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A CONTINUUM THEORY OF EQUILIBRIUM OF IRREVERSIBLE PROCESSES IN --ETC(U)

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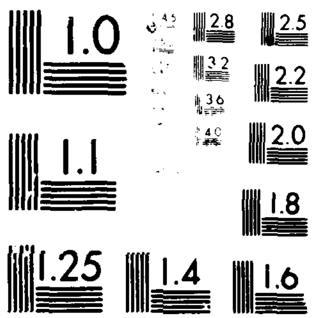

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A CONTINUUM THEORY OF EQUILIBRIUM OF IRREVERSIBLE PROCESSES IN SOLIDS

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PREPARED FOR

OFFICE OF NAVAL RESEARCH
300 N. QUINCY STREET
ARLINGTON, VIRGINIA 22217

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER	2. GOVT ACCESSION NO. AD-A085 800	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) A CONTINUUM THEORY OF EQUILIBRIUM OF IRREVERSIBLE PROCESSES IN SOLIDS		5. TYPE OF REPORT & PERIOD COVERED Final Report 15 Apr 76 - 31 Jan 79
		6. PERFORMING ORG. REPORT NUMBER A.R.A.P. Report No. 395
7. AUTHOR(s) Thomas B. McDonough		8. CONTRACT OR GRANT NUMBER(s) N00014-76-C-0809
9. PERFORMING ORGANIZATION NAME AND ADDRESS Aeronautical Research Associates of Princeton, Inc., 50 Washington Road, Princeton, New Jersey 08540		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research 800 N. Quincy Street Arlington, Virginia 22217		12. REPORT DATE January 1979
		13. NUMBER OF PAGES
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; Distribution Unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Constitutive Equations
Continuum
Coupled Thermal and Mechanical Behavior
Equilibrium Processes
Equilibrium States
Intrinsic Stability
Global Stability

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A theory is presented to describe the coupled thermal and mechanical behavior of solids which exhibit irreversibility. The developments presented are restricted to definitions and analyses of "equilibrium states." The foundations of the theory embody the following assumptions: (1) the local equilibrium state is fully characterized by three state variables - i.e. the deformation gradient, the entropy density, and a substate variable, (2) a caloric equation of state and stress and temperature relations

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) exist, (3) there is a bounded stable equilibrium region in state-space, (4) a stable equilibrium state satisfies a statement of static stability, and (5) an "equilibrium process" may be represented by a continuous path in the stable equilibrium region.

The meaning of "equilibrium state" is clarified by considering global (static) stability. Two types are defined and analyzed: Global Adiabatic Stability and Global Adiabatic Mechanical Stability with Fixed Boundary. An equivalence between the latter and the definition of equilibrium state is proven.

Several ideas for the further development of this theory are discussed in the concluding section.

Letter on file

A

PREFACE

The author wishes to thank Dr. Guido Sandri, A.R.A.P., for several enlightening discussions during the course of this work. His encouragement and penetrating questions and suggestions are appreciated very much.

TABLE OF CONTENTS

LIST OF SYMBOLS

b	body force
C^1	class one function - i.e., differentiable function
D	stable equilibrium region in state-space
∂D	boundary of D
$D^2_{(A^*-A)}\hat{e}(A)$	second directional derivative of e(A) for direction A*-A
DET	determinate
e	internal energy density
F	deformation gradient
GAMSFB	global adiabatic mechanical stability with fixed boundary
GAS	global adiabatic stability
h*	virtual supply of entropy
H	invertible tensor
I	the identity tensor
$L[A_2,A_1]$	straight line in A-space between A_1 and A_2
$N(A_1)$	neighborhood of A_1 in A-space
q	external supply of heating
Q	arbitrary orthogonal tensor
R	rotation tensor
s	an arbitrary scalar parameter
t	time
tr	trace
T	the Cauchy stress tensor
T_R	the first Piola-Kirchhoff stress tensor

U	right stretch tensor
x^i	current coordinates
X^K	material coordinates
α	substate variable
β	substate stretch tensor
δ	variation symbol, or a metric
ϵ	belongs to, or a small parameter
η	entropy density
θ	absolute temperature
κ	a reference configuration
λ	Lagrange multiplier
ρ	material mass density
ρ_R	material mass density in reference configuration
τ	substate tension
ψ	free-energy density (Helmholtz)
(F,η,β)	local state
$\{x,\eta,\beta\}$	global state of finite body
$[\partial^2 e]$	square matrix of second partial derivatives of the function $\hat{e}$
$\frac{d}{ds}(\ ),\ (\dot{\ })$	path derivative of $(\ )$
$(\ )_A$	partial derivative
$(\ )^T$	transpose of second-order tensor
$\{\ \}$	column matrix
$\{\ \}^T$	row matrix
$[\ ]$	square matrix
$\longrightarrow$	maps onto
$\longleftrightarrow$	one-to-one map
$\Rightarrow$	"implies"
$\Leftrightarrow$	"if and only if"

I. INTRODUCTION

The theory presented in this report is intended to represent the behavior of solids which exhibit irreversible straining - e.g. plastic strain. Irreversible straining of a material is a consequence of microstructural mechanisms, the operation of which may be understood but difficult to represent theoretically for a real material. It is presumed that for a continuum theory of thermal statics the effects of the microstructural deformation mechanisms can be represented by a continuous substate variable (or variables).

Only principles of continuum mechanics (including thermal effects) are invoked in the postulates and following analysis; no kinetic models of microstructural mechanisms are assumed. By this approach it is hoped that the results presented will have general utility - within the confines of the theoretical postulates.

At the foundation of the theory are the assumptions that the local equilibrium state of a material is completely characterized by three variables - the deformation gradient, the entropy, and the substate variable - and a caloric equation of state exists. Further, it is assumed that a continuous curve in equilibrium state-space can represent a process; it follows that such processes are independent of rate effects such as viscosity. Also the concept of approach to equilibrium states from nonequilibrium states, used frequently in related theoretical developments, has no place in the present theory. Other researchers have used substate variables in the development of continuum theories, but those theories all differ in some important aspects from the present one. Examples of such developments are represented by the papers of Kratochvil and Dillon(1) and Rice(2) who treat elastic-plastic behavior, and Coleman and Gurtin(3) who treat the thermodynamics of nonlinear material.

The theory has potential for the study of microstructural mechanisms which can be described in terms of metastable states. In the present theory one may interpret the phrase "equilibrium state" with the meaning usually associated with the phrase "metastable state," and an equilibrium process represents a process in a space of metastable states.

Although kinetic models of microstructural mechanisms have not been used in the development of the theory to date, a qualitative description of the mechanisms, which the substate variable may represent, is helpful for intuitive reasoning. Also a quantitative microstructural model (used in conjunction with the present theory) eventually may lead to the improved understanding of the behavior of particular materials. Perhaps the best known examples of such mechanisms in solids are the motion of dislocations in metals[1,2] and the stable growth of microcracks in ceramics.[4]

In Section II (Preliminaries) some concepts of continuum mechanics and mathematical analysis are presented briefly. The postulated theory, titled Definition of Stable Equilibrium, is presented in Section III (Precepts of Equilibrium Theory). The remainder of this section is devoted to further analysis of the caloric equation of state; no new postulates are introduced. In Section IV (Global Stability) two types of stability are defined: Global Adiabatic Stability, and Global Adiabatic Mechanical Stability with Fixed Boundary. Some theorems of dimensional invariance and a relationship between global stability and equilibrium state are proven. There are many more properties of the equilibrium region which can be deduced with no additional postulates. Some of these ideas are discussed in Section V (Concluding Remarks), but their development must be left for the future.

II. PRELIMINARIES

This section is but a brief discussion of a few concepts of continuum mechanics and mathematical analysis which are used in the following sections. Comprehensive presentations of the subjects can be found in the books of Truesdell and Toupin,[5] Truesdell and Noll,[6] Apostol,[7] and Zukerberg.[8]

Generally, direct notation is used throughout this report. A tensor (A_{ij}) is represented by A. The product of two tensors AB^T represents $A_{ik}B_{jk}$ and $\mathrm{tr}AB^T$ represents $A_{ij}B_{ij}$. Symbols $\hat{e}$, $\hat{\theta}$, $\hat{\tau}$, ... represent functions whose values are e, θ, τ.

The deformation gradient (F) is taken as the description of strain in this report. Let X (or X^K) denote material coordinates and x(or x^i) denote current coordinates of a body. In this description X represents a particle and x a position. In particular, let X represent the coordinates of the body in a reference configuration. A deformation of the body is completely described by

$$x^i = \hat{x}^i(X) \qquad \text{(II-1)}$$

and the deformation gradient is

$$F^i{}_L = \hat{F}^i{}_L(X) \equiv \frac{\partial}{\partial X^L}\hat{x}^i(X) \qquad \text{(II-2)}$$

It follows Eq. (II-2) that F transforms as a vector under a change of current coordinates and as a vector under a change of material coordinates. F may also be considered as a second-order tensor with the use of Euclidean shifters (g_k^L); e.g. the components of F relative to material coordinates are

$$F^K{}_L = g^K{}_i\, F^i{}_L \qquad \text{(II-3)}$$

F will be considered a second-order tensor, or equivalently as a linear operator on E^3, and generally the material (coordinate) description will be assumed in this report.

It follows the permanance of matter that

$$0 < \det F < \infty \tag{II-4}$$

and it follows Ineq. (II-4) that F is invertible.

The polar decomposition theorem insures that F has the following unique decomposition:

$$F = RU \tag{II-5}$$

where R (the rotation tensor) is proper orthogonal,

$$R^T R = I$$

$$\det R = +1 \tag{II-6}$$

and U (the right stretch tensor) is symmetric and positive definite.

Furthermore, under a change of reference frame (not a change of coordinates) F,R,U transform as follows

$$\begin{aligned} F' &= QF \\ R' &= QR \\ U' &= U \end{aligned} \tag{II-7}$$

where Q, an orthogonal tensor, characterizes the frame change. Under a change of reference configuration (again not a change of coordinates) F,R,U transform as follows

$$\begin{aligned} F' &= FH^{-1} \\ R' &= RH^T \\ U' &= H^{-1T} U H^{-1} \end{aligned} \tag{II-8}$$

where H , a second-order tensor, $0 < \det H < \infty$, represents the change of reference configuration.

Equations (II-7), (II-8) present precisely the meaning of the following: F and R transform as vectors under a change of reference frame and as vectors under a change of reference configuration; U transforms as a tensor under a change of reference configuration.

The principle of reference frame indifference states that constitutive equations must be indifferent to the reference frame. It is important to note that for constitutive equations, there is no principle of reference configuration indifference.

The measure of stress used in this report is the first Piola-Kirchhoff stress tensor (T_R), which is defined in terms of the Cauchy stress tensor (T) as follows:

$$T_R \equiv JTF^{-1^T} \qquad \text{(II-9)}$$

where J is the ratio of the current material density (ρ) to the material density in the reference configuration. The quantity $(1/\rho_R)(T_R)$ transforms as a vector under a change of reference frame and as a vector under a change of reference configuration.

Consider the set of all second-order tensors in 3-dimensions with elements A (or $A^K{}_L$). It follows that A-space is the space of linear operators defined on E^3 ; therefore A-space is a real vector space with vector product:

$$A_1,\ A_2 \in \text{A-space} \Rightarrow A_1\, A_2 \in \text{A-space}$$
$$\text{and}\quad A_1 \in \text{A-space} \Rightarrow A_1^T \in \text{A-space} \qquad \text{(II-10)}$$

Also define the inner product of A_1, $A_2 \in$ A-space as follows:

$$(A_1,\ A_2) \equiv (A_1)^K{}_L (A_2)_K{}^L \qquad \text{(II-11)}$$

It follows Eq. (II-11) that A-space is a real inner product space. Furthermore A-space is normed and metric by the Euclidian norm and distance functions:

$$\| A_1 \| \equiv (A_1, A_1)^{1/2}$$

$$d(A_1, A_2) \equiv \| A_1 - A_2 \| \qquad \text{(II-12)}$$

for $A_1, A_2 \in$ A-space. One may also show that A-space is closed under the usual matrix product:

$$A_1, A_2 \in \text{A-space} \Rightarrow A_1A_2, A_2A_1 \in \text{A-space} \qquad \text{(II-13)}$$

In other words A-space is a normed linear algebra.

Let $A_1, A_2 \in$ A-space such that

$$0 < \det A_1 < \infty$$

$$\| A_2 \| = 1 \quad \text{a unit vector} \qquad \text{(II-14)}$$

Consider a path in A-space defined as follows:

$$A(s) = A_1 + s A_2 \quad \text{for } s \in R \qquad \text{(II-15)}$$

It follows Eq. (II-14) that A_1 has an inverse in A-space and Eq. (II-15) that

$$A_1^{-1} A(s) = I + s A_1^{-1} A_2 \qquad \text{(II-16)}$$

and

$$\det A_1^{-1}A(s) = 1 + s \operatorname{tr} A_1^{-1}A_2 + s^2 \, II + s^3 \operatorname{Det}(A_1^{-1}A_2) \qquad \text{(II-17)}$$

where II is the second principal invariant of $A_1^{-1}A_2 \in$ A-space. The three invariants of $A_1^{-1}A_2$ are bounded and $0 < \det A_1^{-1} < \infty$; therefore it follows continuity of Eq. (II-17) that there is a neighborhood of $s = 0$, $N(0)$ such that

$$s \in N(0) \Rightarrow 0 < \det A(s) < \infty \qquad \text{(II-18)}$$

Eq. (II-18) was developed for an arbitrary unit vector $A_2 \in$ A-space; therefore it is true for all unit vectors in A-space. It follows

$$A_1 \in \text{A-space},\ 0 < \det A_1 < \infty$$

$\Rightarrow$ there is a neighborhood of A_1 in A-space, $N(A_1)$ such that

$$A \in N(A_1) \Rightarrow 0 < \det A < \infty \qquad \text{(II-19)}$$

Let A^+-space denote the set of all $A \in$ A-space such that $0 < \det A < \infty$. It follows Eq. (II-19) that

$$A^+\text{-space is open} \subset \text{A-space} \qquad \text{(II-20)}$$

Let $Q_1 \in A$ be proper orthogonal:

$$Q_1^T Q_1 = I$$

$$\det Q_1 = +1 \qquad \text{(II-21)}$$

and denote the set of all proper orthogonal tensors by Q^+-space. A proper orthogonal tensor may be interpreted as a rigid rotation of reference frames. It can be demonstrated that given any pair Q_1, $Q_2 \in Q^+$-space , there is a continuous path in A-space, $Q(s)$ on $0 \leq s \leq 1$, such that

$$Q(o) = Q_1,\ Q(1) = Q_2, \text{ and}$$

$$Q(s) \in Q^+\text{-space for } 0 \leq s \leq 1 \qquad \text{(II-22)}$$

Now consider any pair A_1, $A_2 \in A^+$-space. It follows that A_1 has an inverse in A-space and $A_2A_1^{-1} \in A^+$-space. Also for any $Q \in Q^+$-space, $Q\, A_2A_1^{-1} \in A^+$-space. It follows the polar decomposition theorem that there is a $Q_1 \in Q^+$-space such that

$Q_1A_2A_1^{-1} \varepsilon A^+$-space is symmetric and positive definite. Now consider the path in A-space given by the following:

$$\bar{A}(s) = (1-s)A_1 + s\, Q_1A_2 \qquad \text{on } 0 \leq s \leq 1 \tag{II-23}$$

It follows Eq. (II-23) that

$$\bar{A}(s)A_1^{-1} = (1-s)I + s\, Q_1A_2A_1^{-1} \qquad \text{on } 0 \leq s \leq 1 \tag{II-24}$$

and since $Q_1A_2A_1^{-1}$ is symmetric, it follows Eq. (II-24) that $\bar{A}(s)A_1^{-1}$ is symmetric on $0 \leq s \leq 1$. Also it follows Eq. (II-24) that

$$\begin{aligned} \det \bar{A}(s)A_1^{-1} = {} & (1-s)^3 + (1-s)^2 s \operatorname{tr}(Q_1A_2A_1^{-1}) \\ & + (1-s)s^2\, II + s^3 \det(Q_1A_2A_1^{-1}) \\ & \text{on } 0 \leq s \leq 1 \end{aligned} \tag{II-25}$$

where II is the second principal invariant of $Q_1A_2A_1^{-1}$. Because $Q_1A_2A_1^{-1}$ is symmetric positive definite the three principal invariants are strictly positive and it follows Eq. (II-25) that

$$0 < \det \bar{A}(s)A_1^{-1} < \infty \quad \text{on } 0 \leq s \leq 1 \tag{II-26}$$

By hypothesis $0 < \det A_1 < \infty$; therefore $0 < \det A_1^{-1} < \infty$ and it follows Eq. (II-26) that

$$0 < \det \bar{A}(s) < \infty \quad \text{on } 0 \leq s \leq 1 \tag{II-27}$$

In other words the straight path in A-space, represented by Eq. (II-23), is in A^+-space.

It follows Eq. (II-22) that a continuous path $Q(s) \varepsilon$ Q^+-space may be found such that

$$Q(s) \quad \text{on} \quad 0 \leq s \leq 1 \quad \varepsilon \; Q^+\text{-space}$$

$$Q(o) = 1 \; , \; Q(1) = Q_1^T \tag{II-28}$$

Now consider the continuous path in A-space defined by Eqs. (II-23), (II-28):

$$A(s) \equiv Q(s)\,\bar{A}(s) \quad \text{on} \quad 0 \leq s \leq 1$$

$$A(s) = (1-s)\; Q(s)\; A_1 + s\; Q(s) Q_1 A_2$$

$$A(o) = A_1 \qquad\qquad A(1) = A_2 \tag{II-29}$$

Also it follows in Eq. (II-27) and Eq. (II-29) that

$$0 \leq \det A(s) < \infty \quad \text{on } 0 \leq s \leq 1 \tag{II-30}$$

It follows that Eq.(II-29) represents a continuous path between A_1, A_2 which is in A^+-space. But A_1, A_2 were chosen arbitrarily in A^+-space; therefore between any two vectors in A^+-space a continuous path may be found which lies in A^+-space.

In other words A^+-space is a connected subset of A-space. It's convenient to summarize some of the properties:

A-space is the space of second-order tensors on E^3

A-space is an inner product space with matrix multiplication (II-31)

A^+-space $\equiv \{A | A \in$ A-space, $0 < \det A < \infty\}$

A^+-space is a domain (i.e. open connected subset) of A-space (II-32)

A^+-space is a metric space and is closed under matrix multiplication.

It will be useful to characterize further the properties of paths on A^+-space. Let $A_1, A_2 \in A^+$-space be symmetric and positive definite tensors and consider the straight path in A-space connecting them:

$$A(s) = (1-s)A_1 + s\,A_2 \quad \text{on} \quad 0 \leq s \leq 1$$
$$\text{where} \quad A_1, A_2 \in A^+\text{-space are symmetric, positive definite} \tag{II-33}$$

It follows Eq. (II-33) that $A(s) \in$ A-space and symmetric on $0 \leq s \leq 1$. Choose $s = s_1$ on $0 \leq s \leq 1$; since $A(s_1)$ is symmetric it follows that there is an orthonormal basis in E_3 relative to which the matrix $A(s_1)$ is diagonal. Relative to this basis, Eq. (II-33) has the following representation for $A(s_1)$:

$$(A(s_1))_{11} = (1-s_1)(A_1)_{11} + s_1(A_2)_{11}$$

$$(A(s_1))_{22} = (1-s_1)(A_1)_{22} + s_1(A_2)_{22}$$

$$(A(s_1))_{33} = (1-s_1)(A_1)_{33} + s_1(A_2)_{33}$$

$$(A(s_1))_{ij} = 0 \quad \text{for} \quad i \neq j \tag{II-34}$$

Since A_1, A_2 are symmetric positive definite tensors, their diagonal elements (relative to any orthonormal basis in E_3) are strictly positive. It follows Eq. (II-34) easily that

$$0 < A(s_1)_{11} < \infty$$

$$0 < A(s_1)_{22} < \infty$$

$$0 < A(s_1)_{33} < \infty \qquad \text{(II-35)}$$

or equivalently $A(s_1)$ is positive definite. Since s_1 was chosen arbitrarily, Eq. (II-35) applies on $0 \leq s \leq 1$. Hence the following has been proven:

$$\begin{array}{lcl} A_1,\ A_2\ \varepsilon\ A^+\text{-space} & & \\ \text{such that } A_1,\ A_2 & \Longrightarrow & A(s) \text{ is symmetric, positive definite} \\ \text{are symmetric,} & & \text{on } 0 \leq s \leq 1 \text{ where} \\ \text{positive definite} & & A(s) = (1-s)A_1 + s\ A_2 \\ & & \qquad \text{on } 0 \leq s \leq 1 \end{array} \qquad \text{(II-36)}$$

Let $A(s)$ represent a continuous path on A-space - i.e. $A(s)$ represents a function $R \longrightarrow$ A-space. The path derivative is represented by $\dot{A}(s)$:

$$\dot{A}(s) \equiv \frac{d}{ds} A(s) \qquad \text{(II-37)}$$

Let $A(s)$ be straight; it follows that $\dot{A}(s)\ \varepsilon$ A-space:

$$A(s) = (1-s)A_1 + s\ A_2\ ,\quad A_1,\ A_2\ \varepsilon \text{ A-space}$$

$$\dot{A}(s) = A_2 - A_1\ ,\quad A_2 - A_1\ \varepsilon \text{ A-space} \qquad \text{(II-38)}$$

Also note the meaning of tangent. A straight path $(1-s)A_1 + s\ A_2$ is tangent to a continuous path $A(s)$ at A_1 if and only if $A(0) = A_1$ and $\dot{A}(0) = A_2 - A_1$.

Straight paths in A-space will have frequent use; it is convenient to define a brief notation:

$$L\left[A_1,\ A_2\right] = \left\{A(s)\,|\,A(s) = A_1 + s\ A_2,\ s \ \varepsilon\ R,\right.$$
$$\left. A_1,\ A_2\ \varepsilon\ A\text{-space}\right\} \qquad \text{(II-39)}$$

consider a vector A_1 ε A^+-space, and a straight path $L(A_1,\ A_2)$ where A_2 ε A-space.

$$A(s) = A_1 + s\ A_2$$
$$A_1\ \varepsilon\ A^+\text{-space} \quad A_2\ \varepsilon\ A\text{-space} \qquad \text{(II-40)}$$

It follows Eq. (II-18) that

$$A(s)\ \varepsilon\ A^+\text{-space on } s\ \varepsilon\ N(0) \qquad \text{(II-41)}$$

Furthermore it follows the polar decomposition theorem that

$$A(s) = Q(s)\ S(s) \quad \text{on} \quad s\ \varepsilon\ N(0)$$

where $Q(s)$ is proper orthogonal
$S(s)$ is symmetric, positive definite (II-42)

It follows Eqs. (II-40), (II-42) that

$$A(0) = A_1 = Q(0)\ S(0)$$
$$\dot{A}(0) = A_2 = \dot{Q}(0)\ S(0) + Q(0)\ \dot{S}(0) \qquad \text{(II-43)}$$

Eq. (II-43)[2] has the following representation:

$$\dot{A}(0) = A_2 = \dot{Q}(0)\ Q^T(0)\ A_1 + A_1\ S^{-1}(0)\ \dot{S}(0)$$
$$\text{or} \quad \dot{Q}(0),\ \dot{S}(0) \xrightarrow{A_1} A_2 \qquad \text{(II-44)}$$

It follows Eqs. (II-39), (II-44) that a straight path through A_1 is uniquely defined by $\dot{Q}(o)$, $\dot{S}(o)$. Now it will be proven that the function represented by Eq. (II-44) is *one-to-one*.

