(\vec{A}) \cdot \vec{N} d\sigma = \oint_C \vec{A} \cdot \vec{T} ds, \quad (22)$$

where $\vec{N}$ and $\vec{T}$ are, respectively, the unit normals and the unit tangents of S and C .

Proposition 7. If $\vec{A}$ is a vector valued function of $r, \theta,$ and ϕ , then

$$\begin{aligned} \text{curl}(\vec{A}) = & \frac{1}{r \sin \theta} [(\partial/\partial \theta)(\sin \theta A_\phi) - (\partial/\partial \phi)A_\theta] \vec{e}_r \\ & + \frac{1}{r} [(1/\sin \theta)(\partial/\partial \phi)A_r - (\partial/\partial r)(rA_\phi)] \vec{e}_\theta \\ & + \frac{1}{r} [(\partial/\partial r)(rA_\theta) - (\partial/\partial \theta)A_r] \vec{e}_\phi. \end{aligned} \quad (23)$$

Proof. This follows from Stokes' theorem and the fact that in Cartesian coordinates x, y, z , where $\vec{F} = F_x \vec{i} + F_y \vec{j} + F_z \vec{k}$, the curl of vector $\vec{F}$ is defined by

$$\begin{aligned} \text{curl}(\vec{F}) = & \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \vec{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \vec{j} \\ & + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \vec{k}. \end{aligned} \quad (24)$$

Proposition 8. In spherical coordinates if ψ is a function of r , θ , and ϕ , then

$$\begin{aligned}\Delta\psi &= (1/r^2)[(\partial/\partial r)(r^2(\partial/\partial r)\psi)] \\ &+ (1/(r^2\sin\theta))[(\partial/\partial\theta)(\sin\theta(\partial/\partial\theta)\psi)] \\ &+ (1/(r^2\sin^2\theta))(\partial^2/\partial\phi^2)\psi.\end{aligned}\tag{25}$$

Proof. This follows from the fact that in Cartesian coordinates

$$\Delta\psi = (\partial^2/\partial x^2)\psi + (\partial^2/\partial y^2)\psi + (\partial^2/\partial z^2)\psi\tag{26}$$

and the coordinate transforms

$$\begin{aligned}x &= r\sin(\theta)\cos(\phi), \\ y &= r\sin(\theta)\sin(\phi),\end{aligned}\tag{27}$$

and

$$z = r\cos(\theta).$$

Proposition 9. For any C^1 function ψ of r , θ , and ϕ ,

$$\vec{A} = \psi \vec{r} e_r \quad (28)$$

implies that

$$\begin{aligned} \vec{M}_\psi &= \text{curl}(\vec{A}) \\ &= (1/\sin\theta)((\partial/\partial\phi)\psi)\vec{e}_\theta - ((\partial/\partial\theta)\psi)\vec{e}_\phi. \end{aligned} \quad (29)$$

Proof. This follows by direct substitution of equation 28 into equation 23.

Proposition 10. Suppose ψ is a C^2 function satisfying

$$\Delta\psi + k^2\psi = 0, \quad (30)$$

where k is a complex number, then

$$\vec{M}_\psi = \text{curl}(\psi \vec{r} e_r) \quad (31)$$

and

$$\vec{N}_\psi = (1/k)\text{curl}(\vec{M}_\psi) \quad (32)$$

imply that

$$\begin{aligned}\vec{N}_\psi = & [(1/(kr))(\partial/\partial r)(r^2(\partial/\partial r)\psi) + kr\psi]\vec{e}_r \\ & + (1/(kr))(\partial^2/\partial\theta^2)(r\psi)\vec{e}_\theta \\ & + (1/(kr\sin\theta))(\partial^2/\partial r\partial\phi)(r\psi)\vec{e}_\phi .\end{aligned}\quad (33)$$

In the next section we work out some consequences of this proposition when the function ψ is the product of a spherical Bessel function, a Legendre polynomial, and a sine or a cosine.

Expansion of Induced Fields in Terms of Vector Wave Functions

In determining the response to a plane wave of a union of regions of dielectric material delimited by spheres, we use the vector wave functions (of k_p = complex propagation constant for the p -th layer of the dielectric material):

$$\begin{aligned}\vec{M}_{(1,n)}^{(e,j)} = & -\frac{1}{\sin\theta} z_n^j(k_p r) P_n^1(\cos\theta) \sin\phi \vec{e}_\theta \\ & - z_n^j(k_p r) (d/d\theta) (P_n^1(\cos\theta)) \cos\phi \vec{e}_\phi ,\end{aligned}\quad (34)$$

$$\begin{aligned}
\vec{M}_{(1,n)}^{(0,j)} &= \frac{1}{\sin\theta} z_n^j(k_p r) P_n^1(\cos\theta) \cos\phi \vec{e}_\theta \\
&- z_n^j(k_p r) (d/d\theta) (P_n^1(\cos\theta)) \sin\phi \vec{e}_\phi, \quad (35)
\end{aligned}$$

$$\begin{aligned}
\vec{N}_{(1,n)}^{(e,j)} &= \frac{n(n+1)}{k_p r} z_n^j(k_p r) P_n^1(\cos\theta) \cos\phi \vec{e}_r \\
&+ \frac{1}{k_p r} (\partial/\partial r) (r z_n^j(k_p r)) (d/d\theta) (P_n^1(\cos\theta)) \cos\phi \vec{e}_\theta \\
&- \frac{1}{k_p r \sin\theta} (\partial/\partial r) (r z_n^j(k_p r)) P_n^1(\cos\theta) \sin\phi \vec{e}_\phi, \quad (36)
\end{aligned}$$

and

$$\begin{aligned}
\vec{N}_{(1,n)}^{(o,j)} &= \frac{n(n+1)}{k_p r} z_n^j(k_p r) P_n^1(\cos\theta) \sin\phi \vec{e}_r \\
&+ \frac{1}{k_p r} (\partial/\partial r) (r z_n^j(k_p r)) (d/d\theta) (P_n^1(\cos\theta)) \sin\phi \vec{e}_\theta \\
&+ \frac{1}{k_p r \sin\theta} (\partial/\partial r) (r z_n^j(k_p r)) P_n^1(\cos\theta) \cos\phi \vec{e}_\phi. \quad (37)
\end{aligned}$$

Here

$$z_n^1(\rho) = j_n(\rho) = \sqrt{\pi/2\rho} J_{n+\frac{1}{2}}(\rho), \quad (38)$$

$$z_n^3(\rho) = h_n^1(\rho) = \sqrt{\pi/2\rho} H_{n+\frac{1}{2}}^1(\rho), \quad (39)$$

$$H_{n+\frac{1}{2}}^1(\rho) = J_{n+\frac{1}{2}}(\rho) + iY_{n+\frac{1}{2}}(\rho), \quad (40)$$

and $J_{n+\frac{1}{2}}(\rho)$ and $Y_{n+\frac{1}{2}}(\rho)$ are the Bessel and Neuman functions of order half-an-odd integer, respectively.

Proposition 11. For all nonnegative integers m and n and all integers j and j' in $\{1,2,3\}$, we have

$$\langle \vec{M}_{(1,n)}^{(e,j)}, \vec{M}_{(1,m)}^{(o,j')} \rangle = 0, \quad (41)$$

$$\begin{aligned} \langle \vec{M}_{(1,n)}^{(e,j)}, \vec{M}_{(1,m)}^{(e,j')} \rangle &= A \delta_{(n,m)} \frac{2}{2n+1} \frac{(n+1)!}{(n-1)!} \\ &= \langle \vec{M}_{(1,n)}^{(o,j)}, \vec{M}_{(1,n)}^{(o,j')} \rangle, \end{aligned} \quad (42)$$

where

$$A = \pi r^2 z_n^j(kr) z_n^{j'}(kr), \quad (43)$$

$$\langle \vec{N}_{(1,n)}^{(o,j)}, \vec{N}_{(1,m)}^{(e,j')} \rangle = 0, \quad (44)$$

$$\begin{aligned} \langle \vec{N}_{(1,n)}^{(o,j)}, \vec{N}_{(1,m)}^{(o,j')} \rangle &= B \delta_{(n,m)} \frac{2}{2n+1} \frac{(n+1)!}{(n-1)!} n(n+1) + C_n \\ &= \langle \vec{N}_{(1,n)}^{(e,j)}, \vec{N}_{(1,m)}^{(e,j')} \rangle, \end{aligned} \quad (45)$$

$$B = \pi \left(\frac{1}{k_p r} \right)^2 (\partial/\partial r)(r z_n^j(k_p r)) (\partial/\partial r)(r z_n^j(k_p r)) \quad (46)$$

and

$$C_n = \frac{2\pi}{2n+1} \frac{(n+1)!}{(n-1)!} \frac{n^2(n+1)^2}{k_p^2 r^2} z_n^j(k_p r) z_n^j(k_p r) , \quad (47)$$

$$\langle \vec{M}_{(1,n)}^{(o,j)} , \vec{N}_{(1,m)}^{(o,j')} \rangle = 0 , \quad (48)$$

$$\langle \vec{M}_{(1,n)}^{(e,j)} , \vec{N}_{(1,m)}^{(e,j')} \rangle = 0 , \quad (49)$$

$$\langle \vec{M}_{(1,n)}^{(o,j)} , \vec{N}_{(1,m)}^{(e,j')} \rangle = 0 . \quad (50)$$

Proof. This follows from the definitions and the facts that

$$\begin{aligned} & \int_0^\pi [(DP_n^1(\cos\theta))(DP_m^1(\cos\theta)) + \frac{1}{\sin^2\theta} P_n^1(\cos\theta)P_m^1(\cos\theta)] \sin\theta d\theta \\ & = \delta_{(n,m)} \frac{2}{2n+1} \frac{(n+1)!}{(n-1)!} n(n+1) \end{aligned} \quad (51)$$

and

$$\int_0^\pi [P_m^1(\cos\theta)D(P_n^1(\cos\theta)) + P_n^1(\cos\theta)D(P_m^1(\cos\theta))] d\theta = 0 , \quad (52)$$

where $D = d/d\theta$.

Now we want to develop formulas relating the fields. Let us write the fields for the p-th region as

$$\begin{aligned} \vec{E}_p = E_0 \sum_{\ell=1}^{\infty} i^{\ell} \frac{2\ell+1}{\ell(\ell+1)} [a_{(\ell,p)} \vec{M}_{(1,\ell)}^{(o,1)} - ib_{(\ell,p)} \vec{N}_{(1,\ell)}^{(e,1)} \\ + \alpha_{(\ell,p)} \vec{M}_{(1,\ell)}^{(o,3)} - i\beta_{(\ell,p)} \vec{N}_{(1,\ell)}^{(e,3)}] \end{aligned} \quad (53)$$

and

$$\begin{aligned} \vec{H}_p = -\frac{k_p}{\mu_0 \omega} E_0 \sum_{\ell=1}^{\infty} i^{\ell} \frac{2\ell+1}{\ell(\ell+1)} [b_{(\ell,p)} \vec{M}_{(1,\ell)}^{(e,1)} + ia_{(\ell,p)} \vec{N}_{(1,\ell)}^{(o,1)} \\ + \beta_{(\ell,p)} \vec{M}_{(1,\ell)}^{(e,3)} + i\alpha_{(\ell,p)} \vec{N}_{(1,\ell)}^{(o,3)}] \end{aligned} \quad (54)$$

in terms of the spherical vector functions $\vec{M}_{(1,\ell)}^{(i,j)}$ and $\vec{N}_{(1,\ell)}^{(i,j)}$ [cf. Stratton (14, p. 564) for function definitions] and the complex propagation constant k_p . The tangential components of the fields are

$$\begin{aligned} (\vec{E}_p)_{\theta} = E_0 \sum_{\ell=1}^{\infty} i^{\ell} \frac{2\ell+1}{\ell(\ell+1)} [a_{(\ell,p)} \frac{P_{\ell}^1(\cos\theta)}{\sin\theta} (\cos\phi) j_{\ell}(k_p r_p) \\ - ib_{(\ell,p)} (1/k_p r_p) ((\partial/\partial r)(r j_{\ell}(k_p r)))_{r=r_p} ((d/d\theta)(P_{\ell}^1(\cos\theta))) \cos\phi \\ + \alpha_{(\ell,p)} \frac{P_{\ell}^1(\cos\theta)}{\sin\theta} (\cos\phi) h_{\ell}^1(k_p r_p) \\ - i\beta_{(\ell,p)} (1/k_p r_p) ((\partial/\partial r)(r h_{\ell}^1(k_p r)))_{r=r_p} ((d/d\theta)P_{\ell}^1(\cos\theta)) \cos\phi], \end{aligned} \quad (55)$$

$$\begin{aligned}
(\vec{E}_p)_\phi = E_0 \sum_{\ell=1}^{\infty} i^\ell \frac{2\ell+1}{\ell(\ell+1)} [a_{(\ell,p)}(-j_\ell(k_p r_p))((d/d\theta)(P_\ell^1(\cos\theta)))\sin\phi \\
- ib_{(\ell,p)}(-1/k_p r_p)((\partial/\partial r)(r_p j_\ell(k_p r)))\left(\frac{P_\ell^1(\cos\theta)}{\sin\theta}\right)\sin\phi \\
+ \alpha_{(\ell,p)}(-h_\ell^1(k_p r_p))((d/d\theta)(P_\ell^1(\cos\theta)))\sin\phi \\
- i\beta_{(\ell,p)}(-1/k_p r_p)((\partial/\partial r)(r h_\ell^1(k_p r)))\left(\frac{P_\ell^1(\cos\theta)}{\sin\theta}\right)\sin\phi] , \quad (56)
\end{aligned}$$

$$\begin{aligned}
(\vec{H}_p)_\theta = -\frac{k_p}{\mu_0 \omega} E_0 \sum_{\ell=1}^{\infty} i^\ell \frac{2\ell+1}{\ell(\ell+1)} [b_{(\ell,p)}\left(-\frac{P_\ell^1(\cos\theta)}{\sin\theta}\right)(\sin\phi)j_\ell(k_p r_p) \\
+ ia_{(\ell,p)}(1/k_p r_p)((\partial/\partial r)(r j_\ell(k_p r)))\left(\frac{P_\ell^1(\cos\theta)}{\sin\theta}\right)\sin\phi \\
+ \beta_{(\ell,p)}\left(-\frac{P_\ell^1(\cos\theta)}{\sin\theta}\right)(\sin\phi)h_\ell^1(k_p r_p) \\
+ i\alpha_{(\ell,p)}(1/k_p r_p)((\partial/\partial r)(r h_\ell^1(k_p r)))\left(\frac{P_\ell^1(\cos\theta)}{\sin\theta}\right)\sin\phi] \quad (57)
\end{aligned}$$

$$\begin{aligned}
(\vec{H}_p)_\phi = -\frac{k_p}{\mu_0 \omega} E_0 \sum_{\ell=1}^{\infty} i^\ell \frac{2\ell+1}{\ell(\ell+1)} [b_{(\ell,p)}(-j_\ell(k_p r_p))((d/d\theta)(P_\ell^1(\cos\theta)))\cos\phi \\
+ ia_{(\ell,p)}(1/k_p r_p)((\partial/\partial r)(r j_\ell(k_p r)))\left(\frac{P_\ell^1(\cos\theta)}{\sin\theta}\right)\cos\phi \\
+ \beta_{(\ell,p)}(-h_\ell^1(k_p r_p))((d/d\theta)(P_\ell^1(\cos\theta)))\cos\phi \\
+ i\alpha_{(\ell,p)}(-1/k_p r_p)((\partial/\partial r)(r h_\ell^1(k_p r)))\left(\frac{P_\ell^1(\cos\theta)}{\sin\theta}\right)\cos\phi] \quad (58)
\end{aligned}$$

The boundary conditions implying continuity of the tangential component of the electric vector in the θ -direction may be described by the rule

$$\begin{aligned}
& \pi E_0 \sum_{\ell=1}^{\infty} \left[A_{(\ell,p)}^+ \frac{P_{\ell}^1(\cos\theta)}{\sin\theta} - iB_{(\ell,p)}^+ ((d/d\theta)P_{\ell}^1(\cos\theta)) \right. \\
& \quad \left. + A_{(\ell,p)}^+ \frac{P_{\ell}^1(\cos\theta)}{\sin\theta} - iB_{(\ell,p)}^+ ((d/d\theta)P_{\ell}^1(\cos\theta)) \right] \\
& = \pi E_0 \sum_{\ell=1}^{\infty} \left[A_{(\ell,p+1)}^- \frac{P_{\ell}^1(\cos\theta)}{\sin\theta} - iB_{(\ell,p+1)}^- ((d/d\theta)P_{\ell}^1(\cos\theta)) \right. \\
& \quad \left. + A_{(\ell,p+1)}^- \frac{P_{\ell}^1(\cos\theta)}{\sin\theta} - iB_{(\ell,p+1)}^- ((d/d\theta)P_{\ell}^1(\cos\theta)) \right]. \quad (59)
\end{aligned}$$

Here

$$C(\ell) = i^{\ell} \frac{2\ell+1}{\ell(\ell+1)}, \quad (60)$$

$$A_{(\ell,p)}^+ = C(\ell) j_{\ell}(k_p r_p) a_{(\ell,p)}, \quad (61)$$

$$A_{(\ell,p+1)}^- = C(\ell) j_{\ell}(k_{p+1} r_p) a_{(\ell,p+1)}, \quad (62)$$

$$A_{(\ell,p)}^+ = C(\ell) h_{\ell}^1(k_p r_p) \alpha_{(\ell,p+1)}, \quad (63)$$

$$A_{(\ell,p+1)}^- = C(\ell) h_{\ell}^1(k_{p+1} r_p) \alpha_{(\ell,p+1)}, \quad (64)$$

$$B_{(\ell,p)}^+ = C(\ell)(1/k_p r_p)(\partial/\partial r)(rh_\ell^1(k_p r)) \Big|_{r=r_p} B_{(\ell,p)} , \quad (65)$$

$$B_{(\ell,p+1)}^- = C(\ell)(1/k_{p+1} r_p)(\partial/\partial r)(rh_\ell^1(k_{p+1} r)) \Big|_{r=r_p} B_{(\ell,p+1)} , \quad (66)$$

$$B_{(\ell,p)}^+ = C(\ell)(1/k_p r_p)(\partial/\partial r)(rj_\ell(k_p r)) \Big|_{r=r_p} b_{(\ell,p)} , \quad (67)$$

$$B_{(\ell,p+1)}^- = C(\ell)(1/k_{p+1} r_p)(\partial/\partial r)(rj_\ell(k_{p+1} r)) \Big|_{r=r_p} b_{(\ell,p+1)} . \quad (68)$$

Letting

$$S_{(\ell,p)} = A_{(\ell,p+1)}^- + A_{(\ell,p+1)}^- - A_{(\ell,p)}^+ - A_{(\ell,p)}^+ \quad (69)$$

and

$$T_{(\ell,p)} = B_{(\ell,p+1)}^- + B_{(\ell,p+1)}^- - B_{(\ell,p)}^+ - B_{(\ell,p)}^+ , \quad (70)$$

we deduce that

$$\sum_{\ell=1}^{\infty} [S_{(\ell,p)} \frac{p_\ell^1(\cos\theta)}{\sin\theta} - iT_{(\ell,p)}(\partial/\partial\theta)p_\ell^1(\cos\theta)] = 0. \quad (71)$$

Observe that $(E_\phi)_p = (E_\phi)_{p+1}$ implies that

$$\begin{aligned}
& \pi E_0 \sum_{\ell=1}^{\infty} [(-A_{(\ell,p)}^+)(d/d\theta)P_\ell^1(\cos\theta) - i(-B_{(\ell,p)}^+) \frac{P_\ell^1(\cos\theta)}{\sin\theta} \\
& + (-A_{(\ell,p)}^+)(d/d\theta)P_\ell^1(\cos\theta) - i(-B_{(\ell,p)}^+) \frac{P_\ell^1(\cos\theta)}{\sin\theta}] \\
& = \pi E_0 \sum_{\ell=1}^{\infty} [(-A_{(\ell,p+1)}^-)(d/d\theta)P_\ell^1(\cos\theta) - i(-B_{(\ell,p+1)}^-) \frac{P_\ell^1(\cos\theta)}{\sin\theta} \\
& + (-A_{(\ell,p+1)}^-)(d/d\theta)P_\ell^1(\cos\theta) - i(-B_{(\ell,p+1)}^-) \frac{P_\ell^1(\cos\theta)}{\sin\theta}] . \tag{72}
\end{aligned}$$

Thus,

$$\sum_{\ell=1}^{\infty} \{ S_{(\ell,p)} [(d/d\theta)P_\ell^1(\cos\theta)] - iT_{(\ell,p)} [\frac{P_\ell^1(\cos\theta)}{\sin\theta}] \} = 0 \tag{73}$$

and we conclude that $S_{(\ell,p)} = 0$ and $T_{(\ell,p)} = 0$.

