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INTERVAL GRAPHS, CHRONOLOGICAL ORDERINGS, AND RELATED MATTERS

By

Dale Skrien

Technical Report No. 71

June 1980

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DALE J. SKRIEN

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This paper is concerned with special interval representations of interval graphs. The basic notion is that of a chronological ordering of an interval graph, which is an equivalence class of interval representations of the graph in question. Consider a reference set $P = \{\ell_1,\ldots,\ell_n, r_1,\ldots,r_n\}$ whose points are to be associated with the respective left and right endpoints of interval representations of a graph having n nodes. Among the questions that are considered in this paper, and are answered both mathematically and algorithmically, are the following: Given an interval graph G with n nodes, which linear orderings on P_n arise from interval representations of G? Given a partial ordering of P_n, when can it be expanded to a linear ordering associated with an interval representation of G? How many chronological orderings does a given interval graph have? The theorems and algorithms are applicable to a variety of seriation problems.

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circular arc graph
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partial orders
perfect graph
proper interval graph
rigid-circuit graph
semi-order
seriation
topological sorting
transitive orientation
unit interval graph

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INTERVAL GRAPHS, CHRONOLOGICAL ORDERINGS,

AND RELATED MATTERS

by

Dale John Skrien

A dissertation submitted in partial fulfillment
of the requirements for the degree of

Doctor of Philosophy

University of Washington

1980

Approved by ______________________________
(Chairperson of Supervisory Committee)

Program Authorized
to Offer Degree ______________________________

Date ______________________________

Doctoral Dissertation

University of Washington

Abstract

INTERVAL GRAPHS, CHRONOLOGICAL ORDERINGS, AND RELATED MATTERS

By Dale John Skrien

Chairperson of the Supervisory Committee: Professor Victor L. Klee
Department of Mathematics

If a finite, undirected graph is the intersection graph of a set of intervals of the line, it is called an interval graph and the set of intervals is called a representation of the graph. Let $I(G)$ be the set of all representations of an interval graph G in which all of the endpoints of the intervals are distinct.

The set $I(G)$ is divided into a finite number of equivalence classes, called chronological orderings of G, which correspond to the possible relative positions of the intervals in representations of the graph. Consideration of these classes leads to new characterizations of interval graphs, and to algorithms, with worst-case time-complexity $O(n^3)$ where n is the number of vertices of the graph, for solving problems concerning special representations of interval graphs. Such problems include: (1) determining whether a graph has a representation in which certain intervals are properly contained in [extend to the left or right of, are completely to the left or right of] others, and (2) determining whether a graph has a representation in which certain intervals contain given points of the line.

These results are applied to proper interval graphs and proper circular arc graphs, yielding new characterizations of each and displaying an interesting relationship among proper interval graphs,

comparability graphs, triangulated graphs, and a set of graphs called nested interval graphs.

The dissertation concludes with a characterization of interval edge-graphs and a few NP-completeness results.

TABLE OF CONTENTS

LIST OF FIGURES

Number Page

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CHAPTER 1: INTRODUCTION

The subject of interval graphs has been studied for only a little more than 20 years. Such graphs were first mentioned in 1957 by Hajós [15], and were first studied by Benzer [2; 3] because of their application to his work in molecular genetics. Since then, this subject has been an active area of research. See Golumbic [12, Chapters 1 and 8] and Roberts [26, pp. 111-140] for surveys of the many applications of interval graphs that have arisen.

This dissertation is concerned with the possible relative positions of the intervals in a representation of an interval graph. The results presented here partially answer questions raised by Roberts [26, pp. 118-120] concerning the application of interval graphs to archaeological seriation or sequence dating.

1.1 Notation

Our notation for chapters 1-7 will follow that of Golumbic [12 or 13]. A <u>graph</u> (V, E) consists of a finite set V of vertices and an irreflexive relation E on V. The elements of E are called <u>edges</u> and can be thought of as ordered pairs of distinct vertices. Therefore, our graphs have no loops or multiple edges. We define the relation E^{-1} by letting $ab \in E^{-1}$ iff $ba \in E$, and define $\hat{E} = E \cup E^{-1}$. A graph (V, E), or a set of edges E, is said to be <u>undirected</u> if $E = \hat{E} = E^{-1}$, and is said to be <u>oriented</u> if $E \cap E^{-1} = \emptyset$. When we wish to emphasize that a graph is not necessarily oriented or undirected, we will call it <u>mixed</u>.

A graph (V', E') is a subgraph of (V, E) if $V' \subset V$ and $E' \subset E$. (V', E') is called an <u>induced</u> subgraph if

$$E' = \{ab: a, b \in V' \text{ and } ab \in E\}.$$

Let $\underline{V \dot{\times} V}$ denote the irreflexive complete relation on V, i.e.,

$$V \dot{\times} V = \{ab: a, b \in V \text{ and } a \neq b\}.$$

A graph (V, E) is complete if $E = V \dot{\times} V$, and a subgraph (V', E') of a graph G is called a clique of G if (V', E') is complete.

An orientation of a graph (V, E), or of E, is a relation $T \subset E$ such that $T \cap T^{-1} = \emptyset$ and $T + T^{-1} = \hat{E}$ (here "+" will always denote the union of mutually disjoint sets or relations). Thus T contains all $ab \in E$ for which $ba \notin E$, and contains ab or ba (but not both) if $\{ab, ba\} \subset E$.

A relation T is said to be transitive if $T^2 \subset T$, where

$$T^2 = \{xy: xz, zy \in T \text{ for some } z\}.$$

A linear (or total) ordering T of a set V is a relation which satisfies $T^2 \subset T$, $\hat{T} = V \dot{\times} V$, and $T \cap T^{-1} = \emptyset$. Thus a linear ordering of V is just a transitive orientation of the complete graph on V.

A path in (V, E) consists of a sequence of distinct vertices $[v_1, \ldots, v_k]$ such that $v_i v_{i+1} \in E$ for $i = 1, \ldots, k-1$. A cycle (or circuit) is a path $[v_1, \ldots, v_k]$ in which $v_k v_1 \in E$. A chordless path (or cycle) is one for which no other pairs of vertices are joined by an edge. The graph induced by a chordless cycle of m vertices is denoted by $\underline{C_m}$. A graph without any cycles is called acyclic.

For a vertex $x \in V$, define

$$\underline{N(x)} \text{ (or } \underline{N_G(X)}) = \{y \in V: y = x, yx \in E, \text{ or } xy \in E\}.$$

This is the neighborhood of x. Let the open neighborhood $\underline{ON(x)}$ = $N(x)\setminus\{x\}$.

From here until the end of Chapter 7, except for the last section of Chapter 4, we will reserve the letter E to be an undirected set of edges, i.e., $ab \in E \iff ba \in E$. Furthermore V will always be denoted by $\{v_1, \ldots, v_n\}$ and so n will henceforth be the cardinality $|V|$ of V.

When describing an algorithm, we will say that it has (worst-case) complexity $O(p(m))$ for some function p, if there exists a constant $k \geq 0$ such that, for all inputs of "size" m, the number of computational steps the algorithm requires before it halts is at most $kp(m)$. The "size" of a graph (V, E) will usually be $|V| + |E|$ or just $|V|$, and the notion of <u>step</u> should be interpreted in terms of the RAM model of random access computation with the uniform cost criterion, as defined by Aho et al. [1, pp. 5-14].

1.2 Interval Graphs

An (undirected) graph $G = (V, E)$ is called an <u>interval graph</u> if there exists a set of closed intervals $\{I_1, \ldots, I_n\}$ of the real line such that, for $i \neq j$,

$$v_i v_j \in E \iff I_i \cap I_j \neq \emptyset.$$

The set of intervals is called an (<u>interval</u>) <u>representation</u> of G. Thus G is an interval graph iff it has an interval representation.

There are several characterizations of interval graphs, three of which will be described here in some detail because they will be used in the proofs of some of the results of this paper.

In our figures, vertices are drawn as small circles. A line without an arrow connecting vertices a and b indicates that ab and ba are edges. A line with an arrow pointing from a to b indicates that ab is an edge.

Lekkerkerker and Boland first characterized interval graphs in terms of forbidden subgraphs.

Theorem 1.1 (Lekkerkerker and Boland [24])

A graph (V, E) is an interval graph iff it does not contain an induced subgraph which is I, II, III_n, IV_n, V_n shown in Figure 1.1.

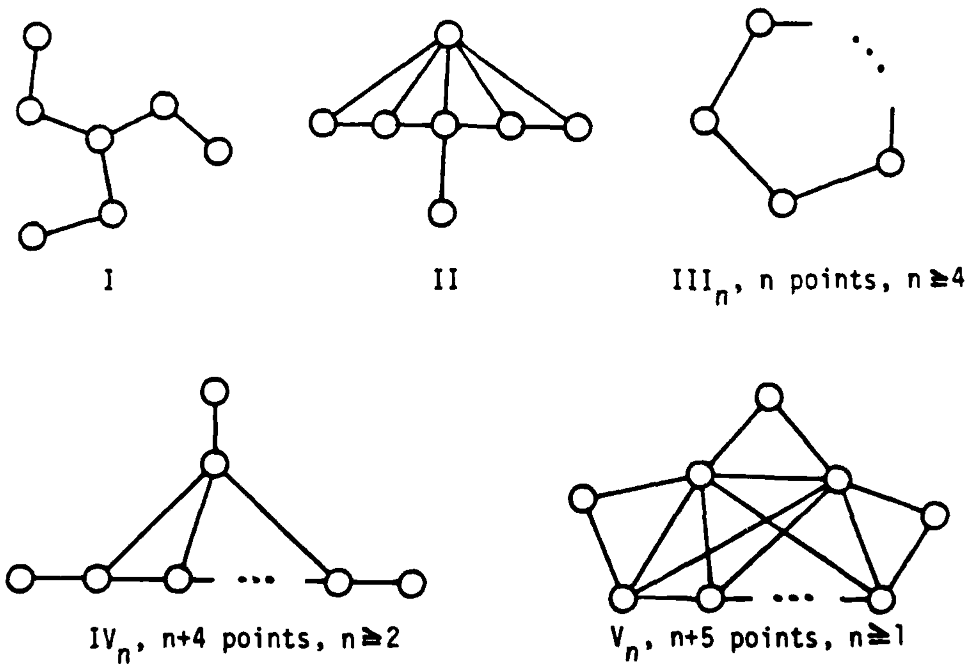


Figure 1.1: Forbidden subgraphs. Note III_n is just C_n.

To present the next characterization, we first give some definitions. A clique is called maximal if it is not properly contained in any other clique. If a graph G has exactly m distinct maximal cliques $\{K_1, \ldots, K_m\}$, then we can construct the $m \times n$ maximal clique-vertex incidence matrix $M = (m_{ij})$ by defining

$$m_{ij} = \begin{cases} 1 & \text{if } v_j \text{ is a vertex of } K_i \\ 0 & \text{otherwise.} \end{cases}$$

Thus the rows correspond to the maximal cliques and the columns to the vertices of V.

A matrix of 0's and 1's is said to have the consecutive ones property if the order of the rows can be permuted so that the 1 entries are arranged consecutively in each column (see Figure 1.2).

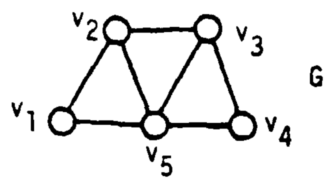
$$A \quad \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \qquad B \quad \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \qquad C \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Figure 1.2: Consecutive ones property. Matrix A does not have the consecutive ones property but matrix B does. Interchanging the first two rows of B yields matrix C, which has consecutive 1's in each column.

Theorem 1.2 (Fulkerson and Gross [9])

An undirected graph is an interval graph iff its maximal clique-vertex incidence matrix has the consecutive ones property.

The following construction shows the sufficiency half of the proof. Suppose the rows of the matrix are arranged so that the 1's in each column are consecutive. If the first 1 in column j is in the a-th row and the last 1 in the b-th row, then represent vertex v_j by the interval $I_j = [a, b]$. Repeat this for every column. It is not hard to see that $\{I_1, \ldots, I_n\}$ is a representation of G. For examples of this, see Figures1.3 and 6.1.



The vertices of the maximal cliques of G:

K_1: $\{v_1, v_2, v_5\}$
K_2: $\{v_3, v_4, v_5\}$
K_3: $\{v_2, v_3, v_5\}$

M

	v_1	v_2	v_3	v_4	v_5
K_1	1	1	0	0	1
K_2	0	1	1	0	1
K_3	0	0	1	1	1

The matrix M has consecutive ones and yields the following representation of G: $\{I_1 = [1,1],\ I_2 = [1,2],\ I_3 = [2,3],\ I_4 = [3,3],\ I_5 = [1,3]\}$.

Figure 1.3: A representation of a graph.

A third characterization of interval graphs concerns the <u>complementary graph</u> $\underline{G}^c = (V, E^c)$ of G, where

$$E^c = \{ab\colon a, b \in V,\ a \neq b, \text{ and } ab \notin E\}.$$

For intervals I_i and I_j, we write $\underline{I_i < I_j}$ to mean $x < y$ for all $x \in I_i$ and $y \in I_j$.

Theorem 1.3 (Gilmore and Hoffman [11])

An undirected graph G is an interval graph iff C_4 is not an induced subgraph of G and there exists a transitive orientation T^c of G^c. Furthermore, if this is the case, then G has a representation $\{I_1, \ldots, I_n\}$ such that $v_i v_j \in T^c \iff I_i < I_j$.

Some other concepts that arise are proper interval graphs (graphs which have representations in which no interval is properly contained in another) and unit interval graphs (graphs which have representations consisting of unit intervals).

The following theorem shows the relationship between these concepts.

Theorem 1.4 (Roberts [28])

Let G be an undirected graph. The following are equivalent:

(a) G is a proper interval graph;

(b) G is a unit interval graph;

(c) G is an interval graph which does not have $K_{1,3}$ as an induced subgraph (see Figure 1.4);

(d) G does not contain an induced subgraph which is $K_{1,3}$, III_n $(n \geq 4)$, IV_2, or V_1 (see Figures 1.1 and 1.4).

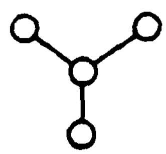


Figure 1.4: $K_{1,3}$.

Efficient algorithms have been developed for recognizing these kinds of graphs. Booth and Lueker [5; 6] in 1975 developed a data structure called a PQ-tree for analyzing matrices of 0's and 1's for the consecutive ones property. Their PQ-tree algorithm has (worst-case) complexity $O(m+n+e)$ for an $m \times n$ matrix with e nonzero entries. Using this algorithm and Theorem 1.2, interval graphs can be recognized in linear time. Furthermore, due to the following theorem, proper (or unit) interval graphs can also be recognized in linear time.

The $n \times n$ augmented adjacency matrix $M = (m_{ij})$ of graph (V, E) is given by

$$m_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in E \text{ or } i = j \\ 0 & \text{otherwise.} \end{cases}$$

<u>Theorem 1.5</u> (Roberts [29]; see also Booth [4, pp. 117-118])
An undirected graph is a proper interval graphs iff its augmented adjacency matrix has the consecutive ones property.

Circular arc graphs form another class of intersection graphs which have been studied. A graph (V, E) is a <u>circular arc graph</u> if there exists a set $\{A_1, \ldots, A_n\}$ of arcs of a circle such that, for $i \neq j$,

$$v_i v_j \in E \iff A_i \cap A_j \neq \emptyset.$$

Tucker [33] has characterized such graphs and has recently [32] developed an algorithm of complexity $O(|V|^3)$ for recognizing them. <u>Proper circular arc</u> graphs have also been characterized, both in terms of forbidden subgraphs [34], and in terms of their augmented adjacency matrices [33], the latter characterization leading to a recognition algorithm of complexity $O(|V| + |E|)$. A characterization of unit circular arc graphs in terms of forbidden subgraphs is also known [34].

1.3 Chapter Summaries

Chapters 2, 3, and 4 give three equivalent formulations of a fundamental concept in this paper, that of a chronological ordering of an interval graph. These are used to construct new characterizations of interval graphs which lead to fairly efficient algorithms for recognizing whether a graph has certain chronological orderings. Chapter 5 applies these results to proper interval graphs and proper circular arc graphs. Chapter 6 discusses the problem of counting how many chronological orderings a graph has. Chapter 7 is concerned with representations which contain certain points. Chapter 8 characterizes interval edge graphs and Chapter 9 provides some NP-completeness results concerning the consecutive ones property of matrices.

1.4 An Archaeological Seriation Problem

We will use the following application of interval graphs to motivate many of the results of this paper. This example is discussed in Roberts [27, pp. 31-37; 26, pp. 118-120]. See also Kendall [18; 19] and Golumbic [12].

While digging in an ancient graveyard, archaeologists often come across quite a variety of styles of pottery or other artifacts. There are many questions about these various styles to which they would like to have answers. For example, were two given styles ever in use at the same time? Was there ever a time at which a certain style "a" was in use but style "b" was not?

To try to answer these questions, let use first assume that:

Each style was in use during a single interval of time.

Under this assumption, there are essentially only three possible relationships between two styles:

(1) Style u appeared after style v disappeared;

(2) Style u appeared after style v and disappeared before style v;

(3) Style u appeared when style v was already in use and disappeared after style v disappeared.

(Of course the role of u and v could be reversed.) These cases correspond to the three possible relationships between two intervals with distinct endpoints:

(1') I_u follows I_v: $\underline{\quad I_v \quad}\ \ \underline{\quad I_u \quad}$

(2') I_u is contained in I_v: $\overline{\quad\quad} \, I_u$ over $\underline{\quad\quad\quad\quad} \, I_v$

(3') I_u overlaps I_v to the right: $\underline{\quad I_v \quad}$ $\overline{\quad\quad} \, I_u$

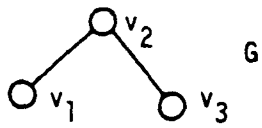
Thus a set of intervals $\{I_u\colon\ u \text{ is a style of pottery}\}$ represents the proper relationships in time between the various styles if: case (1), (2), or (3) is true for the styles u and v $\Leftrightarrow$ case (1'), (2'), (3'), respectively, is true for the intervals I_u and I_v. Such a set of intervals is called a <u>chronological representation</u> of the artifacts. We will consider the problem of trying to find a chronological representation from the data obtained by the archaeologists.

Suppose two different styles of pottery appear in a common grave. Then it is reasonable to assume that their intervals of use intersect. If the collection of graves is quite extensive, it may also be reasonable to assume the converse is true, i.e., if two styles were ever in use at the same time, then artifacts of each style appear in at least one common grave. Thus we are assuming that:

> Two styles appear in a common grave iff their intervals of use intersect.

Now we form a graph $G_p = (V, E)$ in which V is the set of styles of pottery, and in which two distinct vertices are joined by an edge iff those styles appear in a common grave. Then by our assumptions above, G_p must be an interval graph. Furthermore, any interval representation in which all the endpoints of the intervals are distinct is a <u>possible</u> chronological representation of the artifacts. Unfortunately, unless G_p consists of a single vertex, it has at least two representations which differ on containment, overlapping to the right, or following (see Figure 1.5). We will call such representations different <u>chronological orderings</u> of G_p (the name comes from Roberts [26, pp. 118-120]). Thus each chronological ordering of G_p is a possible chronological representation of the artifacts.

Chronological orderings of a graph are studied in Chapters 2, 3, and 4, where the term is more precisely defined.



G has eight chronological orderings, which are represented below.

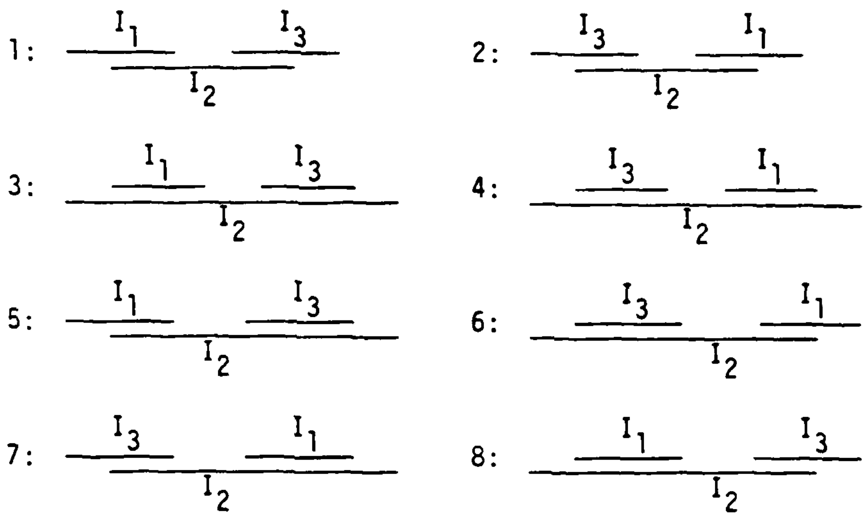


Figure 1.5: Chronological orderings of a graph.

CHAPTER 2: CHRONOLOGICAL ORDERINGS I

One way in which different chronological orderings can be distinguished from each other is by means of the linear order (on the line) of the endpoints of the intervals in a representation of a graph. For this reason, we will restrict our attention to representations in $\underline{I(G)}$, which is defined to be the set of all interval representations of an undirected graph G in which every interval is non-empty and in which the endpoints of the intervals are all distinct real numbers. We remark that G is an interval graph iff $I(G) \neq \emptyset$.

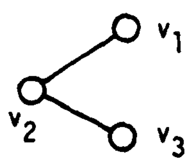
Let $\underline{P_n}$ denote the set of $2n$ elements $\{\ell_1, \ell_2, \ldots, \ell_n, r_1, r_2, \ldots, r_n\}$ and let

$$\underline{O_n} = \{(P_n, T): T \text{ is a linear ordering of } P_n\}.$$

We note that $|O_n| = (2n)!$ Also, let $\underline{L_n} = \{\ell_1, \ldots, \ell_n\}$ and $\underline{R_n} = \{r_1, \ldots, r_n\}$, and so $P_n = L_n \cup R_n$.

The set O_n is related to $I(G)$ by means of the mapping $F_G\colon I(G) \to O_n$ defined as follows. Given a representation $I = \{I_1, \ldots, I_n\} \in I(G)$, we associate the left [resp. right] endpoint of I_k with ℓ_k [resp. r_k] for $k = 1, \ldots, n$. The linear order of the endpoints of the intervals of I on the real line induces a linear order on P_n and hence we get an element of O_n. (See Figure 2.1.)

G



I

$I_1 = [1,4]$

$I_2 = [2,6]$

$I_3 = [5,7]$

The linear order $F_G(I)$ is represented by the following sequential order:

$$\ell_1, \ell_2, r_1, \ell_3, r_2, r_3.$$

Figure 2.1: An example of the mapping F_G.

Using F_G, we construct an equivalence relation $\sim$ on $\mathcal{I}(G)$ by defining

$$I \sim I' \iff F_G(I) = F_G(I')$$

for $I, I' \in \mathcal{I}(G)$. We call an equivalence class under $\sim$ a <u>chronological ordering</u> of G, and call $F_G(I) \in O_n$ the <u>linear ordering of P_n associated with I</u>.

There are many natural questions about chronological orderings. Given an undirected graph G, exactly which linear orders on P_n can be associated with a representation of G? Given a partial order on P_n, when can it be extended to a linear order associated with a representation of G? Is there an efficient algorithm for extending it? How many chronological orderings does a graph have? Most of these questions will be answered here.

<u>2.1 Main Results</u>

<u>Theorem 2.1</u>

A graph $G = (V, E)$ is an interval graph iff there exists $(P_n, T) \in O_n$ with the following properties:

(a) $\ell_i r_i \in T$ for $i = 1, \ldots, n$ and

(b) for $i \neq j$, $v_i v_j \in E \iff \ell_i r_j \in T$ and $\ell_j r_i \in T$.

Furthermore, if this is the case, then (P_n, T) is associated with a representation of G.

<u>Proof</u>: ($\Rightarrow$) Let $I \in \mathcal{I}(G)$ such that $F_G(I) = (P_n, T)$. Clearly, we must have $\ell_i r_i \in T$ for $i = 1, \ldots, n$. If $v_i v_j \notin E$, then, without

loss of generality, $I_i < I_j$. Thus $r_i \ell_j \in T$. Conversely, if $r_j \ell_i \in T$ (or $r_i \ell_j \in T$), then $I_j < I_i$ (or $I_i < I_j$). In particular, $I_i \cap I_j = \emptyset$ and so $v_i v_j \notin E$.