Since $Q(s)$ is orthogonal, it follows that

$$Q^T(s)\,Q(s) = I \quad \text{on} \quad s \in \mathcal{N}(0) \tag{II-45}$$

Differentiating Eq. (II-45) and setting $s = 0$ gives the following for any smooth $Q(s)$:

$$Q^T(0)\dot{Q}(0) + \dot{Q}^T(0)Q(0) = 0$$

$$Q^T(0)\ddot{Q}(0) + \ddot{Q}^T(0)Q(0) = -2(Q^T(0)\dot{Q}(0))^T(Q^T(0)\dot{Q}(0)) \tag{II-46}$$

Also it follows Eq. (II-40) that $\ddot{A}(0) = 0$; therefore it follows Eq. (II-42) that

$$\ddot{Q}(0)\;S(0) + 2\,\dot{Q}(0)\dot{S}(0) + Q(0)\ddot{S}(0) = 0 \tag{II-47}$$

It follows Eq. (II-46)[1] that

$$C \equiv Q^T(0)\dot{Q}(0)$$

$$C = -\,C^T \tag{II-48}$$

Eq. (II-43)[2] has the following representation:

$$A^T(0)A_2S^{-1}(0) = C + \dot{S}(0)S^{-1}(0) \tag{II-49}$$

where C is skew-symmetric, $S(0)$ is symmetric and positive definite, and $\dot{S}(0)$ is symmetric.

Take the transpose of Eq.(II-49) and add to Eq. (II-49):

$$Q^T(0)A_2S^{-1}(0) + (Q^T(0)A_2S^{-1}(0))^T = \dot{S}(0)S^{-1}(0) + S^{-1}(0)\dot{S}(0) \tag{II-50}$$

Eq. (II-50) has a unique solution for $\dot{S}(0)$. To prove it, it is convenient to use matrix representation. Because

$S^{-1}(0)$ is symmetric positive definite, it follows that there is an orthonormal basis in E^3 such that $S^{-1}(0)$ is diagonal:

$$S^{-1}(0) = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}$$

$$\text{where} \quad a_1,\ a_2,\ a_3 > 0 \tag{II-51}$$

It follows that

$$\dot{S}(0)S^{-1}(0)+S^{-1}(0)\dot{S} = \begin{bmatrix} \dot{S}_{11}(2a_1) & \dot{S}_{12}(a_2+a_1) & \dot{S}_{13}(a_3+a_1) \\ & \dot{S}_{22}(2a_2) & \dot{S}_{23}(a_3+a_2) \\ & & \dot{S}_{33}(2a_3) \end{bmatrix} \tag{II-52}$$

The coefficients of the elements of $\dot{S}$ in Eq. (II-52) are strictly positive; therefore Eqs. (II-50) may be solved uniquely for $\dot{S}(0)$, i.e. it follows Eqs. (II-50), (II-52) that

$$A_2 \xrightarrow{A_1} \dot{S}(0) \tag{II-53}$$

It follows Eq. (II-43), (II-53) that

$$A_2 \xrightarrow{A_1} \dot{Q}(0),\ \dot{S}(0) \tag{II-54}$$

Finally, it follows Eqs. (II-44), (II-54) that

$$\dot{Q}(0),\ \dot{S}(0) \xleftrightarrow{A_1} A_2 \tag{II-55}$$

as was to be proved.

Consider a scalar valued function defined on A-space:

$$f = \hat{f}(A) \ , \quad A \in A\text{-space} \ , \quad f \in R \tag{II-56}$$

The gradient of f is a second-order tensor and the second gradient is a fourth-order tensor, which are represented as follows:

$$\hat{f}_A(A) \equiv \frac{\partial}{\partial A} \hat{f}(A)$$

where $f_A \in$ A-space

$$\hat{f}_{AA}(A) \equiv \frac{\partial}{\partial A} \frac{\partial}{\partial A} \hat{f}(A)$$

where f_{AA} is a linear operator on A-space (II-57)

The *first* and *second directional derivatives of* $f(A)$ evaluated at $A = A_1$ for the direction A_2 have the following representations:

$$D_{A_2} \hat{f}(A_1) = \mathrm{tr}\left\{\hat{f}^T_A(A_1)A_2\right\}$$

$$D^2_{A_2} \hat{f}(A_1) = (A_2)_{KL} \ \hat{f}_{A_{KL}A_{MN}}(A_1)(A_1)_{MN} \tag{II-58}$$

Let $f(A)$ be smooth on A-space and $A(s)$ be a smooth path on A-space. It follows Taylor's expansion that

$$\begin{aligned} f(s)-f(0) = {} & s\left(D_{\dot{A}(0)}\hat{f}(A(0))\right) \\ & + \frac{s^2}{2}\left(D_{\ddot{A}(0)}\hat{f}(A(0))+D^2_{\dot{A}(0)}\hat{f}(A(0))\right) \\ & + \cdots \text{ for } s \in N(0) \end{aligned} \tag{II-59}$$

It follows Eq. (II-59) that the path derivative $\dot{f}(0)$ equals the first directional derivative. Also if $A(s)$ is a straight path the second path derivative $\ddot{f}(0)$ equals the second directional derivative. More generally, it follows Eqs. (II-11), (II-58)[1], (II-59) that $\ddot{f}(0)$ equals the second directional derivative if, and only if, $\ddot{A}(0)$ is perpendicular to $\hat{f}_A(A(0))$.

Now consider all symmetric second-order tensors (B):

$$\text{B-space} \equiv \left\{A \mid A \in \text{A-space},\ A = A^T\right\} \tag{II-60}$$

It follows easily the structure of A-space that

$$\begin{aligned}&\text{B-space is a real inner product space}\\&\text{B-space is connected} \subset \text{A-space}\end{aligned} \tag{II-61}$$

Further define B^+-space:

$$B^+\text{-space} \equiv \left\{B \mid B \in \text{B-space},\ B \text{ is positive definite}\right\} \tag{II-62}$$

It follows Eqs. (II-36), (II-62) that

$$B^+\text{-space is connected} \subset \text{B-space} \tag{II-63}$$

Let $B_1 \in B^+$-space and B_2 represent a unit vector in B-space. Consider the straight path in B-space defined as follows:

$$B(0) = B_1 + s\, B_2, \quad 0 \leq s < \infty \tag{II-64}$$

Choose a number s_1 $(0 < s_1 < \infty)$. Since $B(s_1)$ is symmetric, an orthonormal basis in E^3 may be selected so that the matrix is diagonal. Relative to that basis, Eq. (II-64) gives the following:

$$B_{11}(s_1) = (B_1)_{11} + s_1(B_2)_{11}$$

$$B_{22}(s_1) = (B_1)_{22} + s_1(B_2)_{22}$$

$$B_{33}(s_1) = (B_1)_{33} + s_1(B_2)_{33} \tag{II-65}$$

Let b represent the minimum eigenvalue of B_1. Since B_1 is positive definite and B_2 is a unit vector, it follows that

$0 < b < \infty$, and

$$\left.\begin{array}{l} b \leq (B_1)_{11} < \infty \\ b \leq (B_1)_{22} < \infty \\ b \leq (B_1)_{33} < \infty \\ -s \leq s\ (B_2)_{11} \leq s \\ -s \leq s\ (B_2)_{22} \leq s \\ -s \leq s\ (B_2)_{33} \leq s \end{array}\right\} \text{relative to any orthonormal basis in } E^3 \tag{II-66}$$

It follows Eqs. (II-66) that a number $M > 0$ can be found such that

$$\left.\begin{array}{l} 0 < (B_1)_{11} + s\ (B_2)_{11} < \infty \\ 0 < (B_1)_{22} + s\ (B_2)_{22} < \infty \\ 0 < (B_1)_{33} + s\ (B_2)_{33} < \infty \end{array}\right\} \text{relative to any orthornormal basis in } E^3$$

$$\text{for } 0 \leq s < M \tag{II-67}$$

It follows Eqs. (II-67) and (II-65) that

$$B(s) \text{ is positive definite on } 0 \leq s < M \tag{II-68}$$

Since the unit vector B_2 was chosen arbitrarily in B-space, it follows that Eq. (II-68) holds for any unit vector in B-space. It follows that

$$B_1 \in B^+\text{-space} \Longleftrightarrow \begin{array}{l}\text{there is a neighborhood of } B_1 \text{ in} \\ \text{B-space, } N(B_1)\text{ , such that} \\ N(B_1) \subset B^+\text{-space}\end{array} \tag{II-69}$$

In other words

$$B^+\text{-space is open} \subset \text{B-space} \tag{II-70}$$

It is convenient to summarize the properties of the two spaces as follows:

$$\begin{array}{l}\text{B-space is the space of symmetric} \\ \quad\text{second order tensors on } E^3 \\ \text{B-space is a real inner product} \\ \quad\text{space} \\ \text{B-space is connected} \subset \text{A-space}\end{array} \tag{II-71}$$

$$\begin{array}{l}B^+\text{-space} \equiv \left\{B \,|\, B \in \text{B-space, } B \text{ is positive definite}\right\} \\ B^+\text{-space is a domain of B-space} \\ B^+\text{-space is a metric space} \\ B^+\text{-space is connected} \subset \text{A-space}\end{array} \tag{II-72}$$

Now consider the triplet (A, a, B). Define the associated space as follows:

$$(A,\ a,\ B)\text{-space} \equiv \left\{(A, a, B) \,|\, A \in \text{A-space},\ a \in R,\ B \in \text{B-space}\right\} \tag{II-73}$$

Retain all of the operations defined on the three subspaces; e.g. the inner product is

$$(A_1, a_1, B_1) \cdot (A_2, a_2, B_2) = (A_1, A_2) + a_1 a_2 + (B_1, B_2) \qquad \text{(II-74)}$$

and the norm and metric follow easily. It follows that

$$(A, a, B)\text{-space is a real inner product space} \qquad \text{(II-75)}$$

Now define $(A, a, B)^+$-space:

$$(A, a, B)^+\text{-space} \equiv \left\{(A, a, B) \,|\, A \in A^+\text{-space},\ 0 < a < \infty,\ B \in B^+\text{-space}\right\} \qquad \text{(II-76)}$$

It follows easily the properties of the three subspaces that

$$(A, a, B)^+\text{-space is a domain of } (A, a, B)\text{-space, and } (A, a, B)^+\text{-space is a metric space} \qquad \text{(II-77)}$$

Also consider the space defined as follows:

$$(A, a, A)\text{-space} \equiv \left\{(A_1, a, A_2) \,|\, A_1, A_2 \in A\text{-space},\ a \in R\right\} \qquad \text{(II-78)}$$

It is clear that

$$(A, a, A)\text{-space is a real inner product space} \qquad \text{(II-79)}$$

Also

$$(A, a, A)^+\text{-space} \equiv \left\{(A_1, a, A_2) \,|\, A_1, A_2 \in A^+\text{-space},\ 0 < a < \infty\right\} \qquad \text{(II-80)}$$

Then

$$(A, a, A)^+\text{-space is a domain of } (A, a, A)\text{-space}$$
$$(A, a, A)^+\text{-space is a metric space.} \qquad \text{(II-81)}$$

Finally note the following:

$(A, a, B)^+$-space is a connected subset of

$$(A, a, A)^+\text{-space} , \qquad \text{(II-82)}$$

and the mapping from $(A, a, A)^+$-space onto $(A, a, B)^+$-space defined as follows,

$$(A_1, a_1, A_2) \longrightarrow (A_1, a_1, Q_2 A_2)$$

where $Q_2 A_2$ is symmetric, positive definite (II-83)

is open - i.e. any neighborhood of $(A, a, A)^+$-space maps onto a neighborhood of $(A, a, B)^+$-space.

III. PRECEPTS OF EQUILIBRIUM THEORY

Experimental experience indicates that many solids exhibit persistence for restricted ranges of loading. Such phenomena may be described theoretically by criteria of mathematical stability - stability corresponding to persistence and loss of stability corresponding to uncontrolled spontaneous processes. The following theory of equilibrium of solids puts these ideas into precise mathematical statements. Stability is posed in terms of energy considerations; the interaction of both thermal and mechanical sources is included. The following principles of continuum mechanics are the foundations of this theory:

1. Balance of linear momentum, moment of momentum, and energy.
2. The principle of reference frame indifference.
3. The principle of local action.

There is a definition of simple material in the foundations of mechanics for purely mechanical theories. That definition is generalized here to include thermal energy.

Let q represent the external supply of heat (q is a mass density). Then a thermomechanical loading is represented by the pair $(F(t), q(t))$, where $F(t)$ and $q(t)$ are functions of time (t).

<u>Definition of Simple Thermomechanical Material</u>

A simple thermomechanical material is a material whose stress is uniquely determined by the history of its thermomechanical loading:

$$(F(s), q(s)) \text{ for } -\infty < s \leq t \longrightarrow T_R(t) \qquad \text{(III-1)}$$

The idea of "persistent" will be given the following meaning.

Definition of Persistent

Let $F(t_1)$, $T_R(t_1)$ represent the values of the deformation gradient and the stress for $t = t_1$. The pair, $(F(t_1), T_R(t_1))$ is persistent, means there is a thermomechanical loading $(F(t), q(t))$ such that

$$
\begin{aligned}
&(F(t),q(t)) = (F_1(t_1),0) \\
&\text{and} \qquad\qquad\qquad\qquad \text{for } t \geq t_1 \\
&T_R(t) = T_R(t_1)
\end{aligned}
\tag{III-2}
$$

In this report the term persistent will be used to describe experimental observations; the phrase "equilibrium state," which has a similar meaning, will be used in the theory.

A local state, or simply state, is characterized by the values of the deformation gradient (F), the entropy density (η), and the substate (α). Inherent in these measures is a reference configuration from which F is measured and relative to which densities are measured; all densities are measured per unit mass in the reference configuration. The substate variable (α) is a parameter which represents the microstructure. Three additional state variables are assumed primitive concepts: The internal energy density (e), the stress (T_R), and the temperature (θ); e, T_R, θ are ascribed their usual properties. Relative to a reference configuration a state is represented by the triplet (F,η,α). Furthermore, it is assumed that if the triplet (F,η,α) represents a stable state, then it uniquely defines values of all state variables, i.e.

$$(F,\eta,\alpha) \longrightarrow e,T_R,\theta \tag{III-3}$$

So far the substate has been represented vaguely by α. Such a representation is too vague to be of predictive value.

Microstructural changes are represented in the theory of plasticity by plastic strain. A similar measure is used in the present theory - substate stretch tensor.

Definition of the Substate Stretch Tensor (β)

The substate stretch tensor (β) is a symmetric, positive definite second-order tensor which transforms as a second-order tensor under a change of reference configuration:

$$\beta' = H^{-1\,T}\,\beta\,H^{-1} \qquad \text{(III-4)}$$

where H represents the change of reference configuration.

The following postulate makes clear that the theory that follows is restricted to substate processes which may be represented by the substate stretch tensor.

Postulate I

There are substate processes which may be characterized *completely by the effects on the substate stretch*, i.e. there is a map

$$(F,\ \eta,\ \alpha) \longrightarrow (F,\ \eta,\ \beta) \qquad \text{(III-5)}$$

such that if

$$(F_1,\ \eta_1,\ \alpha_1) \longrightarrow e_1,\ T_{R_1},\ \theta_1$$

and

$$(F_1,\ \eta_1,\ \alpha_2) \longrightarrow e_1,\ T_{R_1},\ \theta_1$$

then

$$(F_1,\ \eta_1,\ \alpha_1) \longrightarrow (F_1,\ \eta_1,\ \beta_1)$$
$$(F_1,\ \eta_1,\ \alpha_2) \longrightarrow (F_1,\ \eta_1,\ \beta_1) \qquad \text{(III-6)}$$

It follows that if the triplet (F, η, β) represents a stable state, then there is a function such that

$$(F,\ \eta,\ \beta) \longrightarrow e,\ T_R,\ \theta \qquad \text{(III-7)}$$

Note that, for any proper orthogonal Q, the matrix product $Q\beta$ represents an element in A^+-space (defined in Section II):

$$Q\beta \in A^+\text{-space} \tag{III-8}$$

where Q is proper orthogonal.

Definition of (F,η,β)-space

$$(F,\eta,\beta)\text{-space} \equiv \left\{ (F,\eta,Q\beta) \;\; \begin{array}{l} F \in A^+\text{-space}, \\ 0 < \eta < \infty, \\ \beta \in B^+\text{-space}, \\ Q \text{ is proper orthogonal} \end{array} \right\} \tag{III-9}$$

It follows that (F,η,β)-space is exactly the $(A,a,A)^+$-space described in Section II. The symbol β will always represent the substate stretch tensor; hence it is always symmetric, positive definite. Note in Eq. (III-9) that (F,η,β)-space is isometric to E^{19}. In most of what follows the values of the triplet $(F,\eta,Q\beta)$ will be restricted to the values of the triplet (F,η,β), which defines a connected subset of $F,\eta,\beta)$-space. In other words, even though β is symmetric, its dimension is taken as nine. Also, when convenient, an orthogonal tensor may be introduced into the notation, i.e. a point in (F,η,β)-space may be represented by $(F,\eta,Q\beta)$.

Next follows a definition of stable equilibrium, upon which the analyses which follow rests.

Definition of Stable Equilibrium (S.E.)

There is an open region of (F,η,β)-space (D) for which a triplet $(F,\eta,\beta) \in D$ if and only if:

(a.) There are functions defined on D such that

$$e = \hat{e}(F, \eta, \beta)$$

$$T_R = \hat{T}_R(F, \eta, \beta)$$

$$\theta = \hat{\theta}(F, \eta, \beta) \qquad \text{(III-10)}$$

and

$$\hat{e}(F, \eta, \beta) \in C^2, \; \eta > 0, \; \theta > 0 \qquad \text{(III-11)}$$

(b.) $(F_1, \eta_1, \beta_1), \; (F_1, \eta_2, \beta_1) \in D \Leftrightarrow L\left[(F_1, \eta_1, \beta_1), (F_1, \eta_2, \beta_1)\right] \in D$ (III-12)

(c.) $(F_1, \eta_1, \beta_1) \in D \Rightarrow$ there is a $(F_2, \eta_1, \beta_1) \in D$ such that $\hat{T}_R(F_2, \eta_1, \beta_1) = 0$ (III-13)

(d.) The isotropy group of $\hat{e}(F, \eta, \beta)$ is a proper subset of the unimodular group. (III-14)

(e.) Reference frame indifference requires, for any proper orthogonal Q_1, Q_2, the following:

$$(F_1, \eta_1, \beta_1) \in D \Rightarrow (Q_1F_1, \eta_1, Q_2\beta_1) \in D,$$
$$\hat{e}(F_1, \eta_1, \beta_1) = \hat{e}(Q_1F_1, \eta_1, Q_2\beta_1)$$

(III-15)

(f.) Consider a spherical neighborhood of 0 in E^3 $(N_R(0))$. Then $(F_1\eta_1\beta_1) \in D$ only if there is a neighborhood of the function $F_1X(N(F_1))$, a neighborhood of $\eta_1(N(\eta_1))$, and a neighborhood of $\beta_1(N(\beta_1))$, such that for all homeomorphisms (on $N_R(0)$) $f(X) \in N(F_1)$, all $\eta(X) \in N(\eta_1)$, and all $\beta(X) \in N(\beta_1)$

$$\int_{N_R(0)} \hat{e}(F(X), \eta(X), \beta(X)) \, dV > \hat{e}(F_1, \eta_1, \beta_1)V \qquad \text{(III-16)}$$

where (i) $f(X) = F_1 X$ on $\partial N_R(0)$, and
$F(X) = \nabla f(X)$ on $N_R(0)$

(ii) $$\int_{N_R(0)} \hat{e}_\eta(F_1, \eta_1, \beta_1)(\eta(X) - \eta_1) + \mathrm{tr}\ \hat{e}_\beta^T(F_1, \eta_1, \beta_1)(\beta(X) - \beta_1) \, dV = 0$$

To proceed with analysis, a means of comparing values of state variables on D is required. To distinguish between paths and processes (which will be defined subsequently) the following definition is made.

Definition of an Equilibrium Path

An equilibrium path is any continuous function defined on *an interval* $[a,b]$ *of* R^1*-space with values in* D - i.e.,

$$F = F(s)$$
$$\eta = \eta(s)$$
$$Q\beta = Q(s)\beta(s) \qquad a \le s \le b \qquad \text{(III-17)}$$

- and the state variables all assume their equilibrium values at each point on the path, e.g., $e = \hat{e}(F(s), \eta(s), Q(s)\beta(s))$. Such paths are not restricted to realistic processes - e.g., an equilibrium path may be discussed even if it violates basic principles of irreversible thermodynamics.

A path derivative is represented by a "dot" - e.g., $\dot{F}(s) = (\partial/\partial s)F(s)$ - and, since the analysis in this report is restricted to the "material description," a path derivative is equivalent to the "material derivative" - i.e., following a particle of material.

Theorem 1

$$\left.\begin{aligned}
&\frac{1}{\rho_R}\,\hat{T}_R\,(F,\eta,\beta) = \hat{e}_F(F,\eta,\beta)\\
&\hat{\theta}(F,\eta,\beta) = \hat{e}_\eta(F,\eta,\beta)\\
&F\,\hat{e}_F^{\,T}(F,\eta,\beta)\ \text{is symmetric}\\
&\beta\,\hat{e}_\beta^{\,T}(F,\eta,\beta)\ \text{is symmetric}
\end{aligned}\right\}\quad \text{on } D \tag{III-18}$$

Proof

Consider an equilibrium path through a state (F_1,η_1,β_1) with velocities $(\dot{F},\ \dot{\eta},\ \dot{\beta})$. It follows Eq. (III-10)[1] that

$$\dot{e} = \operatorname{tr}\,\hat{e}_F^{\,T}(F_1,\eta_1,\beta_1)\dot{F} + \hat{e}_\eta(F_1,\eta_1,\beta_1)\dot{\eta} + \operatorname{tr}\,\hat{e}_\beta^{\,T}(F_1,\eta_1,\beta_1)\dot{\beta} \tag{III-19}$$

It follows the foundations of mechanics that $\dot{e}$ may be represented in terms of external energy flux:

$$\dot{e} = \operatorname{tr}\left(\frac{1}{\rho_R}\,T_R^{\,T}\,\dot{F}\right) + q \tag{III-20}$$

where $\operatorname{tr}\left(\frac{1}{\rho_R}\,T_R^{\,T}\,\dot{F}\right)$ is the mechanical working and q represents the heating (addition).