Upon introducing the constants

$$\tilde{A}_{(\ell,p)}^+ = a_{(\ell,p)} (1/k_p r_p) (\partial/\partial r) (r j_\ell(k_p r)) \Big|_{r=r_p} C(\ell), \tag{74}$$

$$\tilde{B}_{(\ell,p)}^+ = b_{(\ell,p)} j_\ell(k_p r_p) C(\ell), \tag{75}$$

$$\chi_{(\ell,p)}^+ = \alpha_{(\ell,p)} (1/(k_p r_p)) (\partial/\partial r) (r h_{\ell}^1(k_p r)) \Big|_{r=r_p} C(\ell), \quad (76)$$

$$\tilde{B}_{(\ell,p)}^+ = \beta_{(\ell,p)} h_{\ell}^1(k_p r_p) C(\ell), \quad (77)$$

$$\tilde{A}_{(\ell,p+1)}^- = a_{(\ell,p+1)} (1/(k_{p+1} r_p)) (\partial/\partial r) (r j_{\ell}(k_{p+1} r)) \Big|_{r=r_p} C(\ell), \quad (78)$$

$$\tilde{B}_{(\ell,p+1)}^- = b_{(\ell,p+1)} j_{\ell}(k_{p+1} r_p) C(\ell), \quad (79)$$

$$\chi_{(\ell,p+1)}^- = \alpha_{(\ell,p+1)} (1/(k_{p+1} r_p)) (\partial/\partial r) (r h_{\ell}^1(k_{p+1} r)) \Big|_{r=r_p} C(\ell), \quad (80)$$

$$\tilde{B}_{(\ell,p+1)}^- = \beta_{(\ell,p+1)} h_{\ell}^1(k_{p+1} r_p) C(\ell), \quad (81)$$

and setting $(H_p)_{\theta} = (H_{p+1})_{\theta}$, we arrive at the equality

$$\begin{aligned} & - \left(\frac{k_p}{u_0 \omega} \right) \pi E_0 \sum_{\ell=1}^{\infty} \left[\tilde{B}_{(\ell,p)}^+ \left(- \frac{P_{\ell}^1(\cos \theta)}{\sin \theta} \right) + i \tilde{A}_{(\ell,p)}^+ \left((d/d\theta) P_{\ell}^1(\cos \theta) \right) \right. \\ & \left. + \tilde{B}_{(\ell,p)}^+ \left(- \frac{P_{\ell}^1(\cos \theta)}{\sin \theta} \right) + i \tilde{A}_{(\ell,p)}^+ \left((d/d\theta) P_{\ell}^1(\cos \theta) \right) \right] \\ & = - \left(\frac{k_{p+1}}{u_0 \omega} \right) \pi E_0 \sum_{\ell=1}^{\infty} \left[\tilde{B}_{(\ell,p+1)}^- \left(- \frac{P_{\ell}^1(\cos \theta)}{\sin \theta} \right) + i \tilde{A}_{(\ell,p+1)}^- \left((d/d\theta) P_{\ell}^1(\cos \theta) \right) \right. \\ & \left. + \tilde{B}_{(\ell,p+1)}^- \left(- \frac{P_{\ell}^1(\cos \theta)}{\sin \theta} \right) + i \tilde{A}_{(\ell,p+1)}^- \left((d/d\theta) P_{\ell}^1(\cos \theta) \right) \right]. \end{aligned} \quad (82)$$

Now set

$$\tilde{S}_{(\ell,p)} = \tilde{B}_{(\ell,p+1)}^{+} k_{p+1} + \tilde{B}_{(\ell,p+1)}^{-} k_{p+1} - \tilde{B}_{(\ell,p)}^{+} k_p - \tilde{B}_{(\ell,p)}^{-} k_p \quad (83)$$

and

$$\tilde{T}_{(\ell,p)} = \tilde{A}_{(\ell,p+1)}^{+} k_{p+1} + \tilde{A}_{(\ell,p+1)}^{-} k_{p+1} - \tilde{A}_{(\ell,p)}^{+} k_p - \tilde{A}_{(\ell,p)}^{-} k_p \quad (84)$$

Equations 82-84 yield

$$\sum_{\ell=1}^{\infty} [\tilde{S}_{(\ell,p)} (-\frac{P_{\ell}^1(\cos\theta)}{\sin\theta}) + i\tilde{T}_{(\ell,p)} (d/d\theta) P_{\ell}^1(\cos\theta)] = 0 \quad (85)$$

The boundary condition

$$(H_p)_{\phi} = (H_{p+1})_{\phi} \quad (86)$$

is now utilized.

We observe that

$$\begin{aligned} & -(\frac{k_p}{\mu\omega})\pi E_0 \sum_{\ell=1}^{\infty} [-\tilde{B}_{(\ell,p)}^{+} ((d/d\theta) P_{\ell}^1(\cos\theta)) + i\tilde{A}_{(\ell,p)}^{+} (\frac{P_{\ell}^1(\cos\theta)}{\sin\theta}) \\ & - \tilde{B}_{(\ell,p)}^{-} ((d/d\theta) P_{\ell}^1(\cos\theta)) + i\tilde{A}_{(\ell,p)}^{-} (\frac{P_{\ell}^1(\cos\theta)}{\sin\theta})] = (H_p)_{\phi}, \end{aligned}$$

and $(H_{p+1})_\phi =$

$$\begin{aligned}
 & - \frac{k_{p+1}}{\mu\omega} \pi E_0 \sum_{\ell=1}^{\infty} [-\tilde{B}_{(\ell,p+1)}((d/d\theta)P_\ell^1(\cos\theta)) + i\tilde{A}_{(\ell,p+1)}(\frac{P_\ell^1(\cos\theta)}{\sin\theta}) \\
 & - \tilde{B}_{(\ell,p+1)}((d/d\theta)P_\ell^1(\cos\theta)) + i\tilde{A}_{(\ell,p+1)}(\frac{P_\ell^1(\cos\theta)}{\sin\theta})] . \quad (87)
 \end{aligned}$$

Thus,

$$\begin{aligned}
 & \sum_{\ell=1}^{\infty} [-\tilde{B}_{(\ell,p+1)}^{k_{p+1}} - \tilde{B}_{(\ell,p+1)}^{k_{p+1}} + \tilde{B}_{(\ell,p)}^{k_p} + \tilde{B}_{(\ell,p)}^{k_p}] (d/d\theta)P_\ell^1(\cos\theta) \\
 & + i \sum_{\ell=1}^{\infty} [\tilde{A}_{(\ell,p+1)}^{k_{p+1}} + \tilde{A}_{(\ell,p+1)}^{k_{p+1}} - \tilde{A}_{(\ell,p)}^{k_p} - \tilde{A}_{(\ell,p)}^{k_p}] (\frac{P_\ell^1(\cos\theta)}{\sin\theta}) \\
 & = 0 . \quad (88)
 \end{aligned}$$

This implies that

$$\sum_{\ell=1}^{\infty} [\tilde{S}_{(\ell,p)}((d/d\theta)P_\ell^1(\cos\theta)) - i\tilde{T}_{(\ell,p)}(\frac{P_\ell^1(\cos\theta)}{\sin\theta})] = 0 . \quad (89)$$

Now equations 85 and 89 are used to determine the values of $\tilde{S}_{(\ell,p)}$ and $\tilde{T}_{(\ell,p)}$. Multiplying both sides of equation 85 by $P_\ell^1(\cos\theta)\sin\theta$ and both sides of equation 89 by $[(d/d\theta)P_\ell^1(\cos\theta)]\sin\theta$ and integrating from 0 to π , results in equation

$$\begin{aligned}
& \sum_{\ell=1}^{\infty} \tilde{S}_{(\ell,p)} \int_0^{\pi} \left[\frac{P_{\ell}^1(\cos\theta) P_{\ell}^1(\cos\theta)}{\sin^2\theta} + ((d/d\theta)P_{\ell}^1(\cos\theta))((d/d\theta)P_{\ell}^1(\cos\theta)) \right] \sin\theta d\theta \\
& - i \sum_{\ell=1}^{\infty} \tilde{T}_{(\ell,p)} \int_0^{\pi} \left[P_{\ell}^1(\cos\theta)((d/d\theta)P_{\ell}^1(\cos\theta)) \right. \\
& \quad \left. + ((d/d\theta)P_{\ell}^1(\cos\theta))P_{\ell}^1(\cos\theta) \right] d\theta = 0
\end{aligned} \tag{90}$$

which implies, in view of equations 51 and 52, that $\tilde{S}_{(\ell,p)} = 0$. By symmetry $\tilde{T}_{(\ell,p)} = 0$. We conclude that

$$\begin{aligned}
& B_{(\ell,p+1)} h_{\ell}^1(k_{p+1} r_p) k_{p+1} + b_{(\ell,p+1)} j_{\ell}(k_{p+1} r_p) k_{p+1} \\
& = B_{(\ell,p)} h_{\ell}^1(k_p r_p) k_p + b_{(\ell,p)} j_{\ell}(k_p r_p) k_p .
\end{aligned} \tag{91}$$

Now the associated relation derived from equating the tangential components of the $\vec{E}$ vector is, from equations 71 and 73, the following:

$$\begin{aligned}
& \frac{1}{k_{p+1} r_p} \left[(\partial/\partial r)(r j_{\ell}(k_{p+1} r)) \right]_{r=r_p} b_{(\ell,p+1)} \\
& + \left(\frac{1}{k_{p+1} r_p} \right) \left[(\partial/\partial r)(r h_{\ell}^1(k_{p+1} r)) \right]_{r=r_p} \beta_{(\ell,p+1)}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{k_p r_p} [(\partial/\partial r)(r j_\ell(k_p r))]_{r=r_p} b_{(\ell,p)} \\
&+ \left(\frac{1}{k_p r_p}\right) [(\partial/\partial r)(r h_\ell^1(k_p r))]_{r=r_p} \beta_{(\ell,p)} .
\end{aligned} \tag{92}$$

Remark. If

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{13} & a_{14} \end{pmatrix}, \tag{93}$$

then the inverse of A is given by

$$A^{-1} = \begin{pmatrix} a_{22}/\Delta & -a_{12}/\Delta \\ -a_{21}/\Delta & a_{11}/\Delta \end{pmatrix}, \tag{94}$$

where $\Delta = a_{11}a_{22} - a_{12}a_{21}$.

We define for the sake of economy several terms, following Shapiro et al. (13). Let

$$\begin{aligned}
\xi_{(\ell,p)}^+ &= \frac{1}{k_p r_p} (\partial/\partial r)(r h_\ell^1(k_p r))_{r=r_p} = \frac{1}{k_p r_p} [h_\ell^1(k_p r_p) + k_p r_p h_\ell^{1'}(k_p r_p)] \\
&= \frac{1}{k_p r_p} (\partial/\partial \rho)(\rho h_\ell^1(\rho))_{\rho=k_p r_p},
\end{aligned} \tag{95}$$

$$\xi_{(\ell, p+1)}^- = \frac{1}{k_{p+1} r_p} (\partial/\partial \rho)(\rho h_{\ell}^1(\rho))_{\rho=k_{p+1} r_p}, \quad (96)$$

$$\begin{aligned} \eta_{(\ell, p)}^+ &= \frac{1}{k_p r_p} (\partial/\partial \rho)(\rho j_{\ell}(\rho))_{\rho=k_p r_p} \\ &= \frac{1}{k_p r_p} (\partial/\partial r)(r j_{\ell}(k_p r))_{r=r_p}, \end{aligned} \quad (97)$$

$$\eta_{(\ell, p+1)}^- = \frac{1}{k_{p+1} r_p} (\partial/\partial \rho)(\rho j_{\ell}(\rho))_{\rho=k_{p+1} r_p}, \quad (98)$$

$$j_{(\ell, p)}^+ = j_{\ell}(k_p r_p), \quad (99)$$

$$j_{(\ell, p+1)}^- = j_{\ell}(k_{p+1} r_p), \quad (100)$$

$$h_{(\ell, p)}^+ = h_{\ell}^1(k_p r_p), \quad (101)$$

and

$$h_{(\ell, p)}^- = h_{\ell}^1(k_{p+1} r_p). \quad (102)$$

Now the relations between the coefficients in matrix form are

$$\begin{pmatrix} j_{(\ell, p+1)}^- k_{p+1} & h_{(\ell, p+1)}^- k_{p+1} \\ \eta_{(\ell, p+1)}^- & \xi_{(\ell, p+1)}^- \end{pmatrix} = B_{-}^{(\ell, p+1)} \quad (103)$$

and

$$\begin{pmatrix} j_{(\ell,p)}^+ k_p & h_{(\ell,p)}^+ k_p \\ n_{(\ell,p)}^+ & \xi_{(\ell,p)}^+ \end{pmatrix} = B_+^{(\ell,p)} . \quad (104)$$

Observe that

$$B_+^{(\ell,p)} \begin{pmatrix} b_{(\ell,p)} \\ \beta_{(\ell,p)} \end{pmatrix} = B_-^{(\ell,p+1)} \begin{pmatrix} b_{(\ell,p+1)} \\ \beta_{(\ell,p+1)} \end{pmatrix} . \quad (105)$$

Thus, letting

$$R_{(\ell,p)} = (B_+^{(\ell,p)})^{-1} B_-^{(\ell,p+1)} , \quad (106)$$

we deduce that since

$$(B_+^{(\ell,p)})^{-1} = \begin{pmatrix} \xi_{(\ell,p)}^+ / k_p \Delta_p & -h_{(\ell,p)}^+ / \Delta_p \\ -n_{(\ell,p)}^+ / k_p \Delta_p & j_{(\ell,p)}^+ / \Delta_p \end{pmatrix} , \quad (107)$$

where

$$\Delta_p = j_{(\ell,p)}^+ \epsilon_{(\ell,p)}^+ - h_{(\ell,p)}^+ \eta_{(\ell,p)}^+ , \quad (108)$$

that

$$\begin{aligned} R_{(\ell,p)} &= \begin{pmatrix} \epsilon_{(\ell,p)}^+ / k_p \Delta_p & -h_{(\ell,p)}^+ / \Delta_p \\ -\eta_{(\ell,p)}^+ / k_p \Delta_p & j_{(\ell,p)}^+ / \Delta_p \end{pmatrix} B_{-}^{(\ell,p+1)} \\ &= \begin{pmatrix} R_{(1,1)}^{(\ell,p)} & R_{(1,2)}^{(\ell,p)} \\ R_{(2,1)}^{(\ell,p)} & R_{(2,2)}^{(\ell,p)} \end{pmatrix} . \end{aligned} \quad (109)$$

Here

$$R_{(1,1)}^{(\ell,p)} = [\epsilon_{(\ell,p)}^+ j_{(\ell,p+1)}^- (\frac{k_{p+1}}{k_p}) - h_{(\ell,p)}^+ \eta_{(\ell,p+1)}^-] / \Delta_p , \quad (110)$$

$$R_{(1,2)}^{(\ell,p)} = [h_{(\ell,p+1)}^- \epsilon_{(\ell,p)}^+ (\frac{k_{p+1}}{k_p}) - h_{(\ell,p)}^+ \epsilon_{(\ell,p+1)}^-] / \Delta_p , \quad (111)$$

$$R_{(2,1)}^{(\ell,p)} = [-\eta_{(\ell,p)}^+ j_{(\ell,p+1)}^- (\frac{k_{p+1}}{k_p}) + \eta_{(\ell,p+1)}^- j_{(\ell,p)}^+] / \Delta_p , \quad (112)$$

$$R_{(2,2)}^{(\ell,p)} = [-h_{(\ell,p+1)}^- \eta_{(\ell,p)}^+ (\frac{k_{p+1}}{k_p}) + \epsilon_{(\ell,p+1)}^- j_{(\ell,p)}^+] / \Delta_p . \quad (113)$$

Now on to the development of the matrices relating the a - α coefficients in layer p to those in layer $p+1$. The first relation is derived from equation 69 and the definitions 60-68, and is expressed, using the notation in equations 95-102, as

$$\begin{aligned} h_{(\ell,p+1)}^{\alpha(\ell,p+1)} + j_{(\ell,p+1)}^{\alpha(\ell,p)} \\ - h_{(\ell,p)}^{\alpha(\ell,p)} - j_{(\ell,p)}^{\alpha(\ell,p)} = 0 . \end{aligned} \quad (114)$$

The next relation, derived from equations 74-81 and equations 83-90, takes the form of

$$\begin{aligned} & \alpha_{(\ell,p+1)} \frac{1}{k_{p+1} r_p} (\partial/\partial r) (r h_{\ell}^1(k_{p+1} r)) \Big|_{r=r_p} C(\ell) k_{p+1} \\ & + \alpha_{(\ell,p+1)} \frac{1}{k_{p+1} r_p} (\partial/\partial r) (r j_{\ell}(k_{p+1} r)) \Big|_{r=r_p} C(\ell) k_{p+1} \\ & - \alpha_{(\ell,p)} \frac{1}{k_p r_p} (\partial/\partial r) (r h_{\ell}^1(k_p r)) \Big|_{r=r_p} C(\ell) k_p \\ & - \alpha_{(\ell,p)} \frac{1}{k_p r_p} (\partial/\partial r) (r j_{\ell}(k_p r)) \Big|_{r=r_p} C(\ell) k_p = 0 \end{aligned} \quad (115)$$

which, after using the notation expressed by equations 95-102, may be written as

$$\begin{aligned}
& \alpha(\ell, p+1) \xi^-(\ell, p+1)^{k_{p+1}} + a(\ell, p+1) \eta^-(\ell, p+1)^{k_{p+1}} \\
& - \alpha(\ell, p) \xi^+(\ell, p)^{k_p} - a(\ell, p) \eta^+(\ell, p)^{k_p} = 0 .
\end{aligned} \tag{116}$$