(<=) Let $A(x) = \{y \in P_n: yx \in T\}$ for each $x \in P_n$. We define a representation $I = \{I_1, \ldots, I_n\}$ of G as follows: Let

$$I_i = [|A(\ell_i)|, \ |A(r_i)|], \quad \text{for} \quad i = 1, \ldots, n.$$

Because of conditions (a) and (b), it is easy to see that I represents G and that $(P_n, T) = F_G(I)$. $\square$

Given $G = (V, E)$ and $(P_n, T) \in O_n$, this theorem provides a way of testing in $O(|V|^3)$ steps whether (P_n, T) is associated with a representation of G. It is natural to ask whether this characterization can be used to construct a recognition algorithm for interval graphs. Such an algorithm will shortly be constructed, but it will be useful to first put the problem in a more general framework. This will enable the algorithm to solve problems which are beyond the ability of the more efficient recognition algorithms that exist for interval graphs.

Let us return to the example of archaeological seriation. As before, suppose that a chronological representation of the artifacts is an interval representation of G_p. Now, let us suppose that we have some extra information on the relative positions of the intervals of use of the artifacts, in the form of knowledge as to which of various endpoints of the intervals are to the left or right of certain other endpoints. For example, suppose it is known, by some other records, that style i disappeared before style j appeared. Or suppose it is known that style i appeared after style j, but it is not known which style disappeared first. We want our representation of G_p to contain this information.

This amounts to putting some restrictions on the linear orders on P_n to be associated with the representation of G_p. For example, knowing that style i appeared after style j disappeared means that we are interested only in those representations of G_p whose associated linear order T on P_n satisfies $r_j \ell_i \in T$.

This raises the following question. We remark that our problems are presented in the format described by Garey and Johnson [10, p. 4]. A <u>problem</u> is a general question to be answered, which is asked of a particular class of objects usually containing several unspecified parameters. An <u>instance</u> is obtained by assigning specific numerical values to the parameters.

<u>Problem 2.2</u>

Instance: Graphs $G = (V, E)$ and (P_n, S).

Question: Does there exist a linear order T on P_n such that $S \subset T$ and $(P_n, T) \in F_G(I(G))$?

We shall construct an alogrithm of complexity $O(|V|^3)$ that solves this problem and constructs the linear order T (if one exists). We first consider the case where $S = \emptyset$.

In the following algorithm, an undirected graph $G(V, E)$ is the input. The output is either "FLAG = 1", which means G is not an interval graph, or "FLAG = 0" and a linear order T of P_n such that $(P_n, T) \in F_G(\mathcal{I}(G))$.

All our algorithms are written in Pidgin ALGOL as described by Aho et al. [1, pp. 33-39].

<u>Algorithm 2.3</u>

<u>begin</u>

initialize: $T \leftarrow \{\ell_i r_i : i = 1,\ldots, n\} \cup \{\ell_i r_j,\ \ell_j r_i : v_i v_j \in E\}$; (1)

<u>if</u> E^c cannot be transitively oriented (2)

<u>then</u> <u>write</u> "FLAG = 1" and halt

<u>else</u> construct such an orientation T^c;

<u>for</u> all $v_i v_j \in T^c$ <u>do</u> $T \leftarrow T \cup \{\ell_i \ell_j,\ \ell_i r_j,\ r_i \ell_j,\ r_i r_j\}$; (3)

<u>for</u> all $v_i v_j \in T^c$ <u>do</u> (4)

<u>for</u> all k such that $v_i v_k \in E$ and $v_k v_j \in E$ <u>do</u> (5)

$T \leftarrow T \cup \{r_i r_k,\ \ell_k \ell_j\}$;

<u>for</u> all $r_i r_j \in R_n \dot{\times} R_n \backslash (T \cup T^{-1})$ with $i < j$ <u>do</u> (6)

$T \leftarrow T \cup \{r_i r_j\}$;

<u>for</u> all $\ell_i \ell_j \in L_n \dot{\times} L_n \backslash (T \cup T^{-1})$ with $i < j$ <u>do</u> (7)

$T \leftarrow T \cup \{\ell_i \ell_j\}$;

<u>if</u> $T \cap T^{-1} \neq \emptyset$ <u>then</u> <u>write</u> "FLAG = 1" (8)

<u>else</u> <u>write</u> "FLAG = 0" and T

<u>end</u>

<u>Theorem 2.4</u>

Algorithm 2.3 solves Problem 2.2 in the case where $S = \emptyset$.

Proof: Suppose G is not an interval graph. Then by Theorem 1.3, either E^c is not transitively orientable or G contains the induced subgraph C_4. In the former case, the algorithm will output "FLAG = 1" from line (2). For the latter case, consider:

Lemma 2.4.1 If G^c has a transitive orientation T^c, then Algorithm 2.3 writes "FLAG = 1" <=> G contains C_4 as an induced subgraph.
Proof of Lemma 2.4.1: Suppose G contains C_4 with vertices v_1, v_2, v_3, v_4. (See Figure 2.2. We draw a dotted line with an arrow from vertex a to vertex b to indicate that $ab \in T^c$.)

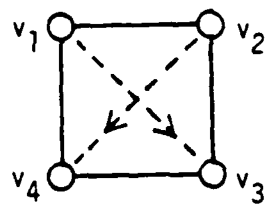


Figure 2.2: Sufficiency in Lemma 2.4.1.

Without loss of generality, $v_1v_3 \in T^c$ and $v_2v_4 \in T^c$. When line (4) considers $v_1v_3 \in T^c$, then line (5) places $\{r_1r_2\}$ in T. But when line (4) is considering $v_2v_4 \in T^c$, then line (5) places $\{r_2r_1\}$ in T. Hence "FLAG = 1" will be written when line (8) is implemented.

Conversely, suppose "FLAG = 1" is the output. Then some x_iy_k and y_kx_i were both added to T, where $x,y \in \{\ell, r\}$. It is easy to see that this must have occurred in line (4) and (5), and therefore that $x = y = r$ or $x = y = \ell$. We will consider only the case in which $x = y = r$; the second follows similarly. Suppose r_ir_k was added to T when $v_i v_j \in T^c$ was being considered in line (4), and $r_k r_i$ was added to T when $v_k v_\ell \in T^c$ was being considered in line (4) (see Figure 2.3).

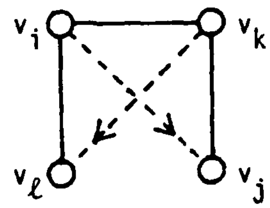


Figure 2.3: Necessity in Lemma 2.4.1.

If $v_\ell v_j \notin E$, then either $v_\ell v_j \in T^c$ or $v_j v_\ell \in T^c$. But each case contradicts the transitivity of T^c. Thus $v_\ell v_j \in E$ and hence $\{v_i, v_k, v_j, v_\ell\}$ induces C_4 in G. This completes the proof of Lemma 2.4.1.

Now suppose G is an interval graph. We must show that the output is "FLAG = 0" and T, and that $(P_n, T) \in F_G(I(G))$. By Theorem 1.3, E^c has a transitive orientation and G does not contain C_4 as an induced subgraph. Hence by Lemma 2.4.1, "FLAG = 0" and T are the output. By steps (1) and (3), the conditions of Theorem 2.1 are satisfied, so all that remains to be shown is that T is a linear order on P_n. To accomplish this, we need only show that $T^2 \subset T$.

Let T' denote the set T just prior to the implementation of line (6).

Lemma 2.4.2 If G is an interval graph, then T' is transitive and each component of $(P_n, (\overline{T'})^c)$ is a complete graph.

Proof of Lemma 2.4.2: We will say that (triangle) $\Delta v_i v_k v_j$ forces $r_i r_k \in T'$ and $\ell_k \ell_j \in T'$ if $v_i v_j \in T^c$, $v_i v_k \in E$, $v_k v_j \in E$ and hence lines (3), (4), and (5) add $\{r_i r_k, \ell_k \ell_j\}$ to T'. The proof will be split into eight cases, each of which has some subcases. We use the symbol "(=><=)" to denote a contradiction, and we use the fact that $v_i v_j \in T^c$ if $v_i v_j \notin E$ and either $r_i r_j \in T'$ or $\ell_i \ell_j \in T'$ (due to line (3)).

CASE 1: Assume $r_i r_j$, $r_j r_k \in T'$. We want to show that $r_i r_k \in T'$.

Subcase 1: $v_i v_j \in E$, $v_j v_k \in E$, $v_i v_k \in E$. Then $r_j r_k$ must have been forced by some $\Delta v_j v_k v_\ell$ with $v_j v_\ell \in T^c$ and $v_k v_\ell \in E$ (see Figure 2.4).

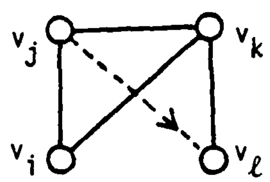


Figure 2.4: Case 1, Subcase 1.

If $v_i v_\ell \in E$, then $\Delta v_j v_i v_\ell$ forces $r_j r_i \in T'$ (=><=). Hence $v_i v_\ell \notin E$ and by the transitivity of T^c, we see that $v_i v_\ell \in T^c$ and so $\Delta v_i v_k v_\ell$ forces $r_i r_k \in T'$.

Subcase 2: $v_i v_j \in E$, $v_j v_k \in E$, $v_i v_k \notin E$. Then $v_i v_k \in T^c$ or else $\Delta v_k v_j v_i$ forces $r_k r_j \in T'$. Hence by line (3), $r_i r_k \in T'$.

<u>Subcase 3</u>: $v_iv_j \in E$, $v_jv_k \notin E$, $v_iv_k \in E$. This cannot happen, because, if so, then $v_jv_k \in T^c$ and hence $\Delta v_jv_iv_k$ forces $r_jr_i \in T'$ (=><=).

<u>Subcase 4</u>: $v_iv_j \notin E$, $v_iv_k \in E$, $v_jv_k \in E$. Then $r_jr_k \in T'$ must have been forced by some $\Delta v_jv_kv_\ell$ with $v_jv_\ell \in T^c$ and $v_kv_\ell \in E$. By the transitivity of T^c, v_iv_j, $v_jv_\ell \in T^c$ $v_iv_\ell \in T^c$. Hence $\Delta v_iv_kv_\ell$ forces $r_ir_k \in T'$.

<u>Subcase 5</u>: $v_iv_j \notin E$, $v_jv_k \notin E$. Then $v_iv_j \in T^c$ and $v_jv_k \in T^c$, so by transitivity of T^c, $v_iv_k \in T^c$ and hence line (3) gives $r_ir_k \in T'$.

<u>Subcase 6</u>: $v_iv_j \notin E$, $v_jv_k \in E$, $v_iv_k \notin E$ or $v_iv_j \in E$, $v_jv_k \notin E$, $v_iv_k \notin E$. By the transitivity of T^c, $v_iv_j \in T^c$ => $v_iv_k \in T^c$ or $v_jv_k \in T^c$ => $v_iv_k \in T^c$. Hence by line (3), $r_ir_k \in T'$.

<u>CASE 2</u>: $\ell_i\ell_j$, $\ell_j\ell_k \in T'$. This is proven just like Case 1.

<u>CASE 3</u>: $r_i\ell_j$, $\ell_j\ell_k \in T'$. First note that $r_i\ell_j \in T'$ => $v_iv_j \in T^c$ => $\ell_i\ell_j \in T'$. If $i = k$, then $\ell_k\ell_j = \ell_i\ell_j \in T'$ (=><=). Thus i, j, k are distinct. Note that $r_i\ell_j \in T'$ => $v_iv_j \in T^c$.

<u>Subcase 1</u>: $v_jv_k \notin E$. Then $v_jv_k \in T^c$ so by the transitivity of T^c, we get $v_iv_k \in T^c$ and hence $r_i\ell_k \in T'$ by line (3).

<u>Subcase 2</u>: $v_jv_k \in E$, $v_iv_k \notin E$. Again by the transitivity of T^c, we get $v_iv_j \in T^c \Rightarrow v_iv_k \in T^c$ and hence $r_i\ell_k \in T'$.

<u>Subcase 3</u>: $v_jv_k \in E$ and $v_iv_k \in E$. This cannot happen because, otherwise $\Delta v_iv_kv_j$ forces $\ell_k\ell_j \in T'$ (=><=).

<u>CASE 4</u>: ℓ_ir_j, $r_j\ell_k \in T'$. Note that $v_jv_k \notin E$ and $v_jv_k \in T^c$. If $i = j$, then $v_iv_k = v_jv_k \in T^c$, so by line (3), we get $\ell_i\ell_k \in T'$. Now assume i, j, k are distinct.

<u>Subcase 1</u>: $v_iv_j \notin E$. Then $v_iv_j \in T^c$ so by the transitivity of T^c, we get $v_iv_k \in T^c$ and hence $\ell_i\ell_k \in T'$.

<u>Subcase 2</u>: $v_iv_j \in E$, $v_iv_k \notin E$. Then by the transitivity of T^c, $v_iv_k \in T^c$ and hence $\ell_i\ell_k \in T'$.

<u>Subcase 3</u>: $v_iv_j \in E$, $v_iv_k \in E$. Then $\Delta v_jv_iv_k$ forces $\ell_i\ell_k \in T'$.

<u>CASE 5</u>: $\ell_i\ell_j$, $\ell_jr_k \in T'$. If $i = k$, then $\ell_ir_k \in T'$ by line (1). If $j = k$ and $v_iv_j \in E$, then $\ell_ir_k \in T'$ by line (1). If $j = k$ and $v_iv_j \notin E$, then $v_iv_j \in T^c$ and so by line (3) we get $\ell_ir_k = \ell_ir_j \in T'$. Now assume i, j, k are distinct.

<u>Subcase 1</u>: $v_iv_j \notin E$, $v_jv_k \notin E$. Then by the transitivity of T^c, $v_iv_j, v_jv_k \in T^c \Rightarrow v_iv_k \in T^c$, so by line (3), $\ell_ir_k \in T'$.

<u>Subcase 2</u>: $v_iv_j \in E$, $v_jv_k \notin E$, $v_iv_k \notin E$. By the transitivity of T^c, $v_jv_k \in T^c \Rightarrow v_iv_k \in T^c$ so $\ell_ir_k \in T'$.

<u>Subcase 3</u>: $v_iv_j \notin E$, $v_jv_k \in E$, $v_iv_k \notin E$. By the transitivity of T^C, $v_iv_j \in T^C \Rightarrow v_iv_k \in T^C$ so $\ell_ir_k \in T'$.

<u>Subcase 4</u>: $v_iv_k \in E$. Then by (1), we get $\ell_ir_k \in T'$.

<u>Subcase 5</u>: $v_iv_k \notin E$, $v_iv_j \in E$, $v_jv_k \in E$. Then $\ell_ir_k \in T'$, or otherwise, $r_k\ell_i \in T' \Rightarrow v_kv_i \in T^C$ and then $\Delta v_kv_jv_i$ forces $\ell_j\ell_i \in T'$ (=><=).

<u>CASE 6</u>: r_ir_j, $r_j\ell_k \in T'$. Note that $v_jv_k \notin E$ and $v_jv_k \in T^C$. If $i = k$, then $v_jv_i = v_jv_k \in T^C \Rightarrow r_jr_i \in T'$ (=><=). Hence i, j, k are distinct.

<u>Subcase 1</u>: $v_iv_j \notin E$. Then by the transitivity of T^C, $v_iv_k \in T^C$ and hence $r_i\ell_k \in T'$ by line (3).

<u>Subcase 2</u>: $v_iv_j \in E$ and $v_iv_k \notin E$. Then by the transitivity of T^C, $v_iv_k \in T^C$ and hence $r_i\ell_k \in T'$ by line (3).

<u>Subcase 3</u>: $v_iv_j \in E$, $v_iv_k \in E$. This cannot happen because, if so, then $\Delta v_jv_iv_k$ forces $r_jr_i \in T'$ (=><=).

<u>CASE 7</u>: ℓ_ir_j, $r_jr_k \in T'$. If $i = k$, then $\ell_ir_k = \ell_ir_i \in T'$ by line (1). If $i = j$ and $v_iv_k \in E$, then $\ell_ir_k \in T'$ by line (1). If $i = j$ and $v_iv_k \notin E$, then $v_iv_k = v_jv_k \in T^C$ so $\ell_ir_k \in T'$ by line (3). Now assume i, j, k distinct.

Subcase 1: $v_iv_k \in E$. Then $\ell_ir_k \in T'$ by line (1).

Subcase 2: $v_iv_k \notin E$ and ($v_jv_k \notin E$ or $v_iv_j \notin E$ or both). Then by the transitivity of T^C, we get $v_iv_k \in T^C$ and so $\ell_ir_k \in T'$ by line (3).

Subcase 3: $v_iv_k \notin E$, $v_jv_k \in E$, $v_iv_j \in E$. Then $v_iv_k \in T^C$ and hence $\ell_ir_k \in T'$ since, if $v_kv_i \in T^C$, then $\Delta v_kv_jv_i$ forces $r_kr_j \in T'$ (=><=).

CASE 8: $r_i\ell_j$, $\ell_jr_k \in T'$. Note that $v_iv_j \notin E$ and $v_iv_j \in T^C$. If $j = k$, then $v_iv_k = v_iv_j \in T^C$ so $r_ir_k \in T'$. So assume i, j, k distinct.

Subcase 1: $v_iv_k \notin E$ or $v_jv_k \notin E$ or both. Then the transitivity of T^C forces $v_iv_k \in T^C$ and so $r_ir_k \in T'$.

Subcase 2: $v_iv_k \in E$, $v_jv_k \in E$. Then $\Delta v_iv_kv_j$ forces $r_ir_k \in T'$.

This shows that T' is transitive. Now let us prove that the components of $(P_n, (\overline{T'})^C)$ are cliques.

Note that each component's vertices are totally contained in R_n or totally contained in L_n, by lines (1) and (3). The two cases are similar, so we will consider only the first. Suppose r_ir_j, $r_jr_k \in (\overline{T'})^C$. Since these edges and their inverses were not added to T' before line (6), it must be true that v_iv_j, v_jv_k, $v_iv_k \in E$. Now suppose $r_ir_k \notin (\overline{T'})^C$. Then without loss of generality, $r_ir_k \in T'$. Thus there must be a triangle $\Delta v_iv_kv_\ell$ which forced $r_ir_k \in T'$ (see

Figure 2.5).

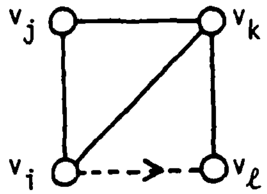


Figure 2.5: The vertices v_i, v_j, v_k, v_ℓ.

If $v_j v_\ell \in E$, then $\Delta v_i v_j v_\ell$ forced $r_i r_j \in T'$ (=><=). But if $v_j v_\ell \notin E$, then the transitivity of T^c forced $v_j v_\ell \in T^c$ and then $\Delta v_j v_k v_\ell$ forced $r_j r_k \in T'$ (=><=). Hence $r_i r_k \in (\overline{T'})^c$. This suffices to show that all the components of $(P_n, (\overline{T'})^c)$ in R_n are cliques.

This completes the proof of Lemma 2.4.2.

Now to finish the proof of Theorem 2.4, we note that, since T' is transitive, any linear order T'' (for example, the one given in lines (6) and (7)) can be given to the vertices in each of the cliques of $(P_n, (\overline{T'})^c)$ and $T' + T''$ will form a linear order on P_n. □

Now let us return to Problem 2.2 and consider the case where $S \neq \emptyset$. The ~~following~~ algorithm accepts as input $G = (V, E)$ and $S \subset P_n \times P_n$. The output, as before, is either "FLAG = 1", indicating there is no solution, or "FLAG = 0" and T, which is the desired linear order.

Algorithm 2.5

begin

initialize: $T^c \leftarrow \emptyset$; $T \leftarrow \{\ell_i r_i : i = 1, \ldots, n\} \cup$

$\{\ell_i r_j, \ell_j r_i : v_i v_j \in E\}$; (1)

for each $x_i y_j \in S$ such that $v_i v_j \in E$ or $i = j$ do

$T \leftarrow T \cup \{x_i y_j\}$; (2)

comment: $x, y \in \{r, \ell\}$;

for each $x_i y_j \in S$ such that $i \neq j$ and $v_i v_j \notin E$ do (3)

$T \leftarrow T \cup \{r_i \ell_j, r_i r_j, \ell_i \ell_j, \ell_i r_j\}$;

for each $r_i r_j \in S$ do (4)

for each $k \neq j$ such that $v_i v_j \in E$, $v_i v_k \in E$, $v_j v_k \notin E$ do (5)

$T \leftarrow T \cup \{r_k r_j, \ell_k \ell_j, r_k \ell_j, \ell_k r_j\}$;

for each $\ell_i \ell_j \in S$ do (6)

for each $k \neq i$ such that $v_i v_j \in E$, $v_j v_k \in E$, $v_i v_k \notin E$ do (7)

$T \leftarrow T \cup \{r_i r_k, \ell_i \ell_k, r_i \ell_k, \ell_i r_k\}$;

for each $r_i r_j \in T$ such that $v_i v_j \notin E$ do $T^c \leftarrow T^c \cup \{v_i v_j\}$; (8)

if there exists a transitive orientation of E^c that contains T^c

then $T^c \leftarrow$ such an orientation (9)

else write "FLAG = 1" and halt; (10)

for each $v_i v_j \in T^c$ do $T \leftarrow T \cup \{\ell_i \ell_j, r_i r_j, \ell_i r_j, r_i \ell_j\}$; (11)

for each $v_i v_j \in T^c$ do

for all k such that $v_i v_k \in E$ and $v_k v_j \in E$ do

$T \leftarrow T \cup \{r_i r_k, \ell_k \ell_j\}$; (12)

if (P_n, T) is not acyclic then write "FLAG = 1" and halt (13)

else $T \leftarrow$ a linear ordering of P_n that contains T; (14)

<u>write</u> "FLAG = 0" and T (15)

<u>end</u>

<u>Theorem 2.6</u>

Algorithm 2.5 solves Problem 2.2.

<u>Proof</u>: Suppose "FLAG = 0" is written as part of the output. Then T is a linear order on P_n by line (14) and, by lines (2) and (3), it contains S. By lines (1) and (11) it satisfies the conditions in Theorem 2.1, so $(P_n, T) \in F_G(\mathcal{I}(G))$ as desired.

Suppose, conversely, that G is an interval graph and there exists an extension T of S such that $(P_n, T) \in F_G(\mathcal{I}(G))$. Then we claim that Algorithm 2.5 will produce such an extension. We show first that the <u>else</u> clause in line (10) is not implemented and that $T \cap T^{-1} = \emptyset$ prior to the implementation of line (13).

Note that any linear order T' which extends S and for which $(P_n, T') \in F_G(\mathcal{I}(G))$ must contain all of the edges added to T in lines (1) - (7), as the following arguments show. This is obvious for lines (1) and (2). For line (3), if an endpoint of I_j is to the right of an endpoint of I_i and $I_j \cap I_i = \emptyset$, then I_j is completely to the right of I_i. For lines (4) and (5), we note that if $\{I_1, \ldots, I_n\} \in \mathcal{I}(G)$ and the right endpoint of I_i is to the left of the right endpoint of I_j, then any interval I_k which intersects I_i but not I_j must be completely to the left of I_j. Lines (6) and (7) follow similarly to (4) and (5). Therefore $T \cap T^{-1} = \emptyset$ prior to the implementation of line (8).

Furthermore, G^c must have a transitive orientation which includes T^c as defined in line (8) by Theorem 1.3. Thus line (9) will not cause "FLAG = 1" to be written.

Let $x_i y_i$ be one of the edges added to T in line (11) when $v_i v_j \in T^c$ is under consideration and suppose that $y_j x_i$ had already been added to T. Then $y_j x_i$ could only have been added when lines (3)-(7) were being implemented. But any of these possibilities would have caused $v_j v_i$ to be added to T^c in line (8), contradicting $v_i v_j \in T^c$. Hence after line (11) is implemented, $T \cap T^{-1} = \emptyset$.

