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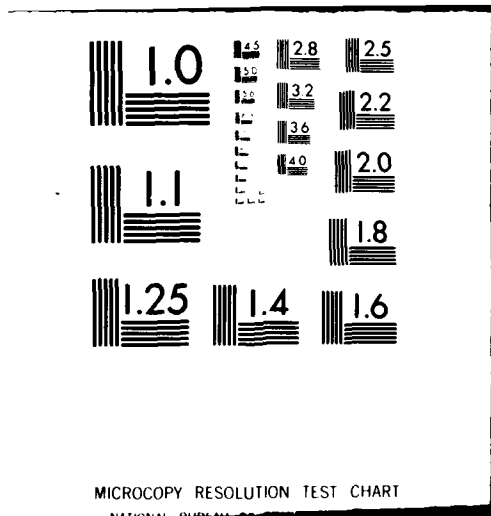
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IN THE ATMOSPHERE

by

A. G. Purcell

June 1979

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6 THE PROPAGATION OF FOCUSED TRUNCATED LASER BEAMS
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10 A. G. Purcell



SUMMARY

Lutomirski's graphical results concerning the intensity distribution of a focussed laser beam in the presence of turbulence have been confirmed and extended to include annular apertures. We have also considered segmented apertures and present some indication of the likely improvement in focal intensity when the segments are optimally adjusted to counteract turbulence.



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1 INTRODUCTION

Lutomirski¹ derived a formula for the mean intensity distribution resulting from a finite laser beam taking turbulence into account. He considered only circular apertures and it was our purpose to confirm his numerical results and then to extend his analysis to include annular and sector shapes. The former was relatively straightforward but the latter proved more awkward in that a simplifying transformation introduced by Lutomirski could not then be used. We shall refer extensively to his paper and many of his equations are reproduced for convenience, their original numbers being distinguished by the prefix L.

2 FORMULATION OF THE PROBLEM (CIRCULAR APERTURE)

The starting point for our work was equation (L5) for the mean intensity distribution.

$$\langle I \rangle(\underline{p}, z) = \left(\frac{k}{2\pi z} \right)^2 \iint \exp \left\{ \frac{ik}{2z} [r_1^2 - r_2^2 - 2\underline{p} \cdot (\underline{r}_1 - \underline{r}_2)] \right\} \\ \times M_s(\underline{r}_1, \underline{r}_2, \underline{p}) U_A(\underline{r}_1) U_A^*(\underline{r}_2) d^2 \underline{r}_1 d^2 \underline{r}_2 \quad (1)$$

Note that equation (L5) was misprinted. Fig 1 explains most of the geometric notation. k is the wave number ($k = 2\pi/\lambda$). U_A describes the field in the aperture and includes a focussing term

$$U_A(\underline{r}_j) = U_0 \exp \left[-\frac{1}{2} r_j^2 (a^{-2} + ikf^{-1}) \right] \quad (2)$$

The amplitude distribution is thus considered to be Gaussian with standard deviation a . Hence a uniform distribution is obtained by setting $a = \infty$.

M_s is the spherical wave mutual coherence function (MCF). Turbulence enters via M_s which essentially describes the loss of coherence due to turbulence. M_s depends on the distance between pairs of points and it is not necessary to consider distances in excess of the aperture diameter, D .

Lutomirski introduces the transformation

$$\underline{p} = \underline{r}_1 - \underline{r}_2 \quad ; \quad \underline{r} = \frac{1}{2}(r_1 + r_2) \quad (3)$$

to get his equation (L6).

$$\langle I \rangle(\rho, z) = \left(\frac{k}{2\pi z} \right)^2 \int d^2 \rho_s M_s(\rho, z) e^{-(ik/z) \rho \cdot \rho} \int U_A(\underline{r} + \frac{1}{2}\rho) \times U_A^*(\underline{r} - \frac{1}{2}\rho) e^{(ik/z) \rho \cdot \underline{r}} d^2 \underline{r} \quad (4)$$

The MCF is defined by equations (L7)-(L10),

$$M_s(\rho, z) = \exp \left\{ -\frac{2z}{z_c} \left[1 - \frac{\int_0^\infty dK K \phi(K) \int_0^1 J_0(K \rho u) du}{\int_0^\infty K \phi(K) dK} \right] \right\} \quad (5)$$

where

$$z_c^{-1} = 2\pi k^2 \int_0^\infty K \phi(K) dK \quad (6)$$

$\phi(K)$ is, to quote Lutomirski and Yura², the spectral density of atmospheric index-of-refraction fluctuations. They suggest a fairly arbitrary modification of the Kolmogorov spectrum to extend its validity to large and small wave numbers, *ie* instead of

$$\phi(K) = 0.033 C_n^2 K^{-11/3} \quad (7)$$

they propose

$$\phi(K) = \frac{0.033 C_n^2 \exp(-K^2 \ell_0^2)}{(K^2 + L_0^{-2})^{11/6}} \quad (8)$$

where ℓ_0 and L_0 are the inner and outer scales of turbulence where respectively viscous damping and inhomogeneity become important. Equation (7) is therefore valid in the so-called inertial subrange $L_0^{-1} \ll K \ll \ell_0^{-1}$. Throughout our work we have assumed $\ell_0 = 0.001$ m and $L_0 = 1$ m. The latter is especially important and is often taken to be the mean distance above the ground which perhaps indicates how imprecise a quantity it is. C_n^2 characterises the strength of the turbulent variations of refractive index. $C_n^2 = 0$ represents no turbulence, while $C_n^2 = 10^{-12} \text{ m}^{-2/3}$ would be strong turbulence. Again, this is clearly a very variable quantity.

Putting equation (8) in (6) and using the fact that $l \ll L_0$ gives

$$z_c^{-1} = 0.39k^2 C_n^2 L_0^{5/3} . \quad (9)$$

We shall return to the detailed calculation of M_s in section 3.2 for it is evident from equations (4) and (5) that it can be performed independently of the main integral. Indeed it can be seen from equation (5) that one could tabulate once and for all the term inside the square brackets as a function of ρ , given C_n^2 .

What is perhaps not so obvious at this moment is that the four-dimensional integral in equation (4) can be simplified. Substituting expression (2) in (4) gives for the inner integral

$$I_1 = \int U_0^2 \exp \left[-\frac{(r^2 + \frac{1}{2}\rho^2)}{a^2} + ik \left(\frac{1}{z} - \frac{1}{f} \right) \underline{\rho} \cdot \underline{r} \right] d^2 \underline{r} .$$

Writing $d^2 \underline{r} = r dr d\theta$ (see Fig 2) and taking the real part of the integral gives

$$I_1 = 4U_0^2 \int_0^{\pi/2} \int_0^{R(\theta)} \exp \left[-\frac{(r^2 + \frac{1}{2}\rho^2)}{a^2} \right] \cos \left[k\rho r \left(\frac{1}{z} - \frac{1}{f} \right) \cos \theta \right] r dr d\theta .$$

If we then redefine the integration variables as shown in Fig 2 and write $x = \rho/D$, the integral over the area of overlap of the two circles becomes

$$I_1(x) = D^2 U_0^2 \int_0^{\cos^{-1}x} d\phi \int_{x/\cos \phi}^1 \exp \left[-\frac{D^2}{4a^2} (u^2 + 2x^2 - 2ux \cos \phi) \right] \\ \times \cos \left[\frac{kD^2 x}{2} \left(\frac{1}{z} - \frac{1}{f} \right) (u \cos \phi - x) \right] u du .$$