Since D is a region, Eq. (III-19) holds for independent $\dot{F}, \dot{\eta}$, and $\dot{\beta}$. Also Eq. (III-20) holds for independent $\dot{F}$ and q. It follows Eqs. (III-19), (III-20) that

$$0 = \operatorname{tr}\left(\frac{1}{\rho_R}\,T_R^{\,T} - \hat{e}_F^{\,T}(F_1,\eta_1,\beta_1)\right)\dot{F} + q - \left(\hat{e}_\eta(F_1,\eta_1,\beta_1)\dot{\eta} + \operatorname{tr}\,\hat{e}_\beta^{\,T}(F_1,\eta_1,\beta_1)\dot{\beta}\right) \tag{III-21}$$

Choose $\dot{F} \neq 0$, $q = 0$, $\dot{\eta} = 0$, $\dot{\beta} = 0$, it follows Eq. (III-21) that

$$0 = \mathrm{tr}\left(\frac{1}{\rho_R} T_R{}^T - \hat{e}_F{}^T(F_1, \eta_1, \beta_1)\right)\dot{F} \qquad \text{(III-22)}$$

must hold for all $\dot{F}$ in the space of second-order tensors. But $\frac{1}{\rho_R} T_R - \hat{e}_F(F_1, \eta_1, \beta_1)$ is a second-order tensor. The only second-order tensor which is orthogonal to all $\dot{F}$ is the zero tensor, i.e.

$$\frac{1}{\rho_R} T_R = \hat{e}_F(F_1, \eta_1, \beta_1) \qquad \text{(III-23)}$$

It follows Eq. (III-21), (III-23) that

$$q = \hat{e}_\eta(F_1, \eta_1, \beta_1)\dot{\eta} + \mathrm{tr}\ \hat{e}_\beta^T(F_1, \eta_1, \beta_1)\dot{\beta} \qquad \text{(III-24)}$$

Now consider $\dot{\beta} = 0$; it follows Eq. (III-24) that

$$\dot{\beta} = 0 \implies q = \hat{e}_\eta(F_1, \eta_1, \beta_1)\dot{\eta} \qquad \text{(III-25)}$$

It follows classical mechanics that for β fixed, the temperature (θ) may be defined from the following equation:

$$q = \theta\ \dot{\eta} \qquad \text{(III-26)}$$

It follows Eqs. (III-25), (III-26) that

$$\theta = \hat{e}_\eta(F_1, \eta_1, \beta_1) \qquad \text{(III-27)}$$

Let $Q(s)$ represent a continuous path in F-space such that $Q(s)$ is proper orthogonal and $Q(0) = I$. It follows Eq. (III-15) that

$$\hat{e}(Q(s)F_1, \eta_1, \beta_1) = \hat{e}(F_1, \eta_1, \beta_1) \qquad \text{(III-28)}$$

Take the path derivative and evaluate for $s = 0$:

$$\operatorname{tr} F_1 \, \hat{e}^T(F_1, \eta_1, \beta_1)\dot{Q}(0) = 0 \qquad \text{(III-29)}$$

Eq. (III-29) indicates that the tensor $\hat{e}_F(F_1, \eta_1, \beta_1)F_1^T$ is orthogonal to $\dot{Q}(0)$. It's easy to show that $\dot{Q}(0)$ is a skew-symmetric tensor. Furthermore any skew-symmetric tensor corresponds to a path $Q(s)$. It follows Eq. (III-29) that $\hat{e}_F(F_1, \eta_1, \beta_1)F_1^T$ is orthogonal to any and all skew-symmetric tensors; hence

$$\hat{e}_F(F_1, \eta_1, \beta_1)\, F_1^T \text{ is symmetric} \qquad \text{(III-30)}$$

Consider the same path $Q(s)$ and it follows Eq. (III-15) that

$$\hat{e}(F_1, \eta_1, Q(s)\beta_1) = \hat{e}(F_1, \eta_1, \beta_1) \qquad \text{(III-31)}$$

Again take the path derivative and evaluate at $s = 0$.

$$\operatorname{tr}\beta_1 \hat{e}_\beta{}^T(F_1, \eta_1, \beta_1)\, \dot{Q}(0) = 0 \qquad \text{(III-32)}$$

Since $\dot{Q}(0)$ covers the space of skew-symmetric tensors, it follows Eq. (III-32) that

$$\hat{e}_\beta(F_1, \eta_1, \beta_1)\, \beta_1^T \text{ is symmetric} \qquad \text{(III-33)}$$

Equation $(III\text{-}18)^1$ follows Eqs. (III-23), $(III\text{-}10)^2$;
Eq. $(III\text{-}18)^2$ follows Eqs. (III-27), $(III\text{-}10)^3$;
Eqs. $(III\text{-}18)^{3,4}$ follow Eqs. (III-30), (III-33).

Q.E.D.

Definition of Substate Tension (τ)

The substate tension tensor (τ) is defined on D by the following:

$$\tau = \hat{\tau}(F,\eta,Q\beta) \equiv \hat{e}_{Q\beta}(F,\eta, Q\beta) \tag{III-34}$$

When $Q\beta$ is restricted to β the following notation will be used.

$$\hat{\tau}(F,\eta, \beta) = \hat{e}_{\beta}(F,\eta, \beta) \tag{III-35}$$

Definition of Adiabatic and Isothermal Paths

Let $(F_1,\eta_1,\beta_1) \in D$ and let $(F(s),\eta(s),\beta(s))$ represent a continuous path in D such that $(F(0),\eta(0),\beta(0)) = (F_1,\eta_1,\beta_1)$. The path is adiabatic if and only if

$$\hat{\theta}(F(s),\eta(s),\beta(s))\dot{\eta}(s)+\mathrm{tr}\,\hat{\tau}^{T}(F(s),\eta(s),\beta(s))\dot{\beta}(s) = 0 \tag{III-36}$$

The path is isothermal if and only if

$$\mathrm{tr}\ \hat{\theta}_F^{T}(F(s),\eta(s),\beta(s))\dot{F}(s) + \hat{\theta}_{\eta}(F(s),\eta(s),\beta(s))\dot{\eta}(s) + \mathrm{tr}\hat{\theta}_{\beta}^{T}(F(s),\eta(s),\beta(s))\dot{\beta}(s) = 0 \tag{III-37}$$

It is convenient to introduce a simple notation. A path $(F(s),\eta(s),\beta(s))$ that is adiabatic will be represented as $(F(s),\eta_A(s),\beta(s))$ and one that is isothermal will be represented as $(F(s),\eta_I(s),\beta(s))$.

Theorem 2

Consider a smooth path $(F(s),\eta(s),\beta(s))$ through the state (F_1,η_1,β_1), i.e. $(F(0),\eta(0),\beta(0)) = (F_1,\eta_1,\beta_1)$.

Then for any smooth $F(s), \beta(s)$ there is a unique $\eta_A(s)$ such that

$$\dot{\eta}_A(s) = - \frac{1}{\hat{\theta}(F(s), \eta_A(s), \beta(s))} \operatorname{tr} \hat{\tau}^T(F(s), \eta_A(s), \beta(s)) \dot{\beta}(s) \tag{III-38}$$

<u>Proof</u>

Choose two smooth functions $F(s)$ and $\beta(s)$. Since θ is invertible Eq., (III-38) follows Eq. (III-36).

Now suppose there is a second function $\eta_1(s)$ which satisfies Eq. (III-38). Let

$$\begin{aligned} \theta_1(s) &\equiv \hat{\theta}(F(s), \eta_1(s), \beta(s)) \\ \tau_1(s) &\equiv \hat{\tau}(F(s), \eta_1(s), \beta(s)) \\ \theta_A(s) &\equiv \hat{\theta}(F(s), \eta_A(s), \beta(s)) \\ \tau_A(s) &\equiv \hat{\tau}(F(s), \eta_A(s), \beta(s)) \end{aligned} \tag{III-39}$$

It follows Eq. (III-38) for both $\eta_1(s)$ and $\eta_A(s)$ that

$$0 = \theta_1(s)\dot{\eta}_1(s) + \tau_1(s)\dot{\beta}(s) = \theta_A(s)\dot{\eta}_A(s) + \tau_A(s)\dot{\beta}(s) \tag{III-40}$$

By definition both paths pass through η_1 :

$$\eta_1(0) = \eta_A(o) = \eta_1 \tag{III-41}$$

It follows Eqs. (III-41), (III-39) that

$$\begin{aligned} \theta_1 &= \theta_1(0) = \theta_A(0) \\ \tau_1 &= \tau_1(0) = \tau_A(0) \end{aligned} \tag{III-42}$$

Evaluate Eq. (III-40) at $s = 0$ and it follows Eq. (III-42)[1] that

$$\dot{\eta}_1(0) = \dot{\eta}_A(0) \qquad \text{(III-43)}$$

Differentiate Eqs. (III-39) and evaluate at $s = 0$ and it follows Eqs. (III-41), (III-42), (III-43) that

$$\ddot{\eta}_1(0) = \ddot{\eta}_A(0) \qquad \text{(III-44)}$$

This process of differentiation may be continued to prove that the n^{th} derivatives are equal:

$$\overset{n}{\eta}_1(0) = \overset{n}{\eta}_A(0) \qquad \text{(III-45)}$$

If $\hat{e}(F,\eta,\beta)$ is smooth in (F,η,β), it follows Eq. (III-38) that $\eta_A(s)$ and $\eta_1(s)$ are analytic on s. Therefore it follows Eq. (III-45) that

$$\eta_1(s) = \eta_A(s) \qquad \text{(III-46)}$$

which proves the uniqueness.

Q.E.D.

<u>Theorem 3</u>

$(F_1,\eta_1,\beta_1) \in D$ only if:

(a.) there is a neighborhood of $\beta_1, N(\beta_1)$, such that for all $\beta_2 \in N(\beta_1)$, $\beta_2 \neq \beta_1$

$$\hat{e}(F_1,\eta_2,\beta_2)-\hat{e}(F_1,\eta_1,\beta_1) > 0 \qquad \text{(III-47)}$$

where

(i) $\hat{\theta}(F_1,\eta_1,\beta_1)(\eta_2-\eta_1)+ \mathrm{tr}\, \hat{\tau}^T(F_1,\eta_1,\beta_1)(\beta_2-\beta_1) = 0$

(b.) there is a neighborhood of η_1, $N(\eta_1)$, such that for all $\eta_2 \in N(\eta_1)$, $\eta_2 \neq \eta_1$.

$$\hat{e}(F_1,\eta_2,\beta_1)-\hat{e}(F_1,\eta_1,\beta_1)-\hat{\theta}(F_1,\eta_1,\beta_1)(\eta_2-\eta_1) > 0 \tag{III-48}$$

(c.) there is a neighborhood of β_1, $N(\beta_1)$, such that for all $\beta_2 \in N(\beta_1)$, $\beta_2 \neq \beta_1$

$$\hat{e}(F_1,\eta_1,\beta_2)-\hat{e}(F_1,\eta_1,\beta_1)-\mathrm{tr}\,\hat{\tau}^T(F_1,\eta_1,\beta_1)(\beta_2-\beta_1) > 0 \tag{III-49}$$

Proof

Consider Ineq. (III-16). Let $F(X) = F_1$ and choose β_2 in a neighborhood of β_1. Let $\beta(X) = \beta_2$. Compute η_2 on the straight path which is tangent to an adiabat at (F_1,η_1,β_1):

$$\eta_2-\eta_1 = -\frac{1}{\hat{\theta}(F_1,\eta_1,\beta_1)}\,\mathrm{tr}\,\hat{\tau}^T(F_1,\eta_1,\beta_1)(\beta_2-\beta_1) \tag{III-50}$$

It follows that the path functions

$$\left.\begin{aligned} f(X) &= F_1X \\ \eta(X) &= \eta_2 \\ \beta(X) &= \beta_2 \end{aligned}\right\} \quad \text{on } N_R(0) \tag{III-51}$$

satisfy the restrictions for Ineq. (III-16); therefore it follows Ineq. (III-16) that

$$\int_{N_R(0)} \hat{e}(F_1,\eta_2,\beta_2)\,dV > \hat{e}(F_1,\eta_1,\beta_1)\,V \tag{III-52}$$

Since $\hat{e}(F_1,\eta_2,\beta_2)$ is constant on $N_R(0)$ and $\int_{N_R(0)} dV = V$ is strictly positive, Ineq. (III-47) follows Ineq. (III-52) and restriction (i) of (III-47) follows restriction (ii) of (III-16).

Now let

$$f(X) = F_1 X$$
$$\beta(X) = \beta_1 \qquad \text{(III-53)}$$

and it follows Ineq. (III-16) that there is a neighborhood of $\eta_1, N(\eta_1)$, such that for all $\eta(X) \in N(\eta_1)$

$$\int_{N_R(0)} \hat{e}(F_1,\eta(X),\beta_1)-\hat{e}(F_1,\eta_1,\beta_1)\ dV > 0 \qquad \text{(III-54)}$$

$$\text{where (i)} \int_{N_R(0)} \hat{\theta}(F_1,\eta_1,\beta_1)(\eta(X)-\eta_1)\ dV = 0$$

By adding the restriction to the inequality, it becomes

$$\int_{N_R(0)} \hat{e}(F_1,\eta(X),\beta_1)-\hat{e}(F_1,\eta_1,\beta_1)-\hat{\theta}(F_1,\eta_1,\beta_1)(\eta(X)-\eta_1)\ dV > 0 \qquad \text{(III-55)}$$

where

$$\text{(i)} \quad \int_{N_R(0)} \hat{\theta}(F_1,\eta_1,\beta_1)(\eta(X)-\eta_1)\ dV = 0$$

It follows easily that

$$\text{Ineq. (III-48)} \Rightarrow \text{Ineq. (III-55)} \qquad \text{(III-56)}$$

independent of restriction (i) of (III-55).

Now suppose not Ineq. (III-48), i.e. suppose in any neighborhood of $\eta_1, N(\eta_1)$, there is an $\eta_2 \neq \eta_1$ such that

$$\hat{e}(F_1,\eta_2,\beta_1)-\hat{e}(F_1,\eta_1,\beta_1)-\hat{\theta}(F_1,\eta_1,\beta_1)(\eta_2-\eta_1) \leq 0 \qquad \text{(III-57)}$$

It follows Taylor's formula with remainder and Ineq. (III-57) that

$$\hat{e}_{\eta\eta}(F_1,\eta,\beta_1) \leq 0 \quad \text{for } \eta \in L[\eta_1,\eta_2] \qquad \text{(III-58)}$$

It follows Taylor's formula with remainder and Ineq. (III-58) that in any neighborhood of η_1, an $\eta_3 \in L[\eta_1,\eta_2]$ and a neighborhood of $\eta_3, N(\eta_3)$ can be found such that for all $\eta \in N(\eta_3)$

$$\hat{e}(F_1,\eta,\beta_1)-\hat{e}(F_1,\eta_3,\beta_1)-\hat{\theta}(F_1,\eta_3,\beta_1)(\eta-\eta_3) \leq 0 \qquad \text{(III-59)}$$

Now it can be shown that $(F_1,\eta_3,\beta_1) \notin D$. Let $a(X)$ be a continuous, scalar-valued, bounded function on $N_R(0)$ such that

$$\int_{N_R(0)} a(X)\, dV = 0 \qquad \text{(III-60)}$$

and $a(X)$ is not constant on $N_R(0)$

Now choose

$$\eta(X) = \eta_3 + \varepsilon\, a(X)$$

$$\text{where } \varepsilon \in R,\ \varepsilon > 0 \qquad \text{(III-61)}$$

It follows that a nonzero constant ε may be chosen small enough so that $\eta(X)$ is in any neighborhood of η_3 on $N_R(0)$. Now consider Ineq. (III-16). Choose

$$\begin{aligned} \hat{x}(X) &= F_1(X) \\ \eta(X) &= \eta_3 + \varepsilon\, a(X) \qquad \text{on} \quad N_R(0) \\ \beta(X) &= \beta_1 \end{aligned} \tag{III-62}$$

which satisfies the restrictions for Ineq. (III-16). It follows Ineq. (III-59) that there is an $\varepsilon > 0$ such that

$$\int_{N_R(0)} \hat{e}(F_1, \eta(X), \beta_1) - \hat{e}(F_1, \eta_3, \beta_1)\, dV \leq 0 \tag{III-63}$$

Ineq. (III-63) is contrary to Ineq. (III-16); therefore

$$(F_1, \eta_3, \beta_1) \notin D \tag{III-64}$$

Then it follows Ineq. (III-57) that not Ineq. (III-48) $\Rightarrow$ in any neighborhood of

$$\begin{aligned} &(F_1, \eta_1, \beta_1) \quad \text{there is a triplet} \\ &(F_1, \eta_3, \beta_1) \notin D\ . \end{aligned} \tag{III-65}$$

But D is open in (F, η, β)-space by definition; therefore it follows Eq. (III-65) that

$$\text{not Ineq.(III-48)} \Rightarrow (F_1, \eta_1, \beta_1) \notin D \tag{III-66}$$

Ineq. (III-47) follows Ineq. (III-52) and Ineq.(III-48) follows Eq. (III-66).

Now consider Ineq. (III-49). It follows Ineq. (III-16) that

$$\int_{N_R(0)} \hat{e}(F_1,\eta_1,\beta(X))-\hat{e}(F_1,\eta_1,\beta_1)-\mathrm{tr}\,\hat{\tau}^T(F_1,\eta_1,\beta_1)(\beta(X)-\beta_1)\,dV > 0$$

for all $\beta(X)$ such that (III-67)

(i) $$\int_{N_R(0)} \mathrm{tr}\ \hat{\tau}^T(F_1,\eta_1,\beta_1)(\beta(X)-\beta_1)\ dV = 0$$

It follows easily that

$$\text{Ineq. (III-49)} \Rightarrow \text{Ineq. (III-67)} \qquad \text{(III-68)}$$

independent of restriction (i).

Now suppose not Ineq. (III-49): in any neighborhood of β_1, $N(\beta_1)$, there is a β_2 such that

$$\hat{e}(F_1,\eta_1,\beta_2)-\hat{e}(F_1,\eta_1,\beta_1)-\mathrm{tr}\,\hat{\tau}^T(F_1,\eta_1,\beta_1)(\beta_2-\beta_1) \leq 0 \qquad \text{(III-69)}$$

It follows Taylor's formula with remainder that Ineq. (III-69) is equivalent to the following:

$$D^2_{0,0,\beta_2-\beta_1}\ \hat{e}\ (F_1\ \eta_1\ \beta) \leq 0 \text{ for all } \beta \in L\ [\beta_1,\beta_2] \qquad \text{(III-70)}$$

It follows Ineq. (III-70) that in any neighborhood of β_1 there is a $\beta_3 \in L\ [\beta_1,\beta_2]$, $\beta_3 \neq \beta_1$ or β_2 such that

$$D^2_{0,0,\beta_1-\beta_3}\ \hat{e}\ (F_1\ \eta_1\ \beta) \leq 0 \text{ for all } \beta \in L\ [\beta_1,\beta_2] \qquad \text{(III-71)}$$

Now to prove that $(F_1, \eta_1, \beta_3) \notin D$. Choose any continuous, scalar-valued, bounded function on $N_R(o)$, $a(X)$, such that

$$\int_{N_R(0)} a(X)\ dV = 0 \tag{III-72}$$

and $a(X)$ is not constant on $N_R(0)$. Let

$$\left.\begin{aligned} f(X) &= F_1 X \\ \eta(X) &= \eta_1 \end{aligned}\right\} \quad \text{on} \quad N_R(0) \tag{III-73}$$

$$\beta(X) = \beta_3 + \varepsilon\ a(X)(\beta_1 - \beta_3)$$

$$\text{where} \quad \varepsilon \in R,\ \varepsilon > 0$$

An $\varepsilon > 0$ may be chosen so that $\beta(X) \in L[\beta_1, \beta_2]$. It follows Ineq. (III-71) that

$$D^2_{0,0,\beta(X)-\beta_3}\ \hat{e}(F_1, \eta_1, \beta_5(X)) \leq 0 \tag{III-74}$$

$$\text{for all} \quad \beta_5(X) \in L[\beta_1, \beta_2]$$

It follows Taylor's formula with remainder that Ineq.(III-74) is equivalent to the following:

$$\hat{e}(F_1 \eta_1, \beta(X)) - \hat{e}(F_1, \eta_1, \beta_3) - \mathrm{tr}\,\hat{\tau}^T(F_1, \eta_1, \beta_3)(\beta(X) - \beta_3) \leq 0 \tag{III-75}$$

and it follows Eqs. (III-72), (III-73)[3] that

$$\int_{N_R(0)} \mathrm{tr}\,\hat{\tau}^T(F_1, \eta_1, \beta_3)(\beta(X) - \beta_3)\ dV = 0 \tag{III-76}$$

It follows Eqs. (III-75), (III-76) and Ineq. (III-16) that

$$(F_1, \eta_1, \beta_3) \notin D \tag{III-77}$$

In other words

not Ineq. (III-49) for (F_1, η_1, β_1) $\Rightarrow$ there is an (F_1, η_1, β_3) in any neighborhood of (F_1, η_1, β_1) such that $(F_1, \eta_1, \beta_3) \notin D$ (III-78)

But D is open by definition; therefore it follows Eq. (III-78) that

not Ineq. (III-49) for (F_1, η_1, β_1) $\Rightarrow$ $(F_1, \eta_1, \beta_1) \notin D$ (III-79)

Ineq. (III-49) follows Eq. (III-79). Q.E.D.

