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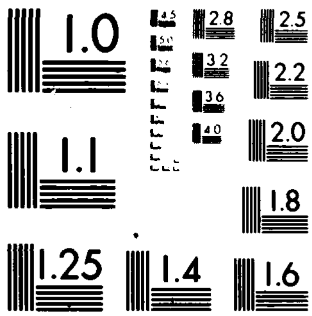

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In the context of a purely mechanical, rate-type theory of elastic-plastic materials and utilizing a strain space formulation ~~introduced in [1]~~, this paper is concerned mainly with developments pertaining to strain-hardening behavior consisting of three distinct types of material response, namely hardening, softening and perfectly plastic behavior. It is shown that such strain-hardening behavior may be characterized by a rate-independent quotient of quantities occurring in the loading criteria of strain space and the

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20. Abstract (continued)

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On the Characterization of Strain-Hardening

in Plasticity

by

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Abstract. In the context of a purely mechanical, rate-type theory of elastic-plastic materials and utilizing a strain space formulation introduced in [1], this paper is concerned mainly with developments pertaining to strain-hardening behavior consisting of three distinct types of material response, namely hardening, softening and perfectly plastic behavior. It is shown that such strain-hardening behavior may be characterized by a rate-independent quotient of quantities occurring in the loading criteria of strain space and the corresponding loading conditions of stress space. With the use of special constitutive equations, the predictive capability of the results obtained are illustrated for strain-hardening response and saturation hardening in a uniaxial tension test.

1. Introduction

Within the scope of a rate-type mechanical theory of elastic-plastic materials, Naghdi and Trapp [1] have recently discussed the advantages of formulating plasticity theory relative to yield (or loading) surfaces in strain space (rather than stress space). We adopt here the loading criteria of the strain space formulation as primary and derive the associated loading conditions in stress space. By comparing the local motion of the loading surface in stress space to that of the loading surface in strain space during loading, we find that three distinct types of material response representing hardening, softening and perfectly plastic behavior can be defined in a natural way. For convenience, these three types of response will be referred to collectively as strain-hardening behavior. The development leading to the latter, as well as illustrative examples of the results for special constitutive equations, are the main objectives of the present paper. As in [1], we confine attention to the purely mechanical theory of elastic-plastic materials, and base our development on the rate-type stress space formulation of Green and Naghdi [2,3][†] and on the alternative strain space formulation introduced by Naghdi and Trapp [1].

By way of motivation, consider the response of a typical ductile metal in a one-dimensional simple tension test in which the strain may be moderately large. Let e and s stand, respectively, for the component e_{11} of the Lagrangian strain tensor and the component s_{11} the symmetric Piola-Kirchhoff stress tensor. Figure 1 shows a plot of the stress s versus the strain e for the one-dimensional homogeneous simple tension test. From the origin 0 to the elastic limit

[†]The theory proposed in [2,3] is a general thermodynamical theory of elastic-plastic materials. The development in [1] is carried out within a purely mechanical framework which can readily be interpreted in terms of the isothermal case of the thermodynamical theory.

(identified by the point 1) the material is elastic, stress strictly increases[‡] with strain, there is no plastic straining and unloading takes place along 1-0. On the rising portion 1-3 (excluding point 3) of the s-e curve both stress and plastic strain strictly increase with strain. Unloading from a point such as 2 takes place along 2-2′ leaving a plastic strain of amount 02′. At point 3, s attains its maximum value[§]. On the falling portion 3-4-5 (excluding point 3) of the s-e curve, stress strictly decreases with strain, but plastic strain continues to strictly increase. Associated with each point of the segment 1-5 in Fig. 1, there is a unique yield point on the s-axis (i.e., in stress space) and a unique yield point on the e-axis (i.e., in strain space). For the points 1,2,3,4,5 these are denoted by A_1,A_2,A_3,A_4,A_5 and B_1,B_2,B_3,B_4,B_5, respectively. The points A_1 and B_1 are the initial yield points. As the segment 1-5 of the stress-strain curve is traversed, the locus of the yield point on the s-axis differs characteristically from that of the yield point on the e-axis, in that the former reverses its direction of motion while the latter does not.

The usual loading criteria of the stress space formulation of plasticity theory, when applied to the one-dimensional case under discussion, require that the plastic strain rate be nonzero whenever the yield point on the s-axis is moving upwards, and be zero when it is stationary. It is further stipulated that the yield point on the s-axis cannot move downwards while tension is being applied. These criteria are consistent with the results of the tensile test for the rising portion 1-3 of the stress-strain curve, both for paths of

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[‡]Recall that a real-valued function f defined on some interval $\mathcal{I}$ of the real line is <u>increasing</u> if $f(x_2) \geq f(x_1)$ whenever x_1 and x_2 belong to $\mathcal{I}$ and $x_2 \geq x_1$. A function f is <u>strictly increasing</u> if $f(x_2) > f(x_1)$ whenever $x_2 > x_1$. Similarly, f is <u>decreasing</u> if $x_2 \geq x_1$ implies $f(x_2) \leq f(x_1)$ and <u>strictly decreasing</u> if $x_2 > x_1$ implies $f(x_2) < f(x_1)$.

[§]As was observed by Naghdi and Trapp [1, p. 789], the maximum of the s-e curve corresponds to a point which is still in the rising portion of the engineering stress (π) versus engineering strain (ϵ) curve. The maximum of the π-ϵ curve, where necking begins, corresponds to a point on the falling portion of the s-e curve.

the type 1-2 and paths of the type 2-2′. They also demand the correct kind of behavior for paths of the type 4-4′ issuing from points on the falling portion 3-5 of the stress-strain curve. However, they are clearly inadequate for paths of the type 3-4 because the yield point on the s-axis does move downwards for any such path; and, as was pointed out in [1], plastic strain is observed to be strictly increasing in this region. On the other hand, again with reference to the one-dimensional case under discussion, the loading criteria of the strain space formulation require that the plastic strain rate be nonzero whenever the yield point on the e-axis is moving outwards and that it be zero whenever this yield point is stationary. It is further required that the yield point on the e-axis cannot move inwards while extension is occurring. These requirements are consistent with the behavior represented in Fig. 1. Thus, the plastic strain is strictly increasing along the paths 1-2 and 3-4 and is constant along the paths 2-2′ and 4-4′.

In order to provide a background for some subsequent developments, it is desirable to make further observations regarding the stress-strain curve in Fig. 1. In the context of the classical infinitesimal theory, we recall the relations

$$e = e_e + e_p \quad , \quad e_e = s/E \quad , \tag{1}$$

where e_e and e_p are abbreviations for the components e^e_{11} and e^p_{11} of the elastic and plastic strains, respectively, and E (>0) is Young's modulus. We note that

$$\frac{de}{ds} = \frac{de_e}{ds} + \frac{de_p}{ds} \quad , \quad \frac{de}{de_e} = \frac{de}{ds}\left(\frac{de_e}{ds}\right)^{-1} = 1 + \frac{de_p}{de_e} \quad . \tag{2}$$

Now with the use of $\frac{de_e}{ds} = \frac{1}{E} > 0$ and $(2)_2$, we have

$$\frac{de}{ds} > 0 \text{ if and only if } \frac{de}{de_e} > 0 \quad , \quad \frac{de}{ds} < 0 \text{ if and only if } \frac{de}{de_e} < 0 \quad . \tag{3}$$

On the rising portion of the s-e curve $\frac{de}{ds} > 0$ (or equivalently $\frac{ds}{de} > 0$), on the falling portion $\frac{de}{ds} < 0$ ($\frac{ds}{de} < 0$) and $\frac{de}{ds}$ at point 3 becomes unbounded. Then, at a point A on the s-e curve, with the help of (2) and (3) it is readily seen that

$$1 + \frac{de_p}{de_e} \begin{cases} > 0 \text{ if and only if A is on the rising portion of the curve,} \\ < 0 \text{ if and only if A is on the falling portion of the curve,} \end{cases} \tag{4}$$

while $1 + de_p/de_e$ becomes unbounded at point 3.

After recalling the main features of the purely mechanical theory of elastic-plastic materials from [1,2,3]* in Section 2, a quotient $\hat{f}/\hat{g}$ of quantities which are derived from the loading functions f in stress space, and g in strain space, is introduced. It is noteworthy that while $\hat{f}$ involves the time rate of the stress tensor and $\hat{g}$ the time rate of the strain tensor, the quotient $\hat{f}/\hat{g}$ is independent of rates. In the latter part of Section 2, using an equation obtained with the help of a physically plausible work assumption introduced by Naghdi and Trapp in [4], we derive a geometrically revealing expression for the quotient $\hat{f}/\hat{g}$ [see Eq. (32)]. Next (Section 3), in terms of the quotient $\hat{f}/\hat{g}$, definitions are provided (see (43)) for strain-hardening behavior, i.e., for hardening, softening and perfectly plastic behavior, and their geometrical implications are examined. It is demonstrated that, while during loading the yield surface in strain space is always moving outwards locally, the corresponding yield surface in stress space may concurrently be moving outwards, inwards or may be stationary depending on whether the material is hardening, softening, or exhibiting perfectly plastic behavior. Because of

*While some of the formulas in Section 2 may appear to be repetitions of those in [1], our starting point and some of our conclusions differ from [1] and for clarity we have repeated these formulas.

our definitions (43), a variety of functions associated with material behavior and deriving from $\hat{f}/\hat{g}$ or $\hat{f}$ are found to be positive, negative or zero according as a material exhibits hardening, softening or perfectly plastic behavior. To avoid undue repetition, we introduce the abbreviation (44) and denote such conditions by the letter H. Any function that satisfies conditions H can be used to characterize strain-hardening behavior. By considering the limiting behavior of $\hat{f}/\hat{g}$, we also examine (in the context of the developments of the present paper) the phenomenon of saturation hardening studied previously by Caulk and Naghdi [5]. Definitions for saturation behavior are given at the end of Section 3.

The results in Sections 2 and 3 hold in the context of the nonlinear theory, but in the remainder of the paper attention is confined to <u>small</u> deformations of elastic-plastic materials. In order to demonstrate the predictive capability of the strain-hardening characterization developed in Section 3, special sets of constitutive equations are utilized in Sections 4 and 5 to discuss, respectively, strain-hardening response and saturation hardening under uniaxial loading.

For the particular constitutive equations utilized in Section 4, a rate-independent characterization of strain-hardening behavior is provided in terms of a certain combination $(2\beta+\psi\phi)$ of material constants. Moreover, it is shown that both the time rate of work-hardening ($\dot{\kappa}$) and the time rate of tension ($\dot{s}$) may be used to characterize strain-hardening behavior. While the quotient $\hat{f}/\hat{g}$ involves the coefficient ψ as well as the derivatives of strains with respect to stress (see Eq. (64)), it is shown that for a certain special case, the quotient $\hat{f}/\hat{g}$ may be expressed (see Eq. (65a)) in terms of quantities appearing in (2), (3) and (4) recorded earlier in this section. An examination of details of the solution in Section 4 shows that in uniaxial tension and in the sense of our definitions, linear elastic behavior is followed for perfectly plastic behavior by a horizontal stress-strain curve, while hardening behavior is represented by a straight line lying above, and softening by a straight line lying below the perfectly plastic line.

Finally in Section 5 we consider another set of constitutive equations having in particular a loading function employed by Caulk and Naghdi [5] in their discussion of hardening response in small deformation of metals. Again it is shown that a number of different functions can be used to characterize strain-hardening behavior. Moreover, it is demonstrated that the quotient $\hat{f}/\hat{g}$ may be calculated in uniaxial tension from a knowledge of the slope de/ds, found from the stress-strain curve, and the elastic constants, namely Young's modulus E and the shear modulus μ, and thus may be easily identified experimentally. Although our characterization of strain-hardening is, in general, different from that discussed previously by Caulk and Naghdi [5], the two sets of results are in agreement for the class of materials for which detailed comparisons with experiments were undertaken in [5]. In this connection see Eqs. (88), which also include a simple expression in terms of material coefficients for the saturation hardening constant.