Let us define

$$A_+^{(\ell, p)} = \begin{pmatrix} j_+^{(\ell, p)} & h_+^{(\ell, p)} \\ \eta_+^{(\ell, p)^{k_p}} & \xi_+^{(\ell, p)^{k_p}} \end{pmatrix} \tag{117}$$

and

$$A_-^{(\ell, p+1)} = \begin{pmatrix} j_-^{(\ell, p+1)} & h_-^{(\ell, p+1)} \\ \eta_-^{(\ell, p+1)^{k_{p+1}}} & \xi_-^{(\ell, p+1)^{k_{p+1}}} \end{pmatrix} \tag{118}$$

Define

$$Q^{(\ell, p)} = (A_+^{(\ell, p)})^{-1} (A_-^{(\ell, p+1)}) \tag{119}$$

and

$$\Delta(\ell, p) = j_{(\ell, p)}^+ \xi_{(\ell, p)}^+ - h_{(\ell, p)}^+ \eta_{(\ell, p)}^+ . \quad (120)$$

Observe that since $(A_+^{(\ell, p)})^{-1}$ is given by

$$\begin{aligned} (A_+^{(\ell, p)})^{-1} &= \begin{pmatrix} \xi_{(\ell, p)}^+ k_p / k_p \Delta_p & -h_{(\ell, p)}^+ / k_p \Delta_p \\ -\eta_{(\ell, p)}^+ k_p / k_p \Delta_p & j_{(\ell, p)}^+ / k_p \Delta_p \end{pmatrix} \\ &= \begin{pmatrix} \xi_{(\ell, p)}^+ / \Delta(\ell, p) & -h_{(\ell, p)}^+ / k_p \Delta(\ell, p) \\ -\eta_{(\ell, p)}^+ / \Delta(\ell, p) & j_{(\ell, p)}^+ / k_p \Delta(\ell, p) \end{pmatrix} , \end{aligned} \quad (121)$$

it follows that

$$\begin{aligned} \det((A_+^{(\ell, p)})^{-1}) &= \frac{1}{k_p \Delta(\ell, p)} \\ &= \frac{1}{\det(A_+^{(\ell, p)})} \end{aligned} \quad (122)$$

and furthermore

$$(A_+^{(\ell, p)})^{-1} A_-^{(\ell, p+1)} = \begin{pmatrix} Q_{(1,1)}^{(\ell, p)} & Q_{(1,2)}^{(\ell, p)} \\ Q_{(2,1)}^{(\ell, p)} & Q_{(2,2)}^{(\ell, p)} \end{pmatrix} , \quad (123)$$

where

$$Q_{(1,1)}^{(\ell,p)} = [\xi_{(\ell,p)}^+ j_{(\ell,p+1)}^- - (\frac{k_{p+1}}{k_p}) h_{(\ell,p)}^+ \eta_{(\ell,p+1)}^-] / \Delta(\ell,p) , \quad (124)$$

$$Q_{(1,2)}^{(\ell,p)} = [\xi_{(\ell,p)}^+ h_{(\ell,p+1)}^- - (\frac{k_{p+1}}{k_p}) h_{(\ell,p)}^+ \xi_{(\ell,p+1)}^-] / \Delta(\ell,p) , \quad (125)$$

$$Q_{(2,1)}^{(\ell,p)} = [(\frac{k_{p+1}}{k_p}) j_{(\ell,p)}^+ \eta_{(\ell,p+1)}^- - \eta_{(\ell,p)}^+ j_{(\ell,p+1)}^-] / \Delta(\ell,p) , \quad (126)$$

and

$$Q_{(2,2)}^{(\ell,p)} = [(\frac{k_{p+1}}{k_p}) j_{(\ell,p)}^+ \xi_{(\ell,p+1)}^- - \eta_{(\ell,p)}^+ h_{(\ell,p+1)}^-] / \Delta(\ell,p) . \quad (127)$$

Now we wish to use the transition matrices $Q^{(\ell,p)}$ and $R^{(\ell,p)}$ to get relations between the internal and the external coefficients. First, note that $\alpha(\ell,1) = \beta(\ell,1) = 0$ and $a(\ell,N) = b(\ell,N) = 1$, where N is the number of regions into which space is subdivided by $N-1$ spheres. Second, observe that

$$\begin{bmatrix} \bar{a}(\ell,1) \\ 0 \end{bmatrix} = Q(\ell,1) Q(\ell,2) \dots Q(\ell,N-1) \begin{bmatrix} 1 \\ \alpha(\ell,N) \end{bmatrix} \quad (128)$$

and

$$\begin{bmatrix} b_{(\ell,1)} \\ 0 \end{bmatrix} = R^{(\ell,1)} R^{(\ell,2)} \dots R^{(\ell,N-1)} \begin{bmatrix} 1 \\ \beta_{(\ell,N)} \end{bmatrix} \quad (129)$$

or setting

$$Q = Q^{(\ell,1)} Q^{(\ell,2)} \dots Q^{(\ell,N-1)} \quad (130)$$

and

$$R = R^{(\ell,1)} R^{(\ell,2)} \dots R^{(\ell,N-1)} , \quad (131)$$

we have the following relations

$$\begin{pmatrix} a_{(\ell,1)} \\ 0 \end{pmatrix} = \begin{pmatrix} Q_{(1,1)} & Q_{(1,2)} \\ Q_{(2,1)} & Q_{(2,2)} \end{pmatrix} \begin{pmatrix} 1 \\ \alpha_{(\ell,N)} \end{pmatrix} \quad (132)$$

and

$$\begin{pmatrix} b_{(\ell,1)} \\ 0 \end{pmatrix} = \begin{pmatrix} R_{(1,1)} & R_{(1,2)} \\ R_{(2,1)} & R_{(2,2)} \end{pmatrix} \begin{pmatrix} 1 \\ \beta_{(\ell,N)} \end{pmatrix} . \quad (133)$$

Thus, we see that

$$\alpha(\ell, n) = -Q_{(2,1)}/Q_{(2,2)} \quad (134)$$

and

$$a(\ell, 1) = Q_{(1,1)} - Q_{(1,2)}Q_{(2,1)}/Q_{(2,2)} \quad (135)$$

Furthermore, once $\alpha(\ell, p)$ and $a(\ell, p)$ are determined, we obtain $\alpha(\ell, p+1)$ and $a(\ell, p+1)$ by the relation

$$\begin{pmatrix} a(\ell, p) \\ \alpha(\ell, p) \end{pmatrix} = \begin{pmatrix} Q_{(1,1)}^{(\ell, p)} & Q_{(1,2)}^{(\ell, p)} \\ Q_{(2,1)}^{(\ell, p)} & Q_{(2,2)}^{(\ell, p)} \end{pmatrix} \begin{pmatrix} a(\ell, p+1) \\ \alpha(\ell, p+1) \end{pmatrix} \quad (136)$$

Also, we deduce from equation 129 that

$$\beta(\ell, N) = R_{(2,1)}/R_{(2,2)} \quad (137)$$

and

$$b(\ell, 1) = R_{(1,1)} - R_{(1,2)}R_{(2,1)}/R_{(2,2)} \quad (138)$$

As before, once $\beta(\ell, p)$ and $b(\ell, p)$ are determined, we obtain $\beta(\ell, p+1)$ and $b(\ell, p+1)$ by the relation

$$\begin{pmatrix} b_{(\ell,p)} \\ \beta_{(\ell,p)} \end{pmatrix} = \begin{pmatrix} R_{(1,1)}^{(\ell,p)} & R_{(1,2)}^{(\ell,p)} \\ R_{(2,1)}^{(\ell,p)} & R_{(2,2)}^{(\ell,p)} \end{pmatrix} \begin{pmatrix} b_{(\ell,p+1)} \\ \beta_{(\ell,p+1)} \end{pmatrix}. \quad (139)$$

By repeated application of matrix equations 136 and 139, we determine all expansion coefficients; and thus, using equations 53 and 54, completely determine electric field $\vec{E}$ and magnetic field $\vec{H}$.

Determination of Total Absorbed Power

The Poynting vector is generally interpreted as a vector having length equal to the power per unit area traveling across a surface normal to the vector and direction of the power flow. One can show by the Gauss divergence theorem that the Poynting vector is given by

$$\vec{S} = \frac{\vec{E} \times \vec{H}^* + \vec{E}^* \times \vec{H}}{2}. \quad (140)$$

Maxwell's equations then imply that

$$\text{div}(\vec{S}) + \frac{\vec{J} \cdot \vec{J}^*}{\sigma} + (\partial/\partial t) \left(\frac{\epsilon \vec{E} \cdot \vec{E}^* + \mu \vec{H} \cdot \vec{H}^*}{2} \right) = 0, \quad (141)$$

Poynting's theorem in differential form in the absence of impressed electromotive forces for linear material media and where ϵ , μ , σ , and $\vec{J}$ are the permittivity, permeability, conductivity, and electric current density, respectively, and the sign $*$ attached to a vector denotes its complex conjugate.

Let us write (in terms of the spherical coordinate base vectors $\vec{e}_r$, $\vec{e}_\theta$, and $\vec{e}_\phi$)

$$\vec{E} = E_r \vec{e}_r + E_\theta \vec{e}_\theta + E_\phi \vec{e}_\phi \quad (142)$$

and

$$\vec{H} = H_r \vec{e}_r + H_\theta \vec{e}_\theta + H_\phi \vec{e}_\phi, \quad (143)$$

where

$$\vec{e}_r = \sin\theta\cos\phi\vec{i} + \sin\theta\sin\phi\vec{j} + \cos\theta\vec{k}, \quad (144)$$

$$\vec{e}_\theta = \cos\theta\cos\phi\vec{i} + \cos\theta\sin\phi\vec{j} - \sin\theta\vec{k}, \quad (145)$$

and

$$\vec{e}_\phi = -\sin\phi\vec{i} + \cos\phi\vec{j}. \quad (146)$$

Observe that

$$\begin{aligned} \vec{e}_r \times \vec{e}_\theta &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \end{vmatrix} \\ &= \vec{i}(-\sin^2\theta\sin\phi - \cos^2\theta\sin\phi) - \vec{j}(-\sin^2\theta\cos\phi - \cos^2\theta\cos\phi) \\ &\quad + \vec{k}(\sin\theta\cos\theta\sin\phi\cos\phi - \sin\theta\cos\theta\sin\phi\cos\phi) = -\sin\phi\vec{i} + \cos\phi\vec{j} \\ &= \vec{e}_\phi. \end{aligned} \quad (147)$$

A similar calculation shows that

$$\vec{e}_\theta \times \vec{e}_\phi = \vec{e}_r \quad (148)$$

and

$$\vec{e}_\phi \times \vec{e}_r = \vec{e}_\theta . \quad (149)$$

Thus

$$\vec{E} \times \vec{H} = (E_\theta H_\phi - H_\theta E_\phi) \vec{e}_r + (E_\phi H_r - H_\phi E_r) \vec{e}_\theta + (E_r H_\theta - H_r E_\theta) \vec{e}_\phi . \quad (150)$$

What we need to compute is

$$\vec{S} \cdot (-\vec{N}) = (\vec{E} \times \vec{H}) \cdot (-\vec{N}) \quad (151)$$

When $\vec{N} = \vec{e}_r$. Now the power going into the sphere is

$$\int_0^\pi \left[\int_0^{2\pi} - (E_\theta H_\phi - H_\theta E_\phi) \sin\theta d\phi \right] d\theta . \quad (152)$$

At this point, we stop to refamiliarize ourselves with the structure of the vector wave functions listed below:

$$\begin{aligned} \vec{M}_{(1,n)}^{(e,j)} &= - \frac{1}{\sin\theta} z_n^j(k_p r) P_n^1(\cos\theta) \sin\phi \vec{e}_\theta \\ &\quad - z_n^j(k_p r) (d/d\theta) (P_n^1(\cos\theta)) \cos\phi \vec{e}_\phi , \end{aligned} \quad (153)$$

$$\begin{aligned} \vec{M}_{(1,n)}^{(o,j)} &= \frac{1}{\sin\theta} z_n^j(k_p r) P_n^1(\cos\theta) \cos\phi \vec{e}_\theta \\ &\quad - z_n^j(k_p r) (d/d\theta) (P_n^1(\cos\theta)) \sin\phi \vec{e}_\phi , \end{aligned} \quad (154)$$

$$\begin{aligned}
\vec{N}_{(1,n)}^{(o,j)} &= \frac{n(n+1)}{k_p r} z_n^j(k_p r) P_n^1(\cos\theta) \sin\phi \vec{e}_r \\
&+ \frac{1}{k_p r} (\partial/\partial r)(r z_n^j(k_p r)) (d/d\theta)(P_n^1(\cos\theta)) \sin\phi \vec{e}_\theta \\
&+ \frac{1}{k_p r \sin\theta} (\partial/\partial r)(r z_n^j(k_p r)) P_n^1(\cos\theta) \cos\phi \vec{e}_\phi, \quad (155)
\end{aligned}$$

and

$$\begin{aligned}
\vec{N}_{(1,n)}^{(e,j)} &= \frac{n(n+1)}{k_p r} z_n^j(k_p r) P_n^1(\cos\theta) \cos\phi \vec{e}_r \\
&+ \frac{1}{k_p r} (\partial/\partial r)(r z_n^j(k_p r)) (d/d\theta)(P_n^1(\cos\theta)) \cos\phi \vec{e}_\theta \\
&- \frac{1}{k_p r \sin\theta} (\partial/\partial r)(r z_n^j(k_p r)) P_n^1(\cos\theta) \sin\phi \vec{e}_\phi; \quad (156)
\end{aligned}$$

to contemplate the use of the fact that in the region ($p=N$) surrounding the biological material the electromagnetic field is

$$\begin{aligned}
\vec{E} &= E_0 \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} [\vec{M}_{(1,n)}^{(o,1)} - i \vec{N}_{(1,n)}^{(e,1)} \\
&+ \alpha_{(m,N)} \vec{M}_{(1,n)}^{(o,3)} - i \beta_{(n,N)} \vec{N}_{(1,n)}^{(e,3)}] \quad (157)
\end{aligned}$$

$$\begin{aligned}
\vec{H} &= - \frac{k_N}{\mu_o \omega} E_0 \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} [\vec{M}_{(1,n)}^{(e,1)} + i \vec{N}_{(1,n)}^{(o,1)} \\
&+ \beta_{(n,N)} \vec{M}_{(1,n)}^{(e,3)} + i \alpha_{(n,N)} \vec{N}_{(1,n)}^{(o,3)}], \quad (158)
\end{aligned}$$

or more compactly

$$\vec{E} = \vec{E}^i + \vec{E}^r \quad (159)$$

and

$$\vec{H} = \vec{H}^i + \vec{H}^r \quad (160)$$

and in accordance with Stratton (14, p. 568), we write (where a factor of $\frac{1}{2}$ has been deleted and now the complex conjugate is indicated by overbar —)

$$\bar{S}_r = E_\theta \bar{H}_\phi - E_\phi \bar{H}_\theta \quad (161)$$

and take

$$W_a = -\text{Re} \int_0^\pi \left[\int_0^{2\pi} \bar{S}_r r^2 \sin\theta d\phi \right] d\theta. \quad (162)$$

Observe that

$$\begin{aligned} \frac{1}{2}(\vec{E} \times \vec{H} + \vec{E} \times \vec{H}) &= \text{Re}(\vec{E} \times \vec{H}) \\ &= \text{Re}(\vec{E}^i \times \vec{H}^i) + \text{Re}(\vec{E}^i \times \vec{H}^r + \vec{E}^r \times \vec{H}^i) \\ &\quad + \text{Re}(\vec{E}^r \times \vec{H}^r). \end{aligned} \quad (163)$$

A direct calculation shows that

$$\oint \text{Re}(\vec{E}^i \times \vec{H}^i) \cdot (-\vec{N}) dA = 0. \quad (164)$$

Thus, to get the total energy absorbed by the sphere, we need only compute

$$W_a = W_t - W_s, \quad (165)$$

where W_t represents the energy dissipated as heat plus the scattered energy, and W_s represents the scattered energy.

