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A MIXED EXPONENTIAL TIME SERIES MODEL. NMEARMA(P,Q).(U)
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A MIXED EXPONENTIAL TIME SERIES MODEL,

NMEARMA(p,q)

by

A. J. Lawrance

and

P. A. W. Lewis

March 1980

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A MIXED EXPONENTIAL TIME SERIES MODEL, NMEARMA(p,q)

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ABSTRACT

A first-order stochastic difference equation with random coefficients is shown to have a solution which makes the marginal distribution of the stationary sequence generated by the equation a convex mixture of two exponential distributions. This Markovian process should be broadly applicable in stochastic modelling in operations analysis. Moreover it is extended quite simply to a mixed exponential process with mixed pth-order autoregressive and qth-order moving average correlation structure. Coupling of the processes to model multivariate situations is also discussed.

1. Introduction

Sequences of independent, exponentially distributed random variables $\{\varepsilon_n\}$ play a central role in stochastic modelling in operations analysis, either to characterize the times-between-events in homogeneous Poisson arrival processes or to characterize, for example, time series of successive service times in a queue or successive response times at a computer terminal. Unfortunately both the assumption of independence and/or the assumption of an exponential distribution can be tenuous in practice. The distributional assumption has long been relaxed, as in the renewal model for arrival processes, but independence has been found difficult to relax in simple tractable models. For instance, the Box-Jenkins or ARMA models, well known in time series analysis, are mainly suitable for modelling time series with marginal Gaussian distributions and are inappropriate for positive variables, such as response times. However, a tractable and flexible time series model has recently been introduced by Lawrance (1980), Lawrance and Lewis (1981), but developed only for time series with exponential marginal distributions. The development here is to extend this new model into mixed exponential variables. Mixed exponential variables are overdispersed relative to exponential variables while retaining their nonmodal distributional aspect. Further, they are often used when data suggests that exponential assumptions are incorrect.

2. The Model to be Developed

A time series of equally spaced observations will be represented by the random variables $\{X_n, n = 0, \pm 1, \pm 2, \ldots\}$. We shall be concerned with the following general model,

$$X_n = \varepsilon_n + \begin{cases} \beta X_{n-1} & \text{w.p. } \alpha \\ 0 & \text{w.p. } 1-\alpha \end{cases}, \tag{2.1}$$

$$= \varepsilon_n + \beta V_n X_{n-1}, \tag{2.2}$$

where the V_n are i.i.d. binary variables with $P(V_n = 1) = 1 - P(V_n=0) = \alpha$, and $\{\varepsilon_n\}$ are assumed to be an i.i.d. sequence. This was the two-parameter model developed in Lawrance and Lewis (1981) for stationary X_n's with exponential marginal distributions, and there called the NEAR(1) model (new exponential autoregressive of first order); here we develop the special one-parameter model, obtained by setting $\beta = 1$ in (2.1),

$$X_n = \varepsilon_n + V_n X_{n-1} , \tag{2.3}$$

for $\{X_n\}$ having a mixed exponential marginal distribution. The first and crucial step is to determine the distribution of ε_n in terms of the marginal distribution of $\{X_n\}$; the model (2.1) or (2.3) provides no guarantee that this can be achieved. The model (2.3) turns out to be very tractable, while still preserving a geometric α^r correlation structure; however, it does restrict sample path behavior to a preference for runs of rising values. Developments of the model in Section 5, 6 and 7 to both higher order autoregressive and higher order moving average dependency overcome this effect to a considerable degree; the more general model (2.2) with mixed exponential marginals has been found to be very limited by tractability.

The marginal distribution function for the mixed exponential variables $\{X_n\}$ will be denoted by $F_X(x)$ where

$$1 - F_X(x) = \pi_1 e^{-x/\mu_1} + \pi_2 e^{-x/\mu_2} \tag{2.4}$$

and $\pi_1, \pi_2 > 0$, $\pi_1 + \pi_2 = 1$ and $\mu_1, \mu_2 \geq 0$, $\mu_1 \neq \mu_2$; since $\pi_1, \pi_2 > 0$ and $\mu_1 \neq \mu_2$ we are considering only a convex mixed exponential and not including the ordinary exponential.

