Report No: NADC-79231-60

LEVEL II

(12)

ADA085873

A TIME RESPONSE APPROACH TO EQUIVALENT AIRCRAFT DYNAMICS

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Interim Report: October, 1978 to September, 1979

AIRTASK No. 62241N/F41400/WF41400000

Work Unit No. ZA602

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Prepared for:
NAVAL AIR SYSTEMS COMMAND
DEPARTMENT OF THE NAVY
Washington, D.C. 20361

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REPORT DOCUMENTATION PAGE

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1. REPORT NUMBER NADC-79231-60	2. GOVT ACCESSION NO. AD-A085 873	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) A Time Response Approach to Equivalent Aircraft Dynamics		5. TYPE OF REPORT & PERIOD COVERED Interim 10/78 through 9/79
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) J. M. Stifel		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Aircraft & Crew Systems Technology Directorate Naval Air Development Center Warminster, PA 18974		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 62241N, F41400 WF41400000, ZA602
11. CONTROLLING OFFICE NAME AND ADDRESS Naval Air Systems Command Department of the Navy Washington, DC 20361		12. REPORT DATE
		13. NUMBER OF PAGES
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office) 11 Sep 79		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

This document has been approved for public release and sale; its distribution is unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

Approved for Public Release, Distribution Limited

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Highly Augmented Aircraft
Equivalent Systems
Pilot Compensation Criteria

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

An analytical investigation into the basic equivalent of high-order and reduced-order aircraft dynamics is presented. The need to consider system response to inputs represented by points in the Laplace domain is explained in terms of the ability of a damped sinusoid series to model general aperiodic pilot inputs. The region of concern in the Laplace domain is related to pilot response time by

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S/N 0102-LF-014-6601

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the time to the initial peak of a single damped sinusoid inputs. Several representative high-order systems and their lower order equivalents are examined for similarity of time response. A new approach for equivalent systems application is suggested based on matching time responses over the Laplace domain region. Implications of the analysis to present methods of applying equivalent systems and to suggested pilot compensation criteria are examined.

The analysis leads to the conclusion that equivalent system parameters are variables dependent on Laplace domain location and that the success of current methods depend on that variation being negligible. The further conclusion is reached that if equivalent system parameters are allowed to vary, artificial time delays are unnecessary and undersirable in achieving time response similarity.

TABLE OF CONTENTS

SUMMARY

An analytical investigation into the basic equivalence of high-order and reduced-order aircraft dynamics is presented. The need to consider system response to inputs represented by points in the Laplace domain is explained in terms of the ability of a damped sinusoid series to model general aperiodic pilot inputs. The region of concern in the Laplace domain is related to pilot response time by the time to the initial peak of a single damped sinusoid input. Several respresentative high-order systems and their lower order equivalents are examined for similarity of time responses. A new approach for equivalent systems application is suggested based on matching time responses over the Laplace domain region. Implications of the analysis to present methods of applying equivalent systems and to suggested pilot compensation criteria are examined.

The analysis leads to the conclusion that equivalent system parameters are variables dependent on Laplace domain location and that the success of current methods depend on that variation being negligible. The further conclusion is reached that if equivalent system parameters are allowed to vary, artificial time delays are unnecessary and undesirable in achieving time response similarity.

LIST OF FIGURES

LIST OF FIGURES, cont.

LIST OF TABLES

LIST OF SYMBOLS

SYMBOL	DESCRIPTION	DIMENSIONS
A	Input Signal Amplitude Coefficient	$x(t)$
B	Output or Free Response Mode Amplitude Coefficient	$y(t)$
C	Arbitrary Constant	arbitrary
F_s	Non-dimensional stick force	none
G	Transfer Function Magnitude	$y(t)/x(t)$
LAHOS	Landing and Approach High Order System Configuration of Reference 4	- - -
HOS	High Order System	- - -
LOS	Low Order System	- - -
L	Fourier Series Half Period	seconds
$\mathcal{L}[\ ]$	Laplace Transform	- - -
$\mathcal{L}^{-1}[\ ]$	Inverse Laplace Transform	- - -
T	Time Delay Constant	seconds
a	Real Part of pole location	sec.^{-1}
b	Imaginary Part of pole location	sec.^{-1} or rad/sec
e	Base of the Natural logarithm or as a subscript for "equivalent"	none
i	Denominator Series Term	none
j	$\sqrt{-1}$, or as a subscript, numerator series term	none
K	Transfer Function Gain	$(\frac{1}{\text{sec}})^{n-m}$
m	Order of transfer function numerator	none
n	Order of transfer function denominator	none
r	Transfer function denominator root or pole location	sec.^{-1}
s	Laplace variable	sec.^{-1}

LIST OF SYMBOLS, cont.

SYMBOL	DESCRIPTION	DIMENSIONS
s^*	Complex conjugate of Laplace Variable	$sec.^{-1}$
t_0	Time to first peak of the input	seconds
t_1	time at first peak of the output	seconds
t_2	time at which output attenuates	seconds
t_H	t_2 for high order system	seconds
t_L	t_2 for low order system	seconds
$x(t)$	input function of time	arbitrary
$x(s)$	Laplace transform of $x(t)$	depends on $x(t)$
$y(t)$	Output function of time	arbitrary
$y(s)$	Laplace transform of $y(t)$	depends on $y(t)$
ζ	Damping ratio	none
θ	Transfer function phase angle	radians$^+$
σ	Real Part of Laplace variable	$sec.^{-1}$
ϕ	Input Function Phase Angle	radians
ψ	Free response mode phase angle	radians
ω	Imaginary part of Laplace variable or input signal frequency	$sec.^{-1}$
ω_n	Undamped Natural frequency	sec^{-1} or rad/sec
T	First Order Lag Time Constant	seconds
θ_c	Commanded Pitch Attitude	radians

$^+$ except in tables 1-6 and figures 2-4 where θ is in degrees.

INTRODUCTION

In the development of the newest fighter aircraft, increased stability augmentation and flight control system sophistication have been employed as a means of maximizing performance capabilities. In doing so, the form of the transfer functions describing the bare airframe and flight control system have grown. In the total system transfer function a clear distinction between roots due to feedback elements and those due to bare airframe dynamics have often been lost. Since the standards of MIL-F-8785B are based on those root locations a problem exists in how to apply the requirements and indeed if the requirements are applicable at all. No extensive body of data currently exists on the acceptability of higher-order systems to pilots. No general consensus exists on proposed methods for dealing with systems of arbitrary order. Thus both designer and evaluator are left without a clear cut means of predicting any given system's acceptability.

Recent attempts to deal with the problem have included the equivalent systems approach of reference (1), and the Neal and Smith criterion of reference (2). Although this investigation deals primarily with equivalent systems, it will show some important implications toward Neal and Smith's criterion as well.

Attempts to verify these criteria have concentrated on statistical sampling of pilot data and correlation of results with the predictions of each method. Since both methods have been used with some success, it seems unlikely that statistical verification alone will resolve the differences between them.

The following methods adopt a much more analytical view defining the concept of equivalency and developing conclusions employing well accepted linear control theory. The concept follows directly from the idea that pilots want an airplane to fly like an unaugmented airplane regardless of its actual dynamics. This is the rationale under which the existing requirements, or any modification based on a fixed system equation, may be applied when the actual system order may vary.

CONCEPT OF AN EQUIVALENT SYSTEM

Whenever attempting to gain insight through analytical development it is useful to begin at the most elementary point. In the case of equivalent systems that point is the concept itself which may be simply stated as follows: It is possible that two linear systems of different order when stimulated by a common input will produce highly similar outputs. If those outputs are sufficiently similar as to be indistinguishable to the human operator of the system, the operator will find each system equally acceptable or unacceptable. In this event the two systems may be considered equivalent, and criteria which define the acceptability of one may be used to determine the acceptability of the other.

Current efforts in equivalent systems have taken the approach that lower-order equivalents may be determined by finding a system of prescribed form whose frequency response (jωBode) closely approximates that of the high-order system over a range of frequencies.

The form is that of unaugmented short period and dutch roll for longitudinal and lateral directional dynamics respectively. Each may be modified by a transport time delay to approximate high frequency lags and obtain a more precise frequency response match. Since much more investigation of longitudinal equivalents has been done, the analysis will concentrate on that form.

Longitudinal Equivalent System Form

$$\frac{\dot{\theta}}{F_S} = \frac{K_e \left(S + \frac{1}{T_{\theta 2e}}\right) e^{-TS}}{\left(s^2 + 2\frac{\xi_{spe}}{\omega_{nspe}} S + \omega^2_{nspe}\right)}$$

T, $\frac{1}{T_{\theta 2e}}$, K_e, ζ_{spe} and ω_{nspe} are determined to minimize the magnitude and phase difference of the high-order system (HOS), and low-order system (LOS), jωBode between frequencies of 0.1 to 10.0 radians. Computer programs have been developed which find the low-order equivalent by comparing magnitude and phase at a number (usually twenty) of frequencies across this range.

The equivalent system parameters are then varied to minimize a squared error function, termed a cost or mismatch function:

$$\text{cost} = \sum_{\omega} (\text{Mag}_{HOS} - \text{Mag}_{LOS})^2 + .02\ (\text{Phase}_{HOS} - \text{Phase}_{LOS})^2$$

Magnitude is in decibels and Phase is in degrees.

The assumption is then made that the resulting low-order system is a true equivalent of the high-order system provided a cost function of 10 or less is obtained. Results of early fixed-base simulation experiments showed considerable pilot dissatisfaction with systems whose cost functions exceeded ten and therefore failure to obtain this level of mismatch was itself considered reason for judging the system unacceptable. Time history responses to step inputs were run to confirm the assumption of equivalency. Attempts to statistically correlate predicted pilot ratings obtained by comparison of the low-order system to the requirements of MIL-F-8785B, with actual pilot ratings of the high-order system have shown good correlation in many cases but unsatisfactory correlation in some. The anomalies may be the result of inconsistent pilot ratings, but may also indicate that the basic assumption linking frequency response match with true equivalency does not always hold for all inputs. The need exists to more clearly understand the link between frequency response matching and similarity of system time response. Since this involves fundamental properties of linear systems the investigation is well suited to analytical treatment.

EXPANSION OF EQUIVALENT SYSTEMS INTO THE LAPLACE DOMAIN

Throughout this analysis several properties of linear systems will be employed which for clarity and convenience have been substantiated in Appendix A. Among these is that a transfer function:

$$\frac{y(s)}{x(s)} = \frac{k\ \pi\{s-(a_m+jb_m)\}}{\pi\{s-(a_n+jb_n)\}}$$

when evaluated at a point in the Laplace domain, $s = \sigma+j\omega$, represents the forced response portion of the complete response:

$$y(t) = \underbrace{GAe^{\sigma t}\sin(\omega t + \theta)}_{\text{forced response}} + \underbrace{\sum_n B_n e^{a_n t}\sin(b_n t + \psi_n)}_{\text{free response}}$$

when the system is subjected to the input:

$$x(t) = Ae^{\sigma t}\sin\omega t\ .$$

Since present methods locate equivalent system parameters based on matching transfer function magnitude, G, and phase shift, θ, along the $j\omega$ axis, they are enforcing only similarity of the forced response for a special class of inputs, undamped sinusoids. It is not immediately clear either that similarity of the free response, or that similarity of the total response to damped simusoids are maintained.

Of course, no low-order system can approximate a higher-order system everywhere in the Laplace domain. Forunately this is not necessary when a human pilot supplies the input. The principle is already recognized by current methods in that the frequency response matching is done only over a limited frequency range. The upper limit, based on human neuromuscular lags, will later be extended into the Laplace domain. First, however, it is desirable to establish why consideration of the system response to Laplace domain inputs is necessary.

The basic concept of equivalency is based on similarity of input and output functions of time. Damped sinusoids within the pilot's operating region are inputs to which the system might be subjected. Therefore a true equivalent must adequately simulate the response to such an input. This reasoning alone is not sufficiently compelling. It could be argued that since any function possesses the Forier series expansion:

$$x(t) = \Sigma A \sin \frac{n\pi}{L} t$$

that any function may be approximated by a sum of sinusoids within the appropriate frequency range. Further, because of the linearity of the system, all that is necessary is to match the total response to each term of the series separately and the total response will have been matched.

In general pilots control aircraft with highly irregular and aperiodic inputs. Although a Fourier series may represent a very arbitrary input signal, it may do so only over the *finite interval*: $0 < t < 2L$. Outside this interval the Fourier series repeats; that is; the function is periodic regardless of the form of the function within the interval.

Consider what is meant by the similarity of two functions. If two functions of time are identical then they will have identical values for the function and all its derivatives at any point in time. If the functions are approximate, only some of these conditions will hold; that is, the values of the functions and/or their derivatives will match at only a finite number of points. The number of such similarity conditions which exist are then a measure of the accuracy of the approximation. A Taylor's series expansion,

$$x(t) = \sum_n \frac{dx^n(t-a)}{dt^n} \frac{(t-a)^n}{n!}$$

approximates the function by equating the value of the function and its derivatives at t=a. The Taylor's series representation is valid only in the vicinity of t=a. To obtain an approximation valid over a wider range of time, the value of the function and its derivatives must be matched at several points over a given time interval. Since it is generally difficult to estimate higher-order derivatives of a time response, it is most practical to require that the value of the function and its first derivative at a point be used as similarity conditions. Obviously guaranteeing that only the value of the function matches the approximating function at a point says very little about the accuracy of the approximation at near-by points.

The general form of the input function which satisfies the differential equation describing the system has already been stated as:

$$x(t) = Ae^{\sigma t} \sin \omega t$$

and the linearity of the system allows simultaneous inputs of the form:

$$x(t) = \Sigma Ae^{\sigma t} \sin (\omega t + \phi)$$

the first derivative of which is:

$$\frac{dx(t)}{dt} = \Sigma Ae^{\sigma t} \{\omega \cos(\omega t + \phi) + \sigma \sin(\omega t + \phi)\}$$

Writing these equations in terms of complex numbers simplifies the form to:

$$x(t) = \Sigma \frac{Ae^{j\phi}}{2j} e^{st} - \frac{Ae^{-j\phi}}{2j} e^{s^*t}$$

or

$$x(t) = \Sigma \frac{Ae^{j\phi}}{2j} e^{st}$$

and

$$x'(t) = \Sigma \frac{Ae^{j\phi}}{2j} se^{st} - \frac{Ae^{-j\phi}}{2j} s^*e^{s+t}$$

or

$$x'(t) = \Sigma \frac{Ae^{-j\phi}}{2j} se^{st}$$

where $s = \sigma + j\omega$ and $s^* = \sigma - j\omega$.

Applying the similarity conditions at i points yields the system of equations:

$$\begin{bmatrix} x(t_1) \\ x(t_2) \\ \vdots \\ x(t_i) \end{bmatrix} = \begin{bmatrix} e^{s_1 t_1} & e^{s_2 t_1} \ldots\ldots e^{s_i t_1} \\ e^{s_1 t_2} & e^{s_2 t_2} \ldots\ldots e^{s_i t_2} \\ \vdots & \vdots \qquad\qquad \vdots \\ e^{s_1 t_1} & e^{s_2 t_1} \ldots\ldots e^{s_2 t_i} \end{bmatrix} \qquad \begin{bmatrix} A_1 e^{j\phi_1}/2j \\ A_2 e^{j\phi_2}/2j \\ \vdots \\ A_i e^{j\phi_i}/2j \end{bmatrix}$$

$$\begin{bmatrix} x'(t_1) \\ x'(t_2) \\ \vdots \\ x'(t_i) \end{bmatrix} = \begin{bmatrix} s_1 e^{s_1 t_1} & s_2 e^{s_2 t_1} \ldots s_i e^{s_i t_1} \\ s_1 e^{s_1 t_2} & s_2 e^{s_2 t_2} \ldots s_i e^{s_i t_2} \\ \vdots & \vdots \qquad\qquad \vdots \\ s_1 e^{s_1 t_i} & s_2 e^{s_2 t_i} \ldots s_i e^{s_i t_1} \end{bmatrix} \qquad \begin{bmatrix} A_1 e^{j\phi_1}/2j \\ A_2 e^{j\phi_2}/2j \\ \vdots \\ A_i e^{j\phi_i}/2j \end{bmatrix}$$

This system of equations may either be viewed as 2i equations in 2i complex variables; s_i and $A_i e^{j\phi_i}/2j$, or as 4i equations in the 4i real variables; σ_i, ω_i, A_i and ϕ_i. Either way unique solutions are obtainable, although the solutions are not straight forward due to the nonlinearities of the equations.

If we consider only values of s for which $s_i = j\omega_i$, $\sigma_i = 0$, we actually eliminate both σ and ω as variables and the preceeding system of equations becomes linear in the constants, $A_i e^{j\phi_i}/2j$.

The general input form,

$$x(t) = \Sigma A e^{\sigma t} \sin(\omega t + \phi)$$

with $\sigma = 0$ becomes:

$$x(t) = \Sigma A \sin(\omega t + \phi).$$

Each individual term of the series repeats its value at $\omega t = n2\pi$. If several terms all repeat at the same time,

$$t = l\frac{2\pi}{\omega_1} = m\frac{2\pi}{\omega_2} = n\frac{2\pi}{\omega_3} \quad - \; - \; - \quad \text{etc.}$$

which implies,

$$\frac{\omega_2}{\omega_1} = \frac{m}{l}, \quad \frac{\omega_3}{\omega_2} = \frac{n}{m}, \quad - - - \text{ etc.}$$

This implies that the ratios of all frequencies in the series are expressable as ratios of integers; i.e., rational numbers. Since any frequency may be approximated by a finite decimal, the periodicity of the series holds to any desired accuracy for any combination of frequencies. If the series repeats over the interval 2L, all frequencies are integral multiples of a single base frequency,

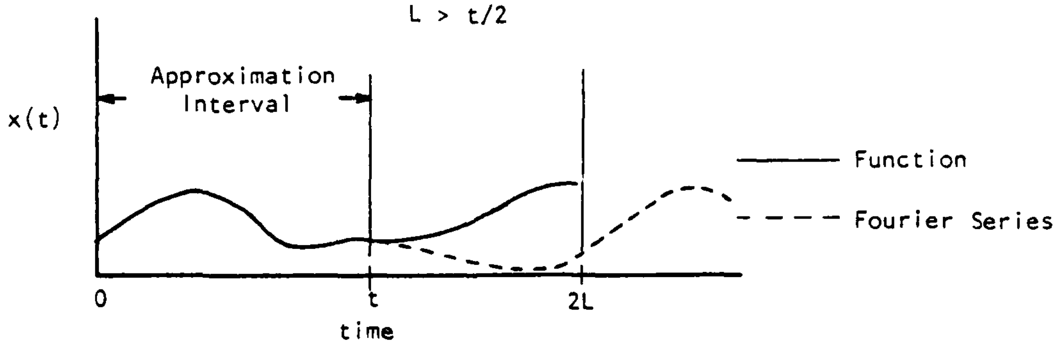
$$\omega_1 = l\frac{\pi}{L}, \quad \omega_2 = m\frac{\pi}{L}, \quad \omega_3 = n\frac{\pi}{L} \quad - - - \text{ etc.}$$

Thus, with $\sigma = 0$ the general input form becomes that of the general Forier series,

$$x(t) = \Sigma A \sin(\frac{n\pi}{L}t + \phi),$$

in order to avoid the Fourier series repeating within the time interval of interest,

$$L > t/2$$

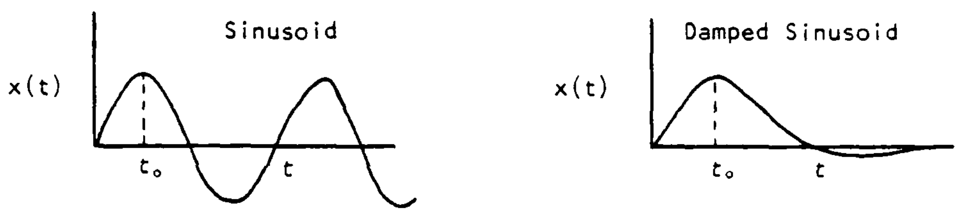


Beyond this the choice of L is arbitrary, but once chosen sets the values of:

$$s = \pm j\frac{n\pi}{L} \qquad 1 \leq n \leq i$$

Therefore, s_i having been determined, half the variables in the matrix equations are eliminated, and only half of the 2i similarity equations may be satisfied. The damped sinusoid series then represents a better approximation than does the Fourier series for a finite number of terms because it can satisfy twice as many similarity conditions. These reasons are considered sufficient to justify consideration of system response to inputs in the Laplace domain. It remains to more clearly define the region of the Laplace domain within which human pilots are capable of operating.

It is generally conceded that a pilot's reaction capabilities provide an upper limit to the frequency range considered for $j\omega$ Bode matching techniques. The problem exists of how to extend this principle into the s plane in order to consider system response to aperiodic inputs. The frequency of a sinusoidal input is easily relatable to any fraction of a cycle deemed to be most significant. The only general similarity between an undamped and highly damped sinusoidal is that both possess an initial maximum or peak.