Now

$$\begin{aligned} W_s &= \text{Re} \int_0^{2\pi} \int_0^\pi (E_\theta \overline{H}_\phi - E_\phi \overline{H}_\theta) r^2 \sin\theta d\theta d\phi \\ &= \text{Re} \int_0^{2\pi} \int_0^\pi \left\{ E_0 \sum_{\ell=1}^{\infty} i^\ell \frac{2\ell+1}{\ell(\ell+1)} [\alpha_{(\ell,N)} (M_{(1,\ell)}^{(0,3)})_\phi - i\beta_{(\ell,N)} (N_{(1,\ell)}^{(e,3)})_\phi] \right. \\ &\quad \cdot \left\{ -\frac{k_N}{\mu_0 \omega} \overline{E}_0 \sum_{s=1}^{\infty} \bar{i}^s \frac{2s+1}{s(s+1)} [\bar{\beta}_{(s,N)} (M_{(1,s)}^{(e,3)})_\theta - i\bar{\alpha}_{(s,N)} (\bar{N}_{(1,s)}^{(0,3)})_\theta] \right\} r^2 \sin\theta d\phi \\ &\quad - \text{Re} \int_0^{2\pi} \int_0^\pi \left\{ E_0 \sum_{\ell=1}^{\infty} i^\ell \frac{2\ell+1}{\ell(\ell+1)} [\alpha_{(\ell,N)} (M_{(1,\ell)}^{(0,3)})_\theta - i\beta_{(\ell,N)} (N_{(1,\ell)}^{(e,3)})_\theta] \right\} \\ &\quad \cdot \left\{ \frac{k_N}{\mu_0 \omega} \overline{E}_0 \sum_{s=1}^{\infty} \bar{i}^s \frac{2s+1}{s(s+1)} [\bar{\beta}_{(s,N)} (M_{(1,s)}^{(e,3)})_\phi - i\bar{\alpha}_{(s,N)} (\bar{N}_{(1,s)}^{(0,3)})_\phi] \right\} r^2 \sin\theta d\phi \\ &= \sum_{s=1}^{\infty} \sum_{\ell=1}^{\infty} \frac{(2s+1)(2\ell+1)}{s(s+1)\ell(\ell+1)} [F_{(\ell,s)}^{(\alpha,\alpha)} \alpha_{(\ell,N)} \bar{\alpha}_{(s,N)} + F_{(\ell,s)}^{(\alpha,\beta)} \alpha_{(\ell,N)} \bar{\beta}_{(s,N)} \\ &\quad + F_{(\ell,s)}^{(\beta,\alpha)} \beta_{(\ell,N)} \bar{\alpha}_{(s,N)} + F_{(\ell,s)}^{(\beta,\beta)} \beta_{(\ell,N)} \bar{\beta}_{(s,N)}] . \end{aligned} \quad (166)$$

Here

$$\begin{aligned} \alpha(\ell, N) \bar{\alpha}(s, N) F_{(\ell, s)}^{(\alpha, \alpha)} = \operatorname{Re} \int_0^{2\pi} \int_0^\pi \frac{k_N}{\mu_0 \omega} E_0^2 i^{s+\ell+1} (-1)^{s+1} \alpha(\ell, N) \bar{\alpha}(s, N) \\ \cdot \left[\overline{(M_{(1, \ell)}^{(0, 3)})_\phi} (N_{(1, s)}^{(0, 3)})_\theta - (M_{(1, \ell)}^{(0, 3)})_\theta \overline{(N_{(1, s)}^{(0, 3)})_\phi} \right] r^2 \sin \theta d\theta d\phi, \end{aligned} \quad (167)$$

$$\begin{aligned} \alpha(\ell, N) \bar{\beta}(s, N) F_{(\ell, s)}^{(\alpha, \beta)} = \operatorname{Re} \int_0^{2\pi} \int_0^\pi \frac{k_N}{\mu_0 \omega} E_0^2 i^{s+\ell} (-1)^s \alpha(\ell, N) \bar{\beta}(s, N) \\ \cdot \left[\overline{(M_{(1, \ell)}^{(0, 3)})_\phi} (M_{(1, s)}^{(e, 3)})_\theta - (M_{(1, \ell)}^{(0, 3)})_\theta \overline{(M_{(1, s)}^{(e, 3)})_\phi} \right] r^2 \sin \theta d\theta d\phi, \end{aligned} \quad (168)$$

$$\begin{aligned} \beta(\ell, N) \bar{\alpha}(s, N) F_{(\ell, s)}^{(\beta, \alpha)} = \operatorname{Re} \int_0^{2\pi} \int_0^\pi \frac{k_N}{\mu_0 \omega} E_0^2 i^{s+\ell} (-1)^s \beta(\ell, N) \bar{\alpha}(s, N) \\ \cdot \left[\overline{-(N_{(1, \ell)}^{(e, 3)})_\phi} (N_{(1, s)}^{(0, 3)})_\theta + (N_{(1, \ell)}^{(e, 3)})_\theta \overline{(N_{(1, s)}^{(0, 3)})_\phi} \right] r^2 \sin \theta d\theta d\phi. \end{aligned} \quad (169)$$

Observe that

$$\begin{aligned}
& \int_0^{2\pi} \int_0^\pi \frac{k_N}{\mu_0 \omega} E_0^2 i^{s+\ell} (-1)^s \beta_{(\ell, N)} \bar{\alpha}_{(s, N)} \\
& \quad [(-\frac{1}{k_N r \sin \theta} (\partial/\partial r) (rh_\ell^1(k_N r)) P_\ell^1(\cos \theta) \sin \phi) \\
& \quad \cdot (\frac{1}{k_N r} \overline{(\partial/\partial r) (rh_s^1(k_N r))} (d/d\theta) P_s^1(\cos \theta) \sin \phi \\
& \quad + (\frac{1}{k_N r} (\partial/\partial r) (rh_\ell^1(k_N r)) (d/d\theta) P_\ell^1(\cos \theta) \cos \phi) \\
& \quad \cdot (\frac{1}{k_N r \sin \theta} \overline{(\partial/\partial r) (rh_s^1(k_N r))} P_s^1(\cos \theta) \cos \phi)] \sin \theta d\theta d\phi \\
& = \beta_{(\ell, N)} \bar{\alpha}_{(s, N)} F_{(\ell, s)}^{(\beta, \alpha)} . \tag{170}
\end{aligned}$$

Since for all positive integers ℓ and s , we have

$$\int_0^\pi [P_\ell^1(\cos \theta) (d/d\theta) P_s^1(\cos \theta) - P_s^1(\cos \theta) (d/d\theta) P_\ell^1(\cos \theta)] d\theta = 0, \tag{171}$$

it, thus, follows from equation 170 that

$$F_{(\ell, s)}^{(\beta, \alpha)} = 0, \tag{172}$$

and an almost identical argument shows that

$$F_{(\ell, s)}^{(\alpha, \beta)} = 0 \quad (173)$$

for all ℓ and s . Note that the value of W_s is independent of r . This enables us to make use of the asymptotic formulas

$$h_n^1(\rho) \approx \frac{1}{\rho} (-i)^{n+1} e^{i\rho} \quad (174)$$

and

$$(d/d\rho) h_n^1(\rho) \approx -\frac{1}{\rho^2} (-i)^{n+1} e^{i\rho} + \frac{i}{\rho} (-i)^{n+1} e^{i\rho} . \quad (175)$$

Further, we have

$$\begin{aligned} F_{(\ell, s)}^{(\alpha, \alpha)} &= \int_0^{2\pi} \int_0^\pi \frac{k_N}{\mu_0 \omega} E_0^2 i^{s+\ell} (-1)^s [(M_{(1, \ell)}^{(0, 3)})_\phi (-i) (N_{(1, s)}^{(0, 3)})_\theta \\ &\quad - (M_{(1, \ell)}^{(0, 3)})_\theta (-i) (N_{(1, s)}^{(0, 3)})_\phi] r^2 \sin \theta d\theta d\phi \\ &= \int_0^\pi \frac{\pi k_N}{\mu_0 \omega} E_0^2 i^{s+\ell} (-1)^s (-i) [(M_{(1, \ell)}^{(0, 3)})_\phi (N_{(1, s)}^{(0, 3)})_\theta \\ &\quad - (M_{(1, \ell)}^{(0, 3)})_\theta (N_{(1, s)}^{(0, 3)})_\phi] r^2 \sin \theta d\theta ; \end{aligned}$$

$$\begin{aligned}
F_{(\ell, s)}^{(\alpha, \alpha)} &= \int_0^\pi \frac{\pi k_N}{\mu_0 \omega} E_0^2 i^{s+\ell} (-1)^s (-i) [h_\ell^1(k_N r) (1/k_N r) (\partial/\partial r) (\overline{r h_\ell^1(k_N r)})] \\
&\quad - ((d/d\theta) P_\ell^1(\cos\theta)) ((d/d\theta) P_s^1(\cos\theta)) - \frac{P_\ell^1(\cos\theta) P_s^1(\cos\theta)}{\sin^2\theta} \big] r^2 \sin\theta d\theta \\
&= \frac{\pi k_N}{\mu_0 \omega} E_0^2 i^{s+\ell+1} (-1)^s \frac{2}{2s+1} \frac{(s+1)!}{(s-1)!} s(s+1) \delta_{(s, \ell)} \\
&\quad \cdot h_\ell^1(k_N r) (1/k_N r) (\partial/\partial r) (\overline{r h_\ell^1(k_N r)}) \quad (176)
\end{aligned}$$

after applying Proposition 6 for $m=1$.

To complete the calculation, we make use of the following lemma.

Lemma 1. For all k ,

$$\lim_{r \rightarrow \infty} h_\ell^1(kr) (1/kr) (\partial/\partial r) (\overline{r h_\ell^1(kr)}) r^2 = -i/k^2. \quad (177)$$

Proof. Equation 177 is equivalent to

$$\lim_{r \rightarrow \infty} h_\ell(kr) \left[\frac{r}{k} h_\ell^1(kr) + r^2 \overline{h_\ell^1(kr)} \right] = -i/k^2. \quad (178)$$

In view of equations 174 and 175, this completes the proof of Lemma 1.

Using Lemma 1 and equation 176 we deduce that

$$\begin{aligned}
F_{(\ell, s)}^{(\alpha, \alpha)} &= \left(\frac{\pi k_N}{\mu_0 \omega} \right) \frac{E_0^2 \delta_{(\ell, s)} i^{2s} (-1)^s i (-i) 2s^2 (s+1)^2}{k_N^2 (2s+1)} \\
&= \left(\frac{\pi k_N}{\mu_0 \omega} \right) 2E_0^2 \delta_{(\ell, s)} s^2 (s+1)^2 / ((2s+1) k_N^2) \quad (179)
\end{aligned}$$

Similarly

$$F_{(\ell,s)}^{(\beta,\beta)} = \left(\frac{\pi k_N}{\mu_0 \omega}\right) 2E_0^2 \delta_{(\ell,s)} s^2 (s+1)^2 / ((2s+1)k_N^2) . \quad (180)$$

Thus

$$W_s = \left(\frac{2\pi k_N}{\mu_0 \omega}\right) E_0^2 \sum_{s=1}^{\infty} (2s+1) (|\alpha_{(s,N)}|^2 + |\beta_{(s,N)}|^2) / k_N^2 . \quad (181)$$

Energy balance is maintained through the introduction of a third term, namely

$$W_t = -\text{Re} \int_0^{2\pi} \int_0^{\pi} (E_{\theta}^r \bar{H}_{\phi}^i + E_{\theta}^i \bar{H}_{\phi}^r - E_{\phi}^r \bar{H}_{\theta}^i - E_{\phi}^i \bar{H}_{\theta}^r) \sin \theta d\theta d\phi . \quad (182)$$

Thus, collecting multiples of the expansion coefficients, we obtain

$$\begin{aligned} W_t = & -\text{Re} \left\{ \int_0^{2\pi} \int_0^{\pi} \left\{ \sum_{\ell=1}^{\infty} i^{\ell} \frac{2\ell+1}{\ell(\ell+1)} [\alpha_{(\ell,N)} \langle M_{(1,\ell)}^{(0,3)} \rangle_{\theta} - i \beta_{(\ell,N)} \langle N_{(1,\ell)}^{(e,3)} \rangle_{\theta}] \right. \right. \\ & \cdot \sum_{s=1}^{\infty} (-i)^s \frac{2s+1}{s(s+1)} [\langle M_{(1,s)}^{(e,1)} \rangle_{\phi} - i \langle N_{(1,s)}^{(0,1)} \rangle_{\phi}] \\ & \left. \left. + \sum_{\ell=1}^{\infty} i^{\ell} \frac{2\ell+1}{\ell(\ell+1)} [\langle M_{(1,\ell)}^{(0,1)} \rangle_{\theta} - i \langle N_{(1,\ell)}^{(e,1)} \rangle_{\theta}] \right\} \right\} \end{aligned}$$

$$\begin{aligned}
& \cdot \left\{ \sum_{s=1}^{\infty} i^s \frac{2s+1}{s(s+1)} [\bar{\beta}_{(s,N)} \overline{(M_{(1,s)}^{(e,3)})}_{\phi}} - i \bar{\alpha}_{(s,N)} \overline{(N_{(1,s)}^{(o,3)})}_{\phi} \right\} \\
& - \sum_{\ell=1}^{\infty} i^{\ell} \frac{2\ell+1}{\ell(\ell+1)} [\alpha_{(\ell,N)} \overline{(M_{(1,\ell)}^{(o,3)})}_{\phi}} - i \beta_{(\ell,N)} \overline{(N_{(1,\ell)}^{(e,3)})}_{\phi}] \\
& \cdot \sum_{s=1}^{\infty} (-i)^s \frac{2s+1}{s(s+1)} [\overline{(M_{(1,\ell)}^{(e,1)})}_{\theta}} - i \overline{(N_{(1,s)}^{(o,1)})}_{\theta}] \\
& - \sum_{\ell=1}^{\infty} \sum_{s=1}^{\infty} [i^{\ell+s} (-1)^s \frac{(2\ell+1)(2s+1)}{\ell(\ell+1)s(s+1)} [\overline{(M_{(1,\ell)}^{(o,1)})}_{\phi}} - i \overline{(N_{(1,\ell)}^{(e,1)})}_{\phi}] \\
& \cdot [\bar{\beta}_{(s,N)} \overline{(M_{(1,s)}^{(e,3)})}_{\theta}} - i \bar{\alpha}_{(s,N)} \overline{(N_{(1,s)}^{(o,3)})}_{\theta}]] \sin \theta d\theta d\phi \left[\frac{E_{oN}^2 k_N}{\mu_o \omega} \right] \\
& = - \operatorname{Re} \left\{ \sum_{\ell=1}^{\infty} \sum_{s=1}^{\infty} i^{\ell+s} (-1)^s \frac{(2\ell+1)(2s+1)}{\ell(\ell+1)s(s+1)} [\alpha_{(\ell,N)} F_{(\ell,s)}^{(\alpha,1)} \right. \\
& \quad \left. + \bar{\alpha}_{(s,N)} F_{(\ell,s)}^{(1,\alpha)} + \beta_{(\ell,N)} F_{(\ell,s)}^{(\beta,1)} + \bar{\beta}_{(s,N)} F_{(\ell,s)}^{(1,\beta)}] \right\} \left[\frac{E_{oN}^2 k_N}{\mu_o \omega} \right], \quad (183)
\end{aligned}$$

where

$$\begin{aligned}
F_{(\ell,s)}^{(\alpha,1)} &= \int_S \{ (M_{(1,\ell)}^{(o,3)})_{\theta} [\overline{(M_{(1,s)}^{(e,1)})}_{\phi}} - i \overline{(N_{(1,s)}^{(o,1)})}_{\phi}] \\
&\quad - (M_{(1,\ell)}^{(o,3)})_{\phi} [\overline{(M_{(1,s)}^{(e,1)})}_{\theta}} - i \overline{(N_{(1,s)}^{(o,1)})}_{\theta}] \} dA, \quad (184)
\end{aligned}$$

$$F_{(\ell,s)}^{(1,\alpha)} = \int_S \{ [(M_{(1,\ell)}^{(0,1)})_{\theta} - i(N_{(1,\ell)}^{(e,1)})_{\theta}] (-i)(N_{(1,s)}^{(0,3)})_{\phi} \\ - [(M_{(1,\ell)}^{(0,1)})_{\phi} - i(N_{(1,\ell)}^{(e,1)})_{\phi}] (-i)(N_{(1,s)}^{(0,3)})_{\theta} \} dA, \quad (185)$$

$$F_{(\ell,s)}^{(\beta,1)} = \int_S \{ (N_{(1,\ell)}^{(e,3)})_{\theta} (-i)[(M_{(1,s)}^{(e,1)})_{\phi} - i(N_{(1,s)}^{(0,1)})_{\phi}] \\ - (N_{(1,\ell)}^{(e,3)})_{\phi} (-i)[(M_{(1,s)}^{(e,1)})_{\theta} - i(N_{(1,s)}^{(0,1)})_{\theta}] \} dA, \quad (186)$$

and

$$F_{(\ell,s)}^{(1,\beta)} = \int_S \{ (M_{(1,s)}^{(e,3)})_{\phi} [(M_{(1,\ell)}^{(0,1)})_{\theta} - i(N_{(1,\ell)}^{(e,1)})_{\theta}] \\ - (M_{(1,s)}^{(e,3)})_{\theta} [(M_{(1,\ell)}^{(0,1)})_{\phi} - i(N_{(1,\ell)}^{(e,1)})_{\phi}] \} dA. \quad (187)$$

First we compute $F_{(\ell,s)}^{(\alpha,1)}$. Observe that if we let

$$A_{(\ell,s)}^{(\alpha,1)} = \int_0^{2\pi} \int_0^{\pi} \frac{1}{\sin\theta} h_{\ell}^1(k_N r) P_{\ell}^1(\cos\theta) \cos\phi \\ [-j_s(k_N r)((d/d\theta)P_s^1(\cos\theta)) \cos\phi \\ - \frac{i}{k_N r \sin\theta} (\partial/\partial r)(r j_s(k_N r)) P_s^1(\cos\theta) \cos\phi] r^2 \sin\theta d\theta d\phi$$

$$A_{(\ell, s)}^{(\alpha, 1)} = \pi r^2 h_{\ell}^1(k_N r) (-j_s(k_N r)) \int_0^{\pi} P_{\ell}^1(\cos \theta) \left(\frac{d}{d\theta} P_s^1(\cos \theta) \right) d\theta \\ - i \pi r^2 h_{\ell}^1(k_N r) \left(\frac{1}{k_N r} \right) \left(\frac{\partial}{\partial r} \right) \overline{(r j_s(k_N r))} \int_0^{\pi} \frac{P_{\ell}^1(\cos \theta) P_s^1(\cos \theta)}{\sin \theta} d\theta \quad (188)$$

and

$$B_{(\ell, s)}^{(\alpha, 1)} = \int_0^{2\pi} \int_0^{\pi} - h_{\ell}^1(k_N r) \left(\frac{d}{d\theta} P_{\ell}^1(\cos \theta) \right) \sin \phi [\\ - \frac{1}{\sin \theta} \overline{j_s(k_N r)} P_s^1(\cos \theta) \sin \phi \\ - \frac{i}{k_N r} \left(\frac{\partial}{\partial r} \right) \overline{(r j_s(k_N r))} \left(\frac{d}{d\theta} P_s^1(\cos \theta) \right) \sin \phi] r^2 \sin \theta d\theta \\ = \pi r^2 h_{\ell}^1(k_N r) \overline{j_s(k_N r)} \int_0^{\pi} \left(\frac{d}{d\theta} P_{\ell}^1(\cos \theta) \right) P_s^1(\cos \theta) d\theta \\ + \frac{\pi i r}{k_N} h_{\ell}^1(k_N r) \left(\frac{\partial}{\partial r} \right) \overline{(r j_s(k_N r))} \int_0^{\pi} \left[\left(\frac{d}{d\theta} P_{\ell}^1(\cos \theta) \right) \right. \\ \left. \cdot \left(\frac{d}{d\theta} P_s^1(\cos \theta) \right) \right] \sin \theta d\theta, \quad (189)$$

then

$$F_{(\ell, s)}^{(\alpha, 1)} = A_{(\ell, s)}^{(\alpha, 1)} - B_{(\ell, s)}^{(\alpha, 1)}. \quad (190)$$

In view of equations 18 and 170, we obtain

$$\begin{aligned}
F_{(\ell,s)}^{(\alpha,1)} &= A_{(\ell,s)}^{(\alpha,1)} - B_{(\ell,s)}^{(\alpha,1)} \\
&= -\pi r^2 h_\ell^1(k_N r) \overline{j_s(k_N r)} \int_0^\pi [P_\ell^1(\cos\theta) (d/d\theta) P_s^1(\cos\theta)] \\
&\quad + P_s^1(\cos\theta) (d/d\theta) P_\ell^1(\cos\theta)] d\theta \\
&\quad - \frac{i\pi r^2 h_\ell^1(k_N r) (\partial/\partial r) (\overline{r j_s(k_N r)})}{k_N r} \int_0^\pi \frac{P_\ell^1(\cos\theta) P_s^1(\cos\theta)}{\sin^2\theta} \\
&\quad + ((d/d\theta) P_\ell^1(\cos\theta)) ((d/d\theta) P_s^1(\cos\theta))] \sin\theta d\theta \\
&= - \frac{i\pi r^2 h_\ell^1(k_N r) (\partial/\partial r) (\overline{r j_s(k_N r)})}{k_N r} \delta_{(\ell,s)} \\
&\quad \cdot \frac{2}{2s+1} \frac{(s+1)!}{(s-1)!} s(s+1) \\
&= - \frac{i\pi r h_\ell^1(k_N r) (\partial/\partial r) (\overline{r j_s(k_N r)})}{k_N} \delta_{(\ell,s)} \frac{2s^2(s+1)^2}{2s+1}. \quad (191)
\end{aligned}$$

