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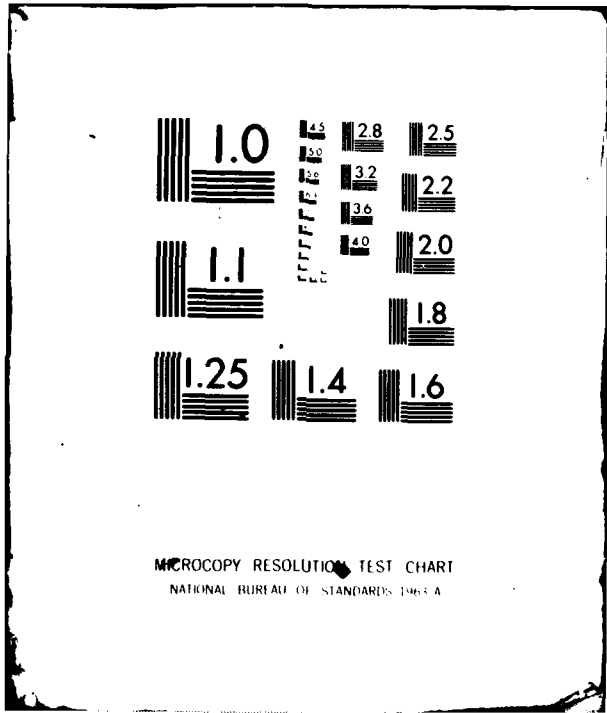
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MULTIPLE COHERENCE

- 3. "Multiple Coherence," by R. D. Trueblood and D. L. Alspach, was presented and appears in the Proceedings of the November 1978 Conference on Systems Science.

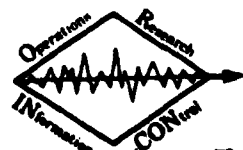
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Summary

The concept of coherence as a measure of the linear relationship between two or more time series is discussed. The definition of coherence in terms of the complex cross power spectral density matrix is given and the relationship between pairwise and multiple coherence and channel signal to noise ratios is discussed. A sample statistic for coherence is given while the details describing the performance which can be obtained from this statistic are contained in a separate report by these authors¹.

1. Introduction

A problem of interest in many different disciplines is that of determining if there is a measurable relationship (physical causality) between two or more time series. In addition, one would often like to obtain a quantitative meaningful measure of the degree of that relationship. This paper describes one possible measure of such a relationship, the coherence.

The most common measure of such a relationship is the pairwise or multiple correlation coefficient. The nature of the correlation coefficient is well documented and will not be discussed here other than to note that it is not a function of frequency and may be affected by linear transformations of either of the time series.

The coherence function (magnitude-squared multiple or pairwise coherence function) is defined as a frequency-dependent quantity that ranges between zero and one²⁻⁴. This coherence function is zero if the two or more ergodic time series are independent (uncorrelated, if Gaussian) and equal to one at any frequency where there is a linear transformation between the one or more input time series and the output or reference time series.

The situation of interest is shown in Figure 1 where v_i indicates the noise contaminating the signal $u_i(t)$ in the i^{th} channel. In general, each transmission channel is composed of linear and nonlinear parts (Figure 2). The sum of the output of the nonlinear system (usually a small part of the total transmission), the measurement noise, and the background noise is grouped into the effective noise term $v_i(t)$ (Figure 3).

We are interested in detecting the presence of a common signal $u(t)$ in two or more channels. The input-output relationship indicated in Figure 1.3 can be written as

$$\underline{x}(t) = \int g(\tau)u(t - \tau)d\tau + \underline{v}(t). \quad (n \times 1) \quad (1.1)$$

Note that because of physical causality requirements $g(\tau)$ is zero for all τ less than zero. In fact, it will be zero for all τ less than some positive time which is the time it takes a signal to travel from the source to a sensor.

