The Glucksterm-Hull Formula for Electron-Nucleus Bremsstrahlung

by Howard E. Brandt

U.S. Army Electronics Research
and Development Command
Harry Diamond Laboratories
Adelphi, MD 20783

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SECURITY CLASSIFICATION OF THIS PAGE (*When Data Entered*)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER HDL-TR-1884	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (*and Subtitle*) The Gluckstern-Hull Formula for Electron-Nucleus Bremsstrahlung		5. TYPE OF REPORT & PERIOD COVERED Technical Report
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Howard E. Brandt		8. CONTRACT OR GRANT NUMBER(s) DA: 1L161101A91A
9. PERFORMING ORGANIZATION NAME AND ADDRESS Harry Diamond Laboratories 2800 Powder Mill Road Adelphi, MD 20783		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Program Ele: 6.11.01.A
11. CONTROLLING OFFICE NAME AND ADDRESS U.S. Army Materiel Development and Readiness Command Alexandria, VA 22333		12. REPORT DATE May 1980
		13. NUMBER OF PAGES 120
14. MONITORING AGENCY NAME & ADDRESS(*if different from Controlling Office*)		15. SECURITY CLASS. (*of this report*) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (*of this Report*)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (*of the abstract entered in Block 20, if different from Report*)

18. SUPPLEMENTARY NOTES

HDL Project: A10826

DRCMS Code: 611101.91A0011

19. KEY WORDS (*Continue on reverse side if necessary and identify by block number*)

Bremsstrahlung	Radiation
X rays	X-ray diagnostics
Gamma rays	Plasma physics
Anisotropic plasmas	Quantum electrodynamics

20. ABSTRACT (*Continue on reverse side if necessary and identify by block number*)

The Gluckstern-Hull formula for the electron-nucleus bremsstrahlung cross section, containing the emitted photon angular dependence with the recoil electron angular dependence integrated out, is shown to be correct. This formula is required to calculate continuum x-ray spectra radiated from anisotropic relativistic beam-plasma systems.

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1. INTRODUCTION

The objective of this work is to calculate the differential cross section for single photon production in the scattering of an unpolarized electron by the Coulomb potential of a nucleus, independent of the final momentum and spin state of the electron and polarization state of the emitted photon. This cross section was calculated some time ago by R. L. Gluckstern and M. H. Hull[1] and is referred to here as the Gluckstern-Hull formula. Because of the inherent difficulty and tediousness of the calculation and complicatedness of the result and because only a brief outline of the calculation was published,[1] this work is commonly only referenced in the literature. This differential cross section is very essential to any fully relativistic and quantum mechanical calculation of bremsstrahlung production as a function of angle in anisotropic relativistic beam-plasma systems. Therefore, it is important that the Gluckstern-Hull formula be both fully understood and verified.

Specifically, the Gluckstern-Hull formula is given by[1]

$$\begin{aligned} d^3\sigma/d^2\Omega\, d\omega = {} & (Z^2e^6/8\pi)(|\vec{p}'|/|\vec{p}|\omega)\{8m^2 \sin^2\theta\,(2E^2 + m^2)/(|\vec{p}|^2x^4) \\ & - 2(5E^2 + 2EE' + 3m^2)/(|\vec{p}|^2x^2) - 2(|\vec{p}|^2 - \omega^2)/(|\vec{T}|^2x^2) \\ & + 4E'/(|\vec{p}|^2x) + (L/|\vec{p}||\vec{p}'|)[4Em^2 \sin^2\theta\,(3\omega m^2 - |\vec{p}|^2E')/(|\vec{p}|^2x^4) \\ & + [4E^2(E^2 + E'^2) - 2m^2(7E^2 - 3EE' + E'^2) + 2m^4]/(|\vec{p}|^2x^2) \\ & + 2\omega(E^2 + EE' - m^2)/(|\vec{p}|^2x)] + (\varepsilon_1/|\vec{p}'||\vec{T}|)[4m^2/x^2 \\ & - 6\omega/x - 2\omega(|\vec{p}|^2 - \omega^2)/(|\vec{T}|^2x)] - 4\varepsilon_2/(|\vec{p}'|x)\}. \end{aligned} \quad (1)$$

Here, $d^3\sigma/d^2\Omega\, d\omega$ is the differential cross section in electron-nucleus bremsstrahlung for the emission of a photon of frequency $\omega/2\pi$ into the solid angle $d^2\Omega = \sin^2\theta\, d\theta\, d\phi$. The angle θ is that between the photon and the incoming electron directions, and ϕ is the photon azimuthal angle. The charge of the nucleus is $-Ze$ where e is the charge of the electron. The electron rest mass is m. The magnitudes of the initial and final momenta of the electron are $|\vec{p}|$ and $|\vec{p}'|$, respectively. The initial and final electron energies are E and E′, respectively. The quantities x, T, L, ε_1, and ε_2 are defined by

$$x = E - |\vec{p}|\cos\theta, \quad (2)$$

$$\vec{T} = \vec{p} - \vec{k}, \quad (3)$$

$$L = \ell n\,[(EE' - m^2 + |\vec{p}||\vec{p}'|)/(EE' - m^2 - |\vec{p}||\vec{p}'|)], \quad (4)$$

$$\varepsilon_1 = \ell n\,[(|\vec{T}| + |\vec{p}'|)/(|\vec{T}| - |\vec{p}'|)], \quad (5)$$

$$\varepsilon_2 = \ell n\,[(E' + |\vec{p}'|)/(E' - |\vec{p}'|)]. \quad (6)$$

[1] *R. L. Gluckstern and M. H. Hull, Physical Review, 90 (1953), 1030.*

where $\vec{k}$ is the photon momentum ($|\vec{k}| = \omega/c$). Relativistic units are used throughout with $\hbar = c = 1$ and $e^2 = 1/137$.

In section 2, the more well-known Bethe-Heitler-Sauter formula[2] is derived for the differential cross section for bremsstrahlung production in which a photon of a given frequency is emitted in a given direction and the secondary electron travels in a prescribed direction, disregarding polarization effects, namely,

$$d^5\sigma/d^2\Omega\, d^2\Omega'\, d\omega = (Z^2e^6m^2/4\pi^2)(|\vec{p}'|/|\vec{p}||\vec{q}|^4\omega)$$

$$\cdot \{(|\vec{q}|^2/\varkappa\varkappa'm^2)(2E^2 + 2E'^2 - |\vec{q}|^2) + |\vec{q}|^2(1/\varkappa - 1/\varkappa')^2$$

$$- 4(E/\varkappa' - E'/\varkappa)^2 + (2\omega|\vec{q}|^2/m^2)(1/\varkappa' - 1/\varkappa)$$

$$- (2\omega^2/m^2)(\varkappa'/\varkappa + \varkappa/\varkappa')\}, \tag{7}$$

where

$$\varkappa = E - (\vec{k}/|\vec{k}|) \cdot \vec{p} = E - |\vec{p}| \cos\theta, \tag{8}$$

$$\varkappa' = E' - (\vec{k}/|\vec{k}|) \cdot \vec{p}' = E' - |\vec{p}'| \cos\theta', \tag{9}$$

$$\vec{q} = \vec{p}' + \vec{k} - \vec{p}. \tag{10}$$

Here, $d^2\Omega' = \sin^2\theta'\, d\theta'\, d\phi'$, where θ' is the angle between the scattered electron and emitted photon directions, and ϕ' is the angle between the component of incident electron momentum transverse to the photon momentum and the component of scattered electron momentum transverse to the photon momentum. In section 3, the integration of the cross section equation (7) over the direction of the secondary electron is performed, and the Gluckstern-Hull formula equation (1) is thereby verified.

2. BETHE-HEITLER-SAUTER FORMULA

The general amplitude for electron-nucleus bremsstrahlung with one photon emission is diagrammatically depicted in figure 1. In this process, an incident electron is scattered by a nucleus of charge -Ze and, as a result of the interaction, a single photon is emitted. In the figure, the light lines, heavy lines, wavy line, and blob represent the initial and final electron, initial and final nucleus, emitted photon, and five-point interaction vertex for the process, respectively. This interaction vertex is exceedingly complicated. However, to lowest order in the electromagnetic coupling constant, neglecting nuclear recoil and treating the nucleus as an external Coulomb field source, the amplitude may be approximated by the sum of two more elementary amplitudes as shown in figure 2. Here, the Coulomb field of the static nucleus is represented by a dashed line, and the lowest

[2] *V. B. Berestetskii, E. M. Lifshitz, and L. P. Pitaevskii, Relativistic Quantum Theory, Part I, Addison-Wesley Publishing Co., Inc., Reading, MA (1971).*

order electromagnetic vertex is represented by a dot. The two parts of the approximate total amplitude correspond to the photon being emitted either before or after the Coulomb interaction. The conditions for validity of the Born approximation are that both the initial and final velocity (v and v′) of the electron be sufficiently big and that the nuclear charge, -Ze, be sufficiently small, namely,

$$Ze^2/v << 1, \tag{11}$$

$$Ze^2/v' << 1, \tag{12}$$

and

$$Z/137 << 1. \tag{13}$$

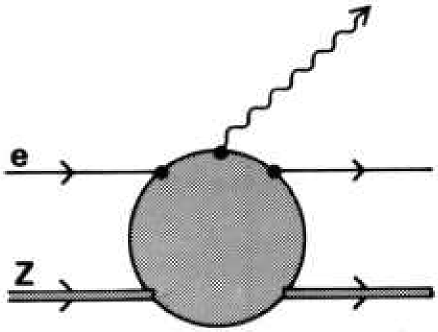

Figure 1. General amplitude for electron-nucleus bremsstrahlung with one photon in final state.

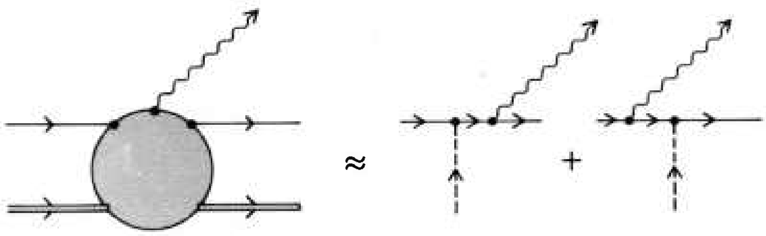

Figure 2. Approximate amplitude in Born approximation (lowest order in electromagnetic coupling constant) neglecting nuclear recoil and treating nucleus as source of external Coulomb field.

By elementary S-matrix considerations[2] the quantum mechanical probability $d^{3N}W$ for the occurrence of a process in which a single particle of energy E is scattered in a constant field in an interaction volume V and produces any number N of particles with energies E'_a and momenta p'_a in the final state is given by[2]

$$d^{3N}W = 2\pi\delta(E_f - E)|M_{fi}|^2(2EV)^{-1} \prod_{a=1}^{N} d^3\vec{p}'_a(2\pi)^{-3}(2E'_a)^{-1}. \tag{14}$$

Here, E_f is the total energy in the final state, and M_{fi} is the scattering amplitude. The latter is normally determined by the Feynman rules. The cross section for the process is given by

$$d^{3N}\sigma = d^{3N}W/j, \tag{15}$$

where j is the incoming particle flux density. The flux density j is given by

$$j = v/V, \tag{16}$$

where v is the velocity of the incoming particle. Clearly,

$$v = |\vec{p}|/E. \tag{17}$$

Substituting equations (14), (16), and (17) into equation (15), then

$$d^{3N}\sigma = 2\pi\delta(E_f - E)|M_{fi}|^2(2|\vec{p}|)^{-1} \prod_{a=1}^{N} d^3\vec{p}'_a(2\pi)^{-3}(2E'_a)^{-1}. \tag{18}$$

For the case of bremsstrahlung with an electron and a photon in the final state, equation (18) becomes

$$d^6\sigma = 2\pi\delta(E' + \omega - E)|M_{fi}|^2(2|\vec{p}|)^{-1}(2\pi)^{-6}(2\omega)^{-1}(2E')^{-1}\, d^3\vec{p}'\, d^3\vec{k} \tag{19}$$

or

$$d^6\sigma = |M_{fi}|^2(8|\vec{p}|E'\omega)^{-1}\delta(E' + \omega - E)(2\pi)^{-5}\, d^3\vec{p}'\, d^3\vec{k}. \tag{20}$$

[2] *V. B. Berestetskiĭ, E. M. Lifshitz, and L. P. Pitaevskiĭ, Relativistic Quantum Theory, Part I, Addison-Wesley Publishing Co., Inc., Reading, MA (1971).*

Since $k_\mu = (\omega, \vec{k})$ and $k^2 = 0$, then

$$|\vec{k}| = \omega. \tag{21}$$

Therefore,

$$d^3\vec{k} = |\vec{k}|^2 \, d|\vec{k}| \, d^2\Omega_k = \omega^2 \, d\omega \, d^2\Omega_k, \tag{22}$$

where $d^2\Omega_k$ is the photon differential solid angle. Also since $p'_\mu = (E', \vec{p}')$ and $p'^2 = m^2$, then

$$|\vec{p}'|^2 = E'^2 - m^2 \tag{23}$$

and

$$d^3\vec{p}' = |\vec{p}'|^2 \, d|\vec{p}'| \, d^2\Omega_{p'} = (E'^2 - m^2) \, d(E'^2 - m^2)^{1/2} \, d^2\Omega_{p'}. \tag{24}$$

But

$$d(E'^2 - m^2)^{1/2} = (E'^2 - m^2)^{-1/2} \, E' \, dE'. \tag{25}$$

Therefore,

$$d^3\vec{p}' = (E'^2 - m^2)^{1/2} \, E' \, dE' \, d^2\Omega_{p'} \tag{26}$$

or

$$d^3\vec{p}' = |\vec{p}'| E' \, dE' \, d^2\Omega_{p'}. \tag{27}$$

Substituting equations (22) and (27) into equation (20), then

$$d^6\sigma = |M_{fi}|^2 (8|\vec{p}|E'\omega)^{-1} \delta(E' + \omega - E)(2\pi)^{-5} |\vec{p}'| E' \, dE' \, d^2\Omega_{p'} \omega^2 \, d\omega \, d^2\Omega_k. \tag{28}$$

Integrating over E′, then

$$d^5\sigma = \int |M_{fi}|^2 (8|\vec{p}|)^{-1} \delta(E' + \omega - E)(2\pi)^{-5} |\vec{p}'| \, dE' \, d^2\Omega_{p'} \omega \, d\omega \, d^2\Omega_k \tag{29}$$

or*

$$d^5\sigma = |M_{fi}|^2 (|\vec{p}'|/8|\vec{p}|)(2\pi)^{-5} \omega \, d\omega \, d^2\Omega_k \, d^2\Omega_{p'}. \tag{30}$$

The matrix element M_{fi} is that corresponding to figure 2 and is redrawn in figure 3 with momentum labels. Here p and p′ are the initial and final four-momenta of the electron, respectively; k is the four-momentum of the emitted photon; and q is the four-vector momentum transfer from the nucleus. By energy and momentum conservation, the last is given by

$$q = p' + k - p. \tag{31}$$

Furthermore, neglecting the change in proton energy due to recoil, then

$$q^0 \approx 0. \tag{32}$$

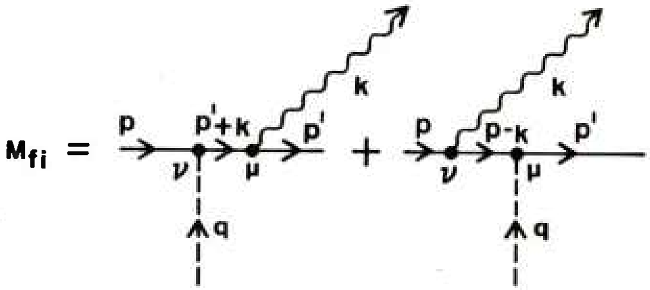

Figure 3. Feynman diagrammatic representation of matrix element M_{fi}.

The subset of Feynman rules to be used in evaluating the matrix element M_{fi} in figure 3 is as follows:[2]

1. Associated with each incoming electron line of four-momentum p is an amplitude u(p), where u(p) is the free electron momentum space wave function.

*Equation (30) corresponds to equation (91.2) in V. B. Berestetskiĭ, E. M. Lifshitz, and L. P. Pitaevskiĭ, Relativistic Quantum Theory, Part I, Addison-Wesley Publishing Co., Inc., Reading, MA (1971). The latter has a typographical error omitting the factor of 1/8.

[2]V. B. Berestetskiĭ, E. M. Lifshitz, and L. P. Pitaevskiĭ, Relativistic Quantum Theory, Part I, Addison-Wesley Publishing Co., Inc., Reading, MA (1971).

2. Each outgoing electron line is associated with the amplitude $\bar{u}(p')$ of the final electron.

3. Each vertex is associated with a four-vector $-i\gamma^\mu$, where γ^μ, $\mu = 0, 1, 2, 3$, are the Dirac gamma matrices.

4. Associated with each incoming photon line of four-momentum k is an amplitude $\sqrt{4\pi}\ \varepsilon_\mu$, where ε_μ is the photon polarization four-vector, and μ is the same as the index μ of the vertex to which it is connected. The Einstein sum convention for repeated indices is understood.

5. Associated with each outgoing photon line is an amplitude $\sqrt{4\pi}\ \varepsilon_\mu^*$, where * denotes complex conjugation.

6. Associated with each internal electron line of four-momentum p and rest mass m is a factor $i(\hat{p} + m)/(p^2 - m^2)$, where $\hat{p}$ denotes $\gamma_\mu p^\mu$.

7. The four-momenta of all lines meeting at an interaction vertex satisfy four-momentum conservation.

8. An external field line is represented by a factor $A^\mu(q)$, where the latter is the spatial Fourier transform of the electromagnetic vector potential and q is the four-momentum transfer from the external source.

9. Diagrams that are identical after removal of all photon lines must have the same sign.

10. Bispinor indices are arranged from right to left, contrary to the arrows.

11. For n vertices, there is an overall factor in M_{fi} of e^n/i.

Therefore,

$$M_{fi} = \bar{u}(p')(-i\gamma^\mu)i(\hat{p}' + \hat{k} + m)[(p' + k)^2 - m^2]^{-1}(-i\gamma^\nu)u(p)(4\pi)^{1/2}\varepsilon_\mu^* A_\nu(q)(e^2/i)$$
$$+ \bar{u}(p')(-i\gamma^\mu)i(\hat{p} - \hat{k} + m)[(p - k)^2 - m^2]^{-1}(-i\gamma^\nu)u(p)(4\pi)^{1/2}\varepsilon_\nu^* A_\mu(q)(e^2/i) \qquad (33)$$

or, by interchanging dummy indices μ and ν in the second term, then

$$M_{fi} = -e^2 A_\nu(q)(4\pi)^{1/2}\varepsilon_\mu^* \bar{u}(p')[\gamma^\mu(\hat{f}' + m)(f'^2 - m^2)^{-1}\gamma^\nu + \gamma^\nu(\hat{f} + m)(f^2 - m^2)^{-1}\gamma^\mu]u(p), \qquad (34)$$

where

$$f = p - k, \qquad (35)$$

$$f' = p' + k. \qquad (36)$$

Using the definition of equation (35),

$$f^2 - m^2 = (p - k)^2 - m^2 = p^2 - 2pk + k^2 - m^2. \qquad (37)$$

But

$$p^2 = m^2, \tag{38}$$

$$k^2 = 0, \tag{39}$$

and equation (37) becomes

$$f^2 - m^2 = -2pk = -2(E\omega - |\vec{p}||\vec{k}| \cos\theta), \tag{40}$$

where θ is the angle that the emitted photon makes with the incoming electron. For the photon,

$$|\vec{k}| = \omega. \tag{41}$$

Equation (40) becomes

$$f^2 - m^2 = -2\varkappa\omega, \tag{42}$$

where

$$\varkappa = E - |\vec{p}| \cos\theta. \tag{43}$$

Similarly,

$$f'^2 - m^2 = 2\varkappa'\omega, \tag{44}$$

where

$$\varkappa' = E' - |\vec{p}'| \cos\theta'. \tag{45}$$

The external field of the assumed static nucleus has a time component only, given by the static Coulomb potential. Hence, neglecting nuclear recoil,

$$A^{\mu}(q) = 4\pi Ze|\vec{q}|^{-2}\delta^{\mu 0}. \tag{46}$$

Substituting equations (42), (44), and (46) into equation (34), then

$$M_{fi} = -e^2 4\pi Ze|\vec{q}|^{-2}(4\pi)^{1/2}\varepsilon_{\mu}^{*}\bar{u}(p')[\gamma^{\mu}(\hat{f}' + m)(2\varkappa'\omega)^{-1}\gamma^0 - \gamma^0(\hat{f} + m)(2\varkappa\omega)^{-1}\gamma^{\mu}]u(p). \tag{47}$$

Defining the matrix variable Q^{μ} by

$$Q^{\mu} = \gamma^{\mu}(\hat{f}' + m)(2\varkappa'\omega)^{-1}\gamma^0 - \gamma^0(\hat{f} + m)(2\varkappa\omega)^{-1}\gamma^{\mu}, \tag{48}$$

then equation (47) becomes

$$M_{fi} = -e^2(4\pi Ze)|\vec{q}|^{-2}(4\pi)^{1/2}\varepsilon_{\mu}^{*}\bar{u}(p')Q^{\mu}u(p). \tag{49}$$

The Dirac matrices γ_{μ} satisfy the anticommutation relations

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu}, \tag{50}$$

where $g^{\mu\nu}$ is the Lorentz space time metric,

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{51}$$

Setting $\mu = \nu = 0$ in equation (50), then

$$\gamma^{0^2} = 1, \tag{52}$$

or setting $\mu = \nu = k$, where $k = 1, 2, 3$,

$$\gamma^{k^2} = -1. \tag{53}$$

Using equations (49) and (52), then

$$M^{+}_{fi} = -e^{2}(4\pi Ze)|\vec{q}|^{-2}(4\pi)^{1/2}\varepsilon_{\nu}u^{+}(p)Q^{\nu +}\gamma^{0+}u(p')$$

$$= -e^{2}(4\pi Ze)|\vec{q}|^{-2}(4\pi)^{1/2}\varepsilon_{\nu}u^{+}(p)\gamma^{0}\gamma^{0}Q^{\nu +}\gamma^{0+}u(p'),$$

or

$$M^{+}_{fi} = -e^{2}(4\pi Ze)|\vec{q}|^{-2}(4\pi)^{1/2}\varepsilon_{\nu}\bar{u}(p)\bar{Q}^{\nu}u(p'), \qquad (54)$$

where

$$\bar{Q}^{\nu} = \gamma^{0}Q^{\nu +}\gamma^{0+} \qquad (55)$$

Therefore, substituting equations (49) and (54) in the expression for the cross section, equation (30), one obtains

$$d^{5}\sigma = Z^{2}e^{6}(4\pi^{2})^{-1}(|\vec{p}'|\omega/|\vec{p}||\vec{q}|^{4})\varepsilon^{*}_{\mu}\varepsilon_{\nu}$$

$$\cdot[\bar{u}(p')Q^{\mu}u(p)][\bar{u}(p)\bar{Q}^{\nu}u(p')]\, d^{2}\Omega\, d^{2}\Omega'\, d\omega. \qquad (56)$$

Defining

$$T(p, p', k) = \varepsilon^{*}_{\mu}\varepsilon_{\nu}[\bar{u}(p')Q^{\mu}u(p)][\bar{u}(p)\bar{Q}^{\nu}u(p')], \qquad (57)$$

equation (56) is

$$d^{5}\sigma = Z^{2}e^{6}(4\pi^{2})^{-1}(|\vec{p}'|\omega/|\vec{p}||\vec{q}|^{4})T(p, p', k)\, d^{2}\Omega\, d^{2}\Omega'\, d\omega. \qquad (58)$$

For an unpolarized incident electron and for the case where the final polarization states of the electron and photon are unspecified, which is the case of interest here, one must average over initial electron spin states and sum over final electron and photon spin states. The cross section is then given by

$$d^{5}\bar{\sigma} = Z^{2}e^{6}(4\pi^{2})^{-1}(|\vec{p}'|\omega/|\vec{p}||\vec{q}|^{4})\bar{T}(p, p', k)\, d^{2}\Omega\, d^{2}\Omega'\, d\omega, \qquad (59)$$

where

$$\overline{T}(p, p', k) = \tfrac{1}{2}\sum_s \sum_{s'} \sum_h T(p, p', k), \tag{60}$$

and $\sum_s$, $\sum_{s'}$, $\sum_h$ denote sums over initial electron spin states, final electron spin states, and emitted photon polarization states, respectively. The factor of ½ arises in taking the average over incident electron spin states because there are two possible spin states for the electron.

Since in the present calculation one is not concerned with the state of polarization of the emitted photon, then in summing over photon polarizations of the final state, one uses the relation[2]

$$\sum_h (\ldots \varepsilon_\mu^* \varepsilon_\nu \ldots) = (\ldots -g_{\mu\nu} \ldots). \tag{61}$$

Also, in summing over the final electron spin states and averaging over the initial electron states (unpolarized), one uses the following relation:[2]

$$\begin{aligned} &\tfrac{1}{2}\sum_s \sum_{s'} [\overline{u}(p')Q^\mu u(p)][\overline{u}(p)\overline{Q}_\mu u(p')] \\ &\qquad = \mathrm{Tr}\, Q^\mu(\hat{p} + m)\overline{Q}_\mu(\hat{p}' + m). \end{aligned} \tag{62}$$

Substituting equation (57) into equation (60) and using equations (61) and (62), then

$$\begin{aligned} \overline{T}(p, p', k) = \tfrac{1}{2}\sum_s \sum_{s'} \sum_h \varepsilon_\mu^* \varepsilon_\nu [\overline{u}(p') \\ \cdot Q^\mu u(p)][\overline{u}(p)\overline{Q}^\nu u(p')] \end{aligned}$$

or

$$\overline{T}(p, p', k) = -\tfrac{1}{2}\mathrm{Tr}\, g_{\mu\nu} Q^\mu(\hat{p} + m)\, \overline{Q}^\nu(\hat{p}' + m) \tag{63}$$

or

$$\overline{T}(p, p', k) = -\tfrac{1}{2}\mathrm{Tr}\, Q_\mu(\hat{p} + m)\overline{Q}^\mu(\hat{p}' + m). \tag{64}$$

To proceed, one must reduce the trace $\overline{T}$ given by equation (64). One first notes according to equations (48) and (55) that

[2] *V. B. Berestetskii, E. M. Lifshitz, and L. P. Pitaevskii, Relativistic Quantum Theory, Part I, Addison-Wesley Publishing Co., Inc., Reading, MA (1971).*

$$\bar{Q}^\nu = \gamma^0 Q^{\nu+}\gamma^{0+} = \gamma^0(\gamma^{0+}\hat{f}'^{+} + m\gamma^{0+})\gamma^{\nu+}\gamma^{0+}(2\varkappa'\omega)^{-1}$$

$$- \gamma^0\gamma^{\nu+}(\hat{f}^{+} + m)\gamma^{0+}\gamma^{0+}(2\varkappa\omega)^{-1}. \quad (65)$$

However, it is true that[2]

$$\gamma^{\nu+} = \gamma^0\gamma^\nu\gamma^0. \quad (66)$$

Using equations (66) and (52), then

$$\gamma^0\gamma^{\nu+}\gamma^0 = \gamma^{02}\gamma^\nu\gamma^{02} = \gamma^\nu. \quad (67)$$

Using equations (66), (52), and (50), it follows that

$$\gamma^{0+} = \gamma^0, \quad (68)$$

and

$$\gamma^{k+} = -\gamma^k. \quad (69)$$

In equation (65), the term $\gamma^0\gamma^{0+}\hat{f}'^{+}\gamma^{\nu+}\gamma^{0+}$ can be simplified by using equations (68), (52), and (67); thus,

$$\gamma^0\gamma^{0+}\hat{f}'^{+}\gamma^{\nu+}\gamma^{0+} = \gamma^{02}\hat{f}'^{+}\gamma^{02}\gamma^{\nu+}\gamma^0, \quad (70)$$

or

$$\gamma^0\gamma^{0+}\hat{f}'^{+}\gamma^{\nu+}\gamma^{0+} = \gamma^0\hat{f}'\gamma^\nu. \quad (71)$$

Similarly,

$$\gamma^0\gamma^{0+}\gamma^{\nu+}\gamma^{0+} = \gamma^{02}\gamma^{\nu+}\gamma^0 = \gamma^0\gamma^\nu, \quad (72)$$

$$\gamma^0\gamma^{\nu+}\hat{f}^{+}\gamma^{0+}\gamma^{0+} = \gamma^0\gamma^{\nu+}\gamma^{02}\hat{f}^{+}\gamma^{02} = \gamma^\nu\hat{f}\gamma^0, \quad (73)$$

[2] *V. B. Berestetskii, E. M. Lifshitz, and L. P. Pitaevskii, Relativistic Quantum Theory, Part I, Addison-Wesley Publishing Co., Inc., Reading, MA (1971).*

and

$$\gamma^0\gamma^{\nu+}\gamma^{0+}\gamma^{0+} = \gamma^0\gamma^{\nu+}\gamma^{02} = \gamma^\nu\gamma^0. \tag{74}$$

Then substituting equations (71) to (74) into equation (65),

$$\bar{Q}^\nu = (\gamma^0\hat{f}'\gamma^\nu + m\gamma^0\gamma^\nu)(2x'\omega)^{-1} - (\gamma^\nu\hat{f}\gamma^0 + m\gamma^\nu\gamma^0)(2x\omega)^{-1}. \tag{75}$$

The function $\bar{T}(p, p', k)$ must be evaluated. Substituting equations (48) and (75) into equation (64),

$$\bar{T}(p, p', k) = -\tfrac{1}{2}\sum_{i=1}^{4} T_i(p, p', k), \tag{76}$$

where

$$T_1(p, p', k) = \mathrm{Tr}\,\gamma_\mu(\hat{f}' + m)(2x'\omega)^{-1}\gamma^0(\hat{p} + m)\gamma^0(\hat{f}' + m)(2x'\omega)^{-1}\gamma^\mu(\hat{p}' + m), \tag{77}$$

$$T_2(p, p', k) = -\mathrm{Tr}\,\gamma_\mu(\hat{f}' + m)(2x'\omega)^{-1}\gamma^0(\hat{p} + m)\gamma^\mu(\hat{f} + m)(2x\omega)^{-1}\gamma^0(\hat{p}' + m), \tag{78}$$

$$T_3(p, p', k) = -\mathrm{Tr}\,\gamma^0(\hat{f} + m)(2x\omega)^{-1}\gamma_\mu(\hat{p} + m)\gamma^0(\hat{f}' + m)(2x'\omega)^{-1}\gamma^\mu(\hat{p}' + m), \tag{79}$$

$$T_4(p, p', k) = \mathrm{Tr}\,\gamma^0(\hat{f} + m)(2x\omega)^{-1}\gamma_\mu(\hat{p} + m)\gamma^\mu(\hat{f} + m)(2x\omega)^{-1}\gamma^0(\hat{p}' + m). \tag{80}$$

Using equations (66), (68), and (69),

$$\gamma^0\hat{p}\gamma^0 = \gamma^0\gamma_\mu p^\mu\gamma^0 = \gamma^0\gamma_\mu\gamma^0 p^\mu = \gamma_\mu^+ p^\mu \tag{81}$$

or

$$\gamma^0\hat{p}\gamma^0 = \gamma_0 p^0 - \gamma_k p^k = \gamma_\mu\tilde{p}^\mu = \hat{\tilde{p}}, \tag{82}$$

where

$$\tilde{p}^{\mu} = (E, -\vec{p}). \tag{83}$$

Substituting equations (52) and (82) into equation (77), then

$$T_1(p, p', k) = \mathrm{Tr}\,\gamma_\mu(\hat{f}' + m)(2x'\omega)^{-1}(\hat{\tilde{p}} + m)(\hat{f}' + m)(2x'\omega)^{-1}\gamma^\mu(\hat{p}' + m). \tag{84}$$

Using the cyclic invariance of the trace with equations (52) and (82), equation (80) can be rewritten as follows:

$$T_4(p, p', k) = \mathrm{Tr}\,\gamma^0(\hat{p}' + m)\gamma^0(\hat{f} + m)(2x\omega)^{-1}\gamma_\mu(\hat{p} + m)\gamma^\mu(\hat{f} + m)(2x\omega)^{-1} \tag{85}$$

or

$$T_4(p, p', k) = \mathrm{Tr}(\hat{\tilde{p}}' + m)(\hat{f} + m)(2x\omega)^{-1}\gamma_\mu(\hat{p} + m)\gamma^\mu(\hat{f} + m)(2x\omega)^{-1} \tag{86}$$

or

$$T_4(p, p', k) = \mathrm{Tr}\,\gamma^\mu(\hat{f} + m)(2x\omega)^{-1}(\hat{\tilde{p}}' + m)(\hat{f} + m)(2x\omega)^{-1}\gamma_\mu(\hat{p} + m) \tag{87}$$

or

$$T_4(p, p', k) = T_1(p', p, -k), \tag{88}$$

where, in the last step, equation (84) with the definitions of equations (35), (36), (43), and (45) have been used. By using equation (88), then

$$T_1(p, p', k) = T_4(p', p, -k). \tag{89}$$

Next, using the cyclic invariance of the trace, equation (79) may be rewritten

$$T_3(p, p', k) = -\mathrm{Tr}\,(\hat{p}' + m)\gamma^0(\hat{f} + m)(2x\omega)^{-1}\gamma_\mu(\hat{p} + m)\gamma^0(\hat{f}' + m)(2x'\omega)^{-1}\gamma^\mu. \tag{90}$$

It is also true that[2]

$$\mathrm{Tr}(\hat{a}_1\hat{a}_2 \ldots \hat{a}_n) = \mathrm{Tr}(\hat{a}_n \ldots \hat{a}_2\hat{a}_1). \tag{91}$$

Thus, using equation (91), equation (90) may be rewritten

$$T_3(p, p', k) = \mathrm{Tr} - \gamma_\mu(\hat{f}' + m)(2\varkappa'\omega)^{-1}\gamma^0(\hat{p} + m)\gamma^\mu(\hat{f} + m)(2\varkappa\omega)^{-1}\gamma^0(\hat{p}' + m). \tag{92}$$

Comparing equation (92) with equation (78), then

$$T_3(p, p', k) = T_2(p, p', k). \tag{93}$$

Substituting equations (89) and (93) into equation (76), then

$$\overline{T}(p, p', k) = -\tfrac{1}{2}[T_4(p', p, -k) + 2T_2(p, p', k) + T_4(p, p', k)] \tag{94}$$

or

$$\overline{T}(p, p', k) = T_{4s}(p, p', k) - T_2(p, p', k), \tag{95}$$

where

$$T_{4s}(p, p', k) = -\tfrac{1}{2}[T_4(p, p', k) + T_4(p', p, -k)], \tag{96}$$

and T_2 and T_4 are given by equations (78) and (87), respectively.