2. The Quotient $\hat{f}/\hat{g}$ of Quantities Occurring in Loading Criteria.

Let the motion of a body be referred to a fixed system of rectangular Cartesian axes and let the position of a typical particle in the present configuration at time t be designated by $x_i = \chi_i(X_A, t)$, where X_A is a reference position of the particle. Throughout the paper, lower case Latin indices are associated with the spatial coordinates x_i and assume the values 1,2,3. Similarly, upper case Latin indices are associated with the material coordinates X_A and take the values 1,2,3. We also adopt the usual convention of summation over repeated indices.

We define a symmetric Lagrangian strain tensor by $e_{KL} = \frac{1}{2}(F_{iK}F_{iL} - \delta_{KL})$, where $F_{iK} = \partial\chi_i/\partial X_A$ is the deformation gradient relative to reference position and δ_{KL} is the Kronecker symbol. The six-dimensional Euclidean vector space formed from the components e_{KL} is called strain space. The components of the symmetric Piola-Kirchhoff stress tensor are denoted by s_{MN} and the six-dimensional Euclidean space formed from these components is called stress space.

We now summarize the main ingredients of the purely mechanical rate-type theory of a finitely deforming elastic-plastic solid and base our treatment on the work of Green and Naghdi [2,3] and Naghdi and Trapp [1]. In addition to the strain tensor e_{KL}, we assume the existence of a symmetric† second order tensor-valued function $e^p_{KL} = e^p_{KL}(X_A, t)$ called the plastic strain at X_A and t, and a scalar-valued function $\kappa = \kappa(X_A, t)$ called a measure of work-hardening. It is assumed that the stress s_{MN} is given by the constitutive equation

$$s_{MN} = \hat{s}_{MN}(\mathfrak{u}) \quad , \quad \mathfrak{u} = \{e_{KL}, e^p_{KL}, \kappa\} \quad , \tag{5}$$

and that for fixed values of e^p_{KL} and κ, $(5)_1$ possesses an inverse of the form

$$e_{MN} = \hat{e}_{MN}(\mathfrak{v}) \quad , \quad \mathfrak{v} = \{s_{KL}, e^p_{KL}, \kappa\} \quad . \tag{6}$$

†In [4], Naghdi and Trapp have shown that the symmetry of e^p_{KL} follows from a physically plausible work assumption which will be discussed at the end of this section.

The response functions $\hat{s}_{MN}$ and $\hat{e}_{MN}$ in (5) and (6) are taken to be smooth.

We admit the existence of a continuously differentiable scalar-valued yield (or loading) function $g(\mathfrak{u})$ such that, for fixed values of e^p_{KL} and κ, the equation

$$g(\mathfrak{u}) = 0 \tag{7}$$

represents a closed orientable hypersurface $\partial\mathcal{E}$ of dimension five enclosing a region $\mathcal{E}$ of strain space. The work-hardening parameter κ is chosen so that $g(\mathfrak{u}) < 0$ for all points in the interior of the region $\mathcal{E}$. The hypersurface $\partial\mathcal{E}$ is called the yield (or loading) surface in strain space. Corresponding to a motion χ_i, we may associate with each particle of the body a continuous oriented curve C_e in strain space. This curve will be called a strain trajectory. The strain trajectories are restricted to lie initially in $\mathcal{E}$ or on its surface $\partial\mathcal{E}$, i.e.,

$$g(\mathfrak{u}) \leq 0 \tag{8}$$

initially on C_e.

The constitutive equations for $\dot{\kappa}$ and $\dot{e}^p_{KL}$ are [1]

$$\dot{\kappa} = c_{KL}\dot{e}^p_{KL} \quad , \tag{9}$$

and

$$\dot{e}^p_{KL} = \begin{cases} 0 & \text{if} \quad g < 0 \quad , & \text{(a)} \\ 0 & \text{if} \quad g = 0 \text{ and } \hat{g} < 0 \quad , & \text{(b)} \\ 0 & \text{if} \quad g = 0 \text{ and } \hat{g} = 0 \quad , & \text{(c)} \\ \lambda\rho_{KL}\hat{g} & \text{if} \quad g = 0 \text{ and } \hat{g} > 0 \quad , & \text{(d)} \end{cases} \tag{10}$$

where $c_{KL} = \bar{c}_{KL}(\mathfrak{u})$ is a symmetric tensor-valued function, a superposed dot indicates material time differentiation,

$$\hat{g} = \frac{\partial g}{\partial e_{MN}} \dot{e}_{MN} \quad , \tag{11}$$

and where $\lambda = \bar{\lambda}(u)$ and $\rho_{KL} = \bar{\rho}_{KL}(u)$ are†, respectively, a scalar-valued function and a symmetric tensor-valued function. The quantity $\hat{g}$ is the inner product of the tangent vector $\dot{e}_{MN}$ to a strain trajectory C_e and the vector $\frac{\partial g}{\partial e_{MN}}$. When $g = 0$ and $\partial g/\partial e_{MN} \neq 0$, $\hat{g}$ gives the inner product of $\dot{e}_{MN}$ and the outward normal vector to the yield surface $\partial\mathcal{E}$, where the notation $\partial g/\partial e_{MN}$ stands for the symmetric form $\frac{1}{2}(\partial g/\partial e_{MN} + \partial g/\partial e_{NM})$. The conditions involving g and $\hat{g}$ in (10) are the loading criteria of the strain space formulation. Using conventional terminology, these four conditions in the order listed correspond to (a) an elastic state (or point in strain space); (b) unloading from an elastic-plastic state, i.e., a point in strain space for which $g = 0$; (c) neutral loading from an elastic-plastic state; and (d) loading from an elastic-plastic state. We assume that the coefficient of $\hat{g}$ in (10d) is nonzero on the yield surface and, without loss in generality, we then set

$$\rho_{KL} \neq 0 \quad , \quad \lambda > 0 \quad . \tag{12}$$

In order to provide a geometrical interpretation of the conditions (10), we need to record the material time derivative of the loading function, namely

$$\dot{g} = \hat{g} + \frac{\partial g}{\partial e^p_{KL}} \dot{e}^p_{KL} + \frac{\partial g}{\partial \kappa} \dot{\kappa} \quad , \tag{13}$$

where (11) has been used. It follows from (7), (9) and (10a) that in an elastic state the strain trajectory C_e lies in the interior of $\mathcal{E}$, which is referred to as the elastic region in strain space, and the yield surface $\partial\mathcal{E}$ remains stationary. Similarly, by (7), (9), (10b) and (13), during unloading the strain trajectory C_e intersects the yield surface $\partial\mathcal{E}$ and is moving in an inwardly direction, with the function g decreasing, while $\partial\mathcal{E}$ itself

†Our notation λ corresponds to $\bar{\lambda}$ in [1].

remains stationary. Likewise, from (7), (9), (10c) and (13) during neutral loading the strain trajectory C_e lies in the yield surface $\partial\mathcal{E}$ while the latter remains stationary and $\dot{g}=0$. Finally, from (7), (9), (10d) and (13) during loading the strain trajectory C_e intersects $\partial\mathcal{E}$ and is moving in an outwardly direction. It is stipulated in this case that $\partial\mathcal{E}$ is locally pushed outwards by the strain trajectory C_e so that†

$$\dot{g} = 0 \quad , \tag{14}$$

if $g=0$, $\hat{g}>0$. Thus, positive values of the function g can never be reached on a strain trajectory and the condition (8) holds for all time. It follows from (9), (10d), (13) and (14) that during loading

$$\hat{g}\{1 + \lambda\rho_{KL}(\frac{\partial g}{\partial e^p_{KL}} + \frac{\partial g}{\partial \kappa} c_{KL})\} = 0 \quad . \tag{15}$$

Therefore, since the coefficient of $\hat{g}$ is independent of $\dot{e}_{MN}$, we have

$$1 + \lambda\rho_{KL}(\frac{\partial g}{\partial e^p_{KL}} + \frac{\partial g}{\partial \kappa} c_{KL}) = 0 \tag{16}$$

at all points on the yield surface $\partial\mathcal{E}$ through which loading can occur. We note that equations $(5)_1$ and $(6)_1$ hold during loading, neutral loading, unloading and in an elastic state.

For a given motion χ_i and an associated strain trajectory C_e we may utilize the constitutive equations $(5)_1$, (9) and (10), together with appropriate initial conditions for e^p_{KL} and κ, to obtain the corresponding stress trajectory C_s, a continuous oriented curve in stress space. In a similar fashion $(6)_1$ may be used to obtain C_e from C_s. Furthermore, for a given loading function $g(\mathfrak{u})$, with the aid of $(6)_1$, we can obtain a corresponding function $f(\mathfrak{v})$ through the formula

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†In the literature on plasticity this is called the "consistency" condition, namely that loading from an elastic-plastic state leads to another elastic-plastic state. For references and background information in the context of a stress space formulation, see for example Naghdi [6, pages 141, 137].

$$g(\mathfrak{u}) = g(\hat{e}_{KL}(\mathfrak{v}), e^p_{KL}, \kappa) = f(\mathfrak{v}) \quad , \tag{17}$$

where the variables $\mathfrak{u}$ and $\mathfrak{v}$ are defined by $(5)_2$ and $(6)_2$, respectively. Conversely, $(5)_1$ may be used to obtain g from f. Because of the assumed smoothness of $(6)_1$, for fixed values of e^p_{KL} and κ, the equation

$$f(\mathfrak{v}) = 0 \tag{18}$$

represents a hypersurface $\partial\mathcal{S}$ in stress space having the same geometrical properties as the hypersurface $\partial\mathcal{E}$ in strain space. The region enclosed by $\partial\mathcal{S}$ is denoted $\mathcal{S}$. It follows from (17) that a point in strain space belongs to the elastic region $\mathcal{E}$ (i.e., $g(\mathfrak{u})<0$) if and only if the corresponding point in stress space satisfies $f(\mathfrak{v})<0$ and hence belongs to $\mathcal{S}$. Similarly, by (17) and (18) a point in strain space belongs to the yield surface $\partial\mathcal{E}$ (i.e., $g=0$) if and only if the corresponding point in stress space belongs to $\partial\mathcal{S}$ (i.e., $f=0$). Hence, we refer to the interior of $\mathcal{S}$ as the elastic region in stress space, and to $\partial\mathcal{S}$ as the yield (or loading) surface in stress space. We have seen that (8) holds for all t. Therefore, by (17), every stress trajectory C_s is restricted to lie in $\mathcal{S}$ or on its surface $\partial\mathcal{S}$ and positive values of f can never be reached. We note that any function of variables $\mathfrak{u}$ can be written as a different function of variables $\mathfrak{v}$ and vice versa, e.g., $C_{KL} = \bar{C}_{KL}(\mathfrak{u}) = \tilde{C}_{KL}(\mathfrak{v})$ which occurs in (9).

