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ASYMPTOTIC BEHAVIOR OF AN INTEGRO-DIFFERENTIAL EQUATION

by

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CONSIDER THE FUNCTIONAL DIFFERENTIAL EQUATION

$$\dot{x}(t) = -\int_{-1}^{0} a(-\theta)g(x(t+\theta))d\theta$$

where $a \geq 0$, g are continuous, $a(1) = 0$. If $a > 0$ and

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$G(x) = \int_0^x g \to \infty$ as $|x| \to \infty$, then it is known that every solution either approaches a zero of g as $t \to \infty$ or has an -limit set which is a 1-periodic solution of the equation $\ddot{x} + a(0)g(x) = 0$. If there are only a finite number of equilibrium points and 1-periodic orbits, then there is a maximal compact invariant set $A_{a,g}$ in $C([-1,0],\mathbb{R})$ which is uniformly asymptotically stable. When a is convex, the topologically structure of $A_{a,g}$ and the flow on $A_{a,g}$ are discussed as a function of g. When g is fixed, $xg(x) > 0$, $x \neq 0$, the complete bifurcation diagram is given for a in a neighborhood of a linear function.

ASYMPTOTIC BEHAVIOR OF AN INTEGRO-DIFFERENTIAL EQUATION

by

Jack K. Hale

__Abstract__: Consider the functional differential equation

$$\dot{x}(t) = -\int_{-1}^{0} a(-\theta)g(x(t+\theta))d\theta$$

where $a \geq 0$, g are continuous, $a(1) = 0$. If $\ddot{a} > 0$ and $G(x) = \int_0^x g \to \infty$ as $|x| \to \infty$, then it is known that every solution either approaches a zero of g as $t \to \infty$ or has an ω-limit set which is a 1-periodic solution of the equation $\ddot{x} + a(0)g(x) = 0$. If there are only a finite number of equilibrium points and 1-periodic orbits, then there is a maximal compact invariant set $A_{a,g}$ in $C([-1,0],\mathbb{R})$ which is uniformly asymptotically stable. When a is convex, the topologically structure of $A_{a,g}$ and the flow on $A_{a,g}$ are discussed as a function of g. When g is fixed, $xg(x) > 0$, $x \neq 0$, the complete bifurcation diagram is given for a in a neighborhood of a linear function.

1. Introduction. Consider the functional differential equation

$$\dot{x}(t) = -\int_{-1}^{0} a(-\theta)g(x(t+\theta))d\theta \tag{1.1}$$

where $a \geq 0$, g are continuous, $a(1) = 0$. If $\ddot{a} > 0$ and $G(x) = \int_0^x g \to \infty$ as $|x| \to \infty$, then every solution approaches a zero of g as $t \to \infty$. If $a(s)$ is linear, then every solution either approaches a zero of g as $t \to \infty$ or has an ω-limit set which is a 1-periodic solution of the euqation $\ddot{x} + a(0)g(x) = 0$ (see Levin and Nohel [6] , Hale [3]).

With some other minor restrictions, there is a maximal compact invariant set $A_{a,g}$ of Equation (1.1) in the space $C = C([-1,0],\mathbb{R})$ which is uniformly asymptotically stable. The primary purpose of this paper is to study the manner in which $A_{a,g}$ and the flow on $A_{a,g}$ induced by Equation (1.1) depend upon the functions a,g. With the kernel a fixed convex function, it appears that the flow on $A_{a,g}$ depends on more than just the zeros of the function g. For a fixed function g, there are linear kernels a_0 where the topological structure of $A_{a,g}$ can change for a in small neighborhood in $C([0,1],\mathbb{R})$ of a_0; that is, a_0 is a bifurcation point. These bifurcation points are analyzed in detail and are shown to be analogous to bifurcation form a focus. However, regardless of the nature of g, the degeneracy at the focus is not of order one; that is, it is not the usual generic Hopf bifurcation.

2. <u>Definitions and background material</u>. If $C = C([-1,0],\mathbb{R})$ and x is a continuous function on $[\sigma-1,\sigma+\alpha]$, $\alpha > 0$, we let $x_t \in C$ be defined by $x_t(\theta) = x(t+\theta)$, $-1 \le \theta \le 0$, $t \in [\sigma,\sigma+\alpha)$. If $g \in C^k(\mathbb{R},\mathbb{R})$, $k \ge 1$, and $a(s) \ge 0$ is continuous on $[0,1]$, let $x(\phi)$ be the solution of the functional differential equation

$$\dot{x}(t) = \int_{-1}^{0} a(-\theta)g(x(t+\theta))d\theta \tag{2.1}$$

with $x_0(\phi) = \phi$. If we assume that $x(\phi)(t)$ for every $\phi \in C$ is defined for $t \ge -1$ and let $T_{a,g}(t)\phi = x_t(\phi)$, $t \ge 0$, then $T_{a,g}(t)\colon C \to C$, is a strongly continuous semigroup with $T_{a,g}(t)\phi$ having continuous derivatives with resp^ct to ϕ up through order k.

<u>Definition 2.1</u>. For Equation (2.1) a set B in C attracts a set H in C if, for any $\varepsilon > 0$, there is a $t_0 = t_0(H,\varepsilon)$ such that $T_{a,g}(t)H$ is in an ε-neighborhood of B for $t \ge t_0$. For Equation (2.1), a set B in C <u>attracts points</u> if B attracts the set $\{\phi\}$ for every $\phi \in C$. Equation (2.1) is <u>point dissipative</u> if there is a bounded set B that attracts points. For Equation (2.1), a set B <u>attracts bounded sets</u> if B attracts every bounded subset H of C. A set B is <u>stable</u> if, for any neighborhood U of B, there is a neighborhood V of B such that $T_{a,g}(t)V \subseteq U$ for $t \ge 0$. A set B is <u>asymptotically stable</u> if it is stable and attracts a neighborhood of B.

<u>Definition 2.2</u>. The <u>positive orbit</u> $\gamma^+(\phi)$ <u>through</u> ϕ is the set $\{T_{a,g}(t)\phi,\ t \ge 0\}$. A function $h\colon (-\infty,0] \to \mathbb{R}$ is said to be a <u>backward extension</u> of $\phi \in C$ if $h_0 = \phi$ and $T_{a,g}(t)h_\tau = h_{t+\tau}$

for $\tau \in (-\infty,0)$, $0 \leq t \leq -\tau$. If h is a backward extension of ϕ, we may define $T_{a,g}(t)\phi$ for $t \leq 0$ as $T_{a,g}(t)\phi = h_t$, $t \leq 0$. A negative orbit $\gamma^-(\phi)$ is the set $\{T_{a,g}(t)\phi,\ t \leq 0\}$ defined by some backward extension of ϕ. A set B in C is invariant if for every $\phi \in B$, we can define $T_{a,g}(t)\phi$ for $t \leq 0$ and $T_{a,g}(t)\phi \in B$ for $t \in \mathbb{R}$. An equilibrium point of (2.1) is an invariant set consisting of a single point $\tilde{\phi}$. This always implies $\tilde{\phi}$ is a constant function.