The true value of the coherence between time series is generally an unknown quantity. In fact, any measure of the relationship between two or more time series generally must be based on time traces of those series. The functional relationship between the time series and the measure or estimate of coherence is a sample statistic for coherence. The assumption that any one infinite length sample of each series will be enough to allow us to estimate the coherence exactly is made implicitly, and all time series are assumed to be stationary and ergodic. Unfortunately, in practice, one is given only a finite amount of data from each of the time series. In this case the sample statistic is a random variable distributed about the "true magnitude-squared coherence". The density function for the sample statistic in this report is given for a number of specific values of degrees of freedom (N) and number of time series (M) in reference 1.

Based on these density functions, receiver operation characteristic (ROC) curves have been developed. These curves define the probability of detection versus probability of false alarm for a signal of a given true coherence. Curves of probability of detection versus true coherence for fixed levels of probability of false alarm have also been developed for many degrees of freedom and number of sensors up to 10. These are all reported in detail by these authors in reference 1. The fundamental question of the relationship between input signal to noise levels and true coherence for the two channel and multi-channel cases of Figure 1 is discussed in this paper.

For a general discussion of the concept of coherence, the reader is directed to reference 5. For a detailed derivation of the distribution of the pairwise and multiple coherence statistic, the reader is referred to reference 7. The densities and derived performance curves in reference 1 are particularly difficult to obtain for low coherence and high values of N (number of samples of the time series), and based on the authors' knowledge are not available elsewhere in the literature.

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2. Cross-Power Spectral Density Matrix and Multiple Coherence

Multiple coherence can be most easily defined in terms of the cross spectral density matrix $S_{xx}(\omega)$, where the elements of this matrix are defined by

$$S_{ij}(\omega) \triangleq S_{x_i x_j}(\omega) = S_{ji}^*(\omega) \quad (2.1)$$

The crosspower spectral density matrix is of course equivalent to the crosscorrelation matrix. Either of these together with the means of the M jointly gaussian stationary processes, $x_1(t), x_2(t), \dots, x_M(t)$, completely specifies the joint distribution function of these processes.

Given M finite length time traces, there are well known techniques for obtaining "sample estimates" of the cross- and autopower spectral elements. These estimates are used to obtain sample estimates for the multiple coherence between the various time series.

The sample estimate for the cross-power spectral density matrix is a function of the basic data $x_1(t), x_2(t), \dots, x_M(t)$ over some finite time record. It does not require knowledge of any of the characteristics of the transmission channels or of the signal-to-noise ratios of the received signals. The meaning of multiple coherence will be discussed in terms of these quantities in later sections in an attempt to illuminate the subject. Here, however, we will define multiple coherence simply in terms of the crosspower spectral density matrix and its elements.

The multiple coherence between $x_1(t)$ and $x_2(t), x_3(t), \dots, x_{j-1}(t), x_{j+1}(t), \dots, x_M(t)$ is defined by (reference 5)

$$|\gamma_{j;1,2,\dots,j-1,j+1,\dots,M}|^2 = 1 - 1 / [S_{jj}(\omega) S^{jj}(\omega)] \quad (2.2)$$

where $S^{jj}(\omega)$ is the j th diagonal element of the inverse of the $S_{xx}(\omega)$. The multiple coherence of the j th sensor with respect to the other sensors represents the proportion of the variance (power) of sensor j that can be explained by a linear combination of the remaining sensors in a minimum mean square sense.

By reducing this definition to the simple two-channel case we can write

$$S_{xx}^{-1}(\omega) = \frac{1}{S_{11}(\omega)S_{22}(\omega) - |S_{12}(\omega)|^2} \begin{bmatrix} S_{22}(\omega) - S_{12}(\omega) \\ -S_{12}^*(\omega)S_{11}(\omega) \end{bmatrix} \quad (2.3)$$

so that

$$S_{11}^{-1}(\omega) = S_{22}(\omega) / [S_{11}(\omega)S_{22}(\omega) - |S_{12}(\omega)|^2] \quad (2.4)$$

This gives

$$|\gamma_{1;2}|^2 = |\gamma_{2;1}|^2 = |\gamma_{1;2}|^2 = \frac{|S_{12}(\omega)|^2}{S_{11}(\omega)S_{22}(\omega)} \quad (2.5)$$

This is the mutual or pairwise coherence between channels one and two.