We next consider step (12). Suppose that, when some $v_i v_j \in T^c$ is under consideration, there is some v_k with $v_i v_k \in E$ and $v_j v_k \in E$ for which $r_k r_i$ or $\ell_j \ell_k$ has already been added to T. Consider the case in which $r_k r_i \in T$; the case where $\ell_j \ell_k \in T$ is similar. Then $r_k r_i$ must have been added to T in line (2) or an earlier loop of line (12). If $r_k r_i$ was added to T in line (2), then lines (4), (5), and (8) would have caused $v_j v_i$ to be in T^c, contradicting $v_i v_j \in T^c$. Thus $r_k r_i$ must have been added to T in an earlier loop of line (12), say, when $v_k v_\ell \in T^c$ was under consideration. Then $v_i v_\ell \in E$ (see Figure 2.6).

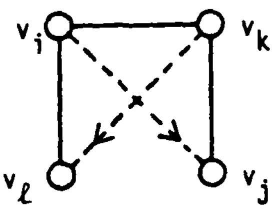


Figure 2.6: The vertices v_i, v_j, v_k, v_ℓ.

If $v_\ell v_j \notin E$, then we obtain a contradiction of the transitivity of T^c regardless of whether $v_j v_\ell \in T^c$ or $v_\ell v_j \in T^c$. Thus $v_\ell v_j \in E$, and so $\{v_i, v_k, v_j, v_\ell\}$ induces C_4 in G, contradicting G's being an interval graph. Thus $r_k r_i$ could not have been added to T and so $T \cap T^{-1} = \emptyset$ prior to the implementation of line (13).

All that remains is to show that (P_n, T) is acyclic. Let T' denote the set of edges added to T in lines (1), (11), and (12). By Lemma 2.4.2 of Theorem 2.4, T' is transitive and $(P_n, (\overline{T'})^c)$ has connected components consisting of cliques. Therefore, since $T \cap T^{-1} = \emptyset$, the set T being tested for cycles in line (13) can be partitioned into $T' + K$, where K is a set of edges from the cliques of $(P_n, (\overline{T'})^c)$. This latter set of edges must have been added to T in line (2) of the algorithm and so $K \subset S$, which means that (P_n, K) is acyclic. When combined with the transitivity of T', this implies that $(P_n, T) = (P_n, T' + K)$ is acyclic, as the following argument shows.

Suppose (P_n, T) has cycles. Let $C = [x_1, x_2, \ldots, x_k]$, $k \geq 3$, be a shortest cycle in (P_n, T). Without loss of generality, $x_1 x_2 \in K$, $x_2 x_3 \in T'$. Now $x_1 x_3 \notin (\overline{T'})^c$ since the components of $(P_n, (\overline{T'})^c)$ are cliques. Therefore, $x_1 x_3 \in T'$ or $x_3 x_1 \in T'$ and so, since T' is transitive, $x_1 x_3 \in T'$. But now $[x_1, x_3, \ldots, x_k]$ is a shorter cycle in (P_n, T), a contradiction.

Therefore (P_n, T) is acyclic, and the theorem is proved. □

An implementation of line (9) is discussed in Chapter 4, where it is shown that it can be done in $O(|V|^3)$ steps. Lines (13) and (14) can be implemented to run in $O(|V| + |T|)$ steps by a "topological sorting" procedure (see Golumbic [12] or Knuth [21, vol. 1, pp. 258-265]). Thus it can readily be seen that Algorithm 2.5 has worst-case complexity $O(|V|^3)$.

2.2 Graphs with Fuzzy Edges

Let us return to the example of archaeological seriation. Consider the case where the graph G_p turns out not to be an interval graph, thus indicating a flaw in one or more of the basic assumptions. The problem could be due to the fact that, for certain pairs of vertices, the data suggesting the inclusion (or omission) of an edge between them is insufficiently compelling. For this reason, some edges may have been added (or left out) that should be left out (or added).

This possibility leads to the following problem. (In this form, those edges for which the information is "fuzzy" are denoted by E_2.)

Problem 2.7

Instance: Graph $G = (V, E_1 + E_2)$, with undirected sets of edges E_1 and E_2.

Question: Does there exist an undirected set E of edges such that $E_1 \subset E \subset E_1 + E_2$ and (V, E) is an interval graph?

This question is also relevant for Seymour Benzer's experiment in molecular genetics [2; 3]. He collected data indicating whether various connected regions of a gene intersect each other, and he hoped this data would support his claim that the gene is linear in structure. If we consider each region as a vertex, and connect two vertices with an edge iff the two regions intersect, then the resulting graph must be an interval graph if Benzer's hypothesis is correct.

For one set of 19 regions, Benzer tested all pairs for intersection and he actually obtained an interval graph, which supported his hypothesis.

However, for a set of 145 regions, he was not able to test all pairs to see if they intersect. In this case, he was in the situation described in Problem 2.7 in which E_2 connects the pairs Benzer was not able to test. In his case, the answer to the question in Problem 2.7 was yes; that is, he was able to find an interval graph as desired, but he did not indicate a method with which we can always efficiently answer the question.

It is not known if there is an efficient means of solving Problem 2.7. It is very closely related to some known NP-complete problems (see Garey and Johnson [10, Problem GT35]), which indicates that the problem might be quite hard to solve. However, a special case of the problem can be solved by an algorithm of complexity $O(|V|^2)$ using the methods developed in this chapter.

Consider once again the example of archaeological seriation. Suppose that, for each pair of intervals which the archaeologists are certain do not intersect, the intervals are so far apart that it can be determined which of the two is older and which is more recent.

We formulate this special case as follows:

Problem 2.8

Instance: Graph $G = (V, E_1 + E_2)$ with undirected sets of edges E_1 and E_2, and an orientation S of all the edges in G^c.

Question: Does there exist a set $I = \{I_1,\ldots, I_n\}$ of closed, non-empty intervals with all endpoints distinct, such that

(a) $v_iv_j \in E_1 \Rightarrow I_i \cap I_j \neq \emptyset$, and

(b) $v_iv_j \in S \Rightarrow I_i < I_j$?

The next theorem characterizes those instances of Problem 2.8 for which the answer is yes. Furthermore it provides a test with $O(|V|^2)$ steps for solving the problem.

Define $T \subset P_n \times P_n$ as follows:

$$T = \{\ell_i r_j, \ell_j r_i : v_iv_j \in E_1\} \cup \{\ell_i r_i : i = 1,\ldots, n\}$$
$$\cup \{\ell_i \ell_j, r_i r_j, \ell_i r_j, r_i \ell_j : v_iv_j \in S\}.$$

Theorem 2.9

There exists a set of intervals as desired in Problem 2.8 iff (P_n, T) is acyclic.

Proof: If there is such a set of intervals, then such a representation will induce a linear order on P_n which clearly contains T. Thus T

must be acyclic.

Conversely, if (P_n, T) is acyclic, T can be extended to a linear order. This linear order is associated with a set of intervals which by Theorem 2.1 has the desired properties. □

Note that this situation could be generalized slightly by allowing some extra restrictions on the ℓ_i's and r_i's in an instance of the problem. This could easily be handled by including them in T, in which case Theorem 2.9 would still hold.

A topological sorting algorithm applied to (P_n, T) will determine if it is acylic, and if so, produce a linear extension in $O(|V| + |T|)$ steps.

As is often the case, not only would it be nice to know when a solution exists, it would also be nice to know how many such solutions there are.

Enumeration of solutions to Problem 2.2 in the special case where $S = \emptyset$ is discussed in Chapter 7. The general case is more complicated, and the author knows of no efficient enumeration scheme for it.

For Problem 2.8, the number of solutions is just the number of linear extensions of T. This subject will be discussed in Chapter 4.

CHAPTER 3: CHRONOLOGICAL ORDERINGS II

It may have become clear in Chapter 2 that to completely describe a chronological ordering of a graph, all that is needed is the linear order of the right endpoints and the linear order of the left endpoints of the intervals. This idea is explored in this chapter.

Recall that $R_n = \{r_1, \ldots, r_n\}$ and $L_n = \{\ell_1, \ldots, \ell_n\}$ $(P_n = R_n \cup L_n)$. Let

$$\mathcal{R}_n = \{(R_n, T_R):\ T_R \text{ is a linear order on } R_n\}$$

and

$$\mathcal{L}_n = \{(L_n, T_L):\ T_L \text{ is a linear order on } L_n\}.$$

As described in Chapter 2, if, for any representation $I = \{I_1, \ldots, I_n\} \in \mathcal{I}(G)$, the left [resp. right] endpoint of interval I_i is associated with ℓ_i [resp. r_i], then the linear order of the endpoints on the real line induces a linear order T_R on R_n and a linear order T_L on L_n. This defines a mapping $F_G^{RL}: \mathcal{I}(G) \to \mathcal{R}_n \times \mathcal{L}_n$. T_R and T_L are said to be <u>associated with I</u>.

The following theorem is the analog of Theorem 2.1.

<u>Theorem 3.1</u>

A graph $G = (V, E)$ is an interval graph iff there exist linear orders T_R on R_n and T_L on L_n with the following properties: For all i, j, and k,

(a) $r_i r_j \in T_R$ and $v_k \in N(v_i)\backslash N(v_j) \Rightarrow \ell_k \ell_j \in T_L$, and

(b) $\ell_i \ell_j \in T_L$ and $v_k \in N(v_j)\backslash N(v_i) \Rightarrow r_i r_k \in T_R$.

Furthermore, if this is the case, then T_R and T_L are associated with some representation of G.

Proof: Let $I \in \mathcal{I}(G)$, $I = \{I_1, \ldots, I_n\}$. It is easy to see that the linear orders T_R and T_L associated with I satisfy conditions (a) and (b). If interval I_j extends to the right of I_i and I_k intersects I_i but not I_j, then I_k must extend to the left of I_j. Also, if I_i extends to the left of I_j and I_k intersects I_j but not I_i, then I_k must extend to the right of I_i.

For the converse, assume there exist linear orders T_R and T_L with properties (a) and (b).

Lemma 3.1.1: *If $v_i v_j \notin E$ and $i \neq j$, then $r_i r_j \in T_R \iff \ell_i \ell_j \in T_L$.*

Proof of Lemma 3.1.1: Let $v_i v_j \notin E$ and $i \neq j$. Assume $r_i r_j \in T_R$. Then $v_i \in N(v_i) \backslash N(v_j)$ so by (a), we have $\ell_i \ell_j \in T_L$. If $\ell_i \ell_j \in T_L$, then since $v_j \in N(v_j) \backslash N(v_i)$, property (b) gives $r_i r_j \in T_R$.

Lemma 3.1.2: G *does not contain* C_4 *as an induced subgraph.*

Proof of Lemma 3.1.2: Suppose $[v_1, v_2, v_3, v_4]$ is a chordless cycle in G. Without loss of generality, we can assume that $r_2 r_4 \in T_R$, $\ell_2 \ell_4 \in T_L$, $r_1 r_3 \in T_R$, $\ell_1 \ell_3 \in T_L$ by Lemma 3.1.1. Furthermore, by symmetry, we can assume that $r_1 r_2 \in T_R$ (see Figure 3.1).

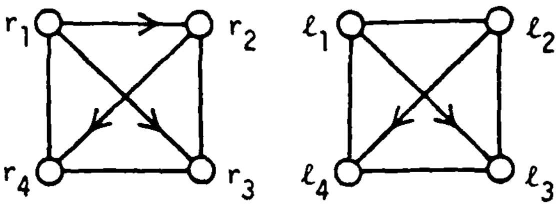


Figure 3.1: Lemma 3.1.2.

But now $r_1r_2 \in T_R$ and $v_4 \in N(v_1)\backslash N(v_2)$, so by property (a) $\ell_4\ell_2 \in T_L$. But this contradicts $\ell_2\ell_4 \in T_L$.

Lemma 3.1.3: Let $T^c = \{v_iv_j \notin E: r_ir_j \in T_R\}$. Then T^c is a transitive orientation of G^c.

Proof of Lemma 3.1.3: Note by Lemma 3.1.1 that $T^c = \{v_iv_j \notin E: \ell_i\ell_j \in T_L\}$. Now let v_iv_j, $v_jv_k \in T^c$. Then r_ir_j, $r_jr_k \in T_R$ and $\ell_i\ell_j$, $\ell_j\ell_k \in T_L$. Hence by linearity of T_R and T_L, we have $r_ir_k \in T_R$ and $\ell_i\ell_k \in T_L$. Thus, $v_iv_k \in T^c$ if $v_iv_k \notin E$. Assume $v_iv_k \in E$. Then since $r_ir_j \in T_R$ and $v_k \in N(v_i)\backslash N(v_j)$, property (a) implies that $\ell_k\ell_j \in T_L$, a contradiction. Hence $v_iv_k \notin E$ and the proof is complete.

Now by Lemmas 3.1.2 and 3.1.3 and Theorem 1.3, G must be an interval graph. We next show that T_R and T_L are associated with a representation in $\mathcal{I}(G)$.

Define a relation T on P_n by

$$T = T_R + T_L + \{\ell_i r_i : i = 1,\dots, n\} + \{\ell_i r_j,\ \ell_j r_i : v_i v_j \in E\}$$
$$+ \{\ell_i r_j,\ r_i \ell_j : v_i v_j \notin E \text{ and } r_i r_j \in T_R\}.$$

Lemma 3.1.4: T is a linear order on P_n which is associated with a chronological ordering of G.

Proof of Lemma 3.1.4: By Theorem 2.1, we are done once we have shown that T is a linear order. We need only check transitivity. Each of eight cases will be considered. We note, first, the following property of T which follows from the definitions of T and T^c:

(*) If $v_i v_j \notin E$, then the following are equivalent:

(1) $r_i r_j \in T$ (3) $r_i \ell_j \in T$ (5) $v_i v_j \in T^c$.

(2) $\ell_i \ell_j \in T$ (4) $\ell_i r_j \in T$

CASE 1: $r_i r_j,\ r_j r_k \in T$. Then $r_i r_k \in T$ by the transitivity of T_R.

CASE 2: $\ell_i \ell_j,\ \ell_j \ell_k \in T$. Then $\ell_i \ell_k \in T$ by the transitivity of T_L.

CASE 3: $\ell_i \ell_j,\ \ell_j r_k \in T$. Assume $r_k \ell_i \in T$. Then $v_k v_i \notin E$ and $r_k r_i \in T$ by (*). If $v_j v_k \in E$, then by property (b), we must have $r_i r_k \in T$ (=><=). Hence $v_j v_k \notin E$, so by (*), $v_j v_k \in T^c$. Since by (*), $v_k v_i \in T^c$, Lemma 3.1.3 says that $v_j v_i \in T^c$. Hence by (*), $\ell_j \ell_i \in T$ (=><=). Thus our assumption that $r_k \ell_i \in T$ is false and so $\ell_i r_k \in T$.

<u>CASE 4</u>: $\ell_i r_j$, $r_j r_k \in T$. The proof that $\ell_i r_k \in T$ is almost identical to the proof of Case 3.

<u>CASE 5</u>: $r_i r_j$, $r_j \ell_k \in T$. By the definition of T, $v_j v_k \notin E$ and so $v_j v_k \in T^c$. If $v_i v_k \in E$, then property (a) gives $\ell_k \ell_j \in E$, contradicting $v_j v_k \in T^c$ and (*). Thus $v_i v_k \notin E$. If $v_i v_j \in E$, then by the transitivity of T^c, $v_i v_k \in T^c$ and hence $r_i \ell_k \in T$ by (*). If $v_i v_j \notin E$, then $v_i v_j \in T^c$ by (*), and hence the transitivity of T^c gives $v_i v_k \in T^c$ and by (*), $r_i \ell_k \in T$.

<u>CASE 6</u>: $r_i \ell_j$, $\ell_j \ell_k \in T$. The proof that $r_i \ell_k \in T$ is almost identical to the proof of Case 5.

<u>CASE 7</u>: $\ell_i r_j$, $r_j \ell_k \in T$. Then by the definition of T, $v_j v_k \in T^c$. If $v_i v_j \notin E$, then $v_i v_j \in T^c$ by (*), so the transitivity of T^c gives $v_i v_k \in T^c$ and therefore $\ell_i \ell_k \in T$ by (*). If $v_i v_j \in E$, then by property (b), $\ell_i \ell_k \in T$. (If $\ell_k \ell_i \in T$, then property (b) implies $r_k r_j \in T$, contradicting $v_j v_k \in T^c$ and (*).)

<u>Case 8</u>: $r_i \ell_j$, $\ell_j r_k \in T$. The proof that $r_i r_k \in T$ is almost identical to the proof of Case 7. This proves the lemma.

Now by Theorem 2.1, (P_n, T) is associated with a representation I of G and clearly, by the definition of T, this implies that T_R and T_L are also associated with I. Thus the theorem is proved. $\square$

Theorem 3.2

Let T_R and T_L be associated with a representation $I = \{I_1, \ldots, I_n\} \in \mathcal{I}(G)$. Then $F_G(I) = (P_n, T)$, where T is defined as in the proof of Theorem 3.1.

Proof: If $F_G(I) = (P_n, T')$, then $T'|_{R_n} = T_R$ and $T'|_{L_n} = T_L$ because T_R and T_L are associated with I. Furthermore, by Theorem 2.1, $\{\ell_i r_i : i = 1, \ldots, n\} \subset T'$ and $\{\ell_i r_j, \ell_j r_i : v_i v_j \in E\} \subset T'$. For each $v_i v_j \notin E$, either $r_i r_j \in T_R$ or $r_j r_i \in T_R$. Since T_R and T_L are associated with I, the former case implies that $I_i < I_j$ and the latter that $I_j < I_i$. Hence

$$\{\ell_i r_j, r_i \ell_j : v_i v_j \notin E \text{ and } r_i r_j \in T_R\} \subset T'.$$

Thus $T \subset T'$ and so $T = T'$ because T is a linear order on P_n. □

Corollary 3.3

Let $I, I' \in \mathcal{I}(G)$. $I \sim I'$ iff $F_G^{RL}(I) = F_G^{RL}(I')$.

Proof: If $I \sim I'$, then $F_G(I) = F_G(I') = (P_n, T')$ for some linear order T' on P_n. Since T_R and T_L are just the restrictions of T' to R_n and L_n, respectively, we obtain $F_G^{RL}(I) = F_G^{RL}(I')$.

Conversely, if $F_G^{RL}(I) = F_G^{RL}(I') = ((R_n, T_R), (L_n, T_L))$, then by Theorem 3.2, $F_G(I) = (P_n, T) = F_G(I')$ where T is defined as in the proof of Theorem 3.1. Therefore $I \sim I'$. □

Thus F_G^{RL} splits $\mathcal{I}(G)$ into the same equivalence classes as F_G, and hence defines the same chronological orderings as F_G.

As for Theorem 2.1, we can ask whether Theorem 3.1 is useful for recognizing interval graphs. More generally, as in Chapter 2, we can ask whether Theorem 3.1 is useful in the situation in which we are given partial orientations on R_n and L_n and we wish to know whether they can be extended to linear orders which are associated with a representation of G.

Theorem 3.1 can be used for recognizing interval graphs by means of an algorithm very similar to Algorithm 2.3. However, it also takes $O(|V|^3)$ steps. Like Algorithm 2.3, it could be modified to take in the more general situation, but a little reflection shows that this is really only a special case of Problem 2.2; it is the case in which the initial conditions in S are restricted to R_n or to L_n. For that reason, such an algorithm will not be discussed here.

CHAPTER 4: CHRONOLOGICAL ORDERINGS III

In this chapter, we formulate the concept of chronological ordering in a third way, which facilitates the construction of algorithms for solving problems not solvable by the earlier algorithms.

4.1 Main Results

Let $\underline{C}$, $\underline{O}$, $\underline{F}$ be irreflexive relations (i.e., edges) on V. Let

$$\underline{\mathcal{D}(V)} = \{(V, C+O+F): C \cap C^{-1} = O \cap O^{-1} = F \cap F^{-1} = \emptyset \text{ and } \bar{C} + \bar{O} + \bar{F} = V \dot{\times} V\}.$$

$\mathcal{D}(V)$ can be thought of as the set of all oriented graphs with vertices V and with exactly one edge between every two vertices (i.e., a tournament), whose edges are partitioned into the three sets C, O, and F.

Let $I = \{I_1, \ldots, I_n\} \in \mathcal{I}(G)$, with $I_i = [a_i, b_i]$ for $i = 1, \ldots, n$. The following process constructs an element $(V, C+O+F)$ of $\mathcal{D}(V)$. Let

$$C = \{v_i v_j : a_j < a_i < b_i < b_j\} \quad (I_j \text{ \underline{c}ontains } I_i)$$
$$O = \{v_i v_j : a_i < a_j < b_i < b_j\} \quad (I_j \text{ \underline{o}verlaps } I_i \text{ on the right})$$
$$F = \{v_i v_j : a_i < b_i < a_j < b_j\} \quad (I_j \text{ \underline{f}ollows } I_i).$$

Since exactly one of these possibilities is true for every pair of intervals in I, it is clear that $(V, C+O+F) \in \mathcal{D}(V)$. This defines a mapping $F_G^{COF}: \mathcal{I}(G) \to \mathcal{D}(V)$. Call $F_G^{COF}(I)$ the <u>tournament associated with I</u>. See Figure 4.1 for an example of this.

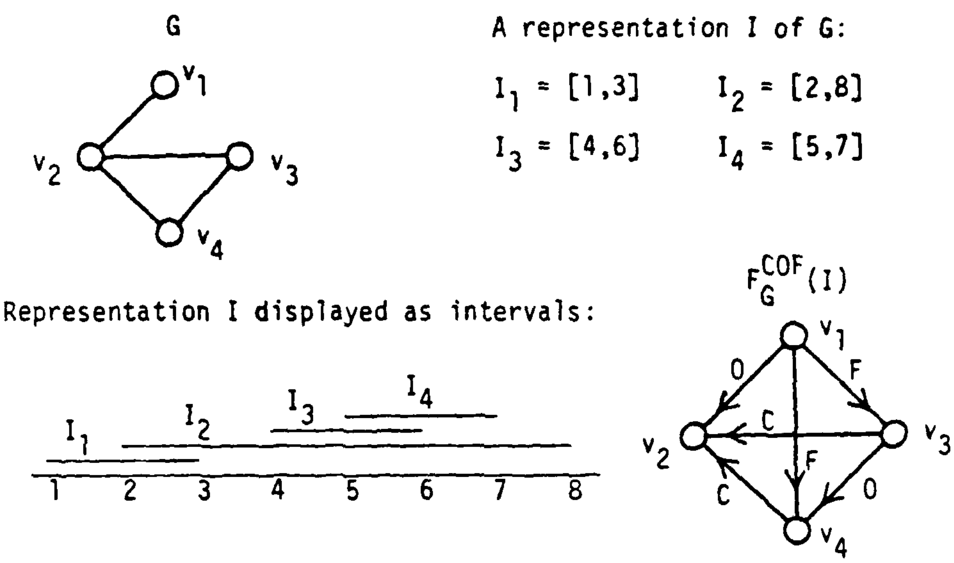


Figure 4.1: An example of the mapping F_G^{COF}.

Theorem 4.1

A graph $G = (V, E)$ is an interval graph iff there exists $D = (V, C+0+F) \in \mathcal{D}(V)$ with the following properties:

(a) $\tilde{C} + \tilde{0} = E$

(b) $C + 0 + F$ and $C^{-1} + 0 + F$ are transitive

(c) $F(C^{-1} + 0 + F) \subset F$.

(d) $(C + 0 + F)F \subset F$.

Furthermore, if this is the case, then D is associated with some representation in $I(G)$.

Proof: Let $I = \{I_1, \ldots, I_n\} \in I(G)$ where $I_i = [a_i, b_i]$, for $i = 1, \ldots, n$. We will show that $F_G^{COF}(I)$ has the desired properties. Let $F_G^{COF}(I) = D = (V, C+0+F)$. Clearly $I_i \cap I_j \neq \emptyset$ iff $v_iv_j \in C$ or $v_jv_i \in C$ or $v_iv_j \in 0$ or $v_jv_i \in 0$. Therefore, property (a) is true.

It can be seen by looking at the definition of F_G^{COF} that $v_iv_j \in C+0+F \iff b_i < b_j$. Since the set $\{b_i: i = 1,\dots, n\}$ is transitively ordered on the real line, so is $C+0+F$. Similarly $v_iv_j \in C^{-1} + 0 + F \iff a_i < a_j$, so $C^{-1} + 0 + F$ is transitive. This proves (b).