In [1] a comparative basis was provided between the two independent sets of loading criteria for the stress space and the strain space formulations. A correspondence between the two sets of loading criteria was established for all conditions except that during loading. The approach in the present paper differs from that of [1] in that the loading criteria of the strain space formulation are regarded as primary and associated loading conditions in stress space are deduced from the former‡. Although in the examination of the loading criteria our starting point and conclusions are different, the arguments employed parallel those of [1]. Thus, taking the material time derivative of (17) and making use of (13) we obtain

$$\dot{f} = \hat{f} + \frac{\partial f}{\partial e^p_{KL}} \dot{e}^p_{KL} + \frac{\partial f}{\partial \kappa} \dot{\kappa} = \hat{g} + \frac{\partial g}{\partial e^p_{KL}} \dot{e}^p_{KL} + \frac{\partial g}{\partial \kappa} \dot{\kappa} \quad , \tag{19}$$

where

$$\hat{f} = \frac{\partial f}{\partial s_{MN}} \dot{s}_{MN} \quad . \tag{20}$$

The quantity $\hat{f}$ is, of course, the inner product of the tangent vector $\dot{s}_{MN}$ to a stress trajectory C_s and the vector $\partial f/\partial s_{MN}$. In view of (19)

$$\hat{f} = \hat{g} \quad \text{if} \quad \dot{e}^p_{KL} = 0 \text{ and } \dot{\kappa} = 0 \quad . \tag{21}$$

Considering an elastic state, $f = g < 0$, (10a) holds, $\dot{\kappa} = 0$ by (9) and hence $\hat{f} = \hat{g}$ by (21). Since the yield surface $\partial\mathcal{E}$ in strain space is stationary so also is the yield surface $\partial\mathcal{S}$ in stress space. The stress trajectories remain in the interior of $\mathcal{S}$. It is clear from (9), (10b), (17) and (21) that* $f = 0$ and $\hat{f} < 0$ if $g = 0$ and $\hat{g} < 0$. In this case ($g = 0$, $\hat{g} < 0$) the stress trajectory C_s intersects $\partial\mathcal{S}$ and is directed inwards, with the function f decreasing in value, while $\partial\mathcal{S}$

‡These derived conditions are not the same as the loading criteria usually assumed in the stress space formulation.

*In [1] it was possible to prove the converse of this statement because of the independent loading criteria that were assumed in the stress-space formulation. It will become clear presently that in the context of this paper the converse statement does not hold.

itself remains stationary. It follows from (9), (10c), (17) and (21) that** $f=0$ and $\hat{f}=0$ if $g=0$ and $\hat{g}=0$. In this case ($g=0$, $\hat{g}=0$) the stress trajectory C_s lies in the surface $\partial\mathcal{S}$ which remains stationary and $\dot{f}=0$.

In the case of loading from an elastic-plastic state, it follows from (9), (10d) and (19) that

$$\frac{\hat{f}}{\hat{g}} = 1 + \lambda\rho_{KL}\{(\frac{\partial g}{\partial e^p_{KL}} - \frac{\partial f}{\partial e^p_{KL}}) + (\frac{\partial g}{\partial \kappa} - \frac{\partial f}{\partial \kappa})C_{KL}\} \quad , \quad (g=0,\ \hat{g}>0) \quad . \tag{22}$$

In the developments that follow, the quotient $\hat{f}/\hat{g}$ can be expressed in a number of different forms. In order to establish one such form we note that by (17), $(5)_1$, $(6)_1$ and the chain rule of differentiation

$$\begin{aligned} \frac{\partial g}{\partial e^p_{KL}} - \frac{\partial f}{\partial e^p_{KL}} &= \frac{\partial f}{\partial s_{MN}} \frac{\partial \hat{s}_{MN}}{\partial e^p_{KL}} = -\frac{\partial g}{\partial e_{MN}} \frac{\partial \hat{e}_{MN}}{\partial e^p_{KL}} \quad , \\ \frac{\partial g}{\partial \kappa} - \frac{\partial f}{\partial \kappa} &= \frac{\partial f}{\partial s_{MN}} \frac{\partial \hat{s}_{MN}}{\partial \kappa} = -\frac{\partial g}{\partial e_{MN}} \frac{\partial \hat{e}_{MN}}{\partial \kappa} \quad . \end{aligned} \tag{23}$$

With the use of (23), (22) can be rewritten as

$$\begin{aligned} \frac{\hat{f}}{\hat{g}} &= 1 + \lambda\rho_{KL} \frac{\partial f}{\partial s_{MN}} \{\frac{\partial \hat{s}_{MN}}{\partial e^p_{KL}} + \frac{\partial \hat{s}_{MN}}{\partial \kappa} C_{KL}\} \\ &= 1 - \lambda\rho_{KL} \frac{\partial g}{\partial e_{MN}} \{\frac{\partial \hat{e}_{MN}}{\partial e^p_{KL}} + \frac{\partial \hat{e}_{MN}}{\partial \kappa} C_{KL}\} \quad , \quad (g=0,\ \hat{g}>0) \quad . \end{aligned} \tag{24a}$$

Another useful form of the quotient $\hat{f}/\hat{g}$ that may be derived from (22) with the help of $(12)_2$ and (16) is

$$\frac{\hat{f}}{\hat{g}} = -\lambda\rho_{KL}\{\frac{\partial f}{\partial e^p_{KL}} + \frac{\partial f}{\partial \kappa} C_{KL}\} = \frac{\rho_{KL}\{\frac{\partial f}{\partial e^p_{KL}} + \frac{\partial f}{\partial \kappa} C_{KL}\}}{\rho_{MN}\{\frac{\partial g}{\partial e^p_{MN}} + \frac{\partial g}{\partial \kappa} C_{MN}\}} \quad , \quad (g=0,\ \hat{g}>0) \quad . \tag{24b}$$

Since the right-hand side of $(24b)_2$ is independent of rates, it is clear that the quotient $\hat{f}/\hat{g}$ is independent of rates and has the same value for all strain trajectories through a given elastic-plastic point on $\partial\mathcal{E}$. Also, in view of (17), $\hat{f}/\hat{g}$ is dimensionless. Clearly a knowledge of all constitutive equations is required

**See the previous footnote.

for the calculation of $\hat{f}/\hat{g}$.

We now turn to the work assumption of Naghdi and Trapp [4,7]. Starting with the assumption that the external work done on an elastic-plastic body in any smooth spatially homogeneous closed strain trajectory is nonnegative, it was demonstrated[†] in [4] that

$$\Big\{\frac{\partial\hat{s}_{MN}}{\partial e^{p}_{KL}}+\frac{\partial\hat{s}_{MN}}{\partial\kappa}\,C_{KL}\Big\}\dot{e}^{p}_{KL}\dot{e}_{MN}\leqq 0 \tag{25}$$

during loading or neutral loading, i.e., when $g=0$, $\hat{g}\geqq 0$. In the case of neutral loading it follows from (10c) that the left-hand side of (25) vanishes and (25) is satisfied trivially, while in the case of loading, it follows from (10d) and $(12)_2$ that (25) becomes

$$\Big\{\frac{\partial\hat{s}_{MN}}{\partial e^{p}_{KL}}+\frac{\partial\hat{s}_{MN}}{\partial\kappa}\,C_{KL}\Big\}\rho_{KL}\dot{e}_{MN}\leqq 0 \tag{26}$$

with $g=0$ and $\hat{g}>0$. The coefficient of $\dot{e}_{MN}$ in (26) is itself independent of $\dot{e}_{MN}$ and the inequality must hold for all choices of $\dot{e}_{MN}$ that satisfy $\hat{g}>0$. Therefore, by the same argument used in Section 5 of [4], we deduce that

$$\Big\{\frac{\partial\hat{s}_{MN}}{\partial e^{p}_{KL}}+\frac{\partial\hat{s}_{MN}}{\partial\kappa}\,C_{KL}\Big\}\rho_{KL}=-\gamma^{*}\,\frac{\partial g}{\partial e_{MN}} \tag{27}$$

evaluated on the yield surface $g=0$, where the scalar function γ^{*} satisfies

$$\gamma^{*}=\bar{\gamma}^{*}(\mathfrak{u})\geqq 0\quad . \tag{28}$$

We emphasize that (27) holds even for a motion that is not homogeneous[‡].

In order to compare (27) with the results of [4] we multiply on both sides of (27) by $\lambda\hat{g}$ and utilize (10d) to obtain

$$\Big\{\frac{\partial\hat{s}_{MN}}{\partial e^{p}_{KL}}+\frac{\partial\hat{s}_{MN}}{\partial\kappa}\,C_{KL}\Big\}\dot{e}^{p}_{KL}=-\lambda\gamma^{*}\hat{g}\,\frac{\partial g}{\partial e_{MN}}\quad,\quad (g=0,\ \hat{g}>0)\quad . \tag{29}$$

[†]See equations (5.2), (5.3) and (4.11) of [4]; the notation H_{MN} in [4] corresponds to C_{MN} in the present paper.

[‡]For a discussion of this point, see [4, p. 40] or [7, p. 63].

Recalling the restrictions $(12)_2$ and (28), we define a function γ by*

$$\gamma = \lambda\gamma^* \hat{g} \geqq 0 \tag{30}$$

and from (29) obtain

$$\left\{\frac{\partial \hat{s}_{MN}}{\partial e^p_{KL}} + \frac{\partial \hat{s}_{MN}}{\partial \kappa} C_{KL}\right\}\dot{e}^p_{KL} = -\gamma \frac{\partial g}{\partial e_{MN}} \quad , \quad (g=0,\ \hat{g}>0) \quad . \tag{31}$$

Equation (31) is the same as (5.4) of [4]. We note that (31) involves rates, while (27) does not. We have shown that (27) implies (31). Conversely, it follows at once from (10d) and $(12)_2$ that (31) implies (27) with $\gamma^* = \gamma/\lambda\hat{g}$, as in (30).

From (27) and $(24a)_1$ follows an expression for $\hat{f}/\hat{g}$ in the form:

$$\frac{\hat{f}}{\hat{g}} = 1 - \lambda\gamma^*\Lambda \quad , \quad (g=0,\ \hat{g}>0) \quad , \tag{32}$$

where

$$\Lambda = \frac{\partial f}{\partial s_{MN}} \frac{\partial g}{\partial e_{MN}} \quad , \quad (g=f=0) \quad . \tag{33}$$

The quantity Λ (when nonzero) represents the inner product of the normal to the yield surface $\partial\mathcal{S}$ in stress space and the normal to the yield surface $\partial\mathcal{E}$ in strain space.

For some purposes it is convenient to express the constitutive equation $(5)_1$ in terms of an equivalent set of kinematical variables in the form

$$s_{MN} = \bar{s}_{MN}(e_{KL}-e^p_{KL}, e^p_{KL}, \kappa) \quad . \tag{34}$$

Suppose that the partial derivatives $\partial\hat{s}_{MN}/\partial e_{KL}$ possess the symmetries†

*The function γ on the right-hand side of (31) depends on the variables e_{MN}, e^p_{MN}, κ and $\dot{e}_{MN}$.

†This is equivalent to the condition that $\hat{s}_{MN}$ be derivable from a potential, as indeed is the case in the general thermodynamical theory (see Section 4 of [3]) of which the present development may be regarded as corresponding to the isothermal case. The existence of a potential in the purely mechanical theory can also be demonstrated by an argument based on the work postulate of [4].

$\frac{\partial \hat{s}_{MN}}{\partial e_{KL}} = \frac{\partial \hat{s}_{KL}}{\partial e_{MN}}$, then in a manner similar to that in [4, Sec. 5] from (27) we obtain‡

$$\{-\delta_{PK}\delta_{QL} + \frac{\partial \hat{e}_{PQ}}{\partial s_{MN}} (\frac{\partial \bar{s}_{MN}}{\partial e^p_{KL}} + \frac{\partial \bar{s}_{MN}}{\partial \kappa} c_{KL})\}\rho_{KL} = -\gamma^* \frac{\partial f}{\partial s_{PQ}} \quad , \quad (g = f = 0) \quad . \tag{35}$$

It is clear from (35), (30), (10d) and (12) that if the response function $\bar{s}_{MN}$ in (34) is independent of its second and third arguments, i.e., if $\frac{\partial \bar{s}_{MN}}{\partial e^p_{KL}} = 0$, $\frac{\partial \bar{s}_{MN}}{\partial \kappa} = 0$, then

$$\rho_{KL} = \gamma^* \frac{\partial f}{\partial s_{KL}} \neq 0 \quad , \quad \dot{e}^p_{KL} = \gamma \frac{\partial f}{\partial s_{KL}} \quad , \quad (g = 0) \tag{36}$$

and ρ_{KL} is directed along the normal to the yield surface $\partial \mathcal{S}$ in stress space, as also is $\dot{e}^p_{KL}$ during loading. It follows from $(36)_1$ and (28) that in this case

$$\gamma^* > 0 \quad , \quad (g = 0) \tag{37}$$

and hence, during loading, in view of (30) and $(12)_2$

$$\gamma > 0 \tag{38}$$

also. When ρ_{KL} satisfies $(36)_1$, (16) can be written as

$$1 + \lambda\gamma^* \frac{\partial f}{\partial s_{KL}} (\frac{\partial g}{\partial e^p_{KL}} + \frac{\partial g}{\partial \kappa} c_{KL}) = 0 \quad , \quad (g = 0) \quad . \tag{39}$$

The last result can be used to solve for the product $\lambda\gamma^*$ and (30) then gives γ. Also, we may set γ^* equal to an arbitrary positive scalar-valued function of the variables $\mathfrak{u}$ and then use (39) to determine λ. Thus, in the special case in which $\bar{s}_{MN}$ in (34) depends only on its first argument, no constitutive equation is needed for ρ_{KL}.