Equations (2.3) and (2.4) constitute a model which will be called NMEAR(1), synonymous for 'new mixed-exponential first-order autoregressive process". The necessary choice for the distribution of ε_n is given in Section 3.

A mixed exponential random variable with distribution function given by (2.4) will be denoted by $ME(\pi_1,\mu_1;\pi_2,\mu_2)$; it can be usefully written as

$$X_n = K_n E'_n \tag{2.5}$$

where $P(K_n = \mu_1) = 1 - P(K_n = \mu_2) = \pi_1$ and E_n is a unit exponential random variable. The customary measure of its dispersion relative to an exponential variable is the coefficient of variation $C(X)$ and

$$C(X) = sd\left[\frac{X}{E(X)}\right] = \frac{sd(X)}{E(X)} = \frac{\{\pi_1\mu_1^2 + \pi_2\mu_2^2 + \pi_1\pi_2(\mu_1-\mu_2)^2\}^{1/2}}{(\pi_1\mu_1 + \mu_2\pi_2)} \quad . \tag{2.6}$$

This is always greater than one, its value in the exponential case.

3. <u>The Innovation Process for the NMEAR(1) Process</u>

To establish the existence and distributional form of the innovation sequence ε_n in (2.3) we prove the following result:

<u>Theorem</u>. For $0 \leq \alpha < 1$ the stationary Markovian sequence $\{X_n\}$ defined by (2.3) has a convex mixed exponential marginal distribution (2.4) if and only if the i.i.d. sequence $\{\varepsilon_n\}$ has the convex mixed exponential distribution function $F_\varepsilon(x)$ given by

$$1 - F_\varepsilon(x) = \eta_1 e^{-x/\gamma_1} + \eta_2 e^{-x/\gamma_2} \;, \qquad (x \geq 0) \;, \tag{3.1}$$

where $\eta_1, \eta_2 > 0$, $\eta_1 + \eta_2 = 1$, $\gamma_1, \gamma_2 > 0$ and

$$\mu = \pi_1\mu_1 + \pi_2\mu_2 \;; \tag{3.2}$$

$$\gamma_1, \gamma_2 = \frac{1}{2}\{\mu_1 + \mu_2 - \alpha\mu \pm \sqrt{[(\mu_1 + \mu_2 - \alpha\mu)^2 - 4(1-\alpha)\mu_1\mu_2]}\} \;; \tag{3.3}$$

$$\gamma_0 = \pi_2\mu_1 + \pi_1\mu_2 \;; \tag{3.4}$$

$$\eta_1 = (\gamma_1-\gamma_0)/(\gamma_1-\gamma_2); \qquad \eta_2 = (\gamma_2-\gamma_0)/(\gamma_2-\gamma_1) \quad . \tag{3.5}$$

To prove the Theorem we need the following preliminary result:

Lemma. For $0 \leq \alpha < 1$ and $0 < \pi_1 < 1$.

$$(\mu_1 + \mu_2 - \alpha\mu)^2 - 4(1-\alpha)\mu_1\mu_2 > \{\mu_1 + \mu_2 - (2-\alpha)\mu\}^2 \geq 0 \ . \qquad (3.6)$$

We have

$$\{\mu_1 + \mu_2 - (2-\alpha)\mu\}^2 = (\mu_1 + \mu_2 - \alpha\mu)^2 - 4(1-\alpha)(\mu_1 + \mu_2 - \alpha\mu)\mu + 4(1-\alpha)^2\mu^2$$

and substituting this into the left-hand side of (3.6) and cancelling like terms, the inequality (3.6) is seen to be equivalent to

$$\mu(\mu_1 + \mu_2) > \mu_1\mu_2 + \mu^2 \quad \text{or} \quad \mu(\mu_1 + \mu_2 - \mu) > \mu_1\mu_2 .$$