Defining further dimensionless quantities

$$\delta = D/2a \quad ; \quad \beta = \frac{1}{2} kD^2 \left(\frac{1}{z} - \frac{1}{f} \right) . \quad (10)$$

results in an expression identical to equation (L15) except that the latter is normalised

$$I_1(\delta, \beta, x) = D^2 U_0^2 \int_0^{\cos^{-1} x} d\phi \int_{x/\cos \phi}^1 u \exp\left[-\delta^2(u^2 + 2x^2 - 2ux \cos \phi)\right] \times \cos\left[2\beta x(u \cos \phi - x)\right] du \quad (11)$$

Writing $d^2\rho = \rho d\rho d\psi = D^2 x dx d\psi$, equation (4) then becomes

$$\langle I \rangle(p, z) = \left(\frac{k}{2\pi z}\right)^2 \int_0^1 dx D^2 x I_1(\delta, \beta, x) M_s(x, z) \int_0^\pi \cos\left(\frac{kDx}{z} p \cos \psi\right) d\psi,$$

$$\langle I \rangle(\alpha, z) = \frac{U_0^2 (kD^2)^2}{4(4z)} \int_0^1 \frac{32}{\pi} x J_0(2\alpha x) M(\delta, \beta, x) M_s(x, z) dx, \quad (12)$$

$$\text{where } M(\delta, \beta, x) = I_1(\delta, \beta, x) / D^2 U_0^2 \quad \text{and} \quad \alpha = \frac{kDp}{2z}. \quad (13)$$

Equation (12) is equivalent to Lutomirski's equation (L16) and is in principle much easier to evaluate because no more than a double integral has to be computed at any stage. In practice, some caution is necessary because as α and $|\beta|$ increase, both $J_0(2\alpha x)$ and $M(\delta, \beta, x)$ become highly oscillatory.

The three parameters α , β , δ allow three of the major features to be separately studied

- (i) α the effect at points off-axis
- (ii) β the effect at points away from the focal plane
- (iii) δ the effect of a Gaussian rather than uniform aperture distribution.

Lutomirski presented only one diagram (half of Fig L5) in which more than one of these is non-zero. We shall see in section 4 that the extension to annular apertures involves the introduction of a fourth parameter.

3 THE EVALUATION OF THE VARIOUS INTEGRALS

3.1 $M(\delta, \beta, x)$

The double integral of equation (11) has been dealt with in a straightforward fashion by a 2-D Simpson procedure. We recognise that the cosine term in the integrand really necessitates a formula such as Filon's and this would be a worthwhile modification if extensive studies were planned of regions well removed from the focal plane. For the present we observe that the largest

argument occurring in the cosine term is $\beta/2$ when $x = \frac{1}{2}$, $\phi = 0$, $u = 1$. With an integration step, $\Delta u \leq 0.05$ we can reasonably trust our simple-minded scheme up to $\beta \approx 20$. We have also adopted $\Delta\phi \leq 0.05$ (3°). These values are empirical and were selected after studying the accuracy of the integration procedure for various different settings. Such an investigation was possible because $\langle I \rangle$ can be obtained analytically on the axis ($J_0(2\alpha x) = 1$) in free space ($M_s = 1$).

To be specific (cf equation (L20)),

$$\int_0^1 \frac{32}{\pi} x M(\delta, \beta, x) dx = \frac{4 \left[\left(1 - e^{-\delta^2/2}\right)^2 + 4e^{-\delta^2/2} \sin^2 \beta/4 \right]}{\delta^4 + \beta^2} \quad (14)$$

Note therefore, that the mean intensity at the focus ($\beta = 0$) is

$$\langle I \rangle(0, f) = \frac{U_0^2}{4} \left(\frac{kD^2}{4f} \right)^2 \left(\frac{1 - e^{-\delta^2/2}}{\delta^2/2} \right)^2 \quad (15)$$

Hence for uniform aperture distributions ($\delta = 0$) the integral in equation (12) is the Strehl ratio (SR) which is why equation (12) was written in that way.

3.2 $M_s(x, z)$

The MCF is defined by equation (5) with (8) and (9). Lutomirski quotes a simple expansion (equation (L11)) which is valid for most cases of practical interest. Writing

$$M_s(x, z) = \exp \left[- \frac{1}{2} D_s(x, z) \right] \quad (16)$$

where D_s is called the wave structure function, Lutomirski's approximate expression is

$$D_s(x, z) = \frac{z}{z_0} \left[2.8 \left(\frac{D_x}{L_0} \right)^{5/3} - 2.0 \left(\frac{D_x}{L_0} \right)^2 \right] \quad (17)$$

Lutomirski and Yura plot $\frac{z_c D_s}{4z}$ against $\frac{D_x}{L_0}$ showing how it tends to unity when

$D_x \gg L_0$. Our investigations suggest that equation (17) is an excellent approximation (<1% error) for $D_x/L_0 \leq 0.3$. Since $x \leq 1$ and $L_0 = 1$ m we can

therefore employ it throughout for aperture diameters not exceeding 30 cm. Note that we are unlikely to require $Dx/L_0 > 1$.

We now explain how equation (17) is obtained and corrected when $Dx > 0.3L_0$. From equations (5), (6) and (16),

$$D_s(x, z) = 8\pi^2 k^2 z \int_0^\infty K \phi(K) dK \int_0^1 \left[1 - J_0(KDxu) \right] du \quad (18)$$

Introducing (8) and using (9) gives

$$D_s(x, z) = \frac{20z}{3z_c} \int_0^\infty \frac{KL_0 e^{-K^2 \ell_0^2}}{\left[(KL_0)^2 + 1 \right]^{11/6}} d(KL_0) \int_0^1 \left[1 - J_0(KDxu) \right] du$$

ie

$$\frac{z_c D_s}{4z} = \frac{5}{6} \int_0^\infty \frac{e^{-y \left(\frac{\ell_0}{L_0} \right)^2}}{(y+1)^{11/6}} dy \int_0^1 \left[1 - J_0(\sqrt{y} Dx u / L_0) \right] du \quad (19)$$

where $y = (KL_0)^2$.