We observe that when ρ_{KL} satisfies $(36)_1$, then $(24b)_1$ may be used to express $\hat{f}/\hat{g}$ as

$$\hat{f}/\hat{g} = \lambda\gamma^*\Gamma \quad , \quad (g = 0, \hat{g} > 0) \quad , \tag{40}$$

‡The symmetry of ρ_{KL} and hence e^p_{KL} follows from (35). See [4, Sec. 5].

where

$$\Gamma = -\frac{\partial f}{\partial s_{KL}} \{\frac{\partial f}{\partial e^p_{KL}} + \frac{\partial f}{\partial \kappa} c_{KL}\} \quad , \quad (g = 0) \quad . \tag{41}$$

Also, in view of (32), (40), (37) and $(12)_2$,

$$0 < \frac{1}{\lambda \gamma^*} = \Gamma + \Lambda \quad , \quad \frac{\hat{f}}{\hat{g}} = \frac{\Gamma}{\Gamma + \Lambda} \quad , \quad (g = 0, \ \hat{g} > 0) \quad . \tag{42}$$

3. Strain-Hardening Response. Geometrical Interpretation.

The quotient $\hat{f}/\hat{g}$ which occurs in $(24b)_2$ and related equations in Section 2, is utilized here to define three distinct types of strain-hardening response for an elastic-plastic material. These definitions are as follows: An elastic-plastic material is said to be hardening, softening or exhibiting perfectly plastic behavior during loading ($g = 0$, $\hat{g} > 0$) according to whether‡

$$\begin{aligned} &(a) \quad \hat{f}/\hat{g} > 0 \quad \text{(for hardening)} \quad , \\ &(b) \quad \hat{f}/\hat{g} < 0 \quad \text{(for softening)} \quad , \\ &(c) \quad \hat{f}/\hat{g} = 0 \quad \text{(for perfectly plastic)} \quad . \end{aligned} \tag{43}$$

We emphasize that a condition of loading, i.e., $g = 0$ and $\hat{g} > 0$, is always presupposed in the definitions (43). It is worth observing from $(24b)_2$ that once ρ_{KL}, C_{KL}, g and $(6)_1$ are specified, then the strain-hardening response is also known.

We now provide a geometrical interpretation of the definitions (43). We recall that during loading, since $g = 0$, $\hat{g} > 0$ and $\dot{g} = 0$, the strain trajectory C_e is intersecting the yield surface $\partial\mathcal{E}$ and locally pushing it outwards. Since $g = 0$ and $\dot{g} = 0$ it follows from (17) and (19) that $f = 0$ and $\dot{f} = 0$ also, and the corresponding stress trajectory C_s is intersecting the yield surface $\partial\mathcal{S}$ in stress space. If the material is hardening, (43a) holds and the stress trajectory C_s is directed outwards and is pushing the surface $\partial\mathcal{S}$ locally outwards. But, (43b) holds if the material is softening and the stress trajectory is directed inwards and is pulling the surface $\partial\mathcal{S}$ locally inwards. In perfectly plastic behavior when (43c) holds, the stress trajectory continues to lie on the yield surface $\partial\mathcal{S}$ which is stationary.

Thus while during loading the stress trajectory C_e is always pushing the

‡Since $\hat{g}$ is always positive in (43), we could use only $\hat{f}$ in providing the above definitions. But the use of the quotient $\hat{f}/\hat{g}$, which is rate-independent, is preferable in general. For certain purposes, however, it is useful to employ only $\hat{f}$ as in $(58)_3$ and (59) of Section 4.

yield surface $\partial\mathcal{E}$ in strain space locally outwards, the corresponding yield surface $\partial\mathcal{S}$ in stress space may be moving concurrently outwards, inwards or may be stationary depending on the type of strain-hardening response being exhibited. The actual occurrence of such behavior has been indicated in Section 1 with reference to the simple tension test. The usual stress space formulation of plasticity theory introduces a priori loading criteria in stress space and stipulates that during loading the yield surface in stress space can never move inwards. Viewed in the context of the present development, the usual stress space formulation of plasticity is seen to include only a hardening type response and to exclude softening and perfectly plastic responses*. Figure 2 illustrates the three types of material behavior discussed above.

The definitions for hardening, softening and perfectly plastic behavior introduced in (43) require the use of yield surfaces both in strain space and stress space. However, it may be noted that our terminology for softening and hardening seems to be consistent with the geometrical sense of these terms employed in a stress space formulation by Edelman and Drucker [8]; see Fig. 5 of their paper. Also, Prager [9] employs the terms hard and soft with reference to material behavior, but his sense of these terms differs from ours: In [9], a hard material is one whose stress-strain curve always lies above a given straight line (representing linear elastic response) with the deviation from linear behavior increasing for larger deformation; a soft material is one whose stress-strain curve always lies below the straight line with the deviation increasing for larger deformation.

In what follows, we frequently need to refer to a set of conditions which must be satisfied by various functions and material coefficients, and which arise from characterization of strain hardening response. To avoid undue

*In the context of the present paper, it is not possible to formulate loading criteria in stress space using only f and $\hat{f}$.

repetition we denote this set of conditions by H and write

$$H : \begin{cases} > 0 \text{ if and only if the material is hardening} \quad , & \text{(a)} \\ < 0 \text{ if and only if the material is softening} \quad , & \text{(b)} \\ = 0 \text{ if and only if the material is exhibiting} \\ \qquad \text{perfectly plastic behavior} \quad . & \text{(c)} \end{cases} \tag{44}$$

Returning to the definitions (43) and recalling $(24b)_1$ and $(12)_2$, it is seen that

$$-\rho_{KL}\{\frac{\partial f}{\partial e^p_{KL}} + \frac{\partial f}{\partial \kappa} c_{KL}\} \text{ satisfies conditions H} \quad . \tag{45}$$

It is worth mentioning that the usual treatment of an elastic-perfectly plastic material (see, for example, [2, Sec. 9]) in stress space requires the use of a yield condition of the form $f(s_{KL}) = \text{const.}$ and the quantity on the left-hand side of (45) indeed vanishes identically in this case.

With the use of the definitions (43a,b), we now obtain an expression for the rate of plastic strain which is valid in regions of hardening and softening behavior only. Thus, by (10d), (12), $(24b)_1$ and (43a), in a region of hardening $\dot{e}^p_{KL}$ can be related to $\hat{f}$ through the expression

$$\dot{e}^p_{KL} = \lambda \rho_{KL} \frac{\hat{f}}{\hat{f}/\hat{g}} = - \frac{\rho_{KL} \hat{f}}{\rho_{MN}\{\frac{\partial f}{\partial e^p_{MN}} + \frac{\partial f}{\partial \kappa} c_{MN}\}} \neq 0 \tag{46}$$

with (43a) and (45a) holding†, while in a region of softening $\dot{e}^p_{KL}$ is again given by (46) but now with (43b) and (45b) holding; in both cases, the sign of

†The equation number (45a) refers to (45) along with part (a) of condition H.

TABLE 1

Summary of loading criteria in strain space and associated conditions in stress space

Elastic	$g<0$	implies	$f<0$
Unloading	$g=0,\ \hat{g}<0$	implies	$f=0,\ \hat{f}<0$
Neutral Loading	$g=0,\ \hat{g}=0$	implies	$f=0,\ \hat{f}=0$
Loading	$g=0,\ \hat{g}>0$ (a) hardening (b) softening (c) perfectly plastic	implies	$f=0,\ \hat{f}>0$ $f=0,\ \hat{f}<0$ $f=0,\ \hat{f}=0$

the coefficient of ρ_{KL} in (46) is positive. For perfectly plastic behavior, it is clear from (10d), $(24b)_1$, (43c) and (45c) that $\dot{e}^p_{KL}$ cannot be expressed as a product involving $\hat{f}$ and must be calculated from (10d). For convenience, a summary of the relationships between the loading criteria in strain space and the associated conditions in stress space is provided in Table 1.

In the remainder of this section, we discuss some special cases of the foregoing results which are of particular interest in view of their simplicity. The first two of these (see cases (a) and (b) below) examine the consequences on strain hardening behavior of certain restrictions on the stress response functions $\hat{s}_{MN}$ in $(5)_1$ and $\bar{s}_{MN}$ in (34). The third (see case (c) below) pertains to a limiting behavior of strain hardening response, i.e., saturation hardening and softening.

(a) Consider the special case of $(5)_1$ for which the stress response is independent of its last two arguments, i.e.,

$$\frac{\partial \hat{s}_{MN}}{\partial e^p_{KL}} = 0 \quad , \quad \frac{\partial \hat{s}_{MN}}{\partial \kappa} = 0 \quad . \tag{47}$$

Then, by (23) we have

$$\frac{\partial g}{\partial e^p_{KL}} = \frac{\partial f}{\partial e^p_{KL}} \quad , \quad \frac{\partial g}{\partial \kappa} = \frac{\partial f}{\partial \kappa}$$

and hence by (24a) or (22)

$$\hat{f}/\hat{g} = 1 \quad . \tag{48}$$

Recalling the definitions (43), it is clear that a material for which $(47)_{1,2}$ hold can never exhibit softening or perfectly plastic behavior. If conditions (47) are satisfied and if $\partial g/\partial e_{MN} \neq 0$, it follows from (27) and (30) that

$$\gamma^* = 0 \quad , \quad \gamma = 0 \quad . \tag{49}$$

Conversely, if $(49)_1$ is satisfied so is $(49)_2$ and then (48) holds by virtue of (32).

(b) Consider the special case of (34) for which the stress response is independent of its last two arguments. In this case, the results (36) to (42) hold. It then follows from (37), (40), (43) and $(12)_2$ that

$$\Gamma \text{ satisfies conditions H} \quad . \tag{50}$$

With the use of $(36)_1$ and (40), in a region of hardening or softening (46) becomes

$$\dot{e}^p_{KL} = \frac{\hat{f}}{\Gamma} \frac{\partial f}{\partial s_{KL}} \neq 0 \quad , \tag{51a}$$

while it follows from (50c), (42), $(36)_1$ and (10d) that in a region of perfectly plastic behavior

$$\dot{e}^p_{KL} = \frac{\hat{g}}{\Lambda} \frac{\partial f}{\partial s_{KL}} \neq 0 \quad . \tag{51b}$$

(c) Caulk and Naghdi [5] have previously introduced a definition of saturation hardening in connection with their discussion of hardening response in cyclic loading of metallic materials (see Eq. (19) in [5]). In view of the definitions (43), it is of interest to reexamine here the notion of saturation hardening. Thus, for our present purpose, an elastic-plastic material is said to exhibit saturation hardening along a strain trajectory C_e (or a stress trajectory C_s) if and only if there exists a constant K_h such that[§]

$$\lim_{t\to\infty} \hat{f}/\hat{g} = K_h > 0 \quad (g = 0,\ \hat{g} > 0) \quad . \tag{52a}$$

Similarly a material exhibits saturation softening along a strain trajectory C_e if and only if there exists a constant K_s such that

$$\lim_{t\to\infty} \hat{f}/\hat{g} = K_s < 0 \quad (g = 0,\ \hat{g} > 0) \quad . \tag{52b}$$

[§]In the definitions (52a,b) we have excluded for convenience the equality sign. If the limit of the left-hand sides of (52a,b) is zero, we say that the material saturates to a perfectly plastic behavior.