In the three-channel case it is easily seen (writing $S_{ij}(\omega)$ as S_{ij}) that

$$|\gamma_{1;2,3}|^2 = \frac{S_{33}|S_{21}|^2 + S_{22}|S_{31}|^2 - 2R_e(S_{12}S_{23}S_{31})}{S_{11}S_{22}S_{33} - S_{11}|S_{23}|^2} \quad (2.6)$$

Using the fact that pairwise coherence is defined by equation 2.5 we can write

$$|\gamma_{1;2,3}|^2 = \frac{|\gamma_{1,2}|^2 + |\gamma_{1,3}|^2 - 2R_e(S_{12}S_{23}S_{31}) / (S_{11}S_{22}S_{33})}{1 - |\gamma_{2,3}|^2} \quad (2.7)$$

$$= \frac{[|\gamma_{1,2}|^2 + |\gamma_{1,3}|^2 - 2R_e(\gamma_{12}\gamma_{23}\gamma_{31})]}{[1 - |\gamma_{2,3}|^2]}$$

Note that if the crosspower spectral density of channels two and three is zero ($S_{23}=0$), the coherence between these two channels is zero and the multiple coherence of channel one, given two and three is,

$$|\gamma_{1;2,3}|^2 = |\gamma_{1,2}|^2 + |\gamma_{1,3}|^2 \quad (2.8)$$

3. Pairwise Coherence and Signal-to-Noise Ratios

If $x_1(t)$ and $x_2(t)$ are generated as indicated in Figure 3, equation 1.1 can be written in the frequency domain by Fourier transform as

$$X_1(\omega) = G_1(\omega)U(\omega) + V_1(\omega) \quad (3.1)$$

$$X_2(\omega) = G_2(\omega)U(\omega) + V_2(\omega) \quad (3.2)$$

Again assuming the noise terms V_1 and V_2 are independent, the cross- and autopower spectral densities can be written as

$$S_{11}(\omega) = G_1(\omega)^2 S_u(\omega) + S_{v_1}(\omega) \quad (3.3)$$

$$S_{22}(\omega) + G_2(\omega)^2 S_u(\omega) + S_{v_2}(\omega) \quad (3.4)$$

$$|S_{12}(\omega)|^2 = G_1(\omega)^2 G_2(\omega)^2 S_u(\omega)^2 \quad (3.5)$$

Noting that $|G_1(\omega)|^2 S_u(\omega)$ and $|G_2(\omega)|^2 S_u(\omega)$ are the output signal power spectra at the receiver in channels one and two, we write the magnitude-squared coherence as

$$|\gamma_{1,2}|^2 = \frac{|G_1(\omega)|^2 |G_2(\omega)|^2 S_u(\omega)^2}{[|G_1(\omega)|^2 S_u(\omega) + S_{v1}(\omega)] [|G_2(\omega)|^2 S_u(\omega) + S_{v2}(\omega)]} \quad (3.6)$$

Defining the signal-to-noise power in the j th channel as

$$(S/N)_j = \frac{|G_j(\omega)|^2 S_u(\omega)}{S_{vj}(\omega)} \quad (3.7)$$

we can write

$$|\gamma_{1,2}|^2 = \frac{(S/N)_1 \cdot (S/N)_2}{\left[1 + \left(\frac{S}{N}\right)_1\right] \left[1 + \left(\frac{S}{N}\right)_2\right]} \quad (3.8)$$

Certain general statements concerning this pairwise magnitude-squared coherence, or just "coherence," can now be made.

The coherence is bounded between zero and one:

$$0 \leq |\gamma_{1,2}|^2 \leq 1. \quad (3.9)$$

If the noise-to-signal power goes to infinity in either channel, the coherence will go to zero. This will happen if the signal power in that channel fades to zero. Also, the noise-to-signal power ratio in both channels must go to zero for the coherence to go to one.