With the interpretation that the initial peak represents the time to initiate removal of a previous control motion, it may be related to response time. The initial peak then represents a convenient means of extending the upper frequency limit over the s plane. The general input form is:

$$x(t) = Ae^{\sigma t} \sin \omega t$$

At the first peak, its first derivative vanishes:

$$\frac{dx(t)}{dt} = Ae^{\sigma t} \{\sigma \sin \omega t_o + \omega \cos \omega t_o\} = 0$$

which reduces to:

$$\sigma \sin \omega t_o + \omega \cos \omega t_o = 0$$

or:

$$t_o = \frac{1}{\omega} \tan^{-1} \left(\frac{-\omega}{\sigma}\right)$$

Note that at $\sigma = 0$ the time to the peak becomes the quarter cycle time;

$$t_o = \frac{\pi}{2\omega}$$

which for the customary upper frequency limit of 10 radians per second is:

$$t_o = \frac{\pi}{20} = .157 \text{ sec.}$$

The pilot operating region is then defined by the contour:

$$\sigma = -\frac{\omega}{\tan(\omega t_o)}$$

Note that as $\omega \to 0$, $\tan \omega t_o \to \sin \omega t_o \to \omega t_o$ and:

$$\lim_{\omega \to o} \sigma = -\frac{1}{t_o} = -6.366 \text{ sec}^{-1}$$

TIME RESPONSE MATCHING

The final test of any equivalent is its ability to approximate the total time response to an arbitrary aperiodic pilot input. This is equivalent to matching time responses over the defined region in the s plane. There is therefore a direct implication that equivalent system parameters should be determined by maximizing the similarity of the total time responses to such inputs. We may consider applying to the output the same similarity conditions we have already applied to the input. The general output form is:

$$y(t) = GAe^{\sigma t} \sin(\omega t + \theta) + \Sigma B_n e^{a_n t} \sin(b_n t + \psi_n)$$

or in terms of complex variables:

$$y(t) = GAe^{j\theta}e^{st} - GAe^{-j\theta}e^{s^* t} + \Sigma B_n e^{j\psi_n} e^{r_n t}$$

and its derivative:

$$y'(t) = sGAe^{j\theta}e^{st} - s^*GAe^{-j\theta}e^{s^* t} + \Sigma r_n B_n e^{j\psi_n} e^{r_n t}$$

where $s = \sigma + j\omega$, $s^* = \sigma - j\omega$, $r_n = a_n \pm jb_n$, and where:

$$Ge^{j\theta} = \frac{K\ \pi\{s - (a_m + jb_m)\}}{\pi\{s - (a_n + jb_n)\}} \quad \text{at } s = \sigma + j\omega$$

The number of variables in these equations is directly dependent on the order of the transfer function and is: Number of variables = 1 + m + 2n. It is possible therefore to enforce this number of similarity conditions on the low-order system output. It is also possible to consider other means, such as a minimum squared error method, of enforcing time response similarity. In addition, considerable judgment may be involved in the choice of times at which to enforce those conditions. These difficulties will not be critical to the most significant conclusion of the analysis* but for the present will be dealt with by making certain assumptions about the pilot's perception of the output.

If it is assumed that human pilots are not sensitive to small differences in the system response but only to certain gross characteristics, we may set similarity conditions which duplicate them. Based on pilot comments on high-order systems contained in references 3 and 4,

* see page 55

the most common judgments made on the systems seem to fall into three catagories:

1: Response magnitude or "sensitivity"

2: Response lags or sluggishness

3: Annoying residual oscillations or tendency to PIO.

For well-damped inputs such as that in Figure 1-a, the pilot will most likely interpret the first peak of the response as the system response to his input and the long term response as characteristic of the residual oscillations. Matching the timing and magnitude of the initial peak would guarantee, under these assumptions, similarity of system sensitivity and speed of reaction as perceived by the pilot. Since the pilot could consider the residual oscillations an annoyance, it is probably most significant to match the time at which they damp out.

The first "n" similarity conditions are always set by the requirement that the system be initially undisturbed. The values of the amplitudes and phasing of the free response modes:

$$B_n e^{j\psi_n} e^{r_n t} \qquad r_n = a_n \pm jb_n$$

are set by requiring that the value of the output function and its first n-1 derivatives be zero at time zero. Considering the first over second order equivalent system form currently in use, these conditions are:

$$GAe^{j\theta} - GAe^{-j\theta} + B_1 e^{j\psi_1} + B_2 e^{j\psi_2} = 0$$
$$sGAe^{j\theta} - s^*GAe^{-j\theta} + r_1 B_1 e^{j\psi_1} + r_2 B_2 e^{j\psi_2} = 0$$

Applying similarity conditions at the peak time, t_1, and the subsidence time, t_2,

$$GAe^{j\theta}e^{st_1} - GAe^{-j\theta}e^{s^*t_1} + B_1 e^{\psi_1} e^{r_1 t_1} + B_2 e^{\psi_2} e^{r_2 t_1} = Y_{pk}$$
$$sGAe^{j\theta}e^{st_1} - s^*GAe^{-j\theta}e^{s^*t_1} + r_1 B_1 e^{\psi_1} e^{r_1 t_1} + r_2 B_2 e^{\psi_2} e^{r_2 t_1} = 0$$
$$GAe^{j\theta}e^{st_2} - GAe^{-j\theta}e^{s^*t_2} + B_1 e^{\psi_1} e^{r_1 t_2} + B_2 e^{\psi_2} e^{r_2 t_2} = Y_{min}$$
$$sGAe^{j\theta}e^{st_2} - s^*GAe^{-j\theta}e^{s^*t_2} + r_1 B_1 e^{\psi_1} e^{r_1 t_2} + r_2 B_2 e^{\psi_2} e^{r_2 t_2} = Y'_{min}$$

where:

- A, s, and s^* are known because the input is known
- Y_{pk}, t_1, and t_2 are determined by the high-order system response which is known
- Y_{min} and Y'_{min} are values of the function and its slope which define the subsidence time t. They are small but not exactly zero.
- G, θ, r_1, r_2, $B_1e^{j\psi_1}$, and $B_2e^{j\psi_2}$ are variables.

The similarity equations may be written more conveniently in Matrix form as:

$$\begin{bmatrix} 0 \\ 0 \\ Y_{pk} \\ 0 \\ Y_{min} \\ Y'_{min} \end{bmatrix} = \begin{bmatrix} A & A & 1 & 1 \\ As & As^* & r_1 & r_2 \\ Ae^{st_1} & Ae^{s^*t_1} & e^{r_1t_1} & e^{r_2t_1} \\ sAe^{st_1} & s^*Ae^{s^*t_1} & r_1e^{r_1t_1} & r_2e^{r_2t_1} \\ Ae^{st_2} & Ae^{s^*t_2} & e^{r_1t_2} & e^{r_2t_2} \\ sAe^{st_2} & s^*Ae^{s^*t_2} & r_1e^{r_1t_2} & r_2e^{r_2t_2} \end{bmatrix} \begin{bmatrix} Ge^{j\theta} \\ Ge^{-j\theta} \\ B_1e^{j\psi_1} \\ B_2e^{j\psi_2} \end{bmatrix}$$

Although highly nonlinear, this system of equations has a solution best obtainable by numerical methods. Having obtained a solution for the transfer function magnitude, phase, and natural response modes (i.e., pole locations), the numerator zero location and gain are determined by:

$$K(s - a_3) = Ge^{j\theta}(s - r_1)(s - r_2)$$

The high order-system could be approximated by lower than a first over second-order system but only if some of the similarity conditions are sacrificed. Higher-order equivalents would produce better time response matches but not different pilot ratings under the assumptions just made.

Figure 1-b represents a hypothetical high-order system response to a highly-damped input, Figure 1-a. Figures 1-c to 1-h represent possible low-order system responses to successively higher-order equivalents. These show how the equivalents, by enforcing progressively more similarity conditions, improve the time response matches. For instance, Figure 1-c demonstrates that a pure gain, having only one variable, is capable only

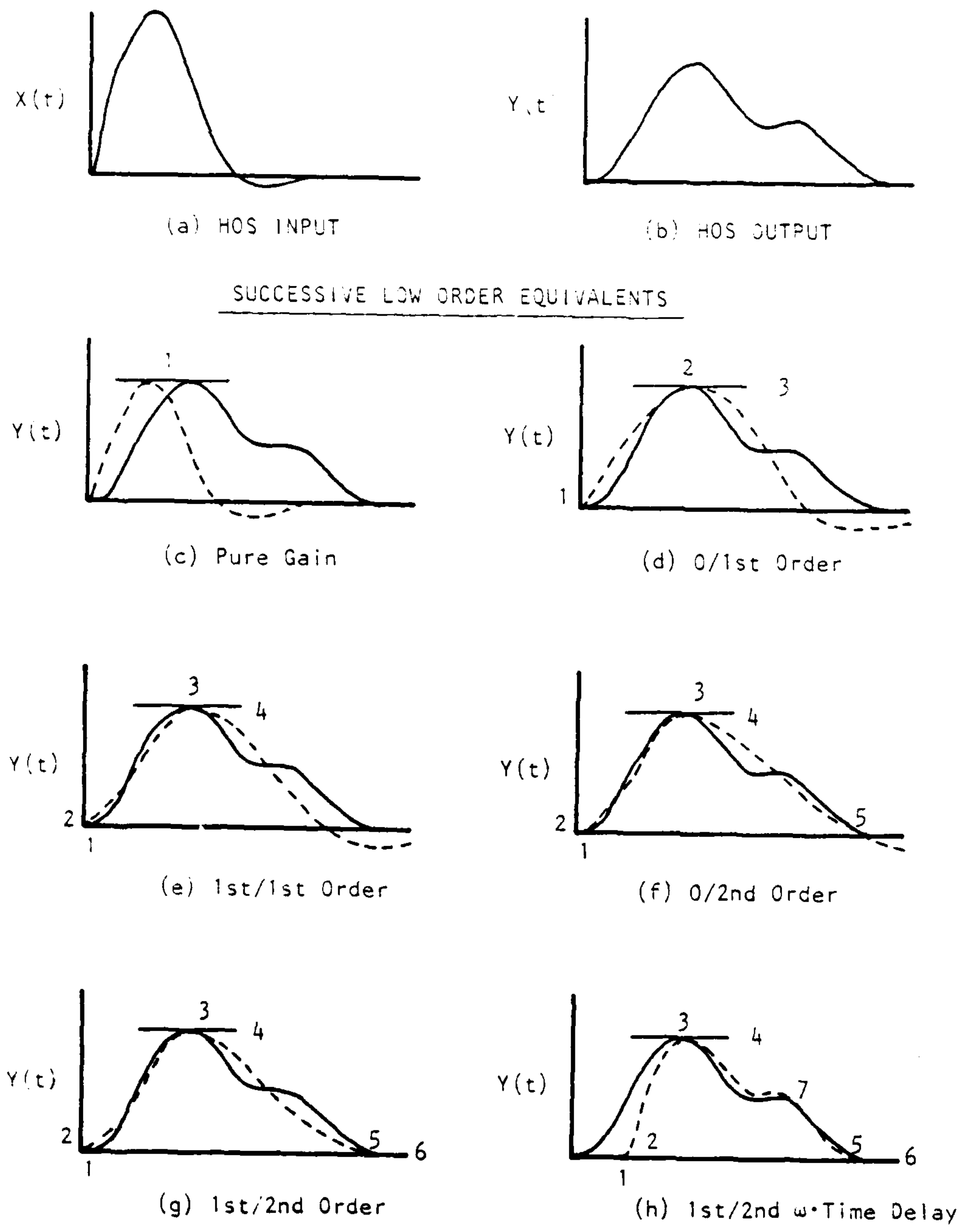


FIGURE 1 - Successive Low Order System Time Response Approximation

of matching the peak magnitude of the high-order system response. Figure 1-g demonstrates that the first over second-order equivalent is the simplest form capable of imposing similarity of the function initially, at the first peak, and at response subsidence simultaneously.

When the time delay is added, as in Figure 1-h, a time scale shift is effected and the similarity conditions which were applied at $t = 0$, $t = t_1$, and $t = t_2$ are now applied at $t' = 0$, $t_1' = t_1 - T$, and $t_2' = t_2 - T$. The result is that although the variable T was acquired, the initial conditions are applied at $t = T$ rather than $t = 0$ and those two similarity conditions are lost. The addition of the time delay results in a net loss of similarity conditions and detracts from the overall similarity of the time response. If it is sufficiently small, the delay may not significantly detract from the initial response similarity but will in general change the other equivalent systems parameters.

SAMPLE HIGH AND LOW ORDER SYSTEMS

To this point in the analysis concepts and principles have been discussed which will serve as a framework for the analysis of specific high-order systems and their equivalents. The high-order systems were selected from among those of the LAHOS experiments of reference 2, while their lower order equivalents were obtained from reference 1. These systems were the LAHOS 1-4, LAHOS 1-C, and LAHOS 6-2 configurations. LAHOS 1-C was considered typical of those in the references, LAHOS 1-4 represents an inadequate form according to the previous development, and LAHOS 6-2 represents the highest order reduction in the references. All possess good j_ω Bode matches with cost functions less than 10. The notation used will be:

$$(TS+1) = [T]; \; (S^2/\omega_n^2+2\zeta/\omega_n S+1) = [\omega_n, \zeta]$$

or:

$$(S - \sigma) = (\sigma); \; (S^2 - 2\sigma S + \sigma^2 + \omega^2) = (\sigma, \omega)$$

CONFIGURATION: LAHOS 1-4: 0/2nd Order Equivalent, Cost Function 9.0:

High Order System:

$$\frac{\dot{\theta}}{F_s} = \frac{[-1.4]}{[26.6,\ 0.6][75.0,\ 0.7][1.0,\ 0.74][-0.5]}$$

or:

$$\frac{\dot{\theta}}{F_s} = \frac{1.0647 \times 10^7 \ (-0.714)}{(-15.6,\ 20.8)(-52.5,\ 53.56)(-0.74,\ 0.6726)(-2.0)}$$

Low Order System:

$$\frac{\dot{\theta}}{F_s} = \frac{1.08}{[1.56,\ 0.74]}$$

or:

$$\frac{\dot{\theta}}{F_s} = \frac{2.628}{(-1.15,\ 1.05)}$$

Time Delay = -0.06 seconds

CONFIGURATION: LAHOS 1-C: L_α Free, Cost Function 8.9:

High Order System:

$$\frac{\dot{\theta}}{F_s} = \frac{[-1.4][-0.2]}{[26.0,\ 0.6][75.0,\ 0.7][1.0,\ 0.74][-0.1]}$$

or:

$$\frac{\dot{\theta}}{F_s} = \frac{1.0647 \times 10^7\ (-0.714)(-5.0)}{(-15.6,\ 20.8)(-52.5,\ 53.56)(-0.74,\ 0.6726)(-10.0)}$$

Low Order System:

$$\frac{\dot{\theta}}{F_s} = \frac{0.99[-2.63]}{[0.86,\ 1.04]}$$

or:

$$\frac{\dot{\theta}}{F_s} = \frac{1.9257\ (-0.38)}{(-1.14)(-0.6487)}$$

Time Delay = -0.037 seconds

CONFIGURATION: LAHOS 6-2: L_α Fixed, Cost Function 0.97

High Order System:

$$\frac{\dot{\theta}}{F_s} = \frac{[0.5][0.43][0.06][1.4]}{[0.2][0.1][1.1][1.9,\ 0.65][26.0,\ 0.6][75.0,\ 0.7]}$$

$$\frac{\dot{\theta}}{F_s} = \frac{1.1269 \times 10^7\ \ (-2.0)(-2.33)(-16.67)(-0.71)}{(-5.0)(-10.0)(-0.91)(-1.235,\ 1.444)(-15.6,\ 20.8)(-52.5,\ 53.56)}$$

Low Order System:

$$\frac{\dot{\theta}}{F_s} = \frac{.97\ [1.4]}{[1.74,\ 0.78]}$$

$$\frac{\dot{\theta}}{F_s} = \frac{4.111(-0.714)}{(-1.357,\ 1.089)}$$

Time Delay = -0.084 seconds

DISSIMILARITY OF THE FORCED RESPONSE

Since current equivalent systems methods match the forced response, transfer function magnitude and phase, along the $j\omega$ axis, we might consider whether this principle is extendable into the S plane. If similarity of the forced response per se is a necessary condition to similarity of the total response, then matching the forced response is a general principle extendable over the appropriate S plane region. Since equivalent systems parameters are currently being determined to minimize the cost function along the $j\omega$ axis, it is reasonable to believe that no such similarity exists away from the $j\omega$ axis. Tables one through six compare high and low order transfer function magnitude and phase angle of the three LAHOS configurations under consideration over a frequency range zero to ten radians and sigma values from zero to -8.5.

The phase angles in these tables occasionally appear positive. The phase angle is calculated by an inverse tangent function which returns angles between ±180°. Positive phase lags then actually represent the angle;

$$\theta = -360° = \theta_{comp} \qquad \theta_{comp} > 0$$

For the low-order equivalents, values greater than -180° occasionally appear because the phase shift due to the time delay is added to the inverse tangent function.

From these tables it is apparent that no similarity exists anywhere but along the line $\sigma = 0$. Therefore if similarity of the forced response is a valid general principle there would be considerable differences in the total time responses to damped sinusoidal inputs. Figures 2, 3, and 4 have been plotted to more clearly show the forced response variation of the three configurations at one radian frequency across this damping range. The sigma limits dictated by extension of pilot reaction times into the S plane are also shown.

CONFIGURATION: LAHOS 1-4 HIGH ORDER SYSTEM

TRANSFER FUNCTION GAIN

SIGMA:OMEGA	.0010	1.0000	3.0000	5.0000	8.0000	10.0000
.0000	.9996	1.0401	.2627	.1056	.0435	.0284
-.5000	.8090	2.3927	.3024	.1139	.0459	.0298
-1.0000	1.6424	3.1051	.3316	.1207	.0480	.0311
-1.5000	4.7048	2.3081	.3441	.1253	.0499	.0324
-2.0000	◆05.4864	1.9023	.3388	.1274	.0514	.0335
-2.5000	3.3024	1.3834	.3191	.1272	.0525	.0345
-3.0000	1.3916	.9308	.2906	.1248	.0533	.0354
-3.5000	.8031	.6393	.2588	.1207	.0537	.0361
-4.0000	.5331	.4605	.2275	.1153	.0537	.0367
-4.5000	.3844	.3471	.1989	.1092	.0535	.0371
-5.0000	.2928	.2716	.1739	.1027	.0529	.0374
-5.5000	.2321	.2191	.1525	.0961	.0521	.0375
-6.0000	.1895	.1811	.1344	.0897	.0511	.0375
-6.5000	.1584	.1527	.1191	.0836	.0500	.0374
-7.0000	.1350	.1310	.1062	.0779	.0488	.0372
-7.5000	.1167	.1139	.0953	.0726	.0474	.0369
-8.0000	.1023	.1001	.0860	.0677	.0460	.0366
-8.5000	.0906	.0890	.0780	.0632	.0446	.0361

TRANSFER FUNCTION PHASE LAG

SIGMA:OMEGA	.0010	1.0000	3.0000	5.0000	8.0000	10.0000
.00	-.04	-65.81	-141.87	-168.02	168.80	156.52
-.50	.17	-95.05	-159.01	-178.87	161.92	151.00
-1.00	179.80	-169.99	-177.58	169.93	154.96	145.45
-1.50	179.89	149.97	163.42	158.61	147.99	139.92
-2.00	90.02	116.11	144.99	147.39	141.07	134.43
-2.50	.14	84.72	127.88	136.49	134.25	129.01
-3.00	.08	62.61	112.51	126.08	127.58	123.70
-3.50	.05	48.46	99.07	116.30	121.12	118.52
-4.00	.04	39.12	87.54	107.23	114.91	113.50
-4.50	.03	32.61	77.75	98.91	108.97	108.66
-5.00	.03	27.84	69.47	91.34	103.33	104.01
-5.50	.02	24.21	62.47	84.51	98.01	99.59
-6.00	.02	21.37	56.53	78.36	93.03	95.40
-6.50	.02	19.09	51.48	72.85	88.38	91.45
-7.00	.02	17.22	47.15	67.92	84.06	87.75
-7.50	.02	15.67	43.44	63.52	80.07	84.30
-8.00	.01	14.37	40.23	59.60	76.41	81.10
-8.50	.01	13.27	37.44	56.10	73.05	78.17

TABLE 1 - TRANSFER FUNCTION EVALUATION IN THE LAPLACE DOMAIN

CONFIGURATION: LAHOS 1-4 0 /2ND ORDER EQUIVALENT

TRANSFER FUNCTION GAIN

SIGMA:OMEGA	.0010	1.0000	3.0000	5.0000	8.0000	10.0000
.0000	1.0837	.9713	.2757	.1037	.0409	.0262
-.5000	1.7233	1.8745	.3117	.1079	.0415	.0265
-1.0000	2.3360	8.0862	.3316	.1099	.0418	.0266
-1.5000	2.1453	3.5742	.3263	.1094	.0417	.0265
-2.0000	1.4400	1.3908	.2985	.1065	.0413	.0264
-2.5000	.8985	.7925	.2596	.1016	.0406	.0261
-3.0000	.5808	.5143	.2196	.0952	.0396	.0257
-3.5000	.3967	.3585	.1838	.0881	.0383	.0251
-4.0000	.2849	.2626	.1537	.0807	.0369	.0245
-4.5000	.2132	.1997	.1290	.0734	.0353	.0238
-5.0000	.1650	.1565	.1090	.0664	.0336	.0231
-5.5000	.1312	.1256	.0928	.0600	.0319	.0222
-6.0000	.1067	.1029	.0796	.0542	.0302	.0214
-6.5000	.0884	.0857	.0688	.0489	.0285	.0205
-7.0000	.0744	.0725	.0599	.0442	.0268	.0197
-7.5000	.0634	.0620	.0525	.0401	.0253	.0188
-8.0000	.0547	.0537	.0464	.0364	.0237	.0179
-8.5000	.0477	.0469	.0412	.0331	.0223	.0171