The term T_2 must first be evaluated. By substituting equation (52) into equation (78) and using equation (50), then

$$T_2 = -\mathrm{Tr}\, \gamma_\mu(\hat{f}' + m)(2\varkappa'\omega)^{-1}\gamma^0(\hat{p} + m)\gamma^0\gamma^0\gamma^\mu(\hat{f} + m)(2\varkappa\omega)^{-1}\gamma^0(\hat{p}' + m) \tag{97}$$

or

[2] *V. B. Berestetskiǐ, E. M. Lifshitz, and L. P. Pitaevskiǐ, Relativistic Quantum Theory, Part I, Addison-Wesley Publishing Co., Inc., Reading, MA (1971).*

$$T_2 = -\mathrm{Tr}\,\gamma_\mu(\hat{f}' + m)(2x'\omega)^{-1}(\hat{\tilde{p}} + m)(2g^{0\mu} - \gamma^\mu\gamma^0)(\hat{f} + m)(2x\omega)^{-1}\gamma^0(\hat{p}' + m) \quad (98)$$

or

$$T_2 = \mathrm{Tr}[-2\gamma^0(\hat{f}' + m)(2x'\omega)^{-1}(\hat{\tilde{p}} + m)(\hat{f} + m)(2x\omega)^{-1}\gamma^0(\hat{p}' + m)$$
$$+ \gamma_\mu(\hat{f}' + m)(2x'\omega)^{-1}(\hat{\tilde{p}} + m)\gamma^\mu\gamma^0(\hat{f} + m)(2x\omega)^{-1}\gamma^0(\hat{p}' + m)]. \quad (99)$$

Using the cyclic invariance of the trace and equation (82), then equation (99) becomes

$$T_2 = \mathrm{Tr}[-2(\hat{f}' + m)(2x'\omega)^{-1}(\hat{\tilde{p}} + m)(\hat{f} + m)(2x\omega)^{-1}(\hat{\tilde{p}}' + m)$$
$$+ \gamma_\mu(\hat{f}' + m)(2x'\omega)^{-1}(\hat{\tilde{p}} + m)\gamma^\mu(\hat{\tilde{f}} + m)(2x\omega)^{-1}(\hat{p}' + m)]. \quad (100)$$

Using the following relations obtained from equation (50),

$$\gamma^\mu\gamma_\mu = 4, \quad (101)$$

$$\gamma^\mu\hat{a}\gamma_\mu = -2\hat{a}, \quad (102)$$

$$\gamma^\mu\hat{a}\hat{b}\gamma_\mu = 4a \cdot b, \quad (103)$$

in equation (100), then

$$T_2 = \mathrm{Tr}[-2(\hat{f}' + m)(2x'\omega)^{-1}(\hat{\tilde{p}} + m)(\hat{f} + m)(2x\omega)^{-1}(\hat{\tilde{p}}' + m)$$
$$+ (4f' \cdot \tilde{p} - 2m\hat{f}' - 2m\hat{\tilde{p}} + 4m^2)(2x'\omega)^{-1}(\hat{\tilde{f}} + m)(2x\omega)^{-1}(\hat{p}' + m)]. \quad (104)$$

Performing the multiplications and recalling that the trace of the product of an odd number of gamma matrices is vanishing,[2] equation (104) becomes

[2] *V. B. Berestetskiĭ, E. M. Lifshitz, and L. P. Pitaevskiĭ, Relativistic Quantum Theory, Part I, Addison-Wesley Publishing Co., Inc., Reading, MA (1971).*

$$
\begin{aligned}
T_2 = (2x'\omega x\omega)^{-1}\mathrm{Tr}[& -\tilde{f}' \, \hat{\tilde{p}}\hat{f}\hat{\tilde{p}}' \\
& - m^2\hat{f}'\hat{\tilde{p}} - m^2\hat{f}'\hat{f} - m^2\hat{f}'\hat{\tilde{p}}' \\
& - m^2\hat{\tilde{p}}\hat{f} - m^2\hat{\tilde{p}}\hat{\tilde{p}}' \\
& - m^2\hat{f}\hat{\tilde{p}}' - m^4 + 2f'\cdot\tilde{p}(\hat{\tilde{f}}\hat{p}' + m^2) \\
& - m\hat{f}'(m\hat{\tilde{f}} + m\hat{p}') - m\hat{\tilde{p}}(m\hat{\tilde{f}} \\
& + m\hat{p}') + 2m^2(\hat{\tilde{f}}\hat{p}' + m^2)].
\end{aligned}
\tag{105}
$$

The trace in equation (105) is evaluated by using the relations[2]

$$
\mathrm{Tr}\,\hat{a}\hat{b} = 4a \cdot b \tag{106}
$$

and

$$
\mathrm{Tr}\,\hat{a}\hat{b}\hat{c}\hat{d} = 4[(a \cdot b)(c \cdot d) - (a \cdot c)(b \cdot d) + (a \cdot d)(b \cdot c)]. \tag{107}
$$

Thus,

$$
\begin{aligned}
T_2 = (2xx'\omega^2)^{-1}[& -4(f' \cdot \tilde{p}\, f \cdot \tilde{p}' - f' \cdot f\, \tilde{p} \cdot \tilde{p}' \\
& + f' \cdot \tilde{p}'\, \tilde{p} \cdot f) - 4m^2 f' \cdot \tilde{p} - 4m^2 f' \cdot f - 4m^2 f' \cdot \tilde{p}' - 4m^2 \tilde{p} \cdot f \\
& - 4m^2 \tilde{p} \cdot \tilde{p}' - 4m^2 f \cdot \tilde{p}' - 4m^4 + 8f' \cdot \tilde{p}\, \tilde{f} \cdot p' \\
& + 8m^2 f' \cdot \tilde{p} - 4m^2 f' \cdot \tilde{f} - 4m^2 f' \cdot p' - 4m^2 \tilde{p} \cdot \tilde{f} \\
& - 4m^2 \tilde{p} \cdot p' + 8m^2 \tilde{f} \cdot p' + 8m^4].
\end{aligned}
\tag{108}
$$

[2] V. B. Berestetskiĭ, E. M. Lifshitz, and L. P. Pitaevskiĭ, *Relativistic Quantum Theory, Part I*, Addison-Wesley Publishing Co., Inc., Reading, MA (1971).

Combining terms, then equation (108) becomes

$$T_2 = (2\varkappa\varkappa'\omega^2)^{-1}[-4(f' \cdot \tilde{p} f \cdot \tilde{p}' - f' \cdot f\tilde{p} \cdot \tilde{p}'$$

$$+ f' \cdot \tilde{p}'\tilde{p} \cdot f - 2f' \cdot \tilde{p}\tilde{f} \cdot p') - 4m^2(f \cdot \tilde{p}$$

$$+ f' \cdot \tilde{p}' + f' \cdot f - f' \cdot \tilde{p} + \tilde{p} \cdot \tilde{p}' + f \cdot \tilde{p}'$$

$$+ f' \cdot \tilde{f} + f' \cdot p' + \tilde{f} \cdot \tilde{p} + \tilde{p} \cdot p' - 2\tilde{f} \cdot p') + 4m^4]. \quad (109)$$

Using the definition of $\tilde{p}$, equation (83), it follows that

$$p \cdot \tilde{p}' = p' \cdot \tilde{p}. \quad (110)$$

Substituting equations (35) and (36) into equation (109), performing the multiplications using equation (110), and combining terms, one obtains

$$T_2 = 2(\varkappa\varkappa'\omega^2)^{-1}[-m^2p \cdot \tilde{p} + m^2k \cdot \tilde{p} - m^2p' \cdot \tilde{p}' - m^2k \cdot \tilde{p}'$$

$$- 2m^2p \cdot p' - m^2\varkappa\omega + m^2k \cdot \tilde{k} - m^4 + m^2k \cdot p$$

$$+ (p' \cdot \tilde{p})^2 - (p' \cdot \tilde{p}k \cdot \tilde{p}') + k \cdot \tilde{p}p \cdot \tilde{p}'$$

$$+ (p \cdot p')^2 - (\varkappa' - \varkappa)\omega p \cdot p' - p' \cdot \tilde{p}'p \cdot \tilde{p}$$

$$+ p' \cdot \tilde{p}'\tilde{p} \cdot k - p \cdot \tilde{p}k \cdot \tilde{p}']. \quad (111)$$

To reduce equation (111) from the expressions involving four-vectors to ones containing explicit space and time components, one proceeds as follows. By defining the unit vector $\vec{n}$ in the direction of photon emission,

$$\vec{n} = \vec{k}/|\vec{k}| = \vec{k}/\omega, \quad (112)$$

and using equation (43), then

$$\vec{n} \cdot \vec{p} = E - \varkappa. \quad (113)$$

Similarly, using equation (45),

$$\vec{n} \cdot \vec{p}' = E' - \varkappa'. \tag{114}$$

Using the definition of $\tilde{p}$ of equation (83), then

$$p \cdot \tilde{p} = E^2 + |\vec{p}|^2. \tag{115}$$

But using equation (38),

$$|\vec{p}|^2 = E^2 - m^2. \tag{116}$$

Substituting equation (116) in equation (115), then

$$p \cdot \tilde{p} = 2E^2 - m^2. \tag{117}$$

Similarly,

$$k \cdot \tilde{p} = \omega(E + \vec{n} \cdot \vec{p}) = \omega(2E - \varkappa), \tag{118}$$

$$p' \cdot \tilde{p}' = E'^2 + |\vec{p}'|^2 = 2E'^2 - m^2, \tag{119}$$

$$k \cdot \tilde{p}' = \omega(E' + \vec{n} \cdot \vec{p}') = \omega(2E' - \varkappa'), \tag{120}$$

$$p \cdot p' = EE' - \vec{p} \cdot \vec{p}', \tag{121}$$

$$k \cdot \tilde{k} = 2\omega^2, \tag{122}$$

$$k \cdot p = \omega(E - \vec{n} \cdot \vec{p}) = \omega\varkappa, \tag{123}$$

$$p' \cdot \tilde{p} = E'E + \vec{p}' \cdot \vec{p}, \tag{124}$$

$$p' \cdot \tilde{p} k \cdot \tilde{p}' = \omega(2E' - \varkappa')(EE' + \vec{p} \cdot \vec{p}'), \tag{125}$$

$$k \cdot \tilde{p} p \cdot \tilde{p}' = (EE' + \vec{p} \cdot \vec{p}')(\omega E + \omega \vec{n} \cdot \vec{p})$$

$$= (EE' + \vec{p} \cdot \vec{p}')(2E - \varkappa)\omega, \tag{126}$$

$$p' \cdot \tilde{p}' p \cdot \tilde{p} = (E'^2 + |\vec{p}'|^2)(E^2 + |\vec{p}|^2) = (2E'^2 - m^2)(2E^2 - m^2), \tag{127}$$

$$p' \cdot \tilde{p}' \tilde{p} \cdot k = (E'^2 + |\vec{p}'|^2)(E + \vec{n} \cdot \vec{p})\omega = (2E'^2 - m^2)(2E - \varkappa)\omega, \tag{128}$$

$$p \cdot \tilde{p} k \cdot \tilde{p}' = (E^2 + |\vec{p}|^2)(E' + \vec{n} \cdot \vec{p}')\omega = (2E^2 - m^2)(2E' - \varkappa')\omega. \tag{129}$$

Also, for the elastic collision at hand, equations (31) and (32) imply

$$E' = E - \omega. \tag{130}$$

Substituting equations (117) to (130) into equation (111) and simplifying, one obtains

$$T_2 = 4(\varkappa\varkappa'\omega^2)^{-1}\Big\{[m^2 + \omega^2 + \omega(\varkappa' - \varkappa)]\vec{p} \cdot \vec{p}' + (\vec{p} \cdot \vec{p}')^2 - m^2EE' + m^2\omega^2$$

$$- E^2E'^2 + \omega EE'^2 - \omega E^2E' - \omega\varkappa E'^2 + \omega\varkappa' E^2\Big\}. \tag{131}$$

The quantity $\vec{p} \cdot \vec{p}'$ appearing in equation (131) may be rewritten in terms of the momentum transfer $\vec{q}$ as follows. Using equation (31), the space component $\vec{q}$ of the four-vector q is given by

$$\vec{q} = \vec{p}' + \vec{k} - \vec{p}. \tag{132}$$

Dotting the vector $\vec{q}$ into itself, then

$$|\vec{q}|^2 = |\vec{p}' + \vec{k} - \vec{p}|^2 = |\vec{p}'|^2 + 2\vec{p}' \cdot \vec{k} + |\vec{k}|^2$$

$$- 2\vec{p} \cdot \vec{p}' - 2\vec{p} \cdot \vec{k} + |\vec{p}|^2. \tag{133}$$

Substituting equations (23), (112), (114), (21), (113), and (116) into equation (133) and solving for $\vec{p} \cdot \vec{p}'$, then

$$\vec{p} \cdot \vec{p}' = \tfrac{1}{2}[E^2 + E'^2 + \omega^2 - 2m^2 - 2\omega(E - E') + 2\omega(\varkappa - \varkappa') - |\vec{q}|^2]. \tag{134}$$

It follows then that

$$\begin{aligned}(\vec{p} \cdot \vec{p}')^2 = \tfrac{1}{4}[&E^4 + 2E^2E'^2 + 6E^2\omega^2 - 4E^2m^2 - 4E^3\omega + 4E^2E'\omega \\ &+ 4(E^2 + E'^2 + \omega^2 - 2m^2 + 2\omega E' - |\vec{q}|^2 - 2\omega E)\omega(\varkappa - \varkappa') \\ &- 2E^2|\vec{q}|^2 + E'^4 + 2E'^2\omega^2 - 4E'^2m^2 - 4EE'^2\omega + 4E'^3\omega \\ &- 2E'^2|\vec{q}|^2 + \omega^4 - 4m^2\omega^2 - 4E\omega^3 + 4E'\omega^3 - 2\omega^2|\vec{q}|^2 + 4m^4 \\ &+ 8m^2\omega E - 8m^2\omega E' + 4m^2|\vec{q}|^2 - 8EE'\omega^2 + 4\omega^2E'^2 \\ &+ 4E\omega|\vec{q}|^2 - 4\omega E'|\vec{q}|^2 + 4\omega^2(\varkappa - \varkappa')^2 + |\vec{q}|^4].\end{aligned} \tag{135}$$

Next, substituting equations (134) and (135) into equation (131), then

$$\begin{aligned}T_2 = (\varkappa\varkappa'\omega^2)^{-1}\{&E^4 + 2E^2E'^2 + 6E^2\omega^2 - 4E^2m^2 - 4E^3\omega + 4E^2E'\omega \\ &+ 4(E^2 + E'^2 + \omega^2 - 2m^2 + 2\omega E' - |\vec{q}|^2 - 2\omega E)\omega(\varkappa - \varkappa') \\ &- 2E^2|\vec{q}|^2 + E'^4 + 2E'^2\omega^2 - 4E'^2m^2 - 4EE'^2\omega + 4E'^3\omega \\ &- 2E'^2|\vec{q}|^2 + \omega^4 - 4m^2\omega^2 - 4E\omega^3 + 4E'\omega^3 - 2\omega^2|\vec{q}|^2 + 4m^4 \\ &+ 8m^2\omega E - 8m^2\omega E' + 4m^2|\vec{q}|^2 - 8EE'\omega^2 + 4\omega^2E'^2 + 4E\omega|\vec{q}|^2 \\ &- 4\omega E'|\vec{q}|^2 + 4\omega^2(\varkappa - \varkappa')^2 + |\vec{q}|^4 \\ &+ 2[m^2 + \omega^2 + \omega(\varkappa' - \varkappa)][E^2 + E'^2 + \omega^2 - 2m^2 - 2\omega(E - E') \\ &+ 2\omega(\varkappa - \varkappa') - |\vec{q}|^2] - 4m^2EE' + 4m^2\omega^2 - 4E^2E'^2 + 4\omega EE'^2 - 4\omega E^2E' \\ &- 4\omega\varkappa E'^2 + 4\omega\varkappa'E^2\}.\end{aligned} \tag{136}$$

Simplifying equation (136) and substituting equation (130), namely,

$$E' = E - \omega, \tag{137}$$

then

$$T_2 = (\varkappa\varkappa'\omega^2)^{-1}(-8m^2E^2 + 8m^2E\omega - 4E^2|\vec{q}|^2 + 4E\omega|\vec{q}|^2 - 2\omega^2|\vec{q}|^2$$

$$+ 2m^2|\vec{q}|^2 + |\vec{q}|^4) - 2(|\vec{q}|^2/\omega)(\varkappa'^{-1} - \varkappa^{-1}) + 4(E\varkappa'^{-1} + E'\varkappa^{-1}). \quad (138)$$

Next one proceeds by evaluating $T_4(p, p', k)$ given by equation (87). By using the cyclic invariance of the trace, equation (87) may be rewritten as

$$T_4(p, p', k) = \mathrm{Tr}(\hat{f} + m)(2\varkappa\omega)^{-1}(\hat{\tilde{p}}' + m)(\hat{f} + m)(2\varkappa\omega)^{-1}\gamma^\mu(\hat{p} + m)\gamma^\mu. \quad (139)$$

Using equation (102) in equation (139), then

$$T_4(p, p', k) = \mathrm{Tr}(\hat{f} + m)(2\varkappa\omega)^{-1}(\hat{\tilde{p}}' + m)(\hat{f} + m)(2\varkappa\omega)^{-1}(-2\hat{p} + 4m), \quad (140)$$

or multiplying terms and recalling that the trace of a product of an odd number of gamma matrices is vanishing, then

$$T_4(p, p', k) = (4\varkappa^2\omega^2)^{-1}\mathrm{Tr}(-2\hat{f}\hat{\tilde{p}}'\hat{f}\hat{p} + 4m^2\hat{f}\hat{\tilde{p}}'$$

$$+ 4m^2\hat{f}\hat{f} - 2m^2\hat{f}\hat{p} + 4m^2\hat{\tilde{p}}'\hat{f}$$

$$- 2m^2\hat{\tilde{p}}'\hat{p} - 2m^2\hat{f}\hat{p} + 4m^4). \quad (141)$$

Again using cyclic invariance of the trace to combine the second and fifth terms, then

$$T_4(p, p', k) = (4\varkappa^2\omega^2)^{-1}\mathrm{Tr}(-2\hat{f}\hat{\tilde{p}}'\hat{f}\hat{p}$$

$$+ 8m^2\hat{f}\hat{\tilde{p}}' + 4m^2\hat{f}\hat{f} - 2m^2\hat{f}\hat{p}$$

$$- 2m^2\hat{\tilde{p}}'\hat{p} - 2m^2\hat{f}\hat{p} + 4m^4). \quad (142)$$

Next, using equations (106) and (107) and combining terms, equation (142) becomes

$$T_4(p, p', k) = (4\varkappa^2\omega^2)^{-1}(-16f \cdot \tilde{p}'\, f \cdot p + 8f^2\, \tilde{p}' \cdot p + 32m^2 f \cdot \tilde{p}'$$

$$+ 16m^2f^2 - 16m^2 f \cdot p - 8m^2\tilde{p}' \cdot p + 16m^4). \quad (143)$$

Substituting equation (35) into equation (143), using equations (38) and (39), and combining terms, then

$$T_4(p, p', k) = (4x^2\omega^2)^{-1}[-16(p \cdot \tilde{p}' - \tilde{p}' \cdot k)(m^2 - k \cdot p)$$
$$+ 32m^2 p \cdot \tilde{p}' - 16p \cdot k\tilde{p}' \cdot p - 32m^2 k \cdot \tilde{p}'$$
$$- 16m^2 p \cdot k + 16m^4]. \tag{144}$$

Again, multiplying and combining terms, equation (144) becomes

$$T_4(p, p', k) = 4(x^2\omega^2)^{-1}(m^2 p \cdot \tilde{p}' - m^2\tilde{p}' \cdot k - m^2 p \cdot k - \tilde{p}' \cdot kk \cdot p + m^4). \tag{145}$$

One notes using equation (83) that

$$p \cdot \tilde{p}' = EE' + \vec{p} \cdot \vec{p}'. \tag{146}$$

Substituting equations (120), (123), and (146) into equation (145), then

$$T_4(p, p', k) = 4(x^2\omega^2)^{-1}[m^2EE' + m^2\vec{p} \cdot \vec{p}' - m^2\omega(2E' - x') - m^2\omega x - \omega^2(2E' - x')x + m^4], \tag{147}$$

or

$$T_4(p, p', k) = 4(x^2\omega^2)^{-1}[m^2\vec{p} \cdot \vec{p}'$$
$$+ m^2EE' - 2m^2\omega E' - m^2\omega(x - x') - 2\omega^2E'x$$
$$+ \omega^2 xx' + m^4]. \tag{148}$$

Substituting equation (134) into equation (148), then

$$T_4(p, p', k) = 4(x^2\omega^2)^{-1}\{(m^2/2)[E^2 + E'^2 + \omega^2 - 2m^2 - 2\omega(E - E') + 2\omega(x - x') - |\vec{q}|^2]$$
$$+ m^2EE' - 2m^2\omega E' - m^2\omega(x - x') - 2\omega^2E'x + \omega^2xx' + m^4\}. \tag{149}$$

Combining terms, equation (149) becomes

$$T_4(p, p', k) = 2m^2(E + E')^2(\varkappa^2\omega^2)^{-1} + 2m^2\varkappa^{-2} - 4m^2(E + E')(\varkappa^2\omega)^{-1}$$
$$- 8E'\varkappa^{-1} + 4\varkappa'\varkappa^{-1} - 2m^2|\vec{q}|^2(\varkappa^2\omega^2)^{-1}. \quad (150)$$

Under the interchange $(p, p', k) \rightarrow (p', p, -k)$, one sees from equations (130), (43), (45), and (132) that $(E, E', \omega, \varkappa, \varkappa', \vec{q}) \rightarrow (E', E, -\omega, \varkappa', \varkappa, -\vec{q})$. Therefore, by using equation (150),

$$T_4(p', p, -k) = 2m^2(E + E')^2/(\varkappa'\omega)^2 + 2m^2/\varkappa'^2 + 4m^2(E + E')/\varkappa'^2\omega$$
$$- 8E/\varkappa' + 4\varkappa/\varkappa' - 2m^2|\vec{q}|^2/\varkappa'^2\omega^2. \quad (151)$$

Substituting equations (150) and (151) into equation (96), then

$$T_{4s} = -\tfrac{1}{2}[2m^2(E + E')^2/\varkappa^2\omega^2 + 2m^2(E + E')^2/\varkappa'^2\omega^2 + 2m^2/\varkappa^2 + 2m^2/\varkappa'^2$$
$$- 4m^2(E + E')/\varkappa^2\omega + 4m^2(E + E')/\varkappa'^2\omega - 8E'/\varkappa - 8E/\varkappa' + 4\varkappa'/\varkappa + 4\varkappa/\varkappa'$$
$$- 2m^2|\vec{q}|^2/\varkappa^2\omega^2 - 2m^2|\vec{q}|^2/\varkappa'^2\omega^2]. \quad (152)$$

Combining terms in equation (152), then

$$T_{4s} = -m^2(E + E' - \omega)^2/\varkappa^2\omega^2 - m^2(E + E' + \omega)^2/\varkappa'^2\omega^2 + 4(E'/\varkappa + E/\varkappa')$$
$$- 2(\varkappa'/\varkappa + \varkappa/\varkappa') + m^2|\vec{q}|^2(\varkappa^{-2} + \varkappa'^{-2})/\omega^2. \quad (153)$$

Next, using equation (130) in the first two terms of equation (153), then

$$T_{4s} = -4m^2E'^2/\omega^2\varkappa^2 - 4m^2E^2/\omega^2\varkappa'^2 + 4(E'/\varkappa + E/\varkappa)$$
$$- 2(\varkappa'/\varkappa + \varkappa/\varkappa') + m^2|\vec{q}|^2(\varkappa^{-2} + \varkappa'^{-2})/\omega^2 \quad (154)$$

or

$$T_{4s} = -(4m^2/\omega^2)(E'/\varkappa - E/\varkappa')^2 - (8m^2/\omega^2)(EE'/\varkappa\varkappa') + 4(E'/\varkappa + E/\varkappa') - 2(\varkappa'/\varkappa + \varkappa/\varkappa')$$
$$+ m^2|\vec{q}|^2(\varkappa^{-1} - \varkappa'^{-1})^2/\omega^2 + 2m^2|\vec{q}|^2/\omega^2\varkappa\varkappa'. \tag{155}$$

Returning to equation (95) and substituting equations (138) and (155), then

$$\overline{T} = -(4m^2/\omega^2)(E'/\varkappa - E/\varkappa')^2 - (8m^2/\omega^2)(EE'/\varkappa\varkappa') + 4(E'/\varkappa + E/\varkappa') - 2(\varkappa'/\varkappa + \varkappa/\varkappa')$$
$$+ (m^2|\vec{q}|^2/\omega^2)(\varkappa^{-1} - \varkappa'^{-1})^2 + 2m^2|\vec{q}|^2/\omega^2\varkappa\varkappa' - (1/\varkappa\varkappa'\omega^2)(-8m^2E^2 + 8m^2E\omega - 4E^2|\vec{q}|^2$$
$$+ 4E\omega|\vec{q}|^2 - 2\omega^2|\vec{q}|^2 + 2m^2|\vec{q}|^2 + |\vec{q}|^4) + (2|\vec{q}|^2/\omega)(\varkappa'^{-1} - \varkappa^{-1}) - 4(E/\varkappa' + E'/\varkappa). \tag{156}$$

One notes using equation (130) that

$$-4E^2|\vec{q}|^2 + 4E\omega|\vec{q}|^2 - 2\omega^2|\vec{q}|^2$$
$$= (-4E^2 + 4E\omega - 2\omega^2)|\vec{q}|^2$$
$$= [-2E^2 - 2(E' + \omega)^2 + 4(E' + \omega)\omega - 2\omega^2]|\vec{q}|^2$$
$$= -(2E^2 + 2E'^2)|\vec{q}|^2. \tag{157}$$

Substituting equation (157) into equation (156), then

$$\overline{T} = (m^2/\omega^2)[(|\vec{q}|^2/\varkappa\varkappa'm^2)(2E^2 + 2E'^2 - |\vec{q}|^2) + |\vec{q}|^2(\varkappa^{-1} - \varkappa'^{-1})^2$$
$$- 4(E/\varkappa' - E'/\varkappa)^2 + (2\omega|\vec{q}|^2/m^2)(\varkappa'^{-1} - \varkappa^{-1}) - (2\omega^2/m^2)(\varkappa'/\varkappa + \varkappa/\varkappa')]. \tag{158}$$

Finally, substituting equation (158) into equation (59), then

$$d^5\sigma = (Z^2\alpha r_e^2/4\pi^2)(|\vec{p}|'m^4/pq^4)(d\omega/\omega)\, d^2\Omega\, d^2\Omega' \cdot [(|\vec{q}|^2/\varkappa\varkappa'm^2)(2E^2 + 2E'^2 - |\vec{q}|^2)$$
$$+ |\vec{q}|^2(\varkappa^{-1} - \varkappa'^{-1})^2 - 4(E/\varkappa' - E'/\varkappa)^2 + (2\omega|\vec{q}|^2/m^2)(\varkappa'^{-1} - \varkappa^{-1})$$
$$- (2\omega^2/m^2)(\varkappa'/\varkappa + \varkappa/\varkappa')], \tag{159}$$

where r_e is the classical radius of the electron,

$$r_e = e^2/m \cdot 1 = e^2/mc^2 = 2.818 \times 10^{-15} \text{ meters}, \tag{160}$$

and α is the fine-structure constant (relativistic units),

$$\alpha = e^2/1 \cdot 1 = e^2/\hbar c = 1/137.04. \tag{161}$$

Equation (159) agrees in form and notation with equation (91.7) of Berestetskiĭ et al.[2] It is the well-known Bethe-Heitler-Sauter formula for the bremsstrahlung cross section in which a photon of given frequency is emitted in a given direction and the incident electron is scattered in a prescribed direction disregarding polarization effects.