It follows continuity of $\hat{e}(F, \eta, \beta)$ and Taylor's formula with remainder that Theorem 3 has the following equivalent representation:

Corollary

$(F_1, \eta_1, \beta_1) \in D$ only if:

(a.) there is a neighborhood of $\beta_1, N(\beta_1)$, such that for all $\beta_2 \in N(\beta_1)$, $\beta_2 \neq \beta_1$

$$D^2_{0\ \eta_2-\eta_1, \beta_2-\beta_1} e(F_1, \eta, \beta) > 0 \tag{III-80}$$

where

(i) $\hat{\theta}(F_1, \eta_1, \beta_1)(\eta_2-\eta_1) + \mathrm{tr}\hat{\tau}^T(F_1, \eta_1, \beta_1)(\beta_2-\beta_1) = 0$

(ii) (η, β) is any pair on $L[(\eta_1, \beta_1), (\eta_2, \beta_2)]$ but $(\eta, \beta) \neq (\eta_1, \beta_1)$

(b.) there is a neighborhood of $\eta_1, N(\eta_1)$, such that for all $\eta_2 \in N(\eta_1)$, $\eta_2 \neq \eta_1$

$$\hat{e}_{\eta\eta}(F_1, \eta_2, \beta_1) > 0 \qquad \text{(III-81)}$$

(c.) there is a neighborhood of $\beta_1, N(\beta_1)$, such that for all $\beta_2 \in N(\beta_1)$, $\beta_2 \neq \beta_1$

$$D^2_{0,0,\beta_2-\beta_1}\hat{e}(F_1, \eta_1, \beta) > 0 \qquad \text{(III-82)}$$

for all $\beta \in L[\beta_1, \beta_2]$, $\beta \neq \beta_1$

Theorem 4

The temperature function, Eq. (III-18)[2], is invertible in η and the inverse is continuous. A state (F, η, β) is uniquely characterized by the triplet (F, θ, η), where θ is the value of the temperature which corresponds to the state (F, η, β). Let $\hat{\eta}(F, \theta, \beta)$ represent the inverse function:

$\hat{\eta}(F, \theta, \beta)$ is the inverse of $\hat{\theta}(F, \eta, \beta)$ on D,

$\hat{\eta}(F, \theta, \beta)$ is continuous on D, therefore the map $(F, \eta, \beta) \longleftrightarrow (F, \theta, \beta)$ is a homeomorphism on D. (III-83)

Proof

Because D is open it follows that in a neighborhood there are states $(F_1, \eta_1, \beta_1)(F_1, \eta_2, \beta_1) \in D$ such that $\eta_1 \neq \eta_2$. Choose two such states, apply Ineq. (III-48) to each state, and add the result.

$$\left(\hat{\theta}(F_1, \eta_2, \beta_1) - \hat{\theta}(F_1 \eta_1 \beta_1)\right)(\eta_2 - \eta_1) > 0 \qquad \text{(III-84)}$$

for $\eta_2 \neq \eta_1$

It follows Ineq. (III-84) that $\hat{\theta}(F_1, \eta, \beta_1)$ is strictly increasing in η and one-to-one. Now let $L(\eta)$ denote the domain of η :

$$L(\eta) = \left\{ \eta \,|\, (F_1, \eta, \beta_1) \in D \right\} \qquad \text{(III-85)}$$

It follows Eq. (III-12) that $L(\eta)$ is connected in R . Ineq. (III-84) applies to any two neighboring states on $L(\eta)$; therefore it follows that

$$\hat{\theta}(F_1, \eta, \beta_1) \text{ is strictly increasing in } \eta \text{ on } L(\eta) \, . \qquad \text{(III-86)}$$

It follows Eq.(III-11)[1] that $\hat{\theta}(F_1, \eta, \beta_1)$ is continuous in η ; therefore it follows Eq. (III-86) that

$\hat{\eta}(F_1, \theta, \eta_1)$ is strictly increasing and continuous on $L(\theta)$ where

$$L(\theta) \equiv \left\{ \theta \,|\, \theta = \hat{\theta}(F_1, \eta, \beta_1), \quad \eta \in L(\eta) \right\} \qquad \text{(III-87)}$$

In other words

$$\eta \xleftrightarrow{F_1, \beta_1} \theta \text{ is a homeomorphism on } D \qquad \text{(III-88)}$$

Since the inverse was defined for an arbitrary pair (F_1, β_1), the definition may be extended onto the image of D :

$$\eta = \hat{\eta}(F, \theta, \beta) \qquad \text{on the image of } D \qquad \text{(III-89)}$$

Let $(F_1, \eta_1, \beta_1) \in D$ and $(F_1, \eta_1, \beta_1) \longrightarrow \theta_1$. Choose a neighborhood of η_1 , $f(\eta_1)$:

$$f(\eta_1) = \left\{ \eta \,|\, \eta_1 < \eta < \eta_U \right\} \qquad \text{(III-90)}$$

where $\eta_L < \eta_1 < \eta_U$

Define θ_L, θ_U as follows

$$\left.\begin{aligned} \theta_L &= \hat{\theta}(F,\eta_L,\beta) \\ \theta_U &= \hat{\theta}(F,\eta_U,\beta) \end{aligned}\right\} \text{ on } D \qquad \text{(III-91)}$$

It follows Eqs. (III-87), (III-90) that

$$\hat{\theta}(F_1,\eta_L,\beta_1) < \theta_1$$

$$\hat{\theta}(F_1,\eta_U,\beta_1) > \theta_1 \qquad \text{(III-92)}$$

It follows continuity of $\hat{\theta}(F,\eta,\beta)$ that a neighborhood of (F_1,β_1), $N(F_1,\beta_1)$ can be found such that

$$(F,\eta) \in N(F_1,\eta_1) \Rightarrow \begin{aligned} \hat{\theta}(F,\eta_L,\beta) &< \theta_1 \\ \hat{\theta}(F,\eta_U,\beta) &> \theta_1 \end{aligned} \qquad \text{(III-93)}$$

Let $\theta_{min.}$, $\theta_{max.}$ represent the inf of θ_L *and the* sup of θ_U respectively:

$$\theta_{min.} = \inf\ \hat{\theta}(F,\eta_L,\beta) \text{ on } N(F_1,\beta_1)$$

$$\theta_{max.} = \sup\ \hat{\theta}(F,\eta_U,\beta) \text{ on } N(F_1,\beta_1) \qquad \text{(III-94)}$$

It follows Eq. (III-94) that

$$\theta_{min.} < \theta_1 < \theta_{max.} \qquad \text{(III-95)}$$

Define a neighborhood of $\theta_1, N(\theta_1)$, as follows:

$$N(\theta_1) = \left\{\theta \mid \theta_{min} < \theta < \theta_{max}\right\} \qquad \text{(III-96)}$$

It follows the above construction that any triplet (F,θ,β) such that $(F,\beta) \in N(F_1,\eta_1)$ and $\theta \in N(\theta_1)$

$$\Longrightarrow (F,\theta,\beta) \longrightarrow \eta \in N(\eta_1) \qquad \text{(III-97)}$$

In other words for any neighborhood of η_1 a neighborhood of (F_1, θ_1, β_1) can be found such that the image of $N(F_1, \theta_1, \beta_1) \subset N(\eta_1)$. It follows that

$$(F, \theta, \beta) \longrightarrow \eta \text{ is continuous} \qquad \text{(III-98)}$$

It follows Eqs. (III-11)[1] and (III-98) that

$$(F, \eta, \beta) \rightarrow (F, \theta, \beta) \text{ is continuous on } D$$

$$(F, \theta, \beta) \rightarrow (F, \eta, \beta) \text{ is continuous on the image of } D \qquad \text{(III-99)}$$

Equation (III-83)[3] follows Eq. (III-99). Equation (III-99)[1] follows Eq. (III-89) and Eq. (III-99)[2] follows Eq. (III-98).

Q.E.D.

It follows Theorem 4 that isothermal paths exist and are unique in some ways.

Corrolary

Consider a continuous path $(F(s), \eta(s), \beta(s))$ in D through the state (F_1, η_1, β_1) with temperature θ_1, i.e. $(F(0), \eta(0), \beta(0)) = (F_1, \eta_1, \beta_1)$ and $(F_1, \eta_1, \beta_1) \longrightarrow \theta$. Then for any continuous pair $F(s), \beta(s)$ there is a unique continuous function $\eta_1(s)$ such that

$$\theta_1 = \hat{\theta}(F(s), \eta_1(s), \beta(s)) \text{ on } D \qquad \text{(III-100)}$$

Theorem 5

Let $(F_1, \eta_1, \beta_1) \in D$ with temperature θ_1; then $\hat{\theta}_\eta(F_1, \eta_1, \beta_1) \neq 0 \Leftrightarrow (F, \theta, \beta) \longrightarrow (F, \eta, \beta)$ is differentiable at (F_1, θ_1, β_1) (III-101)

Proof

Consider the inverse maps:

$$(F, \theta, \beta) \longrightarrow \eta$$
$$(F, \eta, \beta) \longrightarrow \theta \qquad \text{(III-102)}$$

In other words the following equation is an identy in θ :

$$\theta = \hat{\theta}(F, \hat{\eta}(F, \theta, \beta), \beta) \qquad \text{(III-103)}$$

It follows Eq. (III-11)[1] that Eq. (III-102)[2] is differentiable. Assume Eq. (III-102)[1] is differentiable at (F_1, θ_1, β_1); it follows Eq. (III-103) that

$$\begin{aligned} \dot{\theta} = {} & \operatorname{tr}\left[\hat{\theta}_F^T(F_1, \eta_1, \beta_1) + \hat{\theta}_\eta(F_1, \eta_1, \beta_1)\hat{\eta}_F^T(F_1, \theta_1, \beta_1)\right] \dot{F} \\ & + \hat{\theta}_\eta(F_1, \eta_1, \beta_1)\hat{\eta}_\theta(F_1, \theta_1, \beta_1)\ \dot{\theta} \\ & + \operatorname{tr}\left[\hat{\theta}_\beta^T(F_1, \eta_1, \beta_1) + \hat{\theta}_\eta(F_1, \eta_1, \beta_1)\hat{\eta}_\beta^T(F_1, \theta_1, \beta_1)\right]\dot{\beta} = 0 \qquad \text{(III-104)} \end{aligned}$$

for independent $\dot{F}$, $\dot{\theta}$, $\dot{\beta}$. If $\hat{\theta}_\eta(F_1, \eta_1, \beta_1) \neq 0$ it follows Eq. (III-104) that

$$\begin{aligned} \hat{\eta}_F(F_1, \theta_1, \beta_1) &= -\frac{1}{\hat{\theta}_\eta(F_1, \eta_1, \beta_1)}\ \hat{\theta}_F^T(F_1, \eta_1, \beta_1) \\ \hat{\eta}_\theta(F_1, \theta_1, \beta_1) &= \frac{1}{\hat{\theta}_\eta(F_1, \eta_1, \beta_1)} \\ \hat{\eta}_\beta(F_1, \theta_1, \beta_1) &= -\frac{1}{\hat{\theta}_\eta(F_1, \eta_1, \beta_1)}\ \hat{\theta}_\beta(F_1, \eta_1, \beta_1) \qquad \text{(III-105)} \end{aligned}$$

If $\hat{\theta}_\eta(F_1, \eta_1, \beta_1) \neq 0$ it follows Eq. (III-11)[1] that the right-hand side of Eq. (III-105) is continuous; Eq. (III-101) follows Eq. (III-105). Q.E.D.

Corollary

$$\hat{\theta}_{\eta}(F_1,\eta_1,\beta_1) = 0 \Rightarrow \hat{\eta}(F,\theta,\beta) \text{ is not differentiable for } (F,\theta,\beta) = (F_1,\theta_1,\beta_1) \qquad \text{(III-106)}$$

Theorem 6

$(F_1,\eta_1,\beta_1) \in D$ only if there is a neighborhood of (F_1,η_1,β_1) such that for every $(F_1,\eta_2,\beta_2) \in N(F_1,\eta_1,\beta_1)$, $(F_1,\eta_2,\beta_2) \neq (F_1,\eta_1,\beta_1)$

$$\hat{e}(F_1,\eta_2,\beta_2)-\hat{e}(F_1,\eta_1,\beta_1)-\hat{\theta}(F_1,\eta_1,\beta_1)(\eta_2-\eta_1)$$
$$-\mathrm{tr}\hat{\tau}^T(F_1,\eta_1,\beta_1)(\beta_2-\beta_1) > 0 \qquad \text{(III-107)}$$

Proof

Consider the converse of Ineq. (III-107): In any neighborhood of (F_1,η_1,β_1) there is a $(F_1,\eta_2,\beta_2) \neq (F_1,\eta_1,\beta_1)$ such that

$$\hat{e}(F_1,\eta,\beta)-\hat{e}(F_1,\eta_1,\beta_1)-\hat{\theta}(F_1,\eta_1,\beta_1)(\eta-\eta_1)$$
$$-\mathrm{tr}\hat{\tau}(F_1,\eta_1,\beta_1)(\beta-\beta_1) \leq 0 \qquad \text{(III-108)}$$

for all $(F_1,\eta,\beta) \in L\left[(F_1,\eta_1,\beta_1),(F_1,\eta_2,\beta_2)\right]$

It follows Ineq. (III-108) that there is a (F_1,η_3,β_3) in any neighborhood of (F_1,η_1,β_1), $(F_1,\eta_3,\beta_3) \neq (F_1,\eta_1,\beta_1)$, $(F_1,\eta_3,\beta_3) \neq (F_1,\eta_2,\beta_2)$, $(F_1,\eta_3,\beta_3) \in L\left[(F_1,\eta_1,\beta_1)(F_1,\eta_2,\beta_2)\right]$ such that

$$\hat{e}(F_1,\eta,\beta)-\hat{e}(F_1,\eta_3,\beta_3)-\hat{\theta}(F_1,\eta_3,\beta_3)(\eta-\eta_3)$$
$$-\mathrm{tr}\hat{\tau}^T(F_1,\eta_3,\beta_3)(\beta-\beta_3) \leq 0 \qquad \text{(III-109)}$$

for all $(F_1, \eta, \beta) \in L \ (F_1, \eta_1, \beta_1), (F_1, \eta_2, \beta_2)$,

$$(F_1, \eta, \beta) \neq (F_1, \eta_3, \beta_3)$$

Let

$$\begin{aligned} f(X) &= F_1 X \\ \eta(X) &= \eta_3 + \varepsilon\, a(X)(\eta_1 - \eta_3) \qquad \text{on} \quad N_R(0) \\ \beta(X) &= \beta_3 + \varepsilon\, a(X)(\beta_1 - \beta_3) \end{aligned} \tag{III-110}$$

where

$\varepsilon \in R, \ \varepsilon > 0$

$a(X)$ is a scalar-valued continuous bounded function on $N_R(o)$ such that

$$\int_{N_R(0)} a(X)\, dV = 0 \tag{III-111}$$

It follows that for any $a(X)$ an $\varepsilon > 0$ can be found such that

$$(F_1, \eta(X), \beta(X)) \in L \left[(F_1, \eta_1, \beta_1)(F_1, \eta_2, \beta_2)\right] \tag{III-112}$$

It follows Ineq. (III-109), Eq. (III-112) that

$$\begin{aligned} &\hat{e}(F_1, \eta(X), \beta(X)) - \hat{e}(F_1, \eta_1, \beta_1) - \hat{\theta}(F_1, \eta_3, \beta_3)(\beta(X) - \beta_3) \\ &\qquad - \mathrm{tr}\ \hat{\tau}^T(F_1, \eta_3, \beta_3)(\beta(X) - \beta_1) \leq 0 \end{aligned} \tag{III-113}$$

It follows Eqs. (III-110), (III-111) that $(f(X), \eta(X), \beta(X))$ satisfy the constraints for Ineq. (III-16) and it follows Ineqs. (III-113), (III-16) that

$$(F_1, \eta_3, \beta_3) \notin D \tag{III-114}$$

In other words

$$\begin{array}{lll} \text{not (III-107)} & \text{(III-108)} & \text{In any neighborhood of} \\ \text{for } (F_1, \eta_1, \beta_1) \Longleftrightarrow & \text{for} (F_1 \eta_1 \beta_1) \Longrightarrow & (F_1, \eta_1, \beta_1) \text{ there is an} \\ & & (F_1, \eta_3, \beta_3) \notin D \end{array} \quad \text{(III-115)}$$

But D is open by definition; therefore it follows that

$$\begin{array}{l} \text{not (III-117)} \\ \text{for } (F_1, \eta_1, \beta_1) \end{array} \Longrightarrow (F_1, \eta_1, \beta_1) \notin D \qquad \text{(III-116)}$$

Theorem 6 follows Eq. (III-116).

Q.E.D.

An equivalent for Theorem 6 follows easily Taylor's formula with remainder.

Corollary

$(F_1, \eta_1, \beta_1) \in D$ only if there is a neighborhood of (F_1, η_1, β_1) such that for every $(F_1, \eta_2, \beta_2) \in N(F_1, \eta_1, \beta_1)$, $(F_1, \eta_2, \beta_2) \neq (F_1, \eta_1, \beta_1)$

$$D^2_{0, \eta-\eta_1, \beta-\beta_1} \hat{e}(F_1, \eta, \beta) > 0 \qquad \text{(III-117)}$$

for all $(F_1, \eta, \beta) \in L \left[(F_1, \eta_1, \beta_1)(F_1, \eta_2, \beta_2)\right]$,
$(F_1, \eta, \beta) \neq (F_1, \eta_1, \beta_1)$

Theorem 7

$(F_1, \eta_1, \beta_1) \in D$ only if

(a.) $\hat{e}_{\eta\eta}(F_1 \eta_1 \beta_1) = 0 \Rightarrow \hat{e}_{\eta F}(F_1, \eta_1, \beta_1) = 0$

and $\hat{e}_{\eta\beta}(F_1, \eta_1, \beta_1) = 0$ (III-118)

(b.) $\mathrm{tr}\left[(\beta_2-\beta_1)^T \hat{e}_{\beta}\right]_{\beta}(F_1, \eta_1, \beta_1) = 0 \Rightarrow$

$\mathrm{tr}\left[(\beta_2-\beta_1)^T \hat{e}_{\beta}\right]_{\eta}(F_1, \eta_1, \beta_1) = 0$ (III-119)

Proof

It follows Eq. (III-18) that

$$\dot{\theta} = \operatorname{tr}\left[\dot{F}^T \hat{e}_F\right]_\eta (F_1,\eta_1,\beta_1) + \hat{e}_{\eta\eta}(F_1,\eta_1,\beta_1)\dot{\eta} + \operatorname{tr}\left[\dot{\beta}^T \hat{e}_\beta\right]_\eta (F_1,\eta_1,\beta_1)$$

(III-120)

for any path with tangent $(\dot{F},\dot{\eta},\dot{\beta})$ at (F_1,η_1,β_1).
Now assume $\hat{e}_{\eta\eta}(F_1,\eta_1,\beta_1) = 0$ and not both $\hat{e}_{F\eta}(F_1,\eta_1,\beta_1)$, $\hat{e}_{\beta\eta}(F_1,\eta_1,\beta_1)$ are zero. It follows Eq. (III-120) that there is a pair $(\dot{F}_1,\dot{\beta}_1) \neq 0$ such that (i.e. $\dot{\eta} = 0$)

$$\dot{F}_1,\dot{\beta}_1 \longrightarrow \dot{\theta} \neq 0 \qquad \text{(III-121)}$$

Furthermore it follows Eqs. (III-121), (III-120) that

$$\dot{F}_1,\dot{\eta},\dot{\beta}_1 \longrightarrow \dot{\theta} \neq 0 \quad \text{for all } \dot{\eta} \qquad \text{(III-122)}$$

Eq. (III-122) contradicts Eq. (III-100). Eq. (III-118) follows easily.

Now assume the contrary to Eq. (III-119): there is a $\beta_2 \neq \beta_1$ such that

$$\operatorname{tr}\left[(\beta_2-\beta_1)^T \hat{e}_\beta\right]_\beta (F_1,\eta_1,\beta_1) = 0 \text{ , and}$$

$$\operatorname{tr}\left[(\beta_2-\beta_1)^T \hat{e}_\beta^T\right]_\eta (F_1,\eta_1,\beta_1) \neq 0 \qquad \text{(III-123)}$$

Consider the second-directional derivative for $F_2 = F_1$. It follows Eq. (III-123)[1] that

$$D^2_{0,\eta_2-\eta_1,\beta_2-\beta_1}\ \hat{e}(F_1\eta_1\beta_1) = (\eta_2-\eta_1)^2\ \hat{e}_{\eta\eta}(F_1,\eta_1,\beta_1)$$
$$+ 2\operatorname{tr}\left[(\beta_2-\beta_1)^T \hat{e}_\beta\right]_\eta (F_1,\eta_1,\beta_1) \qquad \text{(III-124)}$$

for any (F_1,η_2,β_2) in $N(F_1,\eta_1,\beta_1)$

Let

$$\eta_3 = \eta_1 + \varepsilon(\eta_2 - \eta_1)$$

$$\text{where} \quad \varepsilon \in R, \quad \varepsilon^2 < 1 \ , \ \varepsilon \neq 0 \ . \qquad \text{(III-125)}$$

It follows Eqs. (III-124), (III-125) that

$$D^2_{0,\eta_3-\eta_1,\beta_2-\beta_1} \hat{e}(F_1,\eta_1,\beta_1) = \varepsilon^2(\eta_2-\eta_1)^2 \ \hat{e}_{\eta\eta}(F_1,\eta_1,\beta_1)$$

$$+ 2 \ \varepsilon \ \mathrm{tr} \left[(\beta_2-\beta_1)^T \ \hat{e}_{\beta}\right]_{\eta}(F_1,\eta_1,\beta_1) \qquad \text{(III-126)}$$

The first term on the right-hand side of Eq. (III-126) is non-negative and even in ε , and the second term is odd and linear in ε . If $\mathrm{tr}\left[(\beta_2-\beta_1)^T \ \hat{e}_{\beta}\right]_{\eta}(F_1,\eta_1,\beta_1) \neq 0$ it follows Eq. (III-126) that there is an ε_1 such that

$$D^2_{0,\eta_3-\eta_1,\beta_2-\beta_1} \ \hat{e} \ (F_1,\eta_1,\beta_1) < 0 \qquad \text{(III-127)}$$

for all ε between 0 and ε_1 .

It follows Eqs. (III-126), (III-127) that for any $\beta_2 \neq \beta_1$, such that, Eq. (III-123)[1] holds, there is an $\eta_4 \neq \eta_1$ such that

$$D^2_{0,\eta-\eta_1,\beta_2-\beta_1} \ \hat{e} \ (F_1,\eta_1,\beta_1) < 0 \qquad \text{(III-128)}$$

for all $\eta \in L\left[\eta_1,\eta_4\right]$, $\eta \neq \eta_1, \eta \neq \eta_4$

Ineq. (III-128) violates Ineq. (III-117); therefore $(F_1,\eta_1,\beta_1) \in D$, $\mathrm{tr}\left[(\beta_2-\beta_1)^T \ \hat{e}_{\beta}\right]_{\beta}(F_1\eta_1\beta_1) = 0$

$$\Longrightarrow \mathrm{tr}\left[(\beta_2-\beta_1)^T\hat{e}_{\beta}\right]_{\eta}(F_1\eta_1\beta_1) = 0 \qquad \text{(III-129)}$$

Q.E.D.

Theorem 8

Let $(F_1, \eta_1, \beta_1) \in D$, $\hat{e}_{\eta\eta}(F_1, \eta_1, \beta_1) \neq 0$. Then for any pair (F_2, β_2) in a neighborhood of (F_1, β_1), $(F_2, \beta_2) \neq (F_1, \beta_1)$ there is a unique η_2 in a neighborhood of η_1 such that the second-directional derivative for the direction $(F_2-F_1,\ \eta-\eta_1,\ \beta_2-\beta_1)$ is minimum for $\eta = \eta_2$. Also the straight path $L\left[(F_1, \eta_1, \beta_1), (F_2, \eta_2, \beta_2)\right]$ is tangent to an isotherm at (F_1, η_1, β_1) .

$$(\eta_2-\eta_1) = -\frac{1}{\hat{e}_{\eta\eta}(F_1, \eta_1, \beta_1)}\left\{\mathrm{tr}\left[(F_2-F_1)^T \hat{e}_F\right]_\eta (F_1, \eta_1, \beta_1) + \mathrm{tr}\left[(\beta_2-\beta_1)^T \hat{e}_\beta\right]_\eta (F_1, \eta_1, \beta_1)\right\} \qquad \text{(III-130)}$$

$$D^2_{F_2-F_1, \eta_2-\eta_1, \beta_2-\beta_1}\, \hat{e}(F_1, \eta_1, \beta_1) = D^2_{F_2-F_1, 0, \beta_2-\beta_1}\, \hat{e}\,(F_1 \eta_1 \beta_1) - (\eta_2-\eta_1)^2\, \hat{e}_{\eta\eta}(F_1, \eta_1, \beta_1)$$

$$D^2_{F_2-F_1, \eta-\eta_1, \beta_2-\beta_1}\, \hat{e}(F_1, \eta_1, \beta_1) = D^2_{F_2-F_1, \eta_2-\eta_1, \beta_2-\beta_1}\, \hat{e}\,(F_1, \eta_1, \beta_1) + (\eta-\eta_2)^2\, \hat{e}_{\eta\eta}(F_1, \eta_1, \beta_1)$$

Proof

Consider a straight path $L\left[(F_1, \eta_1, \beta_1)(F_2, \eta_2, \beta_2)\right]$ represented by $(F(s), \eta(s), \beta(s))$ for $0 \leq s \leq 1$. The path derivative is independent of s :

$$(\dot{F}, \dot{\eta}, \dot{\beta}) = (F_2-F_1, \eta_2-\eta_1, \beta_2-\beta_1) \qquad \text{(III-131)}$$

If $\hat{\theta}_\eta\ (F_1, \eta_1, \beta_1) \neq 0$, it follows Eq. (III-37) evaluated for $s = 0$, Eq. (III-131), and Eq. (III-130)[1] that $L\left[(F_1, \eta_1, \beta_1), (F_2, \eta_2, \beta_2)\right]$ is tangent to an isotherm at

(F_1, η_1, β_1) . Now expand the second-directional derivative for the direction $(F_2-F_1, \eta-\eta_1, \beta_2-\beta_1)$:

$$D^2_{F_2-F_1,\eta-\eta_1,\beta_2-\beta_1}\hat{e}(F_1,\eta_1,\beta_1) = D^2_{F_2-F_1,0,\beta_2-\beta_1}\hat{e}(F_1,\eta_1,\beta_1)$$
$$+ 2\left\{tr\left[(F_2-F_1)^T \hat{e}_F\right]_\eta (F_1,\eta_1,\beta_1) + tr\left[(\beta_2-\beta_1)^T e_\beta\right]_\eta (F_1,\eta_1,\beta_1)\right\}(\eta-\eta_1) \tag{III-132}$$
$$+ (\eta-\eta_1)^2\ \hat{e}_{\eta\eta}(F_1,\eta_1,\beta_1)$$

With some algebra it is easy to show

$$-2(\eta_2-\eta_1)(\eta-\eta_1) + (\eta-\eta_1)^2 = -(\eta_2-\eta_1)^2+(\eta-\eta_2)^2 \tag{III-133}$$

Equation $(III-130)^1$ is used to eliminate the second term on the right-hand side of Eq. (III-132) and then Eq. (III-133) is used to rearrange the terms :

$$D^2_{F_2-F_1,\eta-\eta_1,\beta_2-\beta_1}\ \hat{e}\ (F_1,\eta_1,\beta_1) = D^2_{F_2-F_1,0,\beta_2-\beta_1}\hat{e}(F_1,\eta_1,\beta_1)$$
$$-(\eta_2-\eta_1)^2\ \hat{e}_{\eta\eta}(F_1,\eta_1,\beta_1)$$
$$+(\eta-\eta_2)^2\ \hat{e}_{\eta\eta}(F_1,\eta_1,\beta_1) \tag{III-134}$$

Equation $(III-130)^2$ follows Eq. (III-134) for $\eta = \eta_2$. Equation $(III-130)^3$ follows Eqs. $(III-130)^2$ and Eq. (III-134). Also it follows Eq. (III-81) that

$$\hat{e}_{\eta\eta}(F_1,\eta_1,\beta_1) \neq 0 \Rightarrow \hat{e}_{\eta\eta}(F_1,\eta_1,\beta_1) > 0 \tag{III-135}$$

It follows Eqs. (III-130)[3], (III-135) that the second-directional derivative for the direction $(F_2-F_1,\ \eta-\eta_1, \beta_2-\beta_1)$ is minimum for $\eta = \eta_2$.