An interesting and informative interpretation of the coherence between these two channels can be made in the following manner. Assume one of the channels is noise free ($S_{v2}(\omega) = 0$). This channel then becomes the input signal. The coherence between channels one and two is now given by

$$|\gamma_{1,2}|^2 = \frac{|G_1(\omega)|^2 S_u(\omega)}{|G_1(\omega)|^2 S_u(\omega) + S_{v1}(\omega)} = \frac{(S/N)_1}{1 + (S/N)_1} = \frac{S_1}{S_1 + N_1} \quad (3.10)$$

This magnitude-squared coherence is the fraction of the power of the output $x_1(t)$ which comes from the signal input passed through a linear system.

Since the transmitted signal is generally not available, it is useful to look at this physical interpretation of coherence from a different point of view. Let us simply take signal $x_2(t)$ as our basic signal and calculate the coherence between $x_1(t)$ and our given signal $x_2(t)$. From equation 2.7, our definition, this will be the same result as if we took signal $x_1(t)$ as our "given" signal. Thus $U(\omega)$ in equation 3.2 is replaced by $X_2(\omega)$:

$$X_1(\omega) = H_{12}(\omega) X_2(\omega) + V_{e12}(\omega). \quad (3.11)$$

$H_{12}(\omega)$ is the effective linear transfer function between output $x_2(t)$ and output $x_1(t)$. $V_{e12}(t)$ is the

effective noise on the transmission channel. It must be noted that $H_{12}(\omega)$ is no longer necessarily a causal system. Now the coherence between channels one and two can be written as in equation 3.10:

$$|\gamma_{1,2}|^2 = \frac{|H_{12}(\omega)|^2 S_{22}(\omega)}{[|H_{12}(\omega)|^2 S_{22}(\omega) + S_{ve12}(\omega)]} \quad (3.12)$$

This two-channel magnitude-squared coherence is the ratio of the power at output $x_1(t)$, which is caused by the "input $x_2(t)$," transmitted over the effective linear transmission channel $[H_{12}(\omega)]$ to the total power in output x_1 . Consideration of this effective linear transmission channel allows this physical interpretation of the coherence to be easily carried over to multiple coherence.

4. Multiple Coherence and Signal-to-Noise Ratios

In the case of M channels, a relationship between the input signal-to-noise ratios and the multiple coherence, similar to the one in the last section for two channels, can be derived. The output power spectral density can again be written in terms of this input signal spectral density, the unknown channel transfer functions, and the effective channel noise as

$$S_{ii}(\omega) = |G_i(\omega)|^2 S_u(\omega) + S_{vi}(\omega) \quad (4.1)$$

$$|S_{ij}(\omega)|^2 = |G_i(\omega)|^2 |G_j(\omega)|^2 S_u(\omega)^2, \quad i \neq j \quad (4.2)$$

It should be noted that $|G_i(\omega)|^2 S_u(\omega)$ is the signal power density in the output of the i th channel and $S_{vi}(\omega)$ is the noise power spectral density in the channel. The general power spectral density function can then be written as

$$S_{xx}(\omega) = S_u(\omega) G^*(\omega) G^T(\omega) + D(\omega), \quad (4.3)$$

where

$D(\omega)$ is a diagonal matrix with elements $S_{vi}(\omega)$,

and

$$G^T(\omega) = [G_1(\omega) \ G_2(\omega) \ \dots \ G_M(\omega)]. \quad (4.4)$$

The inverse required to calculate the multiple coherence from equation 2.2 can now be calculated by using the following matrix inversion lemma:

$$[A + X^* X^T]^{-1} = A^{-1} - A^{-1} X^* I + X^T A^{-1} X^*^{-1} X^T A^{-1} \quad (4.5)$$

Using this lemma, the inverse of the spectral density matrix can be written (assuming all required inverses exist) as

$$S_{xx}^{-1}(\omega) = D^{-1}(\omega) - S_u(\omega) D^{-1}(\omega) G^*(\omega) [H S_u(\omega) G^T(\omega) D^{-1}(\omega) G^*(\omega)]^{-1} \cdot G^T(\omega) D^{-1}(\omega). \quad (4.6)$$