The cross-section equation (159) can be written in terms of the scattering angles θ and θ' as follows. One first defines

$$\begin{aligned}\Sigma = {} & (|\vec{q}|^2/\varkappa\varkappa' m^2)(2E^2 + 2E'^2 - |\vec{q}|^2) + |\vec{q}|^2(\varkappa^{-1} - \varkappa'^{-1})^2 \\ & - 4(E/\varkappa' - E'/\varkappa)^2 + (2\omega|\vec{q}|^2/m^2)(\varkappa'^{-1} - \varkappa^{-1}) \\ & - (2\omega^2/m^2)(\varkappa'/\varkappa + \varkappa/\varkappa').\end{aligned} \tag{162}$$

Then

$$d^5\sigma = (Z^2\alpha r_e^2/4\pi^2)(|\vec{p}'|m^4/|\vec{p}||\vec{q}|^4)(d\omega/\omega)\ d^2\Omega\ d^2\Omega'\ \Sigma. \tag{163}$$

Equation (162) may be rewritten as

$$\Sigma = \Sigma_1 + \Sigma_2 + \Sigma_3, \tag{164}$$

[2] *V. B. Berestetskiĭ, E. M. Lifshitz, and L. P. Pitaevskiĭ, Relativistic Quantum Theory, Part I, Addison-Wesley Publishing Co., Inc., Reading, MA (1971).*

where

$$\Sigma_1 = (m^2\kappa\kappa')^{-1}[|\vec{q}|^2(2E^2 + 2E'^2 - |\vec{q}|^2) - 2m^2|\vec{q}|^2 + 8m^2EE' + 2\omega|\vec{q}|^2\kappa - 2\omega|\vec{q}|^2\kappa' - 2\omega^2(\kappa'^2 + \kappa^2)], \tag{165}$$

$$\Sigma_2 = \kappa^{-2}(|\vec{q}|^2 - 4E'^2), \tag{166}$$

$$\Sigma_3 = \kappa'^{-2}(|\vec{q}|^2 - 4E^2). \tag{167}$$

The term Σ_1 of equation (165) may be written as

$$\Sigma_1 = (m^2\kappa\kappa')^{-1}\,|\vec{q}|^2(2E^2 + 2E'^2 - |\vec{q}|^2 - 2m^2 + 2\omega\kappa - 2\omega\kappa') + (m^2\kappa\kappa')^{-1}[8m^2EE' - 2\omega^2(\kappa'^2 + \kappa^2)]. \tag{168}$$

Also, Σ_2 of equation (166) can be written

$$\Sigma_2 = (\kappa^2m^2)^{-1}(|\vec{q}|^2 - 4E'^2)m^2. \tag{169}$$

But using equation (116),

$$m^2 = E^2 - |\vec{p}|^2. \tag{170}$$

Substituting equation (170) into equation (169), then

$$\Sigma_2 = (\kappa^2m^2)^{-1}(|\vec{q}|^2 - 4E'^2)(E^2 - |\vec{p}|^2). \tag{171}$$

Substituting

$$1 = \sin^2\theta + \cos^2\theta \tag{172}$$

into equation (171), then

$$\Sigma_2 = (\varkappa^2 m^2)^{-1}(4E'^2 - |\vec{q}|^2)|\vec{p}|^2(\sin^2\theta + \cos^2\theta) + (\varkappa^2 m^2)^{-1}(|\vec{q}|^2 - 4E'^2)E^2 \tag{173}$$

or

$$\Sigma_2 = (p^2/\varkappa^2 m^2)(4E'^2 - |\vec{q}|^2)\sin^2\theta + (1/\varkappa^2 m^2)(4E'^2 - |\vec{q}|^2)(|\vec{p}|^2\cos^2\theta - E^2). \tag{174}$$

Substituting equation (43),

$$|\vec{p}|\cos\theta = E - \varkappa, \tag{175}$$

into equation (174), then

$$\Sigma_2 = (p^2/\varkappa^2 m^2)(4E'^2 - |\vec{q}|^2)\sin^2\theta + (1/\varkappa^2 m^2)(4E'^2 - |\vec{q}|^2)[(E - \varkappa)^2 - E^2] \tag{176}$$

or

$$\Sigma_2 = (p^2/\varkappa^2 m^2)(4E'^2 - |\vec{q}|^2)\sin^2\theta + (1/m^2)(4E'^2 - |\vec{q}|^2) - (2E/\varkappa m^2)(4E'^2 - |\vec{q}|^2). \tag{177}$$

Analogously, Σ_3 of equation (167) can be rewritten in the form

$$\Sigma_3 = (p'^2/\varkappa'^2 m^2)(4E^2 - |\vec{q}|^2)\sin^2\theta' + (1/m^2)(4E^2 - |\vec{q}|^2) - (2E'/\varkappa' m^2)(4E^2 - |\vec{q}|^2). \tag{178}$$

Substituting equations (168), (177), and (178) into equation (164), then

$$\begin{aligned}\Sigma = (1/m^2\varkappa\varkappa')[&|\vec{q}|^2(2E^2 + 2E'^2 - |\vec{q}|^2 - 2m^2 + 2\omega\varkappa - 2\omega\varkappa' - 2\varkappa\varkappa' + 2E\varkappa' + 2E'\varkappa) \\ &+ 8m^2EE' - 2\omega^2(\varkappa'^2 + \varkappa^2) + (4E^2 + 4E'^2)\varkappa\varkappa' - 8EE'^2\varkappa' - 8E'E^2\varkappa] \\ &+ (p'^2/\varkappa'^2 m^2)(4E^2 - |\vec{q}|^2)\sin^2\theta' + (|\vec{p}|^2/\varkappa^2 m^2)(4E'^2 - |\vec{q}|^2)\sin^2\theta\end{aligned} \tag{179}$$

or

$$\Sigma = \overline{\Sigma}/m^2\varkappa\varkappa' + (|\vec{p}'|^2/\varkappa'^2m^2)(4E^2 - |\vec{q}|^2)\sin^2\theta'$$
$$+ (|\vec{p}|^2/\varkappa^2m^2)(4E'^2 - |\vec{q}|^2)\sin^2\theta, \quad (180)$$

where

$$\overline{\Sigma} = \overline{\Sigma}_1 + |\vec{q}|^2(-2m^2 + 2\omega\varkappa - 2\omega\varkappa' - 2\varkappa\varkappa' + 2E\varkappa' + 2E'\varkappa) + 8m^2EE'$$
$$- 2\omega^2(\varkappa'^2 + \varkappa^2) + (4E^2 + 4E'^2)\varkappa\varkappa' - 8EE'^2\varkappa' - 8E'E^2\varkappa, \quad (181)$$

$$\overline{\Sigma}_1 = |\vec{q}|^2(2E^2 + 2E'^2 - |\vec{q}|^2). \quad (182)$$

Using equation (133), $|\vec{q}|^2$ can be expressed in terms of the scattering angles θ and θ' as follows. The momentum vector $\vec{p}$ can be written in terms of components $\vec{p}_\perp$ and $\vec{p}_\parallel$, perpendicular and parallel to the photon momentum vector $\vec{k}$, respectively; thus,

$$\vec{p} = \vec{p}_\perp + \vec{p}_\parallel. \quad (183)$$

Clearly,

$$\vec{p}_\parallel = (\vec{p}\cdot\vec{k}/|\vec{k}|^2)\vec{k}. \quad (184)$$

Substituting equation (184) into equation (183), then

$$\vec{p} = \vec{p}_\perp + (\vec{p}\cdot\vec{k}/|\vec{k}|^2)\vec{k} \quad (185)$$

and by definition

$$\vec{p}_\perp\cdot\vec{k} = 0. \quad (186)$$

Analogously,

$$\vec{p}' = \vec{p}'_\perp + (\vec{p}'\cdot\vec{k}/|\vec{k}|^2)\vec{k} \quad (187)$$

and

$$\vec{p}'_\perp \cdot \vec{k} = 0. \qquad (188)$$

Also, denoting the angle between $\vec{p}_\perp$ and $\vec{p}'_\perp$ by ϕ', then

$$\vec{p}_\perp \cdot \vec{p}'_\perp = |\vec{p}_\perp||\vec{p}'_\perp| \cos \phi'. \qquad (189)$$

Denoting the angle between the incident electron momentum $\vec{p}$ and the emitted photon momentum $\vec{k}$ by θ and that between the scattered electron momentum $\vec{p}'$ and $\vec{k}$ by θ', then

$$|\vec{p}_\perp| = |\vec{p}| \sin \theta, \qquad (190)$$

$$\vec{p} \cdot \vec{k} = |\vec{p}||\vec{k}| \cos \theta, \qquad (191)$$

$$|\vec{p}'_\perp| = |\vec{p}'| \sin \theta', \qquad (192)$$

$$\vec{p}' \cdot \vec{k} = |\vec{p}'||\vec{k}| \cos \theta'. \qquad (193)$$

Using equations (185) and (187), then

$$\vec{p} \cdot \vec{p}' = [\vec{p}_\perp + (\vec{p} \cdot \vec{k}/|\vec{k}|^2)\vec{k}] \cdot [\vec{p}'_\perp + (\vec{p}' \cdot \vec{k}/|\vec{k}|^2)\vec{k}] \qquad (194)$$

or

$$\vec{p} \cdot \vec{p}' = \vec{p}_\perp \cdot \vec{p}'_\perp + (\vec{p} \cdot \vec{k}/|\vec{k}|^2)\vec{p}'_\perp \cdot \vec{k} + (\vec{p}' \cdot \vec{k}/|\vec{k}|^2)\vec{p}_\perp \cdot \vec{k} + (\vec{p} \cdot \vec{k})(\vec{p}' \cdot \vec{k})/|\vec{k}|^2. \qquad (195)$$

Substituting equations (189), (186), (188), and (190) to (193) in equation (195), then

$$\vec{p}\cdot\vec{p}' = |\vec{p}| \sin\theta\, |\vec{p}'| \sin\theta' \cos\phi'$$
$$+ (|\vec{p}|\,|\vec{k}|/|\vec{k}|^2) \cos\theta\, |\vec{p}'|\,|\vec{k}| \cos\theta' \tag{196}$$

or

$$\vec{p}\cdot\vec{p}' = |\vec{p}|\,|\vec{p}'|(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\phi'). \tag{197}$$

Therefore substituting equations (21), (193), (191), and (197) into equation (133), one obtains

$$|\vec{q}|^2 = |\vec{p}|^2 + |\vec{p}'|^2 + \omega^2 - 2|\vec{p}|\omega\cos\theta + 2|\vec{p}'|\omega\cos\theta'$$
$$- 2|\vec{p}|\,|\vec{p}'|(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\phi'). \tag{198}$$

Then substituting equation (198) into equation (182),

$$\bar{\Sigma}_1 = [\, |\vec{p}|^2 + |\vec{p}'|^2 + \omega^2 - 2|\vec{p}|\omega\cos\theta + 2|\vec{p}'|\omega\cos\theta'$$
$$- 2|\vec{p}|\,|\vec{p}'|(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\phi')](2E^2 + 2E'^2 - |\vec{q}|^2) \tag{199}$$

or

$$\bar{\Sigma}_1 = -2|\vec{p}|\,|\vec{p}'| \sin\theta\sin\theta'\cos\phi'(2E^2 + 2E'^2 - |\vec{q}|^2) + \big(|\vec{p}|^2 + |\vec{p}'|^2 + \omega^2$$
$$- 2|\vec{p}|\omega\cos\theta + 2|\vec{p}'|\omega\cos\theta' - 2|\vec{p}|\,|\vec{p}'|\cos\theta\cos\theta'\big)(2E^2 + 2E'^2 - |\vec{q}|^2). \tag{200}$$

Next substituting equation (200) into equation (181), then

$$\bar{\Sigma} = -2|\vec{p}|\,|\vec{p}'| \sin\theta\sin\theta'\cos\phi'\,(2E^2 + 2E'^2 - |\vec{q}|^2) + |\vec{q}|^2\,\bar{\Sigma}_2 + 2\bar{\Sigma}_3, \tag{201}$$

where

$$\bar{\Sigma}_2 = -2m^2 + 2\omega\kappa - 2\omega\kappa' - 2\kappa\kappa' + 2E\kappa' + 2E'\kappa - |\vec{p}|^2 - |\vec{p}'|^2 - \omega^2$$
$$+ 2|\vec{p}|\omega \cos\theta - 2|\vec{p}'|\omega \cos\theta' + 2|\vec{p}||\vec{p}'| \cos\theta \cos\theta', \tag{202}$$

$$\bar{\Sigma}_3 = (E^2 + E'^2)(|\vec{p}|^2 + |\vec{p}'|^2 + \omega^2 - 2|\vec{p}|\omega \cos\theta + 2|\vec{p}'|\omega \cos\theta'$$
$$- 2|\vec{p}||\vec{p}'| \cos\theta \cos\theta') + 4m^2EE' - \omega^2(\kappa'^2 + \kappa^2)$$
$$+ 2(E^2 + E'^2)\kappa\kappa' - 4EE'^2\kappa' - 4E'E^2\kappa. \tag{203}$$

Substituting equations (43) and (45) in equation (202), then

$$\bar{\Sigma}_2 = -2m^2 + 2\omega(E - |\vec{p}| \cos\theta) - 2\omega(E' - |\vec{p}'| \cos\theta') - 2(E - |\vec{p}| \cos\theta)(E' - |\vec{p}'| \cos\theta')$$
$$+ 2E(E' - |\vec{p}'| \cos\theta') + 2E'(E - |\vec{p}| \cos\theta) - |\vec{p}|^2 - |\vec{p}'|^2 - \omega^2$$
$$+ 2|\vec{p}|\omega \cos\theta - 2|\vec{p}'|\omega \cos\theta' + 2|\vec{p}||\vec{p}'| \cos\theta \cos\theta' \tag{204}$$

or

$$\bar{\Sigma}_2 = -2m^2 + 2\omega E - 2\omega|\vec{p}| \cos\theta - 2EE' + 2E|\vec{p}'| \cos\theta' - 2|\vec{p}||\vec{p}'| \cos\theta \cos\theta'$$
$$- |\vec{p}|^2 - 2\omega E' + 2\omega|\vec{p}'| \cos\theta' + 2EE' + 2E'|\vec{p}| \cos\theta$$
$$+ 2|\vec{p}||\vec{p}'| \cos\theta \cos\theta' - |\vec{p}'|^2 + 2\omega|\vec{p}| \cos\theta$$
$$+ 2E'E - 2E|\vec{p}'| \cos\theta' - \omega^2 - 2\omega|\vec{p}'| \cos\theta' - 2E'|\vec{p}| \cos\theta. \tag{205}$$

Substituting equation (130) into the 8th and 15th terms of equation (205), then

$$\bar{\Sigma}_2 = -2m^2 + \omega^2 + 2E^2 - 2\omega E - |\vec{p}|^2 - |\vec{p}'|^2. \tag{206}$$

Further, using equation (130),

$$\omega = E - E', \tag{207}$$

and substituting equation (207) in the second and fourth terms of equation (206), then

$$\bar{\Sigma}_2 = -2m^2 + E^2 + E'^2 - |\vec{p}|^2 - |\vec{p}'|^2. \tag{208}$$

Next, substituting equation (116) and, analogously,

$$|\vec{p}'|^2 = E'^2 - m^2 \tag{209}$$

into equation (208), then finally

$$\bar{\Sigma}_2 = 0. \tag{210}$$

Also, substituting equations (43) and (45) into equation (203) and multiplying terms, then

$$\begin{aligned}
\bar{\Sigma}_3 = {} & E^2|\vec{p}|^2 + E^2|\vec{p}'|^2 + E^2\omega^2 - 2E^2|\vec{p}|\omega \cos\theta \\
& + 2E^2|\vec{p}'|\omega \cos\theta' - 2E^2|\vec{p}|\,|\vec{p}'| \cos\theta \cos\theta' \\
& + E'^2|\vec{p}|^2 + E'^2|\vec{p}'|^2 + E'^2\omega^2 - 2E'^2|\vec{p}|\omega \cos\theta \\
& + 2E'^2|\vec{p}'|\omega \cos\theta' - 2E'^2|\vec{p}|\,|\vec{p}'| \cos\theta \cos\theta' \\
& + 4m^2EE' - \omega^2E'^2 + 2\omega^2E'|\vec{p}'| \cos\theta' - \omega^2|\vec{p}'|^2 \cos^2\theta' \\
& - \omega^2E^2 + 2\omega^2E|\vec{p}| \cos\theta - \omega^2|\vec{p}|^2 \cos^2\theta + 2E^3E' \\
& - 2E^3|\vec{p}'| \cos\theta' - 2E^2E'|\vec{p}| \cos\theta + 2E^2|\vec{p}|\,|\vec{p}'| \cos\theta \cos\theta' \\
& + 2EE'^3 - 2EE'^2|\vec{p}'| \cos\theta' - 2E'^3|\vec{p}| \cos\theta \\
& + 2E'^2|\vec{p}|\,|\vec{p}'| \cos\theta \cos\theta' - 4EE'^3 + 4EE'^2|\vec{p}'| \cos\theta' \\
& - 4E'E^3 + 4E'E^2|\vec{p}| \cos\theta.
\end{aligned} \tag{211}$$

Substituting the trigonometric identities

$$\cos^2\theta' = 1 - \sin^2\theta' \tag{212}$$

and

$$\cos^2\theta = 1 - \sin^2\theta \tag{213}$$

into the 16th and 19th terms of equation (211) and combining terms, then

$$\bar{\Sigma}_3 = \omega^2|\vec{p}|^2 \sin^2\theta + \omega^2|\vec{p}'|^2 \sin^2\theta' + \bar{\Sigma}_4, \tag{214}$$

where

$$\begin{aligned}\bar{\Sigma}_4 = {} & E^2|\vec{p}|^2 + E^2|\vec{p}'|^2 + E'^2|\vec{p}|^2 + E'^2|\vec{p}'|^2 + 4m^2EE' - \omega^2|\vec{p}'|^2 \\ & - \omega^2|\vec{p}|^2 - 2E^3E' - 2EE'^3 + \cos\theta\,(-2E^2|\vec{p}|\omega - 2E'^2|\vec{p}|\omega \\ & + 2\omega^2E|\vec{p}| - 2E^2E'|\vec{p}| - 2E'^3|\vec{p}| + 4E'E^2|\vec{p}|) \\ & + \cos\ \theta'\,(2E^2|\vec{p}'|\omega + 2E'^2|\vec{p}'|\omega + 2\omega^2E'|\vec{p}'| - 2E^3|\vec{p}'| \\ & - 2EE'^2|\vec{p}'| + 4EE'^2|\vec{p}'|).\end{aligned} \tag{215}$$

Next, substituting equation (207) into equation (215) and combining terms, then

$$\bar{\Sigma}_4 = 4m^2EE' + 2EE'|\vec{p}'|^2 + 2EE'|\vec{p}|^2 - 2E^3E' - 2EE'^3. \tag{216}$$

Further, substituting equations (116) and (209) into equation (216), then

$$\bar{\Sigma}_4 = 0. \tag{217}$$

Then substituting equation (217) into equation (214),

$$\bar{\Sigma}_3 = \omega^2|\vec{p}|^2 \sin^2\theta + \omega^2|\vec{p}'|^2 \sin^2\theta'. \tag{218}$$

Therefore, substituting equations (210) and (218) into equation (201), then

$$\begin{aligned}\bar{\Sigma} = {} & -2|\vec{p}|\,|\vec{p}'| \sin\theta \sin\theta' \cos\phi'\,(2E^2 + 2E'^2 - |\vec{q}|^2) \\ & + 2\omega^2|\vec{p}|^2 \sin^2\theta + 2\omega^2|\vec{p}'|^2 \sin^2\theta'.\end{aligned} \tag{219}$$

Then substituting equation (219) into equation (180),

$$\Sigma = m^{-2}[(|\vec{p}'|^2/\varkappa'^2)(4E^2 - |\vec{q}|^2)\sin^2\theta' + (|\vec{p}|^2/\varkappa^2)(4E'^2 - |\vec{q}|^2)\sin^2\theta$$

$$+ (2\omega^2/\varkappa\varkappa')(|\vec{p}|^2\sin^2\theta + |\vec{p}'|^2\sin^2\theta')$$

$$- (2|\vec{p}||\vec{p}'|/\varkappa\varkappa')(2E^2 + 2E'^2 - |\vec{q}|^2)\sin\theta\sin\theta'\cos\phi']. \tag{220}$$

Finally, substituting equation (220) into equation (163),

$$d^5\sigma = (Z^2\alpha r_e^2/4\pi^2)(|\vec{p}'|m^2/|\vec{p}||\vec{q}|^4)(d\omega/\omega)\, d^2\Omega\, d^2\Omega'$$

$$\cdot[(|\vec{p}'|^2/\varkappa'^2)(4E^2 - |\vec{q}|^2)\sin^2\theta' + (|\vec{p}|^2/\varkappa^2)(4E'^2 - |\vec{q}|^2)\sin^2\theta$$

$$+ (2\omega^2/\varkappa\varkappa')(|\vec{p}|^2\sin^2\theta + |\vec{p}'|^2\sin^2\theta')$$

$$- (2|\vec{p}||\vec{p}'|/\varkappa\varkappa')(2E^2 + 2E'^2 - |\vec{q}|^2)\sin\theta\sin\theta'\cos\phi']. \tag{221}$$

The photon and recoil-electron differential solid angles are given by

$$d^2\Omega = \sin\theta\, d\theta\, d\phi \tag{222}$$

and

$$d^2\Omega' = \sin\theta'\, d\theta'\, d\phi', \tag{223}$$

respectively. If one integrates over the photon azimuthal angle ϕ, the corresponding cross section is given by

$$d^4\sigma = \int_0^{2\pi} (d^5\sigma/d\phi)\, d\phi. \tag{224}$$

Substituting equation (221) into equation (224) and using equations (222) and (223), then

$$d^4\sigma = (Z^2\alpha r_e^2/2\pi)(|\vec{p}'|m^2/|\vec{p}||\vec{q}|^4)\,(d\omega/\omega)\sin\theta\, d\theta \sin\theta'\, d\theta'\, d\phi'$$

$$\cdot\big[(|\vec{p}'|^2/\varkappa'^2)(4E^2 - |\vec{q}|^2)\sin^2\theta' + (|\vec{p}|^2/\varkappa^2)(4E'^2 - |\vec{q}|^2)\sin^2\theta$$

$$+ (2\omega^2/\varkappa\varkappa')(|\vec{p}|^2\sin^2\theta + |\vec{p}'|^2\sin^2\theta')$$

$$- (2|\vec{p}||\vec{p}'|/\varkappa\varkappa')(2E^2 + 2E'^2 - |\vec{q}|^2)\sin\theta\sin\theta'\cos\phi'\big]. \quad (225)$$

Equation (225) agrees with equation (91.8) of Berestetskiĭ et al,[2] wherein the notation for ϕ and ϕ' is interchanged from that used here. The differential cross section $d^5\sigma$ of equation (221) is the probability (in Born approximation) of single photon emission into solid angle $d^2\Omega$ and electron recoil into solid angle $d^2\Omega'$ per unit electron flux. The functions $\varkappa$, $\varkappa'$, and $|\vec{q}|^2$ are given by equations (43), (45), and (198), respectively. The quantities E and $|\vec{p}|$ are the energy and the momentum, respectively, of the incident electron; E' and $|\vec{p}'|$ are the energy and the momentum, respectively, of the scattered electron; θ' is the polar angle made by the scattered electron momentum with the photon momentum; $\omega/2\pi$ is the frequency of the emitted photon; θ is the polar angle between the incident electron momentum and the emitted photon; ϕ' is the angle between the component of the incident electron momentum transverse to the photon momentum and the component of the scattered electron momentum transverse to the photon momentum; ϕ is the azimuthal angle of the photon momentum; -Ze is the charge of the nucleus; α is the fine structure constant given by equation (161); and r_e is the classical radius of the electron given by equation (160).

3. GLUCKSTERN-HULL FORMULA

In this section, the Bethe-Heitler cross section, equation (221), is to be integrated over all directions of the recoil electron to obtain the Gluckstern-Hull formula, equation (1). The latter is the differential cross section in Born approximation for single photon emission in the scattering of an electron by the Coulomb potential of a nucleus, independent of the direction of recoil of the final state electron. Also, the calculation here is specialized to the case of an unpolarized incident electron, and the cross section is summed over all polarization states of the emitted photon and all final spin states of the recoil electron.

Thus, the Gluckstern-Hull cross section is given by

$$d^3\sigma = \int (d^5\sigma/d^2\Omega')\, d^2\Omega', \quad (226)$$

where the integral is over all possible angles of electron recoil. By substituting equation (221) into equation (226), then

[2] V. B. Berestetskiĭ, E. M. Lifshitz, and L. P. Pitaevskiĭ, *Relativistic Quantum Theory, Part I*, Addison-Wesley Publishing Co., Inc., Reading, MA (1971).

$$d^3\sigma = (Z^2 \alpha r_e^2 / 2\pi)(|\vec{p}'| m^2 / |\vec{p}|)\ (d\omega/\omega)\ d^2\Omega\ I(|\vec{p}|, \vec{k}, |\vec{p}'|), \tag{227}$$

where

$$I(|\vec{p}|, \vec{k}, |\vec{p}'|) \equiv \int (d^2\Omega'/2\pi)|\vec{q}|^{-4}\Big[(|\vec{p}'|^2/\varkappa'^2) \cdot (4E^2 - |\vec{q}|^2) \sin^2\theta' + (|\vec{p}|^2/\varkappa^2)(4E'^2 - |\vec{q}|^2) \sin^2\theta + (2\omega^2/\varkappa\varkappa')(|\vec{p}|^2 \sin^2\theta + |\vec{p}'|^2 \sin^2\theta') - 2(|\vec{p}|\,|\vec{p}'|/\varkappa\varkappa') \cdot (2E^2 + 2E'^2 - |\vec{q}|^2) \sin\theta \sin\theta' \cos\phi'\Big]. \tag{228}$$

One proceeds to calculate $I(|\vec{p}|, \vec{k}, |\vec{p}'|)$. Equation (228) is first rewritten in the equivalent form

$$I = \int (d^2\Omega'/2\pi)|\vec{q}|^{-4}\Big\{|\vec{q}|^2\big[-(|\vec{p}'|^2/\varkappa'^2) \sin^2\theta' - (|\vec{p}|^2/\varkappa^2) \sin^2\theta + 2(|\vec{p}|\,|\vec{p}'|/\varkappa\varkappa') \sin\theta \sin\theta' \cos\phi\big] + 4(|\vec{p}'|^2 E^2/\varkappa'^2) \sin^2\theta' + 4(|\vec{p}|^2 E'^2/\varkappa^2) \sin^2\theta + 2(\omega^2/\varkappa\varkappa')(|\vec{p}|^2 \sin^2\theta + |\vec{p}'|^2 \sin^2\theta') - 4(|\vec{p}|\,|\vec{p}'|/\varkappa\varkappa')(E^2 + E'^2) \sin\theta \sin\theta' \cos\phi'\Big\}. \tag{229}$$

Next, using equation (45),

$$\cos\theta' = (E' - \varkappa')/|\vec{p}'|. \tag{230}$$

Also, by trigonometry,

$$\sin^2\theta' = 1 - \cos^2\theta'. \tag{231}$$

Substituting equation (230) into equation (231), then

$$\sin^2\theta' = 1 - (E'/|\vec{p}'|)^2 + 2(E'\varkappa'/|\vec{p}'|^2) - \varkappa'^2/|\vec{p}'|^2. \tag{232}$$

Also, from equation (198), it follows that

$$\sin\theta \sin\theta' \cos\phi' = (\tfrac{1}{2}|\vec{p}||\vec{p}'|)(|\vec{p}|^2 + |\vec{p}'|^2 + \omega^2 - |\vec{q}|^2$$
$$- 2|\vec{p}|\omega\cos\theta + 2|\vec{p}'|\omega\cos\theta' - 2|\vec{p}||\vec{p}'|\cos\theta\cos\theta'). \quad (233)$$

Next, substituting equations (232) and (233) into equation (229), then

$$I = \int (d^2\Omega'/2\pi)\{|\vec{q}|^{-2}[-|\vec{p}'|^2/x'^2 + E'^2/x'^2 - 2E'/x' + 1$$
$$- (|\vec{p}|^2/x^2)\sin^2\theta + (xx')^{-1}(|\vec{p}|^2 + |\vec{p}'|^2 + \omega^2 - |\vec{q}|^2$$
$$- 2|\vec{p}|\omega\cos\theta + 2\omega E' - 2\omega x' - 2|\vec{p}|E'\cos\theta + 2|\vec{p}|x'\cos\theta)]$$
$$+ |\vec{q}|^{-4}[4(|\vec{p}'|^2E^2/x'^2) - 4(E'^2E^2/x'^2) + 8(E'E^2/x') - 4E^2$$
$$+ 4(|\vec{p}|^2E'^2/x^2)\sin^2\theta + 2(\omega^2|\vec{p}|^2/xx')\sin^2\theta + 2(\omega^2|\vec{p}'|^2/xx')$$
$$- 2(\omega^2E'^2/xx') + 4(E'\omega^2/x) - 2(\omega^2x'/x) - 2(xx')^{-1}(E^2$$
$$+ E'^2)(|\vec{p}|^2 + |\vec{p}'|^2 + \omega^2 - |\vec{q}|^2 - 2|\vec{p}|\omega\cos\theta$$
$$+ 2\omega E' - 2\omega x' - 2|\vec{p}|E'\cos\theta + 2|\vec{p}|x'\cos\theta)]\}. \quad (234)$$

Equivalently,

$$I = \int (d^2\Omega'/2\pi)\{|\vec{q}|^{-2}[(x'^2)^{-1}(-|\vec{p}'|^2 + E'^2) + (xx')^{-1}(-2E'x + |\vec{p}|^2$$
$$+ |\vec{p}'|^2 + \omega^2 - |\vec{q}|^2 - 2|\vec{p}|\omega\cos\theta + 2\omega E' - 2pE'\cos\theta) + x^{-1}(-2\omega + 2|\vec{p}|\cos\theta)$$
$$+ 1 - (|\vec{p}|^2/x^2)\sin^2\theta] + |\vec{q}|^{-4}[x'^{-2}(4|\vec{p}'|^2E^2 - 4E'^2E^2) + (x'x)^{-1}\{8E'E^2x$$
$$+ 2\omega^2|\vec{p}|^2\sin^2\theta + 2\omega^2|\vec{p}'|^2 - 2\omega^2E'^2 - 2(E^2 + E'^2)(|\vec{p}|^2 + |\vec{p}'|^2 + \omega^2$$
$$- |\vec{q}|^2 - 2|\vec{p}|\omega\cos\theta + 2\omega E' - 2|\vec{p}|E'\cos\theta)\} + 4x^{-1}(E^2 + E'^2)(\omega - |\vec{p}|\cos\theta)$$
$$- 4E^2 + 4(|\vec{p}|^2E'^2/x^2)\sin^2\theta + 4E'\omega^2/x - 2\omega^2x'/x]\}. \quad (235)$$

Using equation (209), one notes that

$$-|\vec{p}'|^2 + E'^2 = m^2 \quad (236)$$

and

$$4|\vec{p}'|^2E^2 - 4E'^2E^2 = -4m^2E^2. \tag{237}$$

Substituting equations (236) and (237) into equation (235), then

$$I = \int (d^2\Omega'/2\pi)\{(m^2(|\vec{q}|^2x'^2)^{-1} + (|\vec{q}|^2xx')^{-1}[-2E'x + |\vec{p}|^2 + |\vec{p}'|^2$$
$$+ \omega^2 - 2|\vec{p}|\omega\cos\theta + 2\omega E' - 2|\vec{p}|E'\cos\theta + 2(E^2 + E'^2)] - (xx')^{-1}$$
$$- |\vec{q}|^{-2}[2x^{-1}(\omega - |\vec{p}|\cos\theta) + |\vec{p}|^2x^{-2}\sin^2\theta - 1] - 4m^2E^2|\vec{q}|^{-4}x'^{-2}$$
$$+ (|\vec{q}|^4xx')^{-1}[8E'E^2x + 2\omega^2|\vec{p}|^2\sin^2\theta + 2\omega^2|\vec{p}'|^2 - 2\omega^2E'^2 - 2(E^2 + E'^2)(|\vec{p}|^2$$
$$+ |\vec{p}'|^2 + \omega^2 - 2|\vec{p}|\omega\cos\theta + 2\omega E' - 2|\vec{p}|E'\cos\theta)] + |\vec{q}|^{-4}[4x^{-1}(E^2 + E'^2)$$
$$\cdot(\omega - |\vec{p}|\cos\theta) - 4E^2 + 4x^{-2}|\vec{p}|^2E'^2\sin^2\theta + 4E'\omega^2x^{-1}] - 2\omega^2x'(|\vec{q}|^4x)^{-1}\}. \tag{238}$$

The terms to be integrated in equation (238) are all of the form

$$\bar{I}_{m,n} = \int (d^2\Omega'/2\pi)|\vec{q}|^{-2m}x'^{-n}, \tag{239}$$

where

$$\{(m, n)\} = \{(0, 1), (1, 0), (1, 1), (1, 2), (2, -1), (2, 0), (2, 1), (2, 2)\}. \tag{240}$$

The integrals of equation (239) can be cast in a more transparent form as follows. Dotting $\vec{q}$ of equation (132) into itself, then

$$\begin{aligned}|\vec{q}|^2 &= |\vec{p} - \vec{p}' - \vec{k}|^2 \\ &= |(\vec{p} - \vec{k}) - \vec{p}'|^2 \\ &= |\vec{p} - \vec{k}|^2 + |\vec{p}'|^2 - 2\vec{p}' \cdot (\vec{p} - \vec{k})\end{aligned} \tag{241}$$

or

$$|\vec{q}|^2 = (|\vec{p} - \vec{k}|^2 + |\vec{p}'|^2)$$

$$\cdot \left[1 - \vec{p}' \cdot 2(\vec{p} - \vec{k})\,(|\vec{p} - \vec{k}|^2 + |\vec{p}'|^2)^{-1}\right]. \tag{242}$$

Defining vectors $\vec{T}$, $\vec{a}$, and $\vec{b}$ by

$$\vec{T} = \vec{p} - \vec{k}, \tag{243}$$

$$\vec{a} = 2\vec{T}(|\vec{T}|^2 + |\vec{p}'|^2)^{-1}, \tag{244}$$

$$\vec{b} = \vec{k}(|\vec{k}|E')^{-1}, \tag{245}$$

then equation (242) is

$$|\vec{q}|^2 = (|\vec{T}^2| + |\vec{p}'|^2)(1 - \vec{p}' \cdot \vec{a}). \tag{246}$$

Also, using equation (45),

$$x' = E' - |\vec{p}'| \cos\theta'$$

$$= E'(1 - |\vec{p}'|E'^{-1} \cos\theta'). \tag{247}$$

Substituting equation (193) into equation (247), then

$$x' = E'[1 - \vec{k} \cdot \vec{p}'(E'|\vec{k}|)^{-1}]. \tag{248}$$

Next, substituting equation (245) into equation (248),

$$x' = E'(1 - \vec{p}' \cdot \vec{b}). \tag{249}$$

Therefore, substituting equations (246) and (249) into equation (239),

$$\bar{I}_{mn} = \int (d^2\Omega'/2\pi)(|\vec{T}|^2 + |\vec{p}'|^2)^{-m}(1 - \vec{p}' \cdot \vec{a})^{-m}$$

$$\cdot E'^{-n}(1 - \vec{p}' \cdot \vec{b})^{-n}. \tag{250}$$

Since T is independent of θ' and ϕ', it can be taken outside the integral, and equation (250) becomes

$$\bar{I}_{mn} = (|\vec{T}|^2 + |\vec{p}'|^2)^{-m} E'^{-n} I_{mn} , \tag{251}$$

where

$$I_{mn} = \int (d^2\Omega'/2\pi)(1 - \vec{p}' \cdot \vec{a})^{-m}(1 - \vec{p}' \cdot \vec{b})^{-n}. \tag{252}$$

One proceeds by evaluating the integrals of equation (252). Begininning with I_{01}, one has

$$I_{01} = \int (d^2\Omega'/2\pi)(1 - \vec{p}' \cdot \vec{b})^{-1}. \tag{253}$$

Since the photon momentum $\vec{k}$ has been chosen as the polar axis in equation (193), then according to the definition of $\vec{b}$ of equation (245), $\vec{b}$ is along the polar axis. Therefore,

$$\vec{p}' \cdot \vec{b} = |\vec{p}'||\vec{b}| \cos \theta'. \tag{254}$$

Expressing the element of solid angle $d^2\Omega'$ in equation (253) explicitly in terms of θ' and ϕ' and substituting equation (254), then

$$I_{01} = \int_0^{2\pi} (d\phi'/2\pi) \int_0^{\pi} \sin \theta' \, d\theta' \, (1 - |\vec{p}'||\vec{b}| \cos \theta')^{-1}. \tag{255}$$