Q.E.D.

Theorem 9

Let $(F_1, \eta_1, \beta_1) \in D$ and there exist a pair $(\eta_2, \beta_2) \neq (\eta_1, \beta_1)$ such that

$$D^2_{0,\eta_2-\eta_1,\beta_2-\beta_1}\ \hat{e}\ (F_1, \eta_1, \beta_1) = 0 \tag{III-136}$$

The following Ineqs. are equivalent:

$$(\eta_2-\eta_1)^2\ \hat{e}_{\eta\eta}(F_1, \eta_1, \beta_1) \neq 0$$

$$D^2_{0,0,\beta_2-\beta_1}\ \hat{e}(F_1, \eta_1, \beta_1) \neq 0$$

$$tr\left[(\beta_2-\beta_1)^T\hat{e}_\beta\right]_\eta(F_1, \eta_1, \beta_1)(\eta_2-\eta_1) \neq 0 \tag{III-137}$$

Also if $(\eta_2-\eta_1)^2\ \hat{e}_{\eta\eta}(F_1, \eta_1, \beta_1) \neq 0$ the straight path $L\left[(F_1, \eta_1, \beta_1), (F_1, \eta_2, \beta_2)\right]$ is tangent to an isotherm at (F_1, η_1, β_1), and

$$- \text{tr}\left[(\beta_2-\beta_1)^T\ \hat{e}_\beta\right]_\eta(F_1, \eta_1, \beta_1)(\eta_2-\eta_1) = (\eta_2-\eta_1)^2\ \hat{e}_{\eta\eta}(F_1, \eta_1, \beta_1)$$

$$= D^2_{0,0,\beta_2-\beta_1}\ \hat{e}(F_1, \eta_1, \beta_1) > 0 \tag{III-138}$$

Proof

Let $(\eta, \beta_2) \neq (\eta_1, \beta_1)$, then expand the second-directional derivative:

$$D^2_{0,\eta-\eta_1,\beta_2-\beta_1}\hat{e}(F_1, \eta_1, \beta_1) = (\eta-\eta_1)^2\ \hat{e}_{\eta\eta}(F_1, \eta_1, \beta_1)$$

$$+ 2\ \text{tr}\left[(\beta_2-\beta_1)^T\ \hat{e}_\beta\right]_\eta(F_1, \eta_1, \beta_1) + D^2_{0,0,\beta_2-\beta_1}\hat{e}(F_1, \eta_1, \beta_1) \tag{III-139}$$

It follows Ineq. (III-117) that

$$(F_1,\eta_1,\beta_1)\ \varepsilon\ D \Rightarrow D^2_{0,\eta-\eta_1,\beta_2-\beta_1}\hat{e}(F_1,\eta_1,\beta_1) \geq 0 \qquad \text{(III-140)}$$

Assume Eq. (III-136) and it follows Eq. (III-140) that

$$D^2_{0,\eta-\eta_1,\beta_2-\beta_1}\ \hat{e}(F_1,\eta_1,\beta_1) \geq D^2_{0,\eta_2-\eta_1,\beta_2-\beta_1}\hat{e}(F_1,\eta_1,\beta_1) = 0 \qquad \text{(III-141)}$$

for all η .

Now assume Eq. (III-137)^1. It follows Eqs. (III-137)^1, (III-81) that

$$\begin{aligned} &\eta_2-\eta_1 \neq 0 \\ &\hat{e}_{\eta\eta}(F_1,\eta_1,\beta_1) > 0 \end{aligned} \qquad \text{(III-142)}$$

It follows Theorem 8 that $L\left[(F_1\eta_2\beta_2),(F_1,\eta_1,\beta_1)\right]$ is tangent to an isotherm at (F_1,η_1,β_1). It follows Eqs. (III-130), (III-141) that

$$\begin{aligned} &-\operatorname{tr}\left[(\beta_2-\beta_1)\hat{e}_{\beta}\right]_{\eta}(F_1,\eta_1,\beta_1)(\eta_2-\eta_1) = (\eta_2-\eta_1)^2\ \hat{e}_{\eta\eta}(F_1,\eta_1,\beta_1) \\ &D^2_{0,0,\beta_2-\beta_1}\hat{e}(F_1,\eta_1,\beta_1) = (\eta_2-\eta_1)^2\ \hat{e}_{\eta\eta}(F_1,\eta_1,\beta_1) \end{aligned} \qquad \text{(III-143)}$$

It follows Eqs. (III-142), (III-143)

$$\text{(III-136)},\text{(III-137)}^1 \Rightarrow \text{(III-137)}^{2,3},\ \text{(III-138)} \qquad \text{(III-144)}$$

Now assume Eq. (III-137)^2. It follows Eqs.(III-118) and (III-137)^2 that

$$\begin{aligned} &\hat{e}_{\beta\eta}(F_1,\eta_1,\beta_1) \neq 0 \Rightarrow \hat{e}_{\eta\eta}(F_1,\eta_1,\beta_1) \neq 0 \\ &\beta_2-\beta_1 \neq 0 \\ &\eta_2-\eta_1 \neq 0 \end{aligned} \qquad \text{(III-145)}$$

It follows Eq. (III-145)^{1,3}, (III-81) that

$$\hat{e}_{\eta\eta}(F_1,\eta_1,\beta_1)(\eta_2-\eta_1)^2 > 0 \qquad \text{(III-146)}$$

It follows Eqs. (III-145), (III-146), (III-144) that

$$\text{(III-136)},\text{(III-137)}^2 \Rightarrow \text{(III-137)}^{1,3},\text{(III-138)} \qquad \text{(III-147)}$$

Now assume Eq. (III-137)^3. It follows

$$\text{(III-136)},\text{(III-139)} \Rightarrow \text{either (III-137)}^1 \text{ or (III-137)}^2 \qquad \text{(III-148)}$$

It follows Eqs. (III-148), (III-144), (III-147) that

$$\text{(III-136)},\text{(III-137)}^3 \Rightarrow \text{(III-137)}^{1,2},\text{(III-138)} \qquad \text{(III-149)}$$

Theorem 9 follows Eqs. (III-144), (III-147), (III-149).

Q.E.D.

Theorem 10

Let $(F_1,\eta_1,\beta_1) \in D$. Then $\hat{e}_{\eta\eta}(F_1,\eta_1,\beta_1) \neq 0$ only if there is a neighborhood of (F_1,η_1,β_1) such that for all $(F_2,\eta_3,\beta_2) \in N(F_1,\eta_1,\beta_1)$

$$\begin{aligned}
&\hat{e}(F_2,\eta_3,\beta_2)-\hat{e}(F_1,\eta_1,\beta_1)-\operatorname{tr}\frac{1}{\rho_R}\hat{T}_R{}^T(F_1,\eta_1,\beta_1)(F_2-F_1)\\
&\quad-\hat{\theta}(F_1,\eta_1,\beta_1)(\eta_3-\eta_1)-\operatorname{tr}\hat{\tau}^T(F_1\eta_1\beta_1)(\beta_2-\beta_1) >\\
&\hat{e}(F_2,\eta_2,\beta_2)-\hat{e}(F_1,\eta_1,\beta_1)-\operatorname{tr}\frac{1}{\rho_R}\hat{T}_R{}^T(F_1,\eta_1,\beta_1)(F_2-F_1)\\
&\quad-\hat{\theta}(F_1,\eta_1,\beta_1)(\eta_2-\eta_1)-\operatorname{tr}\hat{\tau}^T(F_1,\eta_1,\beta_1)(\beta_2-\beta_1) \qquad \text{(III-150)}
\end{aligned}$$

where

$$\text{(i)}\quad \operatorname{tr}\left[(F_2-F_1)^T\hat{e}_F\right]_\eta(F_1,\eta_1,\beta_1)+\hat{e}_{\eta\eta}(F_1,\eta_1,\beta_1)(\eta_2-\eta_1) + \operatorname{tr}\left[(\beta_2-\beta_1)^T\hat{e}_\beta\right]_\eta(F_1,\eta_1,\beta_1) = 0$$

$$\text{(ii)}\quad \eta_3 \neq \eta_2$$

<u>Proof</u>

It follows Ineq. (III-81) that

$$\hat{e}_{\eta\eta}(F_1,\eta_1,\beta_1) \neq 0 \Rightarrow \hat{e}_{\eta\eta}(F_1,\eta_1,\beta_1) > 0 \qquad \text{(III-151)}$$

It follows Theorem 8 and Ineq. (III-151) that for any pair $(F_2,\beta_2) \neq (F_1,\beta_1)$ there is a unique η_2 such that $L[(F_1,\eta_1,\beta_1)(F_2,\eta_2,\beta_2)]$ is tangent to an isotherm at (F_1,η_1,β_1) and

$$D^2_{F_2-F_1,\eta_3-\eta_1,\beta_2-\beta_1}\hat{e}(F_1,\eta_1,\beta_1) > D^2_{F_2-F_1,\eta_2-\eta_1,\beta_2-\beta_1}\hat{e}(F_1\eta_1\beta_1) \qquad \text{(III-152)}$$

where η_2 satisfies restriction (i)

and η_3 satisfies restriction (ii) of Ineq. (III-150).

Represent the two straight paths parametrically:

$$(F(s),\eta(s),\beta(s)) \in L[(F_1\eta_1\beta_1),(F_2,\eta_2,\beta_2)]$$

such that

$$(F(0),\eta(0),\beta(0)) = (F_1,\eta_1,\beta_1)$$

$$(F(1),\eta(1),\beta(1)) = (F_2,\eta_2,\beta_2)$$

and

$$(F(s),\bar{\eta}(s),\beta(s)) \in L[(F_1\eta_1\beta_1),(F_2,\eta_3,\beta_2)] \qquad \text{(III-153)}$$

such that

$$(F(0),\bar{\eta}(0),\beta(0)) = (F_1,\eta_1,\beta_1)$$

$$(F(1),\bar{\eta}(1),\beta(1)) = (F_2,\eta_3,\beta_2)$$

It follows Eq. (III-11)[1] that

$D^2_{F_2-F_1,\eta_3-\eta_1,\beta_3-\beta_1}\hat{e}(F(s)\bar{\eta}(s)\beta(s))$ and

$D^2_{F_2-F_1,\eta_2-\eta_1,\beta_3-\beta_1}\hat{e}(F(s),\eta(s),\beta(s))$ are each continuous

on $0 \leq s \leq 1$, and it follows Ineq. (III -152) that in any neighborhood of $s=0$ there is an $s_1 \neq 0$ such that

$$D^2_{F_2-F_1,\eta_3-\eta_1,\beta_2-\beta_1}\hat{e}(F(s_3),\bar{\eta}(s_3),\beta(s_3)) > D^2_{F_2-F_1,\eta_2-\eta_1,\beta_2-\beta_1}$$

$$\hat{e}(F(s_2),\eta(s_2),\beta(s_2)) \qquad \text{(III-154)}$$

for all s_2, s_3 such that

$$0 \leq s_2 < s_1$$

$$0 \leq s_3 < s_1$$

Let s_4 be any value such that $0 < s_4 < s_3$. It follows Taylor's formula with remainder that

$$\hat{e}(F(s_4),\bar{\eta}(s_4)\beta(s_4)-\hat{e}(F_1,\eta_1,\beta_1)-\frac{1}{\rho_R}\mathrm{tr}\hat{T}_R{}^T(F_1,\eta_1,\beta_1)(F(s_4)-F_1)$$
$$-\hat{\theta}(F_1,\eta_1,\beta_1)(\bar{\eta}(s_4)-\eta_1)-\mathrm{tr}\,\hat{\tau}^{\,T}(F_1,\eta_1,\beta_1)(\beta(s_4)-\beta)$$
$$=\frac{1}{2}\,D^2_{F(s_4)-F_1,\bar{\eta}(s_4)-\eta_1,\bar{\beta}(s_4)-\beta_1}\hat{e}(F(s_5),\bar{\eta}(s_5),\beta(s_5)) \qquad \text{(III-155)}$$

for some s_5 such that $0 \leq s_5 \leq s_4$, and

$$\hat{e}(F(s_4),\eta(s_4),\beta(s_4)-\hat{e}(F_1,\eta_1,\beta_1)-\frac{1}{\rho_R}\,\mathrm{tr}\,\hat{T}_R{}^T(F_1,\eta_1,\beta_1)(F(s_4)-F_1)$$
$$-\hat{\theta}(F_1,\eta_1,\beta_1)(\eta(s_4)-\eta_1)-\mathrm{tr}\,\hat{\tau}^T(F_1,\eta_1,\beta_1)(\beta(s_4)-\beta_1)$$
$$=\frac{1}{2}\,D^2_{F(s_4)-F_1,\eta(s_4)-\eta_1,\beta(s_4)-\beta_1}\hat{e}(F(s_6)\eta(s_6)\beta(s_6)) \qquad \text{(III-156)}$$

for some s_6 such that $0 \leq s_6 \leq s_4$

It follows Eqs. (III-153) and Ineq. (III-154) that

$$D^2_{F(s_4)-F_1,\bar{\eta}(s_4)-\eta_1,\beta(s_4)-\beta_1}\hat{e}(F(s_3),\bar{\eta}(s_3),\beta(s_3) >$$
$$D^2_{F(s_4)-F_1,\eta(s_4)-\eta_1,\beta(s_4)-\beta_1}\hat{e}(F(s_2),\eta(s_2),\beta(s_2)) \qquad \text{(III-157)}$$

for all s_2, s_3 such that

$$0 \leq s_2 < s_1$$
$$0 \leq s_3 < s_1$$

It follows Ineq. (III-157) that the left-hand side of Eq. (III-155) is greater than the left-hand side of Eq. (III-156), and Ineq. (III-150) then follows simply by replacing $(F(s_4),\eta(s_4)\beta(s_4))$ by $(F_2\eta_2\beta_2)$ and $\bar{\eta}(s_4)$ by η_3 .

Q.E.D.

Theorem 11

Let $(F_1,\eta_1,\beta_1) \varepsilon D$ and $\hat{e}_{\eta\eta}(F_1,\eta_1,\beta_1) \neq 0$. Ineq. (III-16) is equivalent to the following: There is a neighborhood of the function F_1X on $N_R(0)(N(F_1))$, a neighborhood of $\eta_1(N(\eta_1))$, and a neighborhood of $\beta_1(N(\beta_1))$, such that for all homeomorphisms (on $N_R(0)$) $f(X) \varepsilon N(F_1)$, and all $\beta(X) \varepsilon N(\beta_1)$

$$\int_{N_R(0)} \hat{e}(F(X),\eta(X),\beta(X))\, dV > \hat{e}(F_1,\eta_1,\beta_1)\, V \tag{III-158}$$

where (i) $f(X) = F_1X$ on $\partial N_R(0)$, and

$F(X) = \nabla f(X)$ on $N_R(o)$

(ii) $\int_{N_R(0)} (\beta(X)-\beta_1)\, dV = 0$

(iii) $\eta(X) = \eta_1 - \dfrac{1}{\hat{e}_{\eta\eta}(F_1,\eta_1,\beta_1)} \Big\{ \mathrm{tr}\left[(F(X)-F_1)^T \hat{e}_{F}\right]_\eta (F_1,\eta_1,\beta_1)$

$+ \mathrm{tr}\left[(\beta(X)-\beta_1)^T \hat{e}_{\beta}\right]_\eta (F_1,\eta_1,\beta_1) \Big\}$ for all $X \varepsilon N_R(0)$

(iv) $(f(X),\eta(X),\beta(X)) \neq (F_1\, X,\eta_1,\beta_1)$ on $N_R(0)$

Proof

It follows Ineq. (III-16) that

$$\int_{N_R(0)} \hat{e}(F(X),\eta(X),\beta(X))\, dV > V\, \hat{e}(F_1\eta_1\beta_1) \tag{III-159}$$

where $(f(X),\eta(X),\beta(X))$ on $N_R(0)$ are restricted as follows

(i) $f(X) = F_1X$ on $\partial N_R(0)$, and

$F(X) = \nabla f(X)$ on $N_R(0)$

(ii) $\int_{N_R(0)} (\beta(X)-\beta_1)\, dV = 0$

$$\text{(iii)} \int_{N_R(0)} (\eta(X)-\eta_1)\ dV = 0$$

$$\text{(iv)} \ (f(X),\eta(X),\beta(X)) \neq (F_1X,\eta_1,\beta_1) \text{ on } N_R(0)$$

Since the restrictions for Ineq. (III-159) satisfy the restrictions for Ineq. (III-16) it follows that

$$\text{(III-16)} \Rightarrow \text{(III-159)} \tag{III-160}$$

Now let $(f(X),\eta(X),\beta(X))$ on $N_R(0)$ be generalized to satisfy the restrictions for Ineq. (III-16). Let

$$\eta_2 \equiv \frac{1}{V} \int_{N_R(0)} \eta(X)\ dV$$

$$\beta_2 \equiv \frac{1}{V} \int_{N_R(0)} \beta(X)\ dV \tag{III-161}$$

Ineq. (III-16) is equivalent to the following:

$$\int_{N_R(0)} \hat{e}(F(X),\eta(X),\beta(X))-\hat{e}(F_1,\eta_2,\beta_2)\ dV$$

$$+ V(\hat{e}(F_1,\eta_2,\beta_2)-\hat{e}(F_1,\eta_1,\beta_1)) > 0 \tag{III-162}$$

where (i) $f(X) = F_1X$ on $\partial N_R(0)$, and

$F(X) = \nabla f(X)$

$$\text{(ii)} \int_{N_R(0)} (\beta(X)-\beta_2)\ dV = 0$$

$$\text{(iii)} \int_{N_R(0)} (\eta(X)-\eta_2)\ dV = 0$$

(iv) $(f(X), \eta(X), \beta(X)) \neq (F_1 X, \eta_1, \beta_1)$ on $N_R(0)$

(v) $\hat{\theta}(F_1, \eta_1, \beta_1)(\eta_2 - \eta_1) + \mathrm{tr}\, \hat{\tau}^T(F_1, \eta_1, \beta_1)(\beta_2 - \beta_1) = 0$

Note that restriction (v) follows Eqs. (III-161) and restriction (ii) for Ineq. (III-16). Note also that the restrictions of $\eta(X), \beta(X)$ on $N_R(0)$ to $N(\eta_1), N(\beta_1)$ and Eq. (III-161) insure that $(F_1, \eta_2, \beta_2) \in D$.