For (c), let $v_iv_j \in F$, $v_jv_k \in C^{-1} + 0 + F$. Then $a_i < b_i < a_j < a_k < b_k$ and so $v_iv_k \in F$. Similarly, if $v_iv_j \in C+0+F$ and $v_jv_k \in F$, then $a_i < b_i < b_j < a_k < b_k$ and hence $v_iv_k \in F$, which proves (d). This proves necessity in Theorem 4.1.

Let $D = (V, C+0+F) \in \mathcal{D}(V)$ have properties (a)-(d). We construct linear orders T_R on R_n and T_L on L_n by letting

(*) $\qquad r_ir_j \in T_R \iff v_iv_j \in C+0+F$, and

(**) $\qquad \ell_i\ell_j \in T_L \iff v_iv_j \in C^{-1} + 0 + F$.

<u>Lemma 4.1.1</u>: T_R and T_L have properties (a) and (b) described in Theorem 3.1.

<u>Proof of Lemma 4.1.1</u>: Let $\ell_i\ell_j \in T_L$ and $v_k \in N(v_j)\backslash N(v_i)$. Then $v_iv_j \in C^{-1} + 0 + F$ and $v_iv_k \notin E$ so by (a) above, $v_iv_k \in F$ or $v_kv_i \in F$. If $v_kv_i \in F$, then (c) above implies that $v_kv_j \in F$ and hence $v_kv_j \notin E$, contradicting $v_k \in N(v_j)$. Thus $v_iv_k \in F$ and hence $r_ir_k \in T_R$ by (*).

Now let $r_ir_j \in T_R$ and let $v_iv_k \in E$, $v_jv_k \notin E$. Then $v_iv_j \in C+0+F$. If $v_jv_k \in F$, then $v_iv_k \in F$ by property (d), which contradicts property (a).

Hence $v_k v_j \in F$ ($v_j v_k \notin \tilde{C} + \tilde{O}$ by property (a)) and $\ell_k \ell_j \in T_L$ by (**). This proves the lemma.

Now by Theorem 3.1, G is an interval graph as desired. To show that D is associated with some representation of G, let $I = \{I_1, \ldots, I_n\} \in I(G)$ be a representation to which T_R and T_L (defined above) are associated and let $I_i = [a_i, b_i]$, $i = 1, \ldots, n$. (One such I is constructed in Theorems 3.1 and 2.1.) We claim: $F_G^{COF}(I) = D$.

If $v_i v_j \in C$, then by (*) and (**), $r_i r_j \in T_R$ and $\ell_j \ell_i \in T_L$. Now T_R and T_L are associated with I, and therefore $a_j < a_i < b_i < b_j$ as desired.

If $v_i v_j \in O$, then $r_i r_j \in T_R$ and $\ell_i \ell_j \in T_L$, as before, and hence $a_i < a_j$ and $b_i < b_j$. Since $v_i v_j \in E$, $I_i \cap I_j \neq \emptyset$. Thus $a_j < b_i$ and therefore $a_i < a_j < b_i < b_j$.

If $v_i v_j \in F$, then $a_i < a_j$ and $b_i < b_j$. Since $v_i v_j \notin E$, it must be true that $b_i < a_j$ or $b_j < a_i$. The latter cannot happen since $a_i < b_i < b_j$. Therefore $b_i < a_j$ and so $a_i < b_i < a_j < b_j$. This proves that D is associated with I and completes the proof of Theorem 4.1. $\square$

Note: An alternate proof of this result is contained in Fournier [8] (see also Golumbic [12]). He provides a more direct construction of a representation I to which D is associated. For any relation R on V, let $\underline{R(v_i)} = \{v \in V: v_i v \in R\}$. Similarly, let $\underline{R^{-1}(v_i)} = \{v \in V: v v_i \in R\}$. Then Fournier shows that D is associated with

$\{I_1, \ldots, I_n\}$ defined as follows. If $I_i = [a_i, b_i]$, then let

$$a_i = 1 + |(C + O^{-1} + F^{-1})(v_i)| + |F^{-1}(v_i)| \text{ and}$$

$$b_i = 2n - |(C+O+F)(v_i)| - |F(v_i)|.$$

Some other properties can be substitued for (b), (c), and (d) in Theorem 4.1, as the next theorem shows.

Theorem 4.2

Let $D = (V, C+O+F) \in \mathcal{D}(V)$. Then the following are equivalent:

(1) D has properties (b), (c), (d) in Theorem 4.1;

(2) The following conditions hold:

(e) $C^2 \subset C$

(f) $(O+F)^2 \subset O + F$

(g) $F^2 \cap O = \emptyset$

(h) $FO \cap O = \emptyset$

(i) $OF \cap O = \emptyset$

(j) $CF \cap O = \emptyset$

(k) $CF \cap O = \emptyset$

(ℓ) $OC \cap F = \emptyset$;

(3) The following conditions hold:

(m) $C^2 \subset C$

(n) $O^2 \subset O + F$

(o) $OF \subset F$

(p) $FO \subset F$

(q) $F^2 \subset F$

(r) $CF \subset F$

(s) $OC \subset O + C$.

Proof: (1)=>(3): By Theorem 4.1, D is associated with a representation $\{I_1, \ldots, I_n\} \in I(G)$ where $G = (V, \bar{C}+\bar{O})$. Let $I_i = [a_i, b_i]$ for $i = 1, \ldots, n$.

If v_iv_j, $v_jv_k \in C$, then

$$a_k < a_j < a_i < b_i < b_j < b_k$$

and so $v_iv_k \in C$. This proves (m).

If v_iv_j, $v_jv_k \in O + F$, then $a_i < a_j < a_k$ and $b_i < b_j < b_k$ and so $v_iv_k \in O + F$. This proves (n).

If $v_iv_j \in O$, $v_jv_k \in F$, then

$$a_i < a_j < b_i < b_j < a_k < b_k$$

and so $v_iv_k \in F$. This proves (o). The proofs for (p), (q), and (r) are similar to these.

To prove (s), we note that if $v_iv_j \in O$ and $v_jv_k \in C$, then $a_i < a_j < b_i < b_j < b_k$. Also $a_k < a_j$. If $a_k < a_i < a_j$, then $v_iv_k \in C$. If $a_i < a_k < a_j$, then $v_iv_k \in O$.

(3) => (2): It is easy to see that the following implications are true: (m) => (e); (n), (o), (p), and (q) => (f); (q) => (g); (p) => (h); (o) => (i); (r) => (j) and (k); (s) => (ℓ).

(2) => (1): It is easy to prove that the union of two complementary partial orders (irreflexive, transitive relations) forms a linear order. Therefore, (e) and (f) => (b). Furthermore, conditions (f), (g), (i), (j), and (k) imply (d), and conditions (f), (g), and (h) imply that $F(O+F) \subset F$. Therefore it only remains to show that $FC^{-1} \subset F$. But if $FC^{-1} \cap O \neq \emptyset$, then $OC \cap F \neq \emptyset$, contradicting (ℓ).

If $FC^{-1} \cap C^{-1} \neq \emptyset$, then $CF \cap C \neq \emptyset$, contradicting (j). Therefore, $FC^{-1} \subset F$ since $FC^{-1} \subset (C^{-1} + O + F)$. This completes the proof. □

Theorem 4.3

Let $I, I' \in \mathcal{I}(G)$. $I \sim I' \iff F_G^{COF}(I) = F_G^{COF}(I')$.

Proof: Let $I = \{I_1, \ldots, I_n\}$ in which $I_i = [a_i, b_i]$, and let $I' = \{I_1', \ldots, I_n'\}$ in which $I_i' = [a_i', b_i']$. Let $F_G^{COF}(I) = (V, C+O+F)$ and $F_G^{COF}(I') = (V, C' + O' + F')$. Then $I \sim I' \iff$ for all i and j, the following are true:

(i) $a_i < a_j$ iff $a_i' < a_j'$

(ii) $a_i < b_j$ iff $a_i' < b_j'$

(iii) $b_i < a_j$ iff $b_i' < a_j'$

(iv) $b_i < b_j$ iff $b_i' < b_j'$

$\iff$ for all i and j, the following are true:

(v) $v_iv_j \in C$ iff $v_iv_j \in C'$

(vi) $v_iv_j \in O$ iff $v_iv_j \in O'$

(vii) $v_iv_j \in F$ iff $v_iv_j \in F'$

$\iff F_G^{COF}(I) = F_G^{COF}(I')$. □

Thus $F_G^{COF}: \mathcal{I}(G) \to \mathcal{D}(V)$ yields the same equivalence classes (i.e., chronological orderings) as $F_G: \mathcal{I}(G) \to O_n$ and $F_G^{RL}: \mathcal{I}(G) \to \mathcal{R}_n \times \mathcal{L}_n$.

Notice that, if $D = (V, C+O+F) \in \mathcal{D}(V)$ is given, it can be determined in $O(|V|^3)$ steps whether D has any of the three equivalent sets of properties in Theorem 4.2 and hence whether D is associated with a representation of $G = (V, \bar{C}+\bar{O})$. Although this does not lead to a very efficient recognition algorithm for interval graphs, it does lead to algorithms which can solve problems beyond the reach of those presented in Chapters 2 and 3.

For example, using the results in Chapters 2 and 3, we can determine whether a graph G has a representation in which certain intervals are properly contained in others. However, if we desire to know whether G has a representation in which those intervals are the <u>only</u> ones properly contained in others, the results obtained earlier are useless. We will shortly present an algorithm which will solve this problem using Theorems 4.1 and 4.2.

Let us return to our problem in archaeological seriation. Under a few basic assumptions, we were able to deduce that one of the chronological orderings of G_p contains a chronological representation of the artifacts.

Now, if the collection of graves is extensive enough, then besides the previously mentioned assumptions, it might be reasonable to make another assumption:

> Style u appeared after style v and disappeared before style v iff every grave containing style u also contains style v.

For simplicity, let us make one additional assumption:

> No two styles appear in exactly the same graves.

If two such styles exist, we simply remove one from under consideration. As there is nothing in the data to distinguish between such styles, it can be assumed that they have the same interval of use.

Now we can represent the data by a mixed graph $G = (V, E+C)$. In this form, V is the set of artifacts and

$v_i v_j \in C$ iff every grave containing style v_i also contains style v_j.

$v_i v_j \in E$ iff styles v_i and v_j appear in a common grave but $v_i v_j \notin C$ and $v_j v_i \notin C$.

Then, according to our assumptions, $G' = (V, E+\bar{C})$ must be an interval graph and have an interval representation $\{I_1, \ldots, I_n\} \in \mathcal{I}(G')$ in which $v_i v_j \in C$ iff $I_i \subset I_j$. Furthermore, any such representation is a possible chronological representation of the artifacts. Thus we have the following problem.

<u>Problem 4.4</u>

Instance: Graph $G = (V, E+C)$.

Question: Does $G' = (V, E+\bar{C})$ have a representation $\{I_1, \ldots, I_n\} \in \mathcal{I}(G')$ in which $v_i v_j \in C \iff I_i \subset I_j$?

Note that we have specified exactly which intervals we want to be contained in others in the representation.

By the preceding discussion and theorems, this question is equivalent to asking whether there exist relations O and F on V such that $\bar{O} = E$, $\bar{F} = V \times V \setminus E \setminus \bar{C}$, and $(V, C+O+F) \in F_{G'}^{COF}(\mathcal{I}(G'))$. This gives us the following theorem.

<u>Theorem 4.5</u>

G' has a representation as desired in Problem 4.4 iff the following conditions hold:

(1) $xy \in C \Rightarrow N_{G'}(x) \subset N_{G'}(y)$;

(2) $C^2 \subset C$;

(3) there exist relations O and $F \subset V\dot{\times}V$ with the following properties:

(a) $\bar{O} = E$

(b) $\bar{F} = V\dot{\times}V \setminus E \setminus \bar{C}$

(c) $(O+F)^2 \subset O + F$

(d) $F^2 \subset F$

(e) $OF \subset F$

(f) $FO \subset F$.

<u>Proof</u>: (=>) Given such a representation I of G' let $D = (V, C+O+F) = F_{G'}^{COF}(I)$. This O and F satisfy the conditions in (3) by Theorems 4.1 and 4.2. Condition (1) is satisfied since if $I_i \subset I_j$, then any interval that intersects I_i must also intersect I_j. Because $I_i \subset I_j \subset I_k \Rightarrow I_i \subset I_k$, condition (2) is satisfied.

(<=) All we need to show is that $D = (V, C+O+F) \in F_{G'}^{COF}(\mathcal{I}(G'))$. By (a), (b), (c), and (2), it is easy to see that $D \in \mathcal{D}(V)$. By Theorems 4.1 and 4.2, we need only show that conditions (e)-(ℓ) are true in Theorem 4.2. But the proofs of (e), (f), (g), (h), and (i) are automatic. Conditions (k), (ℓ), and (j) follow from property (1).

$\square$

This theorem enables us to construct an algorithm that solves Problem 4.4. Conditions (1) and (2) of Theorem 4.5 can be checked in $O(|V|^3)$ steps. We next describe an algorithm of complexity $O(|V|^3)$ for constructing orientations O and F as desired or showing that none exist.

By ignoring $\bar{C}$ and representing $V \times V \backslash E \backslash \bar{C}$ by $\underline{H}$, we can state this problem in the following manner.

Problem 4.6

Instance: Graph $G = (V, E+H)$ with $E = \bar{E}$ and $H = \bar{H}$.

Question: Do there exist orientations O of E and F of H with the following properties:

(1) $(O+F)^2 \subset O + F$ (3) $OF \subset F$

(2) $F^2 \subset F$ (4) $FO \subset F$?

Henceforth, an undirected graph $G = (V, E+H)$ will be called partitioned if E and H consist of (disjoint) undirected sets of edges. Also O will always denote an orientation of E and F an orientation of H.

By property (1), $O+F$ is transitive. For this reason, if orientations O of E and F of H have properties (1)-(4), we will call $O+F$ a strongly transitive orientation (STRO) of G (or of $E+H$), or say that $O+F$ is strongly transitive.

To be able to determine whether a partitioned graph $(V, E+H)$ has a STRO, we need to develop some theory that is a modification of Golumbic's results [12; 13; 14] on transitive orientations of graphs.

It is urged that the reader study these papers simultaneously with the following pages of this chapter (see also Pnueli, Lempel, and Even [25]).

As motivation for the following definitions, we consider some necessary conditions for a graph to have a strongly transitive orientation. If we have a triangle two of whose edges are in H and one is in E, then the two edges in H must both be pointed toward or both away from their common vertex, by property (2). Properties (3) and (4) say that, if we have a triangle two of whose edges are in E and one is in H, then the triangle must be oriented as in Figure 4.2.

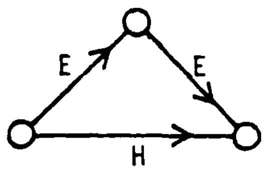


Figure 4.2: Desired orientation.

Define binary relations $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5$ on $E+H$ as follows (see Figure 4.3):

$$bc\,\Gamma_1\,b'c' \iff \begin{cases} \text{either } b = b' \text{ and } cc' \notin E+H \\ \text{or } c = c' \text{ and } bb' \notin E+H \end{cases}$$

$$bc\,\Gamma_2\,b'c' \iff \begin{cases} \text{either } b = b',\ cc' \in E,\ bc \in H,\ b'c' \in H \\ \text{or } c = c',\ bb' \in E,\ bc \in H,\ b'c' \in H \end{cases}$$

$$bc\,\Gamma_3\,b'c' \iff \begin{cases} \text{either } c = b',\ bc' \in H,\ bc \in E,\ b'c' \in E \\ \text{or } c' = b,\ b'c \in H,\ bc \in E,\ b'c' \in E \end{cases}$$

$$bc\Gamma_4 b'c' = \begin{cases} \text{either} & b = b',\ cc' \in E,\ bc \in E,\ b'c' \in H \\ \text{or} & b = b',\ cc' \in E,\ bc \in H,\ b'c' \in E \end{cases}$$

$$bc\Gamma_5 b'c' = \begin{cases} \text{either} & c = c',\ bb' \in E,\ bc \in H,\ b'c' \in E \\ \text{or} & c = c',\ bb' \in E,\ bc \in E,\ b'c' \in H. \end{cases}$$

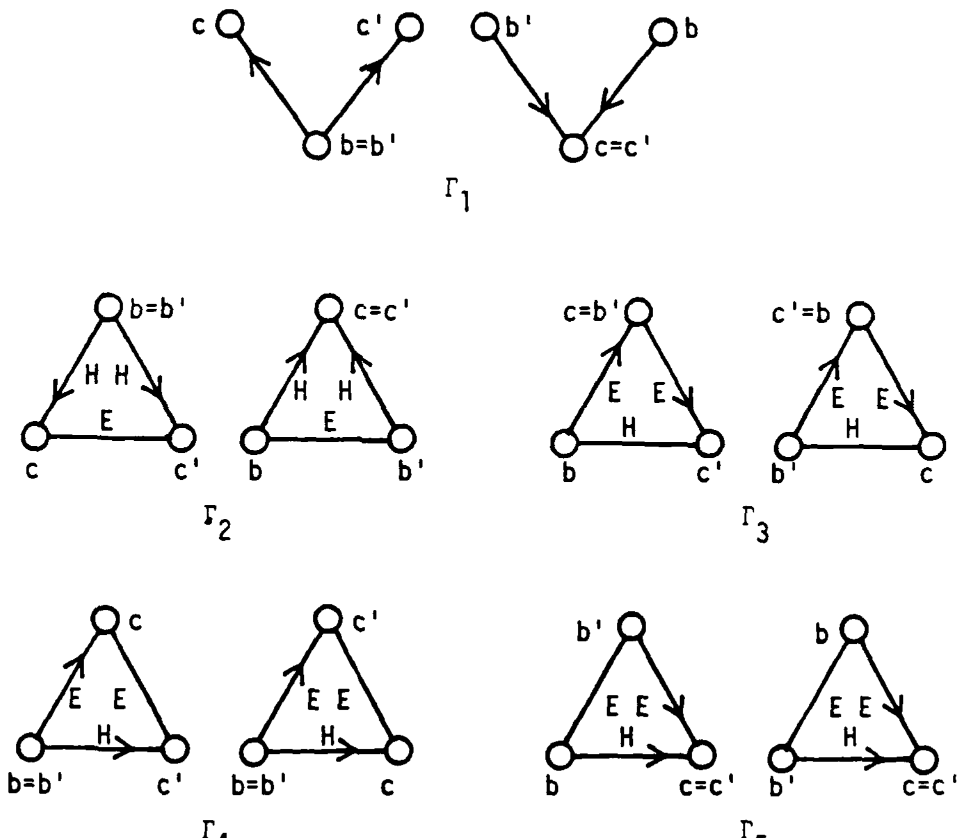


Figure 4.3: Γ_1, Γ_2, Γ_3, Γ_4, Γ_5.

Note that $\Gamma_1, \ldots, \Gamma_5$ are symmetric relations and Γ_1 is reflexive. They each represent a forcing of the orientations of edges in the sense that, if G has a STRO $(V, O+F)$ and $bc \in O+F$, then $b'c'$ must also be in $O+F$. In brief,

Γ_1 - forcing is required for $O+F$ to be transitive,

Γ_2 - forcing is required for F to be transitive, and

$\Gamma_3, \Gamma_4, \Gamma_5$ - forcings are required to ensure that $OF \subset F$ and $FO \subset F$.

The reflexive transitive closure Γ^* of $\Gamma_1, \ldots, \Gamma_5$ is an equivalence relation on $E+H$, partitioning $E+H$ into what we shall call <u>strong implication classes</u> of $E+H$. Thus $ab\Gamma^*a'b'$ iff, for some m,

$$ab = a_1b_1\Gamma_{i_1}a_2b_2\Gamma_{i_2} \cdots \Gamma_{i_{m-1}} a_mb_m = a'b'$$

where $i_1, \ldots, i_{m-1} \in \{1, 2, 3, 4, 5\}$. Such a sequence is called a <u>Γ-chain</u> from ab to $a'b'$. Golumbic defines "implication classes" similarly, but uses only Γ_1 since he does not consider partitioned graphs. Each strong implication class is a union of some of Golumbic's implication classes. What is important here is that virtually all of his results carry through when "implication class" is replaced by "strong implication class".

A fundamental result is the following lemma.

<u>Strong Triangle Lemma (ST Lemma)</u>

Let α, β, γ be strong implication classes of a partitioned graph $G = (V, E+H)$, $\alpha \neq \beta$, $\alpha \neq \gamma^{-1}$, having edges $ab \in \gamma$, $ac \in \beta$, $bc \in \alpha$.

Then

(a) ab, ac, and bc are all in E, all in H, or $ab \in H$, $ac \in H$, $bc \in E$, in which case $\beta = \gamma$;

(b) if $b'c' \in \alpha$, then $ab' \in \gamma$ and $ac' \in \beta$ (see Figure 4.4).

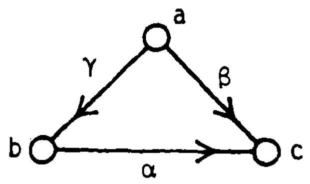


Figure 4.4: ST Lemma.

Proof: (a): The proof consists of checking all possible $2^3 = 8$ ways that the three edges of the triangle could be in E or H. The only ones that don't contradict $\alpha \neq \beta$ and $\alpha \neq \gamma^{-1}$ are when all are in E, all are in H, or $ab \in H$, $ac \in H$, $bc \in E$, in which case $ab\Gamma_2 ac$, so $\beta \cap \gamma \neq \emptyset$ and hence $\beta = \gamma$.

For example, we cannot have $ab \in H$, $ac \in E$, $bc \in H$, because then $bc\Gamma_2 ba$ and so $\alpha \cap \gamma^{-1} \neq \emptyset$, which means $\alpha = \gamma^{-1}$, a contradiction.

(b): If $b'c' \in \alpha$, then by the definition of a strong implication class, there exists a Γ-chain

$$bc = b_1c_1\Gamma_{i_1}b_2c_2\Gamma_{i_2} \cdots \Gamma_{i_{m-1}}b_mc_m = b'c'.$$

The proof proceeds by induction on m. It suffices to consider the case in which $m = 2$, i.e., the case where $bc\Gamma_i b'c'$ for $i = 1, 2, 3, 4$, or 5. We consider each of these possibilities separately.

CASE 1: $bc\Gamma_1 b'c'$. Then $ab' \in \gamma$ and $ac' \in \beta$ by Golumbic's Triangle lemma [13, p. 71].

CASE 2(a): $bc\Gamma_2 b'c'$, $b = b'$ (see Figure 4.5).

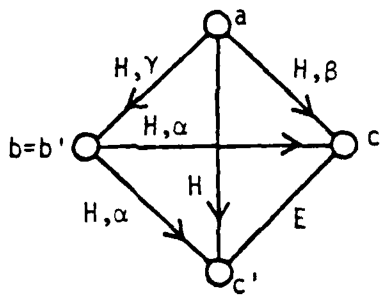


Figure 4.5: Case 2(a).

Then $bc, b'c' \in H$, and $cc' \in E$ *by the definition of* Γ_2 and so $ab, ac \in H$ by part (a). If $ac' \notin E+H$, then $ba\Gamma_1 b'c'$ and therefore $\gamma^{-1} \cap \alpha \neq \emptyset$ (=><=). If $ac' \in E$, then $b'c'\Gamma_2 ba$, so $\gamma^{-1} \cap \alpha \neq \emptyset$ (=><=). Thus $ac' \in H$ and therefore $ac\Gamma_2 ac'$ which means that $ac' \in \beta$. Also $ab' = ab \in \gamma$.

CASE 2(b): $bc\Gamma_2 b'c'$, $c = c'$ (see Figure 4.6).

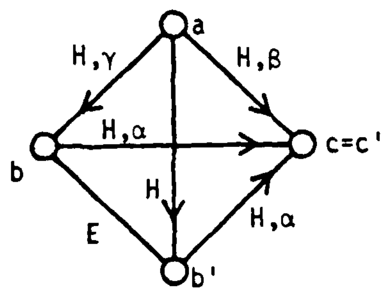


Figure 4.6: Case 2(b).