A periodic orbit of (2.1) is an invariant set which is a closed curve; that is, a periodic orbit is given by $\{p_t,\ t \in \mathbb{R}\}$ where $p(t)$ is a periodic solution of (2.1).

Let

$$(2.2) \qquad A_{a,g} = \{\phi \in C\colon T_{a,g}(t)\phi \text{ is defined and bounded for } t \leq 0\}.$$

The following result gives some of the fundamental properties of the semigroup $T_{a,g}(t)$ and the set $A_{a,g}$ and is essentially contained in Hale [3], [4].

Theorem 2.3. *If Equation (2.1) is point dissipative, then $A_{a,g}$ in (2.2) is the maximal compact invariant set of Equation (2.1) and $A_{a,g}$ is uniformly asymptotically stable. If, in addition, $T_{a,g}(t)$ is one-to-one on $A_{a,g}$, then $T_{a,g}(t)$ is a continuous group on $A_{a,g}$. Also, if $T_{a,g}(t)$ takes bounded sets into bounded sets, then $A_{a,g}$ attracts bounded sets of C.*

In the following discussion, $\mathscr{A} \times \mathscr{G}$ is a topological space which is also a subset of $C([0,1],\mathbb{R}) \times C^k(\mathbb{R},\mathbb{R})$, $k \geq 1$, with

$(a_n, g_n) \to (a,g) \in \mathscr{A} \times \mathscr{G}$ implying that $a_n \to a$ uniformly on $[0,1]$ and $d^r g_n/dx^r \to d^r g/dx^r$, $0 \leq r \leq k$, uniformly on compact sets.

Definition 2.4. $\mathscr{A} \times \mathscr{G}$ is uniformly point dissipative if there is a bounded set $B \subset C$ such that B attracts points of $T_{a,g}(t)$ for every $(a,g) \in \mathscr{A} \times \mathscr{G}$.

The following result is due to Cooperman [1] (see, also, Hale [4]).

Theorem 2.5. *If $\mathscr{A} \times \mathscr{G}$ is uniformly point dissipative, then $A_{a,g}$ is upper semicontinuous in (a,g); that is, for any neighborhood U of $A_{a,g}$, there is a neighborhood V of (a,g) in $\mathscr{A} \times \mathscr{G}$ such that $A_{b,h} \subset U$ for $(b,h) \in V$.*

Definition 2.6. For $(a,g), (b,h) \in \mathscr{A} \times \mathscr{G}$, we say (a,g) is equivalent to (b,h), $(a,g) \sim (b,h)$, if there is a homeomorphism $\tau: A_{a,g} \to A_{b,h}$ such that $\tau T_{a,g}(t) = T_{b,h}(t)\tau$ on $A_{a,g}$. An $(a,g) \in \mathscr{A} \times \mathscr{G}$ is structurally stable in $\mathscr{A} \times \mathscr{G}$ if there is a neighborhood V of (a,g) in $\mathscr{A} \times \mathscr{G}$ such that $(a,g) \sim (b,h)$ for every $(b,h) \in V$.

Definition 2.7. An equilibrium point α of (2.1) is hyperbolic if the solutions of the characteristic equation

$$(2.3) \qquad \lambda + g'(\alpha) \int_{-1}^{0} a(-\theta) e^{\lambda\theta} d\theta = 0$$

have negative real parts. A periodic orbit $\{p_t, t \in \mathbb{R}\}$ is hyperbolic if the linear variational equation

$$\dot{y}(t) + \int_{-1}^{0} a(-\theta)g'(p(t+\theta))y(t+\theta)d\theta = 0 \tag{2.4}$$

has one as a simple characteristic multiplier and no other characteristic multiplier on the unit circle.

If p has period τ, a point $\rho = \exp \lambda\tau$ is a characteristic multiplier of (2.4) if there is a solution of (2.4) of the form $e^{\lambda t}q(t)$ where q is τ-periodic and not identically zero (see Hale [3]).

If a satisfies

$$\int_{-1}^{0} a(-\theta)d\theta > 0 \tag{2.5}$$

then an equilibrium point α is hyperbolic if and only if $g'(\alpha) \neq 0$, asymptotically stable if $g'(\alpha) < 0$ and unstable if $g'(\alpha) > 0$. Therefore, the set of all g for which the equilibrium points are hyperbolic is a residual set in $C^r(\mathbb{R},\mathbb{R})$ for each fixed a satisfying (2.5).

We will see below that there are functions a for which there is no residual set of g for which the periodic orbits are hyperbolic.

3. Convex kernel. In this section, we suppose that $a \in C^2[-1,0]$,

$$a(1) = 0,\ a(s) \geq 0,\ \dot{a}(s) \leq 0,\ \ddot{a}(s) \geq 0,\ s \in [0,1], \tag{3.1}$$

$$G(x) = \int_0^x g \to \infty \quad \text{as} \quad |x| \to \infty. \tag{3.2}$$

The following result may be found in Hale [3, p. 122].

Theorem 3.1. *If the conditions* (3.1), (3.2) *are satisfied and the zeros of* g *are isolated, then every solution of* (2.1) *is bounded, and*

(i) *If there is an* s *such that* $\ddot{a}(s) > 0$, *then the* ω*-limit set of any orbit of* (2.1) *is an equilibrium point of* (2.1); *that is, a constant function whose value is a zero of* g. *An equilibrium point* α *is hyperbolic if and only if* $g'(\alpha) \neq 0$, *asymptotically stable if* $g'(\alpha) > 0$ *and an unstable saddle point, if* $g'(\alpha) < 0$.

(ii) *If* $\ddot{a}(s) = 0$ (*that is, a is linear*) *then, for any* $\phi \in C$, *there is either an equilibrium point or a one-periodic solution* $p = p(\phi)$ *of the ordinary differential equation*

$$\ddot{y} + a(0)g(y) = 0$$

such that the ω*-limit set of the orbit through* ϕ *is* $\{p_t,\ t \in \mathbb{R}\}$.

For any zero α of g, let

$$W^s(\alpha) = \{\phi \in C\colon T_{a,g}(t)\phi \to \text{the constant function } \alpha \text{ as } t \to \infty\}.$$

We may now make the following conjecture

Conjecture 3.2. *If* (3.1), (3.2) *are satisfied and, in addition, the zeros* α_j, $j = 1,2,\ldots,$ *of* g *are simple and there is* $\delta > 0$ *such that*

$$(3.3) \qquad \ddot{a}(s) > 0 \quad \text{for} \quad s \in (1-\delta,1)$$

then the set

$$W = \{W^s(\alpha_j)\colon \ \alpha_j \ \text{is asymptotically stable}\}$$

is dense in C.