Using (3.2) for μ we get

$$[\pi_1\mu_1 + (1-\pi_1)\mu_2][\mu_1 + \mu_2 - (\pi_1\mu_1 + (1-\pi_1)\mu_2)] > \mu_1\mu_2 ,$$

$$[\pi_1\mu_1 + (1-\pi_1)\mu_2][(1-\pi_1)\mu_1 + \pi_1\mu_2] > \mu_1\mu_2 ,$$

$$[(\mu_1 - \mu_2)\pi_1 + \mu_2][(\mu_2-\mu_1)\pi_1 + \mu_1] > \mu_1\mu_2 ,$$

or

$$-(\mu_1-\mu_2)^2\pi_1^2 + (\mu_1-\mu_2)^2\pi_1 + \mu_1\mu_2 > \mu_1\mu_2$$

or

$$0 < \pi_1 < 1$$

which is true by assumption. Thus the Lemma is proved.

Proof of the Theorem

For the stationary sequence given by (2.3) we define

$$\phi_X(t) = E\{\exp(-tX)\}, \qquad \phi_\varepsilon(t) = E\{\exp(-t\varepsilon)\}, \qquad (3.7)$$

so that transforming both sides of (2.3) gives

$$\phi_\varepsilon(t) = \phi_X(t)/\{\alpha\phi_X(t) + (1-\alpha)\} \ . \qquad (3.8)$$

With $\phi_X(t) = (1 + \gamma_0 t)/\{(1 + \mu_1 t)(1 + \mu_2 t)\}$ from (2.4), (3.8) yields

$$\phi_\varepsilon(t) = (1 + \gamma_0 t)/\{1 + (\mu_1 + \mu_2 - \alpha\mu)t + (1-\alpha)\mu_1\mu_2 t^2\}. \quad (3.9)$$

The reciprocal roots of the quadratic denominator are given by (3.3) and are real by the Lemma; furthermore they are distinct and nonnegative since $4(1-\alpha)\mu_1\mu_2 > 0$. A partial fraction expansion of (3.9) then gives

$$\phi_\varepsilon(t) = \eta_1(1 + \gamma_1 t)^{-1} + \eta_2(1 + \gamma_2 t)^{-1} . \quad (3.10)$$

This inverts to the required mixed exponential distribution provided that the weights η_1 and η_2 are both positive and sum to one; the latter is clearly true from (3.5). Now $\eta_1 > 0$ requires, from (3.5), that $\gamma_1 > \gamma_0$, and $\eta_2 > 0$ requires that $\gamma_2 < \gamma_0$. The first of these is equivalent to

$$\frac{1}{2}\{\mu_1 + \mu_2 - \alpha\mu + \sqrt{[(\mu_1 + \mu_2 - \alpha\mu)^2 - 4(1-\alpha)\mu_1\mu_2]}\} > \mu_1 + \mu_2 - \mu$$

or

$$\sqrt{[(\mu_1 + \mu_2 - \alpha\mu)^2 - 4(1-\alpha)\mu_1\mu_2]} > \mu_1 + \mu_2 - (2-\alpha)\mu \quad (3.11)$$

and the second to

$$\sqrt{\{(\mu_1 + \mu_2 - \alpha\mu)^2 - 4(1-\alpha)\mu_1\mu_2\}} > -\{\mu_1 + \mu_2 - (2-\alpha)\mu\} . \quad (3.12)$$

Squaring (3.11) and (3.12) show that the condition for $\gamma_1 > \gamma_0$ and $\gamma_2 < \gamma_0$ is just the condition proved in the Lemma. The key requirement is that π_1 should be a non-zero probability.