Next we compute $F_{(\ell,s)}^{(1,\alpha)}$. As before, we let

$$A_{(\ell,s)}^{(1,\alpha)} = -i \int_S [(M_{(1,\ell)}^{(0,1)})_\theta - i(N_{(1,\ell)}^{(e,1)})_\theta] \overline{(N_{(1,s)}^{(0,3)})_\theta} dA \quad (192)$$

and

$$B_{(\ell,s)}^{(1,\alpha)} = -i \int_S [(M_{(1,\ell)}^{(0,1)})_\phi - i(N_{(1,\ell)}^{(e,1)})_\phi] \overline{(N_{(1,s)}^{(0,3)})_\theta} dA. \quad (193)$$

Use of equations 153-156 yields

$$\begin{aligned}
 A_{(\ell, s)}^{(1, \alpha)} &= -i \int_0^{2\pi} \int_0^\pi \left[\frac{1}{\sin\theta} j_\ell(k_N r) p_\ell^1(\cos\theta) \cos\phi \right. \\
 &\quad - i \frac{1}{k_N r} (\partial/\partial r)(r j_\ell(k_N r)) (d/d\theta) p_\ell^1(\cos\theta) \cos\phi \\
 &\quad \cdot \frac{1}{k_N r \sin\theta} (\partial/\partial r)(r h_s^1(k_N r)) p_s^1(\cos\theta) \cos\phi r^2 \sin\theta d\theta d\phi \\
 &= - \frac{i\pi r}{k_N} j_\ell(k_N r) (\partial/\partial r)(r h_s^1(k_N r)) \int_0^\pi \frac{p_\ell^1(\cos\theta) p_s^1(\cos\theta)}{\sin\theta} d\theta \\
 &\quad - \frac{i\pi}{k_N^2} ((\partial/\partial r)(r j_\ell(k_N r))) ((\partial/\partial r)(r h_s^1(k_N r))) \\
 &\quad \cdot \int_0^\pi ((d/d\theta) p_\ell^1(\cos\theta)) p_s^1(\cos\theta) d\theta .
 \end{aligned} \tag{194}$$

Also, using equations 153-156 and 193, we deduce that

$$\begin{aligned}
 B_{(\ell, s)}^{(1, \alpha)} &= -i \int_0^{2\pi} \int_0^\pi [(-j_\ell(k_N r)) (d/d\theta) p_\ell^1(\cos\theta) \sin\phi \\
 &\quad + \frac{i}{k_N r \sin\theta} (\partial/\partial r)(r j_\ell(k_N r)) p_\ell^1(\cos\theta) \sin\phi \\
 &\quad \cdot \frac{1}{k_N r} (\partial/\partial r)(r h_s^1(k_N r)) (d/d\theta) p_s^1(\cos\theta) \sin\phi] r^2 \sin\theta d\theta d\phi .
 \end{aligned}$$

Carrying out the integration with respect to ϕ we see that

$$\begin{aligned}
 B_{(\ell, s)}^{(1, \alpha)} &= \frac{i\pi r}{k_N} j_\ell(k_N r) (\partial/\partial r) (\overline{rh_s^1(k_N r)}) \\
 &\cdot \int_0^\pi ((d/d\theta) P_\ell^1(\cos\theta)) ((d/d\theta) P_s^1(\cos\theta)) \sin\theta d\theta \\
 &+ \frac{i\pi}{k_N^2} ((\partial/\partial r)(rj_\ell(k_N r))) (\partial/\partial r) (\overline{rh_s^1(k_N r)}) \\
 &\cdot \int_0^\pi P_\ell^1(\cos\theta) (d/d\theta) P_s^1(\cos\theta) d\theta. \tag{195}
 \end{aligned}$$

From equations 185, 194, and 195, it follows that

$$F_{(\ell, s)}^{(1, \alpha)} = A_{(\ell, s)}^{(1, \alpha)} - B_{(\ell, s)}^{(1, \alpha)}. \tag{196}$$

Thus,

$$\begin{aligned}
 F_{(\ell, s)}^{(1, \alpha)} &= - \frac{i\pi r}{k_N} j_\ell(k_N r) (\partial/\partial r) (\overline{rh_s^1(k_N r)}) \\
 &\cdot \int_0^\pi \left[\frac{P_\ell^1(\cos\theta) P_s^1(\cos\theta)}{\sin^2\theta} + ((d/d\theta) P_\ell^1(\cos\theta)) ((d/d\theta) P_s^1(\cos\theta)) \right] \sin\theta d\theta \\
 &- \frac{i\pi}{k_N^2} ((\partial/\partial r)(rj_\ell(k_N r))) (\partial/\partial r) (\overline{rh_s^1(k_N r)}) \\
 &\cdot \int_0^\pi [((d/d\theta) (P_\ell^1(\cos\theta))) P_s^1(\cos\theta) + ((d/d\theta) P_s^1(\cos\theta)) P_\ell^1(\cos\theta)] d\theta \\
 &= - \frac{i\pi r}{k_N} j_\ell(k_N r) (\partial/\partial r) (\overline{rh_s^1(k_N r)}) \left[\delta_{(\ell, s)} \frac{2s^2(s+1)^2}{2s+1} \right]. \tag{197}
 \end{aligned}$$

Furthermore

$$\begin{aligned}
& \operatorname{Re}(\alpha_{(s,N)} F_{(s,s)}^{(\alpha,1)} + \bar{\alpha}_{(s,N)} F_{(s,s)}^{(1,\alpha)}) \\
&= \operatorname{Re} \left[\alpha_{(s,N)} \frac{-\pi r h_s^1(k_N r) (\partial/\partial r) (\overline{r j_s(k_N r)})}{k_N} \delta_{(\ell,s)} \frac{2s^2(s+1)^2}{2s+1} \right. \\
&\quad \left. + \bar{\alpha}_{(s,N)} \frac{-i\pi r j_s(k_N r) (\partial/\partial r) (r h_s^1(k_N r))}{k_N} \delta_{(\ell,s)} \frac{2s^2(s+1)^2}{2s+1} \right] \\
&= \operatorname{Re}(\alpha_{(s,N)}) \frac{2s^2(s+1)^2}{2s+1} .
\end{aligned} \tag{198}$$

Hence, using the fact that k_N is real, we have

$$W_t = \frac{2\pi E_0^2}{k_N^2} \sqrt{\epsilon_0/\mu_0} \operatorname{Re} \sum_{s=1}^{\infty} (2s+1) (\alpha_{(s,N)} + \beta_{(s,N)}) . \tag{199}$$

This follows from an induction argument and the fact that if

$$u + iv = \frac{(-i)^{n+1}}{k_N} [\cos(k_N r) + i \sin(k_N r)] \tag{200}$$

and

$$w = \frac{1}{k_N} \cos[k_N r - (\frac{n+1}{2})\pi] , \tag{201}$$

then for all real numbers A and B,

$$\begin{aligned}
& \operatorname{Re}\{i[w'(u+iv)(A+iB) + w(u'-iv')(A-iB)]\} \\
&= (v'w-vw')A + (u'w-uw')B \\
&= \frac{\Lambda}{k_N}
\end{aligned} \tag{202}$$

for every positive integer n . A prime on u , v , or w denotes differentiation with respect to r .

Thus, time averaging shows that the total absorbed power is given by

$$\begin{aligned}
W_a &= \left| \frac{\pi E_0^2}{k_N^2} \sqrt{\epsilon_0/\mu_0} \operatorname{Re} \sum_{n=1}^{\infty} (2n+1)(\alpha_{(n,N)} + \beta_{(n,N)}) \right| \\
&\quad - \frac{\pi E_0^2}{k_N^2} \sqrt{\epsilon_0/\mu_0} \sum_{n=1}^{\infty} (2n+1)(|\alpha_{(n,N)}|^2 + |\beta_{(n,N)}|^2).
\end{aligned} \tag{203}$$

Summary of Key Equations and Formulas

In summarizing, we set down the key equations and formulas upon which program CSM is based.

Fields for the p -th region:

$$\begin{aligned}
E_p &= E_0 \exp(-\omega t) \sum_{\ell=1}^{\infty} i^{\ell} \frac{2\ell+1}{\ell(\ell+1)} [a_{(\ell,p)} \vec{M}_{(1,\ell)}^{(o,1)} - ib_{(\ell,p)} \vec{N}_{(1,\ell)}^{(e,1)} \\
&\quad + \alpha_{(\ell,p)} \vec{M}_{(1,\ell)}^{(o,3)} - i\beta_{(\ell,p)} \vec{N}_{(1,\ell)}^{(e,3)}],
\end{aligned} \tag{204}$$

$$\begin{aligned}
H_p = & - \frac{k_p}{\mu_0 \omega} E_0 \exp(-\omega t) \sum_{\ell=1}^{\infty} i^{\ell} \frac{2\ell+1}{\ell(\ell+1)} [b_{(\ell,p)} \vec{M}_{(1,\ell)}^{(e,1)} + ia_{(\ell,p)} \vec{N}_{(1,\ell)}^{(o,1)} \\
& + \beta_{(\ell,p)} \vec{M}_{(1,\ell)}^{(e,3)} + ia_{(\ell,p)} \vec{N}_{(1,\ell)}^{(o,3)}] \quad (205)
\end{aligned}$$

where the vector wave functions $\vec{M}_{(1,\ell)}^{(e,3)}$, $\vec{M}_{(1,\ell)}^{(o,3)}$, $\vec{N}_{(1,\ell)}^{(e,3)}$, and $\vec{N}_{(1,\ell)}^{(o,3)}$ are obtained by replacing the spherical Bessel function $j_n(k_p r)$ by the spherical Hankel function $h_n^{(1)}(k_p r)$ in the expressions for the vector wave functions $\vec{M}_{(1,\ell)}^{(e,1)}$, $\vec{M}_{(1,\ell)}^{(o,1)}$, $\vec{N}_{(1,\ell)}^{(e,1)}$, and $\vec{N}_{(1,\ell)}^{(o,1)}$.

Complex propagation constant for the p-th region:

$$k_p = \text{Re}(k_p) + i\text{Im}(k_p), \quad (206)$$

where

$$\text{Re}(k_p) = \frac{\omega}{c} \left\{ \frac{\epsilon_p}{2} \left[\left(1 + \frac{1}{(\epsilon_0 \omega)^2} \left(\frac{\sigma_p}{\epsilon_p} \right)^2 \right)^{1/2} + 1 \right] \right\}^{1/2}, \quad (207)$$

$$\text{Im}(k_p) = \frac{\omega}{c} \left\{ \frac{\epsilon_p}{2} \left[\left(1 + \frac{1}{(\epsilon_0 \omega)^2} \left(\frac{\sigma_p}{\epsilon_p} \right)^2 \right)^{1/2} - 1 \right] \right\}^{1/2}, \quad (208)$$

ϵ_0 = free-space permittivity; $= 8.85 \times 10^{-12} \text{ F/m}$,

ϵ_p = relative dielectric constant of p-th region; $= 1$
for free space,

σ_p = conductivity of the p-th region; $= 0$ for free space,

ω = angular frequency; $= 2\pi \times \text{frequency (in MHz)}$,

c = velocity of light in free space; $= 2.9979 \times 10^8 \text{ m/s}$.

The field expansion coefficients for region one, inner core sphere, and those for the surrounding medium are obtained through the solution of two systems of equations. Utilizing the notation of Shapiro et al. (13), we have, with $a_{(1,N)} = b_{(1,N)} = 1$ and $\alpha_{(1,1)} = \beta_{(1,1)} = 0$,

$$\begin{bmatrix} a_{\ell,1} \\ 0 \end{bmatrix} = \begin{bmatrix} Q_{\ell,T}^{ij} \\ \alpha_{\ell,N} \end{bmatrix} \begin{bmatrix} 1 \\ \alpha_{\ell,N} \end{bmatrix}, \quad (209)$$

$$\begin{bmatrix} b_{\ell,1} \\ 0 \end{bmatrix} = \begin{bmatrix} R_{\ell,T}^{ij} \\ \beta_{\ell,N} \end{bmatrix} \begin{bmatrix} 1 \\ \beta_{\ell,N} \end{bmatrix}, \quad (210)$$

where the product matrices $[Q_{1,T}^{ij}]$ and $[R_{1,T}^{ij}]$ have the representation

$$[Q_{\ell,T}^{ij}] = \prod_{p=1}^{N-1} [Q_{(i,j)}^{(\ell,p)}], \quad (211)$$

$$[R_{\ell,T}^{ij}] = \prod_{p=1}^{N-1} [R_{(i,j)}^{(\ell,p)}], \quad (212)$$

with each factor matrix $[Q_{(i,j)}^{(\ell,p)}]$ and $[R_{(i,j)}^{(\ell,p)}]$ having its (i,j) elements computed by means of the following formulas:

$$Q_{(1,1)}^{(\ell,p)} = (\Delta_{(\ell,p)})^{-1} [\epsilon_{(\ell,p)}^+ j_{(\ell,p+1)}^- - \frac{k_{p+1}}{k_p} h_{(\ell,p)}^+ n_{(\ell,p+1)}^-], \quad (213)$$

$$Q_{(1,2)}^{(\ell,p)} = (\Delta_{(\ell,p)})^{-1} [\epsilon_{(\ell,p)}^+ h_{(\ell,p+1)}^- - \frac{k_{p+1}}{k_p} h_{(\ell,p)}^+ \xi_{(\ell,p+1)}^-], \quad (214)$$

$$Q_{(2,1)}^{(\ell,p)} = (\Delta_{(\ell,p)})^{-1} \left[\frac{k_{p+1}}{k_p} j_{(\ell,p)}^+ \eta_{(\ell,p+1)}^- - \eta_{(\ell,p)}^+ j_{(\ell,p+1)}^- \right], \quad (215)$$

$$Q_{(2,2)}^{(\ell,p)} = (\Delta_{(\ell,p)})^{-1} \left[\frac{k_{p+1}}{k_p} j_{(\ell,p)}^+ \xi_{(\ell,p+1)}^- - \eta_{(\ell,p)}^+ h_{(\ell,p+1)}^- \right], \quad (216)$$

$$R_{(1,1)}^{(\ell,p)} = (\Delta_{(\ell,p)})^{-1} \left[\frac{k_{p+1}}{k_p} \xi_{(\ell,p)}^+ j_{(\ell,p+1)}^- - h_{(\ell,p)}^+ \eta_{(\ell,p+1)}^- \right], \quad (217)$$

$$R_{(1,2)}^{(\ell,p)} = (\Delta_{(\ell,p)})^{-1} \left[\frac{k_{p+1}}{k_p} \xi_{(\ell,p)}^+ h_{(\ell,p+1)}^- - h_{(\ell,p)}^+ \xi_{(\ell,p+1)}^- \right], \quad (218)$$

$$R_{(2,1)}^{(\ell,p)} = (\Delta_{(\ell,p)})^{-1} \left[j_{(\ell,p)}^+ \eta_{(\ell,p+1)}^- - \frac{k_{p+1}}{k_p} \eta_{(\ell,p)}^+ j_{(\ell,p+1)}^- \right], \quad (219)$$

$$R_{(2,2)}^{(\ell,p)} = (\Delta_{(\ell,p)})^{-1} \left[j_{(\ell,p)}^+ \xi_{(\ell,p+1)}^- - \frac{k_{p+1}}{k_p} \eta_{(\ell,p)}^+ h_{(\ell,p+1)}^- \right], \quad (220)$$

Here

$$\Delta_{(\ell,p)} = j_{(\ell,p)}^+ \xi_{(\ell,p)}^+ - h_{(\ell,p)}^+ \eta_{(\ell,p)}^+, \quad (221)$$

$$j_{(\ell,p)}^+ = j_{\ell}(k_p r), \quad (222)$$

$$h_{(\ell,p)}^+ = h_{\ell}(k_p r), \quad (223)$$

$$\eta_{(\ell,p)}^+ = \frac{1}{2\ell+1} [(\ell+1)j_{(\ell-1,p)}^+ - \ell j_{(\ell+1,p)}^+], \quad (224)$$

$$\xi_{(\ell,p)}^+ = \frac{1}{2\ell+1} [(\ell+1)h_{(\ell-1,p)}^+ - \ell h_{(\ell+1,p)}^+], \quad (225)$$

the superscript 1 has been dropped from the spherical Hankel functions $h_n^{(1)}(k_p r) = j_n(k_p r) + iy_n(k_p r)$, and r is the radius of the boundary surface of the p -th region for subscripts p and $p+1$.

The matrix equations

$$\begin{bmatrix} a_{\ell,p} \\ \alpha_{\ell,p} \end{bmatrix} = \begin{bmatrix} Q_{\ell,p}^{(i,j)} \end{bmatrix} \begin{bmatrix} a_{\ell,p+1} \\ \alpha_{\ell,p+1} \end{bmatrix}, \quad (226)$$

$$\begin{bmatrix} b_{\ell,p} \\ \beta_{\ell,p} \end{bmatrix} = \begin{bmatrix} R_{\ell,p}^{(i,j)} \end{bmatrix} \begin{bmatrix} b_{\ell,p+1} \\ \beta_{\ell,p+1} \end{bmatrix}, \quad (227)$$

yield the expansion coefficients for the regions $p = 2, \dots, N-1$ in a recursive manner, starting with derived values of $a_{(1,1)}$ and $b_{(1,1)}$ and known values of $\alpha_{(1,1)}$ and $\beta_{(1,1)}$ as elements in the left-hand members of the matrix equations, and employing equations 213-225 for computing the necessary coefficient matrices $[Q_{\ell,p}^{(i,j)}]$ and $[R_{\ell,p}^{(i,j)}]$.

Absorbed-power density at an interior point of the p -th region:

$$P = 0.5 \sigma_p (\vec{E}_p \cdot \vec{E}_p^*) \quad (228)$$

where

$\vec{E}_p$ = electric vector at an interior point of the p -th region,

σ_p = conductivity of the p -th region,

$*$ = complex conjugate indicator.

Average absorbed-power density:

$$P_{avg} = (3/8\pi)(\epsilon_0/\nu_0)^{1/2} (E_0^2 Q_a / r_{N-1}^3), \quad (229)$$

where

$$Q_a = \left| \frac{2\pi}{k_N^2} \operatorname{Re} \sum_{\ell=1}^{\infty} (2\ell+1)(\alpha_{\ell,N} + \beta_{\ell,N}) \right| - \frac{2\pi}{k_N^2} \sum_{\ell=1}^{\infty} (2\ell+1)(|\alpha_{\ell,N}|^2 + |\beta_{\ell,N}|^2) = Q_t - Q_s, \quad (230)$$

ϵ_0, μ_0 = free-space permittivity and permeability,

k_N = propagation constant of the surrounding medium,

$\alpha_{\ell,N}, \beta_{\ell,N}$ = scattering coefficients.