TRANSFER FUNCTION PHASE LAG

SIGMA:OMEGA	.0010	1.0000	3.0000	5.0000	8.0000	10.0000
.00	-.06	-61.66	-143.93	-170.19	-190.86	-201.11
-.50	-.05	-71.45	-162.76	-181.71	-198.05	-206.86
-1.00	-.02	-70.82	-183.79	-193.59	-205.32	-212.64
-1.50	.03	68.74	154.57	154.44	147.40	141.57
-2.00	.05	60.68	134.28	142.67	140.16	135.80
-2.50	.05	51.07	116.56	131.36	133.02	130.08
-3.00	.04	42.95	101.64	120.71	126.04	124.44
-3.50	.04	36.44	89.25	110.83	119.26	118.90
-4.00	.03	31.28	78.93	101.78	112.72	113.50
-4.50	.03	27.17	70.29	93.54	106.45	108.24
-5.00	.02	23.85	63.00	86.07	100.47	103.14
-5.50	.02	21.14	56.79	79.34	94.78	98.21
-6.00	.02	18.88	51.45	73.25	89.40	93.47
-6.50	.02	16.99	46.84	67.76	84.32	88.92
-7.00	.02	15.38	42.82	62.80	79.53	84.56
-7.50	.01	14.00	39.29	58.31	75.03	80.38
-8.00	.01	12.80	36.17	54.23	70.79	76.40
-8.50	.01	11.76	33.40	50.52	66.81	72.60

TABLE 2 - TRANSFER FUNCTION EVALUATION IN THE LAPLACE DOMAIN

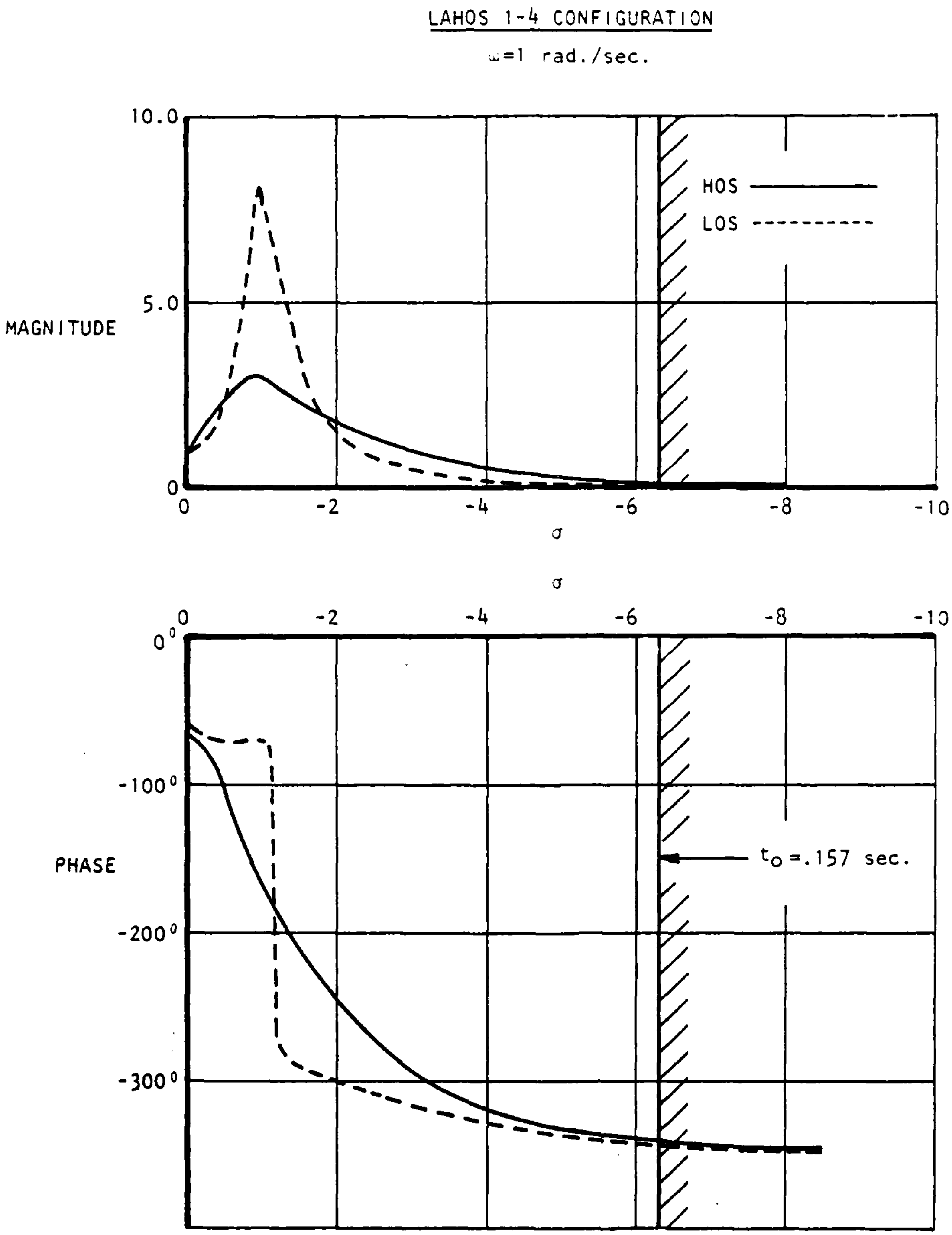


FIGURE 2 - Magnitude and Phase Variations With σ

CONFIGURATION: LAHOS 1-C HIGH ORDER SYSTEM

TRANSFER FUNCTION GAIN

SIGMA:OMEGA	.0010	1.0000	3.0000	5.0000	8.0000	10.0000
.0000	.9996	1.1800	.5290	.3597	.2644	.2288
-.5000	.5748	2.0816	.5506	.3727	.2761	.2396
-1.0000	.7300	1.9994	.5527	.3827	.2876	.2506
-1.5000	.9686	1.0975	.5353	.3895	.2990	.2618
-2.0000	.7521	.7461	.5047	.3938	.3104	.2733
-2.5000	.5504	.5504	.4692	.3964	.3218	.2850
-3.0000	.3976	.4163	.4351	.3984	.3333	.2971
-3.5000	.2780	.3159	.4067	.4011	.3450	.3094
-4.0000	.1777	.2394	.3866	.4055	.3572	.3222
-4.5000	.0874	.1869	.3770	.4127	.3699	.3353
-5.0000	.0002	.1684	.3795	.4234	.3833	.3488
-5.5000	.0902	.1934	.3952	.4383	.3975	.3627
-6.0000	.1895	.2561	.4249	.4576	.4123	.3770
-6.5000	.3055	.3487	.4687	.4812	.4279	.3917
-7.0000	.4499	.4722	.5264	.5089	.4440	.4066
-7.5000	.6421	.6365	.5971	.5397	.4604	.4216
-8.0000	.9205	.8615	.6790	.5727	.4770	.4367
-8.5000	1.3737	1.1813	.7679	.6062	.4934	.4516

TRANSFER FUNCTION PHASE LAG

SIGMA:OMEGA	.0010	1.0000	3.0000	5.0000	8.0000	10.0000
.00	-.00	-33.64	-71.29	-81.39	-95.90	-106.35
-.50	.22	-54.84	-79.41	-85.32	-98.16	-108.23
-1.00	179.86	-117.30	-87.58	-89.09	-100.36	-110.07
-1.50	-179.98	-137.36	-94.88	-92.56	-102.48	-111.87
-2.00	-179.97	-142.58	-100.56	-95.58	-104.49	-113.61
-2.50	-179.96	-144.51	-104.27	-98.06	-106.38	-115.29
-3.00	-179.96	-143.95	-105.94	-99.95	-108.14	-116.91
-3.50	-179.96	-140.28	-105.70	-101.27	-109.78	-118.46
-4.00	-179.94	-131.90	-103.77	-102.09	-111.31	-119.94
-4.50	-179.88	-116.06	-100.52	-102.51	-112.75	-121.36
-5.00	-90.00	-91.91	-96.49	-102.70	-114.11	-122.72
-5.50	-.12	-67.70	-92.36	-102.80	-115.43	-124.03
-6.00	-.06	-51.71	-88.77	-103.01	-116.72	-125.29
-6.50	-.05	-43.08	-86.25	-103.47	-118.02	-126.50
-7.00	-.04	-39.09	-85.12	-104.31	-119.34	-127.68
-7.50	-.04	-38.24	-85.56	-105.62	-120.71	-128.82
-8.00	-.04	-40.09	-87.65	-107.44	-122.13	-129.92
-8.50	-.05	-45.11	-91.37	-109.77	-123.61	-130.99

TABLE 3 - TRANSFER FUNCTION EVALUATION IN THE LAPLACE DOMAIN

CONFIGURATION: LAHOS 1-C LOW ORDER EQUIVALENT L ALF FREE

TRANSFER FUNCTION GAIN

SIGMA:OMEGA	.0010	1.0000	3.0000	5.0000	8.0000	10.0000
.0000	.9895	1.1397	.5912	.3735	.2378	.1911
-.5000	2.4282	1.6158	.6275	.3820	.2399	.1922
-1.0000	24.2751	2.1171	.6503	.3870	.2412	.1928
-1.5000	7.0375	2.0715	.6545	.3881	.2415	.1930
-2.0000	2.6844	1.6535	.6394	.3852	.2408	.1926
-2.5000	1.6215	1.2708	.6092	.3785	.2392	.1918
-3.0000	1.1536	1.0008	.5701	.3688	.2367	.1905
-3.5000	.8929	.8147	.5276	.3566	.2334	.1888
-4.0000	.7273	.6825	.4856	.3428	.2295	.1867
-4.5000	.6131	.5853	.4463	.3281	.2249	.1842
-5.0000	.5297	.5113	.4106	.3131	.2199	.1815
-5.5000	.4661	.4534	.3785	.2982	.2146	.1784
-6.0000	.4161	.4069	.3501	.2837	.2090	.1752
-6.5000	.3758	.3689	.3250	.2698	.2032	.1717
-7.0000	.3425	.3373	.3027	.2566	.1974	.1682
-7.5000	.3147	.3106	.2829	.2442	.1916	.1646
-8.0000	.2910	.2878	.2653	.2326	.1858	.1609
-8.5000	.2706	.2680	.2495	.2217	.1801	.1571

TRANSFER FUNCTION PHASE LAG

SIGMA:OMEGA	.0010	1.0000	3.0000	5.0000	8.0000	10.0000
.00	.01	-31.21	-70.57	-84.71	-96.93	-103.16
-.50	179.05	-44.20	-79.19	-90.23	-100.46	-106.00
-1.00	-.34	-71.71	-88.69	-95.95	-104.04	-108.86
-1.50	-179.83	-104.09	-98.57	-101.75	-107.64	-111.74
-2.00	-179.93	-128.00	-108.24	-107.53	-111.24	-114.61
-2.50	-179.96	-142.67	-117.18	-113.16	-114.79	-117.46
-3.00	-179.97	-151.71	-125.12	-118.54	-118.29	-120.29
-3.50	-179.98	-157.60	-131.97	-123.60	-121.71	-123.06
-4.00	-179.98	-161.68	-137.81	-128.30	-125.01	-125.79
-4.50	-179.98	-164.63	-142.74	-132.62	-128.20	-128.44
-5.00	-179.99	-166.87	-146.92	-136.56	-131.25	-131.02
-5.50	-179.99	-168.61	-150.46	-140.14	-134.16	-133.52
-6.00	-179.99	-170.00	-153.49	-143.39	-136.93	-135.94
-6.50	-179.99	-171.13	-156.09	-146.32	-139.55	-138.26
-7.00	-179.99	-172.08	-158.35	-148.98	-142.02	-140.49
-7.50	-179.99	-172.88	-160.31	-151.38	-144.35	-142.62
-8.00	-179.99	-173.56	-162.03	-153.56	-146.55	-144.66
-8.50	-179.99	-174.14	-163.55	-155.54	-148.61	-146.61

TABLE 4 - TRANSFER FUNCTION EVALUATION IN THE LAPLACE DOMAIN

LAHOS 1-C CONFIGURATION

ω=1 rad./sec.

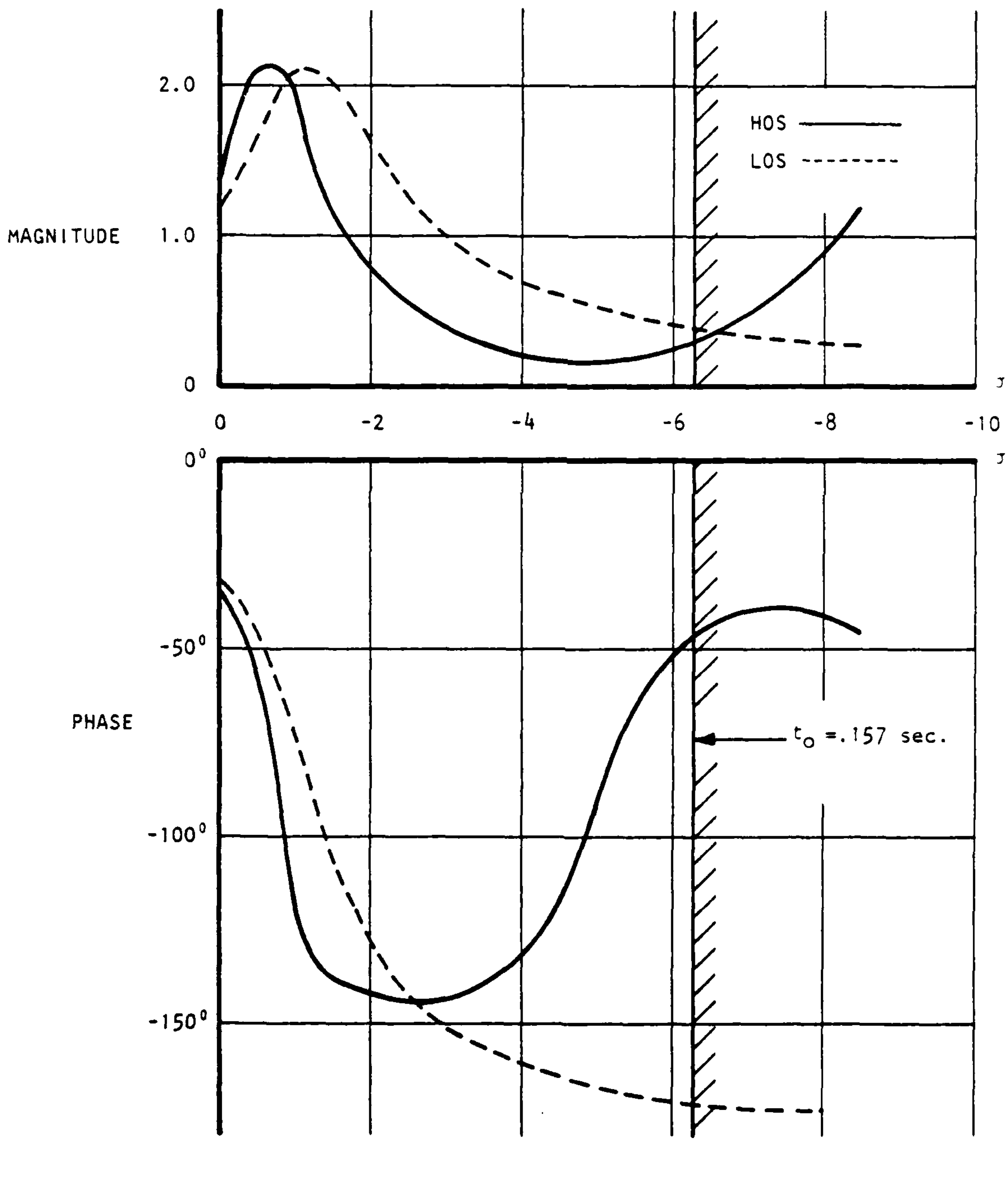


FIGURE 3 - Magnitude and Phase Variations With σ

CONFIGURATION: LAHOS 6-2 HIGH ORDER SYSTEM

TRANSFER FUNCTION GAIN

SIGMA:OMEGA	.0010	1.0000	3.0000	5.0000	8.0000	10.0000
.0000	.9950	1.3827	1.2165	.7874	.5005	.3951
-.5000	.6200	1.8273	1.4169	.8541	.5310	.4166
-1.0000	2.7545	2.6313	1.5894	.9183	.5616	.4382
-1.5000	.4289	1.9988	1.6687	.9780	.5920	.4598
-2.0000	.0003	.9968	1.6576	1.0324	.6216	.4812
-2.5000	.0682	.8133	1.6292	1.0824	.6503	.5022
-3.0000	.4938	1.0400	1.6400	1.1295	.6776	.5226
-3.5000	1.3070	1.5859	1.7053	1.1750	.7033	.5422
-4.0000	2.9369	2.5094	1.8154	1.2191	.7270	.5608
-4.5000	7.7940	3.8597	1.9488	1.2608	.7482	.5781
-5.0000	◆65.6255	5.1428	2.0801	1.2983	.7666	.5939
-5.5000	11.7680	5.4071	2.1876	1.3293	.7819	.6080
-6.0000	7.0070	4.9886	2.2603	1.3515	.7935	.6201
-6.5000	5.5404	4.5611	2.2984	1.3632	.8012	.6301
-7.0000	4.9563	4.3043	2.3081	1.3631	.8046	.6379
-7.5000	4.8063	4.2263	2.2951	1.3504	.8036	.6432
-8.0000	5.0060	4.3238	2.2613	1.3247	.7980	.6459
-8.5000	5.6688	4.6112	2.2031	1.2859	.7879	.6461

TRANSFER FUNCTION PHASE LAG

SIGMA:OMEGA	.0010	1.0000	3.0000	5.0000	8.0000	10.0000
.00	.01	-4.00	-62.66	-88.25	-112.53	-125.59
-.50	.15	11.96	-71.32	-93.64	-116.28	-128.75
-1.00	.51	50.18	-82.43	-99.33	-120.14	-131.96
-1.50	.20	123.64	-94.18	-105.18	-124.09	-135.23
-2.00	90.19	-176.28	-103.77	-111.01	-128.12	-138.53
-2.50	-.44	-126.43	-110.16	-116.75	-132.21	-141.85
-3.00	-.14	-93.18	-114.47	-122.38	-136.34	-145.19
-3.50	-.10	-78.29	-118.36	-127.95	-140.49	-148.52
-4.00	-.09	-77.46	-123.05	-133.56	-144.66	-151.84
-4.50	-.14	-89.08	-129.05	-139.28	-148.83	-155.12
-5.00	-90.02	-111.79	-136.36	-145.17	-152.99	-158.36
-5.50	-179.91	-136.29	-144.57	-151.23	-157.13	-161.54
-6.00	-179.96	-153.88	-153.24	-157.43	-161.22	-164.63
-6.50	-179.98	-165.31	-162.02	-163.74	-165.25	-167.63
-7.00	-179.99	-173.50	-170.77	-170.11	-169.20	-170.51
-7.50	180.00	179.56	-179.52	-176.50	-173.03	-173.26
-8.00	179.99	172.48	171.58	177.14	-176.73	-175.85
-8.50	179.98	163.83	162.39	170.85	179.75	-178.27

TABLE 5 - TRANSFER FUNCTION EVALUATION IN THE LAPLACE DOMAIN

CONFIGURATION: LAHOS 6-2 LOW ORDER EQUIVALENT L ALF FIXED

TRANSFER FUNCTION GAIN

SIGMA:OMEGA	.0010	1.0000	3.0000	5.0000	8.0000	10.0000
.0000	.9696	1.4911	1.2555	.8040	.5102	.4093
-.5000	.4581	2.1610	1.4131	.8357	.5175	.4130
-1.0000	.8952	5.4836	1.5526	.8595	.5228	.4157
-1.5000	2.6785	14.8261	1.6260	.8729	.5259	.4172
-2.0000	3.3055	4.7201	1.6078	.8746	.5267	.4177
-2.5000	2.9459	3.0823	1.5182	.8647	.5251	.4170
-3.0000	2.4188	2.3457	1.3962	.8448	.5213	.4152
-3.5000	1.9821	1.8958	1.2697	.8174	.5154	.4123
-4.0000	1.6532	1.5850	1.1519	.7850	.5077	.4084
-4.5000	1.4067	1.3566	1.0468	.7499	.4983	.4036
-5.0000	1.2187	1.1823	.9546	.7139	.4877	.3980
-5.5000	1.0722	1.0454	.8743	.6781	.4762	.3919
-6.0000	.9555	.9354	.8044	.6435	.4639	.3849
-6.5000	.8607	.8454	.7433	.6105	.4512	.3777
-7.0000	.7824	.7705	.6897	.5793	.4382	.3700
-7.5000	.7167	.7074	.6425	.5501	.4252	.3621
-8.0000	.6610	.6535	.6007	.5229	.4122	.3540
-8.5000	.6131	.6070	.5636	.4977	.3994	.3459

TRANSFER FUNCTION PHASE LAG

SIGMA:OMEGA	.0010	1.0000	3.0000	5.0000	8.0000	10.0000
.00	.02	-3.58	-64.09	-90.49	-114.00	-126.58
-.50	.21	11.34	-72.53	-96.14	-117.58	-129.44
-1.00	179.76	34.84	-83.42	-102.22	-121.25	-132.35
-1.50	179.94	-182.46	-96.04	-108.57	-124.98	-135.29
-2.00	-180.00	-157.67	-108.77	-115.00	-128.73	-138.25
-2.50	-179.99	-157.20	-120.17	-121.33	-132.48	-141.20
-3.00	-179.98	-159.73	-129.71	-127.38	-136.18	-144.13
-3.50	-179.98	-162.67	-137.49	-133.05	-139.80	-147.02
-4.00	-179.99	-165.35	-143.84	-138.27	-143.31	-149.86
-4.50	-179.99	-167.62	-149.08	-143.02	-146.71	-152.64
-5.00	-179.99	-169.51	-153.45	-147.32	-149.96	-155.35
-5.50	-179.99	-171.09	-157.13	-151.20	-153.06	-157.97
-6.00	-179.99	-172.40	-160.27	-154.69	-156.00	-160.51
-6.50	-179.99	-173.50	-162.97	-157.83	-158.79	-162.95
-7.00	-179.99	-174.44	-165.32	-160.66	-161.41	-165.29
-7.50	-180.00	-175.24	-167.36	-163.22	-163.88	-167.53
-8.00	-180.00	-175.94	-169.16	-165.53	-166.21	-169.68
-8.50	-180.00	-176.54	-170.74	-167.62	-168.39	-171.73