4. Strain Hardening Response for Special Constitutive Equations.

We consider now in some detail the nature of the hardening response in small deformation of metals whose behavior is characterized by a simple set of constitutive equations appropriate for elastic-plastic materials which are homogeneous and initially isotropic in their reference state. First, we recall that the infinitesimal elastic strain tensor is defined by $e^e_{KL} = e_{KL} - e^p_{KL}$ and note that with $e^p_{KL} = 0$ in the reference configuration, $e^e_{KL} = 0$ also there. It is convenient to utilize a standard decomposition for second order tensors. Thus, for example, in the case of the stress tensor, we have

$$s_{KL} = \bar{s}\delta_{KL} + \tau_{KL} \quad , \quad \bar{s} = \frac{1}{3} s_{KK} \quad ,$$

where $\bar{s}\delta_{KL}$ is the spherical part of s_{KL}, τ_{KL} is the deviatoric (traceless) part of s_{KL} and $\bar{s}$ is the mean normal stress. In a similar manner we decompose $e_{KL}, e^p_{KL}, e^e_{KL}$ into spherical parts $\bar{e}\delta_{KL}, \bar{e}^p\delta_{KL}, \bar{e}^e\delta_{KL}$ and deviatoric parts γ_{KL}, $\gamma^p_{KL}, \gamma^e_{KL}$.

Let the stress response function in $(5)_1$ be specified by generalized Hooke's law, namely

$$\tau_{KL} = 2\mu\gamma^e_{KL} \quad , \quad \bar{s} = 3k\bar{e}^e \quad , \tag{53}$$

and the coefficient function $\mathcal{C}_{KL}$ for the rate of work-hardening response in (9) in the form [10]

$$\mathcal{C}_{KL} = \beta\tau_{KL} + \phi\bar{s}\delta_{KL} \quad , \tag{54}$$

where μ is the shear modulus, k the bulk modulus and β and ϕ are constants. With the use of the decompositions just noted, the loading functions $f(\mathcal{U})$ and $g(\mathcal{U})$ can be written as different functions $\bar{f}(\tau_{MN}, \bar{s}, \gamma^p_{MN}, \bar{e}^p, \kappa)$ and $\bar{g}(\gamma_{MN}, \bar{e}, \gamma^p_{MN}, \bar{e}^p, \kappa)$. In this section, we restrict attention to special loading functions of the form

$$f(\psi) = \bar{f}(\tau_{MN},\bar{s},\gamma^p_{MN},\bar{e}^p,\kappa) = \tau_{KL}\tau_{KL} + 3\psi\bar{s}^2 - \kappa \quad ,$$
$$g(\mu) = \bar{g}(\gamma_{MN},\bar{e},\gamma^p_{MN},\bar{e}^p,\kappa) = 4\mu^2(\gamma_{KL}-\gamma^p_{KL})(\gamma_{KL}-\gamma^p_{KL}) + 27\psi k^2(\bar{e}-\bar{e}^p)^2 - \kappa \quad , \tag{55}$$

where ψ is a constant and where (17) and (53) have been used[§]. Utilizing formulas of the type[†]

$$\frac{\partial f}{\partial s_{MN}} = \frac{\partial \bar{f}}{\partial \tau_{MN}} - \frac{1}{3}\left(\frac{\partial \bar{f}}{\partial \tau_{KK}} - \frac{\partial \bar{f}}{\partial \bar{s}}\right)\delta_{MN} \tag{56}$$

and recalling (20) and (11), it can be easily shown that

$$\frac{\partial f}{\partial s_{MN}} = 2(\tau_{MN} + \psi\bar{s}\delta_{MN}) \quad , \quad \hat{f} = 2(\tau_{MN}\dot{\tau}_{MN} + 3\psi\bar{s}\dot{\bar{s}}) \quad ,$$
$$\hat{g} = 2(2\mu\tau_{MN}\dot{\gamma}_{MN} + 9\psi k\bar{s}\dot{\bar{e}}) = \hat{f} + 2(2\mu\tau_{MN}\dot{\gamma}^p_{MN} + 9\psi k\bar{s}\dot{\bar{e}}^p) \tag{57}$$

and the expressions for $\partial g/\partial e_{MN}$ and $\partial f/\partial e^p_{MN}$ may be obtained similarly. We recall that during loading $\hat{g}$ is positive while $g = f = \dot{g} = \dot{f} = 0$. Keeping this in mind, it follows from $(55)_1$ and $(57)_2$ that during loading

$$\tau_{KL}\tau_{KL} + 3\psi\bar{s}^2 - \kappa = 0 \quad , \quad 2(\tau_{KL}\dot{\tau}_{KL} + 3\psi\bar{s}\dot{\bar{s}}) - \dot{\kappa} = 0 \quad , \quad \hat{f} = \dot{\kappa} \tag{58}$$

and hence by the definitions (43),

$$\dot{\kappa} \text{ and } (\tau_{MN}\dot{\tau}_{MN} + 3\psi\bar{s}\dot{\bar{s}}) \text{ both satisfy conditions H} \quad . \tag{59}$$

Clearly for the special constitutive equations used in this section, in view of $(58)_3$ and (59), the strain hardening behavior may be characterized by $\dot{\kappa}$. Furthermore, from $(58)_3$ and (10b,c) during neutral loading it is necessary that $\dot{\kappa} = 0$ and during unloading it is necessary that $\dot{\kappa} < 0$. In this connection, recall Table 1 and the discussion following (21).

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§ The loading function $(55)_1$ does not depend explicitly on plastic strain, but includes a dependency on mean normal stress. When $\psi = 0$ and $\kappa = \text{const.}$, $(55)_1$ reduces to the usual von Mises yield function. A loading function of the type $(55)_1$ was previously employed by Green and Naghdi [10].

† It is understood that in line with the summation convention, our notation $\partial\bar{f}/\partial\tau_{KK}$ in (56) stands for the sum $\partial\bar{f}/\partial\tau_{11} + \partial\bar{f}/\partial\tau_{22} + \partial\bar{f}/\partial\tau_{33}$.

The stress response (53) may be regarded as a special case of that in (34) with the last two arguments absent; and, in addition, the symmetry conditions mentioned following (34) are satisfied by (53). Hence, in addition to (36) to (42) the special results obtained at the end of Section 3 [see case (b) following Eq. (49)] remain valid here. Thus, using (54) and (55), from (33), (41) and formulas of the type (56) and $(57)_1$, we obtain

$$\Lambda = 4(2\mu\tau_{KL}\tau_{KL} + 9\psi^2\kappa\bar{s}^2) \quad , \quad \Gamma = 2(\beta\tau_{KL}\tau_{KL} + 3\psi\phi\bar{s}^2) \quad . \tag{60}$$

With the use of (60), $\lambda\gamma^*$ and $\hat{f}/\hat{g}$ may be obtained at once from (42). Also, remembering (50), we observe that in this case the right-hand side of $(60)_2$ provides a rate-independent characterization of strain hardening. Constitutive equations for the rate of plastic strain or equivalently for $\dot{\bar{e}}^p$ and $\dot{\gamma}^p_{KL}$ simplify and may now be obtained from (51a) in a region of hardening or softening and from (51b) in a region of perfectly plastic behavior.

Since our development in Sections 2 and 3 began with the strain space (rather than the stress space) formulation as primary and since the quotient $\hat{f}/\hat{g}$ is used to define strain hardening, it is desirable to examine the predictions of various theoretical results in the case of the familiar one-dimensional tension test. To this end, consider a homogeneous deformation sustained by a uniaxial tension $s_{11} = s = \hat{s}(t)$ along the X_1-axis. Then, using a matrix representation for τ_{KL}, we have

$$\|\tau_{KL}\| = \frac{s}{3}\,\|b_{KL}\| \quad , \quad \bar{s} = \frac{s}{3} \geq 0 \quad , \quad \|b_{KL}\| = \begin{Vmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{Vmatrix} \quad , \tag{61}$$

where for brevity we have introduced the constant matrix $\|b_{KL}\|$. Assuming that initial yield occurs at a value s_o of s and a value $\kappa_o > 0$ of κ, the solution can be obtained in a straightforward manner. We omit details, but record here some of the results of interest*:

$$\kappa = \frac{1}{3} s^2(2+\psi) \quad , \quad \kappa_o = \frac{1}{3} s_o^2(2+\psi) \quad , \quad 2+\psi > 0 \quad , \quad s_o > 0 \quad , \tag{62a}$$

$$s > 0 \quad , \quad \kappa > 0 \quad , \quad \dot{e}^p_{11} > 0 \text{ when } g = 0, \hat{g} > 0 \quad , \tag{62b}$$

$$\text{Both } \dot{s} \text{ and } (2\beta+\psi\phi) \text{ satisfy condition H} \quad . \tag{62c}$$

We postpone a discussion of perfectly plastic behavior until later in this section but consider further calculations for the other two types of behavior: In a region of hardening or softening, the elastic and plastic strains are

$$\bar{e}^e = \frac{s}{9k} \quad , \quad \|\gamma^e_{KL}\| = \frac{s}{6\mu} \|b_{KL}\| \quad , \tag{63a}$$

$$\bar{e}^p = \frac{s-s_o}{9k^*} \quad , \quad \|\gamma^p_{KL}\| = \frac{s-s_o}{6\mu^*} \|b_{KL}\| \quad ,$$

$$\|e^p_{KL}\| = \frac{s-s_o}{E^*} \begin{Vmatrix} 1 & 0 & 0 \\ 0 & -\nu^* & 0 \\ 0 & 0 & -\nu^* \end{Vmatrix} \tag{63b}$$

$$\frac{de^p_{11}}{ds} \text{ and } \frac{d\gamma^p_{11}}{ds} \begin{cases} > 0 \text{ if and only if the material is hardening} & , \\ < 0 \text{ if and only if the material is softening} & , \end{cases} \tag{63c}$$

where the constants ν^*, E^*, μ^* and k^* are defined by

$$\nu^* = \frac{1-\psi}{2+\psi} \quad , \quad E^* = \frac{3(2\beta+\psi\phi)}{2(2+\psi)^2} \quad ,$$

$$\mu^* = \frac{E^*}{2(1+\nu^*)} = \frac{2+\psi}{6} E^* \quad , \tag{63d}$$

$$k^* = \frac{E^*}{3(1-2\nu^*)} = \frac{2+\psi}{9\psi} E^* \quad (\psi \neq 0)$$

and

*It is clear from $(58)_3$, (10), $(62a)_{1,3}$ and $(62b)_1$ that during neutral loading, it is necessary that $\dot{s} = 0$ and during unloading it is necessary that $\dot{s} < 0$.

$$E^* \text{ and } \mu^* \text{ satisfy conditions H} \quad . \tag{63e}$$

The constants in (63d) have been defined analogously to the corresponding constants in linear elasticity, e.g., $\mu = \frac{E}{2(1+\nu)}$, $k = \frac{E}{3(1-2\nu)}$, where ν is Poisson's ratio. In the special case that $\psi = 0$, $\nu^* = \frac{1}{2}$ and the expressions for E^*, μ^* simplify while $k^* \to \infty$ as $\psi \to 0$. Continuing our discussion of hardening and softening behavior, it can be shown that when $\psi \neq 0$ (see Appendix A for details) the quotient $\hat{f}/\hat{g}$ can be written as

$$\frac{\hat{f}}{\hat{g}} = \frac{2+\psi}{\frac{1}{3}\,\mathrm{tr}\left(2\left\|\frac{d\gamma_{KL}}{ds}\right\|\,\left\|\frac{d\gamma^e_{KL}}{ds}\right\|^{-1} + \psi\,\frac{d\bar{e}}{ds}\left(\frac{d\bar{e}^e}{ds}\right)^{-1}\|\delta_{KL}\|\right)} \quad , \tag{64}$$

where tr stands for the trace operator. In the special case when $\nu^* = \nu$, (64) reduces to (see Appendix A for details)

$$\frac{\hat{f}}{\hat{g}} = [E\,\frac{de}{ds}]^{-1} = [1 + \frac{de_p}{de_e}]^{-1} \quad , \tag{65a}$$

and by (43a,b)

$$\frac{de}{ds} \text{ and } (1 + \frac{de_p}{de_e}) \begin{cases} > 0 \text{ if and only if the material is hardening} \quad , \\ < 0 \text{ if and only if the material is softening} \quad , \end{cases} \tag{65b}$$

where as in Section 1, we have again used the notation $e = e_{11}$, $e_e = e^e_{11}$, $e_p = e^p_{11}$.