Total absorbed power:

$$P_{\text{tot}} = \frac{2P_i}{\alpha^2} \sum_{\ell=1}^{\infty} (2\ell+1) [|\operatorname{Re}(\alpha_{\ell,N} + \beta_{\ell,N})| - (|\alpha_{\ell,N}|^2 + |\beta_{\ell,N}|^2)], \quad (231)$$

where

P_i = power incident upon α ; $= \frac{E_0^2}{2\eta} \pi r_{N-1}^2$,

η = intrinsic impedance for free space; = 376.7 ohms,

r_{N-1} = radius of spherical surface adjacent to the surrounding medium,

α = geometrical cross section of the sphere of radius r_{N-1} ; $= 2\pi r_{N-1}/\lambda$,

λ = wavelength of the incident wave.

To complete our summarization, consideration of the formulas used in generating the values of certain functions seems appropriate. The formulas

$$P_{n+1}^1(\cos\theta) = \frac{2n+1}{n} \cos\theta P_n^1(\cos\theta) - \frac{n+1}{n} P_{n-1}^1(\cos\theta), \quad (232)$$

$$\sin\theta(d/d\theta)P_n^1(\cos\theta) = n\cos\theta P_n^1(\cos\theta) - (n+1)P_{n-1}^1(\cos\theta), \quad (233)$$

together with

$$P_1^1(\cos\theta) = \sin\theta, \quad (234)$$

$$P_2^1(\cos\theta) = 3 \sin\theta \cos\theta \quad (235)$$

are used to generate function and derivative values of the associated Legendre functions.

Special limit values are also obtained by

$$\lim_{\theta \rightarrow 0} \frac{P_n^1(\cos\theta)}{\sin\theta} = \frac{n(n+1)}{2}, \quad (236)$$

$$\lim_{\theta \rightarrow \pi} \frac{P_n^1(\cos\theta)}{\sin\theta} = \frac{(-1)^{n+1}n(n+1)}{2}. \quad (237)$$

The forward recurrence relation

$$y_{n+1}(z) + y_{n-1}(z) = \frac{2n+1}{z} y_n(z) \quad (238)$$

is used together with relations

$$y_0(z) = -\frac{\cos z}{z}, \quad (239)$$

$$y_1(z) = -\frac{\cos z}{z^2} - \frac{\sin z}{z} \quad (240)$$

to generate values of the spherical Neuman functions. The generating process is terminated at order N when the following termination criterion

$$|y_n(z)| \geq \text{STOPR} \quad (241)$$

is met. Here STOPR is a number, say 1.0D15. The user's needs will determine whether or not STOPR should retain its presently suggested value. Our own demands were satisfactorily met for complex argument, $\epsilon_p r$, of the spherical Neumann functions for parameter ranges:

$$1.5 \leq |\epsilon_p| \leq 1390.0 \quad \text{and} \quad 0.1 \leq r \leq 10 \text{ cm.}$$

The backward recurrence relation

$$j_{n-1}(z) = \frac{2n+1}{z} j_n(z) - j_{n+1}(z) \quad (242)$$

in combination with an appropriate starting value is used to generate values of the spherical Bessel functions of the first kind, $j_n(z)$. This technique of using the backward relation in place of the forward relation helps to avoid stability problems.

PROGRAM DESCRIPTION

Written in standard FORTRAN IV for the IBM 360/65 system, the Concentric Spherical Model (CSM) is designed to calculate the internal absorbed-power density distributions, average absorbed-power density, and total absorbed power for a spherical shell configuration (simulating the human head) subjected to plane-wave, nonionizing electromagnetic radiation. Five spherical shells plus a brain core sphere are generally treated, but provision is made to allow as many as eight concentric shells to be analyzed. The structural components of the head model are identified by regional designators. Regions 1 through 6 represent the brain, cerebrospinal fluid (CSF), dura, bone, subcutaneous fat, and skin, respectively. Current plotting needs of the USAF School of Aerospace Medicine are met through the use of BGNSTP--an advanced general plotting subroutine package--and the use of the CalComp Model 936 digital incremental plotter. Since the plotting package is not available for distribution, the plotter calls and working arrays are not included in this published version of the program.

Basically, program CSM consists of a driver routine, five sub-routine subprograms, and one function subprogram. These routines are single-entry programs, free of any special machine dependence, and utilize subprograms found in any elementary software library. Types REAL*8 and COMPLEX*16 signify double-precision real and complex variables, respectively, and literal data appearing in a FORMAT statement are enclosed in apostrophes. The arithmetical processes are performed in double-precision, floating-point mode. This feature provides approximately 16.8 decimal digits of precision and numbers with an exponent range of -78 to +75. A list of the driver routine and subprograms, including function, calling sequence, and calling arguments of each member, follows.

Driver routine:

Routine MAIN is used to input/output data; to compute complex propagation constants; to complete the calculations for the absorbed-power density distributions, average absorbed-power density, and total absorbed power; to control the printing activities; and to direct the course of calculations.

Subroutine subprograms:

Subroutine COEF generates the expansion coefficients for the components of the electric-field vectors $\vec{E}_p$, $p = 1, \dots, \text{NOREG}$.

The calling sequence of this subroutine is

COEF(ANP, BNP, ALPNP, BETNP, NMIN)

where the calling arguments are

ANP = array of coefficients for vector functions
 $\vec{M}_{(1,n)}^{(0,1)}$,

BNP = array of coefficients for vector functions
 $\vec{N}_{(1,n)}^{(e,1)}$,

ALPNP = array of coefficients for vector functions

$$\vec{M}_{(1,n)}^{(0,3)},$$

BETNP = array of coefficients for vector functions

$$\vec{N}_{(1,n)}^{(e,3)},$$

NMIN = number of terms in the series expansion of
each component of the electric-field vector,
 $\vec{E}_p$.

The above arrays are double-precision, complex, and each array is dimensioned at 1000.

Subroutine EVEC computes the radial, colatitude, and azimuthal components and the scalar product $\vec{E}_p \cdot \vec{E}_p^*$ for the electric-field vectors $\vec{E}_p, p = 1, \dots, \text{NOREG}$.

The calling sequence of this subroutine is

EVEC(NP,PD)

where the calling arguments are

NP = region identifier,

PD = double-precision, complex, semicompleted
absorbed-power density at an internal point
of the p-th region.

Subroutine TERM computes $(-1)^n$ or $(-1)^{n+1}$ times the appropriate part of the n-th term in the series expansion of each component of the electric-field vectors $\vec{E}_p, p = 1, \dots, \text{NOREG}$.

The calling sequence of this subroutine is

TERM(NCK, T, KEY)

where the calling arguments are

NCK = a counter count,

T = part of the n-th term in the series expansion that is multiplied by the appropriate power of -1,

KEY = 0 for T to be multiplied by $(-1)^n$ and 1 for T to be multiplied by $(-1)^{n+1}$.

The array T is double-precision, complex.

Subroutine BJYH generates the spherical Bessel functions $j_n(k_p r)$, spherical Neumann functions $y_n(k_p r)$, and spherical Hankel functions $h_n^{(1)}(k_p r)$.

The calling sequence of this subroutine is

BJYH(BJNP, BHNP, Z, NN, STOPR)

where the calling arguments are

BJNP = array of spherical Bessel functions for p-th region,

BHNP = array of spherical Hankel functions for the p-th region,

Z = product of complex propagation constant and radius of an internal point or boundary surface of the p-th region,

NN = maximum order of the spherical functions,

STOPR = a test quantity for terminating the generation of the spherical Neumann functions.

The arrays BJNP and BHNP are double-precision, complex, and each array is dimensioned at 100. Variable Z is double-precision, complex.

Subroutine PL generates the associated Legendre functions $P_n^1(\cos\theta)$ and their first derivatives with respect to θ .

The calling sequence for this subroutine is

PL(THETA, N, P, DP)

where the calling arguments are

THETA = value of the colatitude angle expressed
in radians,
N = number of associated Legendre functions
to be generated, starting with the func-
tion of degree one,
P = array of values of the associated Legendre
functions,
DP = array of values of the first derivative
of the associated Legendre functions.

The arrays P and DP are double-precision, real and are dimensioned at 101 and 100, respectively. THETA is a double-precision, real variable.

Function subprogram MINN determines the minimum value of a given array of positive integers.

The calling sequence for this function subprogram is

MINN(NRAY,N)

where the calling arguments are

NRAY = array of positive integers,
N = number of integers.

The array NRAY is single-precision, integer, and dimensioned at 10.

Blank COMMON is used by the driver routine, MAIN, and the sub-routines COEF and EVEC. The list of the arrays and variables stored in this area is

FKP^{*} = wave propagation constants, k_p ,
BJNP^{*} = spherical Bessel functions, $j_n(k_p r)$,
BHNP^{*} = spherical Hankel functions, $h_n^{(1)}(k_p r)$,
CEX^{*} = exponential value, $\exp(-i\omega t)$, for circular frequency ω and time t ,
BDP = spherical surface boundaries,
P = associated Legendre functions, $P_n^1(\cos\theta)$,
DP = first derivative of the associated Legendre functions, $\frac{d}{d\theta} P_n^1(\cos\theta)$,
SIGP = conductivities, σ_p ,
EO = intensity of the incident electric field, $\vec{E}$,
TIME = time,
R = radius of an internal point of the brain core sphere, or spherical shell region, or the radius of a spherical boundary surface,
THETA = colatitude angle, θ ,
PHI = azimuthal angle, ϕ ,
STOPR = a test quantity for terminating the generation of the spherical Neumann functions, $y_n(k_p r)$,
NC = maximum order of the spherical functions minus 2,

The array NRAY is single-precision, integer, and dimensioned at 10.

Blank COMMON is used by the driver routine, MAIN, and the sub-routines COEF and EVEC. The list of the arrays and variables stored in this area is

FKP^{*} = wave propagation constants, k_p ,
BJNP^{*} = spherical Bessel functions, $j_n(k_p r)$,
BHNP^{*} = spherical Hankel functions, $h_n^{(1)}(k_p r)$,
CEX^{*} = exponential value, $\exp(-i\omega t)$, for circular frequency ω and time t ,
BDP = spherical surface boundaries,
P = associated Legendre functions, $P_n^1(\cos\theta)$,
DP = first derivative of the associated Legendre functions, $\frac{d}{d\theta} P_n^1(\cos\theta)$,
SIGP = conductivities, σ_p ,
EO = intensity of the incident electric field, $\vec{E}$,
TIME = time,
R = radius of an internal point of the brain core sphere, or spherical shell region, or the radius of a spherical boundary surface,
THETA = colatitude angle, θ ,
PHI = azimuthal angle, ϕ ,
STOPR = a test quantity for terminating the generation of the spherical Neumann functions, $y_n(k_p r)$,
NC = maximum order of the spherical functions minus 2,

NOREG = number of regions in the Concentric Spherical Model,

NMIN = number of terms in the series expansions of the components of the electric-field vector, $\vec{E}_p$.

The double-precision, complex arrays and variables are flagged with an asterisk (*); while the unflagged arrays and variables are double-precision, real--with the exception of the last three members, of type INTEGER, which are single-precision variables.

A single-labeled common area, COEF, is used by the driver routine, MAIN, and the subroutine EVEC, for values of the expansion coefficients $a_{(n,p)}$, $b_{(n,p)}$, $\alpha_{(n,p)}$, $\beta_{(n,p)}$ stored in the arrays ANP, BNP, ALPNP, and BETNP respectively.

In subroutine BJYH, if variable M, the maximum order of the spherical Neumann functions $y_n(k_p r)$ (complex $k_p r$), tests ≤ 2 , the error message

PROCESS CAN NOT PROCEED SINCE NM2 = 0 FOR Z = ...
is printed out and the computer run is terminated.

Input to program CSM is by keypunched cards. There are two basic input cards with structure and sequential order as follows:

Card No. 1 (control parameters)

Columns: 1-10 FREQ. Frequency in MHz. (E10.3)
11-20 E0. Intensity (field strength) of the incident electric field in volt/meter. (E10.3)
21-30 TIME. Time in seconds. (E10.3)
31-40 STOPR. A test quantity for terminating the generation of the spherical Neumann functions. A suggested value is 1.0E15. (E10.3)
41-45 NORG. Number of regions in the concentric spherical model of the human or animal head. (I5)
46-50 NOCR. Number of cases. (I5)

Card No. 2 (electrical property data)

Columns: 1-10 EPSP(1). Relative dielectric constant for region 1. (E10.3)
11-20 SIGP(1). Conductivity for region 1 in mho/meter. (E10.3)
21-30 EPSP(2). Relative dielectric constant for region 2. (E10.3)
31-40 SIGP(2). Conductivity for region 2 in mho/meter. (E10.3)
41-50 . . .
51-60 . . .
61-70 EPSP(4). Relative dielectric constant for region 4. (E10.3)
71-80 SIGP(4). Conductivity for region 4 in mho/meter. (E10.3)

Card 3 is a similarly structured card for the electrical properties of regions 5 and 6.

Card No. 4 (surface boundary data)

Columns: 1-10 SBDP(1). Radius of the spherical surface for region 1 in centimeters. (E10.3)
11-20 SBDP(2). Radius of the spherical surface for region 2 in centimeters. (E10.3)
21-30 . . .
31-40 . . .
41-50 . . .
51-60 SBDP(6). Radius of the spherical surface for region 6 in centimeters. (E10.3)

Card Nos. 5-(NOCR+4) (coordinate data)

Columns: 1-5 NREG. Region number. (I5)
6-15 R. Radial spherical coordinate of an interior point of region NREG in centimeters.
Range: $0 < R \leq \text{SBDP}(6)$. (E10.3)

- 16-25 THETAD. Colatitude spherical coordinate of an interior point of region NREG in degrees.
Range: $0 \leq \text{THETAD} \leq 180$. (E10.3)
- 26-35 PHID. Azimuthal spherical coordinate of an interior point of region NREG in degrees.
Range: $0 \leq \text{PHID} \leq 360$. (E10.3)

The last card of a single data set must be a termination card with the symbols /* punched in columns 1 and 2. Also program CSM can handle multiple data sets. Each data set [Cards 1-(NOCR+4)] is stacked one behind the other, with the last card in the complete data deck a termination card.

Program printouts consist of the title

ELECTROMAGNETIC ENERGY DEPOSITION IN A CONCENTRIC SPHERICAL MODEL
OF THE HUMAN OR ANIMAL HEAD

followed by such information as

FREQUENCY = MHZ

FIELD STRENGTH = . . . V/M

TIME = . . . SEC

NUMBER OF REGIONS = . . .

RELATIVE DIELECTRIC CONSTANTS = . . .

CONDUCTIVITIES (MHO/M) = . . .

SURFACE BOUNDARIES (CM) = . . .

REGION . . . INTERIOR POINT: RADIUS = . . . CM THETA = . . . DEG

PHI = . . . DEG ABSORBED-POWER DENSITY = . . . W/M**3

REGION . . . INTERIOR POINT: RADIUS = . . . CM THETA = . . . DEG

PHI = . . . DEG ABSORBED-POWER DENSITY = . . . W/M**3

.

AVERAGE ABSORBED-POWER DENSITY = . . . W/M**3

TOTAL ABSORBED POWER = . . . WATT

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APPENDIX A
SAMPLE PROBLEM WITH COMPUTER RESULTS

SAMPLE PROBLEM DECK SETUP

CARD 1 (CONTROL PARAMETERS: PREQ,EO,TIME,STOPF,NOFFG,NOCP)
 1000.0+0 1.0+0 0.0+0 1.0+15 6 20

CARDS 2-5 (ELECTRICAL PROPERTIES: EPSP(I),SIGP(I))
 60.0+0 0.5+0 76.0+0 1.7+0 45.0+0 1.0+0
 5.5+0 0.0+0 45.0+0 1.0+0 9.5+0 0.11+0

CARD 4 (RADII OF SECONDARY SURFACES: SRDP(I))
 5.27+0 5.1+0 5.52+0 5.80+0 5.90+0 6.00+0

CARDS 5-24 (INTERFACIAL POINTS: NREG,P,ZHEAT,PHED)
 1 1.00+0 160.0+0 0.0+0
 1 0.25+0 170.0+0 0.0+0
 1 0.50+0 160.0+0 0.0+0
 1 0.75+0 150.0+0 0.0+0
 1 1.00+0 145.0+0 0.0+0
 1 1.25+0 140.0+0 0.0+0
 1 1.50+0 130.0+0 0.0+0
 1 1.75+0 120.0+0 0.0+0
 1 2.00+0 110.0+0 0.0+0
 1 2.25+0 100.0+0 0.0+0
 1 2.50+0 90.0+0 0.0+0
 1 2.75+0 80.0+0 0.0+0
 1 3.0+0 70.0+0 0.0+0
 1 5.27+0 60.0+0 0.0+0
 2 5.47+0 50.0+0 0.0+0
 3 5.52+0 40.0+0 0.0+0
 4 5.60+0 30.0+0 0.0+0
 4 5.80+0 20.0+0 0.0+0
 5 5.90+0 10.0+0 0.0+0
 6 6.00+0 0.0+0 0.0+0

TERMINATION CARD

/*

ELECTROMAGNETIC ENERGY DEPOSITION IN A CONCENTRIC SPHERICAL MODEL OF THE HUMAN OF ANIMAL HEAD

FREQUENCY = 1000.00 MHZ	FIELD STRENGTH = 1.00 V/M	TIME = 0.0 SEC	NUMBER OF REGIONS = 6
RELATIVE DIELECTRIC CONSTANT = 60.00	76.00	45.00	8.50
CONDUCTIVITIES (MHO/M) = 0.900	1.700	1.000	0.080
CAPACITANCE BOUNDARIES (CM) = 5.270	5.470	5.800	5.900
			6.000