One next changes variables of integration by defining

$$x = |\vec{p}'||\vec{b}| \cos \theta'. \tag{256}$$

Then

$$dx = -|\vec{p}'||\vec{b}| \sin \theta' \, d\theta'. \tag{257}$$

Then equation (255) becomes after integration over ϕ'

$$I_{01} = |\vec{p}'|^{-1}|\vec{b}|^{-1} \int_{-|\vec{p}'||\vec{b}|}^{|\vec{p}'||\vec{b}|} dx\,(1 - x)^{-1}. \tag{258}$$

Performing the elementary integration, then

$$I_{01} = (|\vec{p}'|\,|\vec{b}|)^{-1}\ell n\,(1 - x)^{-1}\,\Big|_{-|\vec{p}'||\vec{b}|}^{|\vec{p}'||\vec{b}|} \tag{259}$$

or finally

$$I_{01} = (|\vec{p}'||\vec{b}|)^{-1}\,\ell n\,[(1 + |\vec{p}'||\vec{b}|)(1 - |\vec{p}'||\vec{b}|)^{-1})]. \tag{260}$$

Similarly evaluating I_{10}, one has

$$I_{10} = \int (d^2\Omega'/2\pi)(1 - \vec{p}'\cdot\vec{a})^{-1} = \int_0^{\pi} d\theta'\,\sin\theta'\,(1 - |\vec{p}'|\,|\vec{a}|\,\cos\theta')^{-1}, \tag{261}$$

or as in equations (255) to (260), then

$$I_{10} = (|\vec{p}'|\,|\vec{a}|)^{-1}\,\ell n[(1 + |\vec{p}'|\,|\vec{a}|)(1 - |\vec{p}'|\vec{a}|)^{-1}]. \tag{262}$$

Next, considering I_{20},

$$\begin{aligned} I_{20} &= \int (d^2\Omega'/2\pi)(1 - \vec{p}'\cdot\vec{a})^{-2} \\ &= \int_0^{2\pi} d\theta'\,\sin\theta'\,(1 - |\vec{p}'|\,|\vec{a}|\,\cos\theta')^{-2} \end{aligned} \tag{263}$$

or

$$\begin{aligned} I_{20} &= (|\vec{p}'|\,|\vec{a}|)^{-1} \int_{-|\vec{p}'||\vec{a}|}^{|\vec{p}'||\vec{a}|} dx\,(1 - x)^{-2} \\ &= (|\vec{p}'|\,|\vec{a}|)^{-1}(1 - x)^{-1}\,\Big|_{-|\vec{p}'||\vec{a}|}^{|\vec{p}'||\vec{a}|} \end{aligned} \tag{264}$$

or

$$I_{20} = 2(1 - |\vec{a}|^2 |\vec{p}'|^2)^{-1}. \tag{265}$$

To evaluate I_{11}, the following identity is useful for arbitrary quantities α and β independent of the integration variable, namely,

$$(\alpha\beta)^{-1} = \int_0^1 dx \left[\alpha x + \beta(1 - x)\right]^{-2}. \tag{266}$$

Equation (266) is true since

$$\begin{aligned} \int_0^1 dx \left[\alpha x + \beta(1 - x)\right]^{-2} &= \int_0^1 dx \left[(\alpha - \beta)x + \beta\right]^{-2} \\ &= -(\alpha - \beta)^{-1}\left[(\alpha - \beta)x + \beta\right]^{-1} \Big|_0^1 \\ &= -(\alpha - \beta)^{-1}\alpha^{-1} + (\alpha - \beta)^{-1}\beta^{-1} \\ &= (\alpha\beta)^{-1}. \end{aligned} \tag{267}$$

Proceeding then with the evaluation of I_{11},

$$I_{11} = \int (d^2\Omega'/2\pi)(1 - \vec{p}' \cdot \vec{a})^{-1}(1 - \vec{p}' \cdot \vec{b})^{-1}. \tag{268}$$

Using the identity equation (266) in the integrand of equation (268), then

$$I_{11} = \int (d^2\Omega'/2\pi) \int_0^1 dx \left[(1 - \vec{p}' \cdot \vec{a})x + (1 - \vec{p}' \cdot \vec{b})(1 - x)\right]^{-2}, \tag{269}$$

or interchanging orders of integration and rewriting the integrand, then

$$I_{11} = \int_0^1 dx \int (d^2\Omega/2\pi)\left\{1 - \vec{p}' \cdot \left[\vec{a}x + (1 - x)\vec{b}\right]\right\}^{-2}. \tag{270}$$

Defining the vector $\vec{g}$ by

$$\vec{g} = x\vec{a} + (1 - x)\vec{b}, \tag{271}$$

then equation (270) may be written

$$I_{11} = \int_0^1 dx \int (d^2\Omega/2\pi)(1 - \vec{p}' \cdot \vec{g})^{-2}. \qquad (272)$$

Clearly, the integrals must be invariant with respect to the choice of the polar axis of integration. By choosing the vector $\vec{g}$ as the polar axis and comparing equation (272) with equations (263) to (265), then

$$I_{11} = 2\int_0^1 dx\, (1 - |\vec{g}|^2 |\vec{p}'|^2)^{-1}. \qquad (273)$$

Substituting equation (271) into equation (273), then

$$I_{11} = 2\int_0^1 dx\, [1 - |\vec{p}'|^2 |\vec{a}x + \vec{b}(1 - x)|^2]^{-1}. \qquad (274)$$

Equation (274) can be rewritten as

$$I_{11} = 2\int_0^1 dx\, [1 - |\vec{b}|^2 |\vec{p}'|^2 - 2x|\vec{p}'|^2(\vec{a} \cdot \vec{b} - |\vec{b}|^2)$$
$$- x^2 |\vec{p}'|^2 (\vec{a} - \vec{b})^2]^{-1}. \qquad (275)$$

Defining

$$A = 1 - |\vec{p}'|^2 |\vec{b}|^2, \qquad (276)$$

$$B = -2|\vec{p}'|^2(\vec{a} \cdot \vec{b} - |\vec{b}|^2), \qquad (277)$$

$$C = -|\vec{p}'|^2 |\vec{a} - \vec{b}|^2, \qquad (278)$$

$$X = A + Bx + Cx^2, \qquad (279)$$

then equation (275) can be written as

$$I_{11} = 2\int_0^1 dx\, X^{-1} \qquad (280)$$

or

$$I_{11} = 2\int_0^1 dx\,(A + Bx + Cx^2)^{-1}. \tag{281}$$

Performing the integration, then

$$I_{11} = 2(B^2 - 4AC)^{-1/2}\,\ell n\left(\frac{2Cx + B - D}{2Cx + B + D}\right)\Big|_0^1, \tag{282}$$

where

$$D = (B^2 - 4AC)^{1/2}. \tag{283}$$

Evaluating the upper and lower limits, equation (282) becomes

$$I_{11} = 2D^{-1}\{\ell n\,[(2C + B - D)(2C + B + D)^{-1}] - \ell n\,[(B - D)(B + D)^{-1}]\} \tag{284}$$

or

$$I_{11} = 2D^{-1}\,\ell n\,[(2C + B - D)(B + D)(2C + B + D)^{-1}(B - D)^{-1}] \tag{285}$$

or

$$I_{11} = 2D^{-1}\,\ell n\,\{[2C(B + D) + B^2 - D^2][2C(B - D) + B^2 - D^2]^{-1}\}. \tag{286}$$

Substituting equation (283) into equation (286),

$$I_{11} = 2D^{-1}\,\ell n\,\{[2C(B + D) + 4AC][2C(B - D) + 4AC]^{-1}\} \tag{287}$$

or

$$I_{11} = 2D^{-1}\,\ell n\,[(2A + B + D)(2A + B - D)^{-1}]. \tag{288}$$

Using equations (276) and (277),

$$2A + B = 2(1 - |\vec{p}'|^2|\vec{b}|^2) - 2|\vec{p}'|^2(\vec{a}\cdot\vec{b} - |\vec{b}|^2) \tag{289}$$

or

$$2A + B = 2(1 - |\vec{p}'|^2\vec{a}\cdot\vec{b}). \tag{290}$$

Substituting equations (244) and (245) into equation (290), then

$$2A + B = 2\{1 - |\vec{p}'|^2(2\vec{k}\cdot\vec{T})[|\vec{k}|E'(|\vec{T}|^2 + |\vec{p}'|^2)]^{-1}\}. \tag{291}$$

Clearly,

$$2\vec{k}\cdot\vec{T} = \vec{k}\cdot\vec{T} + \vec{k}\cdot\vec{T}. \tag{292}$$

By equation (243),

$$\vec{k} = \vec{p} - \vec{T}. \tag{293}$$

Substituting equation (293) into the first term on the right side of equation (292), then

$$2\vec{k}\cdot\vec{T} = (\vec{p} - \vec{T})\cdot\vec{T} + \vec{k}\cdot\vec{T}, \tag{294}$$

or

$$2\vec{k}\cdot\vec{T} = \vec{p}\cdot\vec{T} - |\vec{T}|^2 + \vec{k}\cdot\vec{T}. \tag{295}$$

Substituting equation (243) into the first and last terms on the right of equation (294), then

$$2\vec{k}\cdot\vec{T} = \vec{p}\cdot(\vec{p} - \vec{k}) - |\vec{T}|^2 + \vec{k}\cdot(\vec{p} - \vec{k}) \tag{296}$$

or

$$2\vec{k}\cdot\vec{T} = |\vec{p}|^2 - |\vec{T}|^2 - |\vec{k}|^2. \tag{297}$$

Using equations (116) and (130),

$$|\vec{p}|^2 = (E' + \omega)^2 - m^2. \tag{298}$$

Substituting equations (21), (236), and (298) into equation (297), then

$$2\vec{k} \cdot \vec{T} = (E' + \omega)^2 - (E'^2 - |\vec{p}'|^2) - |\vec{T}|^2 - \omega^2 \tag{299}$$

or

$$2\vec{k} \cdot \vec{T} = |\vec{p}'|^2 + 2E'\omega - |\vec{T}|^2. \tag{300}$$

Then substituting equations (21) and (300) into equation (291),

$$2A + B = 2\left\{1 - |\vec{p}'|^2(|\vec{p}'|^2 + 2E'\omega - |\vec{T}|^2)[\omega E'(|\vec{T}|^2 + |\vec{p}'|^2)]^{-1}\right\}. \tag{301}$$

Simplifying equation (301), then

$$2A + B = 2(|\vec{T}|^2 - |\vec{p}'|^2)(|\vec{T}|^2 + |\vec{p}'|^2)^{-1}(1 + |\vec{p}'|^2/\omega E'). \tag{302}$$

Next, to evaluate D in equation (288), one first evaluates A, B, and C. Substituting equation (245) in equation (276),

$$A = 1 - |\vec{p}'|^2/E'^2 = (E'^2 - |\vec{p}'|^2)E'^{-2}. \tag{303}$$

Using equation (236), equation (303) becomes

$$A = m^2/E'^2. \tag{304}$$

Further, substituting equations (244) and (245) into equation (277),

$$B = -2|\vec{p}'|^2\left\{2\vec{k} \cdot \vec{T}[\omega E'(|\vec{T}|^2 + |\vec{p}'|^2)]^{-1} - E'^{-2}\right\}. \tag{305}$$

Also, substituting equations (244) and (245) into equation (278), then

$$C = -|\vec{p}'|^2[2\vec{T}(|\vec{T}|^2 + |\vec{p}'|^2)^{-1} - \vec{k}(\omega E')^{-1}]^2. \tag{306}$$

Substituting equations (304) to (306) into equation (283), then

$$\begin{aligned} D^2 = {} & 4|\vec{p}'|^4\{(2\vec{k}\cdot\vec{T})^2[\omega^2E'^2(|\vec{T}|^2 + |\vec{p}'|^2)^2]^{-1} \\ & - 2\,(2\vec{k}\cdot\vec{T})[\omega E'^3(|\vec{T}|^2 + |\vec{p}'|^2)]^{-1}\} + 4|\vec{p}'|^4/E'^4 \\ & + 4(m^2|\vec{p}'|^2/E'^2)\{4|\vec{T}|^2(|\vec{T}|^2 + |\vec{p}'|^2)^{-2} \\ & - 2(2\vec{T}\cdot\vec{k})[\omega E'(|\vec{T}|^2 + |\vec{p}'|^2)]^{-1}\} \\ & + 4m^2|\vec{p}'|^2/E'^4. \end{aligned} \tag{307}$$

According to equation (236),

$$|\vec{p}'|^4 = |\vec{p}'|^2(E'^2 - m^2) \tag{308}$$

and

$$m^2 = E'^2 - |\vec{p}'|^2. \tag{309}$$

Substituting equations (308) and (309) into the second and third terms, respectively, of equation (307), then

$$\begin{aligned} D^2 = {} & 4|\vec{p}'|^4(2\vec{k}\cdot\vec{T})^2\,[\omega^2E'^2(|\vec{T}|^2 + |\vec{p}'|^2)^2]^{-1} \\ & + 4|\vec{p}'|^2/E'^2 + 4|\vec{p}'|^2\{4|\vec{T}|^2(|\vec{T}|^2 + |\vec{p}'|^2)^{-2} \\ & - 2(2\vec{T}\cdot\vec{k})[\omega E'(|\vec{T}|^2 + |\vec{p}'|^2)]^{-1}\} \\ & - 16|\vec{p}'|^4|\vec{T}|^2[E'^2(|\vec{T}|^2 + |\vec{p}'|^2)^2]^{-1}. \end{aligned} \tag{310}$$

Simplifying equation (310), then

$$D^2 = 4|\vec{p}'|^2(|\vec{T}|^2 + |\vec{p}'|^2)^{-2}[|\vec{p}'|^2(2\vec{k}\cdot\vec{T})^2(\omega^2E'^2)^{-1}$$

$$+ (|\vec{T}|^2 + |\vec{p}'|^2)^2/E'^2 - 4|\vec{p}'|^2|\vec{T}|^2/E'^2 + 4|\vec{T}|^2$$

$$- 2(2\vec{T}\cdot\vec{k})(|\vec{T}|^2 + |\vec{p}'|^2)(\omega E')^{-1}] \qquad (311)$$

or

$$D^2 = 4|\vec{p}'|^2[\omega^2E'^2(|\vec{T}|^2 + |\vec{p}'|^2)^2]^{-1}[|\vec{p}'|^2(2\vec{k}\cdot\vec{T})^2$$

$$+ \omega^2(|\vec{T}|^2 - |\vec{p}'|^2)^2 + 4\omega^2E'^2|\vec{T}|^2$$

$$- 2\omega E'(2\vec{T}\cdot\vec{k})(|\vec{T}|^2 + |\vec{p}'|^2)]. \qquad (312)$$

Equivalently,

$$D^2 = 4|\vec{p}'|^2[\omega^2E'^2(|\vec{T}|^2 + |\vec{p}'|^2)^2]^{-1}$$

$$\{2\vec{k}\cdot\vec{T}[|\vec{p}'|^2(2\vec{k}\cdot\vec{T}) - 2\omega E'|\vec{T}|^2$$

$$- 2\omega E'|\vec{p}'|^2] + \omega^2(|\vec{T}|^2 - |\vec{p}'|^2)^2$$

$$+ 4\omega^2E'^2|\vec{T}|^2\}. \qquad (313)$$

Next, substituting equation (300) into equation (313),

$$D^2 = 4|\vec{p}'|^2[\omega^2E'^2(|\vec{T}|^2 + |\vec{p}'|^2)^2]^{-1}f_{D^2}, \qquad (314)$$

where

$$f_{D^2} = (|\vec{p}'|^2 + 2E'\omega - |\vec{T}|^2)(|\vec{p}'|^4 + 2|\vec{p}'|^2E'\omega - |\vec{p}'|^2|\vec{T}|^2$$

$$- 2\omega E'|\vec{T}|^2 - 2\omega E'|\vec{p}'|^2) + \omega^2(|\vec{T}|^2 - |\vec{p}'|^2)^2$$

$$+ 4\omega^2E'^2|\vec{T}|^2. \qquad (315)$$

Multiplying terms, equation (315) becomes

$$f_{D^2} = |\vec{p}'|^6 - 4\omega E'|\vec{p}'|^2|\vec{T}|^2 - 2|\vec{p}'|^4|\vec{T}|^2 + 2E'\omega|\vec{p}'|^4$$
$$+ 2\omega E'|\vec{T}|^4 + |\vec{p}'|^2|\vec{T}|^4 + \omega^2(|\vec{T}|^2 - |\vec{p}'|^2)^2 \tag{316}$$

or

$$f_{D^2} = |\vec{p}'|^2(|\vec{p}'|^4 - 2|\vec{p}'|^2|\vec{T}|^2 + |\vec{T}|^4 - 4\omega E'|\vec{T}|^2$$
$$+ 2E'\omega|\vec{p}'|^2) + 2\omega E'|\vec{T}|^4 + \omega^2(|\vec{T}|^2 - |\vec{p}'|^2)^2 \tag{317}$$

or

$$f_{D^2} = |\vec{p}'|^2(|\vec{T}|^2 - |\vec{p}'|^2)^2 + 2\omega E'(|\vec{T}|^2 - |\vec{p}'|^2)^2$$
$$+ \omega^2(|\vec{T}|^2 - |\vec{p}'|^2)^2 \tag{318}$$

or

$$f_{D^2} = (|\vec{T}|^2 - |\vec{p}'|^2)^2(|\vec{p}'|^2 + 2\omega E' + \omega^2). \tag{319}$$

Substituting equation (209) into equation (319), then

$$f_{D^2} = (|\vec{T}|^2 - |\vec{p}'|^2)^2(E'^2 - m^2 + 2\omega E' + \omega^2) \tag{320}$$

or

$$f_{D^2} = (|\vec{T}|^2 - |\vec{p}'|^2)^2[(E' + \omega)^2 - m^2]. \tag{321}$$

Next, using equation (130) in equation (321), then

$$f_{D^2} = (|\vec{T}|^2 - |\vec{p}'|^2)^2(E^2 - m^2). \tag{322}$$

Substituting equation (116) into equation (322), then

$$f_{D^2} = (|\vec{T}|^2 - |\vec{p}'|^2)^2 |\vec{p}|^2. \tag{323}$$

Finally, substituting equation (323) into equation (314), then

$$D^2 = 4|\vec{p}|^2 |\vec{p}'|^2 (|\vec{T}|^2 - |\vec{p}'|^2)^2 \cdot [\omega^2 E'^2 (|\vec{T}|^2 + |\vec{p}'|^2)^2]^{-1}. \tag{324}$$

Therefore,

$$D = 2|\vec{p}|\,|\vec{p}'|\,(|\vec{T}|^2 - |\vec{p}'|^2)\,[\omega E'(|\vec{T}|^2 + |\vec{p}'|^2)]^{-1}. \tag{325}$$

Substituting equations (302) and (325) into equation (288), then

$$I_{11} = \omega E'(|\vec{T}|^2 + |\vec{p}'|^2)[|\vec{p}|\,|\vec{p}'|(|\vec{T}|^2 - |\vec{p}|^2)]^{-1}\, \ell n\, A_{11}, \tag{326}$$

where

$$\begin{aligned} A_{11} = \{&2(|\vec{T}|^2 - |\vec{p}'|^2)(|\vec{T}|^2 + |\vec{p}'|^2)^{-1}(1 + |\vec{p}'|^2/\omega E') \\ &- 2|\vec{p}|\,|\vec{p}'|(|\vec{T}|^2 - |\vec{p}'|^2)[\omega E'(|\vec{T}|^2 + |\vec{p}'|^2)]^{-1}\} \\ &\cdot \{2(|\vec{T}|^2 - |\vec{p}'|^2)(|\vec{T}|^2 + |\vec{p}'|^2)^{-1}(1 + |\vec{p}'|^2/\omega E') \\ &+ 2|\vec{p}|\,|\vec{p}'|(|\vec{T}|^2 - |\vec{p}'|^2)[\omega E'(|\vec{T}|^2 + |\vec{p}'|^2)]^{-1}\}^{-1} \end{aligned} \tag{327}$$

or

$$A_{11} = (1 + |\vec{p}'|^2/\omega E' - |\vec{p}|\,|\vec{p}'|/\omega E') \cdot (1 + |\vec{p}'|^2/\omega E' + |\vec{p}|\,|\vec{p}'|/\omega E')^{-1} \tag{328}$$

or

$$A_{11} = (\omega E' + |\vec{p}'|^2 - |\vec{p}||\vec{p}'|)(\omega E' + |\vec{p}'|^2 + |\vec{p}||\vec{p}'|)^{-1}. \quad (329)$$

Substituting for ω in equation (329) with equation (130), then

$$A_{11} = [(E - E')E' + |\vec{p}'|^2 - |\vec{p}||\vec{p}'|] \cdot [(E - E')E' + |\vec{p}'|^2 + |\vec{p}||\vec{p}'|]^{-1} \quad (330)$$

or

$$A_{11} = (EE' - E'^2 + |\vec{p}'|^2 - |\vec{p}||\vec{p}'|) \cdot (EE' - E'^2 + |\vec{p}'|^2 + |\vec{p}||\vec{p}'|)^{-1}. \quad (331)$$

Using equation (209) in equation (331), then

$$A_{11} = (EE' - m^2 - |\vec{p}||\vec{p}'|)(EE' - m^2 + |\vec{p}||\vec{p}'|)^{-1}. \quad (332)$$

Finally, substituting equation (332) into equation (326), then

$$I_{11} = \omega E'(|\vec{T}|^2 + |\vec{p}'|^2)[|\vec{p}||\vec{p}'|(|\vec{T}|^2 - |\vec{p}'|^2)]^{-1}L, \quad (333)$$

where

$$L = \ell n\,[(EE' - m^2 + |\vec{p}||\vec{p}'|)(EE' - m^2 - |\vec{p}||\vec{p}'|)^{-1}]. \quad (334)$$

Proceeding with the evaluation of I_{12} by using equation (252), then

$$I_{12} = \int(d^2\Omega'/2\pi)(1 - \vec{p}' \cdot \vec{a})^{-1}(1 - \vec{p}' \cdot \vec{b})^{-2}. \quad (335)$$

Using the identity equation (266), one first notes that

$$\partial/\partial\beta(\alpha\beta)^{-1} = \alpha^{-1}\partial/\partial\beta(\beta^{-1}) = -\alpha^{-1}\beta^{-2}, \tag{336}$$

and therefore

$$\alpha^{-1}\beta^{-2} = -\partial/\partial\beta(\alpha\beta)^{-1} = -\partial/\partial\beta \int_0^1 dx[\alpha x + \beta(1 - x)]^{-2} \tag{337}$$

or

$$\alpha^{-1}\beta^{-2} = \int_0^1 dx\, 2(1 - x)[\alpha x + \beta(1 - x)]^{-3}. \tag{338}$$

Using the identity equation (338) in the integrand of equation (335), then

$$I_{12} = \int_0^1 dx \int (d^2\Omega'/2\pi) 2(1 - x)[(1 - \vec{p}' \cdot \vec{a})x + (1 - \vec{p}' \cdot \vec{b})(1 - x)]^{-3} \tag{339}$$

or

$$I_{12} = \int_0^1 dx\, 2(1 - x) \int (d^2\Omega'/2\pi)[1 - \vec{p}' \cdot \vec{g}]^{-3}, \tag{340}$$

where $\vec{g}$ is given by equation (271).

Defining,

$$I_{120}(x) = \int (d^2\Omega'/2\pi)[1 - \vec{p}' \cdot \vec{g}]^{-3}, \tag{341}$$

then equation (340) is written as

$$I_{12} = \int_0^1 dx\, 2(1 - x) I_{120}(x). \tag{342}$$

Again choosing the polar axis of integration to lie along the vector $\vec{g}$, then equation (341) becomes

$$I_{120}(x) = \int_0^\pi d\theta' \sin\theta' \,[1 - |\vec{p}'||\vec{g}|\cos\theta']^{-3}. \tag{343}$$

Changing variables of integration according to

$$y = |\vec{p}'||\vec{g}| \cos \theta', \tag{344}$$

then equation (343) becomes

$$I_{120}(x) = (|\vec{p}'||\vec{g}|)^{-1} \int_{-|\vec{p}'||\vec{g}|}^{|\vec{p}'||\vec{g}|} dy\,(1 - y)^{-3} \tag{345}$$

or

$$I_{120}(x) = (2|\vec{p}'||\vec{g}|)^{-1}(1 - y)^{-2} \Big|_{-|\vec{p}'||\vec{g}|}^{|\vec{p}'||\vec{g}|} \tag{346}$$

or

$$I_{120}(x) = 2(1 - |\vec{p}'|^2|\vec{g}|^2)^{-2}. \tag{347}$$

Substituting equation (347) into equation (342), then

$$I_{12} = 4 \int_0^1 dx\,(1 - x)X^{-2}, \tag{348}$$

where X is defined by equations (276) to (279). The reduction to this form is identical to that of equation (273) to equation (281). Equation (348) is rewritten as

$$I_{12} = 4I_{121} - 4I_{122}, \tag{349}$$

where

$$I_{121} = \int_0^1 dx\,X^{-2}, \tag{350}$$

$$I_{122} = \int_0^1 dx\,xX^{-2}. \tag{351}$$

Performing the integral, equation (350) becomes

$$I_{121} = (2Cx + B)(4AC - B^2)^{-1}(Cx^2 + Bx + A)^{-1} \Big|_0^1 + 2C(4AC - B^2)^{-1} \int_0^1 dx\, (Cx^2 + Bx + A)^{-1}, \qquad (352)$$

or evaluating the first term and using equations (281) and (283), then

$$I_{121} = -(2C + B)[D^2(C + B + A)]^{-1} + B(D^2A)^{-1} - CD^{-2}I_{11}. \qquad (353)$$

Combining terms, equation (353) becomes

$$I_{121} = (B^2 - 2AC + BC)[AD^2(A + B + C)]^{-1} - CD^{-2}I_{11}, \qquad (354)$$

or using equation (283), then

$$I_{121} = (D^2 + 2AC + BC)[AD^2(A + B + C)]^{-1} - CD^{-2}I_{11}. \qquad (355)$$

Next, evaluating equation (351),

$$I_{122} = -(2A + Bx)(4AC - B^2)^{-1}(A + Bx + Cx^2)^{-1} \Big|_0^1 - B(4AC - B^2)^{-1} \int_0^1 dx\, (A + Bx + Cx^2)^{-1}. \qquad (356)$$

Again using equations (281) and (283), equation (356) becomes

$$I_{122} = -(B + 2C)D^{-2}(A + B + C)^{-1} + (B/2D^2)I_{11}. \qquad (357)$$

Next, substituting equations (355) and (357) into equation (349), then

$$I_{12} = 4(D^2 + 2AC + BC)[AD^2(A + B + C)]^{-1} - 4CD^{-2}I_{11} + 4(B + 2C)D^{-2}(A + B + C)^{-1} - 2BD^{-2}I_{11}. \qquad (358)$$

Combining terms, then equation (358) becomes

$$I_{12} = 4(D^2 + 4AC + BC + AB)[AD^2(A + B + C)]^{-1} - 2\,(B + 2C)D^{-2}I_{11}. \tag{359}$$

Using equation (283) in the numerator of the first term, then

$$I_{12} = 4(B^2 + BC + AB)[AD^2(A + B + C)]^{-1} - 2(B + 2C)D^{-2}I_{11} \tag{360}$$

or

$$I_{12} = 4B(AD^2)^{-1} - 2(B + 2C)D^{-2}I_{11}. \tag{361}$$

Next,

$$I_{21} = \int (d^2\Omega'/2\pi)(1 - \vec{p}' \cdot \vec{a})^{-2}(1 - \vec{p}' \cdot \vec{b})^{-1}. \tag{362}$$

Using the identity equation (266), one notes that

$$\alpha^{-2}\beta^{-1} = -\partial/\partial\alpha(\alpha\beta)^{-1} = -\partial/\partial\alpha \int_0^1 dx\,[\alpha x + \beta(1 - x)]^{-2} \tag{363}$$

or

$$\alpha^{-2}\beta^{-1} = 2\int_0^1 dx\,x[\alpha x + \beta(1 - x)]^{-3}. \tag{364}$$

Rewriting the integrand of equation (362) with the identity equation (364) and the definition equation (271), then

$$I_{21} = 2\int_0^1 dx\,x\int (d^2\Omega'/2\pi)(1 - \vec{p}' \cdot \vec{g})^{-3}. \tag{365}$$

Substituting equation (341) into equation (365), then

$$I_{21} = 2 \int_0^1 dx\ xI_{120}(x). \tag{366}$$

Then substituting equation (347) into equation (366),

$$I_{21} = 4 \int_0^1 dx\ x(1 - |\vec{p}'|^2|\vec{g}|^2)^{-2}. \tag{367}$$

Again recalling the reduction of equation (273) to equation (281) and using equations (276) to (279), then equation (367) becomes

$$I_{21} = 4 \int_0^1 dx\ xX^{-2}. \tag{368}$$

Comparing equations (368) and (351), and

$$I_{21} = 4I_{122}. \tag{369}$$

Finally, substituting equation (357) into equation (369), then

$$I_{21} = -4(B + 2C)D^{-2}(A + B + C)^{-1} + 2BD^{-2}I_{11}. \tag{370}$$

Using equation (252), I_{22} is given by

$$I_{22} = \int (d^2\Omega'/2\pi)(1 - \vec{p}' \cdot \vec{a})^{-2}(1 - \vec{p}' \cdot \vec{b})^{-2}. \tag{371}$$

One notes using equation (266) that

$$\alpha^{-2}\beta^{-2} = (\partial/\partial\alpha)(\partial/\partial\beta)(\alpha\beta)^{-1} = \partial^2/\partial\alpha\ \partial\beta \int_0^1 dx\ [\alpha x + \beta(1 - x)]^{-2} \tag{372}$$

or

$$\alpha^{-2}\beta^{-2} = -2\partial/\partial\alpha \int_0^1 dx\ (1 - x)[\alpha x + \beta(1 - x)]^{-3}, \tag{373}$$

or

$$\alpha^{-2}\beta^{-2} = 6 \int_0^1 dx\, x(1 - x)[\alpha x + \beta(1 - x)]^{-4}. \tag{374}$$

Using the identity equation (374) in equation (371), then

$$I_{22} = 6 \int_0^1 dx\, x(1 - x) I_{221}(x), \tag{375}$$

where, analogously,

$$I_{221}(x) = \int (d^2\Omega'/2\pi)(1 - \vec{p}' \cdot \vec{g})^{-4}. \tag{376}$$

Again choosing the polar axis of integration along the vector $\vec{g}$, then

$$I_{221}(x) = \int_0^\pi d\theta' \sin\theta' \, (1 - |\vec{p}'||\vec{g}| \cos\theta')^{-4}, \tag{377}$$

and making the substitution with equation (344), then

$$I_{221}(x) = |\vec{p}'|^{-1} |\vec{g}|^{-1} \int_{-|\vec{p}'||\vec{g}|}^{|\vec{p}'||\vec{g}|} dy\, (1 - y)^{-4}. \tag{378}$$

Performing the integration, then equation (378) becomes

$$I_{221}(x) = (3|\vec{p}'||\vec{g}|)^{-1}[(1 - |\vec{p}'||\vec{g}|)^{-3} - (1 + |\vec{p}'||\vec{g}|)^{-3}]. \tag{379}$$

Simplifying equation (379), then

$$I_{221}(x) = (2/3)(3 + |\vec{p}'|^2 |\vec{g}|^2)(1 - |\vec{p}'|^2 |\vec{g}|^2)^{-3}. \tag{380}$$

Then substituting equation (380) into equation (375),

$$I_{22} = 6 \int_0^1 dx\, x(1 - x)(2/3)[3 + (1 - X)]X^{-3}, \tag{381}$$

where equation (279) has again been used. Rewriting equation (381), then

$$I_{22} = 16I_{221} - 16I_{222} - 4I_{122} + 4I_{223}, \quad (382)$$

where equation (351) has been used, and

$$I_{221} = \int_0^1 dx\, xX^{-3}, \quad (383)$$

$$I_{222} = \int_0^1 dx\, x^2X^{-3}, \quad (384)$$

$$I_{223} = \int_0^1 dx\, x^2X^{-2}. \quad (385)$$

Evaluating the integral of equation (383),

$$I_{221} = -(2A + Bx)[2(4AC - B^2)(A + Bx + Cx^2)^2]^{-1} \Big|_0^1 - 3B[2(4AC - B^2)]^{-1} \int_0^1 dx\, X^{-2}. \quad (386)$$

Evaluating the first term, using equation (283), and substituting equation (350), then

$$I_{221} = (2A + B)[2D^2(A + B + C)^2]^{-1} - D^{-2}A^{-1} + (3/2)BD^{-2}I_{121}. \quad (387)$$

Substituting equation (355) into equation (387), then

$$I_{221} = (2A + B)[2D^2(A + B + C)^2]^{-1} - D^{-2}A^{-1} + (3/2)BD^{-2}$$
$$\cdot [(D^2 + 2AC + BC)A^{-1}D^{-2}(A + B + C)^{-1} - CD^{-2}I_{11}]. \quad (388)$$

Combining terms, then equation (388) becomes

$$I_{221} = [(2A + B)D^2A - 2D^2(A + B + C)^2 + 3B(D^2 + 2AC + BC)(A + B + C)]$$
$$\cdot [2D^4A(A + B + C)^2]^{-1} - (3/2)BCD^{-4}I_{11}. \quad (389)$$