Before closing this section, it is desirable to elaborate briefly on some features of the foregoing results for uniaxial tension, which have been obtained with the use of a special set of constitutive equations. With reference to all three types of strain hardening response defined in (43), it is clear that during loading e^p_{11} is strictly increasing with time by virtue of $(62b)_3$. Moreover, according to (62c) the time rate of stress may be used to characterize strain hardening behavior in uniaxial tension and a characterization of the same behavior is provided by the combination

$(2\beta+\psi\phi)$ of the constitutive coefficients. While the elastic moduli E, μ are always positive, it follows from (63e) that the constants E^*, μ^* are positive in a region of hardening and negative in a region of softening. In the special case of $\nu^* = \nu$, it is clear from (65a) that the quotient $\hat{f}/\hat{g}$ can be expressed in terms of quantities (2) to (4) and indeed (65b) corresponds to the behavior summarized in (4) for uniaxial tension§. Furthermore, with $\psi = 0$ in $(55)_1$, the plastic volume change or equivalently $\dot{e}^p$ vanishes also. The strain-hardening response is then characterized by β, in view of (62c). Also, in a region of hardening or softening the quotient $\hat{f}/\hat{g}$ reduces to (see Appendix A for details)

$$\frac{\hat{f}}{\hat{g}} = [3\mu \frac{d\gamma}{ds}]^{-1} \quad , \tag{66}$$

where we have put $\gamma = \gamma_{11}$ and $\gamma^e = \gamma^e_{11}$.

The significance of the strain space formulation in the case of elastic-perfectly plastic materials was pointed out in [1]. Since the quotient $\hat{f}/\hat{g}$ is used here to define various types of hardening response, it is desirable to indicate the reduction of the present development to the usual perfectly plastic behavior in uniaxial tension. First, we observe that during loading ($g = 0$, $\hat{g} > 0$) for perfectly plastic behavior $\hat{f}/\hat{g} = \Gamma = 0$ by (43c) and (42). It then follows that $\dot{\kappa} = 0$, $\kappa = \kappa_0$, $s = s_0$ by $(58)_3$ and $(62a)_{1,2}$ and that e^p_{11}, although indeterminate‡, is strictly increasing with time in view of $(62b)_3$. Thus, in the context of the present paper, the uniaxial stress-strain curve for elastic-perfectly plastic behavior consists of a linearly elastic portion followed by a horizontal portion and as time progresses the locus of e^p_{11} moves outward along the abscissa of the

§Recall that the special constitutive equations employed in this section are not sufficiently general to predict all details of the stress-strain curve in Fig. 1. Indeed, different choices of the combination $(2\beta+\psi\phi)$ of the coefficients (appropriate for different materials) yield stress-strain curves consisting of straight line segments whose slopes correspond to the rising or falling portions of the curve in Fig. 1.

‡The indeterminacy of e^p_{11} stems from the fact that (51b) in this case reduces to an identity.

s-e curve. This is in agreement with the usual characterization of perfectly plastic behavior in uniaxial tension. We also note that an examination of the solution given by (63a,b) and (63e) easily reveals that hardening (softening) is represented in a stress-strain diagram by a straight line which lies above (below) the horizontal perfectly plastic line. Indeed, since $e_{11} = s/E + (s-s_o)/E^*$, then $de_{11}/ds = 1/E + 1/E^*$ and by standard results for inequalities it follows from (63e) that

$$\infty > \frac{de_{11}}{ds} > \max\{\frac{1}{E}, \frac{1}{E^*}\} \text{ if the material is hardening} \quad ,$$

$$\frac{1}{E} > \frac{de_{11}}{ds} > \frac{1}{E^*} > -\infty \quad \text{ if the material is softening} \quad .$$

Moreover,

$$\infty > \frac{de_{11}}{ds} > \frac{1}{E} \text{ implies that the material is hardening} \quad ,$$

$$-\infty < \frac{de_{11}}{ds} < \frac{1}{E} \text{ implies that the material is softening} \quad .$$

5. Saturation hardening

As in Section 4 we again restrict attention to small deformations of elastic-plastic materials, which are homogeneous and initially isotropic in their reference configuration. We also assume that there is no plastic volume change so that $\bar{e}^p = 0$ in the notation of Section 4. For a fairly large class of metallic materials, it is well known that the stress-strain curves of uniaxial cyclic loading attain -- after several cycles -- saturation hardening. The purpose of this section is to indicate how the development of Sections 2 and 3 can be used to characterize a hardening response that includes saturation behavior and to compare the results with those of Caulk and Naghdi [5].

Starting with a fairly general discussion of loading functions contained in the paper of Green and Naghdi [2], for initially isotropic materials Caulk and Naghdi [5] derived a loading function in the form (see [5, Eqs. $(40)_1$ and $(56)_1$])

$$\begin{gathered} f(\mathfrak{v}) = \bar{f}(\tau_{MN}, \gamma^p_{MN}, \kappa) = \tau_{KL}\tau_{KL} - \alpha\tau_{KL}\gamma^p_{KL} + \sigma\gamma^p_{KL}\gamma^p_{KL} - \kappa \quad , \\ g(\mathfrak{u}) = \bar{g}(\gamma_{MN}, \gamma^p_{MN}, \kappa) = 4\mu^2(\gamma_{KL} - \gamma^p_{KL})(\gamma_{KL} - \gamma^p_{KL}) - 2\alpha\mu(\gamma_{KL} - \gamma^p_{KL})\gamma^p_{KL} + \sigma\gamma^p_{KL}\gamma^p_{KL} - \kappa \quad , \end{gathered} \tag{67}$$

where α and σ are constants and where (53) has been used in writing $(67)_2$. It should be noted that the loading functions $(67)_{1,2}$ depend explicitly on γ^p_{KL} but not on the mean normal stress $\bar{s}$. Here we also adopt $(67)_{1,2}$ but, instead of the hardening response assumed in [5], we specify the coefficient function c_{KL} in (9) by

$$c_{KL} = \hat{\beta}(\kappa)\tau_{KL} + \hat{\eta}(\kappa)\gamma^p_{KL} \quad , \tag{68}$$

which is different from that used in Section 4. The constitutive assumption for c_{KL} in [5] is similar to (68) but with $\hat{\beta}(\kappa)$ and $\hat{\eta}(\kappa)$ specified by

$$\hat{\beta}(\kappa) = \frac{\kappa-\kappa_s}{\kappa_o-\kappa_s}\beta \quad , \quad \hat{\eta}(\kappa) = \frac{\kappa-\kappa_s}{\kappa_o-\kappa_s}\eta \quad , \tag{69}$$

where β and η are constants, κ_o is the value of κ at initial yield and κ_s is the saturation value of κ. Since the stress response (53) is used in this section, in addition to (36) to (42), all the results stated under case (b) at the end of Section 3 are also valid here.

To facilitate the discussion that follows and for later reference, we record the expressions

$$\begin{gathered}
\frac{\partial f}{\partial s_{MN}} = 2\tau_{MN} - \alpha\gamma^p_{MN} \neq 0 \quad , \quad \frac{\partial g}{\partial e_{MN}} = 2\mu\,\frac{\partial f}{\partial s_{MN}} \neq 0 \quad , \quad (g=0) \quad , \\
\hat{f} = (2\tau_{MN}-\alpha\gamma^p_{MN})\dot{\tau}_{MN} \quad , \\
\hat{g} = 2\mu(2\tau_{MN}-\alpha\gamma^p_{MN})\dot{\gamma}_{MN} = \hat{f} + 2\mu(2\tau_{MN}-\alpha\gamma^p_{MN})\dot{\gamma}^p_{MN} \quad ,
\end{gathered} \tag{70}$$

which have been obtained with the use of formulas of the type (56) along with (20), (11), $(36)_1$ and (37). With the help of (70) and recalling the definitions (33) and (41), Λ and Γ are given by

$$\begin{gathered}
\Lambda = 2\mu\,\frac{\partial f}{\partial s_{KL}}\,\frac{\partial f}{\partial s_{KL}} = 2\mu(2\tau_{KL}-\alpha\gamma^p_{KL})(2\tau_{KL}-\alpha\gamma^p_{KL}) > 0 \quad , \\
\Gamma = (2\tau_{KL}-\alpha\gamma^p_{KL})\{(\alpha+\hat{\beta}(\kappa))\tau_{KL} - (2\sigma-\hat{\eta}(\kappa))\gamma^p_{KL}\} \quad ,
\end{gathered} \tag{71}$$

Thus, based on the constitutive equations assumed in this section, Λ is 2μ times the square of the magnitude of the normal to the yield surface $\partial\mathcal{S}$ in stress space. Having obtained the results $(71)_{1,2}$, $\lambda\gamma^*$ can be calculated from $(42)_1$ and it then follows from (32) that the quotient $\hat{f}/\hat{g}$ must satisfy the inequality

$$\hat{f}/\hat{g} < 1 \quad , \tag{72}$$

which limits the extent of the hardening behavior. The restriction (72), in turn, places an upper bound of unity on the value of the saturation constant K_h in (52a) so that

$$0 < K_h \leq 1 \quad . \tag{73}$$

Expressions for $\dot{\gamma}^p_{KL}$ can now be easily calculated from (51a) in a region of hardening or softening and from (51b) in a region of perfectly plastic behavior.

Given the constitutive assumptions employed in this section, the results $(71)_{1,2}$ and the restrictions (72), (73) are valid for any small elastic-plastic deformations. In the rest of this section, however, we again confine attention to a homogeneous deformation sustained by uniaxial tension (61). Since plastic volume change $\bar{e}^p = 0$, $\bar{e}^e = \bar{e}$ is given by $(63a)_1$. Again, as in $(1)_1$, for convenience we use the notation $e = e_{11}$, $e_e = e^e_{11}$, $e_p = e^p_{11}$ and write

$$\|\gamma^p_{KL}\| = \tfrac{1}{2}\, e_p \|b_{KL}\| \quad ,$$

where the constant matrix $\|b_{KL}\|$ is defined by $(61)_3$. Also, from $(70)_{1,3,4}$, (61) and $(1)_2$, we deduce that

$$\hat{f} = (\tfrac{4}{3}\, s - \alpha e_p)\dot{s} \quad , \quad \hat{g} = (\tfrac{4}{3}\, s - \alpha e_p)\{E\dot{e} + (3\mu - E)\dot{e}_p\} \quad , \quad \tfrac{4}{3}\, s - \alpha e_p \neq 0 \quad (g = 0) \quad . \tag{74}$$

At initial yield $e_p = 0$ and $\kappa_0 = \frac{2}{3}\, s_0^2 > 0$ by virtue of $(74)_3$ and $(67)_1$. Hence, on the yield surface $(g = 0)$, $\frac{4}{3}\, s - \alpha e_p$ must be positive. From this last result, along with (30) and $(36)_2$, we have $\dot{e}_p > 0$ during loading and therefore e_p is strictly increasing with time. Further, from the definition (43), and the positivity of the coefficient of $\dot{s}$ in $(74)_1$, it follows that $\dot{s}$ must satisfy the conditions in (44). The above results may be summarized as follows*

*The inequality $(75)_1$, together with $(74)_1$ and (10), imply the following: During neutral loading it is necessary that $\dot{s} = 0$, while during unloading it is necessary that $\dot{s} < 0$.

$$\frac{4}{3}s - \alpha e_p > 0 \quad , \quad \dot{e}_p > 0 \quad ,$$
$$\dot{s} \text{ satisfies conditions H} \quad . \tag{75}$$

While $(75)_2$ holds during all three types of strain-hardening behavior defined in (43), it follows at once from $(75)_3$ and $(1)_2$ that $\dot{e}_e$ also satisfies conditions H.