REGION 1	INTERIOR POINT: RADIUS = 0.001 CM	THETA = 180.00 DEG	PHI = 0.0 DEG	ABSORBED POWER DENSITY = 0.13498334 W/M**2
REGION 1	INTERIOR POINT: RADIUS = 0.250 CM <td>THETA = 170.00 DEG <td>PHI = 0.0 DEG <td>ABSORBED POWER DENSITY = 0.12557081 W/M**2</td> </td></td>	THETA = 170.00 DEG <td>PHI = 0.0 DEG <td>ABSORBED POWER DENSITY = 0.12557081 W/M**2</td> </td>	PHI = 0.0 DEG <td>ABSORBED POWER DENSITY = 0.12557081 W/M**2</td>	ABSORBED POWER DENSITY = 0.12557081 W/M**2
REGION 1	INTERIOR POINT: RADIUS = 0.500 CM <td>THETA = 160.00 DEG <td>PHI = 0.0 DEG <td>ABSORBED POWER DENSITY = 0.10592942 W/M**2</td> </td></td>	THETA = 160.00 DEG <td>PHI = 0.0 DEG <td>ABSORBED POWER DENSITY = 0.10592942 W/M**2</td> </td>	PHI = 0.0 DEG <td>ABSORBED POWER DENSITY = 0.10592942 W/M**2</td>	ABSORBED POWER DENSITY = 0.10592942 W/M**2
REGION 1	INTERIOR POINT: RADIUS = 0.750 CM <td>THETA = 150.00 DEG <td>PHI = 0.0 DEG <td>ABSORBED POWER DENSITY = 0.08574595 W/M**2</td> </td></td>	THETA = 150.00 DEG <td>PHI = 0.0 DEG <td>ABSORBED POWER DENSITY = 0.08574595 W/M**2</td> </td>	PHI = 0.0 DEG <td>ABSORBED POWER DENSITY = 0.08574595 W/M**2</td>	ABSORBED POWER DENSITY = 0.08574595 W/M**2
REGION 1	INTERIOR POINT: RADIUS = 1.000 CM <td>THETA = 145.00 DEG <td>PHI = 0.0 DEG <td>ABSORBED POWER DENSITY = 0.06545893 W/M**2</td> </td></td>	THETA = 145.00 DEG <td>PHI = 0.0 DEG <td>ABSORBED POWER DENSITY = 0.06545893 W/M**2</td> </td>	PHI = 0.0 DEG <td>ABSORBED POWER DENSITY = 0.06545893 W/M**2</td>	ABSORBED POWER DENSITY = 0.06545893 W/M**2
REGION 1	INTERIOR POINT: RADIUS = 1.250 CM <td>THETA = 140.00 DEG <td>PHI = 0.0 DEG <td>ABSORBED POWER DENSITY = 0.04931756 W/M**2</td> </td></td>	THETA = 140.00 DEG <td>PHI = 0.0 DEG <td>ABSORBED POWER DENSITY = 0.04931756 W/M**2</td> </td>	PHI = 0.0 DEG <td>ABSORBED POWER DENSITY = 0.04931756 W/M**2</td>	ABSORBED POWER DENSITY = 0.04931756 W/M**2
REGION 1	INTERIOR POINT: RADIUS = 1.500 CM <td>THETA = 130.00 DEG <td>PHI = 0.0 DEG <td>ABSORBED POWER DENSITY = 0.04162231 W/M**2</td> </td></td>	THETA = 130.00 DEG <td>PHI = 0.0 DEG <td>ABSORBED POWER DENSITY = 0.04162231 W/M**2</td> </td>	PHI = 0.0 DEG <td>ABSORBED POWER DENSITY = 0.04162231 W/M**2</td>	ABSORBED POWER DENSITY = 0.04162231 W/M**2
REGION 1	INTERIOR POINT: RADIUS = 1.750 CM <td>THETA = 120.00 DEG <td>PHI = 0.0 DEG <td>ABSORBED POWER DENSITY = 0.03448622 W/M**2</td> </td></td>	THETA = 120.00 DEG <td>PHI = 0.0 DEG <td>ABSORBED POWER DENSITY = 0.03448622 W/M**2</td> </td>	PHI = 0.0 DEG <td>ABSORBED POWER DENSITY = 0.03448622 W/M**2</td>	ABSORBED POWER DENSITY = 0.03448622 W/M**2
REGION 1	INTERIOR POINT: RADIUS = 2.000 CM <td>THETA = 110.00 DEG <td>PHI = 0.0 DEG <td>ABSORBED POWER DENSITY = 0.02676280 W/M**2</td> </td></td>	THETA = 110.00 DEG <td>PHI = 0.0 DEG <td>ABSORBED POWER DENSITY = 0.02676280 W/M**2</td> </td>	PHI = 0.0 DEG <td>ABSORBED POWER DENSITY = 0.02676280 W/M**2</td>	ABSORBED POWER DENSITY = 0.02676280 W/M**2
REGION 1	INTERIOR POINT: RADIUS = 2.250 CM <td>THETA = 100.00 DEG <td>PHI = 0.0 DEG <td>ABSORBED POWER DENSITY = 0.01938545 W/M**2</td> </td></td>	THETA = 100.00 DEG <td>PHI = 0.0 DEG <td>ABSORBED POWER DENSITY = 0.01938545 W/M**2</td> </td>	PHI = 0.0 DEG <td>ABSORBED POWER DENSITY = 0.01938545 W/M**2</td>	ABSORBED POWER DENSITY = 0.01938545 W/M**2
REGION 1	INTERIOR POINT: RADIUS = 2.500 CM <td>THETA = 90.00 DEG <td>PHI = 0.0 DEG <td>ABSORBED POWER DENSITY = 0.01274409 W/M**2</td> </td></td>	THETA = 90.00 DEG <td>PHI = 0.0 DEG <td>ABSORBED POWER DENSITY = 0.01274409 W/M**2</td> </td>	PHI = 0.0 DEG <td>ABSORBED POWER DENSITY = 0.01274409 W/M**2</td>	ABSORBED POWER DENSITY = 0.01274409 W/M**2
REGION 1	INTERIOR POINT: RADIUS = 2.750 CM <td>THETA = 80.00 DEG <td>PHI = 0.0 DEG <td>ABSORBED POWER DENSITY = 0.00832354 W/M**2</td> </td></td>	THETA = 80.00 DEG <td>PHI = 0.0 DEG <td>ABSORBED POWER DENSITY = 0.00832354 W/M**2</td> </td>	PHI = 0.0 DEG <td>ABSORBED POWER DENSITY = 0.00832354 W/M**2</td>	ABSORBED POWER DENSITY = 0.00832354 W/M**2
REGION 1	INTERIOR POINT: RADIUS = 3.000 CM <td>THETA = 70.00 DEG <td>PHI = 0.0 DEG <td>ABSORBED POWER DENSITY = 0.00649491 W/M**2</td> </td></td>	THETA = 70.00 DEG <td>PHI = 0.0 DEG <td>ABSORBED POWER DENSITY = 0.00649491 W/M**2</td> </td>	PHI = 0.0 DEG <td>ABSORBED POWER DENSITY = 0.00649491 W/M**2</td>	ABSORBED POWER DENSITY = 0.00649491 W/M**2
REGION 2	INTERIOR POINT: RADIUS = 5.470 CM <td>THETA = 50.00 DEG <td>PHI = 0.0 DEG <td>ABSORBED POWER DENSITY = 0.00779011 W/M**2</td> </td></td>	THETA = 50.00 DEG <td>PHI = 0.0 DEG <td>ABSORBED POWER DENSITY = 0.00779011 W/M**2</td> </td>	PHI = 0.0 DEG <td>ABSORBED POWER DENSITY = 0.00779011 W/M**2</td>	ABSORBED POWER DENSITY = 0.00779011 W/M**2
REGION 3	INTERIOR POINT: RADIUS = 5.520 CM <td>THETA = 40.00 DEG <td>PHI = 0.0 DEG <td>ABSORBED POWER DENSITY = 0.00439393 W/M**2</td> </td></td>	THETA = 40.00 DEG <td>PHI = 0.0 DEG <td>ABSORBED POWER DENSITY = 0.00439393 W/M**2</td> </td>	PHI = 0.0 DEG <td>ABSORBED POWER DENSITY = 0.00439393 W/M**2</td>	ABSORBED POWER DENSITY = 0.00439393 W/M**2
REGION 4	INTERIOR POINT: RADIUS = 5.600 CM <td>THETA = 30.00 DEG <td>PHI = 0.0 DEG <td>ABSORBED POWER DENSITY = 0.00432350 W/M**2</td> </td></td>	THETA = 30.00 DEG <td>PHI = 0.0 DEG <td>ABSORBED POWER DENSITY = 0.00432350 W/M**2</td> </td>	PHI = 0.0 DEG <td>ABSORBED POWER DENSITY = 0.00432350 W/M**2</td>	ABSORBED POWER DENSITY = 0.00432350 W/M**2
REGION 5	INTERIOR POINT: RADIUS = 5.800 CM <td>THETA = 20.00 DEG <td>PHI = 0.0 DEG <td>ABSORBED POWER DENSITY = 0.00222757 W/M**2</td> </td></td>	THETA = 20.00 DEG <td>PHI = 0.0 DEG <td>ABSORBED POWER DENSITY = 0.00222757 W/M**2</td> </td>	PHI = 0.0 DEG <td>ABSORBED POWER DENSITY = 0.00222757 W/M**2</td>	ABSORBED POWER DENSITY = 0.00222757 W/M**2
REGION 6	INTERIOR POINT: RADIUS = 5.900 CM <td>THETA = 10.00 DEG <td>PHI = 0.0 DEG <td>ABSORBED POWER DENSITY = 0.00117700 W/M**2</td> </td></td>	THETA = 10.00 DEG <td>PHI = 0.0 DEG <td>ABSORBED POWER DENSITY = 0.00117700 W/M**2</td> </td>	PHI = 0.0 DEG <td>ABSORBED POWER DENSITY = 0.00117700 W/M**2</td>	ABSORBED POWER DENSITY = 0.00117700 W/M**2
REGION 6	INTERIOR POINT: RADIUS = 5.000 CM <td>THETA = 0.0 DEG <td>PHI = 0.0 DEG <td>ABSORBED POWER DENSITY = 0.00693344 W/M**2</td> </td></td>	THETA = 0.0 DEG <td>PHI = 0.0 DEG <td>ABSORBED POWER DENSITY = 0.00693344 W/M**2</td> </td>	PHI = 0.0 DEG <td>ABSORBED POWER DENSITY = 0.00693344 W/M**2</td>	ABSORBED POWER DENSITY = 0.00693344 W/M**2

AVERAGE ABSORBED-POWER DENSITY = 1.60618D-02 W/M**2

TOTAL ABSORBED POWER = 1.45324D-05 WATT

APPROXIMATE EXAGGERATION TEL. = 0.05 CPU MINUTE

APPENDIX B
SOURCE LISTING OF PROGRAM CSM

C	PROGRAM CSM	CSM0001
C	ELECTROMAGNETIC ENERGY DEPOSITION IN A CONCENTRIC	CSM0002
C	SPHERICAL MODEL OF THE HUMAN OR ANIMAL HEAD	CSM0003
C		CSM0004
	IMPLICIT REAL*8 (A-H,O-Z)	CSM0005
	COMMON /COEFF/ANP,BNP,ALPNP,BETNP	CSM0006
	COMMON FKP,BJNP,BHNP,CEX,BDP,P,DP,SIGP,EO,TIME,R,THETA,PHI,STOPR,	CSM0007
	INC,NORG,NMIN	CSM0008
	DIMENSION BDP(9),SBDP(9),EPSP(9),SIGP(9),P(101),DP(100)	CSM0009
	COMPLEX*16 FKP(10),CEX,ANP(1000),BNP(1000),ALPNP(1000),	CSM0010
	1BETNP(1000),BJNP(100),BHNP(100),Z	CSM0011
	CALL ERASESET(208,0,-1,1)	CSM0012
	PIE=3.141592653589793D0	CSM0013
	RAD=180.D0/PIE	CSM0014
	EPSO=8.85416D-12	CSM0015
	VEL=2.997924562D8	CSM0016
C ***	READ CONTROL PARAMETERS	CSM0017
	5 READ (5,10,END=110) FREQ,EO,TIME,STOPR,NORG,NOCP	CSM0018
	10 FORMAT(4E10.0,2I5)	CSM0019
C ***	COMPUTE COMPLEX TIME VARIATION	CSM0020
	OMEGA=2.D6*PIE*FREQ	CSM0021
	ARG=-OMEGA*TIME	CSM0022
	CEX=DCMPLX(DCOS(ARG),DSIN(ARG))	CSM0023
C ***	READ DIELECTRIC PROPERTY PARAMETERS	CSM0024
	READ(5,20) (EPSP(I),SIGP(I),I=1,NORG)	CSM0025
	20 FORMAT(8E10.0)	CSM0026
C ***	COMPUTE COMPLEX PROPAGATION CONSTANTS	CSM0027
	FAC1=OMEGA/VEL	CSM0028
	DO 30 I=1,NORG	CSM0029
	FAC2=EPSP(I)/2.D0	CSM0030
	FAC3=DSQRT(1.D0+(1.D0/(EPSO*OMEGA)**2)*(SIGP(I)/EPSP(I))**2)	CSM0031
	REKP=FAC1*DSQRT(FAC2*(FAC3+1.D0))	CSM0032
	FIMKP=FAC1*DSQRT(FAC2*(FAC3-1.D0))	CSM0033
	FKP(I)=DCMPLX(REKP,FIMKP)	CSM0034
	30 CONTINUE	CSM0035
	FKP(NORG+1)=DCMPLX(FAC1,0.D0)	CSM0036
C ***	READ RADII OF SURFACE BOUNDARIES	CSM0037
	READ(5,20) (SBDP(I),I=1,NORG)	CSM0038
	DO 35 I=1,NORG	CSM0039
	BDP(I)=SBDP(I)/1.D2	CSM0040
	35 CONTINUE	CSM0041
C ***	PRINT OUT TITLE AND BASIC INPUT DATA	CSM0042
	WRITE(6,40) FREQ,EO,TIME,NORG	CSM0043
	40 FORMAT('1ELECTROMAGNETIC ENERGY DEPOSITION IN A CONCENTRIC SPHERICCSM0044	
	1AL MODEL OF THE HUMAN OR ANIMAL HEAD'/'-FREQUENCY =',F9.2,' MHZ	CSM0045
	2 FIELD STRENGTH =',F7.2,' V/M TIME =',F7.2,' SEC NUMBEC	CSM0046
	3P OF REGIONS =',I3)	CSM0047
	WRITE(6,41) (EPSP(I),I=1,NORG)	CSM0048
	41 FORMAT('0RELATIVE DIELECTRIC CONSTANTS =',9(F7.2,2X))	CSM0049
	WRITE(6,42) (SIGP(I),I=1,NORG)	CSM0050
	42 FORMAT('0CONDUCTIVITIES (MHO/M) =',9(F7.3,2X))	CSM0051
	WRITE(6,43) (SBDP(I),I=1,NORG)	CSM0052
	43 FORMAT('0SURFACE BOUNDARIES (CM) =',9(F7.3,2X))	CSM0053
C ***	COMPUTE SERIES EXPANSION COEFFICIENTS FOR ELECTRIC	CSM0054
C ***	FIELDS	CSM0055
	CALL COEF(ANP,BNP,ALPNP,BETNP,NMIN)	CSM0056
	WRITE(6,45)	CSM0057
	45 FORMAT('0')	CSM0058
	DO 70 I=1,NOCP	CSM0059
C ***	READ DEFINING CHARACTERISTICS OF INTERIOR POINTS AT	CSM0060

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C ***      WHICH ABSORBED-POWER DENSITIES ARE TO BE COMPUTED      CSM0061
      READ(5,50) NREG,R,THETAD,PHID      CSM0062
50  FORMAT(I5,3E10.3)      CSM0063
      SAVR=R      CSM0064
      P=F/1.D2      CSM0065
      THETA=THETAD/RAD      CSM0066
      PHI=PHID/RAD      CSM0067
      Z=FKP(NREG)*R      CSM0068
      CALL BJYH(BJNP,BHNP,Z,NC,STOPR)      CSM0069
      NC=NC+2      CSM0070
      IF(NC.GT.NMIN) NC=NMIN      CSM0071
      CALL PL(THETA,NC,P,DP)      CSM0072
C ***      ABSORBED-POWER DENSITY AT GIVEN POINT INTERIOR TO P-TH REGION CSM0073
      CALL EVEC(NREG,PD)      CSM0074
      PD=.5D0*SIGP(NREG)*PD      CSM0075
C ***      PRINT OUT PARTICULARS OF INTERIOR POINT OF REGION P      CSM0076
      WRITE(6,60) NREG,SAVR,THETAD,PHID,PD      CSM0077
60  FORMAT(' REGION',I2,' INTERIOR POINT: RADIUS =',F8.3,' CM THETA = CSM0078
      1',F7.2,' DEG PHI =',F7.2,' DEG ABSORBED POWER DENSITY =',F12.8,' CSM0079
      2 W/M**3')      CSM0080
70  CONTINUE      CSM0081
      NN=NORG*NMIN      CSM0082
      FAC=2.D0*PIE/(FAC1*FAC1)      CSM0083
      QS=0.D0      CSM0084
      QT=0.D0      CSM0085
      DO 90 N=1,NMIN      CSM0086
      FACN=2.D0*N+1.D0      CSM0087
      QT=QT+FACN*DREAL(ALPNP(NN+N)+BETNP(NN+N))      CSM0088
      QS=QS+FACN*(CDABS(ALPNP(NN+N))**2+CDABS(BETNP(NN+N))**2)      CSM0089
90  CONTINUE      CSM0090
      QA=FAC*(DABS(QT)-QS)      CSM0091
C ***      TOTAL ABSORBED POWER      CSM0092
      TOTPOW=2.65441D-3*EO**2*QA/2.D0      CSM0093
C ***      AVERAGE ABSORBED-POWER DENSITY      CSM0094
      PAVG=TOTPOW/(4.D0*PIE*BDP(NORG)**3/3.D0)      CSM0095
C ***      PRINT OUT AVERAGE ABSORBED-POWER DENSITY AND TOTAL ABSORBED CSM0096
C ***      POWER      CSM0097
      WRITE(6,100) PAVG,TOTPOW      CSM0098
100 FORMAT('0',9X,'AVERAGE ABSORBED-POWER DENSITY =',1PD13.5,' W/M**3' CSM0099
      1/'0',9X,'TOTAL ABSORBED POWER =',D13.5,' WATT')      CSM0100
      GO TO 5      CSM0101
110 STOP      CSM0102
      END      CSM0103
      SUBROUTINE COEF(ANP,BNP,ALPNP,BETNP,NMIN)      CSM0104
C      GENERATES EXPANSION COEFFICIENTS      CSM0105
      IMPLICIT REAL*8 (A-H,O-Z)      CSM0106
      COMMON FKP,BJNP,BHNP,CEX,BDP,P,DP,SIGP,EO,TIME,R,THETA,PHI,STOPR, CSM0107
      1NC,NORG      CSM0108
      DIMENSION NTER(10),BDP(9),SIGP(9),P(101),DP(100)      CSM0109
      COMPLEX*16 ANP(1000),BNP(1000),ALPNP(1000),BETNP(1000),BJHP1(1000) CSM0110
      1,BJHP2(1000),BJNP(100),BHNP(100),SJNP1(100),DELNP,SNT11,      CSM0111
      2SNT12,SNT21,SNT22,TNT11,TNT12,TNT21,TNT22,ETAP1,ETAP2,ZEP1,ZEP2, CSM0112
      3SNP11,SNP12,SNP21,SNP22,TNP11,TNP12,TNP21,TNP22,DEL1,DEL2,FKP(10), CSM0113
      4CEX,FATIO,SHNP1(100),Z      CSM0114
C      COMPUTE EXPANSION COEFFICIENTS AN1,BN1,ANN,BNN,ALPN1,BETN1,      CSM0115
C      ALPNN,BETNN      CSM0116
      N1=0      CSM0117
      N2=0      CSM0118
      DO 15 NR=1,NORG      CSM0119
      Z=FKP(NR)*BDP(NR)      CSM0120