$(f(X), \eta(X), \beta(X)) = (F_1 X, \eta_2, \beta_2)$ on $N_R(0)$ satisfies the restrictions to Ineq. (III-159). It follows Ineq. (III-159) that

$$\hat{e}(F_1, \eta_2, \beta_2) - \hat{e}(F_1, \eta_1, \beta_1) > 0 \qquad \text{(III-163)}$$

Also since $(F_1, \eta_2, \beta_2) \in D$ it follows Ineq. (III-159) that

$$\int_{N_R(0)} \hat{e}(F(X), \eta(X), \beta(X)) - \hat{e}(F_1, \eta_2, \beta_2)\, dV > 0 \qquad \text{(III-164)}$$

It follows Ineq. (III-163), (III-164), (III-162) that

$$\text{(III-159)} \Rightarrow \text{(III-16)} \qquad \text{(III-165)}$$

Then it follows Eqs. (III-160), (III-165) that

$$\text{(III-16)} \Leftrightarrow \text{(III-159)} \qquad \text{(III-166)}$$

Now let $(f(X), \eta(X), \beta(X))$ on $N_R(0)$ satisfy the restrictions for Ineq. (III-159), and define $\bar{\eta}(X)$ on $N_R(0)$ as follows:

$$\bar{\eta}(X) = \eta_1 - \frac{1}{e_{\eta\eta}(F_1, \eta_1, \beta_1)} \Big\{ \mathrm{tr} \left[(F(X) - F_1)\hat{e}_F \right]_\eta (F_1, \eta_1, \beta_1)$$

$$+ \mathrm{tr} \left[(\beta(X) - \beta_1)^T \hat{e}_\beta \right]_\eta (F_1, \eta_1, \beta_1) \Big\} \qquad \text{(III-167)}$$

where it is assumed

$$\hat{e}_{\eta\eta}(F_1,\eta_1,\beta_1) \neq 0 \tag{III-168}$$

Let B represent a constant second-order tensor on $N_R(0)$. It follows Green's transformation that

$$\int_{N_R(0)} \operatorname{tr} B(F(X)-F_1)\, dV = \int_{\partial N_R(0)} \operatorname{tr}\left\{B(f(X)-F_1(X))\, dA\right\} \tag{III-169}$$

It follows Eq. (III-169) and restriction (i) for Ineq. (III-159) that

$$\operatorname{tr} B \int_{N_R(0)} (F(X)-F_1)\, dV = 0 \tag{III-170}$$

The integral above is a tensor, and Eq. (III-170) states that it is orthogonal to B. But B was chosen arbitrarily. The only second-order tensor which is orthogonal to all second-order tensors is the zero tensor. It follows that

$$\text{restriction (i)} \Rightarrow \int_{N_R(0)} (F(X)-F_1)\, dV = 0 \tag{III-171}$$

Now integrate Eq. (III-167) on $N_R(0)$ and it follows Eq. (III-171) and restriction (ii) for Ineq. (III-159) that

$$\int_{N_R(0)} (\bar{\eta}(X)-\eta_1)\, dV = 0 \tag{III-172}$$

It follows Eq. (III-172) that $\bar{\eta}(X)$ on $N_R(0)$ defined by Eq. (III-167) satisfies restriction (iii). In other words, for Ineq. (III-159)

$$\left.\begin{array}{l}\text{restrictions (i), (ii)}\\ \text{and Eq. (III-167)}\end{array}\right\} \Rightarrow \text{restriction (iii)} \qquad \text{(III-173)}$$

It follows Ineq. (III-159) that for any $f(X), \beta(X)$ satisfying restrictions (i), (ii) and $\bar{\eta}(X)$ from Eq. (III-167)

$$\int_{N_R(0)} (\hat{e}(F(X),\bar{\eta}(X),\beta(X))-\hat{e}(F_1,\eta_1,\beta_1))\, dV > 0 \qquad \text{(III-174)}$$

It follows Theorem 10 that

$$\begin{aligned}&\hat{e}(F(X),\eta(X),\beta(X))-\hat{e}(F_1,\eta_1,\beta_1)-\mathrm{tr}\frac{1}{\rho_R}\hat{T}_R{}^T(F_1,\eta_1,\beta_1)(F(X)-F_1)\\ &\quad -\hat{\theta}(F_1,\eta_1,\beta_1)(\eta(X)-\eta_1)-\mathrm{tr}\hat{\tau}^T(F_1,\eta_1,\beta_1)(\beta(X)-\beta_1) >\\ &\hat{e}(F(X),\bar{\eta}(X),\beta(X))-\hat{e}(F_1,\eta_1,\beta_1)-\mathrm{tr}\frac{1}{\rho_R}\hat{T}_R{}^T(F_1,\eta_1,\beta_1)(F(X)-F_1)\\ &\quad -\hat{\theta}(F_1,\eta_1,\beta_1)(\bar{\eta}(X)-\eta_1)-\mathrm{tr}\hat{\tau}^T(F_1,\eta_1,\beta_1)(\beta(X)-\beta_1)\end{aligned} \qquad \text{(III-175)}$$

for all $X \in N_R(0)$, and $\eta(X) \neq \bar{\eta}(X)$

Now integrate Ineq. (III-175) on $N_R(0)$, and it follows restrictions (ii), (iii) and Eqs. (III-171), (III-172) that

$$\int_{N_R(0)} (\hat{e}(F(X),\eta(X),\beta(X))-\hat{e}(F_1\eta_1\beta_1))\, dV >$$
$$\int_{N_R(0)} (\hat{e}(F(X),\bar{\eta}(X),\beta(X))-\hat{e}(F_1,\eta_1,\beta_1))\, dV \qquad \text{(III-176)}$$

for all $\eta(X) \neq \bar{\eta}(X)$ on $N_R(0)$

It follows Eqs. (III-176), (III-174) that Ineq. (III-159) for $\bar{\eta}(X)$ on $N_R(0)$ is both necessary and sufficient. In other words

Ineq. (III-159) $\Leftrightarrow$ Ineq. (III-159)
with restrictions with restriction (iii)
(i)(ii)(iii)(iv) replaced by Eq. (III-167) (III-177)

Theorem 11 follows Eqs. (III-166), (III-177).

Q.E.D.

<u>Theorem 12</u>

Let $(F_1, \eta_1, \beta_1) \varepsilon D$, $\beta^* \neq \beta_1$, and L_{β^*} denote the straight path in (η,β)-space, through $(\eta,\beta) = (\eta_1,\beta_1)$, which is tangent to an adiabat for $(F,\eta,\beta) = (F_1,\eta_1,\beta_1)$, i.e.

$$L_{\beta^*} \equiv \left\{(\eta,\beta) \mid \beta\varepsilon L[\beta_1,\beta^*],\ \eta = \eta_1 - \frac{1}{\hat{\theta}(F_1,\eta_1,\beta_1)}\, \mathrm{tr}\hat{\tau}^T(F_1,\eta_1,\beta_1)(\beta-\beta_1)\right\}$$

(III-178)

For any neighborhood of $\beta_1 (N(\beta_1))$ there is a neighborhood of $F_1 (N(F_1))$ such that for any $F_2 \varepsilon N(F_1)$, $F_2 \neq F_1$, and $\beta^* \varepsilon N(\beta_1)$, $\beta^* \neq \beta_1$ there is a pair $(\eta_2,\beta_2)\varepsilon L_{\beta^*}$ such that

$$D^2_{F_2-F_1,\eta_2-\eta_1,\beta_2-\beta_1}\ \hat{e}(F_1,\eta_1,\beta_1) \leq D^2_{F_2-F_1,\eta-\eta_1,\beta-\beta_1}\hat{e}(F_1,\eta_1,\beta_1)$$

(III-179)

for all $(\eta,\beta)\varepsilon L_{\beta^*}$, $(\eta,\beta) \neq (\eta_2,\beta_2)$,
and the equality holds only if

(i) $D^2_{0,\eta_2-\eta_1,\beta_2-\beta_1}\ \hat{e}(F_1,\eta_1,\beta_1) = 0$

Also

$$\begin{Bmatrix} F_2-F_1 \\ 0 \\ 0 \end{Bmatrix}^T \begin{bmatrix} \partial^2\hat{e} \end{bmatrix} \begin{Bmatrix} 0 \\ \eta_2-\eta_1 \\ \beta_2-\beta_1 \end{Bmatrix} + D^2_{0,\eta_2-\eta_1,\beta_2-\beta_1}\ \hat{e}(F_1,\eta_1,\beta_1) = 0$$

(III-180)

$$D^2_{F_2-F_1,\eta_2-\eta_1,\beta_2-\beta_1}\ \hat{e}(F_1,\eta_1,\beta_1) = D^2_{F_2-F_1,o,o}\ \hat{e}(F_1,\eta_1,\beta_1) - D^2_{0,\eta_2-\eta_1,\beta_2-\beta_1}\ \hat{e}(F_1,\eta_1,\beta_1) \tag{III-181}$$

$$D^2_{F_2-F_1,\eta-\eta_1,\beta-\beta_1}\ \hat{e}(F_1,\eta_1,\beta_1) = D^2_{F_2-F_1,\eta_2-\eta_1,\beta_2-\beta_1,}\ \hat{e}(F_1,\eta_1,\beta_1) + (a-1)^2\ D^2_{0,\eta_2-\eta_1,\beta_2-\beta_1}\hat{e}(F_1,\eta_1,\beta_1) \tag{III-182}$$

where a is uniquely defined by

(i) $\beta-\beta_1 = a(\beta_2-\beta_1)$ for $(\beta_2-\beta_1) \neq 0$

$$D^2_{F_2-F_1,\eta-\eta_1,\beta-\beta_1}\ \hat{e}(F_1,\eta_1,\beta_1) = D^2_{F_2-F_1,0,0}\ \hat{e}(F_1,\eta_1,\beta_1) \tag{III-183}$$

if (i) $(\beta_2-\beta_1) = 0$

Proof

Expand the second-directional derivative:

$$D^2_{F_2-F_1,\eta-\eta_1,\beta-\beta_2}\ \hat{e}(F_1,\eta_1,\beta_1) = D^2_{F_2-F_1,0,0}\ \hat{e}(F_1,\eta_1,\beta_1) + D^2_{0,\eta-\eta_1,\beta-\beta_1}\ \hat{e}(F_1,\eta_1,\beta_1) + 2\begin{Bmatrix} F_2-F_1 \\ 0 \\ 0 \end{Bmatrix}^T \begin{bmatrix} \partial^2\hat{e} \end{bmatrix} \begin{Bmatrix} 0 \\ \eta-\eta_1 \\ \beta-\beta_1 \end{Bmatrix} \tag{III-184}$$

It follows Ineq. (III-80) that

$$D^2_{0,\eta-\eta_1,\beta-\beta_1}\ \hat{e}(F_1,\eta_1,\beta_1) \geq 0 \qquad \text{(III-185)}$$

for all $(\eta,\beta)\varepsilon L_{\beta*}$, $(\eta,\beta) \neq (\eta_1,\beta_1)$

Let $(\eta_2,\beta_2) \neq (\eta_1,\beta_1)$ satisfy the following:

$$\begin{Bmatrix} F_2-F_1 \\ 0 \\ 0 \end{Bmatrix}^T \begin{bmatrix} \partial^2\hat{e} \end{bmatrix} \begin{Bmatrix} 0 \\ \eta_2-\eta_1 \\ \beta_2-\beta_1 \end{Bmatrix} + D^2_{0,\eta_2-\eta_1,\beta_2-\beta_1}\ \hat{e}(F_1,\eta_1,\beta_1) = 0 \qquad \text{(III-186)}$$

where $(\eta_2,\beta_2)\varepsilon L_{\beta*}$.

Let $a\varepsilon R$; (η,β) on $L_{\beta*}$ may be represented as follows:

$$\begin{aligned} \eta &= \eta_1 + a(\eta_2-\eta_1) \\ \beta &= \beta_1 + a(\beta_2-\beta_1) \end{aligned} \qquad \text{(III-187)}$$

where $a\varepsilon R$, and the pair (η_2,β_2) satisfies Eq. (III-186).

It follows Eq. (III-187),(III-186) that

$$\begin{aligned} &\begin{Bmatrix} F_2-F_1 \\ 0 \\ 0 \end{Bmatrix}^T \begin{bmatrix} \partial^2\hat{e} \end{bmatrix} \begin{Bmatrix} 0 \\ \eta-\eta_1 \\ \beta-\beta_1 \end{Bmatrix} + D^2_{0,\eta-\eta_1,\beta-\beta_1}\hat{e}(F_1,\eta_1,\beta_1) \\ &= a\begin{Bmatrix} F_2-F_1 \\ 0 \\ 0 \end{Bmatrix}^T \begin{bmatrix} \partial^2\hat{e} \end{bmatrix} \begin{Bmatrix} 0 \\ \eta_2-\eta_1 \\ \beta_2-\beta_1 \end{Bmatrix} + a^2 D^2_{0,\eta_2-\eta_1,\beta_2-\beta_1}\hat{e}(F_1,\eta_1,\beta_1) \end{aligned} \qquad \text{(III-188)}$$

and $a = 1$ is a root for Eq. (III-188). If

$$D^2_{0,\eta_2-\eta_1,\beta_2-\beta_1}\hat{e}(F_1,\eta_1,\beta_1) > 0$$

it follows Eqs. (III-186), (III-188) that

$$\begin{Bmatrix} F_2-F_1 \\ 0 \\ 0 \end{Bmatrix}^T \begin{bmatrix} \partial^2\hat{e} \end{bmatrix} \begin{Bmatrix} 0 \\ \eta_2-\eta_1 \\ \beta_2-\beta_1 \end{Bmatrix} = -D^2_{0,\eta_2-\eta_1,\beta_2-\beta_1}\hat{e}(F_1,\eta_1,\beta_1) < 0 \quad \text{(III-189)}$$

and the root for Eq. (III-188) - i.e. $a = 1$ - is unique.
If $D^2_{0,\eta_2-\eta_1,\beta_2-\beta_1}\ \hat{e}(F_1,\eta_1,\beta_1) = 0$ it follows Eqs. (III-186), (III-188) that

$$\begin{Bmatrix} F_2-F_1 \\ 0 \\ 0 \end{Bmatrix}^T \begin{bmatrix} \partial^2\hat{e} \end{bmatrix} \begin{Bmatrix} 0 \\ \eta_2-\eta_1 \\ \beta_2-\beta_1 \end{Bmatrix} = -D^2_{0,\eta_2-\eta_1,\beta_2-\beta_1}\ \hat{e}(F_1,\eta_1,\beta_1) = 0 \quad \text{(III-190)}$$

and the root for Eq. (III-188) is not unique. It follows Eqs. (III-187), (III-189), (III-190) that Eq. (III-184) has the following representation:

$$D^2_{F_2-F_1,\eta-\eta_1,\beta-\beta_1}\ \hat{e}(F_1,\eta_1,\beta_1) = D^2_{F_2-F_1,0,0}\ \hat{e}(F_1,\eta_1,\beta_1)$$

$$+ a(a-2)D^2_{0,\eta_2-\eta_1,\beta_2-\beta_1}\ \hat{e}(F_1,\eta_1,\beta_1) \quad \text{(III-191)}$$

and

$$D^2_{F_2-F_1,\eta_2-\eta_1,\beta_2-\beta_1}\hat{e}(F_1,\eta_1,\beta_1) = D^2_{F_2-F_1,0,0}\ \hat{e}(F_1,\eta_1,\beta_1)$$

$$- D^2_{0,\eta_2-\eta_1,\beta_2-\beta_1}\ \hat{e}(F_1,\eta_1,\beta_1) \quad \text{(III-192)}$$

It follows Eq. (III-191), (III-192) that

$$D^2_{F_2-F_1,\eta-\eta_1,\beta-\beta_1}\ \hat{e}(F_1,\eta_1,\beta_1) = D^2_{F_2-F_1,\eta_2-\eta_1,\beta_2-\beta_1}\ \hat{e}(F_1,\eta_1,\beta_1)$$

$$+ (a-1)^2\ D^2_{0,\eta_2-\eta_1,\beta_2-\beta_1}\hat{e}(F_1,\eta_1,\beta_1) \quad \text{(III-193)}$$

Equation (III-180) follows Eq. (III-186); Eq. (III-181) follows Eq. (III-192); Eq. (III-182) follows Eq. (III-193); Eq. (III-183) is trivial; Ineq. (III-179) follows Eq. (III-182) easily.

Q.E.D.

<u>Theorem 13</u>

Let $(F_1, \eta_1, \beta_1) \varepsilon D$ and F_2 in some neighborhood of F_1. There is a pair $(\bar{\eta}_2, \bar{\beta}_2)$ and a neighborhood $(N(\eta_1, \beta_1))$ such that

$$(\bar{\eta}_2, \bar{\beta}_2) \varepsilon N(\eta_1, \beta_1)$$

$$D^2_{F_2-F_1, \bar{\eta}_2-\eta_1, \bar{\beta}_2-\beta_1} \hat{e}(F_1, \eta_1, \beta_1) \leq D^2_{F_2-F_1, \eta-\eta_1, \beta-\beta_1} \hat{e}(F_1, \eta_1, \beta_1) \tag{III-194}$$

for all $(\eta, \beta) \varepsilon N(\eta_1, \beta_1)$ such that

(i) $\theta_1(\eta-\eta_1) + \mathrm{tr}\, \tau_1^T(\beta-\beta_1) = 0$

<u>Proof</u>

The proof of this theorem follows theorem 12. Ineq. (III-179) holds for each $L_{\beta*}$. It follows Eq. (III-186) and Eq. (III-11)[1] that the vector $(\eta_2-\eta_1, \beta_2-\beta_1)$ is bounded if (F_2-F_1) is bounded for each $L_{\beta*}$. Also the map $F_2-F_1 \longrightarrow \eta_2-\eta_1, \beta_2-\beta_1$ is linear. Since the vector $(F_2-F_1, \eta_2-\eta_1, \beta_2-\beta_1)$ is bounded, it follows Eq. (III-11)[1] that

$$D^2_{F_2-F_1, \eta_2-\eta_1, \beta_2-\beta_1} \hat{e}(F_1, \eta_1, \beta_1) \text{ is bounded for any } L_{\beta*} \tag{III-195}$$

Now hold F_2 fixed and consider the value of the second directional derivative for different $L_{\beta *}$'s . It follows Eq. (III-190) that there must be a infinum for the values; let $L_{\overline{\beta}*}$ and $(\overline{\eta}_2,\overline{\beta}_2)$ correspond to the infinum. It follows that

$$D^2_{F_2-F_1,\overline{\eta}_2-\eta_1,\overline{\beta}_2-\beta_1}\ \hat{e}(F_1,\eta_1,\beta_1) \leq D^2_{F_2-F_1,\eta_2-\eta_1,\beta_2-\beta_1}\ \hat{e}(F_1,\eta_1,\beta_1)$$

$$\text{for any } L_{\beta *} \qquad \text{(III-196)}$$

Ineq. (III-194) follows Ineqs. (III-196) and (III-179).

Q.E.D.

Theorem 14

The caloric equation of state, Eq. (III-10)[1], is invertible in η and the inverse is differentiable. A state (F,η,β) is uniquely characterized by the triplet (F,e,β), where e is the value of the internal energy density which corresponds to the state (F,η,β). Let $\hat{\hat{\eta}}(F,e,\beta)$ represent the inverse function:

$\hat{\hat{\eta}}(F,e,\beta)$ is the inverse of $\hat{e}(F,\ ,\beta)$ on D,

$\hat{\hat{\eta}}(F,e,\beta)$ is differentiable on the inverse (III-197)

of D, and the map

$$(F,\eta,\beta) \longleftrightarrow (F,e,\beta) \text{ is a diffeomorphism on D} \qquad \text{(III-198)}$$

Proof

Let $(F_1,\eta_1,\beta_1)\varepsilon D$. Because D is open it follows that there are neighborhoods of (F_1,η_1,β_1), $N(F_1,\eta_1,\beta_1)$, such that $N(F_1,\eta_1,\beta_1)\subset D$. Let $(F_1,\eta_1,\beta_1) \longrightarrow \theta_1,\ e_1$. It follows Eqs. (III-18)[2], (III-11)[1,3] that

$$\hat{e}(F,\eta,\beta) \text{ is differentiable, strictly increasing, and convex in } \eta \tag{III-199}$$

Therefore for $(F,\eta) = (F_1,\eta_1)$

$$\eta \xrightarrow{F_1,\beta_1} e \quad \text{differentiable, strictly increasing, and convex on D} \tag{III-200}$$

Since the inverse was defined for an arbitrary pair (F_1,β_1), the definition may be extended onto the image of D:

$$\eta = \hat{\hat{\eta}}\,(F,e,\beta) \quad \text{on the image of D}$$
$$\text{or } (F,e,\beta) \longrightarrow \eta \tag{III-201}$$

Choose a neighborhood of η_1, $f(\eta_1)$:

$$f(\eta_1) = \{\eta \mid \eta_L < \eta < \eta_U\}$$
$$\text{where } \eta_L < \eta_1 < \eta_U \tag{III-202}$$

Define e_L, e_U as follows:

$$\left.\begin{aligned} e_L &= \hat{e}(F,\eta_L,\beta) \\ e_U &= \hat{e}(F,\eta_U,\beta) \end{aligned}\right\} \text{ on D} \tag{III-203}$$

It follows the continuity of $\hat{e}(F,\eta,\beta)$ and Eq. (III-202) that a neighborhood of (F_1,β_1), $N(F_1,\beta_1)$ exists such that

$$(F,\beta)\epsilon N(F_1,\beta_1) \Rightarrow \begin{array}{l} \hat{e}(F,\eta_L,\beta) < e_1 \\ \hat{e}(F,\eta_U,\beta) > e_1 \end{array} \tag{III-204}$$

Let $e_{min.}$, $e_{max.}$ represent the inf of e_L and the sup of e_U respectively:

$$\left.\begin{aligned} e_{min.} &= \inf \ \hat{e}(F,\eta_L,\beta) \\ e_{max.} &= \sup \ \hat{e}(F,\eta_U,\beta) \end{aligned}\right\} \text{ on } N(F_1,\beta_1) \tag{III-205}$$

It follows Eq. (III-205), Ineq. (III-204) that

$$e_{min.} < e_1 < e_{max.} \tag{III-206}$$

Define a neighborhood of e_1 as follows:

$$N(e_1) = \{e | e_{min} < e < e_{max.}\} \tag{III-207}$$

It follows the above construction that

$$(F,e,\beta) | (F,\beta) \varepsilon N(F_1,\beta_1), e \varepsilon N(\eta_1) \Rightarrow$$

$$(F,e,\beta) \longrightarrow \eta \varepsilon N(\eta_1) \tag{III-208}$$

In other words for any neighborhood of η_1 a neighborhood of (F_1,e_1,β_1) can be found such that the image of $N(F_1,e_1,\beta_1) \varepsilon N(\eta_1)$. It follows that

$$(F,e,\beta) \longrightarrow \eta \text{ is continuous on the image of } D \ . \tag{III-209}$$

It follows Eq. (III-209) easily that

$$(F,e,\beta) \longrightarrow (F,\eta,\beta) \text{ is continuous on the image of } D \ . \tag{III-210}$$

Now assume $\hat{\hat{\eta}}(F,e,\beta)$ is differentiable and derive the partial derivatives by implicit differentiation:

$$\hat{\hat{\eta}}_F(F,e,\beta) = -\frac{\hat{e}_F(F,\hat{\hat{\eta}}(F,e,\beta),\beta)}{\hat{e}_\eta(F,\hat{\hat{\eta}}(F,e,\beta),\beta)}$$

$$\hat{\hat{\eta}}_e(F,e,\beta) = \frac{1}{\hat{e}_\eta(F,\hat{\hat{\eta}}(F,e,\beta),\beta)}$$

$$\hat{\hat{\eta}}_\beta(F,e,\beta) = -\frac{\hat{e}_\beta(F,\hat{\hat{\eta}}(F,\hat{\hat{\eta}}(F,e,\beta)\beta)}{\hat{e}_\eta(F,\hat{\hat{\eta}}(F,e,\beta),\beta)} \qquad \text{(III-211)}$$

It follows Ineq.(III-11)^3, Eqs.(III-210), (III-11)^1 that the right-hand sides of Eq. (III-211) are bounded and continuous on the image of D . Therefore Eqs. (III-211) define the partial derivatives of $\hat{\hat{\eta}}(F,e,\beta)$ and it follows Eq. (III-210) that

$$(F,e,\beta) \longrightarrow (F,\eta,\beta) \text{ is diffeomorpic on the image of D .} \qquad \text{(III-212)}$$

Existence of the inverse follows Eq. (III-201); its continuity follows Eq. (III-209); finally its differentiability follows Eq. (III-211).

Q.E.D.

Some topological properties of D follow directly Theorems 4 and 13.