4. Utility of the Theorem

(i) Note that the required distribution for the i.i.d. ε_n's is again the convex mixed exponential; the ε_n's can then be generated as

$$\varepsilon_n = L_n E_n \quad (4.1)$$

where $P(L_n = \gamma_1) = 1 - P(L_n = \gamma_2) = \eta_1$ and E_n is a unit-mean exponential. Thus the model (2.3) gives a means of probabilistically transforming an i.i.d. exponential sequence into a first order autoregressive Markov mixed exponential sequence with autocorrelations $\rho_r = \alpha^r \geq 0$. If $\alpha = 0$, $\{X_n\}$ are i.i.d. $ME(\pi_1,\mu_1;\pi_2,\mu_2)$ random variables. Since the process is Markovian it is easy to show that setting X_0 to be a $ME(\pi_1,\mu_1;\pi_2,\mu_2)$ random variable independent of $\epsilon_1,\epsilon_2,\ldots$ makes the process $\{X_n, n = 1,2,\ldots\}$ stationary. Finally it is easy to see that the regression of X_n on X_{n-1} is linear; from (2.3) $E(X_n|X_{n-1}=x) = E(\epsilon_n) + \alpha x$.

(ii) The theorem also holds when the marginal distribution of X is the zero-jump exponential. In our notation, let $\mu_1 = 0$ and $\mu_2 = \mu'$, so there is a probability π_1 that $X = 0$. This structure is related to the queuing situation where the waiting time can be either zero when no customer is being served, or an exponential length of time when the server is already busy. The theorem gives the following form of ϵ_n;

$$\epsilon_n = \begin{cases} (1-\alpha\pi_2)\mu' E_n & \text{w.p. } (1-\alpha)\pi_2/(1-\alpha\pi_2) \\ 0 & \text{w.p. } \pi_1/(1-\alpha\pi_2) \end{cases} . \qquad (4.2)$$

Thus the ϵ_n variable also has a zero-jump exponential distribution.

(iii) There are two cases in which a mixed exponential marginal distribution $ME(\pi_1,\mu_1;\pi_2,\mu_2)$ of $\{X_n\}$ can reduce to a single exponential distribution; these are when $\mu_1 = \mu_2$ and when either π_1 or π_2 is zero. Neither case is covered by the theorem, but each can be treated directly from the model (2.3). With μ as the mean of the exponential $\{X_n\}$ sequence, $\{\epsilon_n\}$ is an i.i.d. exponential sequence with mean $(1-\alpha)\mu$. This situation is the TEAR(1) process studied in Lawrance and Lewis (1981).

5. Mixed Exponential Moving Average Processes

It is possible to think of (2.3) and its mixed exponential solution as a way of combining independent $ME(\eta_1,\gamma_1;\eta_2,\gamma_2)$ and $ME(\pi_1,\mu_1;\pi_2,\mu_2)$ variables into another $ME(\pi_1,\mu_1;\pi_2,\mu_2)$ variable. This key result allows the constructing (cf. Lawrance and Lewis, 1977) of a first order moving average process

$$X_n = L_n \mathcal{E}_n + V_n K_{n-1} \mathcal{E}_{n-1} , \qquad n = 0, \pm 1, \pm 2, \ldots \quad , \tag{5.1}$$

where V_n, K_n and L_n are defined at (2.2), (2.5) and (4.1), respectively. The dependency parameter is α. By the results of Section 3, X_n will have the $ME(\pi_1,\mu_1;\pi_2,\mu_2)$ distribution (2.4), but the dependency will be that of a first order moving average process; the model will be denoted as NMEMA(1). The moving average aspect is clear because X_n and X_{n+1} have just $\mathcal{E}_n$ in common while X_n and $X_{n+r} (r > 1)$ will have no $\mathcal{E}_n$'s in common and thus be independent. However, the NMEMA(1) process has a very limited range of correlations; $\mathrm{corr}(X_n,X_{n+1})$ is easily derived as

$$\rho_1 = \alpha(1-\alpha)(\pi_1\mu_1 + \pi_2\mu_2)^2/[\pi_1\mu_1^2 + \pi_2\mu_2^2 + \pi_1\pi_2(\mu_1-\mu_2)^2] \tag{5.2}$$

$$= \alpha(1-\alpha)/C^2(X) . \tag{5.3}$$

The $\alpha(1-\alpha)$ is always less than one-fourth and the coefficient of variation $C(X)$ of the mixed exponential distribution is always greater than one; thus an overall bound below one-fourth is implied. Note that the maximum allowable value of ρ_1 decreases as $C(X)$ increases, i.e. as the dispersion of X relative to an exponential random variable for which $C(X) = 1$, increases. A stationary sequence is clearly obtained for $n = 1,2,\ldots$ by starting the moving average over $\mathcal{E}_0$, $\mathcal{E}_1, \ldots$.