From here on we shall assume unity for the exponential factor in (19) which is valid provided that the contribution to the integral from $y > (L_0/\ell_0)^2$ is negligible. The y domain is divided into two regions which are integrated separately. The break point is $y_1 = (L_0/Dx)^2$ so that J_0 can be replaced by a power series for $y < y_1$, and the inner integral can be performed analytically.

$$1 - J_0(w) = \frac{1}{2} w^2 - \frac{\left(\frac{1}{2} w^2\right)^2}{(2!)^2} + \frac{\left(\frac{1}{2} w^2\right)^3}{(3!)^2} - \dots$$

ie

$$\int_0^1 \left[1 - J_0(wu) \right] du = \frac{1}{12} w^2 - \frac{1}{320} w^4 + \frac{1}{16128} w^6 - \dots \quad (20)$$

Two terms are sufficient for >99% accuracy when $w \leq 1$. The integral from $y = 0$ to $y = y_1$ can then be evaluated

$$S_1 = \left[(L_0/Dx)^2 + 1 \right]^{-5/6} \left[\frac{5}{12} - \frac{1}{448} + \left(\frac{Dx}{L_0} \right)^2 \left(\frac{1}{2} + \frac{3}{112} \right) + \frac{9}{280} \left(\frac{Dx}{L_0} \right)^4 \right] - (Dx/L_0)^2 \left[\frac{1}{2} + \frac{9}{280} (Dx/L_0)^2 \right]. \quad (21)$$

When $Dx \ll L_0$ the leading terms suffice

$$S_1' = \frac{5}{12} (Dx/L_0)^{5/3} - \frac{1}{2} (Dx/L_0)^2. \quad (22)$$

Note that equation (21) still holds approximately even when $Dx \gg L_0$; but is then accurate over a smaller range of y .

Now, in order to derive an analytic expression for the second region $y > y_1$, we require $Dx \ll L_0$ so that the denominator of (19) can be simplified, allowing the integral to be written

$$S_2' = \frac{5}{6} \int_0^\infty dy \frac{\int_0^1 [1 - J_0(\sqrt{y}Dxu/L_0)] du}{y^{11/6}} - \frac{5}{6} \int_0^{y_1} \frac{\frac{1}{12} y (Dx/L_0)^2 - \frac{1}{320} y^2 (Dx/L_0)^4}{y^{11/6}} dy. \quad (23)$$

The first integral is transformed by $w = \sqrt{y}Dxu/L_0$; $s = u$, becoming

$$\frac{5}{6} \int_0^\infty \frac{[1 - J_0(w)]}{w^{11/3}} 2w dw \int_0^1 (Dxs/L_0)^{5/3} ds$$

ie

$$\frac{5}{8} \left(\frac{Dx}{L_0} \right)^{5/3} \int_0^\infty \frac{1 - J_0(w)}{w^{8/3}} dw.$$

This integral is a standard form (eg see Luke³, p 486), its value being

$$\frac{\pi}{2^{5/3} \left[\Gamma\left(\frac{11}{16}\right) \right]^2} = 1.1185.$$

Therefore $S_2' = 0.7(Dx/L_0)^{5/3} - \frac{5}{12} (Dx/L_0)^2 + \dots$ (24)

Adding (22) and (24) leads to equation (17).

When $Dx \geq 0.3L_0$ the region $y > y_1$ must be treated numerically because the series expansion is not applicable when $y \gg 1$. We have

$$S_2 = \frac{5}{6} \int_{\left(\frac{L_0}{Dx}\right)^2}^{\infty} \frac{dy}{(y+1)^{11/6}} \int_0^1 \left[1 - J_0(\sqrt{y}Dxu/L_0) \right] du .$$

Substituting $w = (y+1)^{-5/6}$ gives

$$S_2 = \int_0^{w_1} dw \left[1 - \frac{\int_0^{f(w)} J_0(u) du}{f(w)} \right] \quad (25)$$

where $f(w) = (w^{-6/5} - 1)^{1/2} Dx/L_0$

and $w_1 = \left[(L_0/Dx)^2 + 1 \right]^{-5/6}$.

(Hence $f(w_1) = 1$).

The integrand decreases from 1 at $w = 0$ to ≈ 0.08 at $w = w_1$. Note that if the w domain is divided into N equal intervals for the purposes of numerical integration then the largest argument of the Bessel function occurs at the smallest w and is

$$f\left(\frac{w_1}{N}\right) = \left\{ N^{6/5} \left[(L_0/Dx)^2 + 1 \right] - 1 \right\}^{1/2} Dx/L_0 \approx N^{3/5} \left[1 + (Dx/L_0)^2 \right]^{1/2} . \quad (26)$$

One therefore processes the integral from w_1 down to 0 so that the inner integral can be evaluated progressively. The value of $z_c D_s / 4z$ is finally obtained by adding (21) and (25). As $Dx \rightarrow \infty$, $y_1 \rightarrow 0$ so S_1 vanishes while $w_1 \rightarrow 1$, $f(w) \rightarrow \infty$ giving $S_2 = 1$ as expected.

3.3 (I) (α, z)

All numerical integrations are carried out using Simpson's rule. Those described in the previous sections use a constant interval which is written into the computer program. Actually, each range is divided into an even number of equal steps each no wider than the prescribed value.

The integral in equation (12) is tackled adaptively using QAOZA of the Harwell Subroutine Library. This is because of the oscillatory behaviour of $J_0(2\alpha x)$ and $M(\delta, \beta, x)$ and the exponential nature of $M_s(x, z)$ resulting in an awkward integrand. We have included a facility which provides a useful check. D_s is computed at $x = 0.25$ using equation (17) and if it exceeds a chosen value (ELIM), the regions 0-0.25 and 0.25-1 are integrated separately and summed. The program is anyway on the alert for $M_s \rightarrow 0$ and does not integrate beyond x' where $D_s(x', z) = 40$. One can therefore run the program with $ELIM = 0$ and $ELIM = 40$ and compare the results. The former would normally be expected to be the more accurate.

4 EXTENSION TO ANNULAR APERTURES

Annular apertures present no great difficulty, the procedure being essentially the same except that the integration for $M(\delta, \beta, x)$ is over the area of overlap of two annuli. Defining ξ to be the ratio of inner and outer diameters, four cases can be distinguished (see Fig 3).

- (a) $x \geq \frac{1}{2}(1 + \xi)$ only the outer circles overlap
- (b) $\frac{1}{2}(1 + \xi) > x > \frac{1}{2}(1 - \xi)$ inner circle overlaps outer
- (c) $x \leq \frac{1}{2}(1 - \xi)$ whole of inner circle is within both outers
- (d) $x < \xi$ inner circles overlap.

It can be seen from Fig 3 that the range(s) of u -integration depends on the particular choice of ϕ , x and ξ . For example, in case (d) there is no need to carry out u -integration at all at small ϕ while in case (c) at small ϕ there would be two distinct regions to be treated. Consideration of the various configurations leads to the following rules

- (i) No integration necessary when $\cos \phi > \max\left(\frac{x}{\xi}, \frac{1 + 4x^2 + \xi^2}{4x}\right)$.
- (ii) When $\sin \phi \geq \xi/2x$, the full u -range is valid.
- (iii) When $\sin \phi < \xi/2x$, solve

$$v^2 - 4vx \cos \phi + 4x^2 - \xi^2 = 0$$

for v_1 and v_2 ($v_1 < v_2$) and integrate over u from $x/\cos \phi$ to 1 except in the range $v_1 \leq u \leq v_2$ should it overlap.

The formula equivalent to (14) is

$$\int_0^1 \frac{32}{\pi} xM(\delta, \beta, \xi, x) dx$$

$$= 4 \left\{ \left(e^{-\xi^2 \delta^2 / 2} - e^{-\delta^2 / 2} \right)^2 + 4e^{-(1+\xi^2)\delta^2 / 2} \sin^2 \left[\frac{1}{2} \beta (1 - \xi^2) \right] \right\} / (\delta^4 + \beta^2) \quad (27)$$

which is again valuable for verifying numerical procedures. Note that when $\delta = \beta = 0$, the value of the integral reduces to $(1 - \xi^2)^2$.