Then as in Case 2(a), $\{ab, ac, bc, b'c'\} \subset H$, $b'b \in E$, and $ab' \in H$. Hence $ab\Gamma_2 ab'$ and therefore $ab' \in \gamma$ and $ac' = ac \in \beta$.

<u>CASE</u> <u>3(a)</u>: $bc\Gamma_3 b'c'$, $c = b'$ (see Figure 4.7).

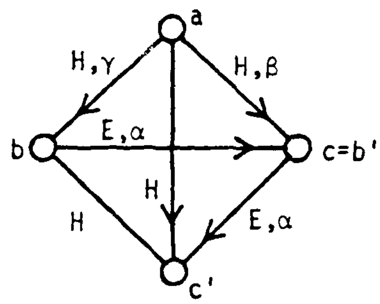


Figure 4.7: Case 3(a).

If $ac' \notin E+H$, then $bc\Gamma_4 bc'$ and $bc'\Gamma_1 ba$, so $\alpha \cap \gamma^{-1} \neq \emptyset$ (=><=). Thus $ac' \in E+H$. We <u>claim</u>: $ab \in H$, $ac \in H$. If this is not true, then by part (a) of the lemma, $ab \in E$, $ac \in E$. Now if $ac' \in E$, then $bc'\Gamma_4 ba$, so $\alpha \cap \gamma^{-1} \neq \emptyset$ (=><=).

If $ac' \in H$, then $ac\Gamma_3 b'c'$, so $\beta \cap \alpha \neq \emptyset$ (=><=). Thus our claim must be true. If $ac' \in E$, then $bc'\Gamma_2 ba$, so $\alpha \cap \gamma^{-1} \neq \emptyset$ (=><=). Thus $ac' \in H$ and so $ac\Gamma_2 ac'$, and therefore $ac' \in \beta$. Also $ab\Gamma_2 ab'$, so $ab' \in \gamma$.

<u>CASE</u> <u>3(b)</u>: $bc\Gamma_3 b'c'$, $c' = b$ (see Figure 4.8).

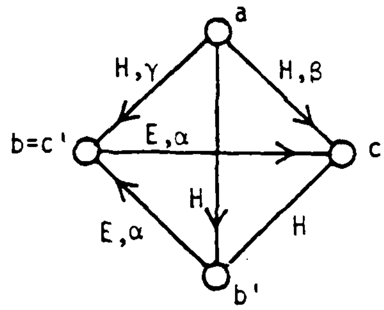


Figure 4.8: Case 3(b).

As in Case 3(a), $ab \in H$, $ac \in H$, and $ab' \in E+H$. If $ab' \in E$, then $bc\Gamma_5 b'c$ and $b'c\Gamma_2 ac$, so $\alpha \cap \beta \neq \emptyset$ (=><=). Hence $ab' \in H$ and so $ab\Gamma_2 ab'$ which gives $ab' \in \gamma$. Also $ac\Gamma_2 ab = ac'$, so $ac' \in \beta$.

<u>CASE 4(a)</u>: $bc\Gamma_4 b'c'$, $b' = b$ (see Figure 4.9)

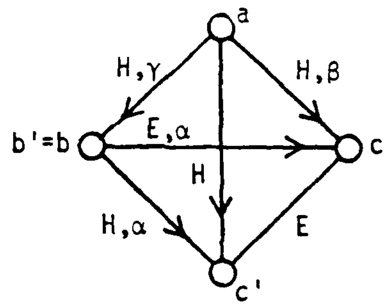


Figure 4.9: Case 4(a).

and $b'c' \in H$, $bc \in E$, $cc' \in E$. As in Case 3(a), $ac' \in H$, $ab \in H$, and $ac \in H$. But $ac\Gamma_2 ac'$, and so $ac' \in \beta$. Also

$ab' = ab \in \gamma$.

<u>CASE</u> <u>4(b)</u>: $bc\Gamma_4 b'c'$, $b' = b$ (see Figure 4.10) and $b'c' \in E$, $bc \in H$, $cc' \in E$.

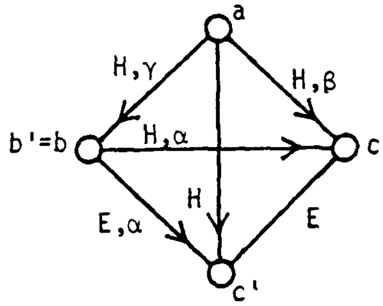


Figure 4.10: Case 4(b).

If $ac \notin E+H$, then $b'c'\Gamma_1 ba$, and so $\alpha \cap \gamma^{-1} \neq \emptyset$ (=><=). If $ac \in E$, then $b'c'\Gamma_1 ba$, so $\alpha \cap \gamma^{-1} \neq \emptyset$ (=><=). Hence $ac \in H$ and therefore $ac\Gamma_2 ac'$ which means $ac' \in \beta$. Also $ab' = ab \in \gamma$.

<u>CASE</u> <u>5(a)</u>: $bc\Gamma_5 b'c'$, $c = c'$ (see Figure 4.11) and $bc \in E$, $b'c' \in H$, $b'b \in E$.

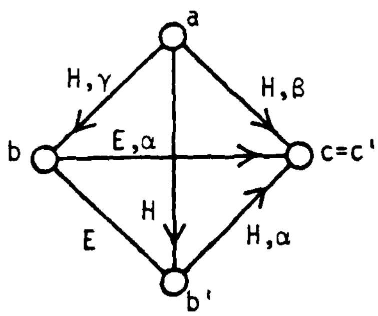


Figure 4.11: Case 5(a).

If $ab' \notin E+H$, then $ac\Gamma_1 b'c'$ and so $\beta \cap \alpha \neq \emptyset$ (=><=). Now, as in Case 3, $ab \in H$, $ac \in H$. If $ab' \in E$, then $b'c'\Gamma_2 ac$, so $\alpha \cap \beta \neq \emptyset$ (=><=). Thus $ab' \in H$ and so $ab\Gamma_2 ab'$ which means $ab' \in \gamma$. Also $ac' = ac \in \beta$.

<u>CASE 5(b)</u>: $bc\Gamma_5 b'c'$, $c = c'$ (see Figure 4.12)

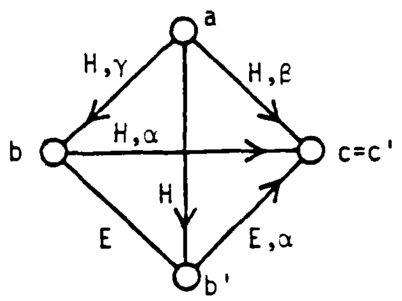


Figure 4.12: Case 5(b).

and $bc \in H$, $b'c' \in E$, $bb' \in E$. Then $ab \in H$, $ac \in H$ by part (a) of the lemma. If $ab' \notin E+H$, then $ac\Gamma_1 b'c'$, so $\beta \cap \alpha \neq \emptyset$ (=><=). If $ab' \in E$, then $ac\Gamma_5 b'c'$, so $\beta \cap \alpha \neq \emptyset$ (=><=). Thus $ab' \in H$ and so $ab\Gamma_2 ab'$ which means $ab' \in \gamma$. Also $ac' = ac \in \beta$. This completes the proof of the ST Lemma. □

<u>Corollary ST</u>

Let α, β, γ be strong implication classes of a partitioned graph $G = (V, E+H)$, $\alpha \neq \beta$, $\alpha \neq \gamma^{-1}$, $\beta \neq \gamma$, having edges $ab \in \gamma$, $ac \in \beta$, $bc \in \alpha$. Then

(a) ab, ac, bc are all in H or all in E; and

(b) if $b'c' \in \alpha$ and $a'b' \in \gamma$, then $a'c' \in \beta$.

Proof: (a) Since $\beta \neq \gamma$, part (a) of the ST Lemma shows that all three edges are in H or all are in E.

(b) We apply the ST Lemma using $b'c' \in \alpha$ to get $ac' \in \beta$, $ab' \in \gamma$. We apply it again to the triangle in Figure 4.13, using $b'a' \in \gamma^{-1}$ to get $c'a' \in \beta^{-1}$, i.e., $a'c' \in \beta$. □

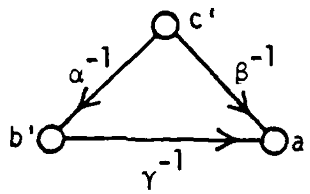


Figure 4.13: Corollary ST.

We need one more fundamental result.

Theorem 4.7

For any union U of strong implication classes of a partitioned graph $G = (V, E+H)$, if U is transitive, then U is also strongly transitive.

Proof: Let O_u and F_u denote the edges of U in E and in H, respectively. We are assuming $(O_u + F_u)^2 \subset O_u + F_u$. All that remains to be shown are $F_u^2 \subset F_u$, $O_u F_u \subset F_u$, and $F_u O_u \subset F_u$.

(a) Let ab, $bc \in F_u$. Then $ac \in O_u + F_u$, by transitivity. If $ac \in O_u$, then $ab\Gamma_2 cb$, so $cb \in F_u$. But then $bc \in F_u$ and $cb \in F_u$ which contradicts the transitivity of $O_u + F_u$. Hence $ac \in F_u$.

(b) Let $ab \in F_u$, $bc \in O_u$. If $ac \in O_u$ then $ac\Gamma_3 cb$, so $cb \in O_u$ and $bc \in O_u$, a contradiction. Hence $ac \in F_u$.

(c) Let $ab \in O_u$, $bc \in F_u$. If $ac \in O_u$, then $ac\Gamma_3 ba$, so $ba \in O_u$ and $ab \in O_u$, a contradiction. Hence $ac \in F_u$. This completes the proof. □

Because of the ST Lemma and Theorem 4.7, virtually all of the results by Golumbic [13] carry over to strongly transitive orientations. The proofs apply almost verbatim upon replacing "implication classes" by "strong implication classes", "transitive orientation" by "strongly transitive orientation", and "E" by "E+H". Thus most such proofs will be omitted here. Basically, Golumbic's methods suffice to prove transitivity for our case, and then Theorem 4.7 gives strong transitivity.

Theorem G.1 (This theorem corresponds to Golumbic's Theorem 1.) If O+F is a STRO of a partitioned graph (V, E+H) and α is a strong implication class of E+H, then $\alpha \cap \alpha^{-1} = \emptyset$ and either $(O+F) \cap \hat{\alpha} = \alpha$ or $(O+F) \cap \hat{\alpha} = \alpha^{-1}$.

Theorem G.2 (This theorem corresponds to Golumbic's Theorem 2.) Let α be a strong implication class of a partitioned graph (V, E+H). Then either

(I) $\alpha = \hat{\alpha} = \alpha^{-1}$, or

(II) $\alpha \cap \alpha^{-1} = \emptyset$, α and α^{-1} are strongly transitive, and they are the only STRO's of $\hat{\alpha}$.

Sketch of proof of Theorem G.2: Using the ST Lemma in Golumbic's proof of his Theorem 2, we obtain the result that α and α^{-1} are transitive, if $\alpha \cap \alpha^{-1} = \emptyset$. Then by Theorem 4.7, α and α^{-1} are strongly transitive. $\square$

A complete undirected subgraph (V_S, S) on $m+1$ vertices of a partitioned graph $G = (V, E+H)$ will be called a simplex of dimension m if each undirected edge $\overline{ab}$ of S is contained in a different member of $\{\hat{\alpha}: \alpha$ is a strong implication class of $E+H\}$. Note that $S \subset E$ or $S \subset H$ by the Corollary ST.

A multiplex of dimension m generated by a simplex S of dimension m is the subgraph (V_M, M) of G defined by

$$M = \{v_i v_j : v_i v_j \Gamma^* xy \text{ for some } xy \in S\}$$
$$= \cup \hat{\alpha}, \text{ the union being over all strong implication classes } \alpha \text{ such that } \alpha \cap S \neq \emptyset.$$

Theorem G.3 (This theorem corresponds to Cor. 8, Thm. 9, and Thm. 10 of Golumbic.)

1. Let $M_1 \subseteq M_2$ be multiplexes.
   (a) Every simplex generating M_1 is contained in a simplex generating M_2.
   (b) Every simplex generating M_2 contains a subsimplex which generates M_1.

2. If M is a multiplex generated by a simplex S, then M is a maximal multiplex iff S is a maximal simplex.

3. Two maximal multiplexes are equal or have disjoint edge sets.

<u>Theorem G.4</u> (This theorem corresponds to Golumbic's Thms. 12 and 13.)

1. Any maximal multiplex M of dimension m which has a STRO, has $(m+1)!$ STRO's.

2. Let $E+H = M_1 + \cdots + M_k$ be a partition of $E+H$ into maximal multiplexes. If $(V, E+H)$ has a STRO, then the number of STRO's that $(V, E+H)$ has is $\Pi_{i=1}^{k}(m_i + 1)!$ where m_i = dimension of M_i.

<u>Sketch of proof of Theorem G.4</u>: Golumbic shows that any of the $(m+1)!$ transitive orientations of a maximal simplex in M extends (via Γ^*) to a transitive orientation of M. Then Theorem 4.7 shows that it extends to a STRO of M.

Golumbic's Theorem 13 shows that any one of the possible orientations can be chosen for each M_i and the sum of all these oriented multiplexes gives a transitive orientation to $(V, E+H)$. Theorem 4.7 again gives us the result that it is actually a STRO. □

Let $(V, E+H)$ be a partitioned graph.

$$E+H = \bar{\beta}_1 + \bar{\beta}_2 + \cdots + \bar{\beta}_k$$

is called a <u>strong decomposition</u> of G if β_i is a strong implication class of $\bar{\beta}_i + \cdots + \bar{\beta}_k$ for $i = 1,\ldots, k$.

Theorem G.5 (STRO Theorem; this theorem corresponds to Golumbic's Thm. 17.)

Let $(V, E+H)$ be a partitioned graph with strong decomposition $E+H = \bar{\beta}_1 + \cdots + \bar{\beta}_k$. The following are equivalent:

(a) $(V, E+H)$ has a STRO.

(b) $\alpha \cap \alpha^{-1} = \emptyset$ for all strong implication classes α of $E+H$.

(c) $\beta_i \cap \beta_i^{-1} = \emptyset$ for $i = 1, \ldots, k$.

Furthermore, when these hold, $\beta_1 + \cdots + \beta_k$ is a STRO of $E+H$.

Again, Golumbic's theorem, with the appropriate changes made, proves the transitivity and Theorem 4.7 then proves the strong transitivity.

The rest of Golumbic's results in [13] also apply to STRO's, but they are not particularly important to the present discussion, so will not be further explored.

Now, similar to the algorithm constructed by Golumbic, Theorems G.4 and G.5 lead to an $O(\delta \cdot |E+H|)$ - time and $O(|E+H| + |V|)$ - space algorithm for determining if a partitioned graph $(V, E+H)$ has a STRO and constructing one if possible. Here δ is the maximum degree of the vertices.

See Golumbic [14] for a parallel presentation to the following algorithm. Our algorithm uses the function

$$\text{CLASS}(i, j) = \begin{cases} 0 & \text{if } v_iv_j \notin E+H \\ k & \text{if } v_iv_j \text{ has been assigned to } \beta_k \\ -k & \text{if } v_iv_j \text{ has been assigned to } \beta_k^{-1} \\ \text{undefined} & \text{if } v_iv_j \text{ has not yet been assigned.} \end{cases}$$

Input: A partitioned graph $G = (V, E+H)$ in the form of adjacency sets $ADJ(i) = \{j: v_i v_j \in E+H\}$.

Output: A strong decomposition of G given by the function CLASS and a variable FLAG which is 0 if G has any STRO's and 1 otherwise. If FLAG is 0, a STRO is given by those edges whose CLASS is positive.

Method: Initially FLAG $\leftarrow$ 0. By the kth iteration, $\beta_1 + \cdots + \beta_{k-1}$ has been determined, and FLAG has been changed to 1 if $\beta_i \cap \beta_i^{-1} \neq \emptyset$ for any $i = 1, \ldots, k-1$. In the kth iteration, an unexplored edge e_k is chosen. The recursive calls of EXPLORE and FORCE cause the exploration of the whole strong implication class of e_k in $E+H\backslash\beta_1\backslash\cdots\backslash\beta_{k-1}$. (This is due to the fact that we ignore edges whose CLASS value is between $-k$ and k.) This yields β_k, and if $\beta_k \cap \beta_k^{-1} \neq \emptyset$, FLAG is set to 1.

Algorithm 4.8

```
begin
   initialize:  k ← 0;  FLAG ← 0;
   for each edge  vivj ∈ E+H  do
      if CLASS (i, j)   is undefined then
         begin
            k ← k+1;
            CLASS (i, j) ← k;   CLASS (j, i) ← -k;
            EXPLORE (i, j)
         end;
```

    <u>end</u>

<u>end</u>

<u>Procedure</u> EXPLORE (i, j):

<u>for</u> each $m \in ADJ(i)$ such that $[m \notin ADJ(j)$ or $|CLASS(j, m)| < k]$ <u>do</u>

        FORCE (i, m);

<u>for</u> each $m \in ADJ(j)$ such that $[m \notin ADJ(i)$ or $|CLASS(i, m)| < k]$

        <u>do</u> FORCE (m, j);

<u>for</u> each m such that v_jv_m, $v_iv_j \in H$, $v_iv_m \in E$ <u>do</u> FORCE (m, j);

<u>for</u> each m such that v_iv_j, $v_iv_m \in H$, $v_mv_j \in E$ <u>do</u> FORCE (i, m);

<u>for</u> each m such that v_iv_j, $v_jv_m \in E$, $v_iv_m \in H$ <u>do</u>

    <u>begin</u>

        FORCE (j, m); FORCE (i, m)

    <u>end</u>;

<u>for</u> each m such that v_iv_j, $v_iv_m \in E$, $v_jv_m \in H$ <u>do</u>

    <u>begin</u>

        FORCE (m, i); FORCE (m, j)

    <u>end</u>;

<u>for</u> each m such that $v_iv_j \in H$, $v_iv_m \in E$, $v_jv_m \in E$ <u>do</u>

    <u>begin</u>

        FORCE (i, m); FORCE (m, j)

    <u>end</u>

<u>return</u>

```
Procedure FORCE (i, j):
if CLASS (i, j)  is undefined then
   begin
      CLASS (i, j) ← k;  CLASS (j, i) ← -k;
      EXPLORE (i, j)
   end
else
   if CLASS (i, j) = -k  then
      begin
         CLASS (i, j) ← k;  FLAG ← 1;
         EXPLORE (i, j)
      end
return
```

We store the adjacency sets as linked lists as described by Golumbic, but use five fields for each element of list ADJ(i) representing edge $v_i v_j$, containing respectively, j, a field indicating whether $v_i v_j \in E$ or $v_i v_j \in H$, CLASS (i, j), pointer to CLASS (j, i), and a pointer to the next element on ADJ (i). Then the storage requirements are $O(|V| + |E+H|)$ and Golumbic's analysis shows that the algorithm takes $O(\delta \cdot |E+H|)$ steps.

As Golumbic mentions, Theorems G.3 and G.4 can be used to count the number of STRO's a graph has. All we need do is make a local search of edges by picking an edge at random and building larger simplices, each containing its predecessor, until we have a maximal simplex. This simplex will generate a maximal multiplex.

By methods very similar to Algorithm 4.8, the enumeration can be done in $O(\delta \cdot |E+H|)$ steps using $O(|V| + |E+H|)$ storage spaces. The details are left as an exercise for the reader.

4.2 A Generalization

Problem 4.4 presents a complete undirected graph whose edges have been partitioned into three sets: $\bar{C}$, E, and $H = V\dot{\times}V \backslash E \backslash \bar{C}$, one of which (C) has been oriented. It asks for orientations of E and H with certain properties.

This view leads naturally to the question: What if one of the sets E or H is initially oriented instead of $\bar{C}$? Or more generally, what if some of the edges of $\bar{C}$, some of E, and some of H are oriented? When can these oriented sets of edges be extended to orientations of $\bar{C}$, E, and H with the desired properties?

For the rest of this chapter, E will not necessarily be an undirected set of edges, as was previously the case.

Problem 4.9

Instance: Relations M, E, H such that $\bar{M}+\bar{E}+\bar{H} = V\dot{\times}V$.

Question: Do there exist orientations C, O, and F of M, E, and H, respectively, so that $(V, C+O+F) \in F_G^{COF}(I(G))$ where $G = (V, \bar{M}+\bar{E})$?

To solve this problem, it is helpful to define:

$C^* = \{v_iv_j \in M\colon\ v_jv_k \in E+M$ and $v_iv_k \in H,$ for some $v_k\},$

$M' = \{v_iv_j \in M;\ v_jv_i \notin M\},$

$E' = \{v_iv_j \in E\colon\ v_jv_i \notin E\},$

$H' = \{v_iv_j \in H\colon\ v_jv_i \notin H\}.$

Theorem 4.10

There exist orientation C, O, F as desired in Problem 4.9 iff

(a) $M' \cup C^*$ can be extended to a transitive orientation C of M, and

(b) $E' + H'$ can be extended to a strongly transitive orientation $O+F$ of $E+H$.

Proof: (<=) Requiring $C^* \subset C$ ensures that conditions (j), (k), and (ℓ) hold in Theorem 4.2. The other conditions (e)-(i) hold by the transitivity of C and strong transitivity of $O+F$. Thus by Theorems 4.1 and 4.2, C, O, and F are the desired orientations.

(=>) If $(V, C+O+F) \in F_G^{COF}(I)$, for some $I = \{I_1, \ldots, I_n\} \in \mathcal{I}(G)$, then by Theorems 4.1 and 4.2, C is transitive and $O+F$ is strongly transitive. Furthermore, $E' \subset O$, $H' \subset F$, and $M' \subset C$. To show that $C^* \subset C$, we assume this is not so. Then there exist v_i, v_j, v_k such that $v_jv_i \in C$, $v_jv_k \in E+M$ and $v_iv_k \in H$. But then $I_j \subset I_i$, $I_k \cap I_j \neq \emptyset$, and $I_k \cap I_i = \emptyset$. Clearly this cannot happen and so $C^* \subset C$. This completes the proof. □

Theorem 4.10 has now reduced the solution of Problem 4.7 to the solution of the following two problems.

Problem 4.11

Instance: Mixed graph $G = (V, E)$.

Question: Does there exist a transitive orientation T of G?

Problem 4.12

Instance: Graph $G = (V, E+H)$, with $\bar{E} \cap \bar{H} = \emptyset$.

Question: Do there exist orientations O of E and F of H such that $(V, O+F)$ is a strongly transitive orientation of $(V, \bar{E}+\bar{H})$?

These problems have nearly identical solutions. Let

$$\alpha(E') = \{v_iv_j \in \bar{E}: \ v_iv_j\Gamma^*xy \text{ for some } xy \in E'\},$$

$$\alpha(E' + H') = \{v_iv_j \in \bar{E}+\bar{H}: \ v_iv_j\Gamma^*xy \text{ for some } xy \in E' + H'\},$$

where in the first set, Γ^* is the transitive closure of Γ_1, whereas in the second set, Γ^* is the transitive closure of Γ_1, Γ_2, Γ_3, Γ_4, and Γ_5, and where E' and H' are defined prior to Theorem 4.10.

Theorem 4.13

There is a relation T as desired in Problem 4.11 iff

(a) $(V, \bar{E})$ is transitively orientable, and

(b) $\alpha(E')$ is acyclic.

Theorem 4.14

There are relations O and F as desired in Problem 4.12 iff

(a) $(V, \bar{E}+\bar{H})$ is strongly transitively orientable, and

(b) $\alpha(E' + H')$ is acyclic.

Proof of Theorem 4.13: Condition (a) is clearly necessary, and (b) is necessary because, if $v_i v_j \Gamma^* xy$ for some $xy \in E'$, then $v_i v_j$ must be in the transitive orientation T as well as xy.

To prove the sufficiency of (a) and (b), we choose a maximal simplex $S_M = (V_M, E_M)$ from each maximal multiplex M of $(V, \bar{E})$. By Theorem G.4, the $m+1$ vertices of a maximal simplex of dimension m can be linearly ordered in an arbitrary manner, and, if this is done for each maximal multiplex, we can extend these orders via Γ^* to form a transitive orientation of $(V, \bar{E})$. Hence if $\alpha(E')$ contains no cycles, $\alpha(E') \cap E_M$ can be extended to form a linear order of V_M for each S_M, which in turn can be extended via Γ^* to an orientation of M. These extensions give the desired orientation of E. □

The proof of Theorem 4.14 is almost the same.