One could try to prove this using an idea of Henry [5]. Let α be a saddle point and for any $r > 0$, $\tau > 0$, let $W^s_{\tau,r}(\alpha) = \{\phi \in C\colon T_{a,g}(\tau)\phi \in W^s(\alpha),\ |T_{a,g}(\tau)\phi - \alpha| < r\}$. The set $W^s_\tau(\alpha)$ is closed. If it contains a ball $B = \{\phi_1 + ph\colon |h| = 1,\ 0 \le p \le r_0\}$, then $T_{a,g}(\tau)B$ has been flattened so that it is contained in a submanifold of codimension one. In particular, $\partial T_{a,g}(t)(\phi_1+ph)/\partial p$ at $p = 0$ is in the tangent space of $W^s(\alpha)$. This function $D_{\phi_1} T_{a,g}(t)h$ is the solution of the linear variational equation

$$\dot{y}(t) = -\int_{-1}^{0} a(-\theta)g'(T_{a,g}(t)\phi_1)y(t)$$

with $y_0 = h$. If the solution operator of the adjoint of this equation is one-to-one, then the assertion that $W^s_{\tau,r}(\alpha)$ contains a ball is false. In fact, take a vector $\eta \neq 0$ such that $\langle\eta,\phi\rangle = 0$ for all ϕ in the tangent space to $W^s(\alpha)$ at $T_{a,g}(\tau)\phi_1$. Integrating the adjoint equation for a function z_t on $[0,\tau]$ with $z_\tau = \eta$, one obtains, for all h,

$$(z_0,h) = (z_\tau,h) = (\eta, D_{\phi_1} T_{a,g}(\tau)h) = 0.$$

Thus, $z_0 = 0$. Since the solution operator for the adjoint equation is one-to-one, it follows that $\eta = 0$. Consequently, the set $W^s_{\tau,r}(\alpha)$ contains no ball, which implies that $W^s_{\tau,r}$ is closed and nowhere dense. Thus $W^s(\alpha)$ has no interior. This will prove the conjecture.

To show the solution operator for the adjoint equation is one-to-one, it is sufficient to show that the solution operator of the linear variational equation is one-to-one and the functions which have backward extensions are dense in C.

If the zeros g are isolated and there is an s in $[0,1]$ such that $\ddot{a}(s) > 0$, then Theorem 3.1 implies every solution of (2.1) approaches an equilibrium point of g. If we assume, in addition, that there are only a finite number of zeros of g, then the method of proof of Theorem 3.1 in Hale [3] shows that $A_{a,g}$ is compact and attracts bounded sets of C. Also,

$$A_{a,g} = \bigcup_{j=1}^{k} W^u(\alpha_j)$$

where α_j is a zero of g and $W^u(\alpha_j)$ is the unstable manifold of α_j. The unstable manifold of any zero has dimension ≤ 1 as is easily seen from Equation (2.5). Thus, $A_{a,g}$ is the union of a finite number of manifolds of dimension ≤ 1. If each α_j is hyperbolic and the ω-limit set of each unstable manifold $W^u(\alpha_j)$ of an unstable equilibrium point is a stable equilibrium point, then $A_{a,g}$ has the one-dimensional structure shown in Figure 1.

Figure 1.

Suppose $\mathscr{A} \times \mathscr{G}$ is a uniformly bounded dissipative set containing (a,g). To see how to construct such a set, consult the proof of Theorem 3.1 in Hale [3]. We can show that (a,g) is structurally stable in $\mathscr{A} \times \mathscr{G}$. In fact, Theorem 2.5 implies $A_{b,h}$ is upper semicontinuous at (a,g). Also, there is a neighborhood U of $A_{a,g}$ such that the flow is given as in Figure 2.

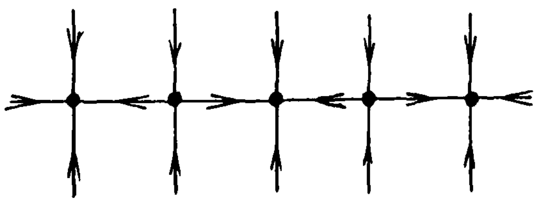

Figure 2.

This neighborhood U can be chosen to be invariant for all $(b,h) \in V$. Thus the new $A_{b,h}$, $(b,h) \in V$ must have the same structure as $A_{a,g}$. We state this as

Theorem 3.3. Suppose there is an s in $(0,1)$ such that $a(s) > 0$ and there are only a finite number of equilibrium points of (2.1), each of which is hyperbolic. If the ω-limit set of an unstable manifold is a stable equilibrium and $\mathscr{A} \times \mathscr{G}$ is a uniformly bounded dissipative set containing (a,g), then (a,g) is structurally stable in $\mathscr{A} \times \mathscr{G}$.

One can now ask the following interesting question.

Question 3.4. Suppose conditions (3.1), (3.2) are satisfied and a is a fixed function satisfying $\ddot{a}(s) > 0$ for $s \in (0,1)$. Let $\mathscr{G}_k = \{g$ which have exactly $2k+1$ zeros which are all simple$\}$. How many different equivalence classes are in $\mathscr{G}_k$?

If $k = 0$, that is, g has only one zero α, then $A_{a,g} = \{\alpha\}$, the constant function α and all $g \in \mathscr{G}_0$ are equivalent.

If $k = 1$, then $g \in \mathscr{G}_1$ has three simple zeros, $\alpha_1 < \alpha_2 < \alpha_3$, with α_1, α_3 asymptotically stable and α_2 a saddle point. Since the unstable manifold at α_2 is smooth, one dimensional and $A_{a,g}$ is uniformly asymptotically stable, it follows that $A_{a,g}$ is a one dimensional manifold with boundary points α_1, α_3. Again, all elements of $\mathscr{G}_1$ are equivalent.

The topological structure of $A_{a,g}$ is not understood for the case when g has given zeros $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 < \alpha_5$. If we consider g depending on some parameters μ with $g(x,\mu) = x^5$ at $\mu = 0$, then $A_{a,g}(\mu)$ is $\{0\}$ for $\mu = 0$. There is a center manifold in a neighborhood of $x = 0$ which is one dimensional and smooth in μ.

As μ varies near $\mu = 0$, $g(x,\mu)$ can have as many as 5 zeros which must lie on this center manifold. Thus, $A_{a,g}(\mu)$ is either a point or a one-dimensional manifold with boundary for μ small. As μ increases, it is conceivable that the topological structure of $A_{a,g}$ changes. Let us give some intuitive reasons for why this is possible. The author wishes to acknowledge conversations with John Mallet-Paret which were of great assitance in the remaining discussion of this section.