Next, evaluating I_{222} of equation (384), then

$$I_{222} = -x[3C(A + Bx + Cx^2)^2]^{-1} \Big|_0^1 - (B/3C) \int_0^1 dx\, x(A + Bx + Cx^2)^{-3}$$

$$+ (A/3C) \int_0^1 dx\, [A + Bx + Cx^2]^{-3}. \tag{390}$$

Evaluating the first term of equation (390) and using equation (383), then

$$I_{222} = -[3C(A + B + C)^2]^{-1} - (B/3C)I_{221} + (A/3C)I_{2221}, \tag{391}$$

where

$$I_{2221} = \int_0^1 dx\, (A + Bx + Cx^2)^{-3}. \tag{392}$$

Evaluating equation (392), then

$$I_{2221} = (2Cx + B)[2(4AC - B^2)(A + Bx + Cx^2)^2]^{-1} \Big|_0^1 + 6C[2(4AC - B^2)]^{-1} \int_0^1 dx\, X^{-2}. \tag{393}$$

Evaluating the first term and using equations (350) and (355), then equation (393) becomes

$$I_{2221} = (2C + B)[2(4AC - B^2)(A + B + C)^2]^{-1} - B[2(4AC - B^2)A^2]^{-1}$$

$$- 3CD^{-2}\{(D^2 + 2AC + BC)[AD^2(A + B + C)]^{-1} - CD^{-2}I_{11}\}. \tag{394}$$

Simplifying equation (394) and using equation (283), then

$$I_{2221} = -\frac{2C + B}{2D^2(A + B + C)^2} + \frac{B}{2D^2A^2} - \frac{3C(D^2 + 2AC + BC)}{AD^4(A + B + C)} + \frac{3C^2}{D^4} I_{11}. \tag{395}$$

Then by substituting equations (389) and (395) into equation (391),

$$I_{222} = -\frac{1}{3C(A+B+C)^2} - \frac{B}{3C}\left[\frac{(2A + B)D^2A - 2D^2(A+B+C)^2 + 3B(D^2+2AC+BC)(A+B+C)}{2D^4A(A+B+C)^2}\right.$$

$$\left. - \frac{3}{2}\frac{BC}{D^4} I_{11}\right] + \frac{A}{3C}\left[-\frac{2C+B}{2D^2(A+B+C)^2} + \frac{B}{2D^2A^2} - \frac{3C(D^2+2AC+BC)}{AD^4(A+B+C)} + \frac{3C^2}{D^4} I_{11}\right]. \tag{396}$$

Next, by evaluating I_{223} of equation (385), then

$$I_{223} = -\frac{x}{C(A + Bx + Cx^2)}\Big|_0^1 + \frac{A}{C}\int dx\, X^{-2}. \tag{397}$$

Evaluating the first term and using equations (350) and (355), then equation (397) becomes

$$I_{223} = -\frac{1}{C(A + B + C)} + \frac{A}{C}\left[\frac{D^2 + 2AC + BC}{AD^2(A + B + C)} - \frac{C}{D^2} I_{11}\right] \tag{398}$$

Simplifying equation (398) and using equation (283), then

$$I_{223} = \frac{2A + B}{D^2(A + B + C)} - \frac{A}{D^2} I_{11}. \tag{399}$$

Finally, substituting equations (389), (396), (357), and (399) into equation (382), then

$$\begin{aligned} I_{22} = {} & 16(1 + B/3C)\{[2D^4A(A + B + C)^2]^{-1}[(2A + B)D^2A - 2D^2(A + B + C)^2 \\ & + 3B(D^2 + 2AC + BC)(A + B + C)] - (3BC/2D^4)I_{11}\} - 16(-[3C(A + B + C)^2]^{-1} \\ & + (A/3C)\{-(2C + B)[2D^2(A + B + C)^2]^{-1} + B/2D^2A^2 - 3C(D^2 + 2AC + BC) \\ & \cdot [AD^4(A + B + C)]^{-1} + (3C^2/D^4)I_{11}\}) - 4\{-(B + 2C)[D^2(A + B + C)]^{-1} \\ & + (B/2D^2)I_{11}\} + 4\{(2A + B)[D^2(A + B + C)]^{-1} - (A/D^2)I_{11}\}. \end{aligned} \tag{400}$$

Combining terms, equation (400) becomes

$$\begin{aligned} I_{22} = {} & [16(1 + B/3C)(-3BC/2D^4) - (16A/3C)(3C^2/D^4) - 2B/D^2 - 4A/D^2]I_{11} + 16(1 + B/3C) \\ & \cdot [2D^4A(A + B + C)^2]^{-1}[(2A + B)D^2A - 2D^2(A + B + C)^2 + 3B(D^2 + 2AC + BC) \\ & \cdot (A + B + C)] - 16(-[3C(A + B + C)^2]^{-1} + (A/3C)\{-(2C + B)[2D^2(A + B + C)^2]^{-1} \\ & + B/2D^2A^2 - 3C(D^2 + 2AC + BC)[AD^4(A + B + C)]^{-1}\}) + 8/D^2. \end{aligned} \tag{401}$$

Next, using equation (252), I_{2-1} is given by

$$I_{2-1} = \int (d^2\Omega'/2\pi)\,(1 - \vec{p}' \cdot \vec{a})^{-2}(1 - \vec{p}' \cdot \vec{b}). \tag{402}$$

Define a unit vector $\hat{n}$ perpendicular to $\vec{a}$ and $\vec{b}$, namely,

$$\hat{n} = \vec{a} \times \vec{b} / |\vec{a}|\,|\vec{b}|, \tag{403}$$

and choose a vector $\hat{n}_\perp$ perpendicular to $\vec{a}$ and $\hat{n}$, namely,

$$\hat{n}_\perp = \vec{a} \times \hat{n} / |\vec{a}|. \tag{404}$$

The vectors $\hat{n}$, $\hat{n}_\perp$, and $\vec{a}$ define orthogonal coordinate axes for the integration. The polar axis is taken along the vector $\vec{a}$. The vector $\vec{p}'$ can be written in terms of its projections along $\hat{n}$, $\hat{n}_\perp$, and $\vec{a}$ as follows:

$$\vec{p}' = (\vec{p}' \cdot \vec{a} / |\vec{a}|^2)\vec{a} + \vec{p}' \cdot \hat{n}\,\hat{n} + \vec{p}' \cdot \hat{n}_\perp\,\hat{n}_\perp. \tag{405}$$

Then

$$\vec{p}' \cdot \vec{b} = \vec{p}' \cdot \vec{a}\,\vec{a} \cdot \vec{b} / |\vec{a}|^2 + \vec{p}' \cdot \hat{n}\,\vec{b} \cdot \hat{n} + \vec{p}' \cdot \hat{n}_\perp\,\vec{b} \cdot \hat{n}_\perp. \tag{406}$$

Using equation (403), one notes that

$$\vec{b} \cdot \hat{n} = 0. \tag{407}$$

Choosing the polar axis of integration to lie along the vector $\vec{a}$ and the azimuthal angle ϕ' to be with respect to $\hat{n}_\perp$, then

$$\vec{p}' \cdot \hat{n}_\perp = |\vec{p}'| \sin\theta' \cos\phi'. \tag{408}$$

Substituting equations (407) and (408) into equation (406), then

$$p' \cdot b = (\vec{p}' \cdot \vec{a}/|\vec{a}|^2)\vec{a} \cdot \vec{b}$$

$$+ \vec{b} \cdot \hat{n}_\perp |\vec{p}'| \sin\theta' \cos\phi'. \tag{409}$$

Using equation (409) in equation (402), then

$$I_{2-1} = \int (2\pi)^{-1} \sin\theta' \, d\theta' \, d\phi' (1 - \vec{p}' \cdot \vec{a})^{-2}$$

$$\cdot (1 - \vec{p}' \cdot \vec{a}\, \vec{a} \cdot \vec{b} |\vec{a}|^{-2} - \vec{b} \cdot \hat{n}_\perp |\vec{p}'| \sin\theta' \cos\phi')$$

$$= \int (2\pi)^{-1} \sin\theta' \, d\theta' \, d\phi' (1 - \vec{p}' \cdot \vec{a})^{-2} (1 - \vec{p}' \cdot \vec{a}\, \vec{a} \cdot \vec{b} |\vec{a}|^{-2}) + I_{2-11}, \tag{410}$$

where

$$I_{2-11} = -\int (2\pi)^{-1} \sin\theta' \, d\theta' \, d\phi' (1 - \vec{p}' \cdot \vec{a})^{-2} \vec{b} \cdot \hat{n}_\perp |\vec{p}'| \sin\theta' \cos\phi'. \tag{411}$$

But clearly

$$I_{2-11} = -\int (2\pi)^{-1} \sin^2\theta' (1 - |\vec{p}'||\vec{a}|\cos\theta')^{-2} d\theta' \, \vec{b} \cdot \hat{n}_\perp |\vec{p}'| \int_0^{2\pi} \cos\phi' \, d\phi' \tag{412}$$

or

$$I_{2-11} = 0. \tag{413}$$

Substituting equation (413) into equation (410), then

$$I_{2-1} = \int (2\pi)^{-1} d^2\Omega' \, (1 - \vec{p}' \cdot \vec{a})^{-2} (1 - \vec{p}' \cdot \vec{a}\, \vec{a} \cdot \vec{b} |\vec{a}|^{-2}). \tag{414}$$

Rewriting equation (414) and then using the above coordinate frame,

$$I_{2-1} = \int (2\pi)^{-1} d^2\Omega' \, (1 - \vec{p}' \cdot \vec{a})^{-2} - \vec{a} \cdot \vec{b} |\vec{a}|^{-2}$$

$$\cdot \int (2\pi)^{-1} d^2\Omega' \, \vec{p}' \cdot \vec{a} (1 - \vec{p}' \cdot \vec{a})^{-2} \tag{415}$$

or

$$I_{2-1} = \int (2\pi)^{-1}\, d^2\Omega'\, (1 - \vec{p}' \cdot \vec{a})^{-2} - \vec{a} \cdot \vec{b} |\vec{a}|^{-2}$$
$$\cdot \int (2\pi)^{-1}\, d^2\Omega'\, (1 - \vec{p}' \cdot \vec{a})^{-2} [1 - (1 - \vec{p}' \cdot \vec{a})] \tag{416}$$

or

$$I_{2-1} = (1 - \vec{a} \cdot \vec{b} |\vec{a}|^{-2}) \int (2\pi)^{-1}\, d^2\Omega'\, (1 - \vec{p}' \cdot \vec{a})^{-2}$$
$$+ \vec{a} \cdot \vec{b} |\vec{a}|^{-2} \int (2\pi)^{-1}\, d^2\Omega'\, (1 - \vec{p}' \cdot \vec{a})^{-1}. \tag{417}$$

Comparing equation (417) with equations (263) and (261), then

$$I_{2-1} = (1 - \vec{a} \cdot \vec{b} |\vec{a}|^{-2}) I_{20} + \vec{a} \cdot \vec{b} |\vec{a}|^{-2} I_{10}. \tag{418}$$

Finally, substituting equations (262) and (265) into equation (418), then

$$I_{2-1} = \vec{a} \cdot \vec{b} |\vec{a}|^{-2} (|\vec{p}'| |\vec{a}|)^{-1} \ln [(1 + |\vec{p}'| |\vec{a}|)$$
$$\cdot (1 - |\vec{p}'| |\vec{a}|)^{-1}] + 2(1 - \vec{a} \cdot \vec{b} |\vec{a}|^{-2})(1 - |\vec{p}'|^2 |\vec{a}|^2)^{-1}. \tag{419}$$

To proceed then, comparing equations (238) and (239), then equation (238) may be rewritten as

$$I = m^2 \bar{I}_{12} + \kappa^{-1} [-2E'\kappa + |\vec{p}|^2 + |\vec{p}'|^2 + \omega^2 - 2|\vec{p}|\omega \cos\theta + 2\omega E' - 2|\vec{p}|\, E' \cos\theta$$
$$+ 2(E^2 + E'^2)] \bar{I}_{11} - \kappa^{-1} \bar{I}_{01} - [2\kappa^{-1}(\omega - |\vec{p}| \cos\theta) + |\vec{p}|^2 \kappa^{-2} \sin^2\theta - 1] \bar{I}_{10}$$
$$- 4m^2 E^2 \bar{I}_{22} + \kappa^{-1} [8E'E^2\kappa + 2\omega^2 |\vec{p}|^2 \sin^2\theta + 2\omega^2 |\vec{p}'|^2 - 2\omega^2 E'^2 - 2(E^2 + E'^2)(|\vec{p}|^2$$
$$+ |\vec{p}'|^2 + \omega^2 - 2|\vec{p}|\omega \cos\theta + 2\omega E' - 2|\vec{p}| E' \cos\theta)] \bar{I}_{21} + [4\kappa^{-1}(E^2 + E'^2)$$
$$\cdot (\omega - |\vec{p}| \cos\theta) - 4E^2 + 4\kappa^{-2} |\vec{p}|^2 E'^2 \sin^2\theta + 4E'\omega^2 \kappa^{-1}] \bar{I}_{20} - 2\omega^2 \kappa^{-1} \bar{I}_{2-1}. \tag{420}$$

Substituting equation (251) into equation (420), then

$$I = m^2(|\vec{T}|^2 + |\vec{p}'|^2)^{-1}E'^{-2}I_{12} + [-2E'\varkappa + |\vec{p}|^2 + |\vec{p}'|^2 + \omega^2$$
$$- 2|\vec{p}|\omega\cos\theta + 2\omega E' - 2|\vec{p}|\cos\theta\, E' + 2(E^2 + E'^2)](|\vec{T}|^2 + |\vec{p}'|^2)^{-1}(E'\varkappa)^{-1}I_{11}$$
$$- \varkappa^{-1}E'^{-1}I_{01} - [2\varkappa^{-1}(\omega - |\vec{p}|\cos\theta) + |\vec{p}|^2\varkappa^{-2}\sin^2\theta - 1](|\vec{T}|^2 + |\vec{p}'|^2)^{-1}I_{10}$$
$$- 4m^2E^2(|\vec{T}|^2 + |\vec{p}'|^2)^{-2}E'^{-2}I_{22} + \varkappa^{-1}[8E'E^2\varkappa + 2\omega^2|\vec{p}|^2\sin^2\theta + 2\omega^2|\vec{p}'|^2$$
$$- 2\omega^2E'^2 - 2(E^2 + E'^2)(|\vec{p}|^2 + |\vec{p}'|^2 + \omega^2 - 2|\vec{p}|\omega\cos\theta + 2\omega E' - 2|\vec{p}|E'\cos\theta)]$$
$$\cdot(|\vec{T}|^2 + |\vec{p}'|^2)^{-2}E'^{-1}I_{21} + [4\varkappa^{-1}(E^2 + E'^2)(\omega - |\vec{p}|\cos\theta) - 4E^2$$
$$+ 4|\vec{p}|^2E'^2\varkappa^{-2}\sin^2\theta + 4E'\omega^2\varkappa^{-1}](|\vec{T}|^2 + |\vec{p}'|^2)^{-2}I_{20} - 2\omega^2\varkappa^{-1}(|\vec{T}|^2$$
$$+ |\vec{p}'|^2)^{-2}E'I_{2-1}. \tag{421}$$

Substituting equations (361), (260), (262), (401), (370), (265), and (419) into equation (421), then

$$I = m^2(|\vec{T}|^2 + |\vec{p}'|^2)^{-1}E'^{-2}[4B/AD^2 - 2(B + 2C)D^{-2}I_{11}] + (|\vec{T}|^2$$
$$+ |\vec{p}'|^2)^{-1}E'^{-1}\varkappa^{-1}[-2E'\varkappa + |\vec{p}|^2 + |\vec{p}'|^2 + \omega^2 - 2|\vec{p}|\omega\cos\theta$$
$$+ 2\omega E' - 2|\vec{p}|E'\cos\theta + 2E^2 + 2E'^2]I_{11} - \varkappa^{-1}E'^{-1}|\vec{p}'|^{-1}|\vec{b}|^{-1}$$
$$\cdot \ell n\,[(1 + |\vec{p}'||\vec{b}|)(1 - |\vec{p}'||\vec{b}|)^{-1}] - [2\varkappa^{-1}(\omega - |\vec{p}|\cos\theta)$$
$$+ |\vec{p}|^2\varkappa^{-2}\sin^2\theta - 1](|\vec{T}|^2 + |\vec{p}'|^2)^{-1}|\vec{p}'|^{-1}|\vec{a}|^{-1}$$
$$\cdot \ell n\,[(1 + |\vec{p}'||\vec{a}|)(1 - |\vec{p}'||\vec{a}|)^{-1}] - 4m^2E^2E'^{-2}(|\vec{T}|^2 + |\vec{p}'|^2)^{-2}$$
$$\cdot [16(1 + B/3C)(-3BC/2D^4) - 16ACD^{-4} - 2BD^{-2} - 4AD^{-2}]I_{11} - 4m^2E^2E'^{-2}(|\vec{T}|^2 + |\vec{p}'|^2)^{-2}$$
$$\cdot (16(1 + B/3C)[(2A + B)D^2A - 2D^2(A + B + C)^2$$
$$+ 3B(D^2 + 2AC + BC)(A + B + C)][2D^4A(A + B + C)^2]^{-1}$$
$$+ 16\,[3C(A + B + C)^2]^{-1} - (16A/3C)\{(2C + B)[-2D^2(A + B + C)^2]^{-1}$$
$$+ B/2D^2A^2 - 3C(D^2 + 2AC + BC)[AD^4(A + B + C)]^{-1}\} + 8D^{-2})$$
$$+ \varkappa^{-1}[8E'E^2\varkappa + 2\omega^2|\vec{p}|^2\sin^2\theta + 2\omega^2|\vec{p}'|^2 - 2\omega^2E'^2 - 2(E^2 + E'^2)(|\vec{p}|^2$$
$$+ |\vec{p}'|^2 + \omega^2 - 2|\vec{p}|\omega\cos\theta + 2\omega E' - 2|\vec{p}|E'\cos\theta)]$$
$$\cdot (|\vec{T}|^2 + |\vec{p}'|^2)^{-2}E'^{-1}\{-4(B + 2C)[D^2(A + B + C)]^{-1}$$

$$+ 2BD^{-2}I_{11}\} + [4(E^2 + E'^2)x^{-1}(\omega - |\vec{p}| \cos\theta) - 4E^2 + 4|\vec{p}|^2E'^2x^{-2}\sin^2\theta$$

$$+ 4E'\omega^2x^{-1}](|\vec{T}|^2 + |\vec{p}'|^2)^{-2}2(1 - |\vec{p}'|^2|\vec{a}|^2)^{-1}$$

$$- 2\omega^2x^{-1}(|\vec{T}|^2 + |\vec{p}'|^2)^{-2}E'\{\vec{a}\cdot\vec{b}|\vec{a}|^{-3}|\vec{p}'|^{-1}$$

$$\cdot \ln[(1 + |\vec{p}'||\vec{a}|)(1 - |\vec{p}'||\vec{a}|)^{-1}]$$

$$+ 2(1 - \vec{a}\cdot\vec{b}|\vec{a}|^{-2})(1 - |\vec{p}'|^2|\vec{a}|^2)^{-1}\}. \tag{422}$$

Separating terms containing I_{11}, then equation (422) may be rewritten in the form

$$I = I_a + I_b, \tag{423}$$

where

$$I_a = \{-2(B + 2C)m^2(|\vec{T}|^2 + |\vec{p}'|^2)^{-1}E'^{-2}D^{-2} + (|\vec{T}|^2 + |\vec{p}'|^2)^{-1}x^{-1}E'^{-1}(-2E'x$$

$$+ |\vec{p}|^2 + |\vec{p}'|^2 + \omega^2 - 2|\vec{p}|\omega\cos\theta + 2\omega E' - 2|\vec{p}|\cos\theta\, E' + 2E^2 + 2E'^2)$$

$$- 4m^2E^2E'^{-2}(|\vec{T}|^2 + |\vec{p}'|^2)^{-2}[-24(1 + B/3C)BCD^{-4} - 16ACD^{-4} - 2BD^{-2} - 4AD^{-2}]$$

$$+ 2Bx^{-1}D^{-2}E'^{-1}(|\vec{T}|^2 + |\vec{p}'|^2)^{-2}[8E'E^2x + 2\omega^2|\vec{p}|^2\sin^2\theta + 2\omega^2|\vec{p}'|^2 - 2\omega^2E'^2 - 2(E^2$$

$$+ E'^2)(|\vec{p}|^2 + |\vec{p}'|^2 + \omega^2 - 2|\vec{p}|\omega\cos\theta + 2\omega E' - 2|\vec{p}|E'\cos\theta)]\}I_{11}, \tag{424}$$

$$I_b = m^2(|\vec{T}|^2 + |\vec{p}'|^2)^{-1}E'^{-2}(4B/AD^2) - (xE'|\vec{p}'||\vec{b}|)^{-1}$$

$$\cdot \ln[(1 + |\vec{p}'||\vec{b}|)(1 - |\vec{p}'||\vec{b}|)^{-1}] - [2x^{-1}(\omega - |\vec{p}|\cos\theta)$$

$$+ |\vec{p}|^2x^{-2}\sin^2\theta - 1](|\vec{T}|^2 + |\vec{p}'|^2)^{-1}|\vec{p}'|^{-1}|\vec{a}|^{-1}$$

$$\cdot \ln[(1 + |\vec{p}'||\vec{a}|)(1 - |\vec{p}'||\vec{a}|)^{-1}] - 4m^2E^2E'^{-2}(|\vec{T}|^2$$

$$+ |\vec{p}'|^2)^{-2}\{16[1 + (B/3C)](2D^4A)^{-1}(A + B + C)^{-2}[(2A + B)D^2A$$

$$- 2D^2(A + B + C)^2 + 3B(D^2 + 2AC + BC)(A + B + C)]$$

$$+ (16/3C)(A + B + C)^{-2} - (16A/3C)[(2C + B)(-2D^2)^{-1}(A + B + C)^{-2}$$

$$+ (B/2D^2A^2) - 3CA^{-1}D^{-4}(A + B + C)^{-1}(D^2 + 2AC + BC)] + 8D^{-2}\}$$

$$+ x^{-1}E'^{-1}(|\vec{T}|^2 + |\vec{p}'|^2)^{-2}[-4(B + 2C)D^{-2}(A + B + C)^{-1}][8E'E^2x + 2\omega^2|\vec{p}|^2\sin^2\theta$$

$$+ 2\omega^2|\vec{p}'|^2 - 2\omega^2E'^2 - 2(E^2 + E'^2)(|\vec{p}|^2 + |\vec{p}'|^2 + \omega^2 - 2|\vec{p}|\omega\cos\theta$$

$$+ 2\omega E' - 2|\vec{p}|E' \cos\theta)] + [4x^{-1}(E^2 + E'^2)(\omega - |\vec{p}| \cos\theta) - 4E^2 + 4x^{-2}|\vec{p}|^2E'^2 \sin^2\theta$$

$$+ 4E'x^{-1}\omega^2](|\vec{T}|^2 + |\vec{p}'|^2)^{-2}2(1 - |\vec{p}'|^2|\vec{a}|^2)^{-1}$$

$$- 2\omega^2x^{-1}(|\vec{T}|^2 + |\vec{p}'|^2)^{-2} E'\{\vec{a} \cdot \vec{b}|\vec{p}'|^{-1}|\vec{a}|^{-3}$$

$$\cdot \ell n [(1 + |\vec{p}'||\vec{a}|)(1 - |\vec{p}'| |\vec{a}|)^{-1}] + 2(1 - \vec{a} \cdot \vec{b}|\vec{a}|^{-2})$$

$$\cdot (1 - |\vec{p}'|^2 |\vec{a}|^2)^{-1}\}. \tag{425}$$

Substituting equation (333) into equation (424), then

$$I_a = \{I_{a1} + x^{-1}E'^{-1}(-2E'x + |\vec{p}|^2 + |\vec{p}'|^2 + \omega^2 - 2|\vec{p}|\omega \cos\theta + 2\omega E' - 2|\vec{p}|E' \cos\theta$$

$$+ 2E^2 + 2E'^2) - 4m^2E^2E'^{-2}(|\vec{T}|^2 + |\vec{p}'|^2)^{-1}F_{a1} + 2Bx^{-1}D^{-2}E'^{-1}(|\vec{T}|^2 + |\vec{p}'|^2)^{-1}$$

$$\cdot [8E'E^2x + 2\omega^2|\vec{p}|^2 \sin^2\theta + 2\omega^2|\vec{p}'|^2 - 2\omega^2E'^2 - 2(E^2 + E'^2)(|\vec{p}|^2 + |\vec{p}'|^2 + \omega^2$$

$$- 2|\vec{p}|\omega \cos\theta + 2\omega E' - 2|\vec{p}|E' \cos\theta)]\}E'\omega|\vec{p}'|^{-1}|\vec{p}|^{-1}(T^2 - |\vec{p}'|^2)^{-1}L, \tag{426}$$

where

$$I_{a1} = -2(B + 2C)m^2E'^{-2}D^{-2}, \tag{427}$$

$$F_{a1} = -24(1 + B/3C)BCD^{-4} - 16ACD^{-4} - 2BD^{-2} - 4AD^{-2}. \tag{428}$$

To reduce equation (427), one first substitutes equation (243) into equation (305), and using equations (191) and (41), then

$$B = -2|\vec{p}'|^2[(2|\vec{p}|\omega \cos\theta - 2\omega^2)\omega^{-1}E'^{-1}(|\vec{T}|^2 + |\vec{p}'|^2)^{-1} - E'^{-2})]. \tag{429}$$

Equivalently,

$$B = -2|\vec{p}'|^2(2E'|\vec{p}|\omega \cos\theta - 2E'\omega^2 - \omega|\vec{T}|^2 - \omega|\vec{p}'|^2)\omega^{-1}E'^{-2}$$

$$\cdot(|\vec{T}|^2 + |\vec{p}'|^2)^{-1}. \tag{430}$$

Substituting equation (243) into the numerator of equation (430) and using equation (41), then

$$B = -2|\vec{p}'|^2\omega^{-1}E'^{-2}(|\vec{T}|^2 + |\vec{p}'|^2)^{-1}(2E'|\vec{p}|\omega \cos\theta - 2E'\omega^2 - \omega|\vec{p}|^2 - \omega^3 + 2|\vec{p}|\omega^2 \cos\theta - \omega|\vec{p}'|^2). \tag{431}$$

Substituting equation (130) into equation (431), then equation (429) becomes

$$B = -2|\vec{p}'|^2E'^{-2}(|\vec{T}|^2 + |\vec{p}'|^2)^{-1}(2|\vec{p}|E \cos\theta - 2E\omega - |\vec{p}|^2 + \omega^2 - |\vec{p}'|^2). \tag{432}$$

Also, by substituting equation (130) into equation (209) and using equation (116), then

$$\begin{aligned} |\vec{p}'|^2 &= (E - \omega)^2 - m^2 \\ &= E^2 - 2E\omega + \omega^2 - m^2 \\ &= |\vec{p}|^2 - 2E\omega + \omega^2. \end{aligned} \tag{433}$$

Substituting equation (433) into equation (432), then

$$B = -2|\vec{p}'|^2E'^{-2}(|\vec{T}|^2 + |\vec{p}'|^2)^{-1}(2|\vec{p}|E \cos\theta - 2|\vec{p}|^2). \tag{434}$$

Furthermore, using equation (43) in equation (434), then

$$B = -2|\vec{p}'|^2E'^{-2}(|\vec{T}|^2 + |\vec{p}'|^2)^{-1}[2E(E - \kappa) - 2|\vec{p}|^2], \tag{435}$$

or again using equation (116), then

$$B = -4|\vec{p}'|^2E'^{-2}(|\vec{T}|^2 + |\vec{p}'|^2)^{-1}(m^2 - \kappa E). \tag{436}$$

Further, using equation (41) in equation (306), then

$$C = -|\vec{p}'|^2[4|\vec{T}|^2 (|\vec{T}|^2 + |\vec{p}'|^2)^{-2} - 2(2\vec{k}\cdot\vec{T}) \cdot \omega^{-1}E'^{-1}(|\vec{T}|^2 + |\vec{p}'|^2)^{-1} + E'^{-2}]. \tag{437}$$

Equation (437) may be rewritten

$$C = -|\vec{p}'|^2\omega^{-1}E'^{-2}(|\vec{T}|^2 + |\vec{p}'|^2)^{-2}[4\omega E'^2\ |\vec{T}|^2 - 2(2\vec{k} \cdot \vec{T})$$

$$\cdot\ E'(|\vec{T}|^2 + |\vec{p}'|^2) + \omega(|\vec{T}|^2 + |\vec{p}'|^2)^2] \tag{438}$$

or

$$C = -|\vec{p}'|^2E'^{-2}(|\vec{T}|^2 + |\vec{p}'|^2)^{-2}F_C, \tag{439}$$

where

$$F_C = 4E'^2|\vec{T}|^2 - 2(2\vec{k} \cdot \vec{T})\omega^{-1}E'(|\vec{T}|^2 + |\vec{p}'|^2)$$

$$+ (|\vec{T}|^2 + |\vec{p}'|^2)^2. \tag{440}$$

Equation (440) can be written

$$F_C = (2\vec{k} \cdot \vec{T}\omega^{-1}E' - |\vec{T}|^2 - |\vec{p}'|^2)^2 - 4(\vec{k} \cdot \vec{T})^2E'^2\omega^{-2}$$

$$+ 4E'^2|\vec{T}|^2. \tag{441}$$

Substituting equation (243) and using equation (191) in equation (441), then

$$F_C = [2\omega^{-1}(\omega|\vec{p}| \cos\theta - \omega^2)E' - (|\vec{p}|^2 - 2|\vec{p}|\omega \cos\theta + \omega^2) - |\vec{p}'|^2]^2$$

$$- 4E'^2[\omega^{-2}(\omega|\vec{p}| \cos\theta - \omega^2)^2 - |\vec{p}|^2 + 2|\vec{p}|\omega \cos\theta - \omega^2]. \tag{442}$$

Substituting equation (130) into equation (442), then

$$F_C = [2E(|\vec{p}| \cos\theta - \omega) + \omega^2 - |\vec{p}|^2 - |\vec{p}'|^2]^2 + 4E'^2|\vec{p}|^2(1 - \cos^2\theta). \tag{443}$$

Substituting equations (116) and (209) and using equations (130) and (213) in equation (443), then

$$F_C = (2E|\vec{p}| \cos\theta - 2E^2 + 2m^2)^2 + 4E'^2|\vec{p}|^2 \sin^2\theta. \tag{444}$$

Substituting equation (43) into equation (444), then

$$F_C = 4[(m^2 - \varkappa E)^2 + E'^2|\vec{p}|^2 \sin^2 \theta]. \quad (445)$$

Then, substituting equation (445) into equation (439),

$$C = -4|\vec{p}'|^2 E'^{-2}(|\vec{T}|^2 + |\vec{p}'|^2)^{-2}[(m^2 - \varkappa E)^2 + E'^2|\vec{p}|^2 \sin^2 \theta]. \quad (446)$$

Also, using equations (243), (191), and (41),

$$|\vec{T}|^2 - |\vec{p}'|^2 = |\vec{p}|^2 - 2|\vec{p}|\omega \cos \theta + \omega^2 - |\vec{p}'|^2. \quad (447)$$

Substituting equations (116) and (209) into equation (447), substituting equation (130), and using equation (43), then

$$|\vec{T}|^2 - |\vec{p}'|^2 = 2\omega\varkappa. \quad (448)$$

Substituting equation (448) into equation (325), then

$$D = 4|\vec{p}|\,|\vec{p}'|\varkappa E'^{-1}(|\vec{T}|^2 + |\vec{p}'|^2)^{-1}. \quad (449)$$

Next, by substituting equations (436), (446), and (449) into equation (427), then

$$I_{a1} = m^2(2E'^2|\vec{p}|^2\varkappa^2)^{-1}[(m^2 - \varkappa E)(|\vec{T}|^2 + |\vec{p}'|^2) + 2(m^2 - \varkappa E)^2 + 2E'^2\,|\vec{p}|^2 \sin^2 \theta]. \quad (450)$$

Using equations (243), (191), and (43)

$$|\vec{T}|^2 + |\vec{p}'|^2 = |\vec{p}|^2 - 2\omega(E - \varkappa) + \omega^2 + |\vec{p}'|^2. \quad (451)$$

Also, using equations (213) and (43),

$$|\vec{p}|^2 \sin^2 \theta = |\vec{p}|^2 - E^2 + 2E\varkappa - \varkappa^2. \quad (452)$$

Then, substituting equations (451) and (452) into equation (450),

$$I_{a1} = m^2(2E'^2|\vec{p}|^2x^2)^{-1}\{(m^2 - xE)[|\vec{p}|^2 - 2\omega(E - x) + \omega^2 + |\vec{p}'|^2]$$
$$+ 2(m^2 - xE)^2 - 2E'^2|\vec{p}|^2 - 2E'^2E^2 + 4E'^2Ex - 2E'^2x^2\}. \tag{453}$$