TABLE 6 - TRANSFER FUNCTION EVALUATION IN THE LAPLACE DOMAIN

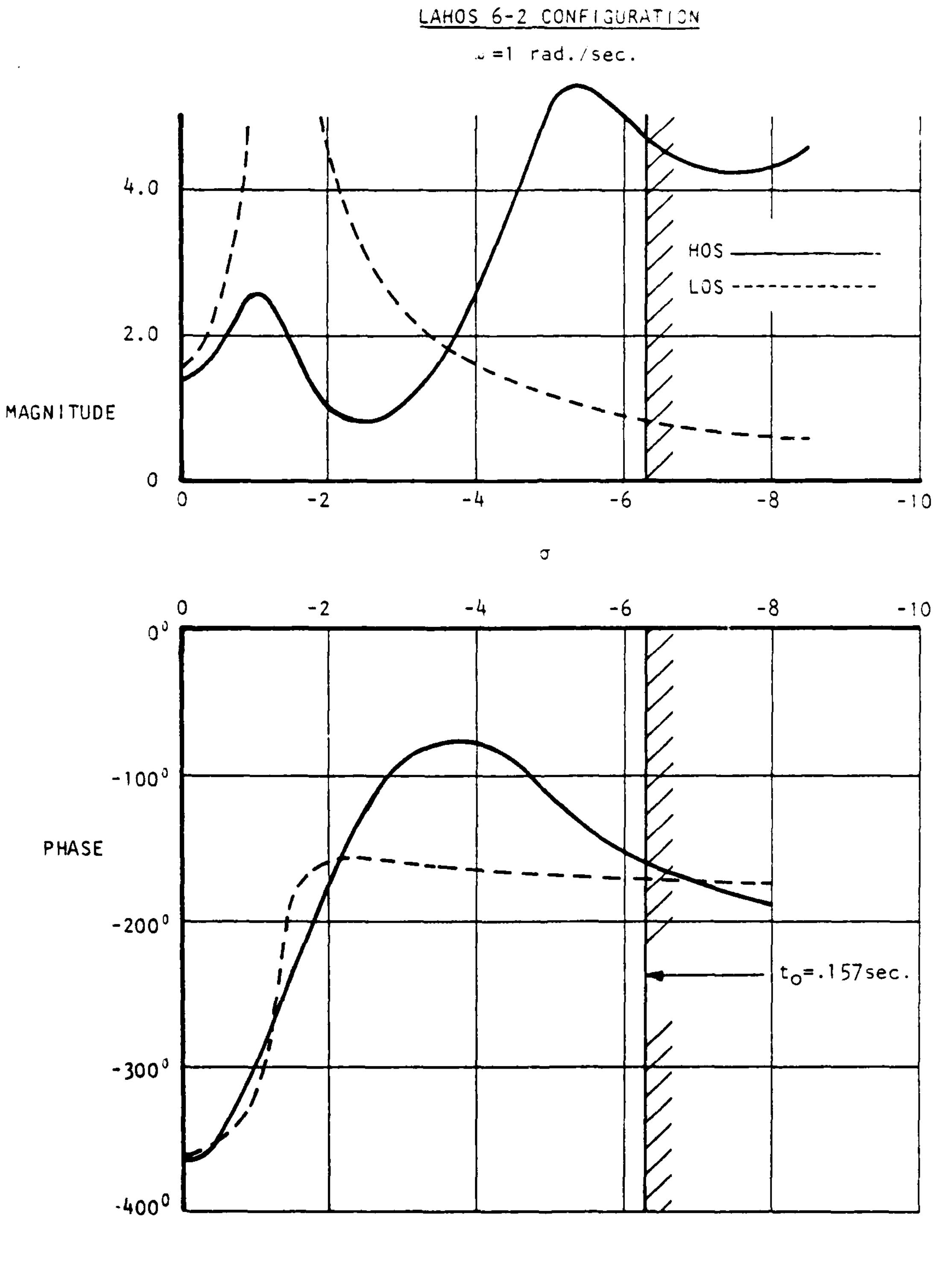


FIGURE 4 - Magnitude and Phase Variations With σ

Figure 3 compares the forced response of the LAHOS 1-C configuration with its first over second-order with time delay equivalent.

At $\sigma = -1.5$:

$$Mag_{HOS} = 1.097 \qquad \theta_{HOS} = -137^\circ$$

$$Mag_{LOS} = 2.071 \qquad \theta_{LOS} = -104^\circ$$

which constitute substantial magnitude and phase errors.

Figure 18 represents the total time response comparison of these two systems for the input,

$$x(t) = Ae^{\sigma t} \sin \omega t, \qquad \sigma = -1.5, \ \omega = 1.0$$

and shows that of the high-order and low-order systems to be highly similar.

Recall that in setting the similarity conditions on the total time response, G and θ were considered variables. If they were set equal to the high-order system values, total variables and therefore similarity conditions would be lost.

Therefore, it is clear that similarity of forced response is not a general principle that holds over the S plane nor in fact should this condition be enforced in determining equivalents. Only the total time response can be the comparison standard over the S plane. It will be shown later that importance of the forced response along the $j\omega$ axis lies in the fact that it also represents the total long-term or steady state response to the general input at those points in the S plane.

TIME HISTORY COMPARISONS

Because the low-order system equivalents were obtained purely by matching the system forced response along the line $\sigma = 0$, it is likely that the six time response similarity conditions are not met at all points within the S plane region of interest, even though good frequency response matches have been obtained. To determine the extent of the mismatching, time history responses were calculated for input points across the defined S plane region. Figure 5 depicts both the region corresponding to a maximum frequency of 10 radians/second, and a time to initial input peak of $t_o = .157$ seconds and the distribution of input test signals across that region. Table 7 relates the frequency damping and amplitude of each input to a letter designation for easy reference.

CASE NUMBER	σ	ω	A
A	0.0	1.0	1.0
B	-1.5		4.355
C	-4.0		10.985
D	-6.0		16.385
E	0.0	5.0	1.0
F	-1.0		1.342
G	-3.0		2.164
H	-5.0		3.102
I	0.0	8.0	1.0
J	-0.5		1.10
K	-1.5		1.319
L	-2.5		1.557

$$x(s) = \frac{\omega A}{[s - (\sigma + j\omega)][s - (\sigma - j\omega)]}$$

$$A = \frac{e^{-\sigma/\omega} \tan^{-1}(\frac{-\omega}{\sigma})}{\sin\{\tan^{-1}(\frac{-\omega}{\sigma})\}}$$

TABLE 7: TEST INPUT CASES

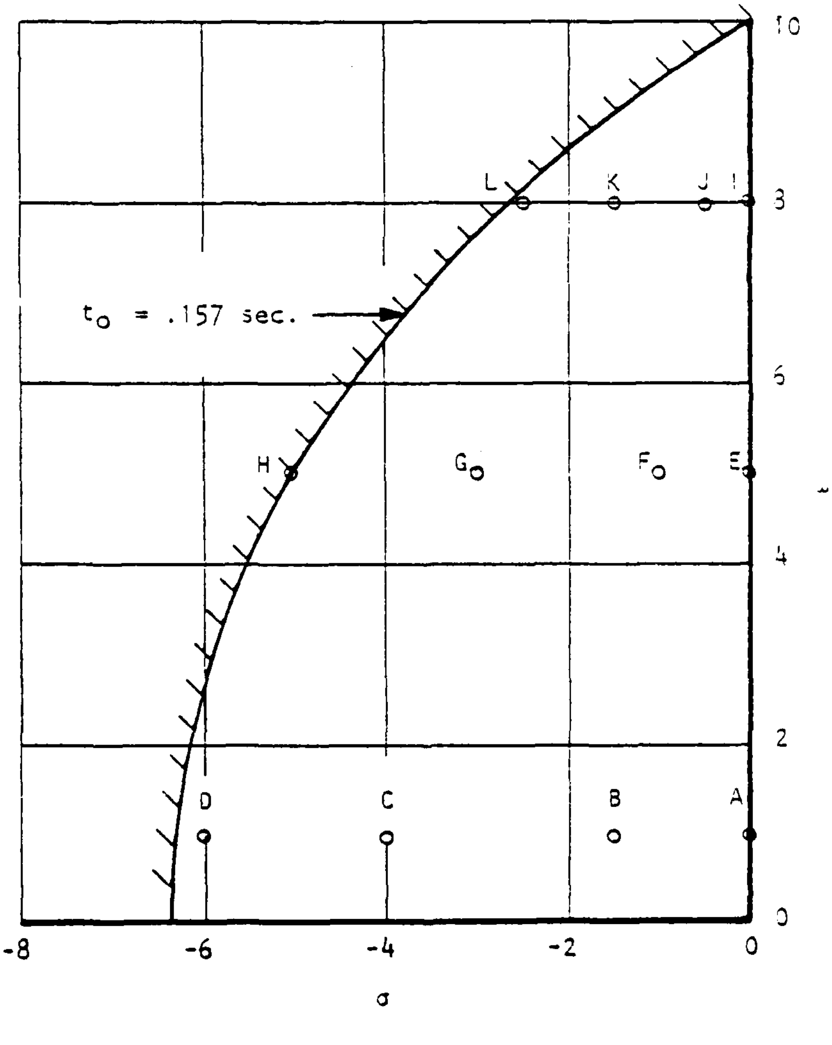


FIGURE 5 - Test Case Distribution

Conditions of constant unity magnitude on the initial peak of the input signal were imposed in order to prevent the input signal from diminishing in magnitude as damping was increased. Time for the initial peak has been shown to be:

$$t_o = \frac{1}{\omega} \tan^{-1} \frac{-\omega}{\sigma}$$

If the input value at this time is unity:

$$x(t_o) = Ae^{\sigma t_o} \sin \omega t_o = 1$$

$$A = \frac{e^{-\sigma/\omega \; \tan^{-1}(\frac{-\omega}{\sigma})}}{\sin \{\tan^{-1}(\frac{-\omega}{\sigma})\}}$$

The time responses were calculated using a digital computer program which employs the Heaviside expansion to calculate inverse Laplace transforms

$$y(t) = \mathcal{L}^{-1}\{F(s)\ x(s)\}$$

where F(s) is the transfer function and $x(s) = \frac{\omega A}{(-\sigma, \omega)}$

Since the author was not responsible for this program, its results were verified by a hand solution of the time response to input case B for the LAHOS 1-C equivalent. This verification appears in Appendix B.

The LAHOS 1-4 configuration was run for the entire range of test signals. The results appear in Figures 6 through 17. Because the total output magnitude decreases with both ω and σ, the absolute importance of the response differences diminishes at higher frequencies. It was decided to examine the LAHOS 1-C, and LAHOS 6-2 configurations only at the lowest frequency cases where the differences between high and low order system responses are most noticeable. These appear in Figures 18 through 23.

Figures 6, 10, and 14 show the LAHOS 1-4 configuration response to the undamped sine wave inputs. The responses show excellent agreement through the initial peak which progresses toward small errors in the magnitude and phaseing of the stady state response as indicated by the small differences in the frequency responses of reference 1. As we proceed into the S plane at each frequency, slight decreases in the accuracy of the initial response peak may be detected along with a quite noticeable

LAHOS 1-4 CONFIGURATION

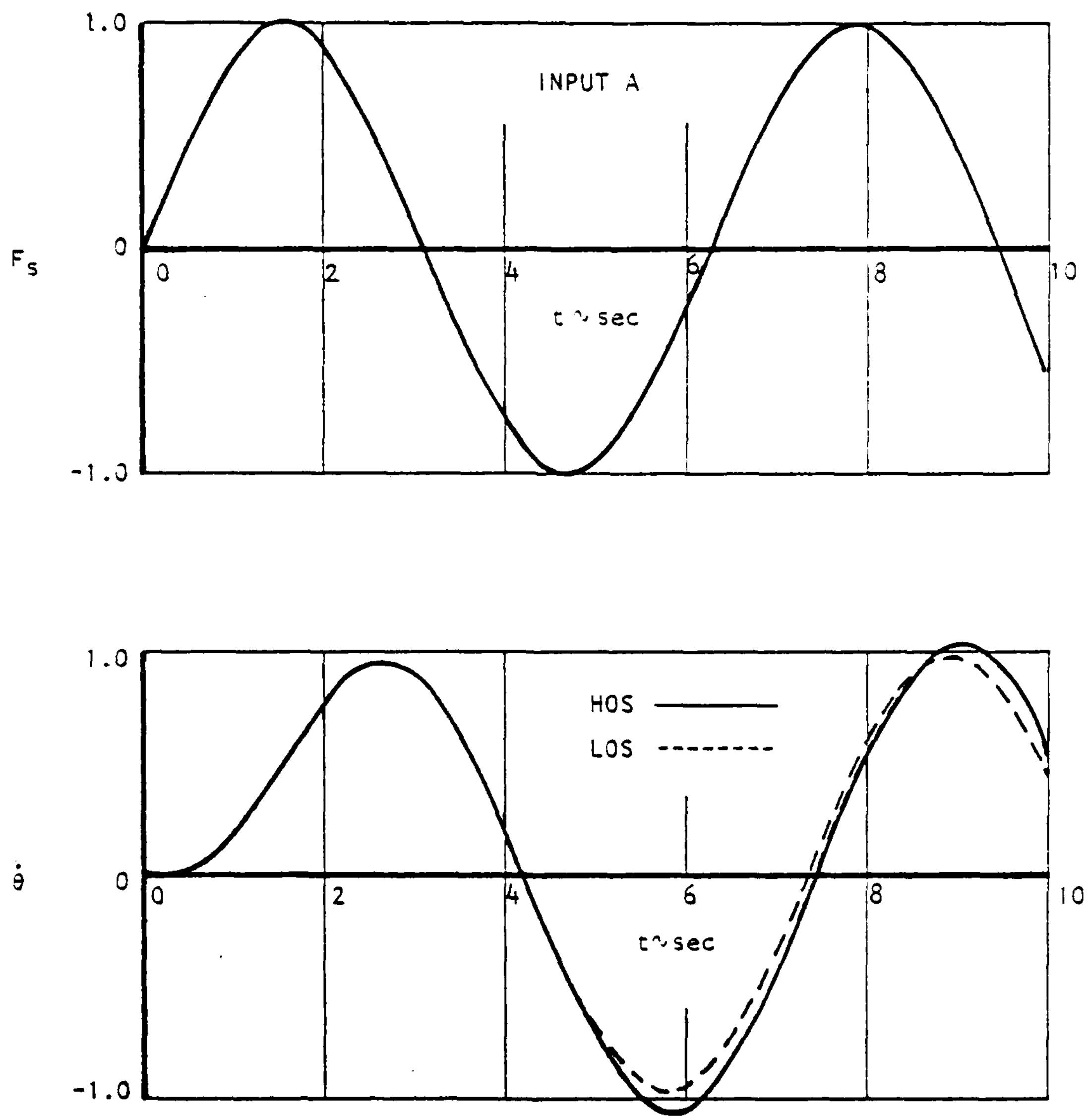
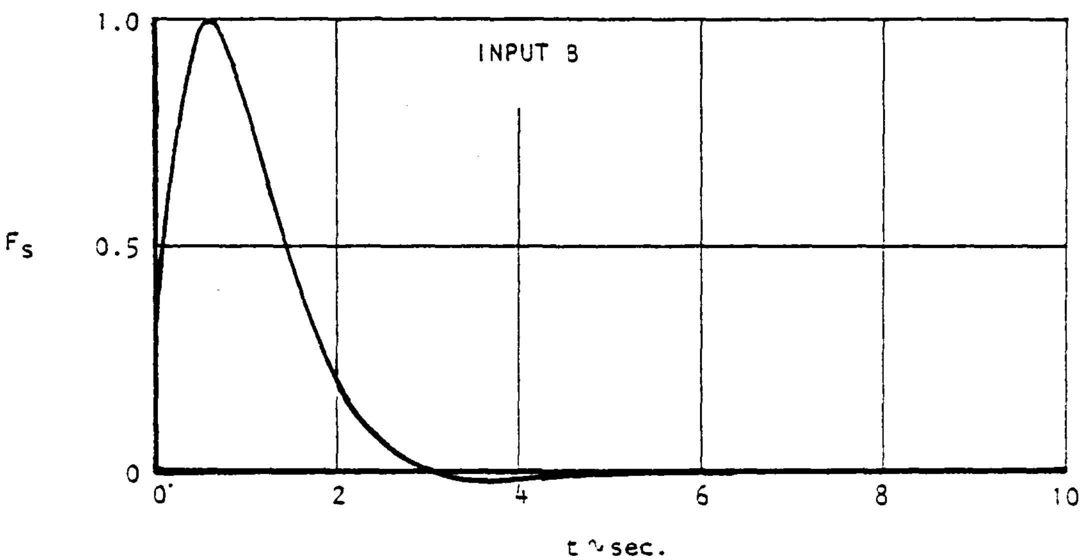