Suppose g has five zeros and the general shape shown in Figure 3. If a is strictly convex and very

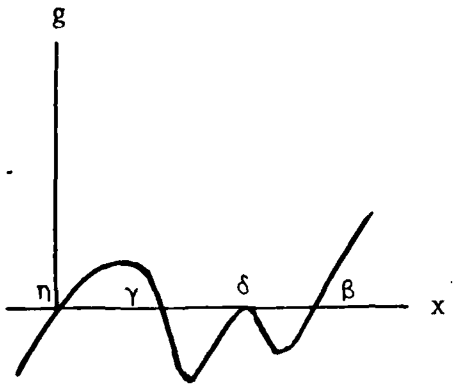

Figure 3.

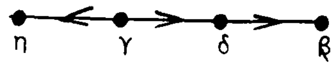

Figure 4.

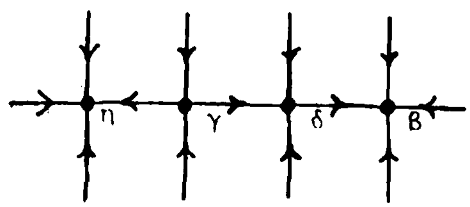

Figure 5.

close to the δ-function at zero, then $A_{a,g}$ is shown in Figure 4. The complete flow near $A_{a,g}$ is shown in Figure 5 with the exponential decay toward $A_{a,g}$ being very rapid and much greater than the convergence of the flow on $A_{a,g}$ toward β.

If $g'(\beta) = \alpha$, then the flow near β is determined by the roots of the characteristic equation

$$\lambda = -\alpha \int_{-1}^{0} a(-\theta) e^{\lambda\theta} d\theta . \tag{3.4}$$

One can show there is an α_0 and a strictly convex function a_0 with the property that the roots of this equation with maximum real part corresponds to a double root root. Furthermore, in any neighborhood of (α_0, a_0) there is an α and a strictly convex function a such that Equation (3.4) has a pair of complex conjugate roots $\lambda_0, \bar{\lambda}_0$ with maximum real parts. For this a and the corresponding g and from the fact that the flow near $A_{a,g}$ is shown in Figure 5 with the stable manifolds transversal to $A_{a,g}$ of codimension one, the flow near $A_{a,g}$ is governed by what happens on a two-dimensional manifold $M_{a,g}$ determined by the invariant manifolds corresponding to the roots $\lambda_0, \bar{\lambda}_0$. This manifold is constructed by finding a local invariant manifold near β which is tangent to the invariant subspace of the linear approximation near β generated by the eigenvalues $\lambda_0, \bar{\lambda}_0$. Then find $M_{a,g}$ by backward extension along the flow. On $M_{a,g}$ the flow is shown in Figure 6.

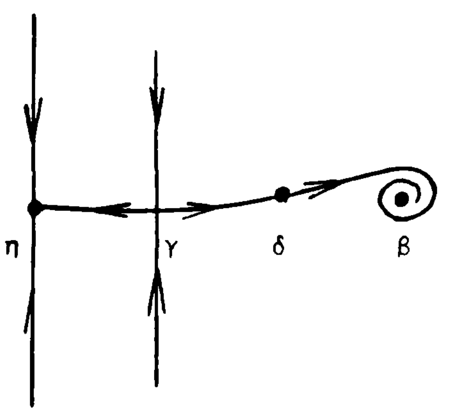

Figure 6.

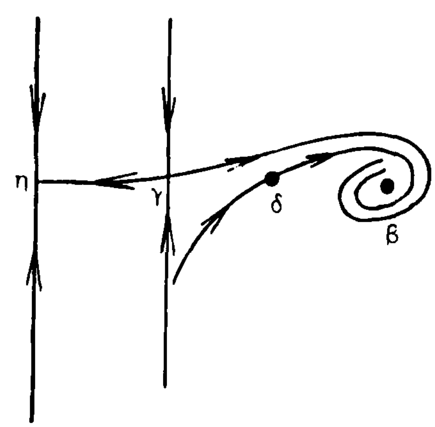

Figure 7.

If it is possible to show that one can break the connection between γ, δ by a small variation in a, then the new picture will be as in Figure 7. This is the point that the author has been unable to verify.

Assuming that the previous step can be done, we have changed the orientation of the points on $A_{a,g}$; namely from Figure 8a) to Figure 8b). Thus, they are in different equivalence classes.

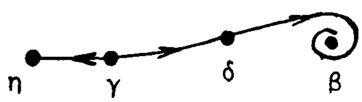

Figure 8a)

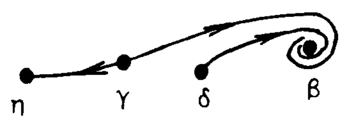

Figure 8b)

A small change in g to make the double zero either disappear or give two simple ones gives the distinct structurally stable systems shown in Figure 9. We draw the spirals as arcs since they are the

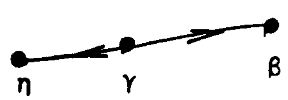

Figure 9a)

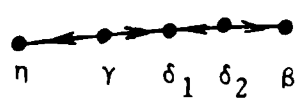

Figure 9b)

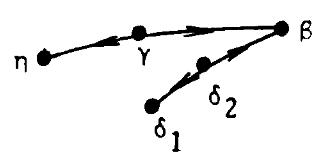

Figure 9c)

same under homeomorphism.

Now let us vary g some more to attempt to go from the structurally stable system in Figure 9c) to the one in Figure 9b) and keep the zeros of g simple. The only possible way to do this is to have a saddle connection; namely, have a function g for which the saddle δ_2 is connected to γ with the configuration shown in Figure 10.

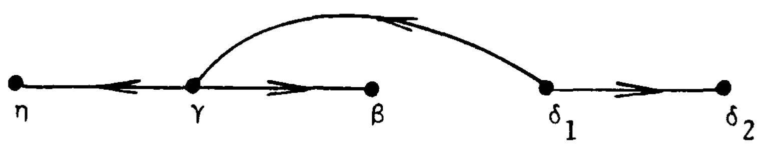

Figure 10.

Conjecture 3.5. There is a saddle connection for some (a,g) with a strictly convex and g having five simple zeros. The set of (a,g) for which this is true is not generic.

Theorem 3.1 can be generalized to an n-vector system of the form

$$\dot{x}(t) = -\int_{-1}^{0} a(-\theta)\,\mathrm{grad}\, G(x(t+\theta))\,d\theta \tag{3.5}$$

where $x \in \mathbb{R}^n$, $G: \mathbb{R}^n \to \mathbb{R}$. The proof will be the same as in Hale [3].

A proof or counterexample to the following conjecture would be interesting:

Conjecture 3.6. If $a(1) = 0$, $\ddot{a}(s) > 0$ for $0 \le s \le 1$, then the set of G for which Equation (3.5) is structurally stable is residual in the set of C^2 functions from $\mathbb{R}^n$ to $\mathbb{R}$ with the Whitney topology.