Corollary

The maps

$$(F,\eta,\beta) \longleftrightarrow (F,\theta,\beta) \longleftrightarrow (F,e,\beta) \qquad \text{(III-213)}$$

are homeomorphic on D . Therefore D preserves its topological properties under these maps. Therefore a state in D is completely characterized by any one of the triplets - (F,η,β) (F,θ,β), or (F,e,β) . All of the state variables may be represented as functions of (F,θ,β) or (F,e,β) in addition to (F,η,β) . The following notation will be used for these representations:

$$\left.\begin{aligned} e &= \hat{\hat{e}}(F,\theta,\beta) \\ T_R &= \hat{\hat{T}}_R(F,\theta,\beta) \\ \tau &= \hat{\hat{\tau}}(F,\theta,\beta) \\ \eta &= \hat{\eta}(F,\theta,\beta) \end{aligned}\right\} \text{ on } D \tag{III-214}$$

$$\left.\begin{aligned} \eta &= \hat{\hat{\eta}}(F,e,\beta) \\ T_R &= \hat{\hat{\hat{T}}}_R(F,e,\beta) \\ \theta &= \hat{\hat{\theta}}(F,e,\beta) \\ \tau &= \hat{\hat{\hat{\tau}}}(F,e,\beta) \end{aligned}\right\} \text{ on } D \tag{III-215}$$

<u>Theorem 15</u>

Let $(F_1,\eta_1,\beta_1)\varepsilon D$, $F_2 \neq F_1$. Let $(\bar{\eta}_2,\bar{\beta}_2)$ and $N(\eta_1,\beta_1)$ correspond to F_2 according to theorem 13. Let (F_2,η,β) represent any triplet in D such that $L[(F_1,\eta_1,\beta_1), (F_2,\eta,\beta)]$ is an adiabatic tangent at (F_1,η_1,β_1). Then there is a neighborhood of $(F_1,\eta_1,\beta_1)(N(F_1,\eta_1,\beta_1))$ such that for all states $(F_2,\eta,\beta)\varepsilon N(F_1,\eta_1,\beta_1)$, which do not correspond to theorem 13,

$$\hat{e}(F_2,\eta,\beta) > \hat{e}(F_2,\bar{\eta}_2,\bar{\beta}_2) \tag{III-216}$$

Proof

Let $N(F_1,\eta_1,\beta_1)\epsilon D$ and $(F_2,\bar{\eta}_2,\bar{\beta}_2),(F_2,\eta,\beta)\epsilon N(F_1,\eta_1,\beta_1)$, $F_2 \neq F_1$, according to the prescription of the above theorem. It follows theorem 13 that

$$D^2_{F_2-F_1,\eta-\eta_1,\beta-\beta_1}\ \hat{e}(F_1,\eta_1,\beta_1) > D^2_{F_2-F_1,\bar{\eta}_2-\eta_1,\bar{\beta}_2-\beta_1}\hat{e}(F_1,\eta_1,\beta_1) \tag{III-217}$$

Let $(F(s),\eta(s),\beta(s))$ represent the state and $e(s)$ represent the internal energy density on $L\left[(F_1,\eta_1,\beta_1)(F_2,\eta,\beta)\right]$ and $(F(s),\bar{\eta}(s),\bar{\beta}(s))$, $\bar{e}(s)$ represent similar parameters on $L\left[(F_1,\eta_1,\beta_1)(F_2,\bar{\eta}_2,\bar{\beta}_2)\right]$, where s is the path parameter:

$$(F(s),\eta(s),\beta(s)) = (1-s)(F_1,\eta_1,\beta_1) + s(F_2,\eta,\beta)$$

$$e(s) = e(F(s),\eta(s),\beta(s)) \quad \text{and}$$

$$(F(s),\bar{\eta}(s),\bar{\beta}(s)) = (1-s)(F_1,\eta_1,\beta_1) + s(F_2,\bar{\eta}_2,\bar{\beta}_2)$$

$$\bar{e}(s) = \hat{e}(F(s),\bar{\eta}(s),\bar{\beta}(s)) \tag{III-218}$$

$$\text{for} \quad 0 \leq s \leq 1$$

It follows Eq. (III-11)[1], (III-18) and Taylor's formula that there is a neighborhood of zero $(N(0))$ such that for $s\epsilon N(0)$

$$\begin{aligned} e(s)-e(0) &= \operatorname{str} \frac{1}{\rho_R} T_R{}^T(F_1,\eta_1,\beta_1)(F_2-F_1) \\ &+ \frac{s^2}{2} D^2_{F_2-F_1,\eta-\eta_1,\beta-\beta_1}\hat{e}(F_1,\eta_1,\beta_1)+\ldots \end{aligned} \tag{III-219}$$

$$\begin{aligned} \bar{e}(s)-e(0) &= \operatorname{str} \frac{1}{\rho_R} \hat{T}_R{}^T(F_1,\eta_1,\beta_1)(F_2-F_1) \\ &+ \frac{s^2}{2} D^2_{F_2-F_1,\bar{\eta}_2-\eta_1,\bar{\beta}_2-\beta_1}\ \hat{e}(F_1,\eta_1,\beta_1)+\ldots \end{aligned}$$

or

$$e(s)-\bar{e}(s) = \frac{s^2}{2}\left[D^2_{F_2-F_1,\eta-\eta_1,\beta-\beta_1}\ \hat{e}(F_1,\eta_1,\beta_1)\right.$$

$$\left. - D^2_{F_2-F_1,\bar{\eta}_2-\eta_1,\beta_2,\beta_1}\hat{e}(F_1,\eta_1,\beta_1)\right]$$

$$+ \ldots \qquad \text{(III-220)}$$

for all $s\varepsilon N(0)$

Ineq. (III-216) follows easily Eqs. (III-217), (III-220).

Q.E.D.

IV. GLOBAL STABILITY

In this report "global stability" will refer to stability of a finite body (B) . Two types of global stability will be defined and analyzed. The definitions are generalizations of the definitions given by Coleman and Noll[9]. In the following, the concept of a global state and a neighborhood will be used. Also the caloric equation of state, Eq. (III-10)[1], and the stress temperature and substate relations, Eqs. (III-18)[1,2] and (III-35), are assumed.

<u>Definition of Global State</u>

A "global state" $\{\hat{x},\eta,\beta\}$ of a body (B) is a configuration $\hat{x}$ of B (i.e., $x = \hat{x}(X)$), an entropy distribution of B (i.e., $\eta = \hat{\eta}(X)$), and a substate distribution of B (i.e., $\beta = \beta(X)$).

A neighborhood in global state-space is defined by the following metric:

$$\delta(\{\hat{x},\eta,\beta\},\{\hat{x}^*\eta^*\beta^*\}) = \sup_{X\epsilon B}\{|\hat{x}^*(X)-\hat{x}(X)|$$

$$+ |F^{*-1}(X)F(X)-I| + |\eta^*-\eta| + |\beta^{*-1}(X)\beta(X)-I \qquad \text{(IV-1)}$$

Note in the above definition of the metric that the rotation tensor of F^* is not restricted.

<u>Corollary 1</u>:

The restriction to a neighborhood in global state-space also restricts each local state to a neighborhood in state-space, i.e.,

$$\{\hat{x}^*,\eta^*,\beta^*\} \ \epsilon \text{ neighborhood of } \{\hat{x},\eta,\beta\} \Longleftrightarrow (F^*(X),\eta^*(X),\beta^*(X))$$

$$\epsilon \text{ neighborhood of } (F(X),\eta(X),\beta(X)) \text{ for each } X\epsilon B$$

$$\text{(IV-2)}$$

One other concept will be required for the definition. Consider the external supply of heat at X .

$$q(X) = \theta(X)\dot{\eta}(X) + tr(\tau^T(X)\dot{\beta}(X)) \qquad \text{(IV-3)}$$

If this equation is integrated on a path while holding the temperature and substate tension fixed, the result has the appearance of a potential for heating.

$$\Delta q(X) = \hat{\theta}(F(X),\eta(X),\beta(X))(\eta^*(X)-\eta(X)) + tr\,\hat{\tau}^T(F(X),\eta(X),\beta(X))(\beta^*(X)-\alpha(X)) \qquad \text{(IV-4)}$$

If $\Delta q(X)$ is divided by $\theta(X)$ the result has the appearance of entropy supply, which will be used for the definitions of global stability.

<u>Definition of the Virtual Supply of Entropy (h^*)</u>

The virtual supply of entropy is defined by the following equation:

$$h^* = \hat{h}^*(X) \equiv (\eta^*(X) - \eta(X)) + tr\,\frac{\tau^T(F(X),\eta(X),\beta(X))}{\hat{\theta}(F(X),\eta(X),\beta(X))}(\beta^*(X)-\beta(X)) \qquad \text{(IV-5)}$$

<u>Definition of Global Adiabatic Stability (GAS)</u>

Let $\{\hat{x},\eta,\beta\}$ be a state of B and let E be the total internal energy corresponding to the state $\{\hat{x},\eta,\beta\}$. The state $\{\hat{x},\eta,\beta\}$ is an "adiabatically stable state of B " if and only if there is a neighborhood of $\{\hat{x},\eta,\beta\}$ such that every global state $\{\hat{x},\eta^*,\beta^*\}$ in the neighborhood with the same configuration and constrained to zero virtual entropy supply has a greater total internal energy than the global state $\{\hat{x},\eta,\beta\}$: .

$$E^* \equiv \int_B \hat{e}(F(X),\eta^*(X),\beta^*(X))\,dM > E \equiv \int_B \hat{e}(F(X),\eta(X),\beta(X))\,dM$$

(IV-6)

where

(i) $\{\hat{x},\eta^*,\beta^*\} \,\varepsilon$ neighborhood of $\{\hat{x},\eta,\beta\}$

(ii) $\int_B \hat{h}^*(X)\,dM = 0$

(iii) $\{\hat{x},\eta^*,\beta^*\} \neq \{\hat{x},\eta,\alpha\}$

A shorthand notation for the above property is $\{\hat{x},\eta,\beta\}$ of B is GAS.

Theorem 1

A necessary condition for $\{\hat{x},\eta,\beta\}$ of B is GAS is that the temperature is uniform on B, i.e.,

$$\theta = \hat{\theta}(F(X),\eta(X),\beta(X)) \text{ is independent of } X \varepsilon B \qquad \text{(IV-7)}$$

Proof

The following variational statement is equivalent to GAS of $\{\hat{x},\eta,\beta\}$ of B:

$$\int_B \hat{e}(F(X),\eta^*(X),\beta^*(X))\,dM \text{ is locally minimum}$$

$$\text{for } \{\hat{x},\eta^*,\beta^*\} = \{\hat{x},\eta,\beta\} \qquad \text{(IV-8)}$$

where the comparison states are constrained by

$$\int_B \left[\eta^*(X) + \operatorname{tr} \frac{\hat{\tau}^T(FX),\eta(X),\beta(X))}{\hat{\theta}(F(X),\eta(X),\beta(X))}\,\beta^*(X)\right] dM = \text{constant} \qquad \text{(IV-9)}$$

It follows that the first variation of

$$\int_B \hat{e}(F(X),\eta^*(X),\beta^*(X) - \lambda\left[\eta^*(X) + \mathrm{tr}\ \frac{\hat{\tau}^T(F(X),\eta(X),\beta(X))}{\hat{\theta}(F(X),\eta(X),\beta(X))}\beta^*(X)\right]dM$$

(IV-10)

subject to restriction (IV-9) vanishes for $(\eta^*(X),\beta^*(X)) = (\eta(X),\beta(X))$. Here λ is a constant Lagrange multiplier.

It follows Eqs. (III-18)[2], (III-35) that the first variation of

$$\hat{e}(F(X),\eta^*(X),\beta^*(X)) - \hat{\theta}(F(X),\eta(X),\beta(X))\eta^*(X) - \mathrm{tr}\ \hat{\tau}^T(F(X),\eta(X),\beta(X))\beta^*(X) \text{ for } X\varepsilon B \qquad \text{(IV-11)}$$

vanishes for $(\eta^*(X),\beta^*(X)) = (\eta(X),\beta(X))$ but for unrestricted variations.

Integrate (IV-11) over B and subtract the result from (IV-10), and it follows that the first variation of

$$\int_B (\hat{\theta}(F(X),\eta(X),\beta(X)) - \lambda)\left[\eta^*(X) + \mathrm{tr}\ \frac{\hat{\tau}^T(F(X),\eta(X),\beta(X))}{\hat{\theta}(F(X),\eta(X),\beta(X))}\beta^*(X)\right]dM$$

(IV-12)

vanishes for $(\eta^*(X),\beta^*(X)) = (\eta(X),\beta(X))$ and the variations are constrained by Eq. (IV-9).

Setting the first variation of (IV-12) to zero it follows that

$$\int_B (\hat{\theta}(F(X),\eta(X),\beta(X)) - \lambda)\phi(X)dM = 0 \qquad \text{(IV-13)}$$

where $\phi(X)$ is any continuous function on B such that

(i) $\int_B \phi(X)dM = 0$

The only functions on B which are orthogonal to all $\phi(X)$'s are constants, i.e., $\hat{\theta}(F(X),\eta(X),\beta(X)) - \lambda$ is a constant. But since λ itself is constant, it follows that the temperature must be constant. Q.E.D.

Definition of Scaled Global State

Let $\{\hat{x},\eta,\beta\}$ be a global state of B with other properties $T_R(X),\theta(X),\tau(X)$, and $e(X)$. Let $\bar{B}$ represent a geometrically similar body with scale a - i.e., there is a map from the reference configuration of B onto the reference configuration of $\bar{B}$:

$$B \xrightarrow{a} \bar{B}$$

$$\text{where} \quad a \varepsilon R,\ a > 0 \tag{IV-14}$$

Let $\bar{X}$ represent the material coordinates of $\bar{B}$, then

$$\bar{X} = aX \tag{IV-15}$$

Also all other properties of the material (i.e. stress, temperature, etc.) at $\bar{X}$ are equal to the same properties of B at X.

Let $\{\bar{x},\bar{\eta},\bar{\beta}\}$ represent a global state of $\bar{B}$. The state $\{\bar{x},\bar{\eta},\bar{\beta}\}$ is called a scaled global state if

$$\begin{aligned} \bar{x}(\bar{X}) &= ax(X) \\ \bar{\eta}(\bar{X}) &= \eta(X) \\ \bar{\beta}(\bar{X}) &= \beta(X) \end{aligned} \tag{IV-16}$$

and

$$\begin{aligned} \bar{T}_R(\bar{X}) &= T_R(X) \\ \bar{\theta}(\bar{X}) &= \theta(X) \\ \bar{\tau}(\bar{X}) &= \tau(X) \\ \bar{e}(\bar{X}) &= e(X) \end{aligned} \tag{IV-17}$$

etc.

Of course it is assumed the two bodies are of the same constitution - i.e., they obey the same caloric equation of state. Then Eqs. (IV-17) follow Eqs. (IV-16).

Theorem 2

Let $\{\hat{x}, \eta, \beta\}$ be a global state of B and let $\{\overline{x}, \overline{\eta}, \overline{\beta}\}$ of $\overline{B}$ be a scaled global state. Then

$$\begin{array}{c}\{\hat{x}, \eta, \beta\} \text{ of } B \\ \text{is GAS}\end{array} \Rightarrow \begin{array}{c}\{\overline{x}, \overline{\eta}, \overline{\beta}\} \text{ of } \overline{B} \\ \text{is GAS}\end{array} \tag{IV-18}$$

Proof

Let Ineq. (IV-6) represent the property for B . It follows implicit differentiation of Eq. (IV-16)[1] that

$$\overline{F}(\overline{X}) = F(X) \tag{IV-19}$$

and it follows Eq. (IV-15) that

$$d\overline{M} = a^3 \, dM \tag{IV-20}$$

Let the comparison states be scaled:

$$\begin{aligned} \overline{\eta}^*(\overline{X}) &= \eta^*(X) \\ \overline{\beta}^*(\overline{X}) &= \beta^*(X) \end{aligned} \tag{IV-21}$$

Substitute Eqs. (IV-16), (IV-19), (IV-20), (IV-21) into Ineq. (IV-6) and it follows that

$$\int_{\overline{B}} \hat{e}(\overline{F}(\overline{X}), \overline{\eta}^*(\overline{X}), \overline{\beta}^*(X)) \, d\overline{M} > \int_{\overline{B}} \hat{e}(\overline{F}(\overline{X}), \overline{\eta}(\overline{X}), \overline{\beta}(\overline{X})) \, d\overline{M} \tag{IV-22}$$

(i) $\{\bar{x}, \bar{\eta}^*, \bar{\beta}^*\}$ ε neighborhood of $\{\bar{x}, \bar{\eta}, \bar{\beta}\}$

(ii) $\int_{\bar{B}} \bar{h}^*(\bar{X})\ d\bar{M} = 0$

(iii) $\{\bar{x}, \bar{\eta}^*, \bar{\beta}^*\} \neq \{\bar{x}, \bar{\eta}, \bar{\beta}\}$

Every comparison state $\{\hat{x}, \eta^*, \beta^*\}$ for B has an image $\{\bar{x}, \bar{\eta}^*, \bar{\beta}^*\}$ for $\bar{B}$ by Eq. (IV-21) and every comparison state $\{\bar{x}, \bar{\eta}^*, \bar{\beta}^*\}$ for $\bar{B}$ has an inverse image $\{\bar{x}, \eta^*, \beta^*\}$ for B . It follows that

$$\text{Ineq. (IV-22)} \Longleftrightarrow \text{Ineq. (IV-6)} \ . \qquad \text{(IV-23)}$$

The theorem follows Eq. (IV-23). Q.E.D.

<u>Theorem 3</u>

Let $\{\hat{x}, \eta, \beta\}$ of B be *uniform* - i.e., F, η, β are uniform on B - and let $\{\hat{x}, \eta, \beta\}$ of $\bar{B}$ represent a uniform state on $\bar{B}$ with the same values of F, η, β . Also $\bar{B}$ is of the same constitution as B but $\bar{B}$ is not related to B through a simple geometric scaling. Then

$$\begin{array}{c} \text{uniform } \{\hat{x}, \eta, \beta\} \text{ of B} \\ \text{is GAS} \end{array} \Longleftrightarrow \begin{array}{c} \text{uniform } \{\hat{x}, \eta, \beta\} \text{ of } \bar{B} \\ \text{is GAS} \end{array} \qquad \text{(IV-24)}$$

<u>Proof</u>

It follows Ineq. (IV-6) that the test for $\{\hat{x}, \eta, \beta\}$ of B is

$$\int_B \hat{e}(F, \eta^*(X), \beta^*(X))\ dM > M\ e(F, \eta, \beta) \qquad \text{(IV-25)}$$

(i) $\{\hat{x}, \eta^*, \beta^*\}$ ε neighborhood of $\{\hat{x}, \eta, \beta\}$

(ii) $\int_B h^*(X)\ dM = 0$

(iii) $\{\hat{x}, \eta^*, \beta^*\} \neq \{\hat{x}, \eta, \beta\}$

where $M = \int_B dm$.

Now let $\overline{B}$ be a subbody of B :

$$X \varepsilon \overline{B} \Rightarrow X \varepsilon B, \; X \not\varepsilon \partial B \qquad \text{(IV-26)}$$

Choose comparison states for B which are zero on $B-\overline{B}$, i.e.,

$$\left.\begin{aligned} \overline{\eta}^*(X) &= \eta \\ \overline{\beta}^*(X) &= \beta \end{aligned}\right\} \text{ for } X \varepsilon B-\overline{B} \qquad \text{(IV-27)}$$

The comparison states represented by Eq. (IV-27) are a subset of the comparison states for Ineq. (IV-25). It follows the substitution of Eq. (IV-27) into Ineq. (IV-25) that a necessary condition for Ineq. (IV-25) is the following:

$$\int_{\overline{B}} \hat{e}(F, \overline{\eta}^*(X)\overline{\beta}^*(X)) \; dm > \overline{M} \cdot \hat{e}(F, \eta, \beta) \qquad \text{(IV-28)}$$

(i) $\{\hat{x}, \overline{\eta}^*, \overline{\beta}^*\} \; \varepsilon$ neighborhood of $\{\hat{x}, \eta, \beta\}$

(ii) $\int_{\overline{B}} \overline{h}^*(X) \; dm = 0$

(iii) $\{\hat{x}, \overline{\eta}^*, \overline{\beta}^*\} \neq \{\hat{x}, \eta, \beta\}$

where $\overline{M} = \int_{\overline{B}} dm$

Since it is a necessary condition it follows that

$$\begin{matrix} \text{uniform } \{\hat{x}, \eta, \beta\} \text{ of } B \\ \text{is GAS} \end{matrix} \Rightarrow \begin{matrix} \text{uniform } \{\hat{x}, \eta, \beta\} \text{ of } \overline{B} \\ \text{is GAS} \end{matrix} \qquad \text{(IV-29)}$$

Now let $\bar{\bar{B}}$ be a body of the same constitution as B and scaled from $\bar{B}$ by a :

$$\bar{B} \xrightarrow{a} \bar{\bar{B}} \qquad \text{(IV-30)}$$

Choose $a > 0$ large enough so that B is a subbody of $\bar{\bar{B}}$:

$$X \epsilon B \Rightarrow X \epsilon \bar{\bar{B}},\ X \not\epsilon\, \partial\bar{\bar{B}} \qquad \text{(IV-31)}$$

The scaling of uniform $\{\hat{x},\eta,\beta\}$ of $\bar{B}$ to $\bar{\bar{B}}$ retains the same uniform triplet (F,η,β). It follows theorem 2 that

$$\begin{array}{c} \text{uniform } \{\hat{x},\eta,\beta\} \text{ of } \bar{\bar{B}} \\ \text{is GAS} \end{array} \Longleftrightarrow \begin{array}{c} \text{uniform } \{\hat{x},\eta,\beta\} \text{ of } \bar{B} \\ \text{is GAS} \end{array} \qquad \text{(IV-32)}$$

Since B is a subbody of $\bar{\bar{B}}$ the roles of B, $\bar{B}$ may be replaced by $\bar{\bar{B}}$, B respectively in Eq. (IV-29):

$$\begin{array}{c} \text{uniform } \{\hat{x},\eta,\beta\} \text{ of } \bar{\bar{B}} \\ \text{is GAS} \end{array} \Longrightarrow \begin{array}{c} \text{uniform } \{\hat{x},\eta,\beta\} \text{ of } B \\ \text{is GAS} \end{array} \qquad \text{(IV-33)}$$

It follows Eqs. (IV-29), (IV-32), (IV-33) that

$$\begin{array}{c} \text{uniform } \{\hat{x},\eta,\beta\} \text{ of } B \\ \text{is GAS} \end{array} \Longleftrightarrow \begin{array}{c} \text{uniform } \{x,\eta,\beta\} \text{ of } \bar{B} \\ \text{is GAS} \end{array} \Longleftrightarrow$$

$$\begin{array}{c} \text{uniform } \{\hat{x},\eta,\beta\} \text{ of } \bar{\bar{B}} \\ \text{is GAS} \end{array} \qquad \text{(IV-34)}$$

The theorem (where $\bar{B}$ represents any body of equal constitution) follows Eq. (IV-34). Q.E.D.

Theorem 4

Let the triplet (F,η,β) represent a state, and let $\{\hat{x},\eta,\beta\}$ represent a uniform state of B which corresponds to (F,η,β). Then

$$(F,\eta,\beta)\varepsilon D \Rightarrow \text{uniform } \{\hat{x},\eta,\beta\} \text{ of } B \text{ is GAS} \tag{IV-35}$$

where B is any body.

Proof

Let $\{\hat{x},\eta,\beta\}$ of B be uniform. It follows theorem 3 that any B may be chosen. Choose B to correspond to a spherical neighborhood of 0 in $E^3(N_R(0))$. It follows that Ineq. (IV-6) for any B is equivalent to the following:

$$\int_{N_R(0)} \hat{e}(F,\eta^*(X),\beta^*(X))\ dm > M\ \hat{e}(F,\eta,\beta) \tag{IV-36}$$

(i) $\{\hat{x},\eta^*,\beta^*\}\ \varepsilon$ neighborhood of $\{\hat{x},\eta,\beta\}$

(ii) $\displaystyle\int_{N_R(0)} \hat{h}^*(X)\ dm = 0$

(iii) $\{\hat{x},\eta^*,\beta^*\} \neq \{\hat{x},\eta,\beta\}$

It follows Eq. (IV-1) that restriction (i) is equivalent to $\eta^*(X)\varepsilon N(\eta)$ and $\beta^*(X)\varepsilon N(\beta)$. Since θ and τ are uniform it follows Eq. (IV-15) that restriction (ii) is equivalent to $\displaystyle\int_{N_R(0)} \theta(\eta^*(X)-\eta) + \operatorname{tr} \tau^T(\beta^*(X)-\beta)\ dM = 0$. Also since ρ_R is uniform, Ineq. (IV-36) has the following equivalent representation:

$$\int_{N_R} \hat{e}(F,\eta^*(X),\beta^*(X))\ dV > V\ \hat{e}(F,\eta,\beta) \tag{IV-37}$$

(i) $\left.\begin{array}{l}\eta^*(X)\varepsilon N(\eta)\\ \beta^*(X)\varepsilon N(\beta)\end{array}\right.$ on $N_R(0)$

(ii) $\displaystyle\int_{N_R(0)} \theta(\eta^*(X)-\eta) + \mathrm{tr}\ \tau^T(\beta^*(X)-\beta)\ dV = 0$

(iii) either $\eta^*(X) \neq \eta$ or $\beta^*(X) \neq \beta$ on $N_R(0)$

The comparison states for Ineq. (IV-37) are a subset of those for Ineq. (III-16). It follows easily upon comparing Ineqs. (III-16) and (IV-37) that

$$\text{Ineq. (III-16)} \Rightarrow \text{Ineq. (IV-37)}\ ,\ \text{or}$$

$$(F,\eta,\beta)\varepsilon\ D \Rightarrow \text{uniform } \{\hat{x},\eta,\beta\} \text{ of } N_R(0) \text{ is GAS} \tag{IV-38}$$

The theorem follows Eq. (IV-38) and theorem 3. Q.E.D.