6. Higher-Order Autoregressive Mixed Exponential Models

Although it is possible to construct higher-order analogs of the autoregressive equation (2.1), as in Lawrance and Lewis (1980), a simple p-th order extension of the model (2.3) is obtained by letting X_{n-1} be a probability mixture of $X_{n-1}, \ldots , X_{n-p}$. If the X_n's are marginally $ME(\pi_1,\mu_1;\pi_2,\mu_2)$ then this mixture will be $ME(\pi_1,\mu_1;\pi_2,\mu_2)$ also, and the error sequence ε_n in the autoregressive extension

$$X_n = \varepsilon_n + \begin{cases} 0 & \text{w.p. } 1-(\alpha_1+\cdots+\alpha_p) \\ X_{n-1} & \text{w.p. } \alpha_1 \\ X_{n-2} & \text{w.p. } \alpha_2 \\ \vdots & \\ X_{n-p} & \text{w.p. } \alpha_p \end{cases} \tag{6.1}$$

is given as at (3.1) to (3.5) to be $ME(\eta_1,\gamma_1;\eta_2,\gamma_2)$ with α replaced by $\alpha_1 + \alpha_2 + \cdots + \alpha_p$. The autocorrelations now follow pth order linear, constant coefficient difference equations paralleling those of the standard autoregressive models.

The second-order situation NMEAR(2) is probably as high as one would want to go in modelling data, in particular to take care of situations where the marginal distribution is mixed exponential but either the process is not first order Markov or the sample paths do not tend to run up. In this case simple computations give

$$\rho_1 = \alpha_1/(1-\alpha_2) \geq \alpha_1 , \qquad \rho_2 = \alpha_2 + \alpha_1\rho_1 . \tag{6.2}$$

The range of ρ_1 and ρ_2 combinations is restricted by $0 \leq \alpha_1, \alpha_2 \leq 1$, and $\alpha_1 + \alpha_2 \leq 1$. In particular any first autocorrelation ρ_1^* between zero and one is attainable for values of α_1 and α_2 satisfying $\alpha_2 = 1-\alpha_1/\rho_1^*$. Since this is linear, α_1 ranges between 0 and ρ_1^* as α_2 ranges from 1 to 0, where $\alpha_2 = 0$ gives the NMEAR(1) case. Note that $\alpha_1 = 0$ implies that $\{X_1, X_3,\ldots\}$ are independent of $\{X_2, X_4,\ldots\}$ and that the odd and the even sequences are governed by separate but identical first order processes. The effect, practically, of being able to choose a small α_1 for a given ρ_1 will be to adjust run-up behavior.

7. Mixed Autoregressive-Moving Average Extensions

It is possible to combine the autoregressive structure (6.1) and the moving average structure (5.1), as was done by Jacobs and Lewis (1977) and Lawrance and Lewis (1980) for exponential variables, to obtain a complete NMEARMA(p,q) process with mixed pth order autoregressive-qth order moving average correlation structure. This yields a much richer process in terms of sample-path behavior but since the process is not Markovian it is difficult to do estimation for the parameters. Thus for the sake of completeness we consider only the (1,1) model, extending the allowable range of marginal behavior of the EARMA(1,1) process of Jacobs and Lewis (1977); higher-order extensions are obvious. The NMEARMA(1,1) process is defined by the equations

$$\begin{aligned} X_n &= L_n(\alpha)E_n + V_n(\alpha)Y_{n-1}\,, \\ Y_n &= L_n'(\alpha')E_n + V_n'(\alpha')Y_{n-1}\,, \end{aligned} \qquad n = 0, \pm 1, \pm 2, \ldots, \qquad (7.1)$$