The structure of this inverse can be seen more clearly by noting that

$$S_u(\omega) G^T(\omega) D^{-1}(\omega) G^*(\omega) = \sum_{j=1}^M \frac{|G_j(\omega)|^2 S_u(\omega)}{S_{v_j}(\omega)} \quad (4.7)$$

This term is the sum of all output signal-to-noise power ratios. Therefore, if, as in the two-channel case, we define

$$(S/N)_i = |G_i(\omega)|^2 S_u(\omega) / S_{v_i}(\omega) \quad (4.8)$$

the inverse of the bracketed term in equation 4.6 can be written as

$$\left[1 + S_u(\omega) G^T(\omega) D^{-1}(\omega) G^*(\omega) \right]^{-1} = 1 / \left[1 + \sum_{j=1}^M (S/N)_j \right] \quad (4.9)$$

With this, the *i*th diagonal element of $S_{uu}^{-1}(\omega)$ is given by

$$S_{ii}^{ii}(\omega) = \frac{1}{S_{v_i}(\omega)} \left[\frac{1 + \sum_{j=1}^M (S/N)_j - (S/N)_i}{1 + \sum_{j=1}^M (S/N)_j} \right] \quad (4.10)$$

Using this and substituting equation 4.10 into equation 2.2, we find

$$\begin{aligned} & |\gamma_{i:1,2,\dots,i-1,i+1,\dots,M}|^2 \\ &= 1 - 1 / S_{ii}^{ii}(\omega) S_{ii}^{ii}(\omega) \\ &= \frac{(S/N)_i \left[\sum_{j=1}^M (S/N)_j - (S/N)_i \right]}{\left[1 + (S/N)_i \right] \left[1 + \sum_{j=1}^M (S/N)_j - (S/N)_i \right]} \quad (4.11) \end{aligned}$$

Several special cases are of interest.

First consider the situation in which the signal-to-noise ratios in all channels are the same:

$$(S/N)_j = (S/N)_i = (S/N) \quad (4.12)$$

This gives the coherence of channel *i* with respect to the other *M*-1 channels as

$$\begin{aligned} & |\gamma_{i:1,2,\dots,i-1,i+1,\dots,M}|^2 \\ &= \frac{(S/N)^2 (M-1)}{[1 + (S/N)] [1 + (M-1)(S/N)]} \quad (4.13) \end{aligned}$$

Note that for *M* equal to two as in section 3 the coherence is given by

$$|\gamma_{1:2}| = |\gamma_{2:1}| = \frac{(S/N)^2}{[1 + (S/N)]^2} \quad (4.14)$$

However, if *M* becomes very large, the coherence goes to

$$|\gamma_{i:1,2,\dots,i-1,i+1,\dots,M}|^2 \approx \frac{S/N}{[1 + (S/N)]} \quad (4.15)$$

The formal requirement for this to be valid is for the signal-to-noise ratio and number of channels to satisfy the following inequality:

$$(M-1)(S/N) \gg 1 \quad (4.16)$$

However, based on data from section 3, equation 4.15 is identical to the coherence of two channels when one has an infinite signal-to-noise ratio and the other has a signal-to-noise ratio (at frequency ω) of (S/N) . In this sense, a large enough number of weak channels [signal-to-noise ratio of (S/N)] is equivalent to the sum of one noise-free channel and one weak channel.

The second special case for equation 4.11 is when the *i*th channel has a very large signal-to-noise ratio. Letting $(S/N)_i$ become large in equation 4.11 and keeping all other signal-to-noise ratios equal to (S/N) we find that

$$\begin{aligned} & |\gamma_{i:1,2,\dots,i-1,i+1,\dots,M}|^2 \\ &= \frac{(M-1)(S/N)}{[1 + (M-1)(S/N)]} \quad (4.17) \end{aligned}$$

Note that for low signal-to-noise ratios, i.e.,

$$(M-1)(S/N) \ll 1 \quad (4.18)$$

the coherence goes up linearly with the number of channels each is considered to have the same signal-to-noise ratio as all others, i.e., (S/N) . As *M* becomes larger or as

$$M(S/N) \gg 1 \quad (4.19)$$

this coherence goes to one as it would in the case of two noise-free channels.