Substituting equations (116) and (209) and using equation (130) in equation (453), then

$$I_{a1} = m^2(2E'^2|\vec{p}|^2x^2)^{-1}[(m^2 - xE)(2E'^2 - 2E'x) - 2E'^2m^2 + 4E'^2Ex - 2E'^2x^2] \tag{454}$$

or

$$I_{a1} = m^2(2E'^2|\vec{p}|^2x^2)^{-1}(-2m^2E'x - 2xEE'^2 + 2x^2EE' + 4E'^2Ex - 2E'^2x^2). \tag{455}$$

Using equations (116) and (130),

$$-2m^2E'x = -2E^2E'x + 2|\vec{p}|^2E'x$$
$$= -2E(E' + \omega)E'x + 2|\vec{p}|^2E'x. \tag{456}$$

Also, using equation (130),

$$2x^2EE' = 2x^2(E' + \omega)E'. \tag{457}$$

Substituting equations (456) and (457) into equation (455), then

$$I_{a1} = m^2(2E'^2|\vec{p}|^2x^2)^{-1}(-2E\omega E'x + 2|\vec{p}|^2E'x + 2x^2E'\omega), \tag{458}$$

or

$$I_{a1} = m^2E'^{-1}|\vec{p}|^{-2}x^{-1}(-E\omega + |\vec{p}|^2 + x\omega) \tag{459}$$

or

$$L_{a1} = m^2E'^{-1}\varkappa^{-1} - m^2\omega EE'^{-1}|\vec{p}|^{-2}\varkappa^{-1} + m^2\omega E'^{-1}|\vec{p}|^{-2}. \qquad (460)$$

Equation (428) can be written as

$$F_{a1} = -2D^{-2}[4D^{-2}(3BC + B^2 + 2AC) + B + 2A]. \qquad (461)$$

Using equation (283) in equation (461), then

$$F_{a1} = -2D^{-2}[4 + 12F_{a11} + B + 2A], \qquad (462)$$

where

$$F_{a11} = (2A + B)CD^{-2}. \qquad (463)$$

Substituting equations (304), (449), (436), and (446) into equation (463), then

$$F_{a11} = -[(m^2 - \varkappa E)^2 + E'^2|\vec{p}|^2 \sin^2\theta][2|\vec{p}|^2\varkappa^2E'^2(|\vec{T}|^2 + |\vec{p}'|^2)]^{-1}F_{a11}, \qquad (464)$$

where

$$F_{a111} = m^2|\vec{T}|^2 + m^2|\vec{p}'|^2 - 2m^2|\vec{p}'|^2 + 2\varkappa E|\vec{p}'|^2. \qquad (465)$$

Substituting equation (451) into equation (465), then

$$F_{a111} = m^2|\vec{p}|^2 - 2\omega m^2(E - \varkappa) + m^2\omega^2 - m^2|\vec{p}'|^2 + 2\varkappa E|\vec{p}'|^2. \qquad (466)$$

Substituting equations (116) and (209) into equation (466), then

$$F_{a111} = m^2E^2 - m^2E'^2 + m^2\omega^2 - 2E\omega m^2 + 2\varkappa(E|\vec{p}'|^2 + \omega m^2). \qquad (467)$$

Substituting equation (130) into equation (467),

$$F_{a111} = 2\varkappa(E|\vec{p}'|^2 + \omega^2m^2). \qquad (468)$$

Then substituting equation (468) into equation (464),

$$F_{a11} = -(E|\vec{p}'|^2 + \omega m^2)|\vec{p}|^{-2}\varkappa^{-1}E'^{-2}(|\vec{T}|^2 + |\vec{p}'|^2)^{-1} \cdot [(m^2 - \varkappa E)^2 + E'^2|\vec{p}|^2 \sin^2\theta]. \quad (469)$$

Substituting equations (469), (449), (436), and (304) into equation (462), then

$$F_{a1} = -E'^2(|\vec{T}|^2 + |\vec{p}'|^2)^2(8|\vec{p}'|^2 |\vec{p}|^2 \varkappa^2)^{-1}\{4 - 12(E|\vec{p}'|^2 + \omega m^2) \cdot [|\vec{p}|^2\varkappa E'^2(|\vec{T}|^2 + |\vec{p}'|^2)]^{-1}[(m^2 - \varkappa E)^2 + E'^2|\vec{p}|^2 \sin^2\theta] - 4|\vec{p}'|^2(m^2 - \varkappa E)E'^{-2}(|\vec{T}|^2 + |\vec{p}'|^2)^{-1} + 2m^2E'^{-2}\}. \quad (470)$$

Simplifying equation (470), then

$$F_{a1} = -(|\vec{T}|^2 + |\vec{p}'|^2)(4|\vec{p}'|^2|\vec{p}|^4\varkappa^3)^{-1}\{|\vec{p}|^2\varkappa(2E'^2 + m^2) \cdot (|\vec{T}|^2 + |\vec{p}'|^2) - 6(E|\vec{p}'|^2 + \omega m^2)[(m^2 - \varkappa E)^2 + E'^2|\vec{p}|^2 \sin^2\theta] - 2|\vec{p}'|^2|\vec{p}|^2\varkappa(m^2 - \varkappa E)\}. \quad (471)$$

Using equation (43) in equation (451),

$$|\vec{T}|^2 + |\vec{p}'|^2 = |\vec{p}|^2 + |\vec{p}'|^2 + \omega^2 - 2|\vec{p}|\omega \cos\theta. \quad (472)$$

Substituting equation (472) into equation (471), then

$$F_{a1} = -(|\vec{T}|^2 + |\vec{p}'|^2)(4|\vec{p}'|^2|\vec{p}|^4\varkappa^3)^{-1}\{(2E'^2 + m^2)\varkappa|\vec{p}|^2 \cdot (|\vec{p}|^2 + |\vec{p}'|^2 + \omega^2 - 2|\vec{p}|\omega \cos\theta) - 6(\omega m^2 + E|\vec{p}'|^2) \cdot [(\varkappa E - m^2)^2 + E'^2|\vec{p}|^2 \sin^2\theta] + 2|\vec{p}'|^2|\vec{p}|^2\varkappa(\varkappa E - m^2)\}. \quad (473)$$

Therefore, substituting equations (448), (449), (460), (473), and (436) into equation (426), then

$$I_a = E'L(2|\vec{p}|\,|\vec{p}'|x)^{-1}(m^2E'^{-1}x^{-1} - m^2\omega EE'^{-1}|\vec{p}|^{-2}x^{-1} + m^2\omega E'^{-1}|\vec{p}|^{-2}$$
$$+ x^{-1}E'^{-1}(-2E'x + |\vec{p}|^2 + |\vec{p}'|^2 + \omega^2 - 2|\vec{p}|\,\omega\cos\theta + 2\omega E' - 2|\vec{p}|\,E'\cos\theta$$
$$+ 2E^2 + 2E'^2) + m^2E^2|\vec{p}'|^{-2}E'^{-2}|\vec{p}|^{-4}x^{-3}\{(2E'^2 + m^2)x|\vec{p}|^2(|\vec{p}|^2 + |\vec{p}'|^2 + \omega^2$$
$$- 2|\vec{p}|\omega\cos\theta) - 6(\omega m^2 + E|\vec{p}'|^2)[(Ex - m^2)^2 + E'^2|\vec{p}|^2\sin^2\theta]$$
$$+ 2|\vec{p}'|^2|\vec{p}|^2x(xE - m^2)\} - (m^2 - xE)(2x^3E'|\vec{p}|^2)^{-1}[8E'E^2x + 2\omega^2|\vec{p}|^2\sin^2\theta$$
$$+ 2\omega^2|\vec{p}'|^2 - 2\omega^2E'^2 - 2(E^2 + E'^2)(|\vec{p}|^2 + |\vec{p}'|^2 + \omega^2 - 2|\vec{p}|\omega\cos\theta$$
$$+ 2\omega E' - 2|\vec{p}|E'\cos\theta)]). \qquad (474)$$

Using equation (43),

$$\cos\theta = (E - x)|\vec{p}|^{-1}. \qquad (475)$$

Substituting equation (475) into equation (213), then

$$\sin^2\theta = 1 - (E - x)^2|\vec{p}|^{-2}. \qquad (476)$$

Then substituting equations (475) and (476) into equation (474),

$$I_a = E'L(2|\vec{p}|\,|\vec{p}'|x)^{-1}(m^2E'^{-1}x^{-1} - m^2\omega EE'^{-1}|\vec{p}|^{-2}x^{-1} + m^2\omega E'^{-1}|\vec{p}|^{-2}$$
$$+ x^{-1}E'^{-1}[-2E'x + |\vec{p}|^2 + |\vec{p}'|^2 + \omega^2 - 2\omega(E - x) + 2\omega E' - 2E'(E - x) + 2E^2 + 2E'^2]$$
$$+ m^2E^2|\vec{p}'|^{-2}E'^{-2}|\vec{p}|^{-4}x^{-3}\{(2E'^2 + m^2)x|\vec{p}|^2[|\vec{p}|^2 + |\vec{p}'|^2 + \omega^2 - 2\omega(E - x)]$$
$$- 6(\omega m^2 + E|\vec{p}'|^2)[(Ex - m^2)^2 + E'^2|\vec{p}|^2 - E'^2(E - x)^2] + 2|\vec{p}'|^2|\vec{p}|^2x(xE - m^2)\}$$
$$- (m^2 - xE)(2x^3E'|\vec{p}|^2)^{-1}\{8E'E^2x + 2\omega^2|\vec{p}|^2 - 2\omega^2(E - x)^2 + 2\omega^2|\vec{p}'|^2 - 2\omega^2E'^2$$
$$- 2(E^2 + E'^2)[|\vec{p}|^2 + |\vec{p}'|^2 + \omega^2 - 2\omega(E - x) + 2\omega E' - 2E'(E - x)]\}). \qquad (477)$$

Rewriting equation (477) as a sum of powers of x^{-1}, then

$$I_a = E'L(2|\vec{p}|\,|\vec{p}'|x)^{-1}(K_1 + K_2x^{-1} + K_3x^{-2} + K_4x^{-3}), \qquad (478)$$

where

$$K_1 = m^2\omega(E'|\vec{p}|^2)^{-1} - 2 + 2\omega E'^{-1} + 2 + E(2E'|\vec{p}|^2)^{-1}(-2\omega^2), \tag{479}$$

$$\begin{aligned} K_2 = {} & m^2E'^{-1} - m^2\omega EE'^{-1}|\vec{p}|^{-2} + E'^{-1}(|\vec{p}|^2 + |\vec{p}'|^2 + \omega^2 - 2\omega E + 2\omega E' - 2EE' + 2E^2 + 2E'^2) \\ & + 2m^2E^2\omega|\vec{p}'|^{-2}|\vec{p}|^{-4}E'^{-2}(2E'^2 + m^2)|\vec{p}|^2 - 6m^2E^2|\vec{p}'|^{-2}|\vec{p}|^{-4}E'^{-2} \\ & \cdot (\omega m^2 + E|\vec{p}'|^2)E^2 + 6m^2E^2|\vec{p}'|^{-2}|\vec{p}|^{-4}E'^{-2}(\omega m^2 + E|\vec{p}'|^2)E'^2 + 2m^2E^3|\vec{p}|^{-2}E'^{-2} \\ & - m^2(2E'|\vec{p}|^2)^{-1}(-2\omega^2) + E(2E'|\vec{p}|^2)^{-1}[8E'E^2 + 4\omega^2E - 2(E^2 + E'^2)(2\omega + 2E')], \end{aligned} \tag{480}$$

$$\begin{aligned} K_3 = {} & m^2E^2|\vec{p}'|^{-2}|\vec{p}|^{-4}E'^{-2}(2E'^2 + m^2)|\vec{p}|^2(|\vec{p}|^2 + |\vec{p}'|^2 + \omega^2 - 2\omega E) \\ & + 12m^2E^2|\vec{p}'|^{-2}|\vec{p}|^{-4}E'^{-2}(\omega m^2 + E|\vec{p}'|^2)m^2E + m^2E^2|\vec{p}'|^{-2}|\vec{p}|^{-4}E'^{-2} \\ & \cdot [-6(\omega m^2 + E|\vec{p}'|^2)(2E'^2E) - 2|\vec{p}'|^2|\vec{p}|^2m^2] - m^2(2E'|\vec{p}|^2)^{-1}[8E'E^2 + 4\omega^2E - 4(E^2 + E'^2)\omega \\ & - 4(E^2 + E'^2)E'] + E(2E'|\vec{p}|^2)^{-1}[2\omega^2|\vec{p}|^2 - 2\omega^2E^2 + 2\omega^2|\vec{p}'|^2 - 2\omega^2E'^2 \\ & - 2(E^2 + E'^2)(|\vec{p}|^2 + |\vec{p}'|^2 + \omega^2 - 2\omega E + 2\omega E' - 2EE')], \end{aligned} \tag{481}$$

$$\begin{aligned} K_4 = {} & -6m^2E^2|\vec{p}'|^{-2}|\vec{p}|^{-4}E'^{-2}(\omega m^2 + E|\vec{p}'|^2)(m^4 + E'^2|\vec{p}|^2 - E'^2E^2) \\ & - m^2(2E'|\vec{p}|^2)^{-1}[2\omega^2|\vec{p}|^2 - 2\omega^2E^2 + 2\omega^2|\vec{p}'|^2 - 2\omega^2E'^2 - 2(E^2 + E'^2) \\ & \cdot (|\vec{p}|^2 + |\vec{p}'|^2 + \omega^2 - 2\omega E + 2\omega E' - 2EE')]. \end{aligned} \tag{482}$$

Simplifying equation (479),

$$K_1 = \omega E'^{-1}|\vec{p}|^{-2}(m^2 + 2|\vec{p}|^2 - E\omega). \tag{483}$$

Using equations (116) and (130) in equation (483), then

$$K_1 = \omega E'^{-1}|\vec{p}|^{-2}[m^2 + 2E^2 - 2m^2 - E(E - E')] \tag{484}$$

or

$$K_1 = \omega E'^{-1}|\vec{p}|^{-2}(E^2 + EE' - m^2). \qquad (485)$$

Further, simplifying equation (480),

$$K_2 = K_{21} + K_{22} + K_{23} + K_{24}, \qquad (486)$$

where

$$K_{21} = 2m^2E^2\omega(2E'^2 + m^2)|\vec{p}'|^{-2}|\vec{p}|^{-2}E'^{-2} - 6m^2E^4(\omega m^2 + E|\vec{p}'|^2)|\vec{p}'|^{-2}|\vec{p}|^{-4}E'^{-2}$$
$$+ 6m^2E^2|\vec{p}'|^{-2}|\vec{p}|^{-4}(\omega m^2 + E|\vec{p}'|^2) + 2m^2E^3|\vec{p}|^{-2}E'^{-2}, \qquad (487)$$

$$K_{22} = |\vec{p}|^{-2}(4E^3 - 2E^3 - 2EE'\omega - 2EE'^2), \qquad (488)$$

$$K_{23} = E'^{-1}|\vec{p}|^{-2}[m^2|\vec{p}|^2 - m^2\omega E + m^2\omega^2 + 2\omega^2E^2 - 2E^3\omega], \qquad (489)$$

$$K_{24} = E'^{-1}(|\vec{p}|^2 + |\vec{p}'|^2 + \omega^2 - 2\omega E + 2\omega E' - 2EE' + 2E^2 + 2E'^2). \qquad (490)$$

Substituting equations (116), (209), and (130) into equation (490), then

$$K_{24} = E'^{-1}[E^2 - m^2 + E'^2 - m^2 + (E - E')^2 - 2(E - E')E$$
$$+ 2(E - E')E' - 2EE' + 2E^2 + 2E'^2] \qquad (491)$$

or

$$K_{24} = 2E'^{-1}(E^2 + E'^2 - m^2). \qquad (492)$$

Substituting equation (130) into equation (489),

$$K_{22} = |\vec{p}|^{-2}[2E^3 - 2E(E - \omega)\omega - 2E(E - \omega)^2] \tag{493}$$

or

$$K_{22} = 2\omega E^2|\vec{p}|^{-2}. \tag{494}$$

Using equation (130) in equation (489), then

$$K_{23} = E'^{-1}|\vec{p}|^{-2}\{m^2|\vec{p}|^2 + \omega[-m^2E + m^2(E - E') + 2E^2(E - E') - 2E^3]\}. \tag{495}$$

Simplifying equation (495), then

$$K_{23} = E'^{-1}|\vec{p}|^{-2}[m^2|\vec{p}|^2 - \omega(m^2E' + 2E^2E')] \tag{496}$$

or

$$K_{23} = m^2E'^{-1}|\vec{p}|^{-2}(|\vec{p}|^2 - \omega E') - 2\omega E^2|\vec{p}|^{-2}. \tag{497}$$

Next, simplifying equation (487),

$$K_{21} = 2|\vec{p}'|^{-2}|\vec{p}|^{-4}E'^{-2}[m^2E^2|\vec{p}|^2\omega(2E'^2 + m^2) - 3m^2E^4(\omega m^2 + E|\vec{p}'|^2)$$
$$+ 3m^2E'^2E^2(\omega m^2 + E|\vec{p}'|^2) + m^2|\vec{p}'|^2|\vec{p}|^2E^3], \tag{498}$$

or rearranging terms, then

$$K_{21} = 2m^2E^2|\vec{p}'|^{-2}|\vec{p}|^{-4}E'^{-2}[2\omega(|\vec{p}|^2 + m^2)E'^2 + m^2\omega E'^2 + m^2\omega|\vec{p}|^2 - 3m^2\omega E^2$$
$$- 3|\vec{p}'|^2E^3 + 3|\vec{p}'|^2EE'^2 + |\vec{p}'|^2|\vec{p}|^2E]. \tag{499}$$

Substituting equation (116) in the first term, then equation (499) becomes

$$K_{21} = 2m^2E^2|\vec{p}'|^{-2}|\vec{p}|^{-4}E'^{-2}[2\omega E^2E'^2 + m^2\omega E'^2 + m^2\omega|\vec{p}|^2 - 3m^2\omega E^2 - 3|\vec{p}'|^2E^3 + 3|\vec{p}'|^2EE'^2 + |\vec{p}'|^2|\vec{p}|^2E]. \quad (500)$$

Substituting equations (130), (116), and (209) into equation (500), then

$$K_{21} = 2m^2E^2|\vec{p}'|^{-2}|\vec{p}|^{-4}E'^{-2}[2(E - E')E^2E'^2 + m^2E'^2(E - E') + m^2(E - E')(E^2 - m^2) - 3m^2E^2(E - E') - 3E^3(E'^2 - m^2) + 3EE'^2(E'^2 - m^2) + (E'^2 - m^2)(E^2 - m^2)E]. \quad (501)$$

Simplifying equation (501),

$$K_{21} = 2m^2E^2|\vec{p}'|^{-2}|\vec{p}|^{-4}E'^{-2}(-2E^2E'^3 - 3m^2EE'^2 - m^2E'^3 + 2m^2E^2E' + m^4E' + 3EE'^4). \quad (502)$$

Combining the first and fourth, second and sixth, and third and fifth terms, then

$$K_{21} = 2m^2E^2|\vec{p}'|^{-2}|\vec{p}|^{-4}E'^{-2}[2E^2E'(-E'^2 + m^2) + 3EE'^2(-m^2 + E'^2) + m^2E'(m^2 - E'^2)]. \quad (503)$$

Using equation (209) in equation (503), then

$$K_{21} = 2m^2E^2|\vec{p}'|^{-2}|\vec{p}|^{-4}E'^{-2}(-2E^2E'|\vec{p}'|^2 + 3EE'^2|\vec{p}'|^2 - m^2E'|\vec{p}'|^2) \quad (504)$$

or

$$K_{21} = 2m^2E^2|\vec{p}|^{-4}E'^{-1}(-2E^2 + 3EE' - m^2). \quad (505)$$

Finally, substituting equations (505), (494), (497), and (492) into equation (486), then

$$K_2 = 2m^2E^2|\vec{p}|^{-4}E'^{-1}(-2E^2 + 3EE' - m^2) + 2\omega E^2|\vec{p}|^{-2} + m^2E'^{-1}|\vec{p}|^{-2}(|\vec{p}|^2 - \omega E') - 2\omega E^2|\vec{p}|^{-2} + 2E'^{-1}(E^2 + E'^2 - m^2) \quad (506)$$

or

$$K_2 = m^2E'^{-1}|\vec{p}|^{-2}(|\vec{p}|^2 - \omega E') + 2E'^{-1}(E^2 + E'^2 - m^2) + 2m^2E|\vec{p}|^{-4}E'^{-1}F_{K2}, \tag{507}$$

where

$$F_{K2} = (-2E^2 + 3EE' - m^2)E. \tag{508}$$

Substituting equation (130) into equation (508), then

$$F_{K2} = (E^2 - 3E\omega - m^2)E. \tag{509}$$

Using equation (116), equation (509) becomes

$$F_{K2} = (|\vec{p}|^2 - 3E\omega)E. \tag{510}$$

Using equation (116), equation (510) becomes

$$F_{K2} = |\vec{p}|^2E - 3(|\vec{p}|^2 + m^2)\omega \tag{511}$$

or

$$F_{K2} = -(3\omega m^2 - |\vec{p}|^2E) - 3|\vec{p}|^2\omega. \tag{512}$$

Substituting equation (512) into equation (507), then

$$K_2 = -2m^2E|\vec{p}|^{-4}E'^{-1}(3\omega m^2 - |\vec{p}|^2E) + f_{K2}E'^{-1}|\vec{p}|^{-2}, \tag{513}$$

where

$$f_{K2} = 2m^2E|\vec{p}|^{-2}(-3|\vec{p}|^2\omega) + m^2(|\vec{p}|^2 - \omega E') + 2(E^2 + E'^2 - m^2)|\vec{p}|^2 \tag{514}$$

or

$$f_{K2} = -6m^2E\omega - m^2|\vec{p}|^2 - m^2\omega E' + 2(E^2 + E'^2)|\vec{p}|^2. \quad (515)$$

Using equations (130) and (116) in equation (515), then

$$f_{K2} = -6m^2E(E - E') - m^2(E^2 - m^2) - m^2(E - E')E' + 2(E^2 + E'^2)(E^2 - m^2) \quad (516)$$

or

$$f_{K2} = -6m^2E^2 + 6m^2EE' - m^2E^2 + m^4 - m^2EE' + m^2E'^2 + 2E^4 - 2m^2E^2 + 2E^2E'^2 - 2m^2E'^2 \quad (517)$$

or

$$\begin{aligned} f_{K2} &= -9m^2E^2 + 5m^2EE' + m^4 - m^2E'^2 + 2E^4 + 2E^2E'^2 \\ &= 2E^2(E^2 + E'^2) + m^4 - m^2(9E^2 - 5EE' + E'^2) \\ &= 2E^2(E^2 + E'^2) + m^4 - m^2(7E^2 + 2E\omega - 3EE' + E'^2) \\ &= 2E^2(E^2 + E'^2) - m^2(7E^2 - 3EE' + E'^2) + m^4 - 2m^2E\omega. \end{aligned} \quad (518)$$

Substituting equation (518) into equation (513), then

$$\begin{aligned} K_2 = &-2m^2E\omega E'^{-1}|\vec{p}|^{-2} - 2m^2E|\vec{p}|^{-4}E'^{-1}(3\omega m^2 - |\vec{p}|^2E) \\ &+ E'^{-1}|\vec{p}|^{-2}[2E^2(E^2 + E'^2) - m^2(7E^2 - 3EE' + E'^2) + m^4]. \end{aligned} \quad (519)$$

Combining the first two terms of equation (519), then

$$\begin{aligned} K_2 = &-2m^2E|\vec{p}|^{-4}E'^{-1}[3\omega m^2 - |\vec{p}|^2(E - \omega)] + E'^{-1}|\vec{p}|^{-2}[2E^2(E^2 + E'^2) \\ &- m^2(7E^2 - 3EE' + E'^2) + m^4]. \end{aligned} \quad (520)$$

Using equation (130) in the second term, then

$$K_2 = -2m^2E|\vec{p}|^{-4}E'^{-1}(3\omega m^2 - |\vec{p}|^2E') + E'^{-1}|\vec{p}|^{-2}[2E^2(E^2 + E'^2) - m^2(7E^2 - 3EE' + E'^2) + m^4]. \quad (521)$$

Equation (481) can be written as

$$K_3 = K_{31} + K_{32} + K_{33} + K_{34} + K_{35}, \quad (522)$$

where

$$K_{31} = m^2E^2|\vec{p}'|^{-2}|\vec{p}|^{-2}E'^{-2}(2E'^2 + m^2)(|\vec{p}|^2 + |\vec{p}'|^2 + \omega^2 - 2\omega E), \quad (523)$$

$$K_{32} = 12m^4E^3|\vec{p}'|^{-2}|\vec{p}|^{-4}E'^{-2}(\omega m^2 + E|\vec{p}'|^2), \quad (524)$$

$$K_{33} = m^2E^2|\vec{p}'|^{-2}|\vec{p}|^{-4}E'^{-2}[-12(\omega m^2 + E|\vec{p}'|^2)E'^2E - 2|\vec{p}'|^2|\vec{p}|^2m^2], \quad (525)$$

$$K_{34} = -m^2(2E'|\vec{p}|^2)^{-1}[8E'E^2 + 4\omega^2E - 4(E^2 + E'^2)\omega - 4(E^2 + E'^2)E'], \quad (526)$$

$$K_{35} = E(2E'|\vec{p}|^2)^{-1}[2\omega^2|\vec{p}|^2 - 2\omega^2E^2 + 2\omega^2|\vec{p}'|^2 - 2\omega^2E'^2 - 2(E^2 + E'^2) \cdot (|\vec{p}|^2 + |\vec{p}'|^2 + \omega^2 - 2\omega E + 2\omega E' - 2EE')]. \quad (527)$$

Using equation (130) in equation (523),

$$K_{31} = m^2E^2|\vec{p}'|^{-2}|\vec{p}|^{-2}E'^{-2}(2E'^2 + m^2)[|\vec{p}|^2 + |\vec{p}'|^2 + (E - E')^2 - 2(E - E')E] \quad (528)$$

or

$$K_{31} = m^2E^2|\vec{p}'|^{-2}|\vec{p}|^{-2}E'^{-2}(2E'^2 + m^2)(|\vec{p}|^2 + |\vec{p}'|^2 - E^2 + E'^2). \quad (529)$$

Substituting equations (116) and (209) into equation (529), then

$$K_{31} = 2m^2E^2|\vec{p}|^{-2}E'^{-2}(2E'^2 + m^2). \tag{530}$$

Using equation (130) in equation (524),

$$K_{32} = 12m^4E^3|\vec{p}'|^{-2}|\vec{p}|^{-4}E'^{-2}[(E - E')m^2 + E|\vec{p}'|^2] \tag{531}$$

or

$$K_{32} = 12m^4E^3|\vec{p}'|^{-2}|\vec{p}|^{-4}E'^{-2}[E(m^2 + |\vec{p}'|^2) - E'm^2]. \tag{532}$$

Substituting equation (209) into equation (532), then

$$K_{32} = 12m^4E^3|\vec{p}'|^{-2}|\vec{p}|^{-4}E'^{-2}(EE'^2 - E'm^2) \tag{533}$$

or

$$K_{32} = 12m^4E^3|\vec{p}'|^{-2}|\vec{p}|^{-4}E'^{-1}(EE' - m^2). \tag{534}$$

Using equation (130) in equation (525),

$$K_{33} = m^2E^2|\vec{p}'|^{-2}|\vec{p}|^{-4}E'^{-2}\{-6[(E - E')m^2 + E|\vec{p}'|^2]\,(2E'^2E) - 2|\vec{p}'|^2|\vec{p}|^2m^2\}, \tag{535}$$

or

$$K_{33} = m^2E^2|\vec{p}'|^{-2}|\vec{p}|^{-4}E'^{-2}[-12E'^2E(Em^2 - E'm^2 + E|\vec{p}'|^2) - 2|\vec{p}'|^2|\vec{p}|^2m^2]. \tag{536}$$

Using equation (209), equation (536) becomes

$$K_{33} = m^2E^2|\vec{p}'|^{-2}|\vec{p}|^{-4}E'^{-2}[-12E'^3E(EE' - m^2) - 2m^2|\vec{p}'|^2|\vec{p}|^2] \tag{537}$$

or

$$K_{33} = 12m^2E^3E'|\vec{p}'|^{-2}|\vec{p}|^{-4}(m^2 - EE') - 2m^4E^2|\vec{p}|^{-2}E'^{-2}. \quad (538)$$

Using equation (130) in equation (526),

$$K_{34} = -m^2(2E'|\vec{p}|^2)^{-1}[8E'E^2 + 4(E - E')^2E - 4(E^2 + E'^2)(E - E') - 4(E^2 + E'^2)E'] \quad (539)$$

or

$$K_{34} = -m^2(2E'|\vec{p}|^2)^{-1}(8E'E^2 + 4E^3 - 8E^2E' + 4E'^2E - 4E^3 - 4E'^2E), \quad (540)$$

or combining terms,

$$K_{34} = 0. \quad (541)$$

Further using equations (116), (209), and (130) in equation (527),

$$K_{35} = E(2E'|\vec{p}|^2)^{-1}[-4\omega^2m^2 - 2(E^2 + E'^2)(|\vec{p}|^2 + |\vec{p}'|^2 - \omega^2 - 2EE')]. \quad (542)$$

Again using equations (130), (209), and (116) in equation (542),

$$K_{35} = E(2E'|\vec{p}|^2)^{-1}[-4m^2(E^2 - 2EE' + E'^2) - 2(E^2 + E'^2)(-2m^2)] \quad (543)$$

or

$$K_{35} = 4m^2E^2|\vec{p}|^{-2}. \quad (544)$$

Then substituting equations (530), (534), (538), (541), and (544) into equation (522),

$$K_3 = 2m^2E^2|\vec{p}|^{-2}E'^{-2}(2E'^2 + m^2) + 12m^4E^3|\vec{p}'|^{-2}|\vec{p}|^{-4}E'^{-1}(EE' - m^2)$$

$$+ 12m^2E^3E'|\vec{p}'|^{-2}|\vec{p}|^{-4}(m^2 - EE') - 2m^4E^2|\vec{p}|^{-2}E'^{-2} + 4m^2E^2|\vec{p}|^{-2}, \quad (545)$$

or combining the second and third terms in equation (545) and using equation (209),

$$K_3 = EE'^{-1}|\vec{p}|^{-2}[2m^2EE'^{-1}(2E'^2 + m^2) - 12m^2E^2|\vec{p}|^{-2}(EE' - m^2) - 2m^4EE'^{-1} + 4m^2EE'] \quad (546)$$

or

$$K_3 = 2m^2EE'^{-1}|\vec{p}|^{-2}[EE'^{-1}(2E'^2 + m^2) - 6E^2|\vec{p}|^{-2}(EE' - m^2) - m^2EE'^{-1} + 2EE'] \quad (547)$$

or

$$K_3 = 2m^2EE'^{-1}|\vec{p}|^{-2}[4EE' - 6E^2(EE' - m^2)|\vec{p}|^{-2}]. \quad (548)$$

Equivalently,

$$K_3 = 4E^2m^2E'^{-1}|\vec{p}|^{-4}[2E'|\vec{p}|^2 - 3E(EE' - m^2)]. \quad (549)$$

Using equations (130) and (116) in equation (549), then

$$K_3 = 4E^2m^2E'^{-1}|\vec{p}|^{-4}[2E'|\vec{p}|^2 + 3m^2(E' + \omega) - 3(|\vec{p}|^2 + m^2)E'] \quad (550)$$

or

$$K_3 = 4E^2m^2E'^{-1}|\vec{p}|^{-4}(3\omega m^2 - E'|\vec{p}|^2). \quad (551)$$

Multiplying terms in equation (482), then

$$K_4 = K_{41} + K_{42}, \quad (552)$$

where

$$K_{41} = -6m^2E^2|\vec{p}'|^{-2}|\vec{p}|^{-4}E'^{-2}(\omega m^6 + \omega m^2E'^2|\vec{p}|^2 - \omega m^2E'^2E^2 + m^4E|\vec{p}'|^2 \cdot$$
$$+ EE'^2|\vec{p}'|^2|\vec{p}|^2 - E^3E'^2|\vec{p}'|^2, \tag{553}$$

$$K_{42} = -m^2\omega^2E'^{-1} + m^2\omega^2E^2E'^{-1}|\vec{p}|^{-2} - m^2\omega^2|\vec{p}'|^2E'^{-1}|\vec{p}|^{-2} + m^2\omega^2E'|\vec{p}|^{-2} + m^2E^2E'^{-1}$$
$$+ m^2E^2|\vec{p}'|^2E'^{-1}|\vec{p}|^{-2} + E^2m^2\omega^2E'^{-1}|\vec{p}|^{-2} - 2m^2\omega E^3E'^{-1}|\vec{p}|^{-2} + 2m^2\omega E^2|\vec{p}|^{-2}$$
$$- 2m^2E^3|\vec{p}|^{-2} + m^2E' + m^2E'|\vec{p}'|^2|\vec{p}|^{-2} + E'm^2\omega^2|\vec{p}|^{-2} - 2m^2\omega EE'|\vec{p}|^{-2}$$
$$+ 2m^2\omega E'^2|\vec{p}|^{-2} - 2m^2E'^2E|\vec{p}|^{-2}. \tag{554}$$