FIGURE 6 - Pitch Rate Response

LAHOS 1-4 CONFIGURATION



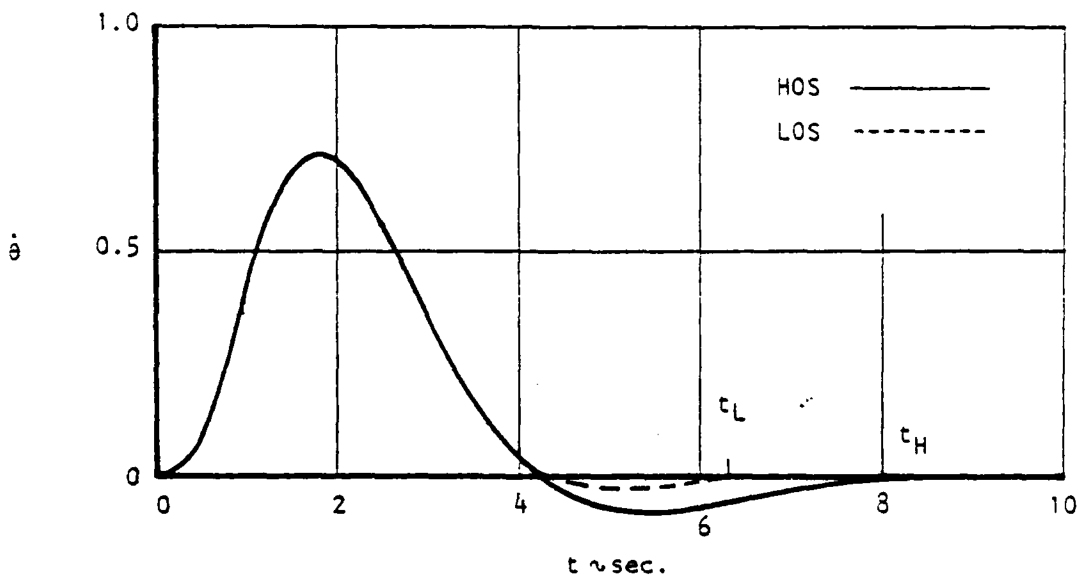


FIGURE 7 - Pitch Rate Response

LAHOS 1-4 CONFIGURATION

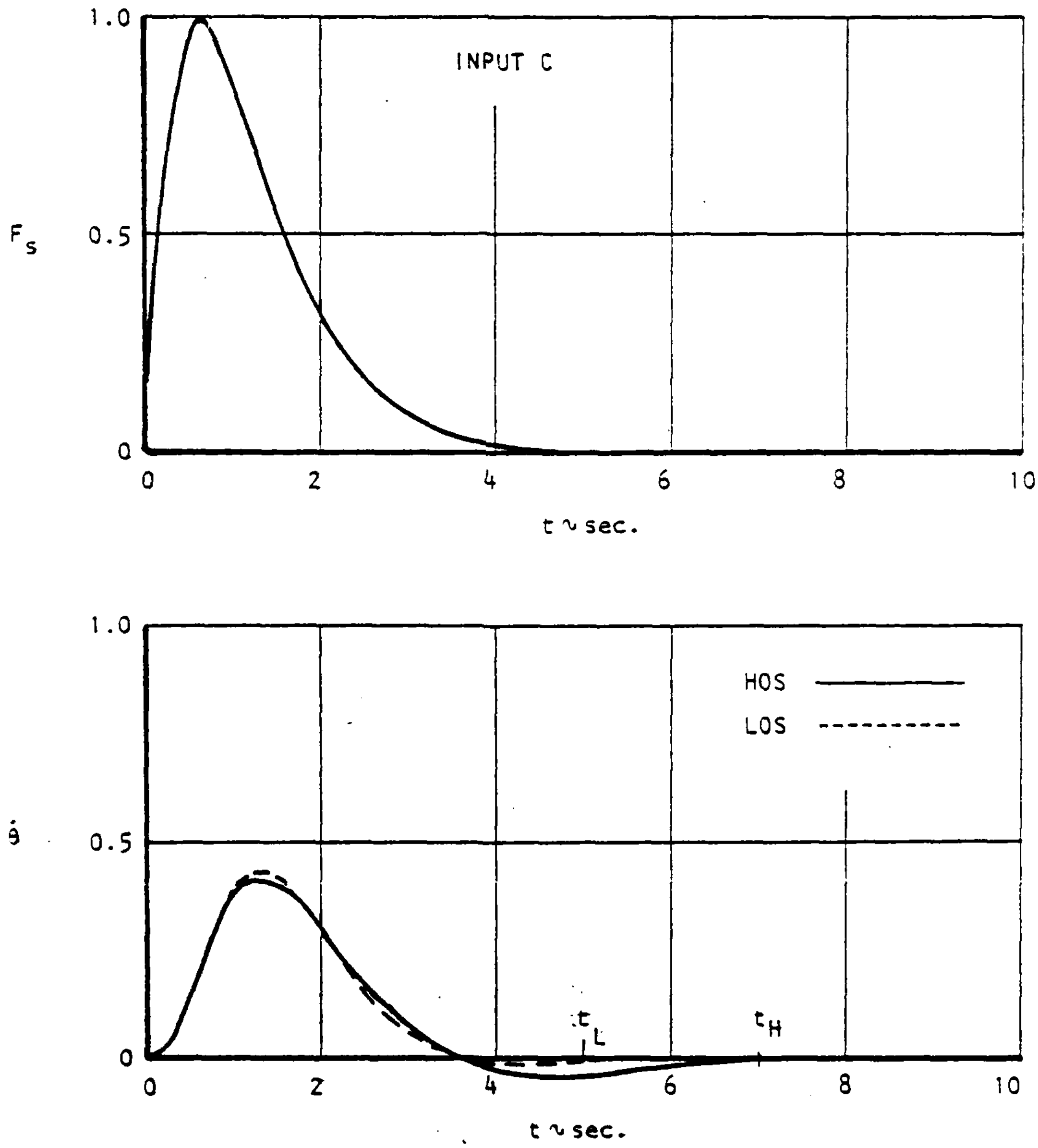


FIGURE 8 - Pitch Rate Response

LAHOS 1-4 CONFIGURATION

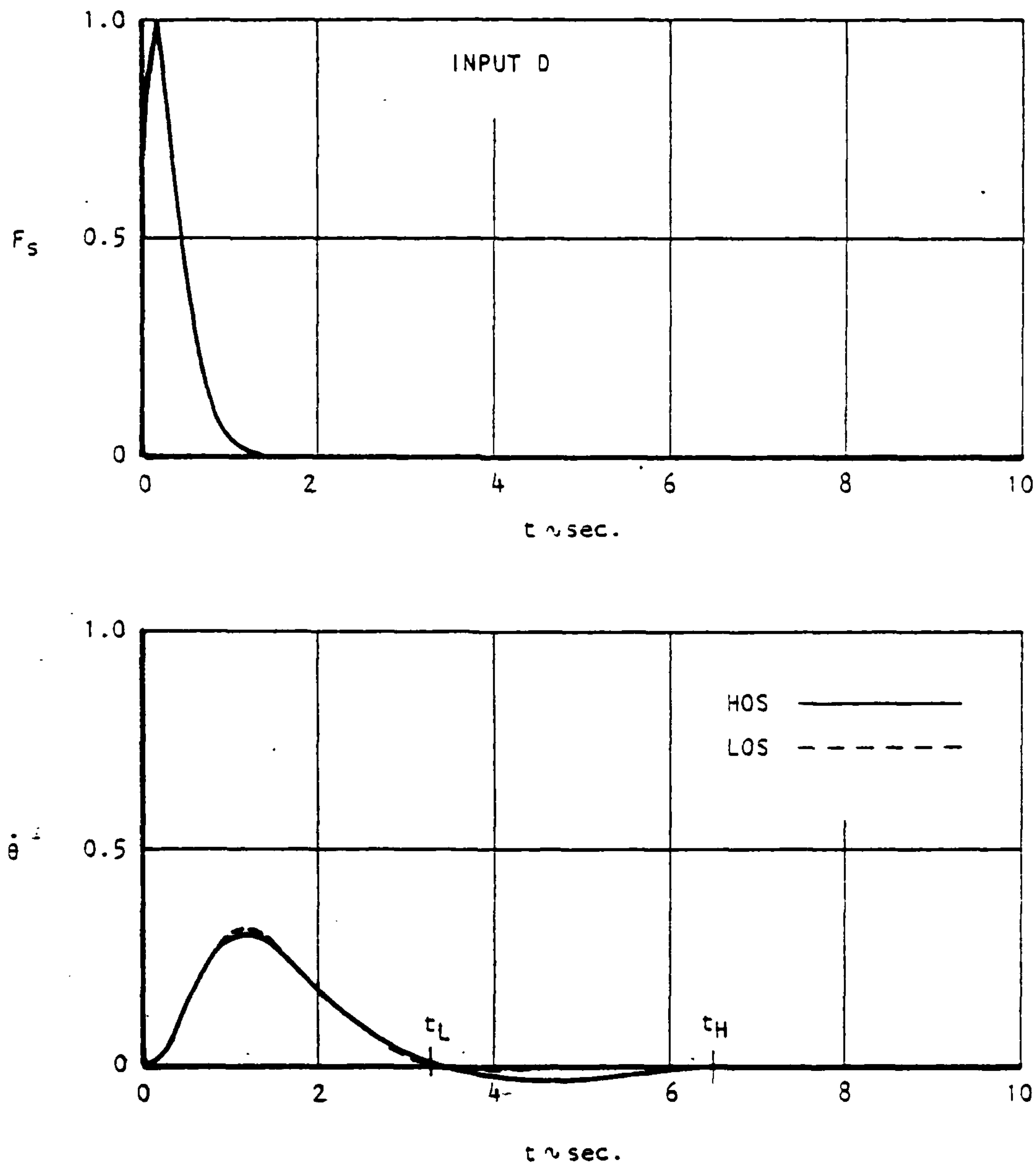


FIGURE 9 - Pitch Rate Response

LAHOS 1-4 CONFIGURATION

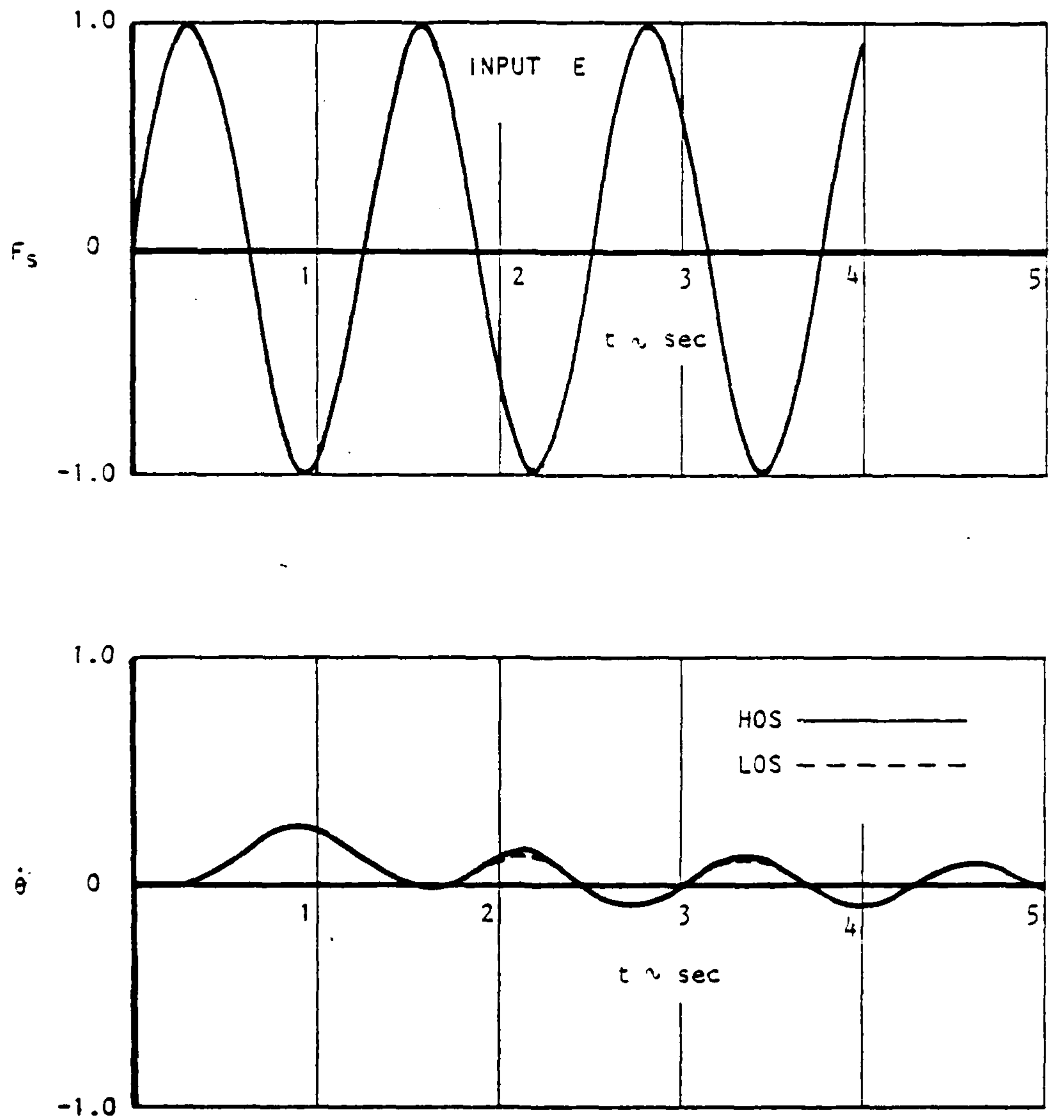


Figure 10 - Pitch Rate Response

LAHOS 1-4 CONFIGURATION

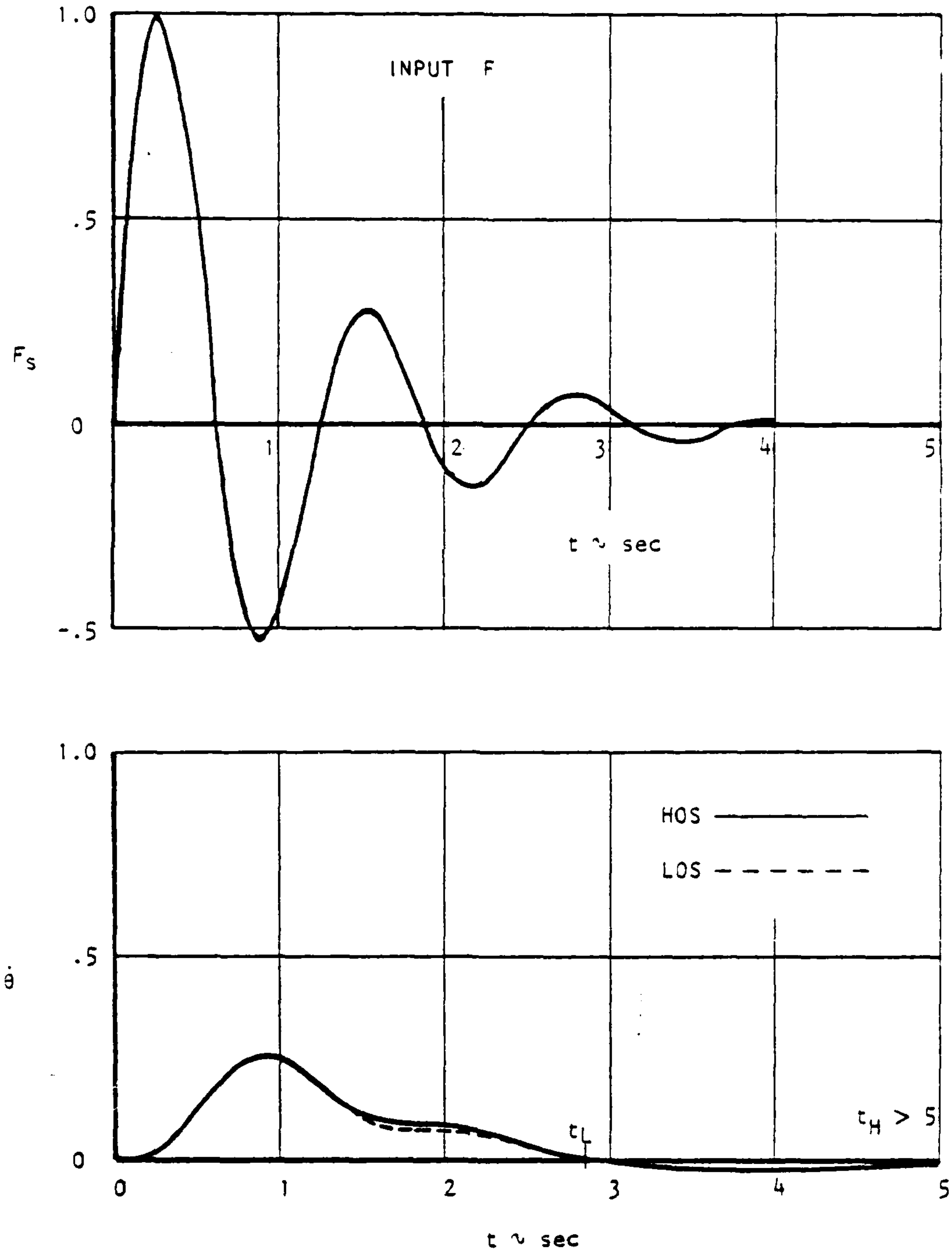
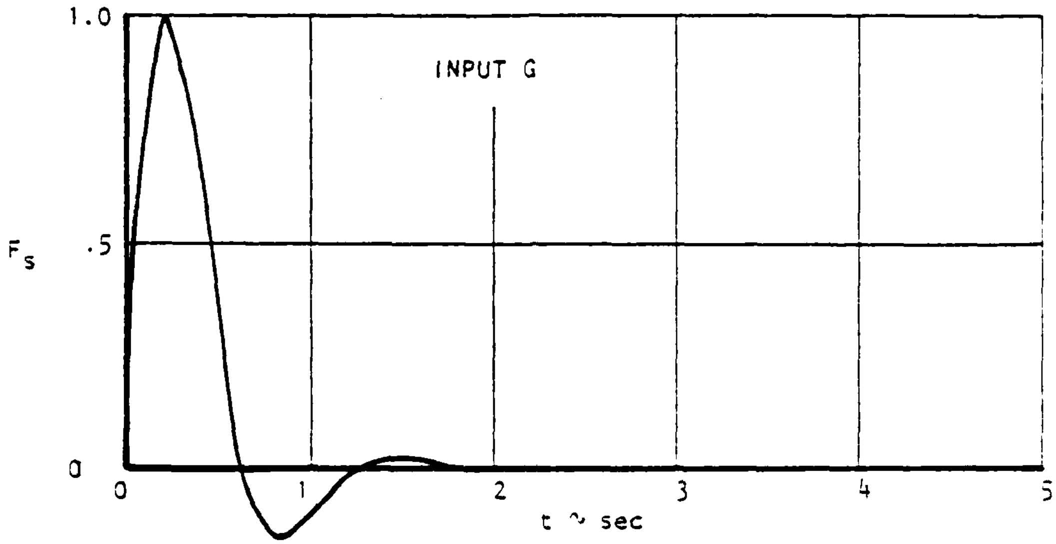