Parts (a) and (b) in Theorems 4.13 and 4.14 can be checked in time $O(\delta \cdot |E|)$ and $O(\delta \cdot |E+H|)$ respectively. One algorithm for Theorem 4.13 consists of first extending E' to $\alpha(E')$. Then $(V, \alpha(E'))$ is checked for cycles. (Topological sorting algorithms can perform this step in time $O(|\alpha(E')| + |V|)$.) The remaining implication classes can then be determined and checked to see if $(V, \bar{E})$ is transitively orientable. If so, we can construct the desired orientation by choosing a maximal simplex $S_M = (V_M, E_M)$ from each maximal multiplex M, and extending $\alpha(E') \cap E_M$ to a linear order of V_M (again this can be done by a topological sorting procedure). These orientations can then be extended to the whole graph. The task of producing a detailed algorithm is left as an exercise for the reader. Minor modifications of this algorithm will yield an algorithm for Theorem 4.14 also.

In the case of Problems 4.9, 4.11, and 4.12, in contrast to Problem 4.4, it is not easy to determine the number of solutions. This is due to the fact that, in these cases, we are extending an acyclic set of edges to a linear ordering in each maximal simplex. Thus to determine the number of solutions, it is necessary and sufficient to have an efficient algorithm for determining how many linear extensions a partial order has. No such algorithm is known by the author. Knuth and Szwarcfiter [22] have constructed an algorithm which determines all such total orders, and which is linear in each output, but this is unfortunately not the algorithm we desire.

CHAPTER 5: APPLICATIONS TO SPECIAL GRAPHS

The results in Chapter 4 lead to interesting characterizations of several types of graphs including proper interval graphs and proper circular arc graphs.

5.1 Proper Interval Graphs

The following theorem is related to the special case of Theorem 4.5 where $C = \emptyset$.

Theorem 5.1

Let $G = (V, E)$ be an (undirected) graph. The following are equivalent:

(a) G is a proper interval graph;

(b) There exist orientations 0 of E and F of E^c such that:

(i) $0F \subset F$,

(ii) $F0 \subset F$, and

(iii) $(0+F)^2 \subset 0+F$;

(c) There exists an acyclic orientation 0 of E such that

(*) $\quad ab, bc \in E$ and $ac \in E^c \Rightarrow ab, bc \in 0$ or $cb, ba \in 0$

(see Figure 5.1).

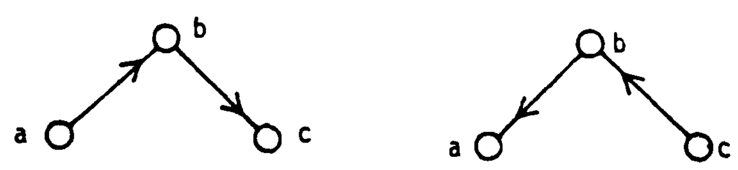


Figure 5.1: Property (*).

Proof: We show (a)=>(b)=>(c)=>(a).

(a)=>(b): Let $I = \{I_1, \ldots, I_n\} \in I(G)$ be a proper interval representation of G in which $I_i = [a_i, b_i]$, $i = 1, \ldots, n$. Construct orientations O and F as follows. For all i and j, let

$$v_i v_j \in O \text{ if } a_i < a_j < b_i < b_j \quad \text{and}$$

$$v_i v_j \in F \text{ if } a_i < b_i < a_j < b_j.$$

If $v_i v_j \in O$ and $v_j v_k \in F$, then $a_i < a_j < b_i < b_j < a_k < b_k$ so $v_i v_k \in F$. This proves (i). Properties (ii) and (iii) are proved similarly.

(b)=>(c): Given orientations O and F as in (b), we claim that O has property (*) in (c) and is acyclic. By (iii) O is acyclic. Now let $ab, bc \in E$ and $ac \in E^c$. If $ac \in F$, then by (i) and (ii), we must have $bc \in O$ and $ab \in O$. Similarly, if $ca \in F$, then $cb, ba \in O$.

(c)=>(a): We show that G can have no induced subgraphs which are $K_{1,3}$, IV_2, V_1, or III_n, $n \geq 4$, (see Figures 1.1 and 1.4) and therefore by Theorem 1.4, G is a proper interval graph. If $K_{1,3}$ is an induced subgraph of G (see Figure 5.2), then without loss of generality $ab \in O$. Hence by property (*), $bc \in O$. But $ab \in O \Rightarrow bd \in O$, and $bc \in O \Rightarrow db \in O$, a contradiction. So $K_{1,3}$ cannot be an induced subgraph of G. Similarly, we can easily show that IV_2, V_1 and III_n, $n \geq 4$, cannot be induced subgraphs of G. □

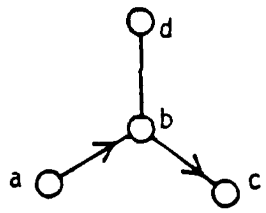


Figure 5.2: Orienting $K_{1,3}$.

A <u>semi-order</u> is a graph (V, P) such that for all $x, y, z, w \in V$:

$$xy \in P \text{ and } zw \in P \Rightarrow xw \in P \text{ or } zy \in P, \text{ and}$$

$$xy \in P \text{ and } yz \in P \Rightarrow xw \in P \text{ or } wz \in P.$$

Note that a semi-order is transitive.

<u>Theorem 5.2</u>

Let F be a transitive relation on a finite set V. Then (V, F) is a semi-order iff F can be extended to a linear order $O+F$ on V such that $OF \subset F$ and $FO \subset F$.

<u>Proof</u>: By a result of Roberts [28, Theorems 3 and 6], an irreflexive relation F on a finite set V forms a semi-order iff there exists a proper interval representation $\{I_1, \ldots, I_n\}$ of $(V, (F)^c)$ such that $v_i v_j \in F \Leftrightarrow \ell_i < r_i < \ell_j < r_j$, where $I_i = [\ell_i, r_i]$, for $i = 1, \ldots, n$.

Given such a representation in $I((V, (\bar{F})^c))$, we define the relation O as follows:

$$v_i v_j \in O \text{ if } \ell_i < \ell_j < r_i < r_j.$$

It is easy to see that $O+F$ is a linear order such that $OF \subset F$ and $FO \subset F$.

Conversely, let $O+F$ be a linear order such that $OF \subset F$ and $FO \subset F$. Since $F^2 \subset F$, Theorem 4.5 states that $D = (V, O+F)$ is associated with a representation of $(V, \bar{O}) = (V, (\bar{F})^c)$, which is the desired proper interval representation. Hence (V, F) is a semi-order. $\square$

Using Theorem 5.1 and the tools developed in Chapter 4, we can construct an algorithm of complexity $O(|V|^3)$ that recognizes proper interval graphs and determines the number of chronological orderings of a graph that consist of proper interval representations (such chronological orderings will be called <u>proper chronological orderings</u>). However, the algorithm described in conjunction with Theorem 1.5 can perform these operations in linear time, so there appears to be little reason to construct an algorithm based on Theorem 5.1. The following discussion shows how the linear-time recognition algorithm can also be used to enumerate the proper chronological orderings of a graph.

Recall that $G = (V, E)$ is a proper interval graph iff its augmented adjacency matrix M has the consecutive ones property. Each consecutive ones form of M gives rise to a unique proper chronological ordering of G. Furthermore, all such chronological orderings arise in this way, so the number of proper chronological orderings of G is equal to the number of consecutive ones forms of M. As described by Booth [4] or Booth and Lueker [5; 6], testing for consecutive ones and counting the number of consecutive ones forms of a matrix can be done in linear time and space, so we obtain the desired result.

Notice that this does not give all chronological orderings of G, because G may well have representations in which one interval is properly contained in another.

It should also be pointed out that the number of orientations O in Theorem 5.1(c) does not correspond to the number of proper chronological orderings of G. This is due to the fact that such orientations say nothing about the order of the connected components of G. However, for connected graphs, it can be shown that the number of such orientations is equal to the number of proper chronological orderings of the graph.

5.2 A Relationship among Some Graphs

Theorems 5.1 and 4.5 lead to an interesting relationship among several types of graphs.

Let $G = (V, E)$ be an undirected graph. G is called a comparability graph if it can be transitively oriented. G is a rigid-circuit graph (or triangulated graph or chordal graph) if it does not contain C_n, $n \geq 4$, as an induced subgraph. G is a nested interval graph if there exists a representation $\{I_1,\ldots, I_n\} \in \mathcal{I}(G)$ such that if $v_iv_j \in E$, then $I_i \subset I_j$ or $I_j \subset I_i$. That is, every edge indicates containment, which makes G a kind of "dual" to a proper interval graph.

Let B_1, B_2, B_3 be the oriented graphs shown in Figure 5.3. G is called a U-graph, where $U \subset \{B_1, B_2, B_3\}$, if it has an acyclic orientation O such that (V, O) does not contain any of the graphs in $\{B_1, B_2, B_3\}\backslash U$ as induced subgraphs. For example, every undirected graph is a $\{B_1, B_2, B_3\}$ - graph, whereas the $\emptyset$-graphs are exactly the complete graphs.

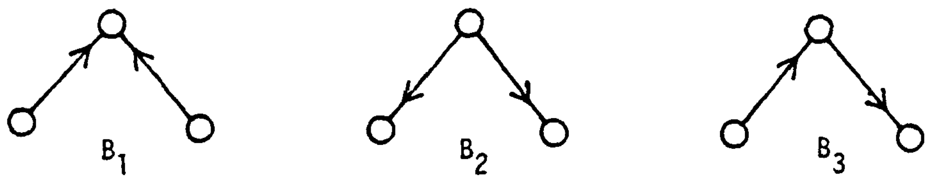


Figure 5.3: B_1, B_2, B_3.

Theorem 5.3

Let $G = (V, E)$ be an undirected graph.

1. G is a $\{B_1, B_3\}$-graph <=> G is a $\{B_2, B_3\}$-graph
   <=> G is a rigid-circuit graph.
2. G is a $\{B_1, B_2\}$-graph <=> G is a comparability graph.
3. G is a $\{B_3\}$-graph <=> G is a proper interval graph.
4. G is a $\{B_1\}$-graph <=> G is a $\{B_2\}$- graph
   <=> G does not contain III_4 or D (see Figure 5.4) as induced subgraphs
   <=> G is a nested interval graph.

III_4 D

Figure 5.4: III_4 and D.

Proof: 1. This is proved by Rose [30] and Kesel'man [20].

2. Any acyclic orientation of G not containing B_3 is clearly transitive. The converse is obvious.

3. This is just the equivalence of (a) and (c) in Theorem 5.1.

4. The first equivalence is obvious.

The second equivalence is the main result of Wolk [36; 37] (see also Jung [17]). For the third equivalence, let $\{I_1, \ldots, I_n\} \in \mathcal{I}(G)$ be a representation of a nested interval graph G in which $v_i v_j \in E \Rightarrow I_i \subset I_j$ or $I_j \subset I_i$. Define an orientation O of E as follows: $v_i v_j \in O$ iff $I_i \subset I_j$. This shows G is a $\{B_1\}$- graph.

Conversely, let G be a $\{B_1\}$- graph. Then by Wolk [37], G^c has a transitive orientation F. Let C be the orientation of E such that (V, C) does not contain B_1 as an induced subgraph. Then this C and F satisfy the conditions in Theorem 4.5 (where the relation $O = \emptyset$), so this theorem gives the desired result that G is a nested interval graph. □

Let $\underline{\omega(G)}$ denote the clique number of a graph G, that is, the size of the largest clique of G. Let $\chi(G)$ denote the chromatic number of G, that is, the minimum number of colors needed to properly color the vertices of G. An undirected graph G is called <u>perfect</u> if $\omega(G') = \chi(G')$ for all induced subgraphs G' of G. See Golumbic [12] for references to many results on perfect graphs.

<u>Corollary 5.4</u>

Every U-graph for which $U \neq \{B_1, B_2, B_3\}$ is perfect.

<u>Proof</u>: Rigid-circuit graphs, comparability graphs, and interval graphs are all perfect. See, for example, Golumbic [12]. □

5.3 Proper Circular Arc Graphs

Can the results of Theorem 5.1 be generalized to circular arc graphs? Equivalence (b) cannot easily be generalized because, if two arcs do not intersect, we cannot say which one is to the "right" or "left" of the other, i.e., there is no relation for arcs on a circle corresponding to the relation F for intervals of the line.

However, property (c) of Theorem 5.1 does generalize, if we restrict our attention to connected graphs. Therefore we will first consider the case where the graph is not connected.

Theorem 5.5

Let G be a disconnected, undirected graph. G is a proper circular arc graph iff G is a proper interval graph.

Proof: (<=) This is obvious. (=>) Any proper circular arc representation of G must miss at least two points of the circle since G is disconnected. We can cut the circle at one of those points and, upon straightening it out, obtain a proper interval representation of G. □

A CORE-cycle of a graph (V, E) is a sequence $[v_1, \ldots, v_m]$ of (not necessarily distinct) vertices in V with the following three properties:

(1) $v_i v_{i+1} \in E$ for $i = 1, \ldots, m-1$, and $v_m v_1 \in E$,

(2) $v_i v_{i+2} \notin E$ for $i = 1, \ldots, m-2$, and $v_{m-1} v_1 \notin E$, $v_m v_2 \notin E$,

(3) $v_i = v_{i+2}$, where $i+2$ is given modulo m, for an odd number of vertices v_i in the sequence.

We remark that a CORE-cycle is not necessarily a cycle or circuit as defined in section 1.1 because the vertices of the sequence need not be distinct.

Theorem 5.6

Let $G = (V, E)$ be a connected graph. The following are equivalent:

(a) G is a proper circular arc graph;

(b) There exists an orientation O of E in which (see Figure 5.1)

$$(*) \quad ab, bc \in E \text{ and } ac \in E^c \Rightarrow ab, bc \in O \text{ or } cb, ba \in O;$$

(c) G has no CORE-cycles;

(d) G does not have LPC_m^* nor $sg1^*$ as induced subgraphs, and G^c does not have EPC_m, OPC_m^*, sg1, sg2, sg3, sg4, sg5 as induced subgraphs (see Figure 5.5).

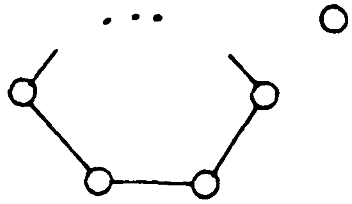
<u>Proof</u>: We will show that (a)=>(b)=>(c)=>(d)=>(a).

(a)=>(b): By a result of Golumbic [12], every proper circular arc graph has a proper circular arc representation in which no two arcs together cover the entire circle. Let $A = \{A_1, \ldots, A_n\}$ be such a representation of G. Furthermore, we can assume that all $2n$ endpoints of the arcs are distinct. Let A_i have counterclockwise endpoint ℓ_i and clockwise endpoint r_i for $i = 1, \ldots, n$.

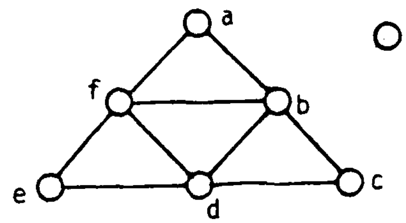
We construct O as follows: If $v_iv_j \in E$, then moving clockwise around the circle, the endpoints of A_i and A_j form the sequence $[\ell_i, \ell_j, r_i, r_j]$ or $[\ell_j, \ell_i, r_j, r_i]$. In the former case, we let $v_iv_j \in O$, and in the latter, we let $v_jv_i \in O$. We do this for all $v_iv_j \in E$. The resulting orientation of G is said to <u>correspond to the representation A</u>. Now we need only show that O has property (*).

Let v_iv_j, $v_jv_k \in E$ and $v_iv_k \in E^c$. If v_iv_j, $v_kv_j \in O$, then $\ell_j \in A_i \cap A_k \neq \emptyset$, contradicting $v_iv_k \in E^c$. If v_jv_i, $v_jv_k \in O$, then $r_j \in A_i \cap A_k \neq \emptyset$, a contradiction. Hence v_iv_j, $v_jv_k \in O$ or v_kv_j, $v_jv_i \in O$.

(b)=>(c): Suppose $C = [v_1, \ldots, v_m]$ is a sequence of vertices in V with properties (1) and (2) in the definition of a CORE-cycle.



LPC_m*
(LPC_m plus isolated vertex)

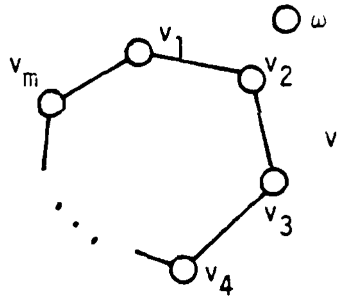


sg1*
(sg1 plus isolated vertex)

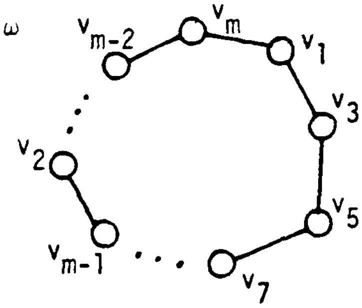
$\underline{LPC_m}$ is a chordless cycle with $m \geq 4$ vertices.

$\underline{OPC_m}$ is a chordless cycle with $m \geq 3$ vertices, m odd.

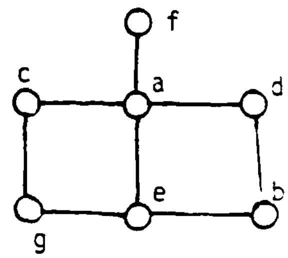
$\underline{EPC_m}$ is a chordless cycle with $m \geq 6$ vertices, m even.



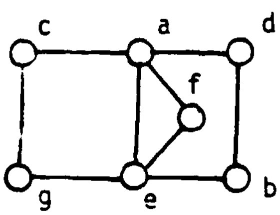
OPC_m*
(OPC_m plus isolated vertex)



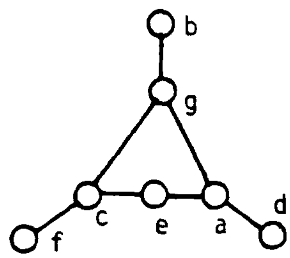
EPC_m



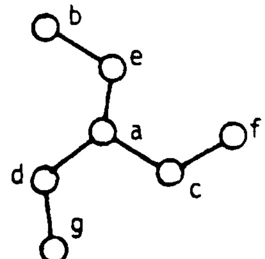
sg2



sg3



sg4



sg5

Figure 5.5: Forbidden subgraphs.

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INTERVAL GRAPHS, CHRONOLOGICAL ORDERINGS, AND RELATED MATTERS.(U)
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Then, in any orientation O of E with property (*), successive edges v_iv_{i+1}, $v_{i+1}v_{i+2}$ in C must be oriented in the same direction (i.e, both $v_iv_{i+1}, v_{i+1}v_{i+2} \in O$ or both $v_{i+2}v_{i+1}, v_{i+1}v_i \in O$), unless $v_i = v_{i+2}$, in which case their orientations are in opposite directions (here i is given modulo m). In other words, the orientation of the edges in the cycle reverses every time a v_{i+1} is encountered for which $v_i = v_{i+2}$. Since we must end up with the same orientation for the edge v_1v_2 with which we started, the cycle can have only an even number of v_i such that $v_i = v_{i+2}$.

(c)=>(d): We will show that each of the forbidden subgraphs of G and the complements of the forbidden subgraphs of G^c contains a CORE-cycle. The labels correspond to those shown in Figure 5.5.

$\underline{OPC_m^*}$: $(OPC_m^*)^c$ contains the CORE-cycle $[v_1, \omega, v_2, \omega, \ldots, v_n, \omega]$.

$\underline{EPC_m}$: $(EPC_m)^c$ contains the CORE-cycle $[v_1, v_2, \ldots, v_{n-1}, v_n, v_2]$.

<u>sg1</u>: (See Figure 5.5, but ignore the isolated vertex.) $(sg1)^c$ contains the CORE-cycle [b,a,b,c,e,f,a,f,d,c,g,a,g,e,d].

<u>sg2</u>: $(sg2)^c$ contains the CORE-cycle [c,b,a,b,f,e,f,g,a,g,d].

<u>sg3</u>: $(sg3)^c$ contains the CORE-cycle [c,b,g,d,g,a,g,f,c,e].

<u>sg4</u>: $(sg4)^c$ contains the CORE-cycle [g,d,c,a,f,g,f,b,c,b,e,b,a,b,d].

<u>sg5</u>: $(sg5)^c$ contains the CORE-cycle [b,a,b,c,e,f,a,f,d,c,g,a,g,e,d].

$\underline{LPC_m^*}$: Let $LPC_m = [v_1, \ldots, v_m]$, $m \geq 4$, and let ω be a vertex that is not adjacent to LPC_m. Since G is connected, there exists a shortest path P from ω to LPC_m. Without loss of generality, $P = [v_1 = \omega_1, \omega_2, \ldots, \omega_k = \omega]$. Clearly we can assume $k = 3$; if $k > 3$, we can replace ω with ω_3. (The vertex ω_3 will be isolated from LPC_m by the minimality of P.)

<u>CASE 1</u>: ω_2 is adjacent to only v_1 in LPC_m. Then $[\omega_2, v_1, v_2, \ldots, v_m, v_1]$ is a CORE-cycle.

<u>CASE 2</u>: ω_2 is adjacent only to v_1 and v_2, or ω_2 is adjacent only to v_1 and v_m. Without loss of generality, ω_2 is adjacent to v_1 and v_2. Then $[\omega, \omega_2, v_2, v_3, \ldots, v_m, v_1, \omega_2]$ is a CORE-cycle.

<u>CASE 3</u>: ω_2 is adjacent to v_1 and some v_i, $i \neq 2$ or m. Then $[\omega, \omega_2, v_1, \omega_2, v_i, \omega_2]$ is a CORE-cycle.

<u>sgl^*</u>: Consider Figure 5.6 below. As before, let $[\omega = \omega_3, \omega_2, \omega_1]$ be the shortest path from ω to sgl (i.e., ω_1 is a vertex of sgl).

<u>CASE 1</u>: ω_2 is not adjacent to a,c or e. Without loss of generality, $\omega_1 = b$. Then $[a,b,\omega_2,b,c,b]$ is a CORE-cycle. Since in all the remaining cases, a, c or e is adjacent to ω_2, we will assume $\omega_1 = a$.

<u>CASE 2</u>: $N(\omega_2) \cap sgl = \{a\}$. Then $[\omega_2,a,f,e,f,b,c,b,a]$ is a CORE-cycle.

<u>CASE 3</u>: $\{a, d\} \subset N(\omega_2)$, $\{a, c\} \subset N(\omega_2)$, or $\{a, e\} \subset N(\omega_2)$. Then $[\omega,\omega_2,a,\omega_2,d,\omega_2]$, $[\omega,\omega_2,a,\omega_2,e,\omega_2]$, or $[\omega,\omega_2,a,\omega_2,c,\omega_2]$ are respective CORE-cycles.

<u>CASE 4</u>: $N(\omega_2) \cap sgl = \{a, b\}$ or $\{a, f\}$. Then $[\omega,\omega_2,b,c,b,f,e,f,a,\omega_2]$ or $[\omega,\omega_2,f,e,f,b,c,b,a,\omega_2]$ are respective CORE-cycles.

<u>CASE 5</u>: $N(\omega_2) \cap sgl = \{a,b,f\}$. Then $[\omega,\omega_2,f,e,f,b,c,b,\omega_2]$ is a CORE-cycle.

(d)=>(a): This is a theorem of Tucker [34, p. 172]. □

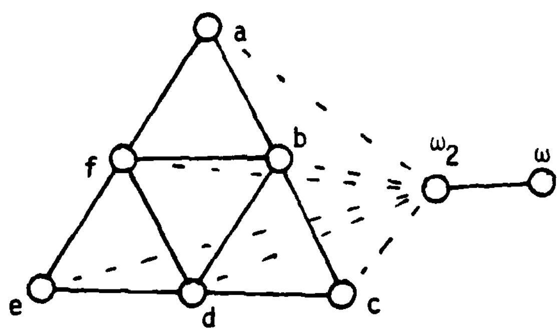


Figure 5.6: The graph sgl*.