Let us now consider in more detail the case when the kernel in Equation (2.1) is a linear function. Consider the equation

$$(3.6) \qquad \dot{x}(t) = -\mu \int_{-1}^{0} a_0(-\theta) g(x(t+\theta)) d\theta, \quad a_0(s) = (1-s), \ \mu > 0$$

and assume that the zeros of g are isolated. From part (ii) of Theorem 3.1, we know that the ω-limit set of every solution of (3.6) is either an equilibrium point of a periodic orbit generated by a 1-periodic solution of the ordinary differential equation

$$(3.7) \qquad \ddot{y} + \mu g(y) = 0.$$

As we shall see below, the structure of $A_{a,g}$ can be very complicated and is not very well understood. However, the following interesting fact is true.

Proposition 3.6. For Equation (3.6), no periodic orbit can be uniformly asymptotically stable relative to perturbations in $A_{\mu a_0, g}$. If $A_{\mu a_0, g}$ contains a periodic orbit, then $(\mu a_0, g)$ cannot be structurally stable.

Proof: Suppose $A_{\mu a_0, g}$ contains a periodic orbit γ which is uniformly asymptotically stable. Then, for any neighborhood U of γ there is a neighborhood V of γ and a neighborhood W of μa_0 in the C^0-topology such that, for any $a \in W$, $\phi \in V$, the positive orbit $\gamma^+(\phi)$ of (2.1) belongs to U. In particular, one can choose the neighborhood U of γ and W of μa_0 so that U contains no equilibrium point of (2.1) for any $a \in W$.

In the neighborhood W of μa_0 there exists a strictly convex function a. Thus, $A_{a,g}$ contains no periodic orbits and every solution of (2.1) approaches an equilibrium point. This contradicts the fact that some positive orbits remain in U. This proves no periodic orbit can be uniformly asymptotically stable. The fact that (a_0,g) cannot be structurally stable if $A_{\mu a_0,g}$ contains a periodic orbit follows by choosing a strictly convex function near μa_0. This proves the proposition.

To understand more about the possible topological structures for $A_{\mu a_0,g}$, we consider some special cases. Suppose

$$yg(y) > 0, \; y \neq 0, \; g'(0) = 1.$$

If $p(t,G)$ is the solution of the equation

$$\ddot{y} + g(y) = 0$$

with $p(0,b) = b > 0$, $\dot{p}(0,b) = 0$, then $p(t,b)$ has period $\omega(b) \to 2\pi$ as $b \to 0$. We assume that $\omega(b)$ is a strictly monotone function of G for $b > 0$.

The solutions of (3.7) are then given by $p(\mu^{1/2}t,b)$ and have period $\omega(b)/\mu^{1/2}$.

The linear variational equation around $x = 0$ is

$$\dot{x}(t) = -\mu \int_{-1}^{0} (1+\theta)x(t+\theta)d\theta \tag{3.8}$$

for which the characteristic equation is

$$\lambda = -\mu \int_{-1}^{0} (1+\theta)e^{\lambda\theta} \tag{3.9}$$

or

$$\lambda + \mu(\lambda-1+e^{-\lambda})\lambda^{-2} = 0. \tag{3.10}$$

Equation (3.7) is a special case of Equation (2.1) and so Theorem 3.1 may be applied. Thus, we see that every solution of (3.7) approaches zero as $t \to \infty$ if

$$\mu \neq 4\pi^2k^2, \quad k = 1,2,3,\ldots\,. \tag{3.11}$$

This implies the roots of Equation (3.9) have negative real parts if μ satisfies (3.10). Consequently, the solution $x = 0$ of Equation (3.6) is uniformly asymptotically stable if μ satisfies Relation (3.10).

For $\mu = 4\pi^2k^2$, Equation (3.9) has the purely imaginary solutions $\pm\lambda$, $\lambda = 2\pi ki$ with all other solutions having negative real parts. Also, the solution $\lambda(\mu)$ with $\lambda(4\pi^2k^2) = 2\pi ki$ satisfies

$$\frac{d\lambda}{d\mu} = \frac{i}{6\pi k} \qquad \text{at} \quad \mu = 4\pi^2 k^2$$

$$\frac{d^2\lambda}{d\mu^2} = \frac{1}{3}\left(1 - \frac{1}{24\pi^4 k^4}\right) \qquad \text{at} \quad \mu = 4\pi^2 k^2.$$

Thus, Re λ behaves as a quadratic function and Im λ as a linear function in a neighborhood of $\mu = 4\pi^2 k^2$.

We can begin to use this information to analyze the set $A_{\mu a_0, g}$ for Equation (3.6). If $0 < \mu < 4\pi^2$, there are no 1-periodic solutions of Equation (3.7) and every solution approaches zero as $t \to \infty$. Thus, $A_{\mu a_0, g} = \{0\}$. If $4\pi^2 < \mu < 16\pi^2$, there is exactly one 1-periodic orbit γ'_μ in $A_{\mu a_0, g}$ generated by the 1-periodic solution of Equation (3.7). Also, $\{0\} \subset A_{\mu a_0, g}$ and the solution $x = 0$ is uniformly asymptotically stable by the previous analysis of Equation (3.7). For μ very close to zero, there is a two dimensional center manifold M'_μ which contains $A_{\mu a_0, g}$. Since $A_{\mu a_0, g}$ is uniformly asymptotically stable, it follows that $A_{\mu a_0, g}$ is the two manifold consisting of the orbit γ'_μ and its interior relative to M'_μ. The flow on $A_{\mu a_0, g}$ is given as in Figure 11. The orbit γ'_μ in the center manifold

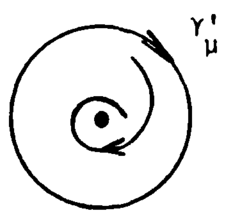

Figure 11.

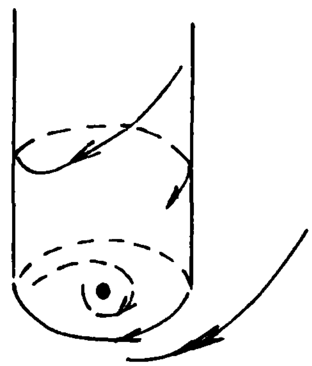

Figure 12.

M'_μ is asymptotically stable from the outside and unstable from the inside. The orbit γ'_μ has an ∞-dimensional stable manifold which is transversal to M'_μ so that locally the flow is similar to the one shown in Figure 12. Everything inside the "cylinder" through γ'_μ consisting of the stable manifold transverse to M'_μ approaches zero and everything outside approaches γ'_μ. If we vary μ in the interval $4\pi^2 < \mu < 16\pi^2$, the same picture will prevail since 0 and γ'_μ must be the attractor for all orbits. Thus, $A_{\mu a_0, g}$ is a "disk" for $4\pi^2 < \mu < 16\pi^2$ and is given by $A_{\mu a_0, g} = \{0\} \cup W^u(\gamma'_\mu)$.