Further, by multiplying in equation (553), then

$$K_{41} = -6\omega m^8E^2|\vec{p}'|^{-2}|\vec{p}|^{-4}E'^{-2} - 6m^4E^2\omega|\vec{p}|^2|\vec{p}'|^{-2}|\vec{p}|^{-4}$$
$$+ 6m^4E^4\omega|\vec{p}'|^{-2}|\vec{p}|^{-4} - 6m^6E^3|\vec{p}|^{-4}E'^{-2} - 6m^2E^3|\vec{p}|^{-2} + 6m^2E^5|\vec{p}|^{-4}. \tag{555}$$

Using equation (116) to reexpress factors of E^2 in the third and sixth terms of equation (555), then

$$K_{41} = -6\omega m^8E^2|\vec{p}'|^{-2}|\vec{p}|^{-4}E'^{-2} - 6m^4E^2\omega|\vec{p}|^2|\vec{p}'|^{-2}|\vec{p}|^{-4} + 6m^4E^2$$
$$\cdot (|\vec{p}|^2 + m^2)\omega|\vec{p}'|^{-2}|\vec{p}|^{-4} - 6m^6E^3|\vec{p}|^{-4}E'^{-2} - 6m^2E^3|\vec{p}|^{-2}$$
$$+ 6m^2E^3(|\vec{p}|^2 + m^2)|\vec{p}|^{-4}. \tag{556}$$

Simplifying and using equation (209), equation (556) becomes

$$K_{41} = -6\omega m^8E^2|\vec{p}'|^{-2}|\vec{p}|^{-4}E'^{-2} + 6m^6E^2\omega|\vec{p}'|^{-2}|\vec{p}|^{-4} - 6m^4$$
$$\cdot (E'^2 - |\vec{p}'|^2)E^3|\vec{p}|^{-4}E'^{-2} + 6m^4E^3|\vec{p}|^{-4} \tag{557}$$

or

$$K_{41} = -6m^4E^2|\vec{p}'|^{-2}|\vec{p}|^{-4}E'^{-2}(\omega m^4 - m^2\omega E'^2 - E|\vec{p}'|^4). \tag{558}$$

Using equation (209) in the second term of equation (558), then

$$K_{41} = 6m^4E^2|\vec{p}|^{-4}E'^{-2}(\omega m^2 + E|\vec{p}'|^2). \tag{559}$$

Next, combining terms, equation (554) becomes

$$\begin{aligned} K_{42} = &-m^2(E - E')^2E'^{-1} + 2m^2\omega^2E^2E'^{-1}|\vec{p}|^{-2} - m^2\omega^2|\vec{p}'|^2E'^{-1}|\vec{p}|^{-2} + 2m^2\omega^2E'|\vec{p}|^{-2} + m^2E^2E'^{-1} \\ &+ m^2(E' + \omega)^2|\vec{p}'|^2E'^{-1}|\vec{p}|^{-2} - 2m^2\omega E^2(E' + \omega)E'^{-1}|\vec{p}|^{-2} + 2m^2\omega E^2|\vec{p}|^{-2} - 2m^2E^3|\vec{p}|^{-2} \\ &+ m^2E' + m^2E'|\vec{p}'|^2|\vec{p}|^{-2} - 2m^2\omega(E' + \omega)E'|\vec{p}|^{-2} + 2m^2\omega E'^2|\vec{p}|^{-2} - 2m^2E'^2E|\vec{p}|^{-2}, \end{aligned} \tag{560}$$

where equation (130) has been used in the 1st, 6th, 7th, and 12th terms. Further, combining terms in equation (560), then

$$\begin{aligned} K_{42} = &2m^2E - m^2\omega^2|\vec{p}'|^2E'^{-1}|\vec{p}|^{-2} + m^2E^2|\vec{p}'|^2E'^{-1}|\vec{p}|^{-2} - 2m^2E^3|\vec{p}|^{-2} \\ &+ m^2E'|\vec{p}'|^2|\vec{p}|^{-2} - 2m^2E'^2E|\vec{p}|^{-2} \end{aligned} \tag{561}$$

or

$$\begin{aligned} K_{42} = &m^2(2E - \omega^2|\vec{p}'|^2E'^{-1}|\vec{p}|^{-2} + E^2|\vec{p}'|^2E'^{-1}|\vec{p}|^{-2} - 2E^3|\vec{p}|^{-2} \\ &+ E'|\vec{p}'|^2|\vec{p}|^{-2} - 2E'^2E|\vec{p}|^{-2}). \end{aligned} \tag{562}$$

Using equation (130) and substituting for ω in the second term of equation (562), then

$$K_{42} = 2m^2E|\vec{p}|^{-2}(|\vec{p}|^2 + |\vec{p}'|^2 - E^2 - E'^2). \tag{563}$$

Substituting equations (116) and (209) into equation (563), then

$$K_{42} = -4m^4E|\vec{p}|^{-2}. \tag{564}$$

Finally, substituting equations (559) and (564) into equation (552), then

$$K_4 = 6m^4E^2|\vec{p}|^{-4}E'^{-2}(\omega m^2 + E|\vec{p}'|^2) - 4m^4E|\vec{p}|^{-2} \tag{565}$$

or

$$K_4 = 2m^4E|\vec{p}|^{-4}E'^{-2}(3E\omega m^2 + 3E^2|\vec{p}'|^2 - 2|\vec{p}|^2E'^2). \tag{566}$$

Substituting equation (209) into the second term of equation (566) and using equation (130), then

$$K_4 = 2m^4E|\vec{p}|^{-4}E'^{-2}[3E\omega m^2 + 3E^2E'^2 - 3E(E' + \omega)m^2 - 2|\vec{p}|^2E'^2] \tag{567}$$

or

$$K_4 = 2m^4E|\vec{p}|^{-4}E'^{-1}(-3Em^2 + 3E^2E' - 2|\vec{p}|^2E'). \tag{568}$$

Using equation (130) in the first term of equation (568), then

$$K_4 = 2m^4E|\vec{p}|^{-4}E'^{-1}[-3\omega m^2 + 3E'(E^2 - m^2) - 2|\vec{p}|^2E']. \tag{569}$$

Finally, substituting equation (116) into equation (569), then

$$K_4 = -2m^4E|\vec{p}|^{-4}E'^{-1}(3\omega m^2 - |\vec{p}|^2E'). \tag{570}$$

Then by substituting equations (485), (521), (551), and (570) into equation (478),

$$\begin{aligned} I_a = {} & E'L(2|\vec{p}|\,|\vec{p}'|\varkappa)^{-1}(\omega E'^{-1}|\vec{p}|^{-2}(E^2 + EE' - m^2) + \varkappa^{-1}\{-2m^2E|\vec{p}|^{-4}E'^{-1} \\ & \cdot (3\omega m^2 - |\vec{p}|^2E') + E'^{-1}|\vec{p}|^{-2}[2E^2(E^2 + E'^2) - m^2(7E^2 - 3EE' + E'^2) + m^4]\} \\ & + \varkappa^{-2}[4E^2m^2E'^{-1}|\vec{p}|^{-4}(3\omega m^2 - E'|\vec{p}|^2)] + \varkappa^{-3}[-2m^4E|\vec{p}|^{-4}E'^{-1} \\ & \cdot (3\omega m^2 - |\vec{p}|^2E')]). \end{aligned} \tag{571}$$

Collecting terms with the common factor $(3\omega m^2 - |\vec{p}|^2 E')$, equation (571) can be rewritten as

$$I_a = L(2|\vec{p}|\,|\vec{p}'|)^{-1}\{(3\omega m^2 - |\vec{p}|^2E')|\vec{p}|^{-4}x^{-4}F_{a1} + |\vec{p}|^{-2}x^{-1}\omega(E^2 + EE' - m^2)$$

$$+ |\vec{p}|^{-2}x^{-2}[2E^2(E^2 + E'^2) - m^2(7E^2 - 3EE' + E'^2) + m^4]\}, \tag{572}$$

where

$$F_{a1} = -2m^2Ex^2 + 4E^2m^2x - 2m^4E. \tag{573}$$

Equation (573) may be rewritten as

$$F_{a1} = -2m^2E(x^2 - 2Ex + m^2)$$

or

$$F_{a1} = -2m^2E[(E - x)^2 - E^2 + m^2]. \tag{574}$$

Using equations (43) and (116) in equation (574), then

$$F_{a1} = -2m^2E(|\vec{p}|^2 \cos^2\theta - |\vec{p}|^2). \tag{575}$$

Using equation (213), equation (575) becomes

$$F_{a1} = 2m^2E|\vec{p}|^2 \sin^2\theta. \tag{576}$$

Substituting equation (576) into equation (572), then

$$I_a = L(4|\vec{p}|\,|\vec{p}'|)^{-1}\{4Em^2\sin^2\theta(3\omega m^2 - |\vec{p}|^2E')|\vec{p}|^{-2}x^{-4} + [4E^2(E^2 + E'^2) - 2m^2$$

$$\cdot (7E^2 - 3EE' + E'^2) + 2m^4]\,|\vec{p}|^{-2}x^{-2} + 2\omega(E^2 + EE' - m^2)|\vec{p}|^{-2}x^{-1}\}. \tag{577}$$

Proceeding also to reduce equation (425), one first rewrites equation (425) in the form

$$I_b = I_{b1} + I_{b2} + I_{b3} + I_{b4} + I_{b5} + I_{b6}, \tag{578}$$

where

$$I_{b1} = F_{b1}L_{b1}, \tag{579}$$

$$\begin{aligned} F_{b1} = &-[2\varkappa^{-1}(\omega - |\vec{p}| \cos \theta) + |\vec{p}|^2\varkappa^{-2} \sin^2 \theta - 1]|\vec{p}'|^{-1}|\vec{a}|^{-1} \\ &\cdot (|\vec{T}|^2 + |\vec{p}'|^2)^{-1} - 2\omega^2 E'\varkappa^{-1}|\vec{p}'|^{-1}|\vec{a}|^{-1} \\ &\cdot (|\vec{T}|^2 + |\vec{p}'|^2)^{-2}\, \vec{a}\cdot\vec{b}|\,\vec{a}|^{-2}, \end{aligned} \tag{580}$$

$$L_{b1} = \ell n\, [(1 + |\vec{p}'|\, |\vec{a}|)(1 - |\vec{p}'|\, |\vec{a}|)^{-1}], \tag{581}$$

$$I_{b2} = -\varkappa^{-1}E'^{-1}|\vec{p}'|^{-1}|\vec{b}|^{-1}\, \ell n\, [(1 + |\vec{p}'|\, |\vec{b}|)(1 - |\vec{p}'|\, |\vec{b}|)^{-1}], \tag{582}$$

$$I_{b3} = m^2(|\vec{T}|^2 + |\vec{p}'|^2)^{-1}E'^{-2}(4BA^{-1}D^{-2}), \tag{583}$$

$$I_{b4} = I_{b41} + I_{b42} + I_{b43} + I_{b44}, \tag{584}$$

$$\begin{aligned} I_{b41} = &-4m^2E^2E'^{-2}(|\vec{T}|^2 + |\vec{p}'|^2)^{-2}(A + B + C)^{-2}[8D^{-2}(1 + B/3C)(2A + B) \\ &+ 16/3C + (8A/3C)(2C + B)D^{-2}], \end{aligned} \tag{585}$$

$$\begin{aligned} I_{b42} = &-4m^2E^2E'^{-2}(|\vec{T}|^2 + |\vec{p}'|^2)^{-2}(A + B + C)^{-1}\, [24D^{-4}A^{-1}(1 + B/3C)B \\ &\cdot (D^2 + 2AC + BC) + 16D^{-4}(D^2 + 2AC + BC)], \end{aligned} \tag{586}$$

$$I_{b43} = -4m^2E^2E'^{-2}(|\vec{T}|^2 + |\vec{p}'|^2)^{-2}[-16D^{-2}A^{-1}(1 + B/3C) - 8B(3CD^2A)^{-1} + 8D^{-2}], \tag{587}$$

$$I_{b44} = -4(B + 2C)\varkappa^{-1}E'^{-1}(|\vec{T}|^2 + |\vec{p}'|^2)^{-2}D^{-2}(A + B + C)^{-1}[8E'E^2\varkappa + 2\omega^2|\vec{p}|^2 \sin^2\theta + 2\omega^2|\vec{p}'|^2 - 2\omega^2E'^2 - 2(E^2 + E'^2)(|\vec{p}|^2 + |\vec{p}'|^2 + \omega^2 - 2|\vec{p}|\omega\cos\theta + 2\omega E' - 2|\vec{p}|E'\cos\theta)], \quad (588)$$

$$I_{b5} = 2(|\vec{T}|^2 + |\vec{p}'|^2)^{-2}(1 - |\vec{p}'|^2\,|\vec{a}|^2)^{-1}[4\varkappa^{-1}(E^2 + E'^2)(\omega - |\vec{p}|\cos\theta) - 4E^2 + 4|\vec{p}|^2E'^2\varkappa^{-2}\sin^2\theta + 4E'\omega^2\varkappa^{-1}], \quad (589)$$

$$I_{b6} = -4\omega^2E'\varkappa^{-1}(|\vec{T}|^2 + |\vec{p}'|^2)^{-2}(1 - \vec{a}\cdot\vec{b}\,|\vec{a}|^{-2})(1 - |\vec{p}'|^2\,|\vec{a}|^2)^{-1}, \quad (590)$$

Substituting equation (244) into equation (581),

$$L_{b1} = \ell n\{[1 + 2|\vec{p}'|\,|\vec{T}|(|\vec{T}|^2 + |\vec{p}'|^2)^{-1}][1 - 2|\vec{p}'|\,|\vec{T}| \cdot (|\vec{T}|^2 + |\vec{p}'|^2)^{-1}]^{-1}\} \quad (591)$$

or

$$L_{b1} = 2\,\ell n\,[(|\vec{T}| + |\vec{p}'|)(|\vec{T}| - |\vec{p}'|)^{-1}]. \quad (592)$$

To reduce equation (580), one first notes that, using equations (244), (245), and (41),

$$\vec{a}\cdot\vec{b}\,|\vec{a}|^{-2} = \vec{T}\cdot\vec{k}\,(|\vec{T}|^2 + |\vec{p}'|^2)(2\omega E'|\vec{T}|^2)^{-1}. \quad (593)$$

Substituting equation (300) into equation (593), then

$$\vec{a}\cdot\vec{b}\,|\vec{a}|^{-2} = (4\omega E'|\vec{T}|^2)^{-1} \cdot (|\vec{p}'|^2 + 2E'\omega - |\vec{T}|^2)(|\vec{T}|^2 + |\vec{p}'|^2). \quad (594)$$

Substituting equations (244) and (594) into equation (580), then

$$F_{b1} = F_{b11} + F_{b12}, \quad (595)$$

where

$$F_{b11} = -[2\varkappa^{-1}(\omega - |\vec{p}| \cos\theta) + |\vec{p}|^2\varkappa^{-2}\sin^2\theta - 1]|\vec{p}'|^{-1}|\vec{a}|^{-1}$$
$$\cdot (|\vec{T}|^2 + |\vec{p}'|^2)^{-1} + \omega(4\varkappa|\vec{p}'|\,|\vec{T}|)^{-1}, \quad (596)$$

$$F_{b12} = -\omega(|\vec{p}'|^2 + 2E'\omega)(4\varkappa|\vec{p}'|\,|\vec{T}|^3)^{-1}. \quad (597)$$

Substituting equations (209) and (130) into equation (597),

$$F_{b12} = -\omega(E'^2 - m^2 + 2E\omega - 2\omega^2)(4\varkappa|\vec{p}'|\,|\vec{T}|^3)^{-1}. \quad (598)$$

Substituting equation (130) into equation (598), then

$$F_{b12} = -\omega(E^2 - m^2 - \omega^2)(4\varkappa|\vec{p}'|\,|\vec{T}|^3)^{-1}, \quad (599)$$

or again using equation (116),

$$F_{b12} = -\omega(|\vec{p}|^2 - \omega^2)(4\varkappa|\vec{p}'|\,|\vec{T}|^3)^{-1}. \quad (600)$$

Further, using equations (43) and (244) in equation (596), then

$$F_{b11} = -\{2\varkappa^{-1}(\omega - E + \varkappa) + |\vec{p}|^2\varkappa^{-2}[1 - (E - \varkappa)^2|\vec{p}|^{-2}] - 1\}(2|\vec{p}'|\,|\vec{T}|)^{-1}$$
$$+ \omega(4\varkappa|\vec{p}'|\,|\vec{T}|)^{-1} \quad (601)$$

or

$$F_{b11} = -(2|\vec{p}'|\,|\vec{T}|)^{-1}[|\vec{p}|^2\varkappa^{-2} - E^2\varkappa^{-2} + 3\omega(2\varkappa)^{-1}], \quad (602)$$

or using equation (116),

$$F_{b11} = -(2|\vec{p}'|\,|\vec{T}|)^{-1}[3\omega(2\varkappa)^{-1} - m^2\varkappa^{-2}]. \quad (603)$$

Substituting equations (603) and (600) into equation (595), then

$$F_{b1} = -(2|\vec{p}'|\,|\vec{T}|)^{-1}[3\omega(2\varkappa)^{-1} - m^2\varkappa^{-2}] - \omega(|\vec{p}|^2 - \omega^2)(4\varkappa|\vec{p}'|\,|\vec{T}|^3)^{-1} \quad (604)$$

or

$$F_{b1} = (8|\vec{p}'|\,|\vec{T}|)^{-1}[4m^2\varkappa^{-2} - 6\omega\varkappa^{-1} - 2\omega(|\vec{p}|^2 - \omega^2)\varkappa^{-1}|\vec{T}|^{-2}]. \quad (605)$$

Then substituting equations (605) and (592) into equation (579),

$$I_{b1} = (4|\vec{p}'|\,|\vec{T}|)^{-1}[4m^2\varkappa^{-2} - 6\omega\varkappa^{-1} - 2\omega\varkappa^{-1}|\vec{T}|^{-2}(|\vec{p}|^2 - \omega^2)]$$
$$\cdot\, \ell n[(|\vec{T}| + |\vec{p}'|)(|\vec{T}| - |\vec{p}'|)^{-1}]. \quad (606)$$

Next substituting equation (245) into equation (582), then

$$I_{b2} = -\varkappa^{-1}E'^{-1}E'|\vec{p}'|^{-1}\,\ell n\,[(1 + |\vec{p}'|E'^{-1})(1 - |\vec{p}'|E'^{-1})^{-1}] \quad (607)$$

or

$$I_{b2} = -\varkappa^{-1}|\vec{p}'|^{-1}\,\ell n\,[(E' + |\vec{p}'|)(E' - |\vec{p}'|)^{-1}]. \quad (608)$$

Next, substituting equations (304), (436), and (449) into equation (583),

$$I_{b3} = -m^2|\vec{p}|^{-2}\varkappa^{-2} + E|\vec{p}|^{-2}\varkappa^{-1}. \quad (609)$$

Equation (585) may be rewritten as

$$I_{b41} = -4m^2E^2E'^{-2}(|\vec{T}|^2 + |\vec{p}'|^2)^{-2}[8(B + 3C)(2A + B) + 16D^2$$
$$+ 8A(2C + B)][3CD^2(A + B + C)^2]^{-1} \quad (610)$$

or

$$I_{b41} = -4m^2E^2E'^{-2}(|\vec{T}|^2 + |\vec{p}'|^2)^{-1}(24AB + 8B^2 + 64AC + 24BC + 16D^2)[3CD^2(A + B + C)^2]^{-1}. \quad (611)$$

Substituting equation (283) in the numerator of equation (611), then

$$I_{b41} = -32m^2E^2E'^{-2}(|\vec{T}|^2 + |\vec{p}'|^2)^{-2}BC^{-1}D^{-2}(A + B + C)^{-1}. \quad (612)$$

Next, simplifying equation (587),

$$I_{b43} = -4m^2E^2E'^{-2}(|\vec{T}|^2 + |\vec{p}'|^2)^{-2}F_{b43}, \quad (613)$$

where

$$F_{b43} = -16(1 + B/3C)D^{-2}A^{-1} - 8B(3CD^2A)^{-1} + 8D^{-2} \quad (614)$$

or

$$F_{b43} = 8D^{-2}A^{-1}(-2 - BC^{-1} + A). \quad (615)$$

Substituting equation (615) into equation (613), then

$$I_{b43} = -32m^2E^2E'^{-2}(|\vec{T}|^2 + |\vec{p}'|^2)^{-2}D^{-2}A^{-1}(-2 - BC^{-1} + A). \quad (616)$$

Equation (588) may be rewritten as

$$I_{b44} = -4(B + 2C)x^{-1}E'^{-1}(|\vec{T}|^2 + |\vec{p}'|^2)^{-2}D^{-2}(A + B + C)^{-1}F_{b44}, \quad (617)$$

where

$$F_{b44} = 8E'E^2x + 2\omega^2|\vec{p}|^2 \sin^2\theta + 2\omega^2|\vec{p}'|^2 - 2\omega^2E'^2 - 2(E^2 + E'^2)(|\vec{p}|^2 + |\vec{p}'|^2 + \omega^2 - 2|\vec{p}|\omega\cos\theta + 2\omega E' - 2|\vec{p}|E'\cos\theta). \quad (618)$$

Using equations (130), (43), (116), (213), and (209) in equation (618), then

$$F_{b44} = 8E'E^2x + 2(E - E')^2(E^2 - m^2)[1 - (E - x)^2(E^2 - m^2)^{-1}] - 2(E - E')^2m^2 - 2(E^2 + E'^2)[E^2 + E'^2 - 2m^2 + E^2 - 2EE' + E'^2 - 2E(E - x) + 2(E - E')E'], \tag{619}$$

or simplifying equation (619), then

$$F_{b44} = 8m^2EE' - 2(E - E')^2x^2. \tag{620}$$

Substituting equation (620) into equation (617), then

$$I_{b44} = -4(B + 2C)[8m^2EE' - 2(E - E')^2x^2][xE'(|\vec{T}|^2 + |\vec{p}'|^2)D^2(A + B + C)]^{-1}. \tag{621}$$

Using equations (612) and (616), then

$$I_{b41} + I_{b43} = -32m^2E^2B[E'^2(|\vec{T}|^2 + |\vec{p}'|^2)^2CD^2(A + B + C)]^{-1} - 32m^2E^2(-2 - BC^{-1} + A)[E'^2(|\vec{T}|^2 + |\vec{p}'|^2)^2D^2A]^{-1} \tag{622}$$

or

$$I_{b41} + I_{b43} = I_{b413} - 32m^2E^2(-2 + A)[E'^2(|\vec{T}|^2 + |\vec{p}'|^2)^2D^2A]^{-1}, \tag{623}$$

where

$$I_{b413} = -32m^2E^2B[E'^2(|\vec{T}|^2 + |\vec{p}'|^2)^2CD^2]^{-1}[(A + B + C)^{-1} - A^{-1}]. \tag{624}$$

Equation (624) may be rewritten as

$$I_{b413} = 32m^2E^2B(B + C)E'^{-2}(|\vec{T}|^2 + |\vec{p}'|^2)^{-2} \cdot [AD^2C(A + B + C)]^{-1}. \tag{625}$$

Substituting equation (625) into equation (623), then

$$I_{b41} + I_{b43} = 32m^2E^2B(B + C)[E'^2(|\vec{T}|^2 + |\vec{p}'|^2)^2AD^2C(A + B + C)]^{-1}$$
$$- 32m^2E^2(A - 2)[E'^2(|\vec{T}|^2 + |\vec{p}'|^2)^2D^2A]^{-1}. \quad (626)$$

Substituting equations (626) and (586) into equation (584), then

$$I_{b4} = I_{b4a} - 32m^2E^2(A - 2)[E'^2(|\vec{T}|^2 + |\vec{p}'|^2)^2D^2A]^{-1}, \quad (627)$$

where

$$I_{b4a} = I_{b4b} + I_{b44}, \quad (628)$$

$$I_{b4b} = 32m^2E^2B(B + C)[E'^2(|\vec{T}|^2 + |\vec{p}'|^2)^2AD^2C(A + B + C)]^{-1}$$
$$- 32m^2E^2[E'^2(|\vec{T}|^2 + |\vec{p}'|^2)^2]^{-1}\{(1 + B/3C)3B(D^2 + 2AC + BC)[D^4A(A + B + C)]^{-1}$$
$$+ 2(D^2 + 2AC + BC)[D^4(A + B + C)]^{-1}\}. \quad (629)$$

Combining terms in equation (629), then

$$I_{b4b} = 32m^2E^2[E'^2(|\vec{T}|^2 + |\vec{p}'|^2)^2ACD^4(A + B + C)]^{-1}F_{b4b}, \quad (630)$$

where

$$F_{b4b} = BD^2(B + C) - 3BC(1 + B/3C)(D^2 + 2AC + BC) - 2AC(D^2 + 2AC + BC). \quad (631)$$

Equivalently, equation (631) is

$$F_{b4b} = B^2D^2 + BCD^2 - (3BC + B^2)(D^2 + 2AC + BC) - 2AC(D^2 + 2AC + BC) \quad (632)$$

or

$$F_{b4b} = B^2D^2 + BCD^2 - (D^2 + 2AC + BC)(3BC + B^2 + 2AC) \tag{633}$$

or

$$F_{b4b} = D^2(-2BC - 2AC) - (2AC + BC)(3BC + B^2 + 2AC) \tag{634}$$

or

$$F_{b4b} = -2CD^2(A + B + C) + 2CD^2 - (2AC + BC)(3BC + B^2 + 2AC). \tag{635}$$

Substituting equation (283) into the second term of equation (635), then

$$F_{b4b} = -2CD^2(A + B + C) + C(-B^2C - 8AC^2 - 8ABC - 2AB^2 - 4A^2C - B^3), \tag{636}$$

or

$$F_{b4b} = -2CD^2(A + B + C) - B^2C(A + B + C) - ACD^2 - 8AC^2(A + B + C). \tag{637}$$

Substituting equation (637) into equation (630), then

$$I_{b4b} = 32m^2E^2[E'^2(|\vec{T}|^2 + |\vec{p}'|^2)^2ACD^4(A + B + C)]^{-1}[-2CD^2(A + B + C) - B^2C(A + B + C) - ACD^2 - 8AC^2(A + B + C)]. \tag{638}$$

Substituting equations (638) and (621) into equation (628), then

$$I_{b4a} = -64m^2E^2E'^{-2}(|\vec{T}|^2 + |\vec{p}'|^2)^{-2}A^{-1}D^{-2} - 32m^2E^2B^2E'^{-2} \cdot (|\vec{T}|^2 + |\vec{p}'|^2)^{-2}A^{-1}D^{-4} - 256m^2E^2CE'^{-2}(|\vec{T}|^2 + |\vec{p}'|^2)^{-2}D^{-4} - I_{b4c}, \tag{639}$$

where

$$I_{b4c} = 32m^2E^2E'^{-2}(|\vec{T}|^2 + |\vec{p}'|^2)^{-2}D^{-2}(A + B + C)^{-1} + 4(B + 2C)[8m^2EE' - 2(E - E')^2\varkappa^2] \cdot \varkappa^{-1}E'^{-1}(|\vec{T}|^2 + |\vec{p}'|^2)^{-2}D^{-2}(A + B + C)^{-1}. \tag{640}$$

Rewriting equation (640), then

$$I_{b4c} = 4\{8\varkappa m^2E^2 + E'(B + 2C)[8m^2EE' - 2(E - E')^2\varkappa^2]\} \cdot \varkappa^{-1}E'^{-2}[(|\vec{T}|^2 + |\vec{p}'|^2)^2D^2(A + B + C)]^{-1} \tag{641}$$

or

$$I_{b4c} = 4\{8\varkappa m^2E^2 - E'[(2A + B) - 2(A + B + C)][8m^2EE' - 2(E - E')^2\varkappa^2]\} \cdot \varkappa^{-1}E'^{-2}(|\vec{T}|^2 + |\vec{p}'|^2)^{-2}D^{-2}(A + B + C)^{-1}. \tag{642}$$

Then equation (642) becomes

$$I_{b4c} = I_{b4d} + 8E'[8m^2EE' - 2(E - E')^2\varkappa^2]\varkappa^{-1}E'^{-2}(|\vec{T}|^2 + |\vec{p}'|^2)^{-2}D^{-2}, \tag{643}$$

where

$$I_{b4d} = 4\{8\varkappa m^2E^2 - E'(2A + B)[8m^2EE' - 2(E - E')^2\varkappa^2]\} \cdot [\varkappa E'^2(|\vec{T}|^2 + |\vec{p}'|^2)^2D^2(A + B + C)]^{-1}. \tag{644}$$

Using equations (276) to (278),

$$A + B + C = 1 - |\vec{p}'|^2|\vec{b}|^2 - 2|\vec{p}'|^2(\vec{a} \cdot \vec{b} - |\vec{b}|^2) - |\vec{p}'|^2\, |\vec{a} - \vec{b}|^2 \tag{645}$$

or

$$A + B + C = 1 - |\vec{p}'|^2|\vec{a}|^2. \tag{646}$$

Substituting equation (244) into equation (646), then

$$A + B + C = 1 - 4|\vec{p}'|^2|\vec{T}|^2(|\vec{T}|^2 + |\vec{p}'|^2)^{-2}. \tag{647}$$

Simplifying equation (647), then

$$A + B + C = (|\vec{T}|^2 - |\vec{p}'|^2)^2(|\vec{T}|^2 + |\vec{p}'|^2)^{-2}. \tag{648}$$

Substituting equation (448) into equation (648), then

$$A + B + C = (2\omega x)^2(|\vec{T}|^2 + |\vec{p}'|^2)^{-2}. \tag{649}$$

Next, substituting equations (304), (436), (449), and (649) into equation (644), then

$$I_{b4d} = 4\{8xm^2E^2 - E'[8m^2EE' - 2(E - E')^2x^2][2m^2E'^{-2} - 4|\vec{p}'|^2(m^2 - xE)E'^{-2} \cdot (|\vec{T}|^2 + |\vec{p}'|^2)^{-1}]\}\{xE'^2(16|\vec{p}'|^2|\vec{p}|^2x^2)[E'^2(|\vec{T}|^2 + |\vec{p}'|^2)^2]^{-1}4\omega^2x^2\}^{-1}. \tag{650}$$

Using equation (130) and simplifying the result, then equation (650) becomes

$$I_{b4d} = (|\vec{T}|^2 + |\vec{p}'|^2)(4|\vec{p}'|^2|\vec{p}|^2x^5E'\omega^2)^{-1}\{2xm^2E'E^2 \cdot (|\vec{T}|^2 + |\vec{p}'|^2) - (4m^2EE' - \omega^2x^2)[m^2(|\vec{T}|^2 + |\vec{p}'|^2) - 2|\vec{p}'|^2(m^2 - xE)]\}. \tag{651}$$

Using equation (448) in equation (651), then

$$I_{b4d} = (\omega x + |\vec{p}'|^2)(2|\vec{p}'|^2|\vec{p}|^2x^5E'\omega^2)^{-1}F_{b4d}, \tag{652}$$

where

$$F_{b4d} = (2\omega x + 2|\vec{p}'|^2)(2xm^2E^2E' - 4m^4EE' + m^2\omega^2x^2) + 2|\vec{p}'|^2(m^2 - xE)(4m^2EE' - \omega^2x^2). \tag{653}$$

Multiplying in equation (653), then

$$F_{b4d} = 4\omega x^2m^2E^2E' - 8\omega m^4EE'x + 2m^2\omega^3x^3 + 4|\vec{p}'|^2m^2E^2E'x - 8m^4|\vec{p}'|^2EE' + 2m^2|\vec{p}'|^2\omega^2x^2 + 8m^4EE'|\vec{p}'|^2 - 2|\vec{p}'|^2m^2\omega^2x^2 - 8m^2|\vec{p}'|^2E^2E'x + 2|\vec{p}'|^2\omega^2Ex^3 \tag{654}$$

or

$$F_{b4d} = 4\omega m^2E^2E'\varkappa^2 + 2m^2\omega^3\varkappa^3 + 2|\vec{p}'|^2\omega^2E\varkappa^3 - 8\omega m^4EE'\varkappa - 4|\vec{p}'|^2m^2E^2E'\varkappa. \quad (655)$$

Substituting equation (655) into equation (652), then

$$I_{b4d} = (\omega\varkappa + |\vec{p}'|^2)(2|\vec{p}'|^2|\vec{p}|^2\varkappa^5E'/\omega^2)^{-1}(4\omega m^2E^2E'\varkappa^2 + 2m^2\omega^3\varkappa^3 + 2|\vec{p}'|^2\omega^2E\varkappa^3 - 8\omega m^4EE'\varkappa - 4|\vec{p}'|^2m^2E^2E'\varkappa). \quad (656)$$

Substituting equations (652) and (655) into equation (643), then

$$I_{b4c} = (\omega\varkappa + |\vec{p}'|^2)(2|\vec{p}'|^2|\vec{p}|^2\varkappa^5E'/\omega^2)^{-1}(4\omega m^2E^2E'\varkappa^2 + 2m^2\omega^3\varkappa^3 + 2|\vec{p}'|^2\omega^2E\varkappa^3 - 8\omega m^4EE'\varkappa - 4|\vec{p}'|^2m^2E^2E'\varkappa) + 8E'[8m^2EE' - 2(E - E')^2\varkappa^2] \cdot [\varkappa E'^2(|\vec{T}|^2 + |\vec{p}'|^2)^2D^2]^{-1}. \quad (657)$$