Next consider the case where one channel other than the *i*th channel has a very high signal-to-noise ratio relative to the others:

$$(S/N)_k \gg \sum_{j=1}^M (S/N)_j - (S/N)_i - (S/N)_k \quad (4.20)$$

Under these conditions, equation 4.11 is approximately given by

$$|\gamma_{i:1,2,\dots,i-1,i+1,\dots,M}|^2 \frac{(S/N)_i (S/N)_k}{[1+(S/N)_i][1+(S/N)_k]} \quad (4.21)$$

or

$$|\gamma_{i:1,2,\dots,i-1,i+1,\dots,M}|^2 \sim |\gamma_{i;k}|^2, \quad (4.22)$$

as if all other channels were not used. If $(S/N)_i$ is approximately equal to $(S/N)_k$, this means that all weaker channels could be neglected and only the two-channel coherence between the two stronger channels could be used. Also consider the case when all channels including the i th have a much lower signal-to-noise ratio than the k th channel, i. e.,

$$(S/N)_k \gg (S/N)_i = (S/N) \text{ all } i \neq k \quad (4.23)$$

Then, while the coherence of the i th channel given the others is provided by equation 4.21, the coherence of the k th channel given the others is

$$|\gamma_{k:1,2,\dots,k-1,k+1,\dots,M}|^2 = \frac{(M-1)(S/N)_k (S/N)}{[1+(S/N)_k][1+(M-1)(S/N)]} \quad (4.24)$$

Giving for this case

$$|\gamma_{k:1,2,\dots,k-1,k+1,\dots,M}|^2 = \frac{(M-1)[1+(S/N)]}{[1+(M-1)(S/N)]} |\gamma_{i;k}|^2 \quad (4.25)$$

For the case of weak signals in the other channels (from equation 4.22):

$$|\gamma_{k:1,2,\dots,k-1,k+1,\dots,M}|^2 = \frac{(M-1)[1+(S/N)]}{[1+(M-1)(S/N)]} |\gamma_{i;k}|^2 \quad (4.26)$$

Further simplify equation 4.26 to the special case of

$$(M-1)(S/N) \ll 1, \quad (4.27)$$

we have

$$|\gamma_{k:1,2,\dots,k-1,k+1,\dots,M}|^2 \sim (M-1) |\gamma_{i;k}|^2 \quad (4.28)$$

The coherence between the strong signal and the weaker ones goes up linearly with the number of weaker channels. This means that the largest of the M multiple coherence values will be the one in which the largest signal-to-noise ratio channel is used as the reference, which is as expected.

5. A Sample Statistic for Multiple Coherence

The true multiple coherence of a set of time series is a function of the underlying statistics of these processes. The statistics are generally unknown and must be estimated from sample realizations of the processes. The estimates of the basic statistics can then be used to provide estimates of the multiple coherence of the M underlying stochastic processes.

The method of obtaining estimates for true multiple coherence is as follows. Using well documented techniques^{1,3,4,5,6}, obtain sample estimates for each element of the crosspower spectral density matrix. From these sample estimates

$$\hat{S}_{\underline{xx}}(\omega) = \begin{bmatrix} \hat{S}_{11}(\omega) & \hat{S}_{12}(\omega) & \dots & \hat{S}_{1M}(\omega) \\ \hat{S}_{21}(\omega) & \hat{S}_{22}(\omega) & \dots & \hat{S}_{2M}(\omega) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{S}_{M1}(\omega) & \hat{S}_{M2}(\omega) & \dots & \hat{S}_{MM}(\omega) \end{bmatrix} \quad (5.1)$$

one calculates the sample estimate for multiple coherence in the following manner

$$|\hat{\gamma}_{i:1,2,\dots,i-1,i+1,\dots,M}|^2 = 1 - 1/[\hat{S}_{11}(\omega) \hat{S}^{ii}(\omega)], \quad (5.2)$$

where $\hat{S}^{ii}(\omega)$ is the i th diagonal element of the inverse of the M -by- M sample spectral density matrix $\hat{S}_{\underline{xx}}(\omega)$. Details of how to form such estimates are discussed at length in the literature. Since these estimates are random variables there has been considerable study of their distribution. The distributions of these cross- and autopower spectral estimates are known in closed form.