where the independent i.i.d. sequences $V_n(\alpha)$ and $V_n'(\alpha')$ have $P\{V_n(\alpha) = 1\} = \alpha$, $P\{V_n'(\alpha') = 1\} = \alpha'$. Also $L_n(\alpha)$ and $L_n'(\alpha')$ are independent with distributions given at (4.1). By the Theorem of Section 3 the $\{X_n\}$ sequence has a $ME(\pi_1,\mu_1;\pi_2,\mu_2)$ marginal distribution and a correlation structure typical of standard ARMA(1,1) models. Other properties of the model go through in complete analogy with the EARMA(1,1) process. In fact they will be simpler since that process is based on the EAR(1) model of Gaver and Lewis (1980) whose zero-defect gives most of the problems in the limit theorems of Jacobs and Lewis (1977).

8. Modelling and Multivariate Extensions

The mixed exponential time series generated by (2.3) and written, using (4.1) and (2.5), as

$$X_n = L_n E_n + V_n X_{n-1}, \qquad n = 0, \pm 1, \pm 2, \ldots$$

is a simple, random linear combination of random variables. As with the ordinary ARMA models, this makes it ideal for modelling complex systems and for extension to multivariate time series. The use of EARMA models in modelling queues has been discussed in Lewis and Shedler (1977) and Jacobs (1980); the NEAR(1) structure used here is actually better for this purpose since, unlike the EAR(1) structure, the error term does not disappear. These applications of the NMEAR(1) process will be discussed elsewhere. Here we only discuss direct multivariate extensions.

Thus let (E'_n, E''_n) be a bivariate pair with unit exponential marginals and let X'_n and X''_n be

$$\begin{aligned} X'_n &= L'_n E'_n + V'_n X'_{n-1} , \\ & \qquad\qquad\qquad\qquad n = 0, \pm 1, \pm 2, \ldots \qquad (8.1) \\ X''_n &= L''_n E''_n + V''_n X''_{n-1} , \end{aligned}$$

where L'_n is chosen to make X'_n be $ME(\pi'_1,\mu'_1;\pi'_2,\mu'_2)$ and L''_n is chosen to make X''_n be $ME(\pi''_1,\mu''_1;\pi''_2,\mu''_2)$. The i.i.d. squences $\{L'_n,L''_n\}$ and $\{V'_n,V''_n\}$ could be dependent within pairs. The sequences $\{X'_n\}$ and $\{X''_n\}$ will have negative cross-correlation if the $\{E'_n,E''_n\}$ pairs are negatively correlated. A cross-coupled version of this process, as in Gaver and Lewis (1980), will create alternating and posibly negative autorcorrelation in X'_n and X''_n.

Many other multivariate extensions are possible.

9. Discussion

This paper has focussed on replacing the 'Poisson process' assumptions of exponentiality and independence which are often made in modelling sequences of identically distributed positive random variables. The exponential assumption has been replaced by a mixed exponential marginal distribution, allowing more variable behavior, while the independence has been replaced by autogressive and moving average dependence. Utility of the model has been referred to in queuing and multivariate situations. Simulation of the model is possible from easily obtainable i.i.d. exponential sequences.

Various aspects of the model are currently being explored or are open to development. Extensions incorporating negative dependency can be treated by the method of cross-coupling used in Gaver and Lewis (1980) and Lawrance and Lewis (1981). Estimation via the likelihood is available in the pure autoregressive models, but the standard assumptions are not satisfied because of discontinuities in the likelihood. This aspect is investigated in Raftery (1980a,b) under the assumption of an exponential marginal distribution. Extensions to nonconvex mixed exponential solutions may be useful when the marginal distribution is less dispensed relative to the exponential distribution; one such case is from the sum of two independent non-identical exponentials. However, the model has no Gamma solution. A more flexible two-parameter autoregressive model is available, but its development is limited by mathematical tractability.

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