An interesting feature of the double integral for $M(\delta, \beta, x)$ (or $M(\delta, \beta, \xi, x)$) is that when $\delta = \beta = 0$ it reduces to an expression which is proportional to the area of overlap of two identical circles (annuli) and hence related to the probability density function, $P(x)$ for the separation of two points placed randomly in a circle (annulus) *eg* consider equation (11) with $\delta = \beta = 0$,

$$D^2 M(0, 0, x) = \int_0^{\cos^{-1} x} d\phi \int_{x/\cos \phi}^1 u du$$

$$= \frac{1}{2} D^2 \left(\cos^{-1} x - x \sqrt{1 - x^2} \right) \quad (28)$$

and

$$P(x) = \frac{16}{\pi} \left(x \cos^{-1} x - x^2 \sqrt{1 - x^2} \right)$$

$$= \frac{32}{\pi} xM(0, 0, x) \quad (29)$$

Hence for $\delta = \beta = 0$, the integral in equation (12) for points in the focal plane can be written

$$\int_0^1 P(x) J_0(2\alpha x) M_g(x, f) dx \quad (30)$$

Gilbey⁴ has derived the same result (for $\alpha = 0$, in which case the integral is the Strehl ratio) from first principles without following Lutomirski's line of development.

Note that it is possible to integrate (30) analytically in free space.

$$\frac{16}{\pi} \int \left(x \cos^{-1} x - x^2 \sqrt{1 - x^2} \right) J_0(2\alpha x) dx = \left[\frac{2J_1(\alpha)}{\alpha} \right]^2 . \quad (31)$$

This is the well-known Airy pattern for the focal plane intensity.

5 SEGMENTED APERTURES AND ADAPTIVE CONTROL

The Strehl ratio denotes the reduction of intensity due to turbulence. It is clear from equations (16) and (17) that for given z_c , M_s will be larger when D is small. Thus, to maximise the Strehl ratio the smallest possible aperture should be used to reduce the maximum separation of pairs of points.

However, the reduced aperture will give rise to a reduction of diffraction-limited intensity at focus in free space. This effect can be overcome by synthesising a large aperture from a number of smaller apertures or 'segments' (see Fig 4a). The wave launched from each segment must be independently controlled in phase so that the contributions from the various segments are in phase when they reach the focus. In other words the objective would then be to adjust the phase of each segment to enhance the intensity, particularly at the focus. In this approach, it must be remembered that the Strehl ratio obtainable cannot exceed that due to a single element, *ie* the degradation of intensity on account of turbulence will be at best as if the total power was being channelled through a single segment. If the aperture contains segments of different shapes and sizes (see Fig 4b) then the maximum attainable Strehl ratio will be some intermediate value. Therefore we need only consider one segment of each type and to combine the results appropriately in order to predict the optimum focal intensity. Note that it is not possible to calculate the corresponding intensity at other points because the relative phases of the aperture segments are not actually determined.

Since circular symmetry is now lost, transformation (3) is no longer helpful and we must return to equation (1) (as indeed was preferable for the derivation of equation (14) or (27)). Considering only the focus, the mean intensity is, using (2) ,

$$\begin{aligned}
\langle I \rangle(f) &= \left(\frac{k}{2\pi f} \right)^2 U_0^2 \iint \exp \left[- \frac{(r_1^2 + r_2^2)}{2a^2} \right] M_s(r_1, r_2, f) d^2 r_1 d^2 r_2 \\
&= \left(\frac{kU_0}{2\pi f} \right)^2 \int_{R_1}^{R_2} \int_0^\theta \int_{R_1}^{R_2} \int_0^\theta \exp \left[- \frac{(r_1^2 + r_2^2)}{2a^2} \right] M_s(r_1, r_2, \theta_1, \theta_2, f) r_1 r_2 d\theta_2 dr_2 d\theta_1 dr_1 \quad \dots (32)
\end{aligned}$$

The integration limits can be reduced because M_s only involves the separation of pairs of points (see Fig 4c) and not the direction in which that separation is measured. This is analogous to the previous treatment involving overlapping circles where only one quarter of the region of overlap needed to be considered. Hence

$$\langle I \rangle(f) = \left(\frac{kU_0}{2\pi f} \right)^2 4 \int_{R_1}^{R_2} \int_0^\theta \int_{R_1}^{R_2} \int_0^\theta \exp \left[- \frac{(r_1^2 + r_2^2)}{2a^2} \right] M_s(r_1, r_2, \theta_1, \theta_2, f) r_1 r_2 d\theta_2 dr_2 d\theta_1 dr_1 \quad \dots (33)$$

Introducing $x_1 = (2r_1/D)^2$; $x_2 = (2r_2/D)^2$ and, as before, $\xi = R_1/R_2$, $\delta = D/2a$, gives

$$\langle I \rangle(f) = \frac{1}{4} \left(\frac{kU_0 D^2}{4\pi f} \right)^2 \int_{\xi^2}^1 \int_0^\theta \int_{x_1}^1 \int_0^\theta e^{-(x_1 + x_2)\delta^2/2} M_s(v, f) d\theta_2 dx_2 d\theta_1 dx_1 \quad (34)$$

where $v^2 = \frac{1}{4} D^2 [x_1 + x_2 - 2\sqrt{x_1 x_2} \cos(\theta_2 - \theta_1)]$.

It is then easily verified that in free space ($M_s \equiv 1$),

$$\langle I \rangle(f) = \left(\frac{\theta}{2\pi} \right)^2 \frac{1}{4} \left(\frac{kU_0 D^2}{4f} \right)^2 \left(\frac{e^{-\xi^2 \delta^2/2} - e^{-\delta^2/2}}{\delta^2/2} \right)^2 \quad (35)$$

so by setting $\theta = 2\pi$ the intensity due to an annular aperture is regained.

A computer program using a four-dimensional Simpson scheme has been written to evaluate (34) for an aperture of the type depicted in Fig 46 and some results are discussed in the next section. We now explain how the Strehl ratio for the

complete, optimally adjusted aperture is determined from the values for the individual sectors.

When the aperture is a solid circle or annulus one may still think of it as being made up of sectors which, in the presence of turbulence, are non-optimally adjusted so that at the focus, when the complex disturbances are summed, the amplitude of the sum is less than the sum of the amplitudes. Clearly, the latter represents the best possible performance.

Consider an aperture consisting of K annuli such that the outermost circle is of diameter $D_0 = D$ and the innermost is D_k and define

$$\xi_i = D_i/D_{i-1} \quad ; \quad \xi_0 = 1 \quad ; \quad \eta_i = SR$$

for any sector in the i th annulus.