The similarities between equivalence (b) in Theorem 5.6 and Golumbic's work [13] on transitive orientations of graphs should be noted. If we attempt to orient a graph so that it has property (*), the orientation of one edge may force neighboring edges into particular orientations. In this way, we obtain equivalence classes of edges, in which the orientation of any one edge forces the orientations of all the other edges in its class. Furthermore, for any such class β, it is true that $\bar{\beta} = \bar{\alpha}$ for some implication class α as defined by Golumbic.

We remark that there appears to be little purpose in using Theorem 5.6 for constructing a recognition algorithm for proper circular arc graphs, due to the fact that a linear-time recognition algorithm already exists (see Booth [4, p. 120]).

Furthermore, equivalence (b) is not very useful for the enumeration of "chronological orderings" of proper circular arc graphs. This is due to the fact that not all orientations O of E with property (*) correspond to a proper circular arc representation of G as defined in the proof of Theorem 5.6 (see Figure 5.7). The following theorem characterizes those orientations which do. A Hamiltonian path (or circuit) in a graph (V, E) is one which contains all the vertices of V.

Theorem 5.7

Let $G = (V, E)$ be a connected graph and let O be an orientation of E. Then G has a proper circular arc representation to which the ordering O corresponds iff (V, O) contains a Hamiltonian path or circuit $P = [v_1, \ldots, v_n]$ in which, for all i, $V_i = \{v \in V: v_i v \in O\}$ consists of consecutive members of P, and for which $|V_i| \le |V_{i+1}| + 1$, where i is given modulo n if P is a circuit.

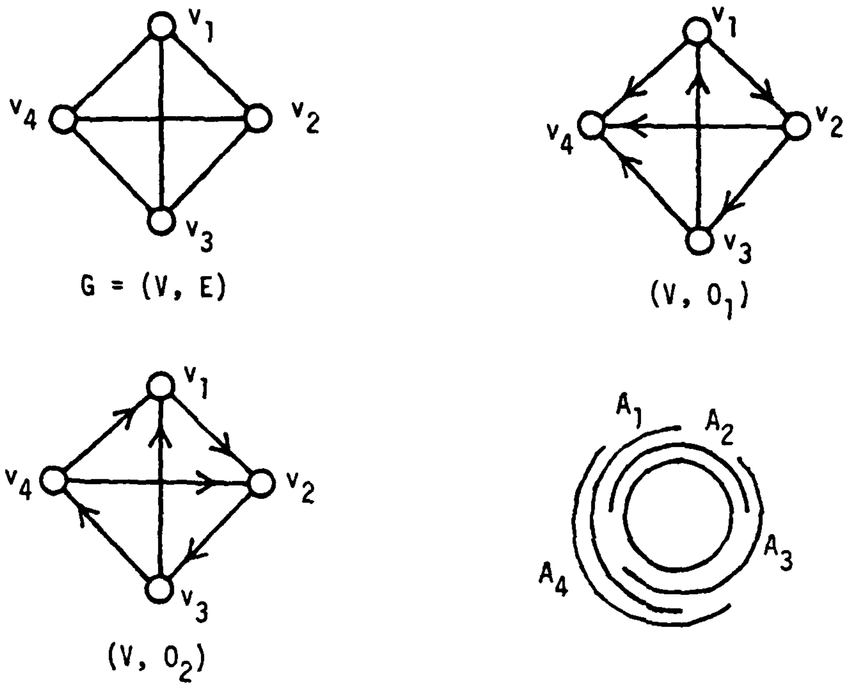


Figure 5.7: An example for Theorem 5.7. The graph (V, 0_1) has no corresponding representation. The graph (V, 0_2) has the corresponding representation in the lower right figure.

<u>Proof</u>: (=>) Let $\{A_1, \ldots, A_n\}$ be a representation of G to which 0 corresponds and let A_i have counterclockwise endpoint ℓ_i. If $[\ell_1, \ell_2, \ldots, \ell_n]$ is the clockwise order of the endpoints of the A_i, then it is easy to see that $P = [v_1, \ldots, v_n]$ (or some cyclic permutation of this) is the desired path or circuit.

(<=) We construct the desired representation as follows. We consider an n-hour clock and initially represent v_i by the clockwise arc A_i starting at i o'clock and ending at $i + |V_i|$ o'clock. However, some arcs may have the same clockwise endpoint. If $\{A_k, A_{k+1}, \ldots, A_{k+s}\}$ (the indices being given modulo n) all end at p o'clock, then we replace A_{k+i} by the clockwise arc A'_{k+i} which starts at $k+i$ o'clock and ends at $p + \frac{i}{s+1}$ o'clock. The set $\{A'_1, \ldots, A'_n\}$ is the desired representation. $\square$

CHAPTER 6: ENUMERATION OF CHRONOLOGICAL ORDERINGS

The preceding chapters have described various ways of determining whether an undirected graph has a representation (or chronological ordering) which satisfies certain restrictions. But in very few cases were we also able to determine <u>how many</u> chronological orderings satisfy those restrictions. The following problem is the main topic of this chapter. We remark here that our discussion will be mainly expository in nature and will contain no major new results.

<u>Problem 6.1</u>

Instance: Interval graph G.

Question: How many chronological orderings does G have?

Fred Roberts [27, pp. 36-37; 26, pp. 118-122] discusses this question and notes that it was only recently determined how many chronological orderings a graph G has that differ in the partial order they induce on G^c. This number can be counted using the tools of Booth and Lueker [5; 6] or Golumbic [12; 13]. The following discussion gives a more complete answer to Problem 6.1, but in a somewhat unsatisfactory way.

Let M be a maximal clique-vertex incidence matrix of G in consecutive ones form. As discussed in the introduction, we can easily construct a representation of G from M. (If the first 1 in column i is in the a_i-th row and the last 1 is in the b_i-th row of M, then we represent vertex v_i by $I_i = [a_i, b_i]$.) Note however that not all endpoints will be distinct. Nevertheless, we can stretch each interval a certain amount to make them distinct, without changing the intersection properties of the intervals. To determine how many different ways we can do this, we let

$$\underline{V_R^M(i)} \quad [\text{resp. } \underline{V_L^M(i)}] = \{v_j \in V: \text{the right [resp. left] endpoint of } I_j \text{ is equal to } i\}.$$

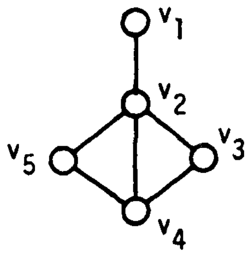
(See Figure 6.1.) We can arbitrarily linearly order the elements in each $V_R^M(i)$ and then stretch the right ends of each interval whose vertex is in $V_R^M(i)$ so that they are ordered along the line in this way. For each $V_R^M(i)$, there are $|V_R^M(i)|!$ ways of doing this. A similar thing can be done for all $V_L^M(i)$. Each resulting representation belongs to a different chronological ordering of G. Thus starting from the matrix M, we can construct representations corresponding to

$$\prod_{i=1}^{m} (|V_R^M(i)|!)(|V_L^M(i)|!)$$

chronological orderings of G, where m is the number of maximal cliques in G.

Furthermore, we can repeat this procedure for every consecutive ones form of the maximal clique-vertex incidence matrix. It is not hard to show that all the chronological orderings represented by modifying one consecutive ones form of the matrix will be distinct from those formed by modifying another consecutive ones form of the matrix, and further, that all possible chronological orderings of G will arise in one of these situations. Thus we have the following:

G



The vertices of the maximal cliques:

A: $\{v_1, v_2\}$

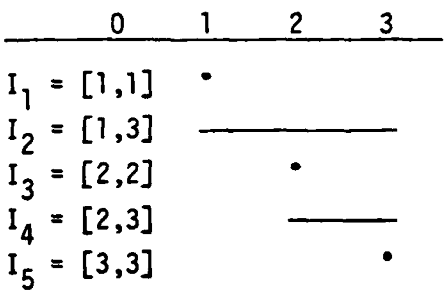
B: $\{v_2, v_3, v_4\}$

C: $\{v_2, v_4, v_5\}$

The matrix M:

$$\begin{array}{c|ccccc|} & v_1 & v_2 & v_3 & v_4 & v_5 \\ \hline A & 1 & 1 & 0 & 0 & 0 \\ B & 0 & 1 & 1 & 1 & 0 \\ C & 0 & 1 & 0 & 1 & 1 \\ \hline \end{array}$$

Representation derived from M:



V_R^M and V_L^M :

$V_R^M(1) = \{v_1\}$ $\quad$ $V_L^M(1) = \{v_1, v_2\}$

$V_R^M(2) = \{v_3\}$ $\quad$ $V_L^M(2) = \{v_4, v_3\}$

$V_R^M(3) = \{v_2, v_5, v_4\}$ $\quad$ $V_L^M(3) = \{v_5\}$

New representation with the endpoints ordered as written in V_R and V_L

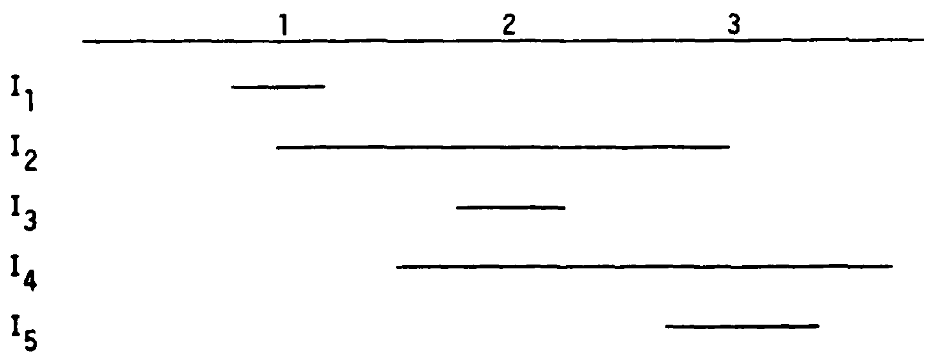


Figure 6.1: An example for Proposition 6.1.

Proposition 6.2

Let $G = (V, E)$ be an interval graph and let $\{M(1), M(2),\ldots, M(\ell)\}$ be the set of consecutive ones forms of the maximal clique-vertex incidence matrix of G. If m is the number of maximal cliques of G, then the number of chonological orderings of G is

$$\sum_{k=1}^{\ell} \prod_{i=1}^{m} |V_R^{M(k)}(i)|! \cdot |V_L^{M(k)}(i)|!.$$

We can state this another way. Let $I(M)$ be the set of all chronological orderings of G for which the matrix is of the form M. Then

$$|I(M)| = \prod_{i=1}^{m} |V_R^{M}(i)|! \cdot |V_L^{M}(i)|!$$

and the number of chronological orderings of G is equal to $\sum|I(M)|$, the sum being over all consecutive ones forms M of the maximal clique-vertex incidence matrix. In the notation of Proposition 6.2, this is $\sum_{k=1}^{\ell} |I(M(k))|$. In other words, the set $\{M(1),\ldots, M(\ell)\}$ partitions the set of chronological orderings of G into $I(M(1))+ \cdots + I(M(\ell))$.

Each consecutive ones form of the matrix corresponds to exactly one transitive orientation of G^c, so we can use these orientations to partition the set of chronological orderings of G and we obtain the same result. Furthermore, with a little reflection, it can be seen that each $V_R(i)$ and $V_L(i)$ corresponds to exactly one of the vertex-disjoint cliques of $(P_n, Pn \dot{x} P_n\backslash\overline{T})$ as described in Lemma

2.4.2 in Chapter 2. (The particular transitive orientation of G^c chosen in Algorithm 2.3 forces the orientation of all the edges in T'. The remaining choices are exactly those described above.)

Thus for each consecutive ones form M of the matrix for G (or for each transitive orientation of G^c), it is easy to count the number of chronological orderings which arise. Furthermore, by Booth and Lueker [6, pp. 367-372] (or Golumbic [13, pp. 73-78]), it is easy to count how many consecutive ones forms G's matrix has (or how many transitive orientations G^c has). Unfortunately, these do not combine to give us an efficient way of determining the total number of chronological orderings of G, because the consecutive ones forms may have varying numbers of associated chronological orderings. That is, $|I(M(i))|$ must be computed individually for each i. Since the number of consecutive ones forms of the matrix of G can easily be of size exponential in $|V|$, this does not lead to a polynomial-time algorithm for determining the total.

The method of counting used in Chapter 4 does not seem to help here either. We are able to determine the number of chronological orderings, given which edges indicate inclusion, but, of course, there are an exponential number of ways we can choose a subset of the edges to indicate inclusion.

It is possible that Problem 6.1 is #P-complete (for definitions and some results in this area, see Garey and Johnson [10], or Valiant [35]), but the author has not been able to prove or disprove this conjecture.

CHAPTER 7: SPECIAL REPRESENTATIONS

7.1 Representations containing Given Points

Once again, let us return to our example from archaeology. Suppose that, due to the availability of additional information, it could be precisely determined when some of the graves were dug. Then it seems reasonable that the points in time at which the graves had been dug are contained in all the intervals related to the artifacts found in those graves. This leads to several interesting questions concerning interval graphs.

Problem 7.1

Instance: Graph $G = (V, E)$, a set of real numbers $\{p_1, p_2, \ldots, p_m\}$ for which $p_1 < p_2 < \cdots < p_m$, and a collection $\{V_1, \ldots, V_m\}$ of subsets of V.

Question: Does G have an interval representation $\{I_1, \ldots, I_n\}$ such that, for $i = 1, \ldots, m$, $p_i \in \bigcap_{v_j \in V_i} I_j$?

Theorem 7.2

G has a representation as desired in Problem 7.1 iff the following are true:

(a) C_4 is not an induced subgraph of G,

(b) for $k = 1, \ldots, m$, the subgraph of G induced by V_k is a clique,

(c) G^c has a transitive orientation T^c such that

$$\{v_i v_j \in E^c : v_i \in V_k,\ v_j \in V_\ell,\ \text{for some } k < \ell\} \subset T^c.$$

Proof: The necessity of (a) and (b) is evident. If G has the desired representation, then we can give G^c the transitive orientation T^c defined by

$$v_i v_j \in T^c \iff I_i < I_j.$$

Furthermore, if $v_i \in V_k$, $v_j \in V_\ell$, $v_i v_j \in E^c$ and $k < \ell$, then $I_i < I_j$ since $p_k < p_\ell$, $p_k \in I_i$, $p_\ell \in I_j$ and $I_i \cap I_j = \emptyset$. Thus $v_i v_j \in T^c$.

To prove sufficiency, we note that conditions (a) and (c) imply, by means of Theorem 1.3, that G is an interval graph with a representation $I' = \{I_1', I_2', \ldots, I_n'\}$ of closed intervals such that $v_i v_j \in T^c \iff I_i' < I_j'$. Let $I_j' = [a_j, b_j]$ for $j = 1, \ldots, n$. Without loss of generality, $I' \in I(G)$, i.e., $A = \{a_j, b_j : j = 1, \ldots, n\}$ is a set of $2n$ distinct real numbers.

Let $\underline{I'(V_k)}$ denote the set $\{I_j' : v_j \in V_k\}$. Because each V_k generates a clique, Helly's Theorem (see Danzer, Grünbaum, and Klee [7]) states that $\cap I'(V_k) \neq \emptyset$. Let ω_k denote the left endpoint of $\cap I'(V_k)$ and let

$$\varepsilon = \min \{|x-y| : x, y \in A \text{ and } x \neq y\} > 0.$$

That is, ε is the distance between the two closest members of A.

We now define a set of points $\{p_1', \ldots, p_m'\}$ from which we will derive the desired representation of G. Let $p_1' = \omega_1$. For $1 < i \leq m$, inductively define

$$p'_i = \max\{\omega_i, p_{i-1} + \varepsilon/m\}.$$

Then $p'_1 < p'_2 < \cdots p'_m$.

Lemma 7.2.1 For all i, $p'_i \in \cap I'(V_i)$.

Proof: It is true for $i = 1$. Let $i > 1$ and assume $p'_i \notin \cap I'(V_i)$. Then, by the definition of ω_i, $p'_i = p'_{i-1} + \varepsilon/m > \omega_i$ and hence p'_i is completely to the right (along the real line) of some $I'_j \in I'(V_i)$, i.e., $I'_j < \{p'_i\}$.

Since $p'_1 = \omega_1$, there exists some $k > 0$ such that

$$p'_i = p'_{i-1} + \varepsilon/m = \cdots = p'_{i-k} + k\varepsilon/m = \omega_{i-k} + k\varepsilon/m.$$

Now, by the definition of ε, $\omega_{i-k} = \max\{x \in A: x < p'_i\}$ and so $b'_j < \omega_{i-k}$. But ω_{i-k} is the left endpoint of some interval $I'_s \in I'(V_{i-k})$, which means that $I'_j < I'_s$. But this contradicts the fact that $v_j \in V_i$, $v_s \in V_{i-k}$, $v_jv_s \in E^c$ and $i > i-k$, which implies that $v_sv_j \in T^c$ by property (c), and hence $I'_s < I'_j$. Therefore, $p'_i \in \cap I'(V_i)$ and the lemma is proved.

We can now form the desired representation as follows (see Figure 7.1). Translate and dilate that part of each interval in I' between p'_i and p'_{i+1} so that it stretches from p_i to p_{i+1} (i.e., affinely transform $[p'_i, p'_{i+1}]$ onto $[p_i, p_{i+1}]$), for all i. The resulting representation $\{I_1,\ldots, I_n\}$ has the desired property. $\square$

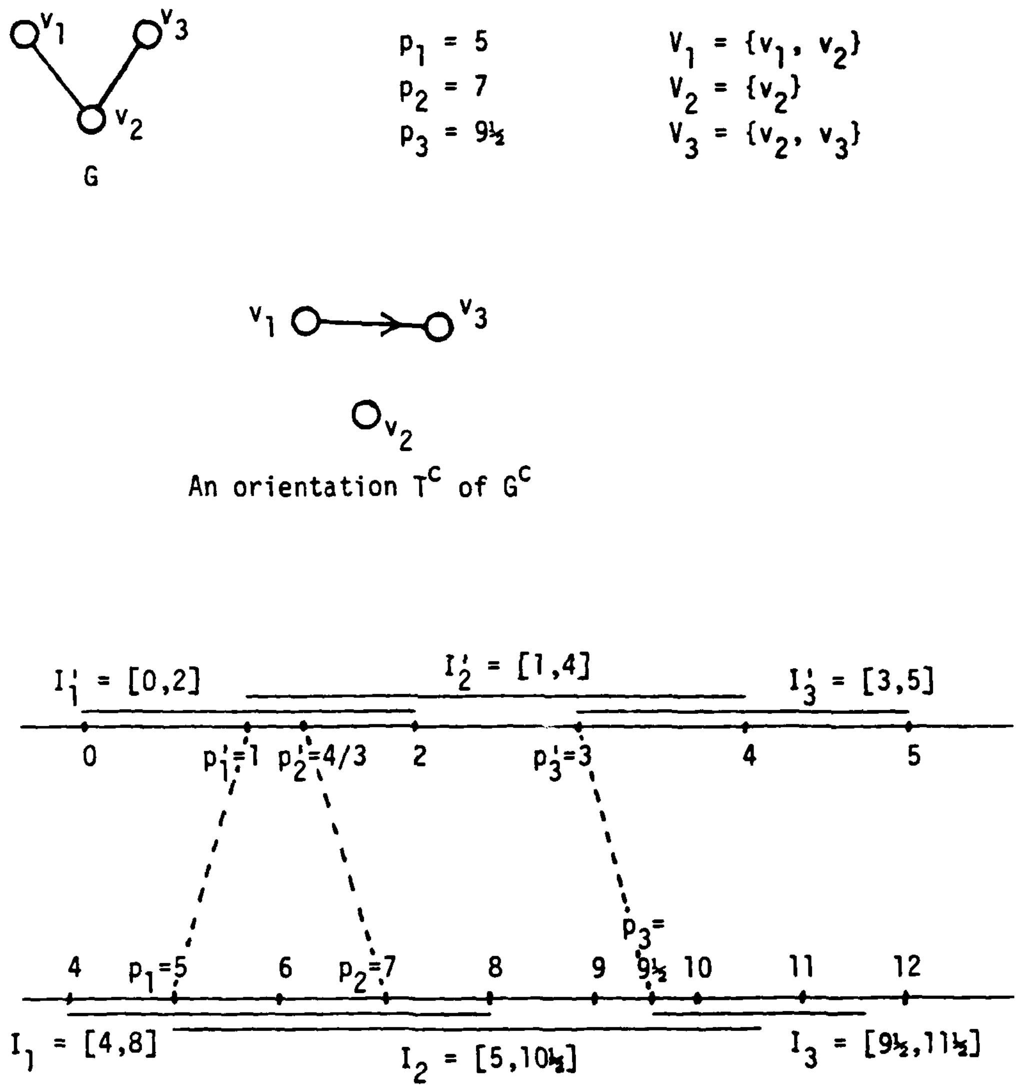


Figure 7.1: An example for Theorem 7.2.

Corollary 7.3

G has a representation as desired in Problem 7.1 with the added property that no interval in the representation is properly contained in another iff the conditions in Theorem 7.2 hold and $K_{1,3}$ (see Figure 1.4) is not an induced subgraph of G.

Sketch of proof: If $K_{1,3}$ is not an induced subgraph of G, then the initial representation $\{I_1', \ldots, I_n'\}$ of G can easily be modified (by stretching some endpoints if necessary) so that there is no proper containment between any two intervals. The rest of the proof then follows as in Theorem 7.2. □

Notice that it is possible for intervals other than those in $I(V_i)$ to contain p_i. The following theorem shows that we can also determine whether G has a representation in which those and only those intervals of $I(V_i)$ contain p_i.

Problem 7.4

Instance: Graph $G = (V, E)$, a set of real numbers $\{p_1, \ldots, p_m\}$ for which $p_1 < p_2 < \cdots < p_m$, and a collection $\{V_1, \ldots, V_m\}$ of subsets of V.

Question: Does G have a representation $\{I_1, \ldots, I_n\}$ in which $p_i \in I_j \iff v_j \in V_i$?

Define $G' = (V + \{p_i : i = 1,\dots, m\},\ E + E_p)$ where xy, $yx \in E_p$ iff, for some i, $x = p_i$ and $y \in V_i$. That is, we enlarge G by adding m new vertices corresponding to the points $\{p_1,\dots,p_m\}$, and we connect each new vertex p_i with the vertices in V_i.

<u>Theorem 7.5</u>

G has a representation as desired in Problem 7.4 iff the following are true:

(a) C_4 is not an induced subgraph of G',

(b) For all k, the subgraph of G induced by V_k is a clique, and

(c) $(G')^c$ has a transitive orientation T^c such that $\{p_i p_j : i < j\} \subset T^c$.

<u>Proof:</u> If Problem 7.4 has a solution in which G is represented by $\{I_1,\dots, I_n\}$, then G' is represented by the set $\{I_1,\dots, I_n, \{p_1\}, \dots, \{p_m\}\}$, where vertex p_i is represented by the real number p_i (an interval of length 0). Therefore G' is an interval graph and so, by Theorem 1.3, conditions (a) and (c) hold. Condition (b) is obvious.

Conversely, suppose conditions (a), (b), (c) are true. Then G' is an interval graph. In the representation I of G' constructed by means of Gilmore and Hoffman's procedure [11] using T^c, each vertex p_i is represented by a point p_i', which is contained in exactly the intervals in $I(V_i)$. If we now affinely transform each interval $[p_i', p_{i+1}']$ onto the interval $[p_i, p_{i+1}]$, we obtain the desired representation. □

Corollary 7.6

G has a representation as desired in Problem 7.4 with the added property that no interval representing a vertex of G is properly contained in another iff the conditions of Theorem 7.5 are satisfied and $K_{1,3}$ is not an induced subgraph of G.

Proof: The proof is similar to the proof of Corollary 7.3. □

Using the algorithm described in Chapter 4 for extending the oriented edges of a mixed graph to a transitive orientation of the whole graph (Problem 4.11), it is easy to construct an algorithm of complexity $O(|V|^3 + m^3)$ which solves Problems 7.1 and 7.4. The task of producing a detailed algorithm is left as an exercise for the reader.