For uniaxial tension under discussion, the quantities Λ and Γ in $(71)_{1,2}$ reduce to

$$\Lambda = 3\mu\left(\frac{4}{3}s - \alpha e_p\right)^2 > 0 \quad , \quad \Gamma = \frac{3}{2}\left(\frac{4}{3}s - \alpha e_p\right)\overline{\Gamma} \quad ,$$
$$\overline{\Gamma} = \frac{2}{3}(\alpha + \hat{\beta}(\kappa))s - (2\sigma - \hat{\eta}(\kappa))e_p \quad , \tag{76}$$

where for later convenience we have introduced the quantity $\overline{\Gamma}$ defined by $(76)_3$. Further, from (50), $(76)_2$ and $(75)_1$ follows the result

$$\overline{\Gamma} \text{ satisfies conditions H} \quad . \tag{77}$$

Also, the expression for plastic strain rate $\dot{e}_p$ in a region of hardening or softening can be written as

$$\dot{e}_p = \frac{\frac{2}{3}\left(\frac{4}{3}s - \alpha e_p\right)}{\overline{\Gamma}}\dot{s} = \frac{\frac{2}{3}E\left(\frac{4}{3}s - \alpha e_p\right)}{\overline{\Gamma} + \frac{2}{3}E\left(\frac{4}{3}s - \alpha e_p\right)}\dot{e} \quad , \tag{78}$$

which is similar in form to that obtained in [5] and where the relation $s = E(e - e_p)$ has been used in deriving $(78)_2$. In fact, if the coefficient functions $\hat{\beta}$ and $\hat{\eta}$ which occur in $\overline{\Gamma}$ are specialized to those given by (69), then (78) reduces to that in [5, Eq. (80)].

The result $(75)_3$ enables us to calculate the slopes de/ds, de_p/ds explicitly as functions of s, e_p, κ. Thus, with the use of (1), $(78)_1$ and chain rule of differentiation, in a region of hardening or softening we have

$$\frac{de}{ds} = \frac{1}{E} + \frac{de_p}{ds} \quad , \quad \frac{de_p}{ds} = \frac{\frac{2}{3}(\frac{4}{3} s - \alpha e_p)}{\bar{\Gamma}} \tag{79}$$

It follows from (79), $(75)_1$ and (77) that

$$\infty > \frac{de_p}{ds} > 0 \quad , \quad \infty > \frac{de}{ds} > \max\{\frac{1}{E}, \frac{de_p}{ds}\} > 0 \text{ if the material is hardening} \quad ,$$
$$-\infty < \frac{de_p}{ds} < 0 \quad , \quad \frac{1}{E} > \frac{de}{ds} > \frac{de_p}{ds} \text{ if the material is softening} \quad . \tag{80a}$$

Moreover,

$$\infty > \frac{de}{ds} > \frac{1}{E} \text{ (or equivalently } \infty > \frac{de_p}{ds} > 0) \text{ implies hardening} \quad ,$$
$$-\infty < \frac{de}{ds} < \frac{1}{E} \text{ (or equivalently } -\infty < \frac{de_p}{ds} < 0) \text{ implies softening} \quad . \tag{80b}$$

Since $de_e/ds = \frac{1}{E}$ which is always positive, we may write $de_p/de_e = E\, de_p/ds$, $de/de_e = E\, de/ds$ and then obtain explicit expressions for these derivatives from (79). It is evident that conditions of the type indicated in (80) for de_p/ds also hold for de_p/de_e. It follows from (42), (76) and $(79)_2$ that in a region of hardening or softening

$$1 > \frac{\hat{f}}{\hat{g}} = [1 + 3\mu \frac{de_p}{ds}]^{-1} = [1 + \frac{d\gamma^p_{11}}{ds} (\frac{d\gamma^e_{11}}{ds})^{-1}]^{-1} \quad . \tag{81}$$

In view of (80a) and (43), (81) implies that $de_p/ds < -1/3\mu$ in a region of softening. It is clear from $(81)_1$ and (1) that a knowledge of μ,E and the slope de/ds suffices to determine $\hat{f}/\hat{g}$. If the material saturates to perfectly plastic behavior, the left-hand side of $(81)_1$, i.e., $\hat{f}/\hat{g}$ must tend to zero and hence in this case de_p/ds must become unbounded.

We now turn to a brief discussion of saturation hardening usually observed under uniaxial cyclic loading. Recalling the definitions (52a,b), from (81) we deduce that saturation hardening occurs if there exists a constant K_h such that

$$[1+3\mu \lim_{t\to\infty} \frac{de_p}{ds}]^{-1} = K_h \quad , \quad \lim_{t\to\infty} \frac{de}{ds} = \frac{1}{E} + \frac{1-K_h}{3\mu K_h} \quad , \quad (0 < K_h \leq 1) \quad . \tag{82}$$

In order to exploit the implications of (82), we first observe that $\bar{\Gamma}$ defined by $(76)_3$ can be rewritten as

$$\bar{\Gamma} = \frac{1}{2} \left(\frac{4}{3} s - \alpha e_p\right)(\alpha + \hat{\beta}(\kappa)) + \left(\frac{\alpha^2}{2} - 2\sigma + \hat{\eta}(\kappa) + \frac{\alpha}{2} \hat{\beta}(\kappa)\right) e_p \tag{83}$$

and then express $(79)_2$ in the form

$$\frac{de_p}{ds} = \left[\frac{3}{4} (\alpha + \hat{\beta}(\kappa)) + \frac{(\alpha^2 - 2\sigma + \hat{\eta}(\kappa) + \frac{\alpha}{2} \hat{\beta}(\kappa)) e_p}{\frac{2}{3} (\frac{4}{3} s - \alpha e_p)}\right]^{-1} \quad . \tag{84}$$

Consider now a special material response which corresponds to the vanishing of the numerator of the second term in the square brackets in (84), i.e.,

$$\frac{\alpha^2}{2} - 2\sigma + \hat{\eta}(\kappa) + \frac{\alpha}{2} \hat{\beta}(\kappa) = 0 \quad , \tag{85}$$

which has the same form as a particular case discussed in [5]. From $(75)_1$, $(77)_1$, (83) to (85) and (81), it can be readily concluded that

$$\begin{gathered} \alpha + \hat{\beta}(\kappa) \text{ satisfies conditions H} \quad , \\ \frac{de_p}{ds} = \frac{4/3}{\alpha + \hat{\beta}(\kappa)} \quad , \quad \frac{\hat{f}}{\hat{g}} = \left[1 + \frac{4\mu}{\alpha + \hat{\beta}(\kappa)}\right]^{-1} < 1 \quad . \end{gathered} \tag{86}$$

Also, in view of (43b) and $(86)_2$, in a region of softening:

$$0 > \alpha + \hat{\beta}(\kappa) > -4\mu \quad .$$

If saturation hardening occurs with $0 < K_h < 1$, then from $(82)_1$, (86) and the condition (85) we have

$$\lim_{t\to\infty} \hat{\beta}(\kappa) = \frac{4\mu K_h}{1-K_h} - \alpha \quad , \quad \lim_{t\to\infty} \hat{\eta}(\kappa) = -\frac{2\alpha\mu K_h}{1-K_h} + 2\sigma \quad , \quad (0 < K_h < 1) \quad , \tag{87}$$

while $\hat{\beta}(\kappa)$ becomes unbounded for $K_h = 1$.

We further examine saturation hardening by adopting the special coefficients (69) subject to the condition (85). When saturation is assumed to occur, the limit of the coefficients (69) as $t \to \infty$ is zero and from (85), (82) and $(86)_2$ we obtain

$$\alpha^2 = 4\sigma \quad , \quad \alpha\beta + 2\eta = 0 \quad , \quad \lim_{t\to\infty} \frac{de}{ds} = \frac{1}{E} + \frac{4}{3\alpha} \quad , \quad 0 < K_h = [1 + \frac{4\mu}{\alpha}]^{-1} < 1 \quad , \quad \alpha > 0 \quad , \quad (88)$$

the first three of which are the same as those derived in [5, Eqs. (70) and (86)].

By way of illustration, consider the 304 stainless steel whose behavior in cyclic tension-compression is discussed in [5, Sec. 7]. As in [5], for the 304 stainless steel, we take the values of $E = 123$ GPa and $de/ds = (3.85 \text{ GPa})^{-1}$ at initial yield and also assume the value $\nu = 0.3$ for Poisson's ratio†. With these values, the expressions $(79)_1$ and (81) predict that the quotient $\hat{f}/\hat{g}$ at initial yield is approximately equal to 0.027. Again using the above values, as well as $\alpha = 1.5$ (for tension), $(88)_3$ gives an approximate value of 0.008 for K_h. Thus, $\hat{f}/\hat{g}$ decreases from a value of 0.027 at initial yield to a value of 0.008 at saturation. It is clear from (82) that the definition of saturation hardening given by (52a) implies that the slopes de/ds or de_p/ds tend to constant limits at saturation. In this connection, it should be noted that when $\hat{\beta}$ and $\hat{\eta}$ are of the form (69), the definition of saturation hardening used in [5] also gives constant limiting slopes.

We return once more to the perfectly plastic case, and first observe that the expression for‡ $\dot{\gamma}^p_{KL}$ can be obtained from (51b) with the use of $(70)_{1,4}$ and $(71)_1$. In view of $(75)_3$, $\dot{s} = 0$ for perfectly plastic behavior and s retains its

†A value for Poisson's ratio was not needed in the calculation given in [5, Sec. 7]. With $\nu = 0.3$ and $E = 123$ GPa, μ is calculated to be 47.31 GPa.

‡In fact, in the case of uniaxial tension, the resulting expression is an identity.

initial yield value s_o and, in accordance with $(75)_2$, e_p is strictly increasing with time during loading. The work-hardening parameter κ may then be obtained as a function of e_p from $g = f = 0$ with f given by $(67)_1$:

$$\kappa = \frac{2}{3} s_o^2 - \alpha s_o e_p + \frac{3}{2} \sigma e_p^2 \quad . \tag{89}$$

By (76) and (77c), for perfectly plastic behavior it is necessary that

$$\frac{2}{3} (\alpha + \hat{\beta}(\kappa)) s_o - (2\sigma - \hat{\eta}(\kappa)) e_p = 0 \tag{90}$$

for all e_p. We observe, however, that in view of (76) and (77c) the constant values

$$\hat{\beta}(\kappa) = -\alpha \quad , \quad \hat{\eta}(\kappa) = 2\sigma \tag{91}$$

are sufficient for perfectly plastic behavior. It should be noted that the values (91) satisfy the condition (85).