The optimised Strehl ratio is given by

$$\eta_{opt} = \left(\sum_{i=1}^K \sqrt{\eta_i I_i} \right)^2 / \left(\sum_{i=1}^K \sqrt{I_i} \right)^2 \quad (36)$$

where I_i is the focal intensity in free space due to the i th annulus. Thus for the configuration of Fig 4b and a uniform aperture distribution

$$\sqrt{\eta_{opt}} = \frac{\sqrt{\eta_1} (1 - \xi_1^2) + \sqrt{\eta_2} \xi_1^2 (1 - \xi_2^2)}{1 - \xi_1^2 \xi_2^2}$$

ie

$$\eta_{opt} = (\gamma \sqrt{\eta_1} + (1 - \gamma) \sqrt{\eta_2})^2$$

where

$$\gamma = \frac{1 - \xi_1^2}{1 - \xi_1^2 \xi_2^2} \quad (37)$$

If the central circle had also been illuminated, $\xi_3 = 0$

$$\text{and} \quad \sqrt{\eta_{opt}} = \sqrt{\eta_1} (1 - \xi_1^2) + \sqrt{\eta_2} \xi_1^2 (1 - \xi_2^2) + \sqrt{\eta_3} \xi_1^2 \xi_2^2$$

6 DISCUSSION OF RESULTS

Two FORTRAN computer programs have been written to evaluate the integrals in expressions (12) and (34). The first has been used to verify the graphical

results of Lutomirski¹ (Figs L4-L8), excellent agreement being obtained. However, it is to be expected that relative accuracy will be lost as the Strehl ratio decreases or as α and/or β increase (ie as the integrand in (12) and/or (11) becomes more oscillatory). This is not particularly disturbing since one is mainly interested in the intensity at the focus and its immediate vicinity while we are confident that, actually at the focus, discrepancies are negligible for $\eta \geq 0.001$. It is unlikely that cases with $\eta < 0.001$ have any practical significance so that our neglect of the exponential term in $\phi(k)$ (equations (8) and (9)) can be justified as follows.

Ignoring the inner scale of turbulence implies that $M_s(\ell_0, z) = 1$ and hence that separations less than ℓ_0 do not contribute significantly to $\langle I \rangle$. Clearly, the region $0 \leq x \leq \ell_0/D$ will only be important if the integrand of (12) decreases rapidly with x or if $D \approx \ell_0$. We do not envisage even the smallest element of a segmented aperture having an incircle less than $10\ell_0$ diameter while the first eventuality implies a very small Strehl ratio. It can be expressed as a limitation on range from equation (17)

$$z \ll \left(0.39k^2 C_n^2 \ell_0^{5/3} \right)^{-1} = 6.3\lambda^2 / C_n^2 \quad (38)$$

where $\ell_0 = 0.001$ m, λ is in μm and C_n^2 is in units of $10^{-12} \text{ m}^{-2/3}$.

Some results obtained with the main program (see Appendix A for details of its implementation) for an annular aperture are presented in Figs 5 and 6 and are compared with those for a circular aperture of the same outer diameter (see also Figs L5b and L6b). Fig 7 may be compared with Fig L8a.

The second program has been used to predict the maximum Strehl ratio attainable with the aperture shown in Fig 4b, the results appearing in Table 1. Appreciable enhancement of the Strehl ratio is clearly theoretically possible although in this example the factor of 15 achieved at $C_n^2 = 10^{-12} \text{ m}^{-2/3}$ would still be considered insufficient and one would have to divide the same size aperture into many more segments in order to reach $\eta_{\text{opt}} \approx 0.5$. Gilbey⁴ has confirmed these calculations using equation (30) by approximating the sectors by rectangles, for which $P(x)$ is available in an analytical form. Appendix B explains how to use our program while Appendix C contains analytic expressions for $P(x)$ for an annulus.

Neither program consumes more than a few seconds of central processor time on the ICL 1906S (and run especially rapidly when $D/L_0 < 0.3$) and they are also more accurate than Monte-Carlo multi-dimensional quadrature procedures.

7 CONCLUSIONS

Two FORTRAN programs have been written

- (i) to calculate the mean intensity distribution due to a focussed laser beam emanating from a circular or annular aperture in the presence of turbulence,
- (ii) to estimate the improvement in mean focal intensity obtainable by adaptive control of segmented apertures.

The programs are based on expressions taken or developed from Lutomirski¹. It must be emphasised that the theory is far from exhaustive and in particular is only applicable to low-power laser sources. Hence, as is often the case, the accuracy of the numerical solution exceeds that merited by the theory.

The programs provide close agreement with Lutomirski's graphs and with exact expressions available for free space propagation. The first program loses accuracy as the longitudinal distance (characterised by β) from the focus increases but this could be overcome by implementing a more appropriate integration method (Filon's would be adequate) for the inner integral in equation (11). The second program only predicts the mean intensity at the focal point.

Appendix A

USING PROGRAM M21LASER

This program evaluates the integral in (12) using QAOZA of the Harwell Subroutine Library. J_0 is obtained via the ICL subroutine F4J0 while the calculation of $M_s(x,z)$ is described in section 3.2 and $M(\delta,\beta,x)$ is covered in section 3.1 and extended to include annular apertures ($\xi \neq 0$) in section 4. The data consists of (NRUN + 4) cards.

- (1) NRUN: gives the number of different values of α which will be specified. NRUN = 0 stops the program.
- (2) z/z_c , D/L_0 , ELIM:
 $z/z_c = 0$ for free space. ELIM determines whether the x -range will be split into two ($0 \leq x \leq 0.25$, $0.25 \leq x \leq 1$). ELIM = 0 forces this to occur, otherwise ELIM = 4 is recommended so that the range will be split if M_s is decreasing quickly (see section 3.3).
- (3) β , δ , ξ
- (4) ERR: The absolute accuracy required in the value of the integral. ERR = 0.0001 is recommended but it can of course be reduced if the integral unexpectedly turns out to be small.
- (5) α : The program will calculate the integral and then ask for the next value of α until NRUN values have been processed after which a new value for NRUN is expected.

The output consists of a single line

VALUE OF INTEGRAL = X.XXXEXXX ESTIMATED ERROR = X.XXXEXXX XXX INTEGRAND EVALUATIONS .

The Strehl ratio can then be obtained by dividing by expression (27) if $\alpha = 0$ or by (31) if $\alpha \neq 0$ but $\delta = \beta = \xi = 0$. Otherwise, the intensity *in vacuo* has to be evaluated numerically by running the program with $z/z_c = 0$.

All integrals are evaluated by Simpson's method, the maximum interval width (m.i.w.) having been chosen empirically. $M(\delta,\beta,\xi,x)$ is found from equation (11), the range of the inner integral being modified for annular apertures as described in section 4. The m.i.w. are specified in DATA statements and are

$$\text{HODPHI} = \frac{1}{2\Delta\phi_{\max}} = 10 \quad ; \quad \text{HODUM} = \frac{1}{2\Delta u_{\max}} = 10 .$$

Likewise, the m.i.w. for the integrals in (25) involved in the determination of $M_s(x,z)$ when $Dx/L_0 > 3$, are specified in DATA statements.

$$HODU = \frac{1}{2\Delta u_{\max}} = 10 \quad ; \quad N = 50 \quad (\text{hence } \Delta w = 0.02w_1) .$$

Finally, QAOZA needs to know, when evaluating (12), the smallest permissible interval, Δx_{\min} and this has been set at $DXMIN = 0.007$. QAOZA begins by dividing the range into 16 equal intervals and then successively halves the step locally to obtain the desired accuracy. Thus when the interval (0,1) is treated, a maximum of 127 integrand evaluations will be performed ($\Delta x = 0.0078125$) and when it is divided into (0,0.25) and (0.25,1) the largest number possible is $32 + 63$ ($\Delta x = 0.01171875$) = 95, where we have used the fact that the integrand is zero at $x = 0$ and $x = 1$.