The closed-form expression for the multiple-coherence statistic is available^{7, 8}. This represents the range of values of the multiple-coherence test statistic and the relative probability of its being in a particular band. All values are of course bounded by zero and one. The density function is conditioned on the total number of different time records, or different stochastic processes, available (p). It is also conditioned on the number of independent samples available from each of the time records (N). Thus the density function of the sample estimate for coherence given the true coherence is given by⁹

$$\frac{p}{|y|^2} (y/N, p, |y|^2) = \frac{\Gamma(N)}{\Gamma(p-1)\Gamma(N-p+1)} (1-|y|^2)^N y^{p-2} (1-y)^{N-p} {}_2F_1(N, N, p-1; y |y|^2)$$

where

$$\frac{p}{|y|^2} (y/N, p, |y|^2) = 0 \quad y > 1 \text{ or } y < 0. \quad (5.3)$$

In equation 5.3, ${}_2F_1(\)$ is the hypergeometric function.

This expression for the density function of multiple coherence is both expensive to calculate and generally numerically ill conditioned. Thus to evaluate the density numerically, additional manipulations are required. Great difficulty can be encountered in attempting to use computer library expressions for the hypergeometric function.

For low values of N and p we use a transformation given in reference 12:

$${}_2F_1(N, N, p-1; |y|^2 y) = (1-|y|^2 y)^{p-1-2N} {}_2F_1(p-1-N, p-1; |y|^2 y). \quad (5.4)$$

For the cases of interest, $(p-1-N)$ is a negative integer so that a finite series expansion for this latter hypergeometric function is available:

$${}_2F_1(p-1-N, p-1; |y|^2 y) = (1-|y|^2 y)^{1-1-2N} \sum_{j=0}^{N-p+1} \frac{(-N+p-1)_j^2}{(p-1)_j} (|y|^2 y)^j \quad (5.5)$$

where

$$(p-1)_j = (p-1)(p-2)\dots(p-1-j) \quad (5.6)$$

$$(-N+p-1)_j = (-N+p-1)(-N+p-2)\dots(-N+p-2+j). \quad (5.7)$$

Using these expressions for the hypergeometric function we can write for y between zero and one:

$$\frac{p}{|y|^2} (y/N, p, |y|^2) = \frac{\Gamma(N)(1-|y|^2)^N}{\Gamma(p-1)\Gamma(N-p+1)} \sum_{j=0}^{N-p+1} \frac{(-N+p-1)_j^2}{(p-1)_j} (|y|^2 y)^j \quad (5.8)$$

Other expressions for this density valid to large values of N and specific ranges of true coherence $|y|^2$ and y are described in reference 1. These curves of probability of detection versus probability of false alarm are presented as are curves of probability of detection versus true coherence for fixed values of probability of false alarm.

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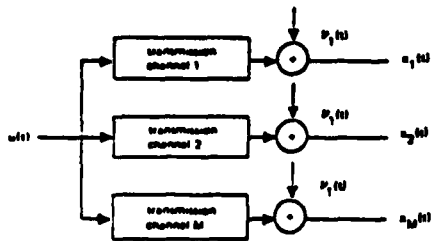


Figure 1. One input-output system.

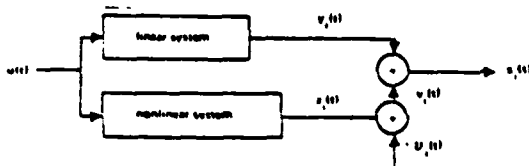


Figure 2. Linear and nonlinear parts of a single channel.

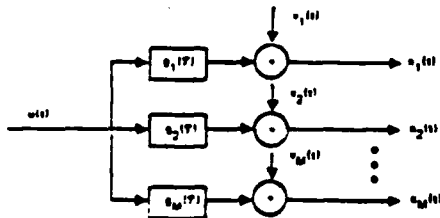


Figure 3. Effective noise with M linear transmission channels.

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