Figure 11 - Pitch Rate Response

LAHOS 1-4 CONFIGURATION



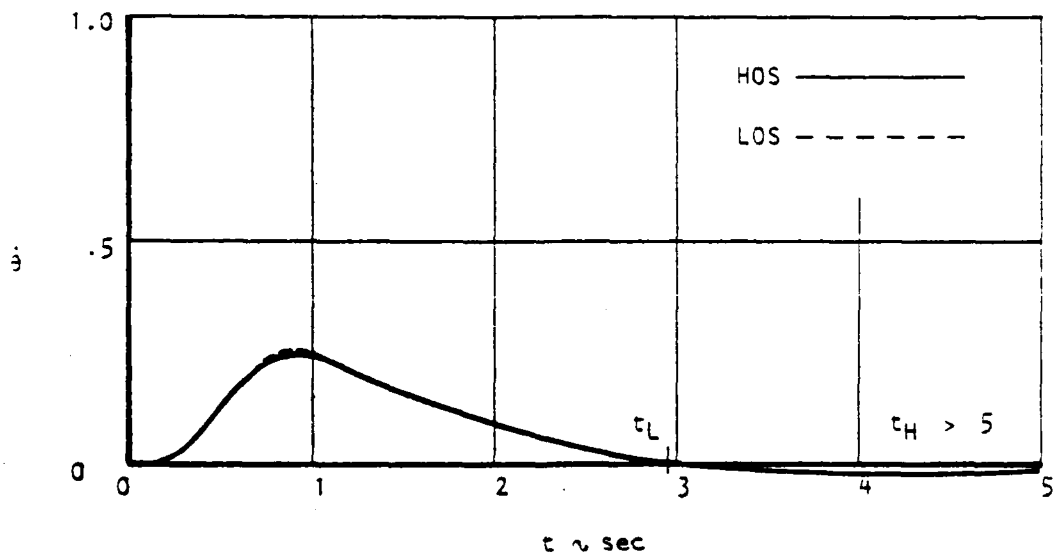


Figure 12 - Pitch Rate Response

LAHOS 1-4 CONFIGURATION

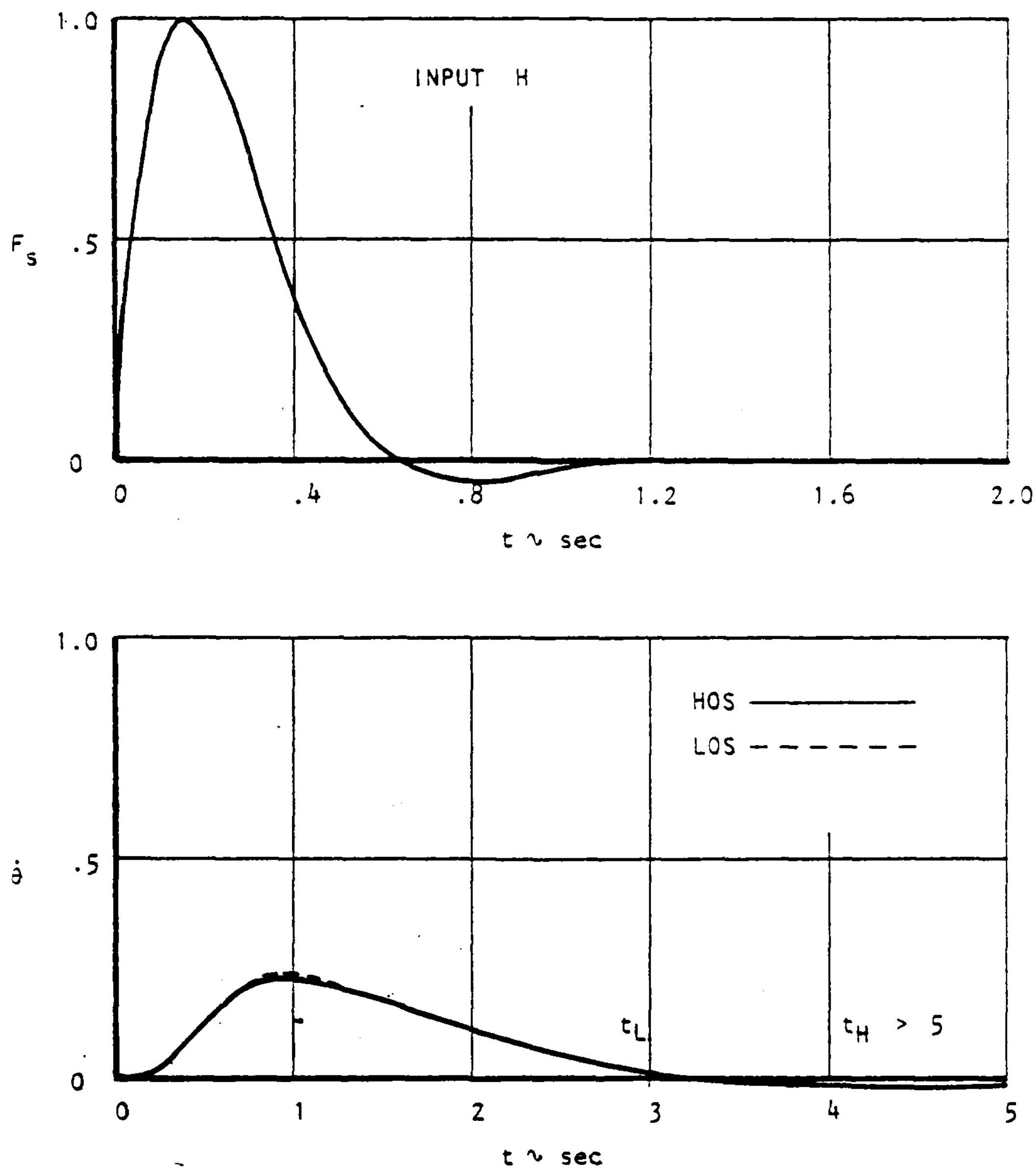


Figure 13 - Pitch Rate Response

LAHOS 1-4 CONFIGURATION

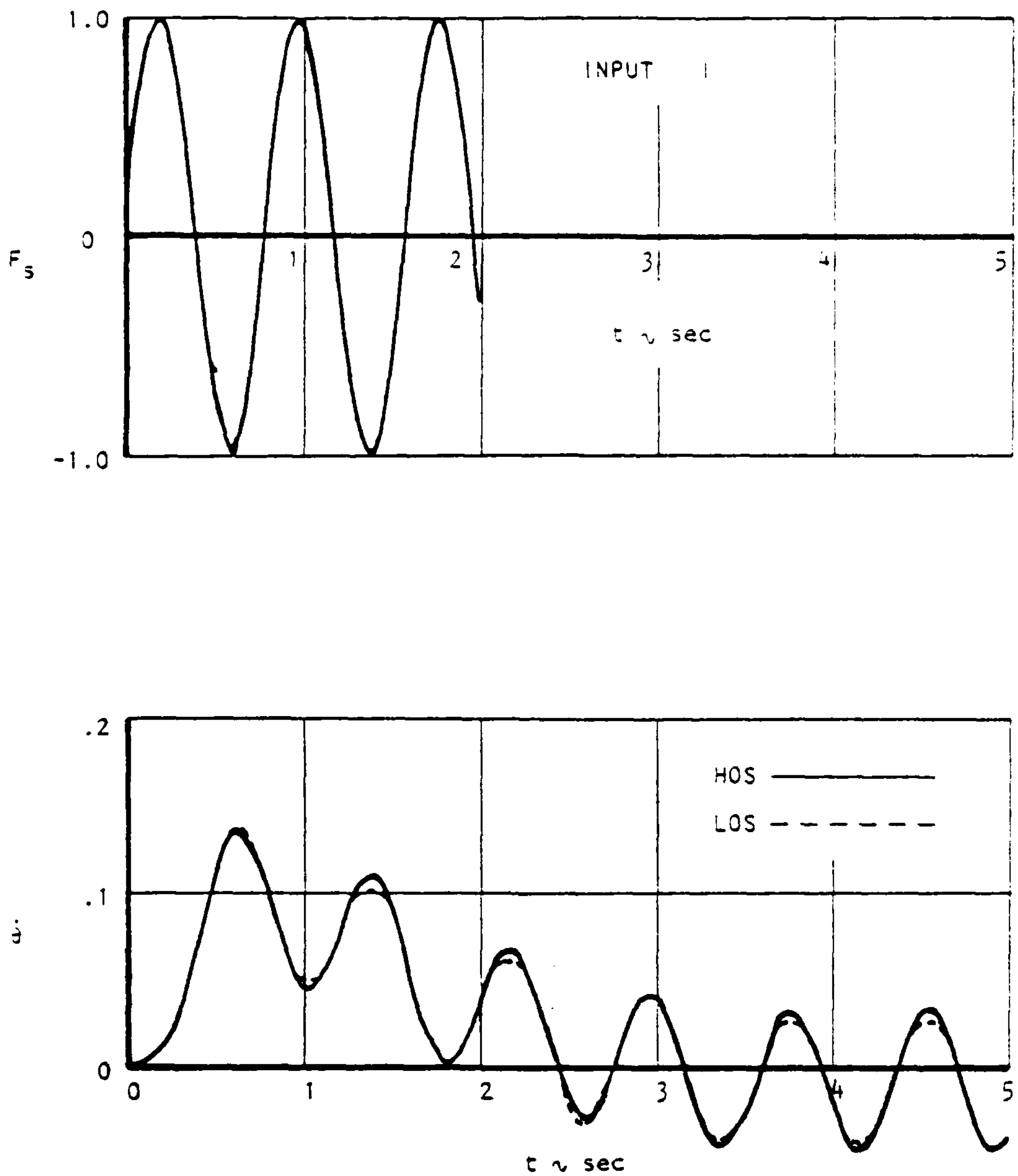


Figure 14 - Pitch Rate Response

LAHOS 1-4 CONFIGURATION

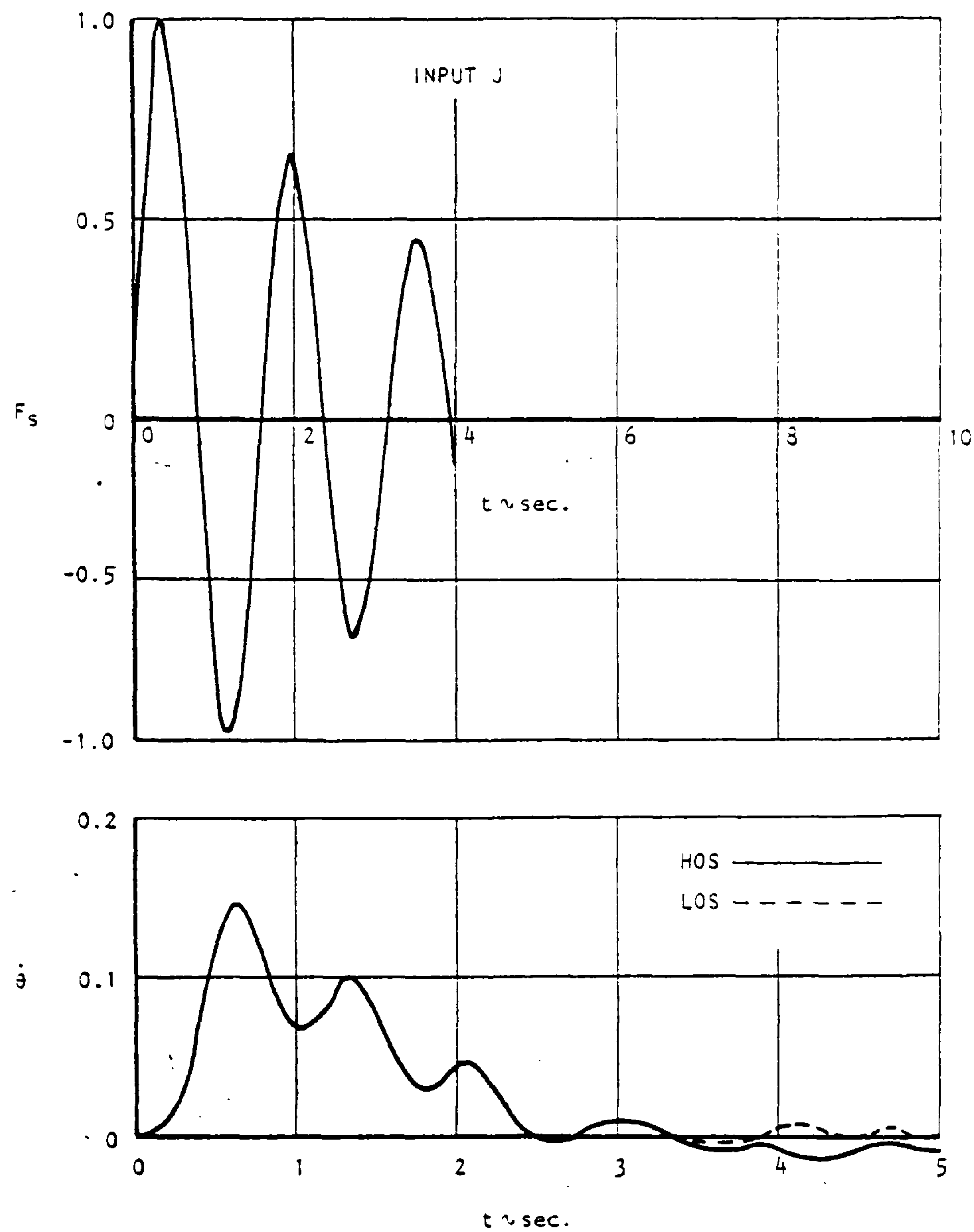


FIGURE 15 - Pitch Rate Response

LAHOS 1-4 CONFIGURATION

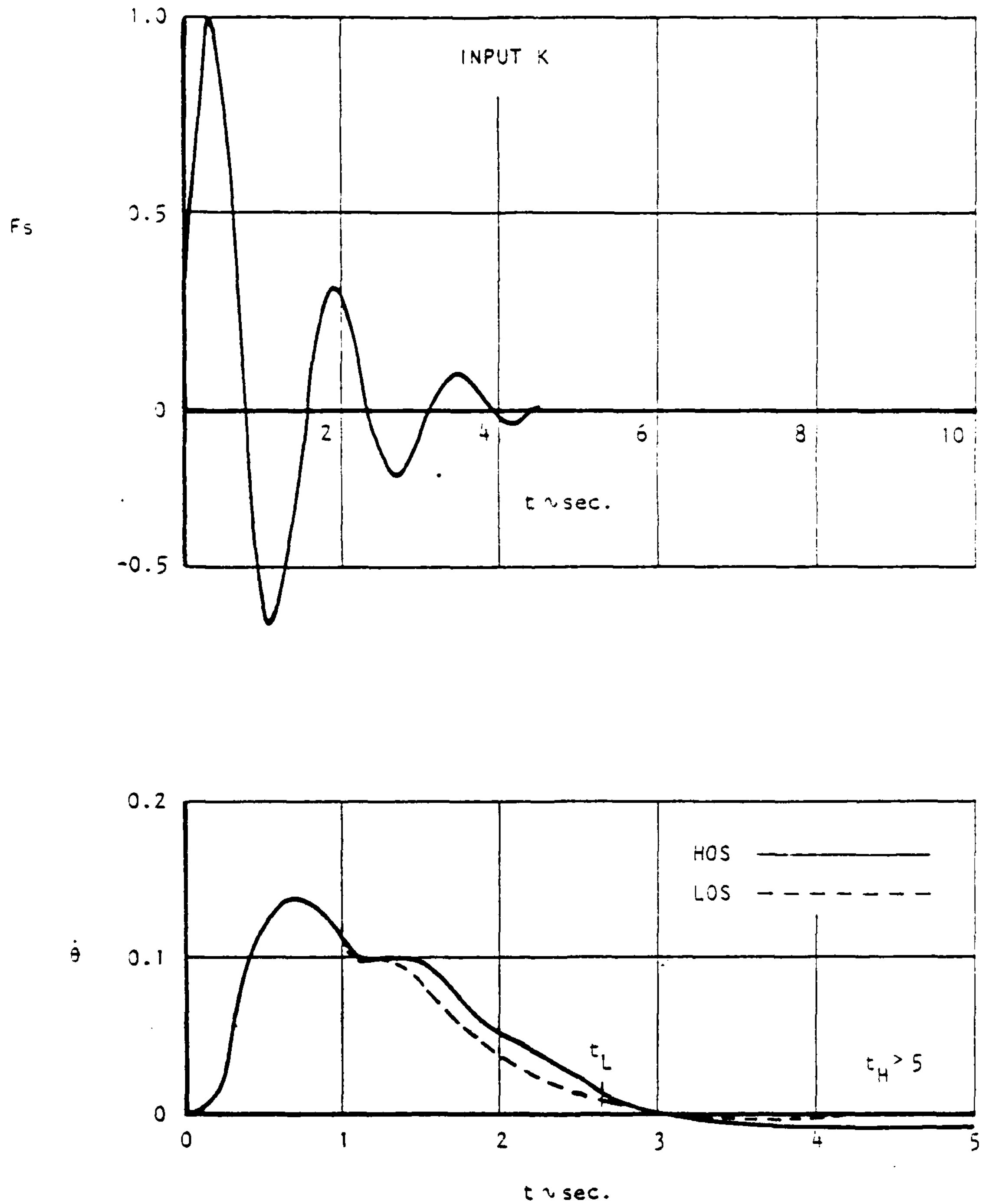


FIGURE 16 - Pitch Rate Response

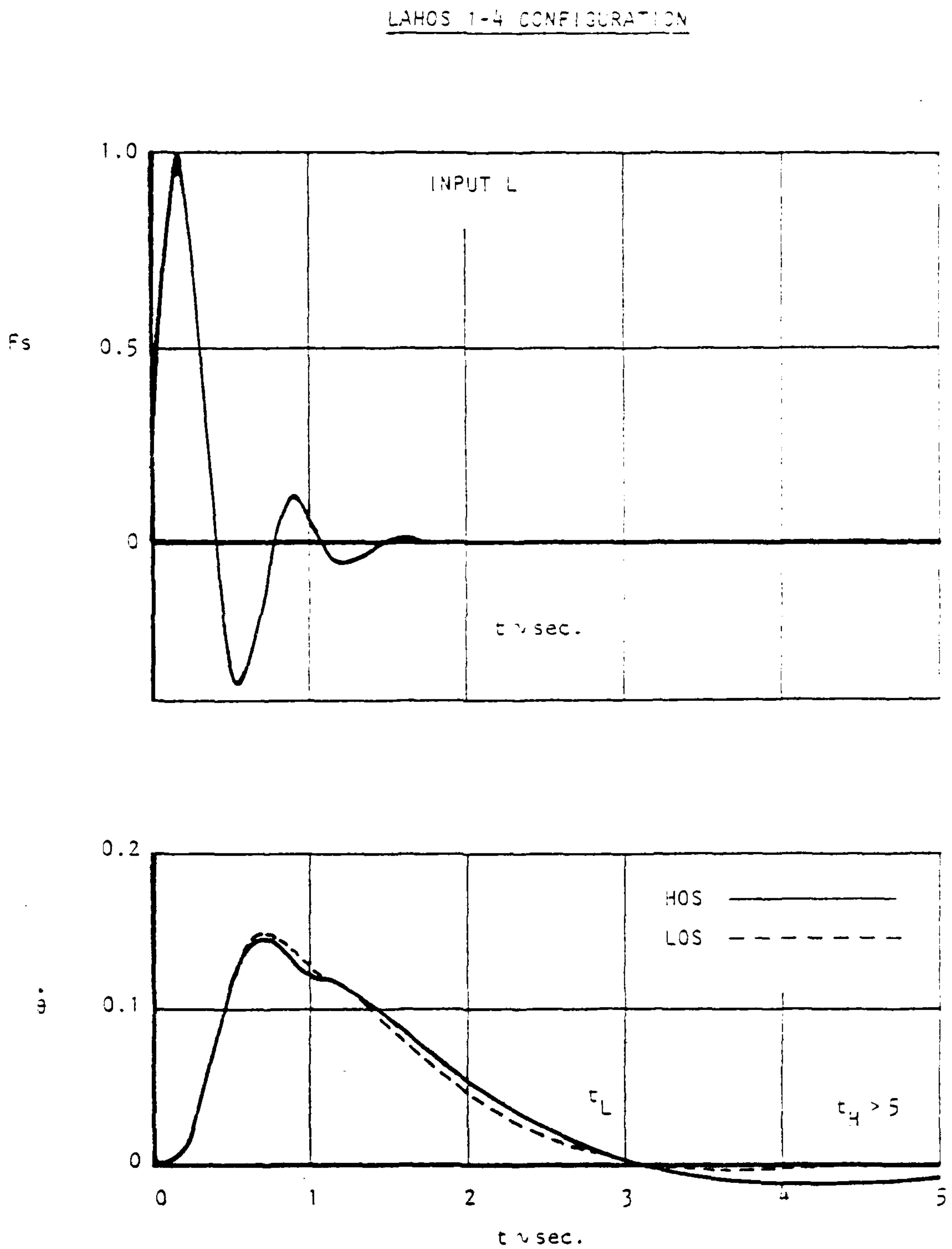


FIGURE 17 - Pitch Rate Response

LAHOS 1-C CONFIGURATION

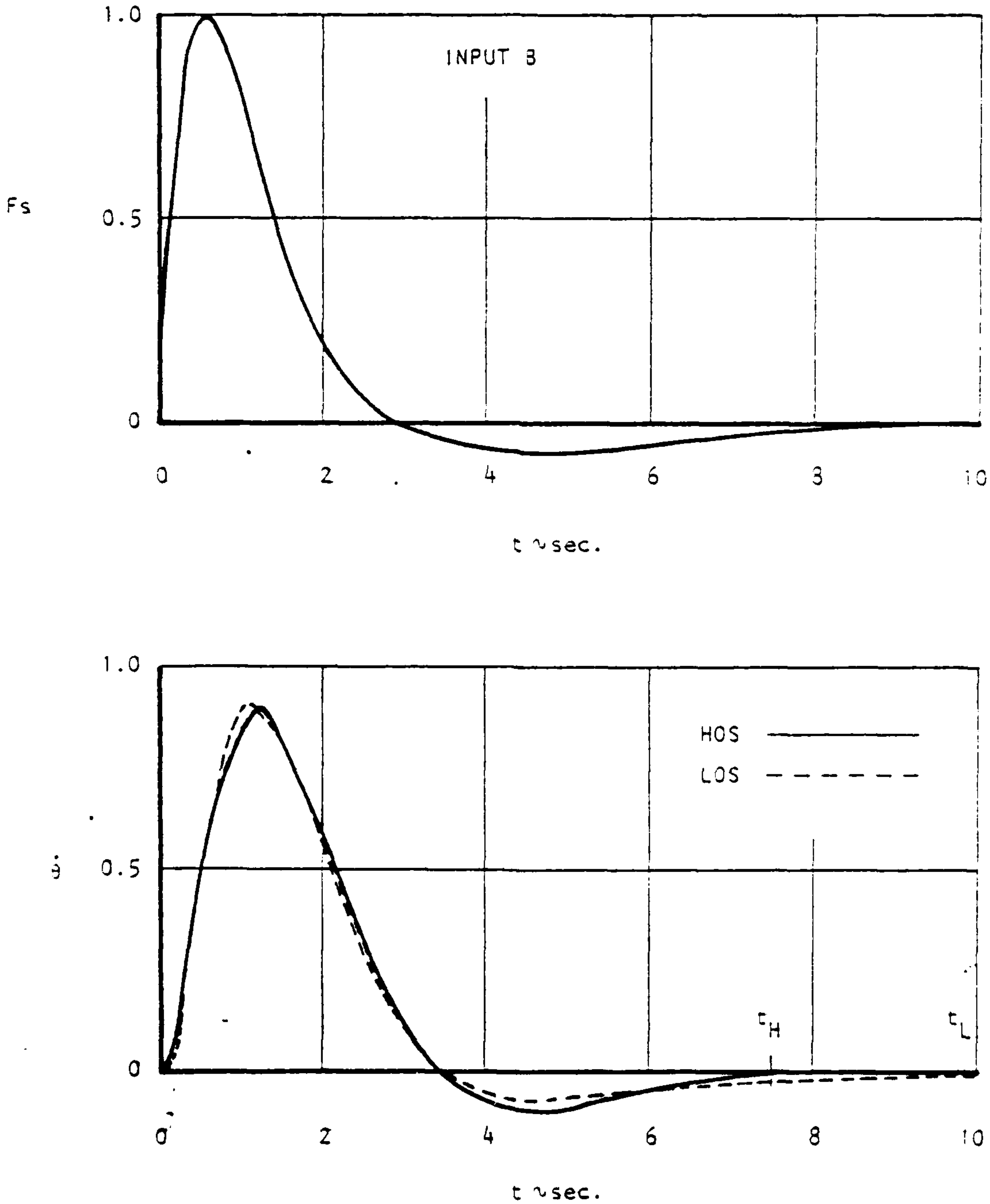


FIGURE 18 - Pitch Rate Response

LAHOS 1-C CONFIGURATION

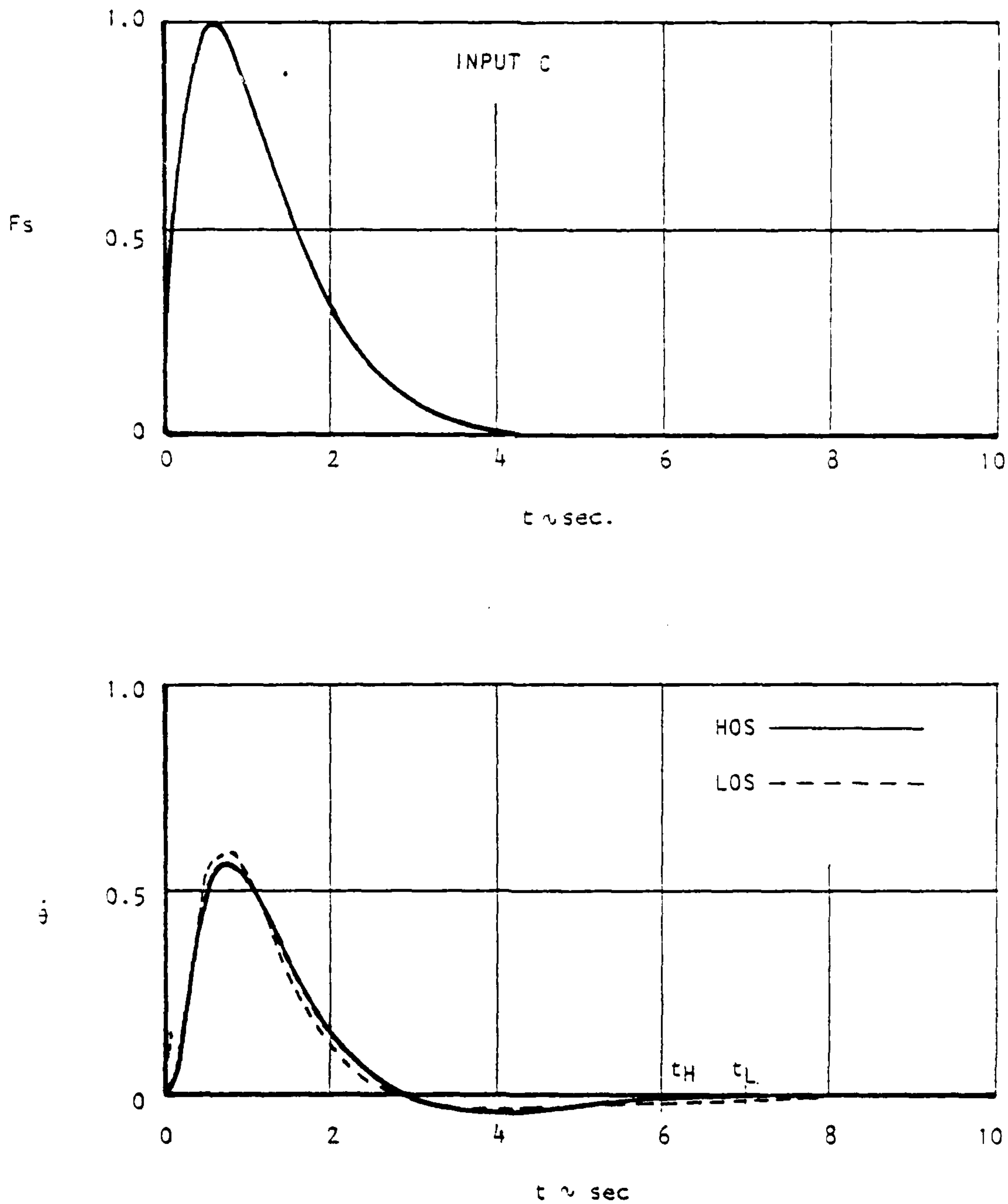
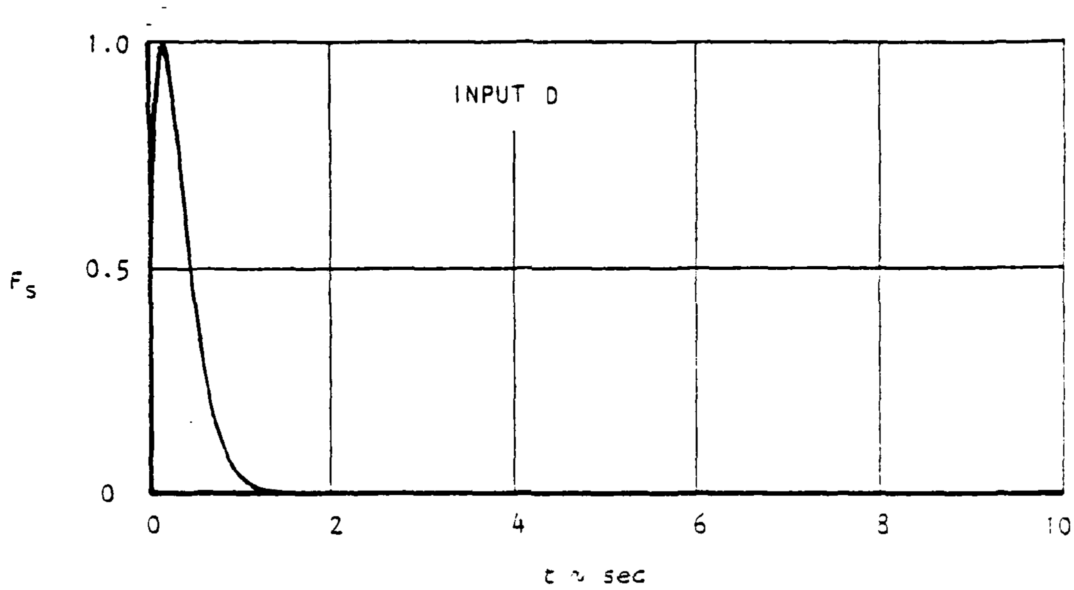