When $NRUN > 1$ we need to recalculate only J_0 in equation (12) when α changes. We therefore store $M(\delta, \beta, \xi, x)$ and $M_s(x, z)$ (in arrays AXMDB, AEMS) and record which values of x have occurred (in array LST) in order to avoid unnecessary work. Thus the time taken to treat $NRUN$ values of α is much less than $NRUN$ times the time for one α . However, should one wish to reduce $DXMIN$ then the size of these arrays must be increased accordingly.

Appendix B

USING PROGRAM M21OPTSR

This program evaluates the integral in equation (34), M_s being calculated as described in section 3.2 (with HODU = 10 and N = 50 when applicable, as in Appendix A). The data consists of three records as follows

- (1) f/z_c , D/L_0 : $f/z_c = 0$ for free space and $f/z_0 < 0$ stops the program.
- (2) HMAX, HF, DTH: recommended values 0.1, 1.0, 0.05
- (3) δ , ξ , NS: NS is the number of equal sectors comprising the annulus (ie $\theta = 2\pi/NS$).

Each range of integration is divided into an even number of intervals and Simpson's rule applied. The maximum interval widths are

$$\Delta\theta_{\max} = DTH \quad ; \quad \Delta x_{\max} = \min \left[HMAX, \frac{HF}{\frac{f}{z_c} \left(\frac{D}{L_0} \right)^{5/3}} \right] .$$

The output from the program consists of a single line

STREHL RATIO FOR SECTOR = X.XXXXEXXX

where the answer is obtained by dividing (34) by (35).

Appendix C

THE PROBABILITY DENSITY FUNCTION FOR THE SEPARATION
OF TWO POINTS PLACED RANDOMLY WITHIN AN ANNULUS

Let the outer diameter be D and the inner diameter be ξD . The p.d.f. $p(Dx)$ is related to the area of overlap, A , of two identical annuli whose centres are separated by Dx .

$$A = A_1 + A_3 - 2A_2 \quad . \quad (C-1)$$

A_1 is the area of overlap of the outer circles

$$A_1 = \frac{1}{2}D^2(\cos^{-1}x - x(1-x^2)^{\frac{1}{2}}) \quad . \quad (C-2)$$

A_2 is the area enclosed between an inner circle and the outer circle which is not concentric with it.

$$2A_2 = \frac{1}{2}D^2 \left\{ \xi^2 \left[\cos^{-1}b - b(1-b^2)^{\frac{1}{2}} \right] + \left[\cos^{-1}a - a(1-a^2)^{\frac{1}{2}} \right] \right\} \quad . \quad (C-3)$$

where

$$a = \frac{1 - \xi^2}{4x} + x$$

$$b = \left(x - \frac{(1 - \xi^2)}{4x} \right) / \xi \quad (C-4)$$

A_3 is the area of overlap of the inner circles

$$A_3 = \frac{1}{2}(\xi D)^2 \left[\cos^{-1}(x/\xi) - (x/\xi) \left(1 - (x/\xi)^2 \right)^{\frac{1}{2}} \right] \quad . \quad (C-5)$$

The 'normalised' p.d.f. is then

$$P(Dx) = \frac{32x}{\pi D^2} A \quad (C-6)$$

so that

$$\int_0^1 P(Dx) dx = (1 - \xi^2)^2 \quad . \quad (C-7)$$

Note that $A_3 = 0$ when $x > \xi$ while from (C-4) one can deduce that

$$a, |b| > 1 \quad \text{when} \quad x > \frac{1}{2}(1 + \xi) \quad \text{or} \quad x < \frac{1}{2}(1 - \xi)$$

giving

$$2A_2 = 0 \quad \text{when} \quad x > \frac{1}{2}(1 + \xi) \quad ;$$

$$2A_2 = \frac{1}{2}\pi\xi^2 D^2 \quad \text{when} \quad x < \frac{1}{2}(1 - \xi) \quad .$$

The latter implies that the inner circle is wholly contained within the other outer circle when $x < \frac{1}{2}(1 - \xi)$ (as in Fig 3c).

Table 1

SHOWING THE IMPROVEMENT IN STREHL RATIO PREDICTED
BY OPTIMISING THE SEGMENTED APERTURE OF FIG 4b

C_n^2	f/z_c	η_1	η_2	η_{opt}	$\eta_{annulus}$
10^{-12}	3000	0.0882	0.0864	0.0876	0.0059
10^{-13}	300	0.588	0.580	0.585	0.0874
10^{-14}	30	0.941	0.940	0.941	0.600
10^{-15}	3	0.994	0.994	0.994	0.945
10^{-16}	0.3	0.9994	0.9994	0.9994	0.995

C_n^2 is in units of $m^{-2/3}$. Columns 3 and 4 were obtained with M21OPTSR, the fifth column being derived from them using (37). M21LASER produced column 6. Hence in this example the optimally adjusted segmented aperture performs as well as the full aperture would in ten times weaker turbulence.

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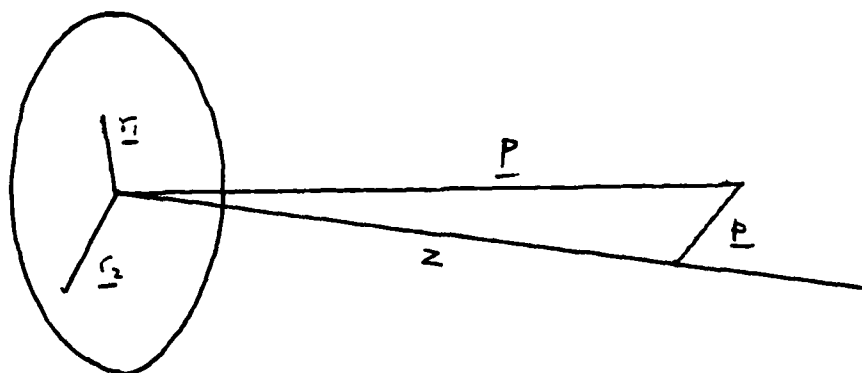


Fig 1 Basic geometry showing two typical points in the aperture and the point, P at which the intensity is required