At $\mu = 16\pi^2$, there is a solution of (3.9) on the imaginary axis given by $\lambda = 4\pi i$ and for $16\pi^2 < \mu < 36\pi^2$, there is a unique periodic orbit γ^2_μ of (3.6) of period 1/2 (and therefore of period 1) generated by the (1/2-periodic solution of (3.7). For $16\pi^2 < \mu < 36\pi^2$, the origin is uniformly asymptotically stable from the above analysis of Equation (3.8). Again, there is a two-dimensional center manifold M^2_μ through $x = 0$ which is transversal to $\{0\} \cup W^u(\gamma'_{16\pi^2})$ and asymptotically stable. The orbit γ^2_μ must be on M^2_μ. Reasoning as before, we have for $\mu > 16\pi^2$ but close to $16\pi^2$ that $A_{\mu a_0, g} \cap M^2_\mu = \{0\} \cup W^u(\gamma^2_\mu)$. Since $W^u(\gamma^2_\mu)$ is transversal to $W^u(\gamma'_\mu)$, the set $A_{\mu a_0, g}$ is two-dimensional and is the union of a finite number of manifolds. Locally near 0, the set $A_{\mu a_0, g}$ is like two disks in 4-dimensional space intersecting at zero.

As μ passes through $36\pi^2$, another priodic orbit of period 1/3 appears and something similar happens, but the geometry becomes more complicated.

If the function g in (3.6) has three simple zeros, say $\alpha < \beta < \gamma$, then β must always be an unstable saddle point with $W^u(\beta)$ one-dimensional. The equilibrium points α,γ will be uniformly asymptotically stable as long as $\mu g'(\alpha) \neq 4\pi^2k^2$, $\mu g'(\beta) \neq 4\pi^2k^2$ for all $k = 1,2,\ldots$. The phase portrait for the second order equation (3.7) is shown in Figure 13. If there is no

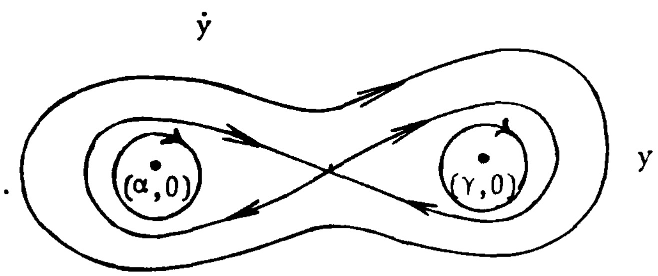

Figure 13

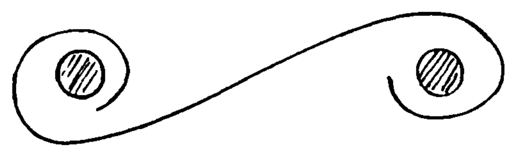

Figure 14a)

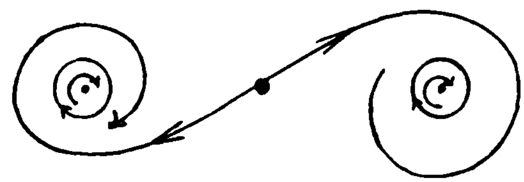

Figure 14b)

periodic orbit of period 1, then $A_{\mu a_0,g}$ is a one-dimensional manifold with boundary points being the constant functions α,γ. Suppose now that there is no periodic solution of period 1 whose orbit encircles all three equilibrium points and only one encircling each of the points $(\alpha,0),(\gamma,0)$. Arguing as for the previous case, one sees that $A_{\mu a_0,g}$ has the structure shown in Figure 14a) with the flow on $A_{\mu a_0,g}$ shown if Figure 14b). Each of the periodic orbits is unstable from the inside on $A_{\mu a_0,g}$. Suppose now that there is no 1-periodic orbit of (3.7) encircling one equilibrium point and exactly one 1-periodic orbit γ encircling all three equilibriums. We do not know the structure of $A_{\mu a_0, g}$ but conjecture that it is a two-dimensional manifold with boundary γ, the equilibrium points α, γ are asymptotically stable and β, γ are unstable on $A_{\mu a_0, g}$.

By choosing different functions g, one can certainly make the set $A_{\mu a_0, g}$ have a large number of periodic orbits and equilibrium points. The way these periodic orbits and equilibrium points fit together in the flow on $A_{\mu a_0, g}$ is not known. However, from Proposition 3.5, we know that no periodic orbit in $A_{a_0, f}$ can be uniformly asymptotically stable.

Fron the above analysis, we see that Equation (2.1) with $a(s)$ strictly convex has a very nice structure which appears to be structurally stable for a residual set of functions g. Also, one can show that (a,g) structurally stable for $\bar{a}$ strictly convex will imply that (a,g) is structurally stable if g is close to $\bar{g}$ and a is close to $\bar{a}$; that is, the kernel also can be cahnged slightly.

If $a_0(s)$ is linear, it is probably true that (a_0, g) is structurally stable for a residual set of g. However, (a_0, g_0) structurally stable subject to variations in g does not imply that (a,g) will be structurally stable if one changes a_0 slightly to a new function a (Proposition 3.5). In fact, the structure of $A_{a,g}$ can be drastically different for a strictly convex function a which is close to a_0 in the C^2-topology. In fact, all periodic orbits in $A_{\mu a_0, g}$ will disappear. Thus, it becomes interesting to analyze the behavior of the solutions of (2.1) for an arbitrary function a close to the linear function μa_0 in the C^2-topology. This is the topic disuccsed in the next section.

4. Bifurcation near linear kernels. In this section, we consider equation (2.1) with

$$(4.1)\qquad \begin{aligned} &xg(x) > 0 \quad \text{for} \quad x \neq 0,\ g'(0) = 1, \\ &G(x) = \int_0^x g \to \infty \quad \text{as} \quad |x| \to \infty, \end{aligned}$$

where g is continuous together with all derivative up through order five and a is in a C^0-neighborhood of $4\pi^2 a_0$, $a_0(s) = 1 - s$. We also suppose the period $\omega(b)$ of the solution through $(b,0)$, $b > 0$, of the equation

$$(4.2)\qquad \ddot{x} + g(x) = 0$$

satisfies

$$(4.3)\qquad \omega'(b) \neq 0.$$

Under an additional conditional on g which will be specified later, we determine the behavior of the solutions of (2.1) in a neighborhood of $x = 0$ for all a in a neighborhood of the bifurcation point $4\pi^2 a_0$. The analysis is based on the application of the method of Liapunov-Schmidt for periodic orbits near $x = 0$ together with properties of the bifurcation function from deOliveira and Hale [2].

We need the following lemma.