Figure 19 - Pitch Rate Response

LAHOS 1-C CONFIGURATION



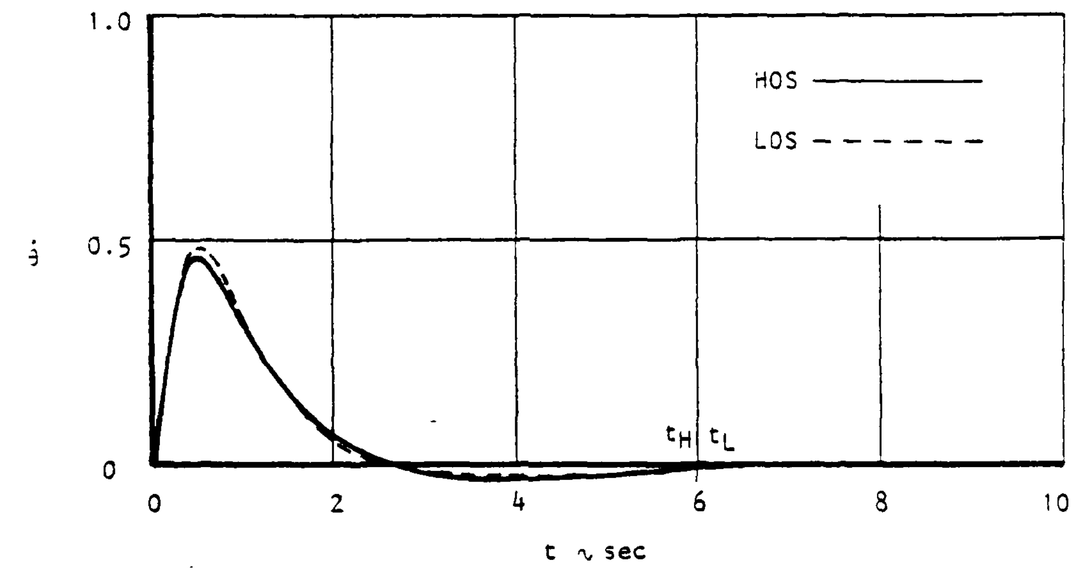
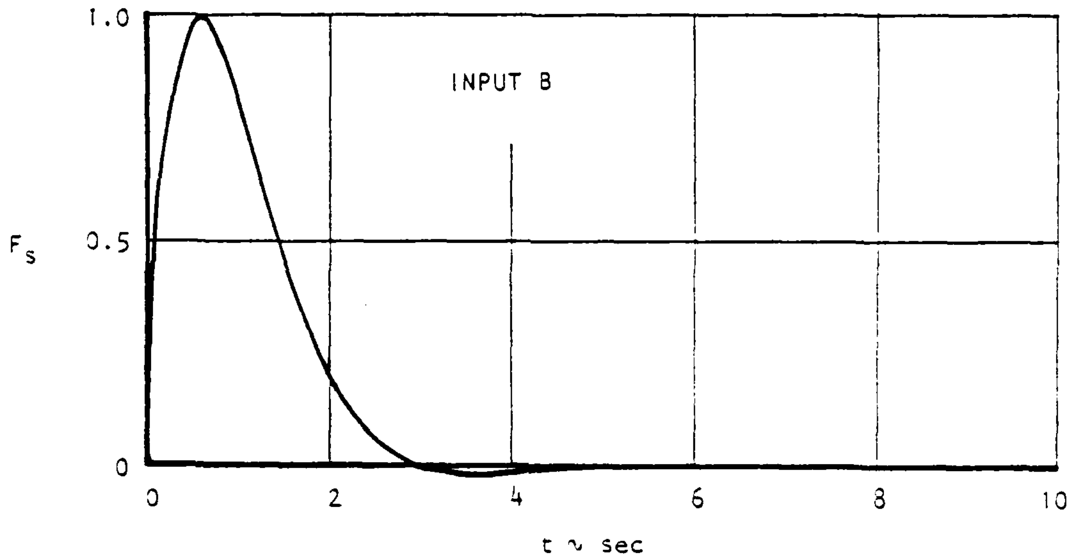


Figure 20 - Pitch Rate Response

LAHOS 6-2 CONFIGURATION



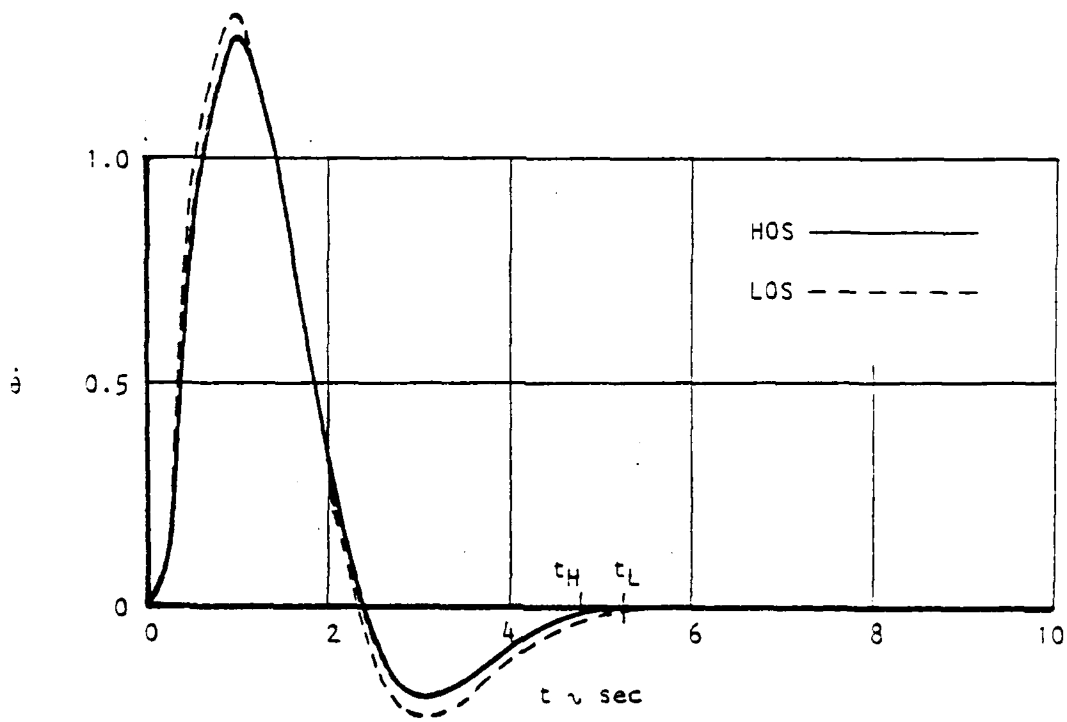


Figure 21 - Pitch Rate Response

LAHOS 6-2 CONFIGURATION

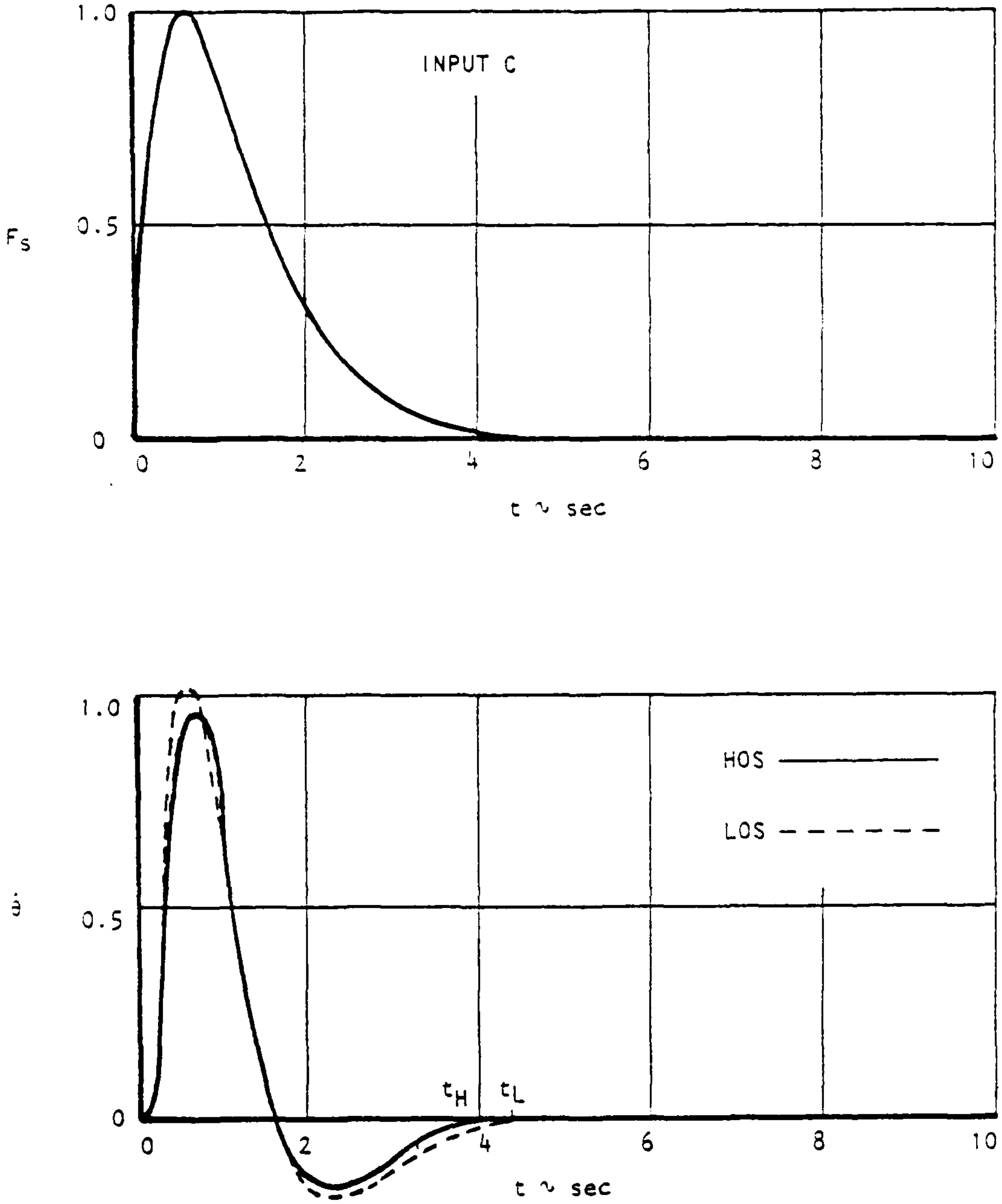


Figure 22 - Pitch Rate Response

LAHOS 6-2 CONFIGURATION

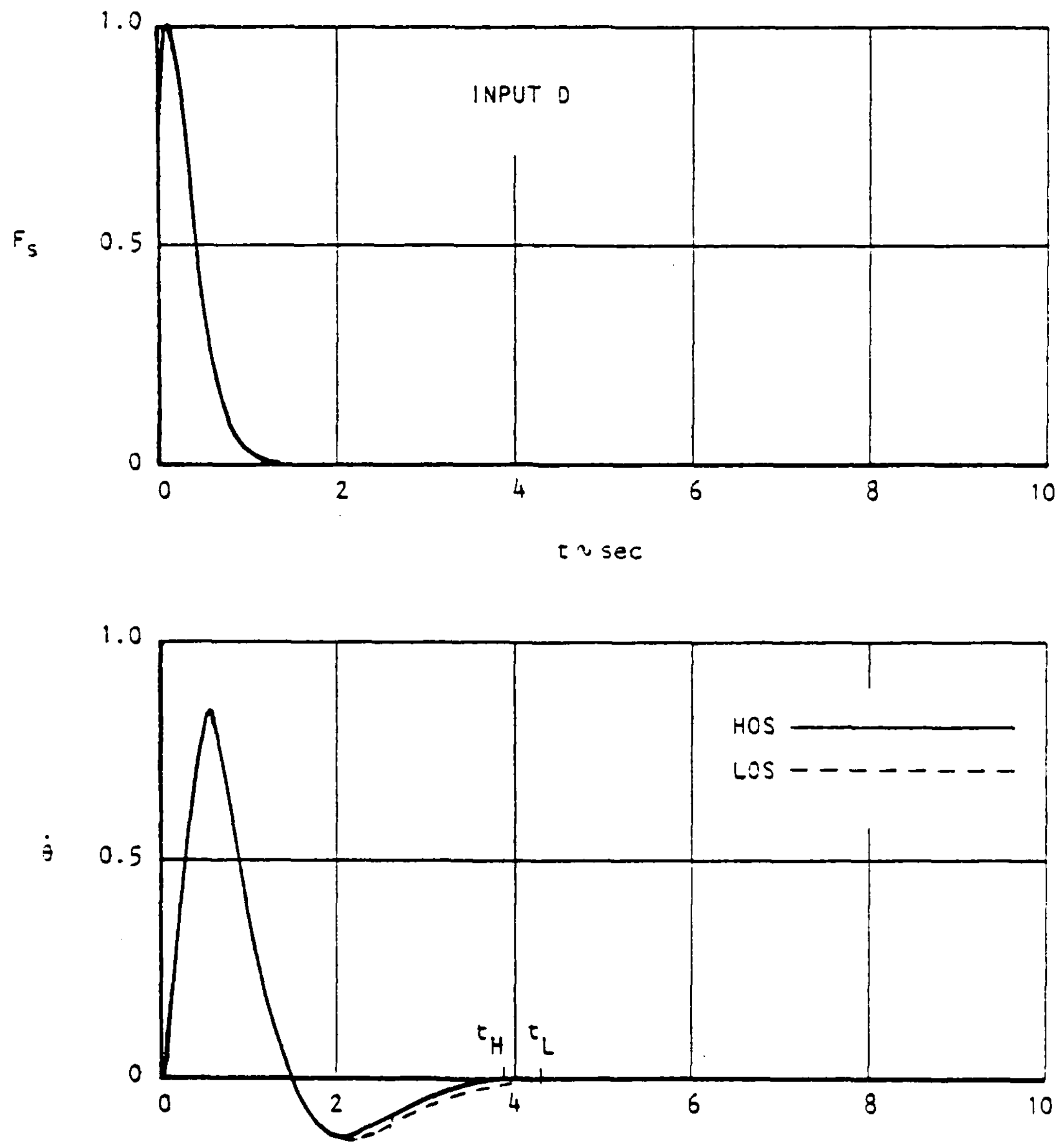


FIGURE 23 - PITCH RATE RESPONSE

difference in the subsidence times of the high order system (t_H) and low order system (t_L). The relative trend persists at high frequencies but is less noticeable due to the decreased magnitude of the output.

Figures 18 to 20 show the LAHOS 1-C high and low-order system responses to the low frequency well-damped inputs. The higher order equivalent shows a much better overall match of the time response by improving the subsidence time match while incurring slightly greater initial peak errors. Similar statements hold for the time responses of LAHOS 6-2 in Figures 21 to 23.

While it may be doubtful that these differences (for LAHOS 1-6, and LAHOS 6-2) would be significant to the pilot observing them, they do show that the equivalent's ability to represent the high-order system varies with the input signal and is not optimized for any input or S plane point. The LAHOS 1-4 configuration, while not using the recommended form for the equivalent, demonstrates that $j\omega$ Bode matching alone does not guarantee adequate similarity of response. Further, the theory prediction that this form could not satisfy all the similarity conditions is confirmed.

IMPLICATIONS TO $j\omega$ BODE MATCHING METHODS

As the damping of the input signal is decreased to zero, the long-term output does not attenuate but approaches the forced response. Similarity of the forced response to undamped inputs is the same condition as matching the subsidence time for well-damped inputs. They both match the long-term or steady-state response.

The six non-linear equations which enforce the total response similarity conditions are dependent on the value of S; that is, the input chosen. In general the equivalent systems parameters will vary over the Laplace domain. The exception is the case where the high-order system and its equivalent are of the same form.

In the case where the equivalent system parameter variation over the region of interest is small, they might be considered constants. The parameters could equally well be determined by satisfying a single similarity condition for several different inputs as by satisfying several similarity conditions for a single input. To within acceptable accuracy the same parameters would be obtained. Locating equivalent system parameters by matching $j\omega$ Bodes in effect assumes they are approximately constant over the S plane and imposes the similarity of the long term response along the line $\sigma = 0$. The initial similarity conditions are always matched in the determination of the free response mode coefficients. If a very good frequency response match is obtainable, the assumption of invariance is good at least along the $j\omega$ axis, and adequate matching of other similarity conditions (with sinusoidal inputs) is implied. As shown by the LAHOS 1-4 time histories, similarity of the $j\omega$ Bode alone does not guarantee time response similarity elsewhere.

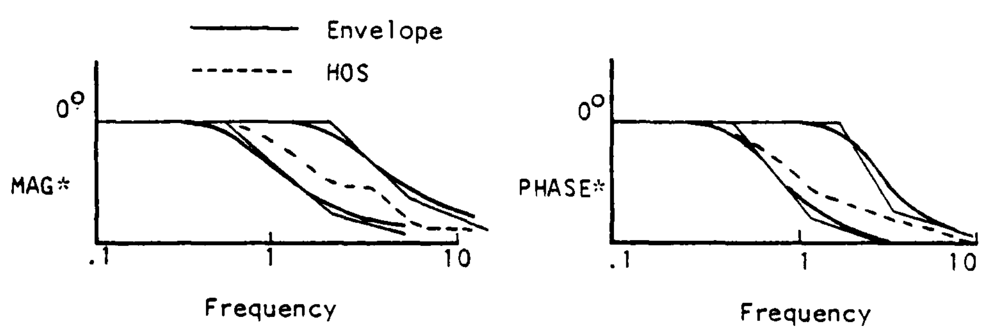
One method of obtaining time response matched equivalents has been suggested here. Others might be possible. The important point is that the equivalent system parameters vary with both σ and ω and that $j\omega$ Bode matching techniques depend on that variation being negligible. This implication remains regardless of the method used to match the time response.

Failure to obtain an acceptable $j\omega$ Bode match is a direct indication that the equivalent system parameters vary significantly. The requirements of MIL-8785B allow these parameters to vary within certain tolerances. The fact that they vary does not mean that they exceed the specified tolerances. Failure to obtain a good $j\omega$ Bode match is therefore an insufficient reason for discarding the system.

It is too restrictive to require that for a system to be judged acceptable that a single low-order equivalent be capable of representing it over the entire S plane region. Rather, it is necessary only that the high-order system behave like some acceptable lower-order system.