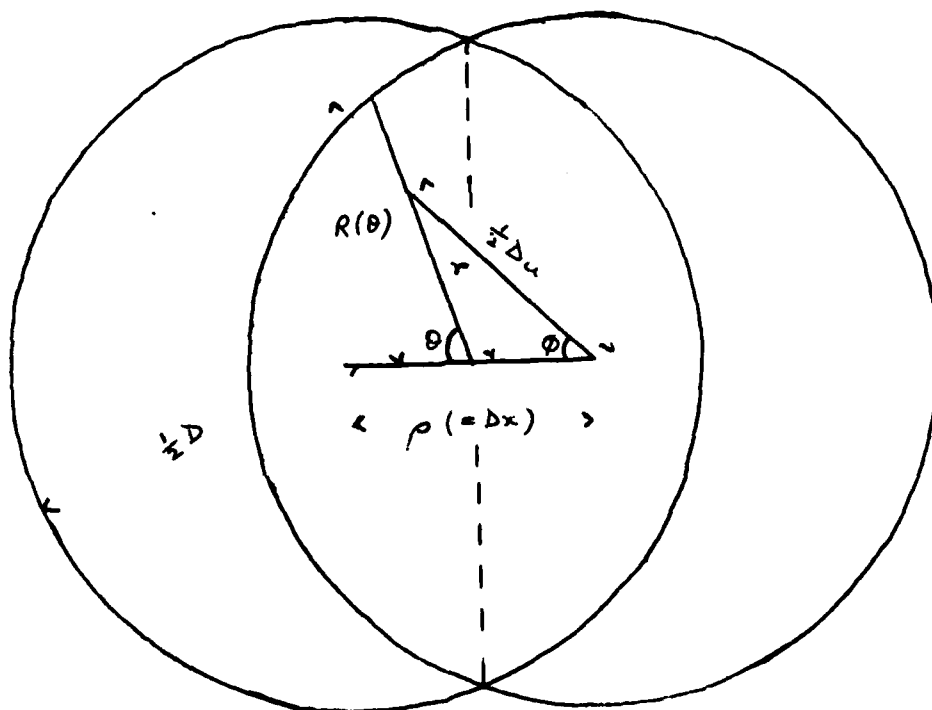
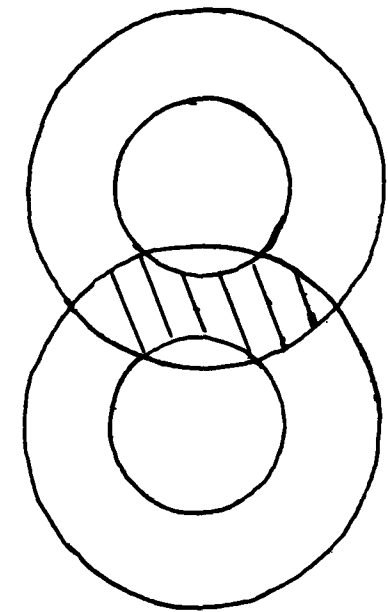
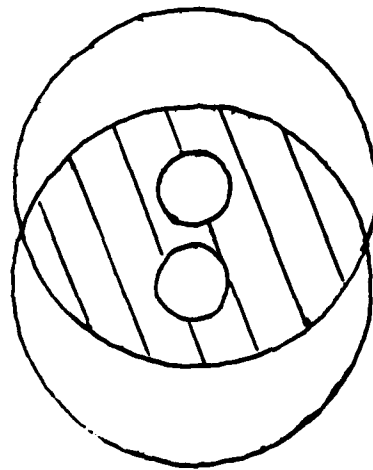


Fig 2 Two alternative parametrisations of the domain of the inner integral in equation (4)

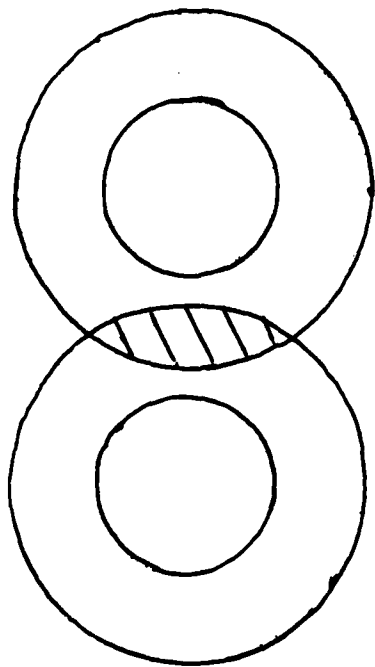
Fig 3



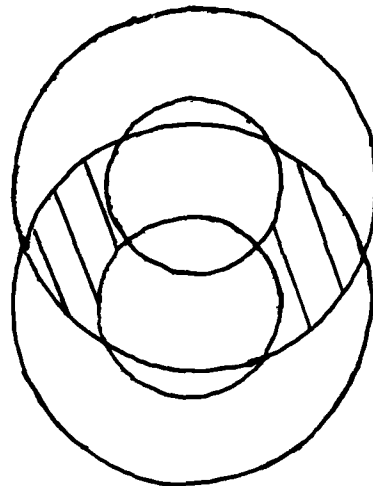
(a) $x \geq \frac{1 + \sqrt{3}}{2}$



(b) $x \leq \frac{1 - \sqrt{3}}{2}$

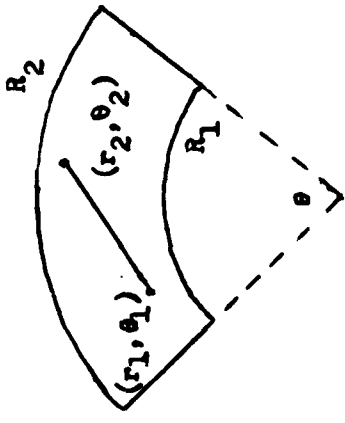


(c) $\frac{1 + \sqrt{3}}{2} > x > \frac{1 - \sqrt{3}}{2}$



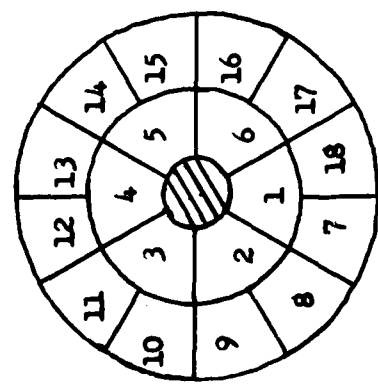
(d) $x < \sqrt{3}$

Fig 3 The area enclosed between two identical overlapping annuli



(c) Fundamental sector geometry.

$$\zeta = R_1 / R_2$$

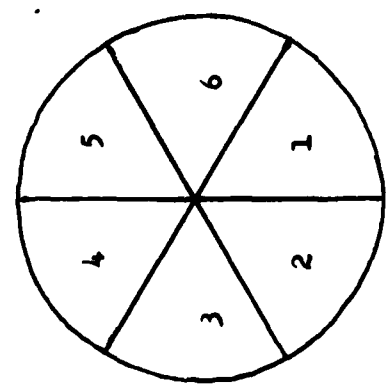


(b) 18 - element aperture.