Substituting equation (657) into equation (639), then

$$I_{b4a} = -64m^2E^2[E'^2(|\vec{T}|^2 + |\vec{p}'|^2)^2AD^2]^{-1} - 32m^2E^2B^2 \cdot [E'^2(|\vec{T}|^2 + |\vec{p}'|^2)^2AD^4]^{-1} - 256m^2E^2C[E'^2(|\vec{T}|^2 + |\vec{p}'|^2)^2D^4]^{-1} - (\omega\varkappa + |\vec{p}'|^2) \cdot (2|\vec{p}'|^2|\vec{p}|^2\varkappa^5E'/\omega^2)^{-1}(4\omega m^2E^2E'\varkappa^2 + 2m^2\omega^3\varkappa^3 + 2|\vec{p}'|^2\omega^2E\varkappa^3 - 8\omega m^4EE'\varkappa - 4|\vec{p}'|^2m^2E^2E'\varkappa) - 8E'[8m^2EE' - 2(E - E')^2\varkappa^2] \cdot [\varkappa E'^2(|\vec{T}|^2 + |\vec{p}'|^2)^2D^2]^{-1}. \quad (658)$$

Substituting equation (658) into equation (627), then

$$I_{b4} = -64m^2E^2[E'^2(|\vec{T}|^2 + |\vec{p}'|^2)^2AD^2]^{-1} - 32m^2E^2B^2$$

$$\cdot [E'^2(|\vec{T}|^2 + |\vec{p}'|^2)^2AD^4]^{-1} - 256m^2E^2C[E'^2(|\vec{T}|^2 + |\vec{p}'|^2)^2D^4]^{-1}$$

$$- 8E'[8m^2EE' - 2(E - E')^2\varkappa^2][\varkappa E'^2(|\vec{T}|^2 + |\vec{p}'|^2)^2D^2]^{-1} - (\omega\varkappa + |\vec{p}'|^2)$$

$$\cdot (2|\vec{p}'|^2|\vec{p}|^2\varkappa^5E'\omega^2)^{-1}(4\omega m^2E^2E'\varkappa^2 + 2m^2\omega^3\varkappa^3 + 2|\vec{p}'|^2\omega^2E\varkappa^3$$

$$- 8\omega m^4EE'\varkappa - 4|\vec{p}'|^2m^2E^2E'\varkappa) - 32m^2E^2(A - 2)$$

$$\cdot [E'^2(|\vec{T}|^2 + |\vec{p}'|^2)^2D^2A]^{-1}. \qquad (659)$$

To reduce equation (589), one first notes using equation (244) that

$$1 - |\vec{p}'|^2|\vec{a}|^2 = 1 - (2|\vec{p}'|\,|\vec{T}|)^2(|\vec{T}|^2 + |\vec{p}'|^2)^{-2}, \qquad (660)$$

or substituting equation (448), then

$$1 - |\vec{p}'|^2|\vec{a}|^2 = (2\omega\varkappa)^2(|\vec{T}|^2 + |\vec{p}'|^2)^{-2}. \qquad (661)$$

Substituting equations (661) and (43) into equation (589), then

$$I_{b5} = 2(2\omega\varkappa)^{-2}\{4E^2\omega\varkappa^{-1} + 4E'^2\omega\varkappa^{-1} - 4(E^2 + E'^2)\varkappa^{-1}(E - \varkappa) - 4E^2 + 4|\vec{p}|^2E'^2\varkappa^{-2}$$

$$\cdot [1 - (E - \varkappa)^2|\vec{p}|^{-2}] + 4E'\omega^2\varkappa^{-1}\}. \qquad (662)$$

Simplifying equation (662), then

$$I_{b5} = 2\omega^{-2}\varkappa^{-2}(E^2\omega\varkappa^{-1} + E'^2\omega\varkappa^{-1} - E^3\varkappa^{-1} - E'^2E\varkappa^{-1} + |\vec{p}|^2E'^2\varkappa^{-2} - E'^2E^2\varkappa^{-2}$$

$$+ 2E'^2E\varkappa^{-1} + E'\omega^2\varkappa^{-1}). \qquad (663)$$

Substituting equation (116) into equation (663), then

$$I_{b5} = 2\omega^{-2}\varkappa^{-3}(E^2\omega + E'^2\omega + E'\omega^2 - E^3 + E'^2E - m^2E'^2\varkappa^{-1}). \qquad (664)$$

Using equation (130) in equation (664), then

$$I_{b5} = -2m^2E'^2\omega^{-2}x^{-4}. \tag{665}$$

Next substituting equations (594) and (660) into equation (590), then

$$I_{b6} = -4\omega^2E'x^{-1}(|\vec{T}|^2 + |\vec{p}'|^2)^{-2}[1 - (4\omega E'|\vec{T}|^2)^{-1}(|\vec{p}'|^2 + 2E'\omega - |\vec{T}|^2)(|\vec{T}|^2 + |\vec{p}'|^2)][1 - 4|\vec{p}'|^2|\vec{T}|^2 \cdot (|\vec{T}|^2 + |\vec{p}'|^2)^{-2}]^{-1} \tag{666}$$

or

$$I_{b6} = -\omega[x(|\vec{T}|^2 - |\vec{p}'|^2)^2|\vec{T}|^2]^{-1}(-|\vec{p}'|^4 + 2E'\omega|\vec{T}|^2 - 2E'\omega|\vec{p}'|^2 + |\vec{T}|^4). \tag{667}$$

Substituting equation (448) into equation (667), then

$$I_{b6} = [|\vec{p}'|^2(2x^2|\vec{T}|^2)^{-1}(|\vec{p}'|^2 + 2E'\omega) - E'\omega x^{-2} - |\vec{T}|^2(2x^2)^{-1}](2\omega x)^{-1}. \tag{668}$$

Using equation (451) in equation (668), then

$$I_{b6} = (2\omega x)^{-1}[|\vec{p}'|^2(2x^2|\vec{T}|^2)^{-1}(|\vec{p}'|^2 + 2E'\omega) - E'\omega x^{-2} - [|\vec{p}|^2 - 2\omega(E - x) + \omega^2](2x^2)^{-1}]. \tag{669}$$

Substituting equation (130) into the second term of equation (669), then

$$I_{b6} = (2\omega x)^{-1}[|\vec{p}'|^2(2x^2|\vec{T}|^2)^{-1}(|\vec{p}'|^2 + 2E'\omega) - \omega x^{-1} - (|\vec{p}|^2 - \omega^2)(2x^2)^{-1}] \tag{670}$$

or

$$I_{b6} = I_{b61} - (2x^2)^{-1} - (|\vec{p}|^2 - \omega^2)(4\omega x^3)^{-1}, \tag{671}$$

where

$$I_{b61} = |\vec{p}'|^2(|\vec{p}'|^2 + 2E'\omega)(4\omega\varkappa^3|\vec{T}|^2)^{-1}. \quad (672)$$

Using equations (448), (209), (130), and (116) in equation (672), then

$$I_{b61} = (|\vec{T}|^2 - 2\omega\varkappa)(|\vec{p}|^2 - \omega^2)(4\omega\varkappa^3|\vec{T}|^2)^{-1} \quad (673)$$

or

$$I_{b61} = -(|\vec{p}|^2 - \omega^2)(2\varkappa^2|\vec{T}|^2)^{-1} + (|\vec{p}|^2 - \omega^2)(4\omega\varkappa^3)^{-1}. \quad (674)$$

Substituting equation (674) into equation (671), then

$$I_{b6} = -(|\vec{p}|^2 - \omega^2)(2\varkappa^2|\vec{T}|^2)^{-1} - (2\varkappa^2)^{-1}. \quad (675)$$

Then using equations (609), (659), (665), and (675), one obtains

$$I_{b3} + I_{b4} + I_{b5} + I_{b6} = I_{b36}, \quad (676)$$

where

$$I_{b36} = -(|\vec{p}|^2 - \omega^2)(2\varkappa^2|\vec{T}|^2)^{-1} - (2\varkappa^2)^{-1} - m^2|\vec{p}|^{-2}\varkappa^{-2} + E|\vec{p}|^{-2}\varkappa^{-1} - 2m^2E'^2\omega^{-2}\varkappa^{-4}$$
$$+ I_{b361} + I_{b362} + I_{b363} + I_{b364} + I_{b365} + I_{b366}, \quad (677)$$

$$I_{b361} = -32m^2E^2(A - 2)[E'^2(|\vec{T}|^2 + |\vec{p}'|^2)^2D^2A]^{-1}, \quad (678)$$

$$I_{b362} = -64m^2E^2[E'^2(|\vec{T}|^2 + |\vec{p}'|^2)^2AD^2]^{-1}, \quad (679)$$

$$I_{b363} = -32m^2E^2B^2[E'^2(|\vec{T}|^2 + |\vec{p}'|^2)^2AD^4]^{-1}, \quad (680)$$

$$I_{b364} = -256m^2E^2C[E'^2(|\vec{T}|^2 + |\vec{p}'|^2)^2D^4]^{-1}, \quad (681)$$

$$I_{b365} = -8E'(8m^2EE' - 2\omega^2\varkappa^2)[\varkappa E'^2(|\vec{T}|^2 + |\vec{p}'|^2)^2D^2]^{-1}, \tag{682}$$

where equation (130) has been used, and

$$I_{b366} = -(2\omega\varkappa + 2|\vec{p}'|^2)(4|\vec{p}'|^2|\vec{p}|^2\varkappa^5E'\omega^2)^{-1}(4\omega m^2E^2E'\varkappa^2 + 2m^2\omega^3\varkappa^3 + 2|\vec{p}'|^2\omega^2E\varkappa^3$$
$$- 8\omega m^4EE'\varkappa - 4|\vec{p}'|^2m^2E^2E'\varkappa). \tag{683}$$

Substituting equations (304) and (449) into equation (678), then

$$I_{b361} = -32m^2E^2(m^2E'^{-2} - 2)E'^{-2}[16|\vec{p}'|^2E'^{-2}|\vec{p}|^2\varkappa^2m^2E'^{-2}]^{-1} \tag{684}$$

or

$$I_{b361} = -2E^2m^2[|\vec{p}'|^2|\vec{p}|^2\varkappa^2]^{-1} + 4E^2E'^2(|\vec{p}'|^2|\vec{p}|^2\varkappa^2)^{-1}. \tag{685}$$

Substituting equations (304) and (449) into equation (679),

$$I_{b362} = -4E^2E'^2(|\vec{p}'|^2|\vec{p}|^2\varkappa^2)^{-1}. \tag{686}$$

Substituting equations (304), (449), and (436) into equation (680),

$$I_{b363} = -2m^4E^2(|\vec{p}|^4\varkappa^4)^{-1} + 4m^2E^3(|\vec{p}|^4\varkappa^3)^{-1} - 2E^4(|\vec{p}|^4\varkappa^2)^{-1}. \tag{687}$$

Substituting equation (446) and (449) into equation (681) and using equations (43) and (213), then

$$I_{b364} = 4m^6E^2(|\vec{p}'|^2|\vec{p}|^4\varkappa^4)^{-1} - 8m^4E^3(|\vec{p}'|^2|\vec{p}|^4\varkappa^3)^{-1} + 4m^2E^4$$
$$\cdot (|\vec{p}'|^2|\vec{p}|^4\varkappa^2)^{-1} + 4m^2E^2E'^2(|\vec{p}'|^2|\vec{p}|^2\varkappa^4)^{-1} - 4m^2E'^2E^4$$
$$\cdot (|\vec{p}'|^2|\vec{p}|^4\varkappa^4)^{-1} + 8m^2E^3E'^2(|\vec{p}'|^2|\vec{p}|^4\varkappa^3)^{-1} - 4m^2E^2E'^2$$
$$\cdot (|\vec{p}'|^2|\vec{p}|^4\varkappa^2)^{-1}. \tag{688}$$

Next, substituting equation (449) into equation (682),

$$I_{b365} = -4m^2EE'^2(|\vec{p}'|^2|\vec{p}|^2\varkappa^3)^{-1} + E'\omega^2(|\vec{p}'|^2\,|\vec{p}|^2\varkappa)^{-1}. \tag{689}$$

Equation (683) is

$$\begin{aligned} I_{b366} = {} & -2m^2E^2(|\vec{p}'|^2|\vec{p}|^2\varkappa^2)^{-1} - m^2\omega^2(|\vec{p}'|^2|\vec{p}|^2E'\varkappa)^{-1} - E\omega(|\vec{p}|^2E'\varkappa)^{-1} \\ & + 4m^4E(|\vec{p}'|^2|\vec{p}|^2\varkappa^3)^{-1} + 2m^2E^2(|\vec{p}|^2\omega\varkappa^3)^{-1} - 2m^2E^2(|\vec{p}|^2\omega\varkappa^3)^{-1} \\ & - m^2\omega(|\vec{p}|^2E'\varkappa^2)^{-1} - |\vec{p}'|^2E(|\vec{p}|^2E'\varkappa^2)^{-1} + 4m^4E(|\vec{p}|^2\omega\varkappa^4)^{-1} \\ & + 2|\vec{p}'|^2m^2E^2(|\vec{p}|^2\omega^2\varkappa^4)^{-1}. \end{aligned} \tag{690}$$

Substituting equations (685) to (690) in equation (677) and collecting terms containing various powers of $\varkappa$, then

$$I_{b36} = I_{b\varkappa1}\varkappa^{-1} + I_{b\varkappa2}\varkappa^{-2} + I_{b\varkappa3}\varkappa^{-3} + I_{b\varkappa4}\varkappa^{-4}, \tag{691}$$

where

$$\begin{aligned} I_{b\varkappa1} = {} & E|\vec{p}|^{-2} - m^2\omega^2(|\vec{p}'|^2|\vec{p}|^2E')^{-1} - E\omega(|\vec{p}|^2E')^{-1} \\ & + E'\omega^2(|\vec{p}'|^2|\vec{p}|^2)^{-1}, \end{aligned} \tag{692}$$

$$\begin{aligned} I_{b\varkappa2} = {} & -|\vec{p}|^2(2|\vec{T}|^2)^{-1} + \omega^2(2|\vec{T}|^2)^{-1} - \tfrac{1}{2} - m^2|\vec{p}|^{-2} - 4E^2m^2 \\ & \cdot (|\vec{p}'|^2|\vec{p}|^2)^{-1} - 2E^4|\vec{p}|^{-4} + 4m^2E^4(|\vec{p}'|^2|\vec{p}|^4)^{-1} \\ & - 4m^2E^2E'^2(|\vec{p}'|^2|\vec{p}|^4)^{-1} - m^2\omega(|\vec{p}|^2E')^{-1} - |\vec{p}'|^2E(|\vec{p}|^2E')^{-1}, \end{aligned} \tag{693}$$

$$\begin{aligned} I_{b\varkappa3} = {} & 4m^2E^3|\vec{p}|^{-4} - 8m^4E^3(|\vec{p}'|^2|\vec{p}|^4)^{-1} + 8m^2E^3E'^2(|\vec{p}'|^2|\vec{p}|^4)^{-1} \\ & + 4m^4E(|\vec{p}'|^2|\vec{p}|^2)^{-1} + 2m^2E^2(|\vec{p}|^2\omega)^{-1} - 2m^2E^2(|\vec{p}|^2\omega)^{-1} \\ & - 4m^2EE'^2(|\vec{p}'|^2|\vec{p}|^2)^{-1}, \end{aligned} \tag{694}$$

$$\begin{aligned} I_{b\varkappa4} = {} & -2m^2E'^2\omega^{-2} - 2m^4E^2|\vec{p}|^{-4} + 4m^6E^2(|\vec{p}'|^2|\vec{p}|^4)^{-1} + 4m^2E^2E'^2(|\vec{p}'|^2|\vec{p}|^2)^{-1} \\ & - 4m^2E'^2E^4(|\vec{p}'|^2|\vec{p}|^4)^{-1} + 4m^4E(|\vec{p}|^2\omega)^{-1} + 2|\vec{p}'|^2m^2E^2(|\vec{p}|^2\omega^2)^{-1}. \end{aligned} \tag{695}$$

Rewriting equation (692),

$$I_{b\times 1} = (E|\vec{p}'|^2E' - m^2\omega^2 - |\vec{p}'|^2E\omega + E'^2\omega^2)(|\vec{p}'|^2|\vec{p}|^2E')^{-1} \tag{696}$$

or

$$I_{b\times 1} = [E|\vec{p}'|^2(E' - \omega) + \omega^2(E'^2 - m^2)](|\vec{p}'|^2|\vec{p}|^2E')^{-1}. \tag{697}$$

Using equations (130) and (209) in equation (697), then

$$I_{b\times 1} = E'|\vec{p}|^{-2}. \tag{698}$$

Rewriting equation (693),

$$I_{b\times 2} = -(|\vec{p}|^2 - \omega^2)(2|\vec{T}|^2)^{-1} + I_{b\times 21}, \tag{699}$$

where

$$I_{b\times 21} = -\tfrac{1}{2} - m^2|\vec{p}|^{-2} - 4E^2m^2|\vec{p}'|^{-2}|\vec{p}|^{-2} - 2E^4|\vec{p}|^{-4} + 4m^2E^4|\vec{p}'|^{-2}|\vec{p}|^{-4}$$
$$- 4m^2E^2E'^2|\vec{p}'|^{-2}|\vec{p}|^{-4} - m^2\omega|\vec{p}|^{-2}E'^{-1} - |\vec{p}'|^2E|\vec{p}|^{-2}E'^{-1}. \tag{700}$$

Using equations (130) and (209) in the last two terms of equation (700), equation (209) in the sixth term, and equation (116) in the fourth and fifth terms, then

$$I_{b\times 21} = -\tfrac{1}{2} - m^2|\vec{p}|^{-2} - 4E^2m^2|\vec{p}'|^{-2}|\vec{p}|^{-2} - 2(m^2 + |\vec{p}|^2)^2|\vec{p}|^{-4} + 4m^2E^2$$
$$\cdot (m^2 + |\vec{p}|^2)|\vec{p}'|^{-2}|\vec{p}|^{-4} - 4m^2E^2(|\vec{p}'|^2 + m^2)|\vec{p}'|^{-2}|\vec{p}|^{-4}$$
$$- EE'|\vec{p}|^{-2} + m^2|\vec{p}|^{-2} \tag{701}$$

or

$$I_{b\times 21} = -\tfrac{1}{2} - 4E^2m^2|\vec{p}'|^{-2}|\vec{p}|^{-2} - 2m^4|\vec{p}|^{-4} - 4m^2|\vec{p}|^{-2} - 2 + 4m^4E^2|\vec{p}'|^{-2}|\vec{p}|^{-4}$$
$$+ 4m^2E^2|\vec{p}'|^{-2}|\vec{p}|^{-2} - 4m^2E^2|\vec{p}|^{-4} - 4m^4E^2|\vec{p}'|^{-2}|\vec{p}|^{-4} - EE'|\vec{p}|^{-2} \tag{702}$$

or

$$I_{b\times21} = -(5/2)(1 + m^2|\vec{p}|^{-2}) - EE'|\vec{p}|^{-2} - (3/2)m^2|\vec{p}|^{-2} - 4m^2E^2|\vec{p}|^{-4} - 2m^4|\vec{p}|^{-4}. \quad (703)$$

Then using equation (116) in the first term of equation (703),

$$I_{b\times21} = -(5/2)E^2|\vec{p}|^{-2} - EE'|\vec{p}|^{-2} - (3/2)m^2|\vec{p}|^{-2} - 4m^2E^2|\vec{p}|^{-4} - 2m^4|\vec{p}|^{-4} \quad (704)$$

or

$$I_{b\times21} = -(5E^2 + 2E'E + 3m^2)(2|\vec{p}|^2)^{-1} - 2m^2(2E^2 + m^2)|\vec{p}|^{-4}. \quad (705)$$

Substituting equation (705) into equation (699), then

$$I_{b\times2} = -(½)(|\vec{p}|^2 - \omega^2)|\vec{T}|^{-2} - (2|\vec{p}|^2)^{-1}(5E^2 + 2EE' + 3m^2) - 2m^2|\vec{p}|^{-4}(2E^2 + m^2). \quad (706)$$

Using equation (209) in the second and third terms of equation (694), then

$$I_{b\times3} = 4m^2E^3|\vec{p}|^{-4} - 8m^2(E'^2 - |\vec{p}'|^2)E^3(|\vec{p}'|^2|\vec{p}|^4)^{-1} + 8m^2E^3(|\vec{p}'|^2 + m^2)$$
$$(|\vec{p}'|^2|\vec{p}|^4)^{-1} + 4m^4E(|\vec{p}|^2|\vec{p}|^2)^{-1} - 4m^2EE'^2(|\vec{p}'|^2|\vec{p}|^2)^{-1} \quad (707)$$

or

$$I_{b\times3} = 20m^2E^3|\vec{p}|^{-4} - 8m^2(E'^2 - m^2)E^3|\vec{p}'|^{-2}|\vec{p}|^{-4} + 4m^2E|\vec{p}'|^{-2}|\vec{p}|^{-2}(m^2 - E'^2). \quad (708)$$

Using equation (209) in the second and third terms, equation (708) becomes

$$I_{b\times3} = 20m^2E^3|\vec{p}|^{-4} - 8m^2E^3|\vec{p}|^{-4} - 4m^2E|\vec{p}|^{-2} \quad (709)$$

or

$$I_{b\times3} = 4m^2E|\vec{p}|^{-4}[2E^2 + (E^2 - |\vec{p}|^2)]. \quad (710)$$

Then using equation (116) in equation (710),

$$I_{b\kappa 3} = 4m^2E|\vec{p}|^{-4}(2E^2 + m^2). \tag{711}$$

Next, rewriting equation (695),

$$I_{b\kappa 4} = (\omega^2|\vec{p}|^4|\vec{p}'|^2)^{-1}F_{b\kappa 4}, \tag{712}$$

where

$$F_{b\kappa 4} = -2m^2E'^2|\vec{p}|^4|\vec{p}'|^2 - 2m^4E^2\omega^2|\vec{p}'|^2 + 4m^6E^2\omega^2 + 4m^2E^2E'^2|\vec{p}|^2\omega^2 - 4m^2\omega^2E'^2E^4$$
$$+ 4m^4\omega|\vec{p}'|^2E|\vec{p}|^2 + 2|\vec{p}'|^4m^2E^2|\vec{p}|^2. \tag{713}$$

Using equation (116) in the fifth term of equation (713), then

$$F_{b\kappa 4} = -2m^2E'^2|\vec{p}|^4|\vec{p}'|^2 - 2m^4E^2\omega^2|\vec{p}'|^2 + 4m^6E^2\omega^2 + 4m^2E^2E'^2|\vec{p}|^2\omega^2$$
$$- 4m^2\omega^2E'^2E^2(|\vec{p}|^2 + m^2) + 4m^4\omega|\vec{p}'|^2E|\vec{p}|^2 + 2|\vec{p}'|^4m^2E^2|\vec{p}|^2 \tag{714}$$

or

$$F_{b\kappa 4} = -2m^2E'^2|\vec{p}|^4|\vec{p}'|^2 - 2m^4E^2\omega^2|\vec{p}'|^2 + 4m^6E^2\omega^2 - 4m^4\omega^2E'^2E^2$$
$$+ 4m^4\omega E|\vec{p}'|^2|\vec{p}|^2 + 2|\vec{p}'|^4m^2E^2|\vec{p}|^2. \tag{715}$$

Using equation (116) in the second term, equation (209) in the fourth and sixth terms, and equation (130) in the fifth term, equation (715) becomes

$$F_{b\kappa 4} = -2m^2E'^2|\vec{p}|^4|\vec{p}'|^2 - 2m^4(|\vec{p}|^2 + m^2)\omega^2|\vec{p}'|^2 + 4m^6E^2\omega^2 - 4m^4\omega^2$$
$$\cdot (m^2 + |\vec{p}'|^2)E^2 + 4m^4(E - E')|\vec{p}'|^2E|\vec{p}|^2 + 2|\vec{p}'|^2(E'^2 - m^2)m^2E^2|\vec{p}|^2, \tag{716}$$

or simplifying and using equations (116) and (130), then

$$F_{bx4} = -2m^2E'^2|\vec{p}|^2(E^2 - m^2)|\vec{p}'|^2 - 2m^6\omega^2|\vec{p}'|^2 - 2m^4|\vec{p}|^2(E - E')^2|\vec{p}'|^2$$
$$- 4m^4\omega^2|\vec{p}'|^2E^2 + 4m^4E^2|\vec{p}'|^2|\vec{p}|^2 - 4m^4|\vec{p}'|^2EE'|\vec{p}|^2$$
$$+ 2m^2|\vec{p}'|^2E^2E'^2|\vec{p}|^2 - 2m^4|\vec{p}'|^2E^2|\vec{p}|^2. \quad (717)$$

Equation (717) reduces to

$$F_{bx4} = -2m^6\omega^2|\vec{p}'|^2 - 4m^4\omega^2|\vec{p}'|^2E^2. \quad (718)$$

Substituting equation (718) into equation (712), then

$$I_{bx4} = -2m^4|\vec{p}|^{-4}(m^2 + 2E^2). \quad (719)$$

One next recalls, substituting equation (691) into equation (676), that

$$I_{b3} + I_{b4} + I_{b5} + I_{b6} = I_{bx1}x^{-1} + I_{bx2}x^{-2} + I_{bx3}x^{-3} + I_{bx4}x^{-4}. \quad (720)$$

Then substituting equation (720) into equation (578),

$$I_b = I_{b1} + I_{b2} + I_{bx1}x^{-1} + I_{bx2}x^{-2} + I_{bx3}x^{-3} + I_{bx4}x^{-4}. \quad (721)$$

Next substituting equations (606), (608), (698), (706), (711), and (719) into equation (721), then

$$I_b = (4|\vec{p}'|\,|\vec{T}|)^{-1}\,\ell n\,[(|\vec{T}| + |\vec{p}'|)(|\vec{T}| - |\vec{p}'|)^{-1}][4m^2x^{-2}$$
$$- 6\omega x^{-1} - 2\omega(|\vec{p}|^2 - \omega^2)x^{-1}|\vec{T}|^{-2}] - x^{-1}|\vec{p}'|^{-1}\,\ell n[(E' + |\vec{p}'|)(E' - |\vec{p}'|)^{-1}]$$
$$+ E'x^{-1}|\vec{p}|^{-2} + x^{-2}[-(|\vec{p}|^2 - \omega^2)(2|\vec{T}|^2)^{-1} - (5E^2 + 2EE' + 3m^2)(2|\vec{p}|^2)^{-1}$$
$$- 2m^2(2E^2 + m^2)|\vec{p}|^{-4}] + 4m^2E(2E^2 + m^2)x^{-3}|\vec{p}|^{-4}$$
$$- 2m^4x^{-4}|\vec{p}|^{-4}(m^2 + 2E^2). \quad (722)$$

Collecting terms with the common factor $(2E^2 + m^2)$, equation (722) can be rewritten as

$$I_b = (2E^2 + m^2)|\vec{p}|^{-2}x^{-4}G_b - (5E^2 + 2EE' + 3m^2)(2|\vec{p}|^2x^2)^{-1} - (|\vec{p}|^2 - \omega^2)(2|\vec{T}|^2x^2)^{-1}$$
$$+ E'x^{-1}|\vec{p}|^{-2} + (4|\vec{p}'|\,|\vec{T}|)^{-1}\,\ell n\,[(|\vec{T}| + |\vec{p}'|)$$
$$\cdot (|\vec{T}| - |\vec{p}'|)^{-1}][4m^2x^{-2} - 6\omega x^{-1} - 2\omega(|\vec{p}|^2 - \omega^2)x^{-1}|\vec{T}|^{-2}]$$
$$- x^{-1}|\vec{p}'|^{-1}\,\ell n\,[(E' + |\vec{p}'|)(E' - |\vec{p}'|)^{-1}], \qquad (723)$$

where

$$G_b = -2m^2x^2|\vec{p}|^{-2} + 4m^2Ex|\vec{p}|^{-2} - 2m^4|\vec{p}|^{-2}. \qquad (724)$$

Equation (724) can be rewritten

$$G_b = 2m^2|\vec{p}|^{-2}(-x^2 + 2Ex - m^2). \qquad (725)$$

Substituting equation (43) into equation (725), then

$$G_b = 2m^2|\vec{p}|^{-2}[-(E - |\vec{p}|\cos\theta)^2 + 2E(E - |\vec{p}|\cos\theta) - m^2] \qquad (726)$$

or

$$G_b = 2m^2|\vec{p}|^{-2}(E^2 - m^2 - |\vec{p}|^2\cos^2\theta). \qquad (727)$$

Using equations (116) and (213) in equation (727), then

$$G_b = 2m^2\sin^2\theta. \qquad (728)$$

Substituting equation (728) into equation (723), then

$$I_b = (1/4)\{8m^2 \sin^2 \theta(2E^2 + m^2)|\vec{p}|^{-2}\varkappa^{-4} - 2(5E^2 + 2EE' + 3m^2)|\vec{p}|^{-2}\varkappa^{-2} - 2(|\vec{p}|^2 - \omega^2)$$
$$\cdot |\vec{T}|^{-2}\varkappa^{-2} + 4E'|\vec{p}|^{-2}\varkappa^{-1} + |\vec{p}'|^{-1}|\vec{T}|^{-1}$$
$$\cdot \ell n\, [(\vec{T} + |\vec{p}'|)(|\vec{T}| - |\vec{p}'|)^{-1}][4m^2\varkappa^{-2} - 6\omega\varkappa^{-1}$$
$$- 2\omega(|\vec{p}|^2 - \omega^2)|\vec{T}|^{-2}\varkappa^{-1}] - 4|\vec{p}'|^{-1}\varkappa^{-1}\, \ell n\, [(E' + |\vec{p}'|)(E' - |\vec{p}'|)^{-1}]\}. \qquad (729)$$

Finally, substituting equation (423) into equation (227),

$$d^3\sigma = Z^2\alpha r_e^2(2\pi)^{-1}|\vec{p}'|\, |\vec{p}'|^{-1}m^2\omega^{-1}\, d\omega\, d^2\Omega\, (I_a + I_b). \qquad (730)$$

Therefore, substituting equations (577) and (729) into equation (730) and using equations (160), (161), (334), (243), and (43), one obtains

$$d^3\sigma/d^2\Omega\, d\omega = (Z^2e^6/8\pi)(|\vec{p}'|/|\vec{p}|\omega)\{8m^2 \sin^2 \theta\, (2E^2 + m^2)/(|\vec{p}|^2\varkappa^4)$$
$$- 2(5E^2 + 2EE' + 3m^2)/(|\vec{p}|^2\varkappa^2) - 2(|\vec{p}|^2 - \omega^2)/(|\vec{T}|^2\varkappa^2)$$
$$+ 4E'/(|\vec{p}|^2\varkappa) + (L/|\vec{p}||\vec{p}'|)[4Em^2 \sin^2 \theta\, (3\omega m^2 - |\vec{p}|^2E')/(|\vec{p}|^2\varkappa^4)$$
$$+ [4E^2(E^2 + E'^2) - 2m^2(7E^2 - 3EE' + E'^2) + 2m^4]/(|\vec{p}|^2\varkappa^2)$$
$$+ 2\omega(E^2 + EE' - m^2)/(|\vec{p}|^2\varkappa)] + (\varepsilon_1/|\vec{p}'||\vec{T}|)[4m^2/\varkappa^2$$
$$- 6\omega/\varkappa - 2\omega(|\vec{p}|^2 - \omega^2)/(|\vec{T}|^2\varkappa)] - 4\varepsilon_2/(|\vec{p}'|\varkappa)\}. \qquad (731)$$

where

$$\varkappa = E - |\vec{p}|\cos \theta, \qquad (732)$$

$$\vec{T} = \vec{p} - \vec{k}, \qquad (733)$$

$$L = \ell n\, [(EE' - m^2 + |\vec{p}||\vec{p}'|)/(EE' - m^2 - |\vec{p}||\vec{p}'|)], \qquad (734)$$

$$\varepsilon_1 = \ell n\, [(|\vec{T}| + |\vec{p}'|)/(|\vec{T}| - |\vec{p}'|)], \qquad (735)$$

$$\varepsilon_2 = \ell n\, [(E' + |\vec{p}'|)/(E' - |\vec{p}'|)]. \qquad (736)$$

Equation (731) verifies the Gluckstern-Hull formula, equation (1), in agreement with equation (4.1) of Gluckstern and Hull's paper.[1] The differential cross-section equation (731) is that of electron-nucleus bremsstrahlung for photon emission into solid angle $d^2\Omega$ and frequency interval $d\omega/2\pi$. The angle θ is that between the direction of the emitted photon and that of the incident electron; e is the charge of the electron; -Ze is the charge of the scattering nucleus; $\vec{p}$ and $\vec{p}'$ are the initial and scattered electron momenta, respectively; E and E' are the initial and scattered electron energies, respectively; m is the mass of the electron; $\omega/2\pi$ is the frequency of the emitted photon; $\vec{k}$ is the momentum of the emitted photon; and $\vec{T}$ is the difference in momentum between the incident electron and the emitted photon. The Gluckstern-Hull formula is to be used in calculations of continuum x-ray spectra radiated by anisotropic relativistic beam plasma systems.

[1] *R. L. Gluckstern and M. H. Hull, Physical Review,* 90 *(1953), 1030.*

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