The success of $j\omega$ Bode matched equivalents here and in the references indicates that in many cases invariance of the equivalent system parameters is a good assumption. But uncertainty for each individual case will always exist until the assumption is checked. Further, time delays are required in many cases to enforce the invariance assumption. As discussed on page 17, this alters the parameters and decreases the similarity of the time response. It is uncertain that a low-order system with a small time delay will always be evaluated in the same manner as an identical system without any time delay. Yet that assumption is made in applying the current equivalent system parameters to the requirements of MIL-8785B. If the time delay is sufficiently small this is not a serious objection, but no such difficulties are encountered with time response matching.

Section 3.2.2.1 of MIL-8785B describes tolerances for ζ_{sp} and $\omega_{n_{sp}}$ as a function of N_z/α (L_α or $1/T_{\theta 2}$) Given a low frequency magnitude of 0 db and low frequency phase angle of 0°, these tolerances define a magnitude and phase envelope in the frequency domain:



* (Not plots of actual envelopes)

If a high-order system has a frequency response within that envelope, then there is some acceptable low-order system which has the same steady state response. If the high-order system response violates the envelope it may be discarded. If it possesses a good lower order system match, the equivalent system parameters are approximately constant for undamped inputs within the frequency range. Significant variations may still exist in the σ direction. If the high-order system response lies within the envelope but does not possess an acceptable frequency response matched equivalent the only way to determine the equivalent system parameters and therefore its acceptability is by total time response matching.

Current Bode matching techniques are a good test of the high-order system only if the implied assumption of parameter invariance holds over the S plane. The assumption works well in many cases as long as time delays are included in the equivalent system form, but statistical data can only establish the probability of its holding in any given application.

IMPLICATIONS TO PILOT COMPENSATION CRITERIA

Reference 3 developes a flying qualities criteria based on pilot compensation of the form,

$$\frac{\delta s}{\theta c} = K_p \frac{(T_1 S + 1)}{(T_2 S + 2)} e^{-.3s}$$

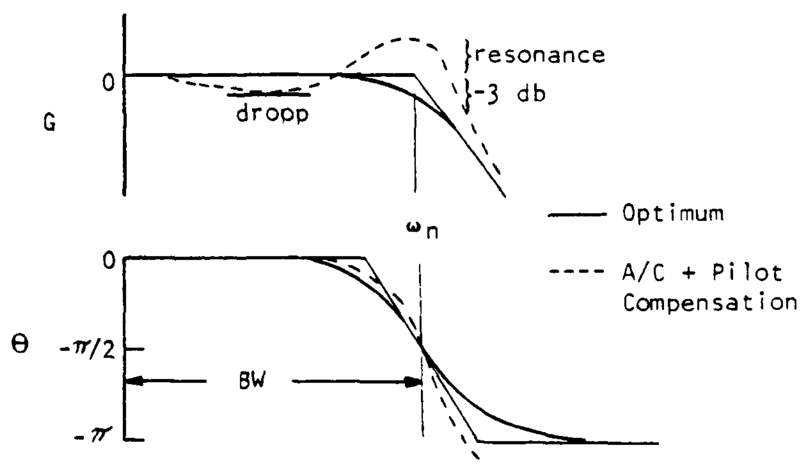
The basic philosophy is that pilots desire a certain "optimum" response and are willing to provide limited amounts of compensation to achieve that response before down-rating the system. The final pilot rating is then dependent on how much compensation must be provided and the nearness of the best result to the optimum.

If it is assumed the pilot wants direct pitch control with no phase lags the implication is that at low frequency:

$$\text{Mag } (\theta/\theta_c) = 1$$
$$\text{Mag } (\theta/\theta_c)_{Db} = 0$$
$$\measuredangle\ (\theta/\theta_c) = 0$$

Deviations from this (droop) should be minimized.

Furthermore, if oscillatory overshoot is to be minimized, a minimum zero-over-second-order form for the optimum must be considered in order for the system to have an oscillaroty free response mode. Maintaining a certain bandwidth maintains the desired response up to a certain speed of input. For a well-damped (ζ = .707) 0/2nd system the -3 db bandwidth occurs at the break frequency, ω_n, where the phase angle passes through $-\pi/2$.



The compensation parameters K_p, T_1, T_2, are determined by providing phase compensation to hold $\theta = -\pi/2$ at the specified bandwidth, while matching the low frequency magnitude and minimizing droop. The amount of phase compensation is then a measure of how hard the pilot must work to obtain the optimum and the resonance is a measure of how close to the optimum he is able to get. Since the object is to come as close as possible to the simple "optimum" Bode, this method is providing the same results as matching the system Bode to obtain an equivalent. In this case we are attempting to match the aircraft and flight control system dynamics plus pilot compensation with the optimum zero over second order system. The main difference is that we are attempting to match the low-order system by varying root locations in the high-order system rather than vice-versa.

If we view this criteria in terms of the high-order system Bode, we see that by allowing the pilot to provide certain magnitude and phase compensation, the high-order system magnitude and phase may be allowed to vary from the optimum by a like amount. This is the same as allowing the high-order system Bode to lie within a certain envelope similar to that already discussed. This technique then is not as restrictive as current j_ω Bode matching techniques because it allows magnitude and phase to vary within that envelope.

The pilot compensation approach, however, is sensitive to the bandwidth chosen. Like the current equivalent systems methods it does not consider what differences in total response to highly damped inputs might occur. Thus the pilot compensation approach is, in principle, doing much the same thing as current equivalent systems approaches and suffers from some of the same deficiencies.

CONCLUSIONS AND RECOMMENDATIONS

Current methods of applying equivalent systems as well as proposed pilot compensation criteria are based on sound concepts. Both have been shown to have highly similar implications and some common limitations. Current equivalent systems methods are limited by the implied assumption of invariance of the equivalent systems parameters. Since they deal only with $j\omega$ Bodes neither method considers total system response to aperiodic pilot inputs. Provided the recommended equivalent system form which includes artificial time delays is employed, the invariance assumption is valid in many cases. But because it is not assured, straightforward application of either method may not always be reliable. Determination of equivalent systems parameters based on time response matching not only eliminates the uncertainty, but is directly implied both by the equivalent system concept and by the idea of an optimum response basic to pilot compensation methods. As such it represents the most logical extension of current specifications.

The analysis suggests that equivalent systems parameters be considered variables over the region of the Laplace domain within which human pilots may operate; that their values be determined by similarity of total time responses to inputs within that range; and that the acceptability of the system be judged by comparing the variations with the current MIL-F-8785B requirements.

It is recognized that application of this criteria presents some problems. Programming simulators for direct comparison of high-order systems with their equivalents is difficult because of the variance of equivalent systems parameters with input. No design methods currently exist to meet the criteria. It is unclear what judgement should be made about a high-order system if it is acceptable over only a portion of the S plane region. Additional verification of the implications are necessary before high levels of confidence in the method are achieved. In particular some of the anomalies which have occured with present methods should be examined under the suggested criteria.

It is believed that most of these problems may be either solved or avoided as further attempts to apply the methods are made.

REFERENCES

1. Hodgkinson, J., and Lamanna, W. J. "Equivalent Systems Approaches to Handling Qualities Analysis and Design Problems of Augmented Aircraft." McDonnell-Douglas Corporation Paper: MCAIR 77-016, August 1977.

2. Neal, T. P., and Smith, R. E. "An In-Flight Investigation to Develop Control System Design Criteria for Fighter Airplanes." AFFDL TR-70-74, December 1970.

3. Johnston, K. A., and Hodgkinson, J. "Flying Qualities Analysis of an In-Flight Simulation of High-Order Control System Effects on Fighter Aircraft Approach and Landing." McDonnell-Douglas Corporation Report: MDC-A5596, 22 December 1978.

4. Smith, R. E. "Effects of Control System Dynamics on Fighter Approach and Landing Longitudinal Flying Qualities." Calspan Corporation Report Number AK-5280-F-12, March 1978.

5. Hogkinson, J.; Berger, R.; and Bear, R. L. "Analysis of High-Order Aircraft Flight Control System Dynamics Using and Equivalent Systems Approach." McDonnel-Douglas Corporation Paper: MCAIR 76-009, April 1976.

6. Hodgkinson, J. "The Application of Equivalent Systems to MIL-F-8785B." McDonnell-Douglas Corporation Paper: MCAIR 79-015, September 1978.

7. McRuer, D.; Ashkenas, I.; and Graham, D. Aircraft Dynamics and Automatic Control. Princeton University Press, 1973.

APPENDIX A

APPLICABLE PROPERTIES OF DIFFERENTIAL EQUATIONS

All linear control theory considers solutions to a linear ordinary differential equation with constant coefficients:

$$\sum_{i=0}^{n} B_i \frac{d^i y(t)}{dt^i} = \sum_{j=0}^{m} A_j \frac{d^j x(t)}{dt^j} \qquad \text{(A-1)}$$

where the output function, $y(t)$, is initially zero and for times greater than zero is a consequence of the input function, $x(t)$. For any given input function there may be many solutions for the output function, the complete solution being the sum of all such solutions. It is a direct consequence of the system's linearity that any sum of solutions for the output is also a solution and that the complete response to any linear combination of inputs is the sum of the complete responses to each individual input.

IF: $y_1(t)$ and $y_2(t)$ are solutions,

THEN: $y_1(t) + y_2(t)$ is also a solution.

and

IF: $x_1(t)$ produces $y_1(t)$,

and: $x_2(t)$ produces $y_2(t)$,

THEN: $x_1(t) + x_2(t)$ produces $y_1(t) + y_2(t)$.

Since zero may be added to either side of equation (A-1) without changing the equality, any functions which make:

$$\sum_{i=0}^{n} B_i \frac{d^i y(t)}{dt^i} = 0 \qquad \text{(A-2)}$$

$$\sum_{j=0}^{m} A_j \frac{d^j x(t)}{dt^j} = 0 \qquad \text{(A-3)}$$

are always solutions. The functions for $y(t)$ implied above are called the homogeneous or free response solutions since they do not depend directly on the input function (s). All other solutions depend explicitly on the input function (s) and are termed the forced response. The total response is the sum of both: $y(t) = y(t)_{forced} + y(t)_{free}$.

Equations of the form (A-1) are easily solved using the Laplace transform which changes the operation of differentiation into multiplication by the Laplace variable, S.

$$\mathcal{L}\left[\frac{d^i y(t)}{dt^i}\right] = S^i y(s)$$

changes (A-1) to:

$$y(s) \sum_{i=0}^{n} B_i S^i = x(s) \sum_{j=0}^{m} A_j S^j \qquad \text{(A-4)}$$

which when factored and rearranged yields the transfer function:

$$\frac{y(s)}{x(s)} = \frac{\prod_{j=0}^{m} \{S - (a_j + jb_j)\}}{\prod_{i=0}^{n} \{S - (a_i + ib_i)\}} \qquad \text{(A-5)}$$

The above function in the complex variable $s = \sigma + j\omega$ may be evaluated as a single complex number at any point in the Laplace domain:

$$\frac{y(s)}{x(s)} = Ge^{j\theta}$$

Choosing a value for s is equivalent to setting the operation of differentiation equal to multiplication by a complex constant.

IF: $s = \sigma + j\omega$

and: $\mathcal{L}\left[\frac{dx(t)}{dt}\right] = sx(s)$

THEN: $\mathcal{L}\left[\frac{dx(t)}{dt}\right] = (\sigma + j\omega)\ x(s)$,

inverse Laplace transforming:

$$\frac{dx(t)}{dt} = (\sigma + j\omega)\ x(t)$$

$$\int \frac{dx(t)}{x(t)} = (\sigma + j\omega) \int dt + C$$

$$\ln x(t) = (\sigma + j\omega)t + C$$

or $$x(t) = Ae^{(\sigma + j\omega)t} \qquad A = e^C$$

The transfer function equation becomes:

$$y(s) = Ge^{j\theta}\, x(s)$$

inverse Laplace transforming

$$y(t) = Ge^{j\theta}\, x(t)$$

or

$$y(t) = GAe^{\sigma t}\, e^{j(\omega t + \theta)}$$

For $x(t)$ and $y(t)$ to be real functions of time, σ and ω must appear in conjugate pairs:

$$x(t) = Ae^{\sigma t}\, e^{j\omega t}$$

or

$$x(t) = Ae^{\sigma t}\,(\cos \omega t + j \sin \omega t)$$

A pair of inputs gives:

$$x_1(t) + x_2(t) = A_1e^{\sigma_1 t}(\cos \omega_1 t + j \sin \omega_1 t) + A_2e^{\sigma_2 t}\,(\cos \omega_2 t + j \sin \omega_2 t)$$

For the imaginary part to vanish:

$$A_1e^{\sigma_1 t} \sin \omega_1 t + A_2e^{\sigma_2 t} \sin \omega_2 t = 0$$

which directly implies:

$$A_1 = A_2, \quad \sigma_1 = \sigma_2, \quad \omega_1 = -\omega_2$$

and

$$x_1(t) + x_2(t) = Ae^{\sigma t}\,(e^{j\omega t} + e^{-j\omega t})$$

half of which is:

$$\frac{x_1(t)}{2} + \frac{x_2(t)}{2} = Ae^{\sigma t} \cos \omega t$$

Realizing that the arbitrary constant of integration might be complex makes A in general a complex coefficient, allowing phasing of the input.

$$x(t) = Ae^{\sigma t} \sin(\omega t + \phi)$$

$$y(t) = GAe^{\sigma t} \sin(\omega t + \phi + \theta)$$

represent the general form of the input and that part of the output directly dependent on the input, the forced response.

If equation (A-2) is Laplace transformed it is identical to setting the denominator of the transfer function (A-5) to zero. The roots of the characteristic equation then represent the free response solutions,

$$y(t) = \sum_{i=1}^{n} B_i e^{r_i t} \qquad \text{where} \qquad r_i = a_i + jb_i$$

Since the free response is also a real function of time the same arguments about conjugate roots and complex coefficients hold as for the forced response. The completely general form of the complete response, allowing for simultaneous input functions, is:

$$y(t) = \underbrace{\Sigma GAe^{\sigma t} \sin\{\omega t + (\phi+\theta)\}}_{\text{Forced Response}} + \underbrace{\Sigma Be^{\sigma t} \sin(bt + \psi)}_{\text{Free Response}}$$

Since the free response represents all solutions to the homogeneous and characteristic equations it is complete. Some question might remain as to the completeness of the forced response. To examine this question we may seek additional solutions for a simple case using the known forced response solution.

Consider the zero over first order system described by the differential equation:

$$\frac{dy(t)}{dt} + ay(t) = x(t)$$

whose transfer function is:

$$\frac{y(s)}{x(s)} = \frac{1}{s + a}$$

Choosing the input $x(t) = e^{-\sigma t}$ is equivalent to setting $s = -\sigma$. The known forced solution is of the form $y(t) = Ge^{-\sigma t}$.

With this function of the output the differential equation may be written as:

$$-\sigma Ge^{-\sigma t} + aGe^{-\sigma t} = e^{-\sigma t} \qquad \text{or} \qquad G = \frac{1}{(-\sigma + a)}$$

which is the magnitude of the transfer function at $s = -\sigma$. Additional solutions may be found by multiplying the known solution by another function of time, $h(t)$, or by adding an additional function of time, $g(t)$:

$$y(t) = h(t)\ Ge^{-\sigma t} + g(t)$$

where neither $h(t)$ nor $g(t)$ equal $Ge^{-\sigma t}$.

Returning again to the differential equation,

$$y'(t) = h'(t)\ Ge^{\sigma t} + g'(t) - \sigma\ G\ h(t)e^{-\sigma t}$$

and

$$h'(t)\ Ge^{-\sigma t} + (a-\sigma)G\ h(t)e^{-\sigma t} + \{g'(t) + a\ g(t)\} = e^{-\sigma t}$$

equating the coefficients of $e^{-\sigma t}$ gives the homogeneous equation:

$$g'(t) + a\ g(t) = 0$$

and

$$G\ \{h'(t) + (a-\sigma)\ h(t)\} = 1$$

Making the substitution:

$$u(t) = h(t) - 1$$
$$u'(t) = h'(t)$$

the second equation becomes:

$$u'(t) + (a-\sigma)\ u(t) = 0 \qquad \frac{1}{G} = (a-\sigma).$$

which is the same form as the homogeneous equation and has the solution:

$$u(t) = Ce^{-(\sigma-a)t}$$
$$h(t) = 1 + Ce^{-(\sigma-a)t}$$

The total solution is:

$$y(t) = Ge^{-\sigma t}\ \{1 + Ce^{-(-\sigma+a)t}\} + Be^{-at}$$

or

$$y(t) = Ge^{-\sigma t} + (GC + B)\ e^{-at}$$

Applying the condition that the output is initially zero:

$$y(0) = G + (GC + B) = 0$$

or

$$y(t) = \frac{1}{(a-\sigma)} \{e^{-\sigma t} - e^{-at}\}$$

and both h(t) and g(t) have collapsed back into the free response solution. With some additional complexity the same principle holds for a zero over second order system. Although more rigorous proof will not be attempted here it may be taken that equation (A-6) represents the complete solution except at the poles of the transfer function.

In the expression above as σ approaches a the output becomes undefined.

$$\lim_{\sigma \to a} \frac{1}{a-\sigma} (e^{-\sigma t} - e^{-\ t}) = \frac{0}{0}$$

In this case we may use the free reponse solution to find the complete solution because it is always part of the complete solution.

$$y(t)_{free} = Be^{-at}$$

Additional solutions are:

$$y(t) = h(t)\ Be^{-at} + g(t)$$

Replacing y(t) in the differential equation with this expression:

$$h'(t)\ Be^{-at} - a\ h(t)\ Be^{-at} + g'(t) + a\ h(t)\ Be^{-at} + a\ g(t) = Ae^{-at}$$

which yields again the homogeneous equation: $g'(t) + a\ g(t) = 0$, and

$$h'(t)\ Be^{-at} = Ae^{-at}$$

$$h'(t)\ B/A = 1$$

$$h(t) = (A/B)t + c$$

Then:

$$y(t) = (At + BC)e^{-at} + Ae^{-at}$$

$$y(t) = \{At + (A+BC)\}\ e^{-at}$$

For the initial condition:

$$y(0) = (A+BC) = 0$$

the complete solution at $\sigma = -a$ is:

$$y(t) = Ate^{-at}$$

SUMMARY

The properties of linear systems used in this analysis are then:

1. The total time response of the system is the sum of the free response and forced response.

2. The transfer function magnitude and phase angle at any point in the Laplace domain represents the complete forced response:

$$y(t) = GAe^{\sigma t} \sin (\omega t + \phi + \theta)$$

to the input signal:

$$x(t) = Ae^{\sigma t} \sin (\omega t + \phi)$$

3. The free response is determined by the transfer function pole locations:

$$y(t) = \Sigma B_n e^{a_n t} \sin (b_n t + \psi_n)$$

and the complex coefficients, $B_n e^{i\psi n}$ are determined by satisfying initial conditions (usually zero) on the total time response and its first $n - 1$ derivatives.

4. The total time response to the sum of any such inputs is the sum of the total response to each input separately.

5. The only exception to the general output form is at the pole locations where:

$$y(t) = Ate^{-at}$$

APPENDIX B

CHECK OF INVERSE LAPLACE TRANSFORM PROGRAM

The LAHOS 1-C equivalent system from page 20 is:

$$\frac{y(s)}{x(s)} = \frac{1.9257\ (S + .38)\ e^{-.037s}}{(s + 1.14)(s + .6487)}$$

For input case B, page 33:

$$x(t) = 4.355\ e^{-1.5t} \sin (t)$$

Table 5 shows: $G = 2.071$, $\theta = -104°$, 1.815 radians, and the output is:

$$y(t) = GAe^{\sigma t} \sin (\omega t + \theta') + B_1 e^{a_1 t} + B_2 e^{a_2 t}$$

To determine the free response mode coefficients set,

$$y(0) = GA \sin \theta + B_1 + B_2 = 0$$

$$y'(0) = GA\ (\omega \cos\theta' + \sigma \sin \theta') + B_1 a_1 + B_2 a_2 = 0$$

which may be written:

$$GA\ (\omega\cos\theta' + (\sigma - a\ )\ \sin \theta') + B_2\ (a_2 - a_1) = 0$$

$$GA\ (\omega\cos\theta' + (\sigma - a_2)\ \sin \theta') + B_1\ (a_1 - a_2) = 0$$

or

$$\frac{B_1}{GA} = \frac{\omega\cos\theta' + (\sigma - a_2)\ \sin \theta'}{(a_1 - a_2)}$$

$$\frac{B_2}{GA} = -\frac{\omega\cos\theta' + (\sigma - a_1)\ \sin \theta'}{(a_1 - a_2)}$$

The above phase angle includes an increment due to the time delay of:

$$\Delta\theta = -\omega T = .037 \text{ radians}$$

so

$$\theta' = \theta - \Delta\theta\ , \quad \theta' = 1.778 \text{ radians}$$

with $\omega = 1$, $\sigma = -1.5$, $a_1 = -1.14$, $a_2 = -.6487$,

$$\frac{B_1}{GA} = 1.2717$$

$$\frac{B_2}{GA} = -.2935$$

$$GA = 9.019$$

and the output function is:

$$y(t') = 9.019\ (e^{-1.5t'}\ \sin\ (t' - 1.778) + 1.2717\ e^{-1.14t'} - .2935e^{-.6487t'}$$

by inserting values of $t' = t - T = t - .037$ sec. into this equation values for $y(t)$ may be calculated. This was done and compared to the values computed by the program in Table B-1 and were found to agree within the error shown.

TABLE B-1

time (sec.)	Program	Equation	Error
1	.8677	.8608	.0069
2	.5467	.5701	.0239
3	.0900	.1022	.0122
4	-.0606	-.0579	.0027
5	-.0658	-.0660	.0002

The worst error is: $e_{max} = .024$, and the average error is $\bar{e} = .009$

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