Lemma 4.1. *There is a neighborhood U of $4\pi^2 a_0$ in the C^0-topology, a $\delta > 0$ and an analytic function $\lambda^*: U \to \mathbb{C}$ such that* $\lambda^*(4\pi^2 a_0) = 2\pi i$

and, for every $a \in U$, the equation

$$\lambda + \int_{-1}^{0} a(\theta)e^{\lambda\theta}d\theta = 0 \tag{4.4}$$

has exactly one solution in each of the circles $|\lambda \pm 2\pi i| < \delta$ given respectively by $\lambda^*(a), \bar{\lambda}^*(a)$ and all other solutions with real parts $\leq -\delta$. Furthermore, if

$$\Gamma^- = \{a \in U: \ \text{Re}\ \lambda^*(a) < 0\}$$

$$\Gamma^0 = \{a \in U: \ \text{Re}\ \lambda^*(a) = 0\}$$

$$\Gamma^1 = \{a \in U: \ \text{Re}\ \lambda^*(a) > 0\}$$

then each of these sets is nonempty and Γ^0 is a submanifold of codimension one.

Proof. If

$$F(\lambda, a) = \lambda - \int_{-1}^{0} a(\theta)e^{\lambda\theta}d\theta$$

then $F(2\pi i, 4\pi^2 a_0) = 0$ and $\partial f/\partial\lambda = 1 - 8\pi^2$ at $(2\pi i, 4\pi^2 a_0)$. The Implicit Function Theorem implies the existence of a function $\lambda^*(a) \in \mathbb{C}$ analytic in a neighborhood of a_0 with $\lambda^*(a_0) = 2\pi i$. The other properties of λ^* follow essentially from Rouché's Theorem.

To show Γ^0 has codimension one, consider the family of functions $4\pi^2(a_0+\nu b_0)$, $b_0(s) = s(1-s)$, $\nu \in \mathbb{R}$. Then the derivative of $\lambda^*(4\pi^2(a_0+\nu b_0)$ with respect to ν at $\nu = 0$ is easily seen to satisfy

$$\left.\frac{\partial\lambda^*}{\partial\nu}\right|_{\nu=0} = -4\pi^2 \int_{-1}^{0} \theta(1+\theta)e^{2\pi i\theta}d\theta$$

Thus,

$$\left.\frac{\partial \mathrm{Re}\ \lambda^*}{\partial\nu}\right|_{\nu=0} = -4\pi^2 \int_{-1}^{0} \theta(1-\theta)\cos\ 2\pi\theta d\theta > 0.$$

This shows that Γ^0 has codimension one and also that Γ^-,Γ^+ are not empty. This proves the lemma.

Remark 4.2. We knew from the previous section that $\mathrm{Re}\ \lambda^*(a) < 0$ if $a \in U$ is strictly convex. The above proof shows that $\mathrm{Re}\ \lambda^*(a) > 0$ if $a \in U$ is of the form $a = 4\pi^2 a_0 + b$ where b is strictly concave, $b(0) = b(1) = 0$.

For $a = 4\pi^2 a_0$, the characteristic equation for the linear variational equation around zero has two purely imaginary roots. For a near $4\pi^2 a_0$, and a neighborhood W of zero let $B(r,a,g)$ be the scalar bifurcation function obtained by applying the usual method of Liapunov-Schmidt for the periodic solutions of (2.1) in W which for $a = 4\pi^2 a_0$ are equal to $r \cos 2\pi t$ (see, for example, deOliveira and Hale [2]). This function has the property that the periodic solutions of the type specified are in one to one correspondence with the nonnegative zeros of $B(r,a,g)$. Furthermore, the stability properties of the periodic solution corresponding to a zero r_0 of $B(r,a,g)$ when restricted to a center manifold at $x = 0$ are the same as the stability properties of the equilibrium point r_0 of the scalar equation

$$(4.5) \qquad \dot{r} = B(r,a,g)$$

(see deOliveira and Hale [2]). The function $B(r,a,g)$ is an odd function of r and has five continuous derivatives. Let

$$(4.6) \qquad B(r,a,g) = \alpha_1(a,g)r + \alpha_3(a,g)r^3 + \alpha_5(a,g)r^5 + o(|r|^5)$$

as $r \to 0$.

If $\lambda^*(a)$ is the function given in Lemma 4.1, the manner in which the bifurcation function is constructed implies that

$$(4.7) \qquad \begin{aligned} &\alpha_1(a,g) = 0 \quad \text{if and only if} \quad \operatorname{Re} \lambda^*(a) = 0 \\ &\operatorname{sign} \alpha_1(a,g) = \operatorname{sign} \operatorname{Re} \lambda^*(a) \end{aligned}$$

Thus, $\alpha_1(4\pi^2 a_0, g) = 0$.

Let $\alpha_3^0 = \alpha_3(4\pi^2 a_0, g)$. We now show that $\alpha_3^0 = 0$. Since the solution $x = 0$ of Equation (2.1) for $a = 4\pi^2 a_0$ is asymptotically stable, it follows that the zero solution of (4.5) for $a = 4\pi^2 a_0$ is asymptotically stable. Thus, $\alpha_3^0 \leq 0$. If $\alpha_3^0 < 0$, then the function $B(r,a,g)$ for a varying in a small neighborhood of $4\pi^2 a_0$ can have only three possible shapes in a neighborhood of $r = 0$ as shown in Figure 15.

Now consider the special function $a = \mu a_0$ for μ varying in small neighbhood of $4\pi^2$, $|\mu - 4\pi^2| < \varepsilon$. If $\mu \neq 4\pi^2$, we know that $\operatorname{Re} \lambda^*(\mu a_0) < 0$ from the previous section. Thus, $\alpha_1(\mu a_0, g) < 0$ if $\mu \neq 4\pi^2$. For definiteness, suppose $\omega'(b) < 0$ in (4.3). Then,

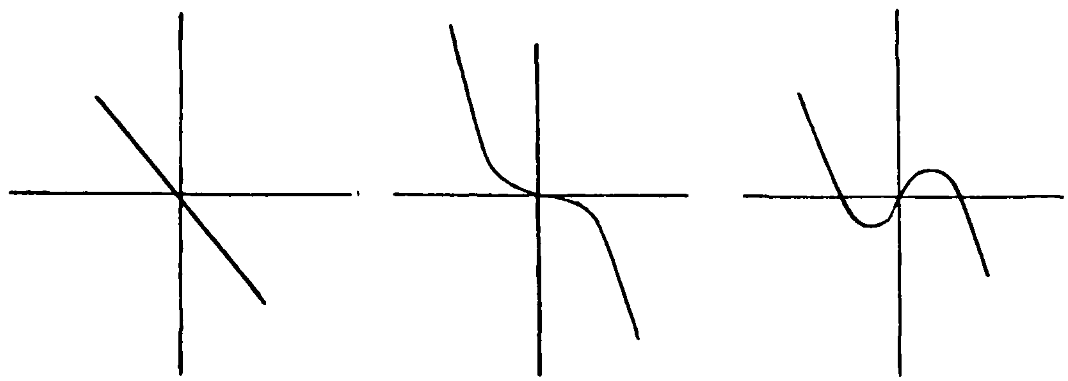

Figure 15.