Data for example :

$$D = 0.2m, D_1 = 0.12m, D_2 = 0.036m$$

$$i.e. \zeta_1 = 0.6, \zeta_2 = 0.3.$$

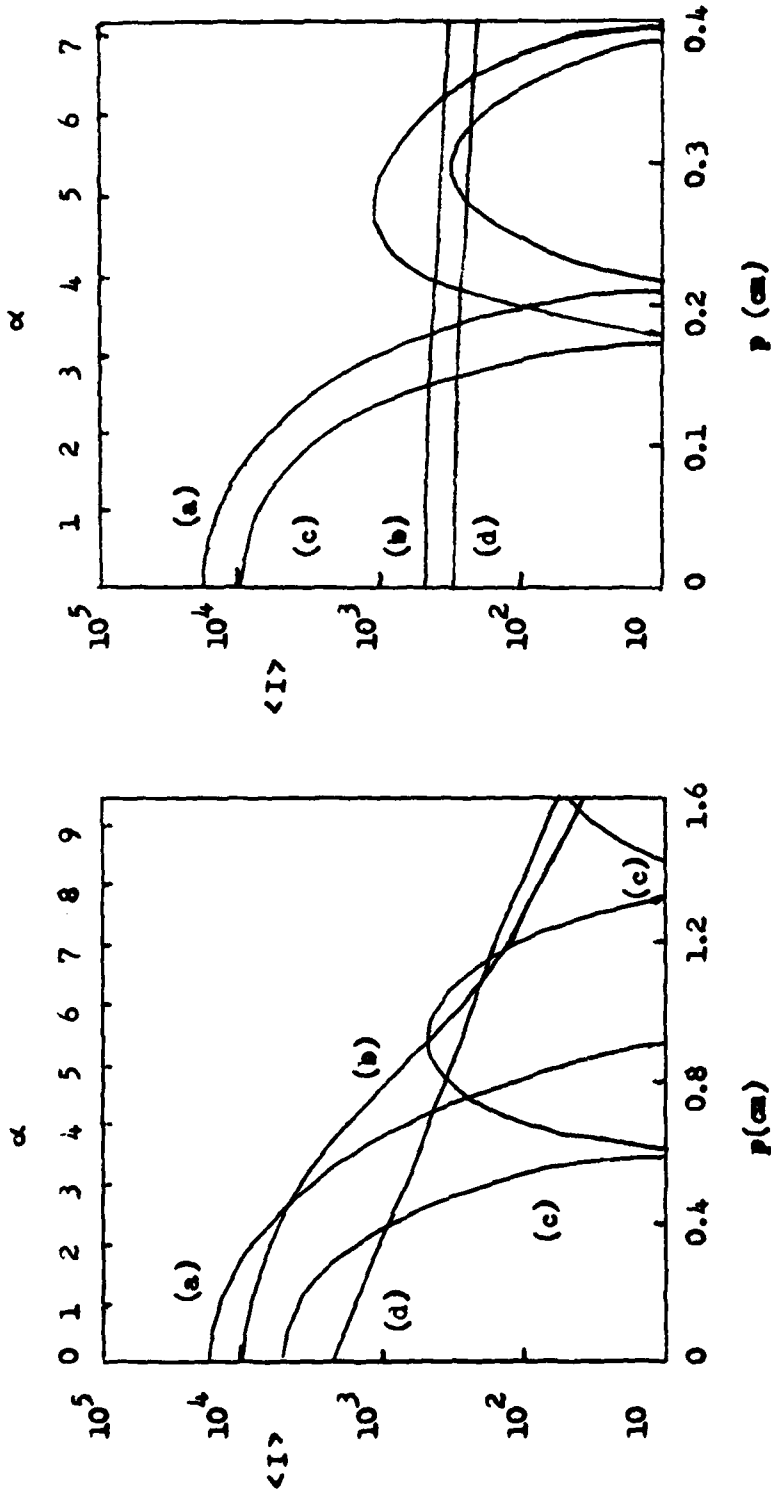


(a) Simple segmented aperture.

$$\zeta = 0, \theta = 60^\circ.$$

Fig 4

Fig 4 Segmented apertures

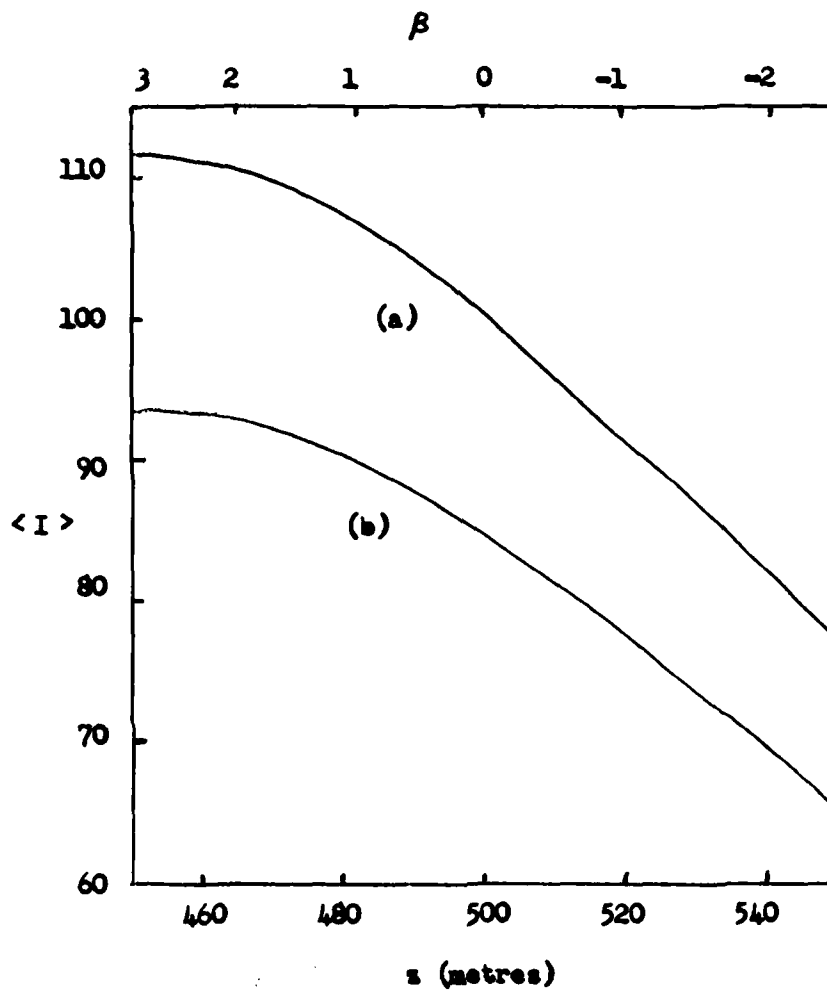


- (a) No turbulence ; full aperture
- (b) $C_n^2 = 3 \times 10^{-15} \text{ cm}^{-2/3}$; full aperture
- (c) No turbulence ; $z = 0.5$
- (d) $C_n^2 = 3 \times 10^{-15} \text{ cm}^{-2/3}$; $z = 0.5$

Fig 5 Comparison of focal plane patterns at $\lambda = 10.6\mu$ for $D = 1 \text{ m}$ focussed at 0.5 km for a truncated Gaussian ($\delta = 2$) aperture distribution

Fig 6 Comparison of focal plane patterns at $\lambda = 1.06\mu$ for $D = 30 \text{ cm}$ focussed at 0.5 km for a uniform ($\delta = 0$) aperture distribution

Fig 7



- (a) No turbulence ; $\bar{\epsilon} = 0.5$
- (b) $C_n^2 = 3 \times 10^{-15} \text{ cm}^{-2/3}$; $\bar{\epsilon} = 0.5$

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Fig 7 Comparison of on-axis intensities at $\lambda = 10.6\mu$ for $D = 30$ cm focussed at 0.5 km for a uniform aperture distribution (cf Fig L8a).

REPORT DOCUMENTATION PAGE

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17. Abstract Lutomirski's graphical results concerning the intensity distribution of a focussed laser beam in the presence of turbulence have been confirmed and extended to include annular apertures. We have also considered segmented apertures and present some indication of the likely improvement in focal intensity when the segments are optimally adjusted to counteract turbulence.					

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