7.2 A Class of Graphs of Interval Count 2

In the preceding section, we characterized those graphs which have a proper interval representation in which certain intervals contain specified points. These results cannot easily be extended to the case in which we desire G to have a unit interval representation, because in this case, the distances between the p_i's become important, a factor we were able to ignore completely until now.

In this section, we characterize those graphs which have representations containing only points and unit intervals. Here we do not care which points on the line they are. However, a better framework for Theorem 7.7 below is the subject of interval counts.

Given a finite set J of intervals of the line, let $\underline{IC(J)}$ be the number of different size intervals in J. For an interval graph G, define the <u>interval count</u> of G

$$\underline{IC(G)} = \min\{IC(J)\colon\ J \text{ is a representation of } G\}.$$

Thus $IC(G) = 1$ iff G is a unit interval graph. Leibowitz [23] has proved some results about interval counts. She showed that for any interval graph $G = (V, E)$, if $G \backslash x$ (the subgraph of G induced by $V \backslash \{x\}$) has interval count 1, then $IC(G) \leq 2$. She also described some other classes of graphs with interval count 2.

Theorem 7.7 presents another class. It characterizes those graphs of interval count 2 or less for which one of the two lengths of the intervals is 0.

We give a few more definitions, using the notation of Roberts [28, p. 140] and Scott and Suppes [31, p. 118]. Define an equivalence relation $\underline{EQ}$ on the vertices V of a graph $G = (V, E)$ by defining $xEQy$ iff $N(x) = N(y)$. Thus two vertices are equivalent iff they are adjacent to exactly the same vertices in V (including themselves). Let the <u>reduced graph</u> $\underline{G^*}$ of G be the graph obtained by cancelling this equivalence relation, i.e., the vertices of G^* are equivalence classes of vertices of G and two classes are adjacent if adjacency holds between the representatives from the two classes.

In Theorem 7.7, it suffices to restrict our attention to reduced graphs, because if two vertices are equivalent, they can be represented by the same interval.

Call a vertex v a simplicial vertex if $N(v)$ generates a clique in G. Also, for $S \subset V$, let $E(S) = \{xy \in E\colon x \in S \text{ or } y \in S\}$.

Theorem 7.7

Let $G = (V, E)$ be a reduced graph. Then G has a representation in which all intervals are closed and of length 1 or 0 (i.e., are unit intervals or points) iff $E \setminus E(S)$ and E^c have orientations 0 and F, respectively, so that $0+F$ is strongly transitive, where S is the set of simplicial vertices of V.

Proof: Suppose G has such a representation. Clearly those vertices represented by points must be in S. Conversely, if any vertex $s \in S$ is represented by a unit interval, note that, since $\bigcap_{v_i \in N(s)} I_i \neq \emptyset$, we obtain an equally valid representation by G by representing s by any one of the points in $\bigcap_{v_i \in N(s)} I_i$. Thus we may assume that the set of vertices represented by points is exactly S.

It is not hard to see that all the unit intervals can be translated, if necessary, so that all endpoints are distinct, without changing the intersection properties of the intervals. Furthermore, each point that represents a vertex in S can be lengthened slightly to form an interval properly contained in the unit intervals with which it intersects, so that there are 2n distinct endpoints in the representation. In this way , we obtain a representation of G in

$I(G)$ whose associated tournament $(V, C+O+F)$ satisfies the condition that $xy \in C$ iff $xy \in E$ and $x \in S$. Thus by Theorem 4.5, we have proven the necessity of the existence of the orientations O and F.

Conversely, suppose $E \setminus E(S)$ and E^c have orientations O and F with the properties described. Then by Theorem 4.5, G has a representation $I = \{I_1, \ldots, I_n\} \in I(G)$ such that $I_i \subset I_j$ iff $v_i v_j \in E$ and $v_i \in S$. (That is, we are defining $C = \{v_i v_j : v_i v_j \in E$ and $v_i \in S\}$ for use in Theorem 4.5. It is clear that $C^2 \subset C$ and $xy \in C \Rightarrow N(x) \subset N(y)$ since G is reduced and since S is the set of simplicial vertices of G.) Now, since all the neighbors of $v_i \in S$ are adjacent, no intersections are created or obliterated if I_i is shrunk to a point for each $v_i \in S$. All that remains is to expand or shrink each interval that is not a point into a unit interval. This can be done without changing any intersections because no such interval is properly contained in another. The task of checking all the details is left as an exercise for the reader. $\square$

We remark that this can be checked algorithmically in $O(|V|^3)$ steps using the methods of Chapter 4.

CHAPTER 8: INTERVAL EDGE-GRAPHS

In this chapter and Chapter 9, the elements of E will be unordered pairs $\{x, y\}$ of distinct vertices. We will still call the elements of E edges, which hopefully will not cause any confusion. All graphs are undirected and there are no loops or multiple edges.

Let $G = (V, E)$ be a graph. We construct a new graph $G' = (E, F)$ called the edge-graph (or line-graph) of G by letting E be the set of vertices of G' and letting F be the set of all unordered pairs of (distinct) edges of G which have a common endpoint (see Figure 8.1).

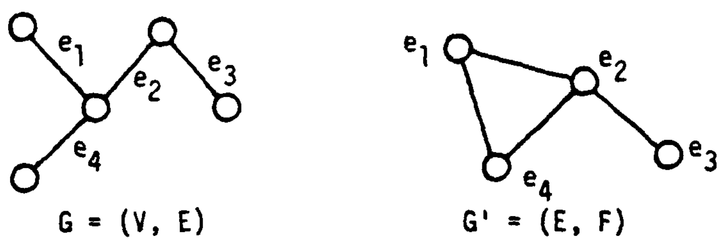


Figure 8.1: A graph and its edge-graph.

The following theorem characterizes those graphs whose edge-graphs are interval graphs. Given a graph $G = (V, E)$, define a simple circuit of G to be a sequence $[v_1, v_2, \ldots, v_m]$ of distinct vertices with $m \geq 3$, such that $\{v_i, v_{i+1}\} \in E$ for $i = 1, \ldots, m-1$ and $\{v_m, v_1\} \in E$. A simple circuit $[v_1, \ldots, v_m]$ is called long if $m \geq 4$. A cactus is a graph with no long simple circuits.

Theorem 8.1

Let $G' = (E, F)$ be the edge-graph of $G = (V, E)$. The following are equivalent:

(a) G' is an interval graph,

(b) G' is a unit interval graph,

(c) G is a cactus which does not contain I nor IV_2 (see Figure 1.1) as (not necessarily induced) subgraphs,

(d) G is a cactus in which each component C of G has a chordless path P with the property that, for each vertex v in C but not in P, $ON(v)$ consists of exactly one vertex of P or exactly two adjacent vertices of P.

Before we prove this, we need a lemma.

Lemma 8.1.1 $K_{1,3}$ is not an induced subgraph of any edge-graph.

Proof of Lemma: Suppose an edge-graph $G' = (E, F)$ of a graph $G = (V, E)$ has such a subgraph with the "center" vertex labelled v and the other three vertices labelled v_1, v_2, v_3. Then v must be an edge of G which has a common vertex with three other edges in E, but none of them has a common vertex with each other. Since v has only two endpoints, this cannot happen.

Proof of Theorem 8.1: (a) <=> (b) By Lemma 8.1.1 above and Theorem 1.4.

(b)=>(c) If $[v_1, \ldots, v_m]$, $m \geq 4$, is a long simple circuit in G, then $[\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_m, v_1\}]$ is a chordless cycle in G', contradicting the fact that G' is an interval graph. Thus G is a cactus.

If G contains I, then G' must contain IV_2 as an induced subgraph since IV_2 is the edge-graph of I. This contradicts Theorem 1.4. If G contains IV_2, then G' must contain V_1 (see Figure 1.1) as an induced subgraph since V_1 is the edge-graph of IV_2. This again contradicts Theorem 1.4.

(c)=>(d) Without loss of generality, G is connected. The proof will be by induction on $n = |V|$. Clearly, it is true for $n = 1$. Consider the case where $n > 1$. Let x be any vertex in V such that $G \backslash x$ (the subgraph of G induced by $V \backslash \{x\}$) is connected. Then $G \backslash x$ has a path with the desired properties. Let $P = [v_1, \ldots, v_k]$ be the longest such path.

Note that x is adjacent to at most two vertices of P since G is a cactus. We'll consider each possibility separately.

<u>CASE 1</u>: x is adjacent to two vertices v_i, v_j of P. Without loss of generality, $i < j$. If $j \neq i+1$, then $[x, v_i, v_{i+1}, \ldots, v_j]$ is a long simple circuit. Thus x must be adjacent to v_i and v_{i+1} for some i. The proof will be complete when we show that x is not adjacent to any other vertex of G. To show this, we suppose not, i.e., we suppose some vertex y is adjacent to x, with $y \notin P$. Since $G \backslash x$ is connected, y is adjacent to some v_k in P. Without loss of generality, $k \leq i$. But then $[x, y, v_k, v_{k+1}, \ldots, v_i, v_{i+1}]$ is a

long simple circuit, a contradiction.

CASE 2: x is adjacent to one vertex v_i of P. If $ON_G(x) = \{v_i\}$, then P is the desired path in G. Thus assume there exists a $y \in V \setminus P$ such that $\{x, y\} \in E$. Since $G \setminus x$ is connected, $\{y, v_j\} \in E$ for some $v_j \in P$. Without loss of generality, $j \le i$. If $j < i$, then $[y, v_j, v_{j+1}, \ldots, v_i, x]$ is a long simple circuit. Hence $ON_G(y) = \{x, v_i\}$, and $ON_G(x) = \{y, v_i\}$. (If there were another vertex ω adjacent to x, then $[x, \omega, v_i, y]$ would be a long simple circuit.) If $3 \le i \le k-2$, then $\{v_{i-2}, v_{i-1}, v_i, v_{i+1}, v_{i+2}, x, y\}$ generates a graph which contains I. Thus $i = 1, 2, k-1$, or k. Without loss of generality, $i = 1$ or 2 (see Figure 8.2).

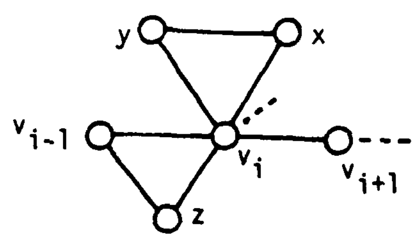


Figure 8.2: Case 2.

If $i = 1$, then $[y, v_1, v_2, \ldots, v_k]$ is a longer path than P with the desired properties, contradicting the maximality of P in $G\backslash x$. Thus $i = 2$. Now if $ON(v_1) = \{v_2\}$, then $P' = [x, v_2, \ldots, v_k]$ is the desired path. Therefore, consider the case in which $\{z, v_1\} \in E$ for some vertex $z \neq v_2$. Then $\{z, v_2\} \in E$ by the maximality of P in $G\backslash x$. Now if $k > 3$, then $\{x,y,z,v_1,v_2,v_3,v_4\}$ generates a graph which contains I. Hence it must be true that $k = 2$ or 3. If $k = 2$, then $[v_1, v_2, x]$ is the desired path. If $k = 3$, then we claim $ON(v_3) = \{v_2\}$, in which case $[v_1, v_2, x]$ is still the desired path. But if $\{\omega, v_3\} \in E$ for some $\omega \neq v_2$, then $\{x,y,z,\omega, v_1, v_2,v_3\}$ generates a graph which contains I. Thus $ON(v_3) = \{v_2\}$ and Case 2 is proved.

<u>CASE 3</u>: x is not adjacent to P. Then $\{x, y\} \in E$ for some vertex $y \notin P$, and $\{y, v_i\} \in E$ for some $v_i \in P$. Furthermore, this is x's only neighbor since any other neighbor would also have to be adjacent to P, producing a long simple circuit.

Now y can also be adjacent to v_{i+1} (or equivalently v_{i-1}) in P. We'll consider these possibilities separately.

<u>Subcase 1</u>: $ON(y) = \{x, v_i, v_{i+1}\}$ (see Figure 8.3). If $k \geq 4$, then $i = 1$ or $k-1$ because otherwise $\{x,y,v_{i-1},v_i,v_{i+1},v_{i+2}\}$ generates IV_2.

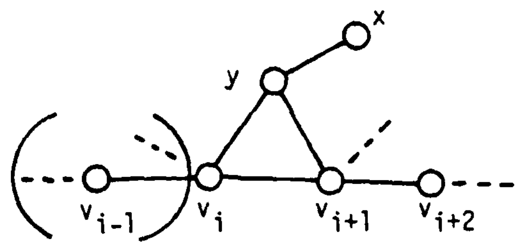


Figure 8.3: Case 3, Subcase 1.

Thus for any $k \geq 2$, we can assume that $i = 1$. Due to the maximality of P in $G\backslash x$ and the fact that there does not exist a vertex z (other than y) such that $ON(z) \supset \{v_1, v_2\}$, v_1 can have no neighbors other than y and v_{i+1}. Now it is easy to see that $P' = [x, y, v_2, v_3, \ldots, v_k]$ is the desired path.

<u>Subcase 2</u>: $ON(y) = \{x, v_i\}$ (see Figure 8.4).

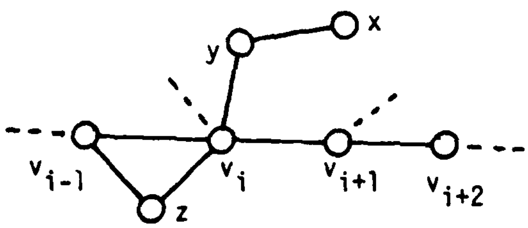


Figure 8.4: Case 3, Subcase 2.

If $3 \le i \le k-2$, then $\{x, y, v_{i-2}, v_{i-1}, v_i, v_{i+1}, v_{i+2}\}$ generates I. Thus without loss of generality, $i = 1$ or 2. If $i = 1$, then $[y, v_1, v_2, \ldots, v_k]$ is a path that contradicts the maximality of P in $G\backslash x$. Thus $i = 2$. If $ON(v_1) = \{v_2\}$, then $[x, y, v_2, \ldots, v_k]$ is the desired path. If $\{v_1, z\} \in E$ for some vertex $z \neq v_2$, then by the maximality of P, $\{v_2, z\} \in E$. In this case, it must be true that $k \le 3$, since $k > 3$ implies that $\{x, y, z, v_1, v_2, v_3, v_4\}$ generates a graph that contains I. If $k = 2$, then $[x, y, v_2, v_1]$ is the desired path. If $k = 3$, then v_3 can have no neighbors besides v_2 (any such vertex ω would cause $\{x, y, z, \omega, v_1, v_2, v_3\}$ to generate a graph containing I). Thus $[x, y, v_2, v_1]$ is the desired path. This finishes Case 3 and the proof that (c)=>(d).

(d)=>(c) It is clear that there can be no such path P if G contains I or IV_2.

(c)=>(b) G' cannot contain $K_{1,3}$ as an induced subgraph by Lemma 8.1.1. G' does not contain IV_2 or V_1 as an induced subgraph because, if so, then G contains I or IV_2. If G' contains III_n with $n \ge 4$, then G has a long simple circuit. Hence G' has no induced subgraphs of the form III_n, $n \ge 4$. Thus by Theorem 1.4, G' is a unit interval graph.

This completes the proof of the theorem. $\square$

CHAPTER 9: SOME NP-COMPLETENESS RESULTS

In this chapter, we use the notation described in the first paragraph of Chapter 8, i.e., E denotes a set of unordered pairs of distinct vertices.

For all definitions and background material on NP-completeness, the reader is referred to Garey and Johnson [10] or Aho et al. [1].

Recall that, according to Theorem 1.2, a graph is an interval graph iff its maximal clique-vertex incidence matrix has the consecutive ones property. Therefore, if we want to measure how "close" a graph is to being an interval graph, one way is to measure how "close" its maximal clique-vertex incidence matrix is to being a matrix with the consecutive ones property. There are four measures of "closeness" that are discussed here, each of which leads to an NP-complete problem. This is also true when "consecutive ones property" is replaced by "circular ones property". A matrix of 0's and 1's has the <u>circular ones property</u> if the rows can be permuted, so that, when the matrix is rolled into a cylinder that makes the first and last rows adjacent, all the 1 entries are consecutive in each column. This is equivalent to requiring that there exist a permutation of the rows of the matrix so that all the 1 entries or all the 0 entries are consecutive in each column. For related NP-complete problems, see Garey and Johnson [10, pp. 229-230].

The four measures of closeness are:

(1) The minimum number of 1 entries that need to be changed to 0's to give the matrix the consecutive ones property.

(2) The minimum number of 0 entries that need to be changed to 1's

to give the matrix the consecutive ones property.

(3) The minimum number of rows that need to be removed so that the remaining matrix has the consecutive ones property.

(4) The minimum number of columns that need to be removed so that the remaining matrix has the consecutive ones property.

Measure (1) corresponds to the following problem:

<u>Consecutive Ones Matrix Diminution (COMD)</u>

Instance: An $m \times n$ matrix M of 0's and 1's and a non-negative integer K.

Question: Is there a matrix M', obtained from M by changing K or fewer 1 entries to 0's, such that M' has the consecutive ones property?

<u>Theorem 9.1</u>

COMD is NP-complete.

<u>Proof</u>: A proof requires showing that COMD is in NP and that there is an NP-complete problem which is polynomially transformable into COMD. It is easy to see that COMD is in NP because, as was previously mentioned, there is a linear-time algorithm for checking for the consecutive ones property.

The remainder of this proof shows that "Hamiltonian Path Completion" (number [GT34] in Garey and Johnson's book [10]), which is NP-complete, is polynomially transformable into COMD.

Hamiltonian Path Completion (HPC)

Instance: Graph $G = (V, E)$ and positive integer $L \le |V|$.

Question: Is there a superset E' containing E such that $|E' \setminus E| \le L$ and the graph $G' = (V, E')$ has a Hamiltonian path?

Given an instance of HPC, we construct the $|V| \times |E|$ vertex-edge incidence matrix M of G, where $M = (m_{ij})$ is defined by

$$m_{ij} = \begin{cases} 1 & \text{if vertex } v_i \text{ is an endpoint of edge } e_j \\ 0 & \text{otherwise.} \end{cases}$$

Let $K = |E| - |V| + L+1$. This matrix M and integer K specify an instance of COMD (if $|V| > |E| + L+1$, then there is clearly no Hamiltonian path in any such G').

Claim: The desired superset E' exists for the instance of HPC iff the desired matrix M' exists for the instance of COMD.

Proof of Claim: If the minimum number of edges we need to add to G' to give it a Hamiltonian path P is $\ell \le L$, then G must already have $|V| - \ell - 1$ of the edges connecting the vertices of the path P, since the total number of such edges is $|V| - 1$. Now by changing one 1 to a 0 in each of the $|E| - |V| + \ell+1$ columns of M corresponding to the edges of E not connecting the vertices in P, we obtain a matrix M' with the consecutive ones property (we can permute the rows of M' to correspond to the order of the vertices along P to get M'

into consecutive ones form).

Conversely, suppose that $k \le K$ is the minimum number of 1's that can be changed to 0's in M to give M' the consecutive ones property. Clearly we need to change at most one 1 in each column, since a column contains only two 1's. Let F be the set of edges corresponding to the columns of M in which no 1 was changed. It is easy to see that (V, F) can have no vertex of degree ≥ 3 nor any simple circuits. Thus it is possible to add $|V| - |F| - 1$ edges to G to form a Hamiltonian path which uses all of the edges in F. Hence the minimum number ℓ of edges that need to be added to G so that G' has a Hamiltonian path satisfies:

$$\begin{aligned}
\ell &\le |V| - |F| - 1 \\
&= |V| - |E| + k - 1 \quad (\text{since } |F| = |E| - k) \\
&\le |V| - |E| + K - 1 \\
&= |V| - |E| + (|E| - |V| + L + 1) - 1 \\
&= L.
\end{aligned}$$

This proves the claim and hence the theorem. □

Corollary 9.2

Circular Ones Matrix Diminution is NP-complete.

Sketch of proof: We can prove that Hamiltonian Circuit Completion (Gary and Johnson's [GT34]) is polynomially transformable into it by making minor modifications to the proof above.

An alternate proof is obtained by noting that, by changing all the 0's to 1's and 1's to 0's, we obtain a polynomial transformation of Circular Ones Matrix Augmentation (mentioned below) into it. □

The corresponding problems of Consecutive Ones Matrix Augmentation and Circular Ones Matrix Augmentation (Garey and Johnson's [SR16]) were proved to be NP-complete by Booth [4, pp. 106-108]. These correspond to measure (2) of closeness.

The following problem corresponds to measure (3).

Maximum Row Consecutive Ones Matrix (MRCOM)

Instance: An $m \times n$ matrix M of 0's and 1's and a positive integer $K \leq m$.

Question: Does there exist a $k \times n$ submatrix M' of M with $k \geq K$ such that M' has the consecutive ones property?

The statements of Garey and Johnson [10, p. 229] on this problem are confusing because their definition of "consecutive ones property" asks whether the columns can be permuted so that the 1's in each row are consecutive. Problem [SR14] was not proved to be NP-complete by Booth [4, p.111] as they describe it. Booth actually proved that "Maximum Column Consecutive Ones Matrix" (the same problem as MRCOM except that it asks for an $m \times k$ submatrix with the consecutive ones property) is NP-complete. This corresponds to measure (4) of closeness.

We now proceed to prove that MRCOM (or equivalently, [SR14] in Garey and Johnson's book) is NP-complete.

Theorem 9.3

MRCOM is NP-complete.

Proof: It is clear that MRCOM is in NP. To finish the proof, we show that the problem "Induced subgraph which contains only paths" (problem [GT21] in Garey and Johnson's book), which is NP-complete, is polynomially transformable into MRCOM.

> Induced subgraph which contains only paths (ISCOP)
>
> Instance: Graph $G = (V, E)$ and a positive integer $K \leq |V|$.
>
> Question: Is there a subset $V' \subseteq V$ with $|V'| \geq K$ such that the subgraph G' induced by V' contains only paths, i.e., G' has no vertices of degree > 2 and no simple circuits?

Note: This property qualifies for property Π in [GT21] as stated in Garey and Johnson's book, because this problem is in NP, and this property holds for arbitrarily large graphs, does not hold for all graphs, and is hereditary.

Now, given an instance of ISCOP, we construct the vertex-edge incidence matrix M of G. Then K and M give us an instance of MRCOM.

<u>Claim</u>: The desired subset exists for the instance of ISCOP iff the desired submatrix exists for the instance of MRCOM.

<u>Proof of Claim</u>: Suppose there exists a subset V' as desired in ISCOP. Let M' be the $|V'| \times |E|$ submatrix of M whose rows correspond to the vertices in V'. Now rearrange the rows of M' so that the vertices we encounter as we travel along the paths of G' are consecutive. This permutation puts M' into a consecutive ones form, so M' must have the consecutive ones property.

Conversely, suppose that there exists a $k \times |E|$ submatrix M' of M with $k \geq K$ such that M' has the consecutive ones property. Consider the subgraph G' of G induced by the vertices of V corresponding to the rows of M'. Since M' has the consecutive ones property, it is easy to see that no vertex of G' can have degree > 2, because the matrix

$$\begin{array}{|ccc|} \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ \hline \end{array}$$

cannot be a submatrix of M'. G' cannot have any simple circuits for a similar reason. Thus G' has only paths, which proves the claim and also the theorem. □

<u>Corollary 9.4</u>

Maximum Row Circular Ones Matrix is NP-complete.

Sketch of proof: By minor modifications of the proof above, we can show that the following NP-complete problem is polynomially transformable into it. This problem also corresponds to problem [GT21] in Garey and Johnson's book.

Induced Subgraph with only paths or a Hamiltonian circuit

Instance: Graph $G = (V, E)$ and a positive integer $K \le |V|$.

Question: Is there a subset $V' \subset V$ with $|V'| \ge K$ such that the subgraph induced by V' has no vertex of degree > 2 and no simple circuits except possibly a Hamiltonian circuit?

This gives the desired result. □

Corollary 9.5

The problems proven to be NP-complete in Theorems 9.1 and 9.3 and Corollaries 9.2 and 9.4 remain NP-complete when restricted to matrices with at most two 1 entries in each column.

Proof: This is clear from the proofs of the theorems. □

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