the analysis in the previous section implies there is an $\varepsilon > 0$ and a sufficiently small neighborhood W of $x = 0$ such that there is no periodic orbit of (2.1) in W for $0 < \mu - 4\pi^2 < \varepsilon$ and a unique periodic solution γ_μ in W for $-\varepsilon < \mu - 4\pi^2 < 0$, $\gamma_\mu \to 0$ as $\mu \to 4\pi^2$ and γ_μ is unstable in $A_{a,g}$. Thus on a center manifold, γ_μ is stable from the outside and unstable from the inside. This means the same is true for the corresponding equilibrium point r_μ of Equation 4.5. Consequently, the bifurcation function must have the shape shown in Figure 16. This is impossible if $\alpha_3^0 < 0$. Thus,

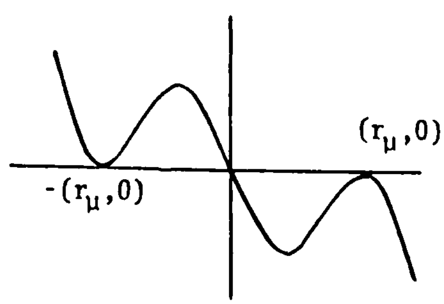

Figure 16.

Thus, $\alpha_3^0 = 0$.

Again, the stability of the zero solution of Equation (2.1) for $a = 4\pi^2 a_0$ implies that $\alpha_5(4\pi^2 a_0, g) \leq 0$. We make the hypothesis that

$$\alpha_5^0(g) \overset{\text{def}}{=} \alpha_5(4\pi^2 a_0, g) < 0. \tag{4.8}$$

This implies that

$$B(r, 4\pi^2 a_0, g) = \alpha_5^0(g) r^5 + o(r^5), \quad \alpha_5^0 < 0. \tag{4.9}$$

We have not made the computations (which would be extremely complicated) to obtain the constant $\alpha_5^0(g)$. However, it certainly seems plausible that the set of g for which $\alpha_5^0(g) < 0$ is open in the space $C^5(U, \mathbb{R})$ for a given bounded neighborhood U of $x = 0$.

If $B(r,a,g) = rP(r^2,a,g)$, then

$$P(\rho,a,g) = \alpha_1(a,g) + \alpha_3(a,g)\rho + \alpha_5(a,g)\rho^2 + o(\rho^2) \tag{4.10}$$

as $\rho \to 0$. This function has a unique maximum $\eta(a,g)$ in a neighborhood U of $a = 4\pi^2 a_0$ which occurs at a value $\rho^*(a,g)$ and $\eta(4\pi^2 a_0, g) = 0$. If

$$
(4.11)\qquad
\begin{aligned}
SN^0 &= \{a \in U:\ \eta(a,g) = 0,\ \rho^*(a,g) \geq 0\} \\
SN^{+(-)} &= \{a \in U:\ \eta(a,g) > (<) 0,\ \rho^*(a,g) \geq 0\}.
\end{aligned}
$$

One can show that every tangent vector to SN^0 is a tangent vector to Γ^0. We suppose that

$$
(4.12)\qquad SN^0 \text{ is a submanifold of codimension } 1,\ S^+ \neq \phi.
$$

It is possible to show that hypothesis (4.12) is satisfied for an open set of $g \in C^5(W,\mathbb{R})$ for a bounded neighborhood W of zero.

We can now prove the following result.

Theorem 5.3. With hypotheses (4.1), (4.3), (4.8) and (4.12), there is a neighborhood U of $4\pi^2 a_0$ in the C^0-topology and a neighborhood W of $x = 0$ such that U is subdivided into regions as shown in Figure 17, the set $A_{a,g}$ is a disk for each $a \in U$ with boundary being a periodic orbit and the flow on a two dimensional manifold in $A_{a,g}$ is shown in Figure 17.

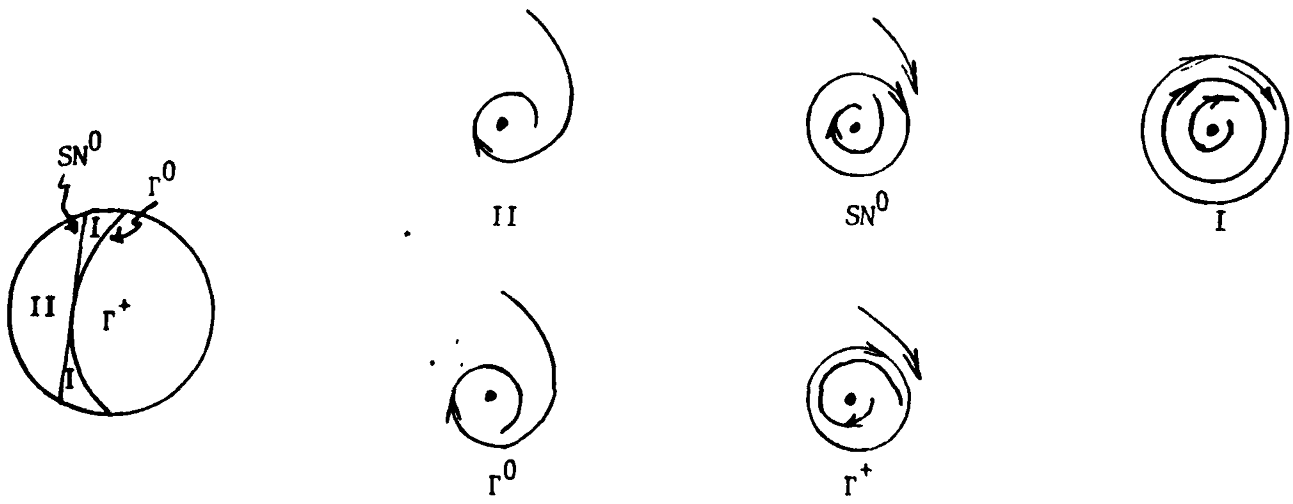

Figure 17.

<u>Proof</u>. If $a \in \Gamma^+$, then $\alpha_1(a,g) > 0, \alpha_3(a,g) < 0$ implies that $P(p,a,g)$ in (4.10) has a unique positive zero. Thus, there is a unique periodic solution in a small neighborhood of zero and it is asymptotically stable as shown in the flow for Γ^+. This shows that $SN \subseteq \Gamma^-$. By hypothesis (4.12) and the fact that the stability properties of the priodic orbits are determined by (4.5), we have that the flow on SN^0 is the one shown in Figure 17. Also, the flow in the other two regions must be one of those shown in Figure 17. We only need to verify that the regions are ordered as shown. In the proof of Lemma 4.1 we showed that the curve $4\pi^2(a_0+\nu b_0)$, $b_0(s) = s(1-s)$ was transversal to Γ^0 at $\nu = 0$. For $\nu < 0$ this function is in Γ^- and stictly convex. Thus, the origin is uniformly asymptotically stable. This proves that the flow in Region II in Figure 17 is the one that is depicted. This proves the theorem.

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