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Appendix A

We provide here details of the calculations leading to (64), (65a) and (66), and also record alternative useful forms of the quotient $\hat{f}/\hat{g}$ associated with the constitutive equations of Section 4. From (63), in a region of hardening or softening we have

$$\frac{d\bar{e}}{ds}\left(\frac{d\bar{e}^e}{ds}\right)^{-1} = 1 + \frac{d\bar{e}^p}{ds}\left(\frac{d\bar{e}^e}{ds}\right)^{-1} = 1 + \frac{k}{k^*} \quad , \quad (\psi \neq 0) \quad , \tag{A1}$$

$$\left\|\frac{d\gamma_{KL}}{ds}\right\| \left\|\frac{d\gamma^e_{KL}}{ds}\right\|^{-1} = \|\delta_{KL}\| + \left\|\frac{d\gamma^p_{KL}}{ds}\right\| \left\|\frac{d\gamma^e_{KL}}{ds}\right\|^{-1} = (1 + \frac{\mu}{\mu^*})\|\delta_{KL}\| \quad , \tag{A2}$$

$$\left\|\frac{de_{KL}}{ds}\right\| \left\|\frac{de^e_{KL}}{ds}\right\|^{-1} = \|\delta_{KL}\| + \left\|\frac{de^p_{KL}}{ds}\right\| \left\|\frac{de^e_{KL}}{ds}\right\|^{-1} = \|\delta_{KL}\| + \frac{E}{E^*}\left\|\begin{matrix} 1 & 0 & 0 \\ 0 & \nu^*/\nu & 0 \\ 0 & 0 & \nu^*/\nu \end{matrix}\right\| , \tag{A3}$$

In a region of hardening or softening $\Gamma \neq 0$ by (50) and using $(42)_2$ we may write $\hat{f}/\hat{g} = (1 - \frac{\Lambda}{\Gamma})^{-1}$. Then, by (60) and (61) we have:

$$\begin{aligned} \frac{\hat{f}}{\hat{g}} &= [1 + \frac{2(4\mu + 3\psi^2 k)}{2\beta + \psi\phi}]^{-1} \\ &= [1 + \frac{E}{3E^*}\{\frac{2(1+\nu^*)^2}{1+\nu} + \frac{(1-2\nu^*)^2}{1-2\nu}\}]^{-1} \\ &= [1 + \frac{E}{E^*}\{1 + \frac{2(\nu-\nu^*)^2}{(1+\nu)(1-2\nu)}\}]^{-1} \end{aligned} \tag{A4}$$

and we may recall that $1+\nu > 0$, $1-2\nu > 0$. If $\psi \neq 0$, then $\hat{f}/\hat{g}$ may also be written as

$$\frac{\hat{f}}{\hat{g}} = [1 + \frac{4\mu + 3\psi^2 k}{4\mu^* + 3\psi^2 k^*}]^{-1} = \frac{2+\psi}{2(1 + \frac{\mu}{\mu^*}) + \psi(1 + \frac{k}{k^*})} \quad . \tag{A5}$$

The result (64) follows at once from (A1), (A2) and (A5). Similarly, if $\nu^* = \nu$ (or $\psi = \frac{1-2\nu}{1+\nu}$), then from (A3) and (A4) we obtain

$$\frac{\hat{f}}{\hat{g}} = [1 + \frac{E}{E^*}]^{-1} = [\frac{1}{3}\,\mathrm{tr}\{\|\delta_{KL}\| + \|\frac{de^p_{KL}}{ds}\|\,\|\frac{de^e_{KL}}{ds}\|^{-1}\}]^{-1} \quad . \tag{A6}$$

The result (65a) follows from (A6), (63a) and (64). With the use of (A3) it is also possible to write (A6) in terms of $\|\frac{de_{KL}}{ds}\|\,\|\frac{de^e_{KL}}{ds}\|^{-1}$ but we do not record this here.

In the special case that $\psi = 0$, we note that by $(63d)_{2,3}$ and (A2),

$$1 + 4\,\frac{\mu}{\beta} = \frac{1}{3}\,\mathrm{tr}\{\|\frac{d\gamma_{KL}}{ds}\|\,\|\frac{d\gamma^e_{KL}}{ds}\|^{-1}\} = \frac{d\gamma}{ds}\,(\frac{d\gamma^e}{ds})^{-1} \quad . \tag{A7}$$

The relations $(A4)_1$, (A7), $(63a)_2$ and $(63b)_2$ lead to the expression (66).

Captions for Figures

Fig. 1. Idealized stress-strain diagram for a typical ductile metal. As the points 1,2,3,4,5 of the stress-strain curve are successively traversed, the locus of the yield point on the e-axis moves outwards through B_1,B_2,B_3,B_4 and B_5, respectively, while the corresponding locus of the yield point on the s-axis first moves upwards through A_1,A_2 to A_3, and it then moves downwards through A_4 and A_5. All unloading curves are drawn parallel to the linear elastic segment 1-0 and hysteresis is ignored.

Fig. 2. A sketch indicating the motion of yield surfaces in stress space and strain space. During loading the yield surface $\partial\mathcal{E}$ in strain space moves outwards with the strain trajectory C_e through positions such as B_1,B_2, B_3,B_4,B_5. The corresponding yield surface $\partial\mathcal{S}$ in stress space moves outwards through positions such as A_1 and A_2 during hardening behavior, is stationary in positions of the type A_3 during perfectly plastic behavior, and moves inwards through positions such as A_4 and A_5 during softening behavior.

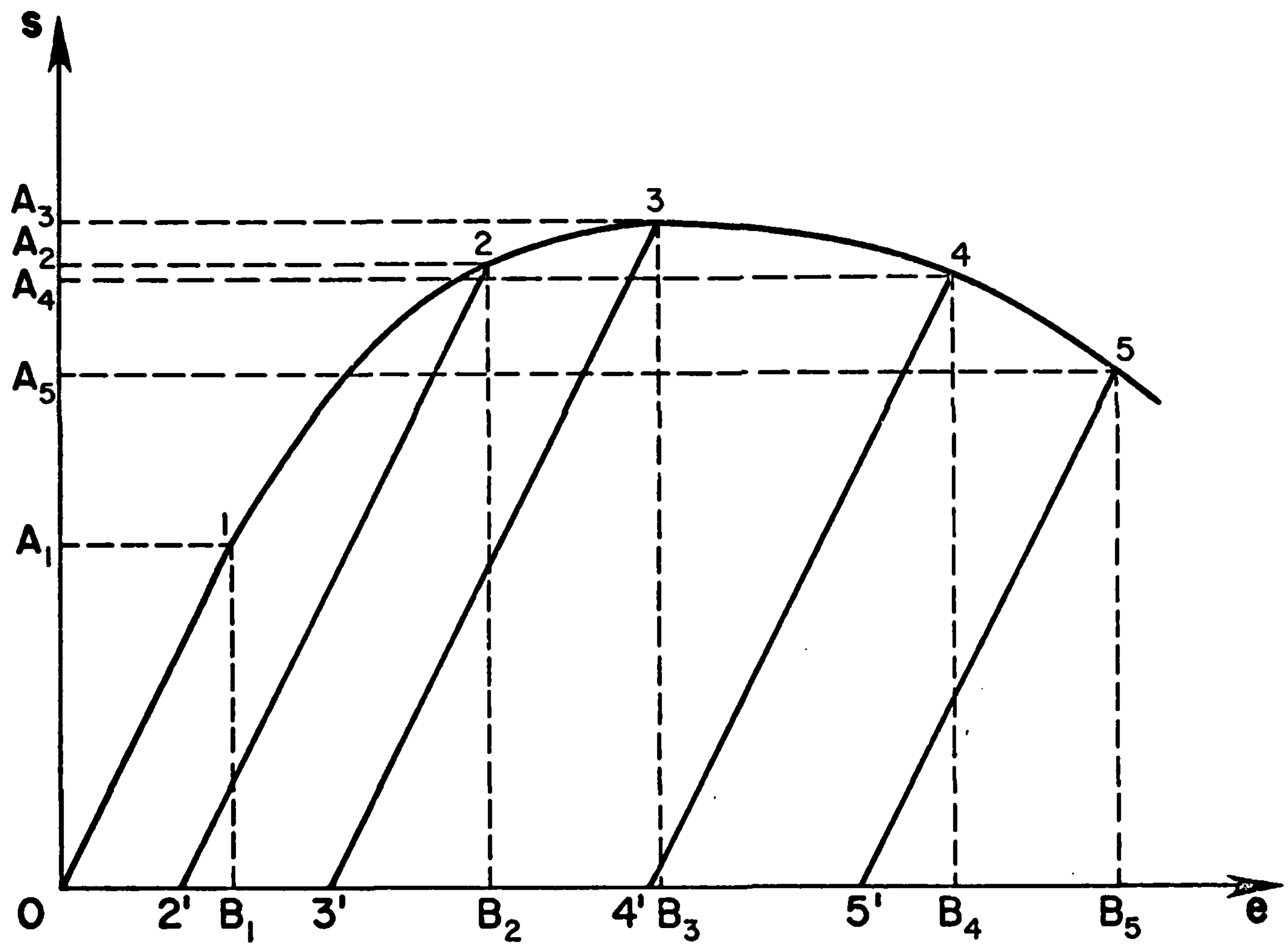

Fig. 1

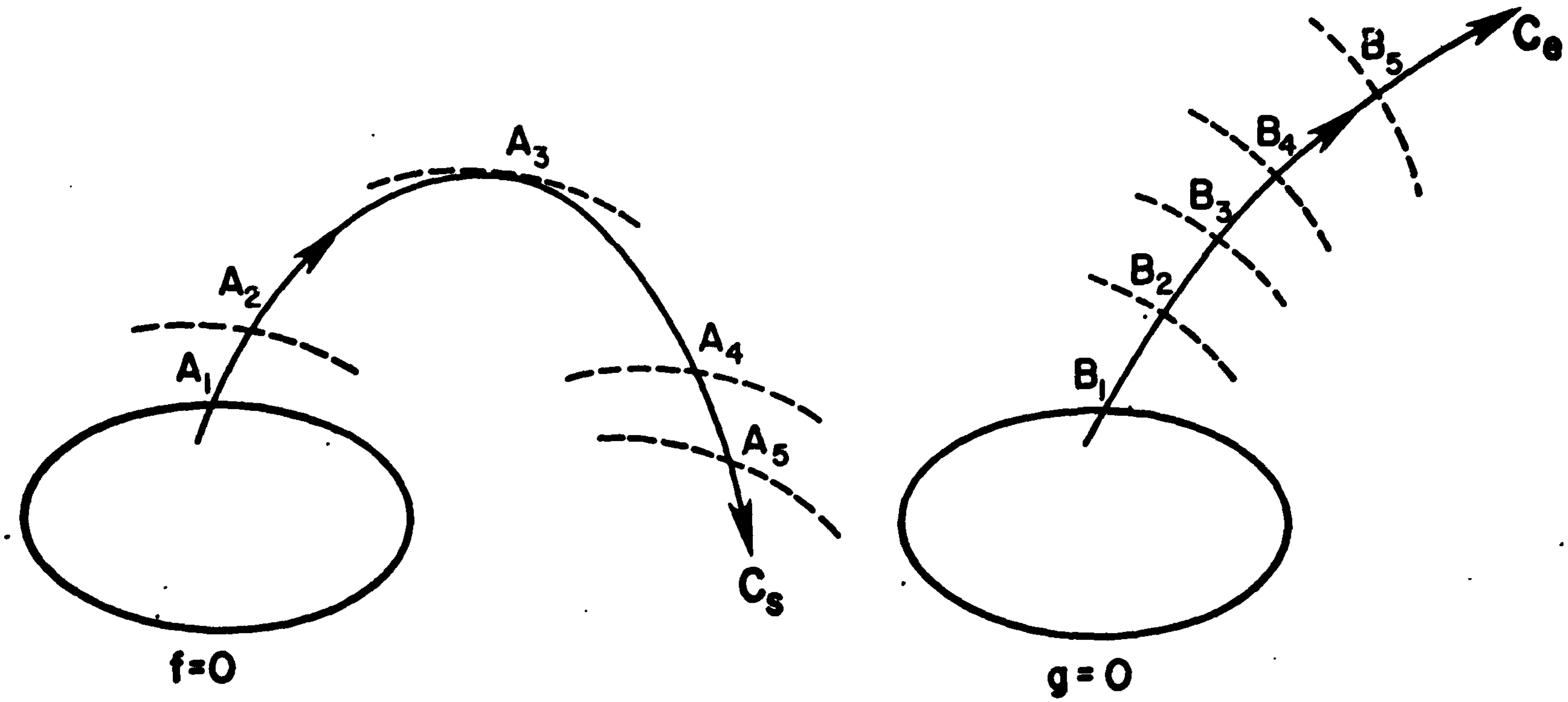

Fig. 2

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