Theorem 5

The substate tension (τ) is the "force of constrain" necessary to prevent spontanious substate processes.

Proof

Consider uniform $\{\hat{x},\eta,\beta\}$ of B which is GAS. Restrict the comparison states so that $\eta^*(X) = \eta$ and $\int_B (\beta^*(X)-\beta)dm = 0$. It follows Ineq. (IV-6) that

$$\int_B \hat{e}(F,\eta,\beta^*(X)) - \hat{e}(F,\eta,\beta)\ dM > 0 \tag{IV-39}$$

where

(i) $\{\hat{x},\eta,\beta^*\}\varepsilon$ neighborhood of $\{x,\eta,\beta\}$

(ii) $\int_B (\beta^*(X)-\beta)\ dM = 0$

(III) $\{\hat{x},\eta,\beta^*\} \neq \{\hat{x},\eta,\beta\}$

Now multiply restriction (ii) by a constant Lagrange multiplier (λ) and add to the above integral:

$$\int_B \left[\hat{e}(F,\eta,\beta^*(X)) - \hat{e}(F,\eta,\beta) + \mathrm{tr}\ \lambda(\beta^*(X)-\beta)\right]dM > 0 \qquad \text{(IV-40)}$$

where restriction (ii) still applies.

Now if λ is chosen equal to $-\tau^T$, it can be proven that Ineq. (IV-40) holds independent of restriction (ii). Following concepts in classical mechanics, τ is interpreted as the "force-of-constraint" necessary to insure the persistence of β. Q.E.D.

One type of global mechanical stability will be discussed, but first certain conditions of mechanical equilibrium will be reviewed.

<u>Definition of Mechanical Equilibrium</u>

A state $\{\hat{x},\eta,\beta\}$ of B is a state of mechanical equilibrium if the stress $(\hat{T}_R(X) \equiv \hat{T}_R(F(X),\eta(X),\beta(X))$ corresponding to $\{\hat{x},\eta,\beta\}$ satisfies the following equations:

$$\left.\begin{aligned} \mathrm{DIV}\ \hat{T}_R(X) + \rho_R b(X) &= 0 \\ \hat{T}_R(X)F^T(X) &= F(X)\hat{T}_R{}^T(X) \end{aligned}\right\} \text{ on B} \qquad \text{(IV-41)}$$

where $b(X)$ is a body force field and DIV is computed relative to the material coordinates.

Theorem 6

$\{x,\eta,\beta\}$ of B is a state of mechanical equilibrium is equivalent to the following:

$$\int_B \operatorname{tr} \frac{1}{\rho_R} \hat{T}_R{}^T(X)(F^*(X)-F(X))dM$$
$$= \int_S (\hat{x}^*(X)-\hat{x}(X))\hat{T}_R(X)dA + \int_B b(X)(x^*(X)-x(X))dM \qquad \text{(IV-42)}$$

where $x^* = \hat{x}^*(X)$ is any differentiable configuration of B, and S is the surface of B.

Proof

Multiply Eq. (IV-41)[1] by $\hat{x}^*(X)-\hat{x}(X)$ and integrate over B.

$$\int_B (\hat{x}^*(X)-\hat{x}(X))\mathrm{DIV}\ \hat{T}_R(X)dv + \int_B (\hat{x}^*(X)-\hat{x}(X))b(X)dM = 0 \qquad \text{(IV-43)}$$

Green's theorem gives

$$\int_B (\hat{x}^*(X)-\hat{x}(X))\mathrm{DIV}\ \hat{T}_R(X)dv = \int_S (\hat{x}^*(X)-\hat{x}(X))\hat{T}_R(X)dA$$
$$- \int_B \operatorname{tr} \frac{1}{\rho_R} \hat{T}_R{}^T(X)(F^*(X)-F(X))dM \qquad \text{(IV-44)}$$

Eq. (IV-42) follows easily Eqs. (IV-43) and (IV-44). Q.E.D.

Definition of Global Adiabatic Mechanical Stability with Fixed Boundary

A state $\{\hat{x},\eta,\beta\}$ of B is called GAMSFB if and only if

(a.) $\{\hat{x},\eta,\beta\}$ of B is a state of mechanical equilibrium for zero body force, (b.) the temperature is uniform, and

(c.) there is a neighborhood of $\{\hat{x},\eta,\beta\}$ such that for all $\{\hat{x}^*,\eta^*,\beta^*\}$ in the neighborhood

$$\int_B \hat{e}(F^*(X),\eta^*(X),\beta^*(X))\ dM > \int_B \hat{e}(F(X),\eta(X),\beta(X))\ dM$$

$$\text{(i)}\quad \int_B \hat{h}^*(X)\ dM = 0$$

$$\text{(ii)}\quad \hat{x}^*(X) = \hat{x}(X),\ \text{for}\ \ X\varepsilon\partial B$$

$$\text{(iii)}\quad \{\hat{x}^*,\eta^*,\beta^*\} \neq \{\hat{x},\eta,\beta\} \qquad \text{(IV-45)}$$

Theorem 7

$$\begin{array}{ccc} \{\hat{x},\eta,\beta\}\ \text{of}\ \ B & \Rightarrow & \{\hat{x},\eta,\beta\}\ \ \text{of}\ \ B \\ \text{is GAMSFB} & & \text{is GAS} \end{array} \qquad \text{(IV-46)}$$

Proof

The comparison states for Ineq. (IV-6) are a subset of those for Ineq (IV-37). It follows easily that

$$\text{Ineq. (IV-37)} \Rightarrow \text{Ineq. (IV-6)} \qquad \text{(IV-47)}$$

The theorem follows Eq. (IV-47). Q.E.D.

Theorem 8

Let $\{\hat{x},\eta,\beta\}$ be a global state of B and let $\{\bar{x},\bar{\eta},\bar{\beta}\}$ of $\bar{B}$ be a scaled global state. Then

$$\begin{array}{ccc} \{\hat{x},\eta,\beta\}\ \text{of}\ \ B & \Leftrightarrow & \{\bar{x},\bar{\eta},\bar{\beta}\}\ \text{of}\ \ \bar{B} \\ \text{is GAMSFB} & & \text{is GAMSFB} \end{array} \qquad \text{(IV-48)}$$

Proof

Let $\{\hat{x},\eta,\beta\}$ of B be GAMSFB. It follows the definition of GAMSFB and theorem 6 that the following are necessary and sufficient.

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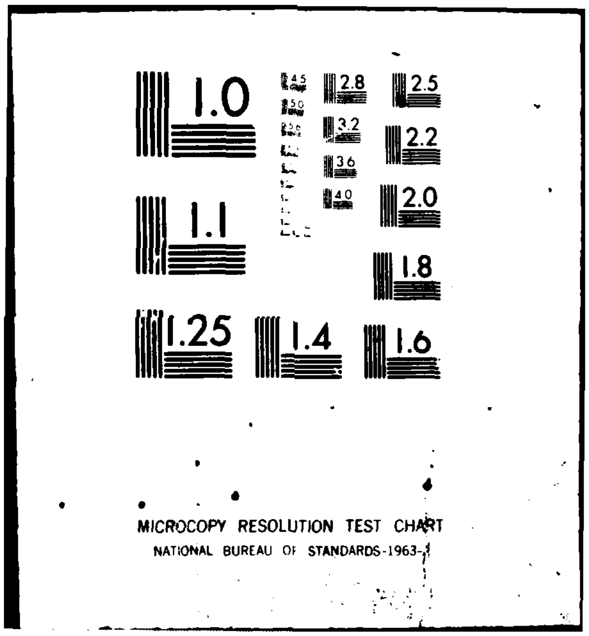
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$$\int_B \mathrm{tr}\, \frac{1}{\rho_R} T_R^T(X)(F^*(X)-F(X))\, dM$$

$$= \int_S (\hat{x}^*(X)-\hat{x}(X))\, T_R(X)\, dA$$

(IV-49)

and

$$\int_B \hat{e}(F^*(X),\eta^*(X),\beta^*(X)\, dM > \int_B \hat{e}(F(X),\eta(X),\beta(X))\, dM \quad \text{(IV-50)}$$

(i) $\int_B \hat{h}^*(X)\, dM = 0$

(ii) $\hat{x}^*(X) = \hat{x}(X)$ for $X \varepsilon \partial B$

(iii) $\{\hat{x}^*,\eta^*,\beta^*\} \neq \{\hat{x},\eta,\beta\}$

It follows Eq. (IV-15) and implicit differentiation of Eqs. (IV-16)[1], (IV-17)[1] that

$$\partial B \xrightarrow{a} \partial \bar{B}$$

$$d\bar{M} = a^3 dM$$

$$\bar{F}(\bar{X}) = F(X)$$

$$\bar{\nabla}\, \bar{T}_R(\bar{X}) = \nabla\, T_R(X) \cdot \frac{1}{a} \quad \text{(IV-51)}$$

It follows Eqs. (IV-51)[3,4], (IV-17)[1] and Eq. (IV-41) for zero body force that

$\{\hat{x},\eta,\beta\}$ of B is a state of mechanical equilibrium $\Longleftrightarrow$ $\{\bar{x},\bar{\eta},\bar{\beta}\}$ of $\bar{B}$ is a state of mechanical equilibrium

(IV-52)

Choose comparison states for $\bar{B}$ as follows:

$$\bar{x}^*(\bar{X}) = a\hat{x}^*(X)$$
$$\bar{\eta}^*(\bar{X}) = \eta^*(X)$$
$$\bar{\beta}^*(X) = \beta^*(X) \qquad \text{(IV-53)}$$

Substitute Eqs. (IV-53), (IV-51)[3], (IV-16)[2,3], (IV-17) into Ineq. (IV-50) and it follows Eq. (IV-51)[1,2] that

Ineq. (IV-50) $\Rightarrow$

$$\int_{\bar{B}} \hat{e}(\bar{F}^*(\bar{X}),\bar{\eta}^*(\bar{X}),\bar{\beta}^*(\bar{X}))\, d\bar{M} > \int_{\bar{B}} \hat{e}(\bar{F}(X),\bar{\eta}(X),\bar{\beta}(X))\, dM \qquad \text{(IV-54)}$$

$$\text{(i)} \quad \int_{\bar{B}} \bar{h}^*(\bar{X})\, d\bar{M} = 0$$

$$\text{(ii)} \quad \bar{x}^*(\bar{X}) = \bar{x}(\bar{X}) \quad \text{for} \quad \bar{x} \in \bar{B}$$

$$\text{(iii)} \quad \{\bar{x}^*,\bar{\eta}^*,\bar{\beta}^*\} \neq \{\bar{x},\bar{\eta},\bar{\beta}\}$$

Every comparison state $\{x^*,\eta^*,\beta^*\}$ for B has an image $\{\bar{x}^*,\bar{\eta}^*,\bar{\beta}^*\}$ for $\bar{B}$ by Eq. (IV-52) and every comparison state $\{\bar{x}^*,\bar{\eta}^*,\bar{\beta}^*\}$ for $\bar{B}$ has an inverse image $\{\hat{x}^*,\eta^*,\beta^*\}$ for B. It follows that

$$\text{Ineq. (IV-54)} \Leftrightarrow \text{Ineq. (IV-50)} = \text{Ineq. (IV-45)} \qquad \text{(IV-55)}$$

The theorem follows Eq. (IV-52), (IV-55) and the definition of GAMSFB.

Q.E.D.

<u>Theorem 9</u>

Let $\{\hat{x},\eta,\beta\}$ of B be uniform and let $\{\hat{x},\eta,\beta\}$ of $\bar{B}$ represent a uniform state on $\bar{B}$ with the same values of

F, η, β . Also $\bar{B}$ is of the same constitution as B but $\bar{B}$ is not related to B through a simple geometric scaling. Then

$$\begin{array}{ccc} \text{uniform } \{\hat{x}, \eta, \beta\} \text{ of } B & \Leftrightarrow & \text{uniform } \{\hat{x}, \eta, \beta\} \text{ of } \bar{B} \\ \text{is GAMSFB} & & \text{is GAMSFB} \end{array} \tag{IV-56}$$

Proof

Body forces are assumed zero. It follows easily that uniform states are states of mechanical equilibrium.

It follows Ineq. (IV-45) that the test of $\{\hat{x}, \eta, \beta\}$ of B is

$$\int_B \hat{e}(F^*(X), \eta^*(X), \beta^*(X))\, dM > M\, \hat{e}(F, \eta, \beta) \tag{IV-57}$$

(i) $\int_B \hat{h}^*(X)\, dM = 0$

(ii) $\hat{x}^*(X) = \hat{x}(X)$ for $X \epsilon \partial B$

(iii) $\{\hat{x}^*, \eta^*, \beta^*\} \neq \{\hat{x}, \eta, \beta\}$

where $M = \int_B dM$

Let $\bar{B}$ be a subbody of B :

$$x \epsilon \bar{B} \Rightarrow X \epsilon B,\ X \not\epsilon \partial B \tag{IV-58}$$

Choose comparison states for B which are zero on $B-\bar{B}$' i.e.

$$\begin{aligned} \bar{x}^*(X) &= \hat{x}(X) \\ \bar{\eta}^*(X) &= \eta(X) \qquad \text{for } X \epsilon B-\bar{B} \\ \bar{\beta}^*(X) &= \beta(X) \end{aligned} \tag{IV-59}$$

These comparison states are a subset of the comparison states for Ineq. (IV-57). It follows the substitution of Eq. IV-59) into Ineq. (IV-57) that a necessary condition for Ineq. (IV-57) is the following:

$$\int_{\bar{B}} \hat{e}(\bar{F}^*(X), \bar{\eta}^*(X), \beta^*(X)\, dM > \bar{M}\ \hat{e}(F, \eta, \beta) \tag{IV-60}$$

(i) $\int_{\bar{B}} \bar{h}^*(X)\ dM = 0$

(ii) $\bar{x}^*(X) = \hat{x}(X)$ for $X \epsilon \partial \bar{B}$

(iii) $\{\bar{x}^*, \bar{\eta}^*, \bar{\beta}^*\} \neq \{\hat{x}, \eta, \beta\}$ on $\bar{B}$

where $\bar{M} = \int_{\bar{B}} dM$

Clearly Ineq. (IV-60) is the application of Ineq. (IV-45) to $\bar{B}$. Since Ineq. (IV-60) is a necessary condition for Ineq. (IV-57) it follows that

$$\begin{matrix} \text{uniform } \{\hat{x}, \eta, \beta\} \text{ of } B \\ \text{is GAMSFB} \end{matrix} \Rightarrow \begin{matrix} \text{uniform } \{\hat{x}, \eta, \beta\} \text{ of } \bar{B} \\ \text{is GAMSFB} \end{matrix} \tag{IV-61}$$

Now let $\bar{\bar{B}}$ be a body of the same constitution as B and scaled from $\bar{B}$ by a:

$$\bar{B} \xrightarrow{a} \bar{\bar{B}} \tag{IV-62}$$

Choose $a > 0$ large enough so that B is a subbody of $\bar{\bar{B}}$:

$$X \epsilon B \Rightarrow X \epsilon \bar{\bar{B}},\quad X \notin \partial \bar{\bar{B}} \tag{IV-63}$$

The scaling of uniform $\{\hat{x}, \eta, \beta\}$ of $\bar{B}$ to $\bar{\bar{B}}$ retains the same uniform triplet (F, η, β). It follows theorem 8

$$\begin{array}{c} \text{uniform } \{\hat{x},\eta,\beta\} \text{ of } \bar{B} \\ \text{is GAMSFB} \end{array} \Longleftrightarrow \begin{array}{c} \text{uniform } \{\hat{x},\eta,\beta\} \text{ of } \bar{\bar{B}} \\ \text{is GAMSFB} \end{array} \qquad \text{(IV-64)}$$

Since B is a subbody of $\bar{\bar{B}}$ the roles of B, $\bar{B}$ may be replaced by $\bar{\bar{B}}$, B respectively in Eq. (IV-61):

$$\begin{array}{c} \text{uniform } \{\hat{x},\eta,\beta\} \text{ of } \bar{\bar{B}} \\ \text{is GAMSFB} \end{array} \Longrightarrow \begin{array}{c} \text{uniform } \{\hat{x},\eta,\beta\} \text{ of } B \\ \text{is GAMSFB} \end{array} \qquad \text{(IV-65)}$$

It follows Eqs. (IV-61), (IV-64), (IV-65) that

$$\begin{array}{c} \text{uniform } \{\hat{x},\eta,\beta\} \text{ of } B \\ \text{is GAMSFB} \end{array} \Longleftrightarrow \begin{array}{c} \text{uniform } \{\hat{x},\eta,\beta\} \text{ of } \bar{B} \\ \text{is GAMSFB} \end{array} \Longleftrightarrow$$

$$\begin{array}{c} \text{uniform } \{\hat{x},\eta,\beta\} \text{ of } \bar{\bar{B}} \\ \text{is GAMSFB} \end{array} \qquad \text{(IV-66)}$$

The theorem (where $\bar{B}$ represents any body of equal constitution) follows Eq. (IV-66) . Q.E.D.

Theorem 10

Let the triplet (F,η,β) correspond to a uniform $\{\hat{x},\eta,\beta\}$ of B . Then

$$(F,\eta,\beta)\varepsilon D \Longleftrightarrow \begin{array}{c} \text{uniform } \{\hat{x},\eta,\beta\} \text{ of } B \\ \text{is GAMSFB} \end{array} \qquad \text{(IV-67)}$$

where B is any body.

Proof

Let $\{\hat{x},\eta,\beta\}$ of B be uniform. It follows theorem 9 that the property GAMSFB is independent of the body. Choose B to correspond to a spherical neighborhood of 0 in E^3 $(N_R(0))$. It follows that Ineq. (IV-45) for any B is equivalent to the following:

$$\int_{N_R(0)} \hat{e}(F^*(X),\eta^*(X),\beta^*(X)\ dM > M\ \hat{e}(F,\eta,\beta) \tag{IV-68}$$

(i) $\displaystyle\int_{N_R(0)} \hat{h}^*(X)\ dM = 0$

(ii) $\hat{x}^*(X) = \hat{x}(X)$ for $X\varepsilon\partial N_R(0)$

(iii) $\{\hat{x}^*,\eta^*,\beta^*\} \neq \{\hat{x},\eta,\beta\}$

It follows Eq. (IV-1) that the restriction of $\{x^*,\eta^*,\beta^*\}$ to a neighborhood of $\{x,\eta,\beta\}$ is equivalent to $\hat{x}^*(X)\varepsilon N(\hat{x}(X))$, $F^*(X)\varepsilon N(F)$, $\eta^*(X)\varepsilon N(\eta)$, $\beta^*(X)\varepsilon N(\beta)$ on B . Since θ,τ are uniform, it follows Eq. (IV-15) that restriction (i) is equivalent to $\int_{N_{R(0)}} \theta(\eta^*(X)-\eta) + \mathrm{tr}\ \tau^T(\beta^*(X)-\beta)\ dM = 0$.

Also since ρ_R is uniform Ineq. (IV-67) has the following equivalent representation:

$$\int_{N_R(0)} \hat{e}(F^*(X),\eta^*(X),\beta^*(X))\ dV > V\ \hat{e}(F,\eta,\beta) \tag{IV-69}$$

(i) $\displaystyle\int_{N_R(0)} \theta(\eta^*(X)-\eta) + \mathrm{tr}\ \tau^T(\beta^*(X)-\beta)\ dV = 0$

(ii) $\hat{x}^*(X) = \hat{x}(X)$ for $X\varepsilon\partial N_R(0)$

(iii) either $F^*(X) \neq F$ or $\eta^*(X) \neq \eta$ or $\beta^*(X) \neq \beta$ on $N_R(0)$

and

$$\left.\begin{array}{l} \hat{x}^*(X)\varepsilon N(\hat{x}(X)) \\ F^*(X)\varepsilon N(F) \\ \eta^*(X)\varepsilon N(\eta) \\ \beta^*(X)\varepsilon N(\beta) \end{array}\right\} \text{ on } N_R(0)$$

In spite of notational differences for the neighborhoods, it is clear that Ineq. (IV-69) is equivalent to Ineq. (III-16):

$$\text{Ineq. (IV-68)} \Leftrightarrow \text{Ineq. (III-16)} \qquad \text{(IV-70)}$$

A uniform state is a state of mechanical equilibrium. It follows the definition of GAMSFB that for a uniform state

$$\text{Ineq (IV-69)} \Leftrightarrow \begin{array}{l}\text{GAMSFB}\\ \text{for a uniform state}\end{array} \qquad \text{(IV-71)}$$

The theorem follows Eqs. (IV-70), (IV-71) .

Q.E.D.

Theorem 10 shows clearly an important property of an equilibrium state. Only triplets (F, η, β) in D may correspond to stable (i.e. GAMSFB) uniform global states.

V. CONCLUDING REMARKS

Obviously the postulated theory is based on classical developments of continuum mechanics and thermal statics. One of the most important properties assigned to an equilibrium state is the stability property - Ineq. (III-16). It has been shown in Section IV that the stability property is equivalent to the static Global Adiabatic Mechanical Stability with Fixed Boundary. The equivalence is reassuring; this property of static global stability is what was sought for the theory of equilibrium states.

A previous attempt at the theory was made by this author.[10] The stability property assigned in that attempt, it is clear only now, is necessary but not sufficient for Ineq. (III-16). It follows that more predictive capability can be expected of this theory than was possible with the previous version.

The development presented in this report falls far short of that presented previously[10] in several respects. There hasn't been time to modify all of the previous development consistent with the theory reported here, but it appears that most of that work will hold for the present theory.

There are additional developments which further characterize the equilibrium region. Characteristic states (e.g. natural states and ultrastable states) and state functions (e.g. natural state internal energy density, isentropic recoverable internal energy density, and isothermal recoverable internal energy density) may be defined, which are useful in the further analysis. There are also other kinds of global stability which may be defined (global mechanical stability for fixed tractions - adiabatic and isothermal), and associated subsets of equilibrium state-space may be found. For the description of processes some ideas are available. An equilibrium process may be defined jointly with an activation criterion. The equilibrium process has similarity to the quasistatic process for classical thermodynamics, and the activation criterion has similarity to yield criterion and flow laws for the theory of plasticity. The intent in this theory is to define the activation criterion in terms of the properties of equilibrium state-space. In other words, given a caloric equation of state as a function of (F, η, β), everything about equilibrium state-space and equilibrium processes follows.

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