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CALL BJYH(BJNP,BHNP,Z,N,STOPR)	CSM0121
DO 5 I=1,N	CSM0122
SJNP1(I)=BJNP(I)	CSM0123
5 SHNP1(I)=BHNP(I)	CSM0124
Z=FKP(NR+1)*BDP(NR)	CSM0125
CALL BJYH(BJNP,BHNP,Z,NN,STOPR)	CSM0126
NMIN=MINO(N,NN)	CSM0127
NTER(NR)=NMIN	CSM0128
N2=N2+NMIN	CSM0129
DO 10 I=1,NMIN	CSM0130
BJHP1(N1+I)=SJNP1(I)	CSM0131
BJHP1(N2+I)=SHNP1(I)	CSM0132
BJHP2(N1+I)=BJNP(I)	CSM0133
BJHP2(N2+I)=BHNP(I)	CSM0134
10 CONTINUE	CSM0135
N1=N1+2*NMIN	CSM0136
N2=N2+NMIN	CSM0137
15 CONTINUE	CSM0138
NMIN=MINN(NTER,NORG)	CSM0139
NMIN=NMIN-2	CSM0140
DO 17 I=1,NMIN	CSM0141
ALPNP(I)=DCMPLX(0.D0,0.D0)	CSM0142
17 BETNP(I)=DCMPLX(0.D0,0.D0)	CSM0143
NSUM=NORG*NMIN	CSM0144
DO 30 I=1,NMIN	CSM0145
JJ=0	CSM0146
KK=0	CSM0147
II1=I+1	CSM0148
II2=2*I+1	CSM0149
SNT11=DCMPLX(1.D0,0.D0)	CSM0150
SNT12=DCMPLX(0.D0,0.D0)	CSM0151
SNT21=SNT12	CSM0152
SNT22=SNT11	CSM0153
TNT11=SNT11	CSM0154
TNT12=SNT12	CSM0155
TNT21=SNT12	CSM0156
TNT22=SNT11	CSM0157
DO 27 J=1,NORG	CSM0158
KK=KK+NTER(J)	CSM0159
ETAP1=(II1*BJHP1(JJ+I)-I*BJHP1(JJ+I+2))/II2	CSM0160
ETAP2=(II1*BJHP2(JJ+I)-I*BJHP2(JJ+I+2))/II2	CSM0161
ZEP1=(II1*BJHP1(KK+I)-I*BJHP1(KK+I+2))/II2	CSM0162
ZEP2=(II1*BJHP2(KK+I)-I*BJHP2(KK+I+2))/II2	CSM0163
DELNP=BJHP1(JJ+I+1)*ZEP1-BJHP1(KK+I+1)*ETAP1	CSM0164
RATIO=FKP(J+1)/FKP(J)	CSM0165
SNP11=(ZEP1*BJHP2(JJ+I+1)-RATIO*BJHP1(KK+I+1)*ETAP2)/DELNP	CSM0166
SNP12=(ZEP1*BJHP2(KK+I+1)-RATIO*BJHP1(KK+I+1)*ZEP2)/DELNP	CSM0167
SNP21=(RATIO*BJHP1(JJ+I+1)*ETAP2-ETAP1*BJHP2(JJ+I+1))/DELNP	CSM0168
SNP22=(RATIO*BJHP1(JJ+I+1)*ZEP2-ETAP1*BJHP2(KK+I+1))/DELNP	CSM0169
Z=SNT11	CSM0170
SNT11=SNT11*SNP11+SNT12*SNP21	CSM0171
SNT12=Z*SNP12+SNT12*SNP22	CSM0172
Z=SNT21	CSM0173
SNT21=SNT21*SNP11+SNT22*SNP21	CSM0174
SNT22=Z*SNP12+SNT22*SNP22	CSM0175
TNP11=(RATIO*ZEP1*BJHP2(JJ+I+1)-BJHP1(KK+I+1)*ETAP2)/DELNP	CSM0176
TNP12=(RATIO*ZEP1*BJHP2(KK+I+1)-BJHP1(KK+I+1)*ZEP2)/DELNP	CSM0177
TNP21=(BJHP1(JJ+I+1)*ETAP2-RATIO*ETAP1*BJHP2(JJ+I+1))/DELNP	CSM0178
TNP22=(BJHP1(JJ+I+1)*ZEP2-RATIO*ETAP1*BJHP2(KK+I+1))/DELNP	CSM0179
Z=TNT11	CSM0180

TNT11=TNT11*TNP11+TNT12*TNP21	CSM0181
TNT12=Z*TNP12+TNT12*TNP22	CSM0182
Z=TNT21	CSM0183
TNT21=TNT21*TNP11+TNT22*TNP21	CSM0184
TNT22=Z *TNP12+TNT22*TNP22	CSM0185
JJ=JJ+2*NTER(J)	CSM0186
KK=KK+NTEF(J)	CSM0187
27 CONTINUE	CSM0188
ANP(I)=SNT11-(SNT12*SNT21)/SNT22	CSM0189
BNP(I)=TNT11-(TNT12*TNT21)/TNT22	CSM0190
LL=NSUM+I	CSM0191
ANP(LL)=DCMLPX(1.D0,0.D0)	CSM0192
BNP(LL)=DCMLPX(1.D0,0.D0)	CSM0193
ALPNP(LL)=-SNT21/SNT22	CSM0194
BETNP(LL)=-TNT21/TNT22	CSM0195
30 CONTINUE	CSM0196
IF (NORG.EQ.1) RETURN	CSM0197
COMPUTE EXPANSION COEFFICIENTS AN2,...,AN(N-1);BN2,...,	CSM0198
BN(N-1);ALPN2,...,ALPN(N-1);BETN2,...,BETN(N-1)	CSM0199
JJ=0	CSM0200
KK=0	CSM0201
MM1=0	CSM0202
MM2=NMIN	CSM0203
NRGM1=NORG-1	CSM0204
DO 45 J=1,NRGM1	CSM0205
KK=KK+NTER(J)	CSM0206
DO 40 I=1,NMIN	CSM0207
II1=I+1	CSM0208
II2=2*I+1	CSM0209
ETAP1=(II1*BJHP1(JJ+I)-I*BJHP1(JJ+I+2))/II2	CSM0210
ETAP2=(II1*BJHP2(JJ+I)-I*BJHP2(JJ+I+2))/II2	CSM0211
ZEP1=(II1*BJHP1(KK+I)-I*BJHP1(KK+I+2))/II2	CSM0212
ZEP2=(II1*BJHP2(KK+I)-I*BJHP2(KK+I+2))/II2	CSM0213
DELNP=BJHP1(JJ+I+1)*ZEP1-BJHP1(KK+I+1)*ETAP1	CSM0214
RATIO=FKP(J+1)/FKP(J)	CSM0215
SNP11=(ZEP1*BJHP2(JJ+I+1)-RATIO*BJHP1(KK+I+1)*ETAP2)/DELNP	CSM0216
SNP12=(ZEP1*BJHP2(KK+I+1)-RATIO*BJHP1(KK+I+1)*ZEP2)/DELNP	CSM0217
SNP21=(RATIO*BJHP1(JJ+I+1)*ETAP2-ETAP1*BJHP2(JJ+I+1))/DELNP	CSM0218
SNP22=(RATIO*BJHP1(JJ+I+1)*ZEP2-ETAP1*BJHP2(KK+I+1))/DELNP	CSM0219
DEL1=SNP11*SNP22-SNP12*SNP21	CSM0220
TNP11=(RATIO*ZEP1*BJHP2(JJ+I+1)-BJHP1(KK+I+1)*ETAP2)/DELNP	CSM0221
TNP12=(RATIO*ZEP1*BJHP2(KK+I+1)-BJHP1(KK+I+1)*ZEP2)/DELNP	CSM0222
TNP21=(BJHP1(JJ+I+1)*ETAP2-RATIO*ETAP1*BJHP2(JJ+I+1))/DELNP	CSM0223
TNP22=(BJHP1(JJ+I+1)*ZEP2-RATIO*ETAP1*BJHP2(KK+I+1))/DELNP	CSM0224
DEL2=TNP11*TNP22-TNP12*TNP21	CSM0225
NN1=MM1+I	CSM0226
NN2=MM2+I	CSM0227
ANP(NN2)=(ANP(NN1)*SNP22-ALPNP(NN1)*SNP12)/DEL1	CSM0228
BNP(NN2)=(BNP(NN1)*TNP22-BETNP(NN1)*TNP12)/DEL2	CSM0229
ALPNP(NN2)=(-ANP(NN1)*SNP21+ALPNP(NN1)*SNP11)/DEL1	CSM0230
BETNP(NN2)=(-BNP(NN1)*TNP21+BETNP(NN1)*TNP11)/DEL2	CSM0231
40 CONTINUE	CSM0232
JJ=JJ+2*NTER(J)	CSM0233
KK=KK+NTER(J)	CSM0234
MM1=MM1+NMIN	CSM0235
MM2=MM2+NMIN	CSM0236
45 CONTINUE	CSM0237
RETURN	CSM0238
END	CSM0239
SUBROUTINE EVEC(NP,PD)	CSM0240

AD-A085 082

SCHOOL OF AEROSPACE MEDICINE BROOKS AFB TX
ELECTROMAGNETIC ENERGY DEPOSITION IN A CONCENTRIC SPHERICAL MOD—ETC(U)
DEC 79 E L BELL, D K COHOON, J W PENN
SAM-TR-79-6

F/G 6/18

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C	COMPUTES THE RADIAL, COLATITUDE, AND AZIMUTHAL	CSM0241
C	COMPONENTS OF ELECTRIC FIELD VECTOR E FOR	CSM0242
C	REGION P AND SCALAR PRODUCT E.E*	CSM0243
	IMPLICIT REAL*8 (A-H,O-Z)	CSM0244
	COMMON /COEFF/ANP,BNP,ALPNP,BETNP	CSM0245
	COMMON PKP,BJNP,BHNP,CEX,BDP,P,DP,SIGP,EO,TIME,R,THETA,PHI,STOPR,	CSM0246
	1NC,NORG,NMIN	CSM0247
	DIMENSION BDP(9),SIGP(9),P(101),DP(100)	CSM0248
	COMPLEX*16 ERAD,ETHETA,EPHI,PKP(10),ANP(1000),BNP(1000),	CSM0249
	1ALPNP(1000),BETNP(1000),BJNP(100),BHNP(100),CEX,T,T1,PROD	CSM0250
	ERAD=DCMPLX(0.D0,0.D0)	CSM0251
	ETHETA=DCMPLX(0.D0,0.D0)	CSM0252
	EPHI=DCMPLX(0.D0,0.D0)	CSM0253
	NCK=0	CSM0254
	NN=(NP-1)*NMIN	CSM0255
	DO 40 N=1,NC	CSM0256
	FAC1=2.D0*N+1.D0	CSM0257
	FAC2=N*(N+1.D0)	CSM0258
	RATIO=FAC1/FAC2	CSM0259
	T=FAC1*P(N)*(BNP(NN+N)*BJNP(N+1)+BETNP(NN+N)*BHNP(N+1))	CSM0260
	NCK=NCK+1	CSM0261
	CALL TERM(NCK,T,1)	CSM0262
	ERAD=ERAD+T	CSM0263
	T=ANP(NN+N)*BJNP(N+1)+ALPNP(NN+N)*BHNP(N+1)	CSM0264
	CALL TERM(NCK,T,0)	CSM0265
	T1=BNP(NN+N)*((N+1.D0)*BJNP(N)-N*BJNP(N+2))/FAC1+BETNP(NN+N)*	CSM0266
	1((N+1.D0)*BHNP(N)-N*BHNP(N+2))/FAC1	CSM0267
	CALL TERM(NCK,T1,1)	CSM0268
	IF((THETA.GE.1.D-6).AND.(THETA.LT.3.141592D0))GO TO 20	CSM0269
	IF(THETA.GE.3.141592D0)GO TO 10	CSM0270
	ETHETA=ETHETA+FAC1/2.D0*T-RATIO*DP(N)*T1	CSM0271
	EPHI=EPHI-RATIO*DP(N)*T+FAC1/2.D0*T1	CSM0272
	GO TO 30	CSM0273
10	ETHETA=ETHETA+(-1.D0)**(N+1)*FAC1/2.D0*T-RATIO*DP(N)*T1	CSM0274
	EPHI=EPHI-RATIO*DP(N)*T+(-1.D0)**(N+1)*FAC1/2.D0*T1	CSM0275
	GO TO 30	CSM0276
20	ETHETA=ETHETA+RATIO*P(N)/DSIN(THETA)*T-RATIO*DP(N)*T1	CSM0277
	EPHI=EPHI-RATIO*DP(N)*T+RATIO*P(N)/DSIN(THETA)*T1	CSM0278
30	IF(NCK.EQ.4)NCK=0	CSM0279
40	CONTINUE	CSM0280
	PROD=EO*CEX	CSM0281
	ERAD=-PROD*DCOS(PHI)/(FKP(NP)*R)*ERAD	CSM0282
	ETHETA=PROD*DCOS(PHI)*ETHETA	CSM0283
	EPHI=PROD*DSIN(PHI)*EPHI	CSM0284
	PD=DPEAL(ERAD*DCONJG(ERAD))+DREAL(ETHETA*DCONJG(ETHETA))+DREAL(EPHI	CSM0285
	1I*DCONJG(EPHI))	CSM0286
	FRETURN	CSM0287
	END	CSM0288
	SUBROUTINE TERM(NCK,T,KEY)	CSM0289
C	COMPUTES I**NCK*(N-TH TERM IN SERIES)	CSM0290
	IMPLICIT REAL*8 (A-H,O-Z)	CSM0291
	COMPLEX*16 T	CSM0292
	IF(KEY.EQ.1)GO TO 20	CSM0293
	GO TO (5,10,15,45),NCK	CSM0294
20	GO TO (10,15,45,5),NCK	CSM0295
5	T=DCMPLX(-DIMAG(T),DPEAL(T))	CSM0296
	GO TO 45	CSM0297
10	T=-T	CSM0298
	GO TO 45	CSM0299
15	T=DCMPLX(DIMAG(T),-DPEAL(T))	CSM0300

45	RETURN	CSM0301
	END	CSM0302
	SUBROUTINE BJJH(BJNP,BHNP,Z,NN,STOPR)	CSM0303
C	GENERATES SPHERICAL BESSEL FUNCTIONS JN(KR) AND YN(KR)	CSM0304
C	AND SPHERICAL HANKEL FUNCTIONS OF THE FIRST KIND HN(KR)	CSM0305
	IMPLICIT REAL*8 (A-H,O-Z)	CSM0306
	COMPLEX*16 BJNP(100),BYNP(100),BHNP(100),QP,Z	CSM0307
	BYNP(1)=-CDCOS(Z)/Z	CSM0308
	BYNP(2)=(BYNP(1)-CDSIN(Z))/Z	CSM0309
	DO 5 M=3,100	CSM0310
	BYNP(M)=(2*M-3)/Z*BYNP(M-1)-BYNP(M-2)	CSM0311
	IF(CDABS(BYNP(M)).GE.STOPR)GO TO 10	CSM0312
5	CONTINUE	CSM0313
10	IF(M.GT.3)GO TO 25	CSM0314
C ***	PRINT OUT ERROR MESSAGE	CSM0315
	WRITE(6,20)Z	CSM0316
20	FORMAT('0**** PROCESS CAN NOT PROCEED SINCE NM2 = 0 FOR Z =',1P2D1	CSM0317
	15.7)	CSM0318
	STOP	CSM0319
25	BJNP(M)=DCMPLX(0.D0,0.D0)	CSM0320
	BJNP(M-1)=-1.D0/(Z*Z*BYNP(M))	CSM0321
	NM2=M-2	CSM0322
	DO 30 I=1,NM2	CSM0323
	L=M-I	CSM0324
	BJNP(L-1)=(2*L-1)/Z*BJNP(L)-BJNP(L+1)	CSM0325
30	CONTINUE	CSM0326
	QP=CDSIN(Z)/(Z*BJNP(1))	CSM0327
	NM1=M-1	CSM0328
	DO 35 N=1,NM1	CSM0329
	NN=N	CSM0330
	BJNP(N)=QP*BJNP(N)	CSM0331
	IF(CDABS(BJNP(N)).LT.1.D-25)GO TO 40	CSM0332
35	CONTINUE	CSM0333
40	DO 45 I=1,NN	CSM0334
	REJN=DREAL(BJNP(I))	CSM0335
	FIMJN=DIMAG(BJNP(I))	CSM0336
	REYN=DREAL(BYNP(I))	CSM0337
	FIMYN=DIMAG(BYNP(I))	CSM0338
	REHN=REJN-FIMYN	CSM0339
	FIMHN=REYN+FIMJN	CSM0340
	BHNP(I)=DCMPLX(REHN,FIMHN)	CSM0341
45	CONTINUE	CSM0342
	RETURN	CSM0343
	END	CSM0344
	SUBROUTINE PL(THETA,N,P,DP)	CSM0345
C	GENERATES ASSOCIATED LEGENDRE FUNCTIONS OF THE FIRST	CSM0346
C	KIND, ORDER 1 AND DEGREE N, AND THEIR FIRST DERIVATIVES	CSM0347
	IMPLICIT REAL*8 (A-H,O-Z)	CSM0348
	DIMENSION P(101),DP(100)	CSM0349
	SNJ=DSIN(THETA)	CSM0350
	CNJ=DCOS(THETA)	CSM0351
	P(1)=SNJ	CSM0352
	P(2)=3.D0*SNJ*CNJ	CSM0353
	DP(1)=CNJ	CSM0354
	DO 10 M=2,N	CSM0355
	A=M	CSM0356
	MP1=M+1	CSM0357
	P(MP1)=(2.D0*A+1.D0)/A*CNJ*P(M)-(A+1.D0)/A*P(M-1)	CSM0358
	IF((THETA.GE.1.D-6).AND.(THETA.LT.3.141592D0))GO TO 5	CSM0359
	DP(M)=M*MP1/2	CSM0360

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      IF (THETA.GE.3.141592D0) DP (M) = (-1.D0) **M*DP (M)
      GO TO 10
5     DP (M) = (A*CNJ*P (M) - (A+1.D0) *P (M-1) ) /SNJ
10    CONTINUE
      RETURN
      END
      FUNCTION MINN (NRAY,N)
C      DETERMINES MINIMUM POSITIVE INTEGER VALUE
      DIMENSION NRAY (10)
      IF (N.EQ.1) GO TO 20
      NMIN=NRAY (1)
      DO 10 I=2,N
      NTEMP=NRAY (I)
      IF (NTEMP.LT.NMIN) NMIN=NTEMP
10    CONTINUE
      MINN=NMIN
      GO TO 30
20    MINN=NRAY (1)
30    RETURN
      END

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CSM0361
CSM0362
CSM0363
CSM0364
CSM0365
CSM0366
CSM0367
CSM0368
CSM0369
CSM0370
CSM0371
CSM0372
CSM0373
CSM0374
CSM0375
CSM0376
CSM0377
CSM0378
CSM0379
CSM0380

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