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ON A GENERALIZATION OF THE NEWTON-SYLVESTER INEQUALITIES.(U)
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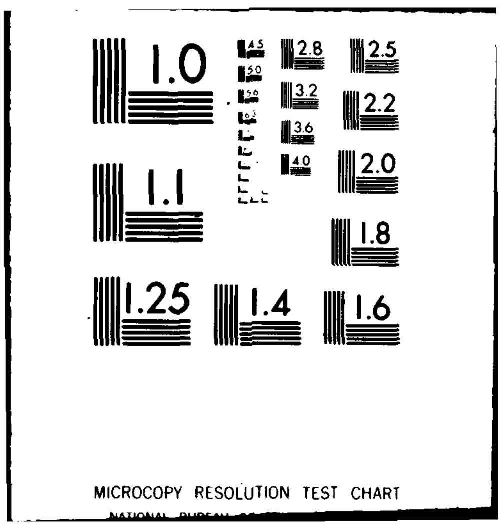
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BASEL MATHEMATICAL NOTES

On a generalisation of the Newton-Sylvester inequalities

by A. M. Ostrowski

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§ 1. Introduction

1.1. As a generalization of the Boudan-Fourier Theorem Sylvester found, in 1865, two theorems giving bounds for the number of real roots of algebraic equations. Writing

(I.1) $$f_o(x) = a_o x^n + a_1 x^{n-1} + \dots + a_n \ , \ a_o = 1 \ ,$$

form for a convenient sequence of positive constants $r_o, \dots, r_{n-1}$ the expressions

(I.2)
$$F_\nu(x) := r_\nu f^{(\nu)^2}(x) - r_{\nu-1} f^{(\nu-1)}(x) f^{(\nu+1)}(x) \quad (\nu = 1,2,\dots,n-1),$$
$$F_o(x) := f_o(x)^2 \quad , \quad F_n(x) := 1 \quad ,$$

assuming that the r_ν satisfy the relations

(I.3) $$r_{\nu+1} = 2r_\nu - r_{\nu-1} \quad (\nu = 1,2,\dots,n-2) \ .$$

For simplicity sake we write $f_\nu(x)$ for the ν-th derivative of $f_o(x)$ and leave out the notation of the argument in $f_\nu(x)$ and $F_\nu(x)$ if the argument is x.

We put generally, for an x_o, $VP(x_o)$ for the number of indices $=1,2,\dots,n$ for which $\operatorname{sgn}\left[f_{\nu-1}(x_o) f_\nu(x_o)\right] = -1$ and simultaneously $\operatorname{sgn}\left[F_{\nu-1}(x_o) F_\nu(x_o)\right] = 1$. We speak then of <u>variation permanences</u>, VP, corresponding to such ν.

Further we denote by $PP(x_o)$ the number of indices $\nu = 1,\dots,n$ for which $\operatorname{sgn}\left[f_{\nu-1}(x_o) f_\nu(x_o)\right] = \operatorname{sgn}\left[F_{\nu-1}(x_o) F_\nu(x_o)\right] = 1$. In this case we speak of <u>permanence permanences</u>, PP, corresponding to these values of ν.

Then, denoting by $N_o(a,b)$ the number of roots of $f_o(x)$ in the open interval (a,b), we can formulate both theorems of

Sylvester as saying that the expressions

$$(I.4) \qquad VP(a) - VP(b) - N_o(a,b) ,$$

$$(I.5) \qquad PP(b) - PP(a) - N_o(a,b) ,$$

are non-negative and even.

We assumed in these formulations that none of the f_ν and F_ν vanish in a or b.

1.2. Sylvester's theorems are also historically of particular interest, as the so-called Newton's Rule could be deduced from them and in this way was for the first time completely proved.

However, Sylvester carried out his proofs completely only under special assumptions on non-existence of multiple roots of the function f_ν, F_ν. Thus, still in 1898 H. Weber [6] wrote that in this case the question remains open.

As a matter of fact, Sylvester (and later de Jonquières [5] and Marchand [7]) considered as the main difficulty finding the right interpretation of the zeros in the double sequences

$$(I.6) \qquad \begin{matrix} f_o(a) , f_1(a) , \ldots , f_n(a) \\ F_o(a) , F_1(a) , \ldots , F_n(a) \end{matrix} ,$$

$$(I.7) \qquad \begin{matrix} f_o(b) , f_1(b) , \ldots , f_n(b) \\ F_o(b) , F_1(b) , \ldots , F_n(b) \end{matrix} .$$

The problem was taken up and treated in a very detailed way in E. Marchand's doctoral dissertation [7].

It must be however said that the best results are obtained if the rules describing the attribution of signs to zeros in the double sequences (I.6) differs from the corresponding rules concerning (I.7). Namely, it is more appropriate to consider the sequences of the signs of

$$\text{(I.8)}\quad \begin{array}{l} f_o(a+0)\ ,\ f_1(a+0)\ ,\ \dots\ ,\ f_n(a+0) \\ F_o(a+0)\ ,\ F_1(a+0)\ ,\ \dots\ ,\ F_n(a+0) \end{array} \quad ,$$

$$\text{(I.9)}\quad \begin{array}{l} f_o(b-0)\ ,\ f_1(b-0)\ ,\ \dots\ ,\ f_n(b-0) \\ F_o(b-0)\ ,\ F_1(b-0)\ ,\ \dots\ ,\ F_n(b-0) \end{array} \quad .$$

However, even this rule still does not sufficiently account for the attribution of signs if some of the F_ν vanish identically and in this case an additional discussion becomes necessary.

Thus, asking for convenient attribution of the signs at the points a, b themselves is practically asking the wrong question.

1.3. We derive in this paper a generalisation of Sylvester's theorems assuming that $f_\nu(x)=f_o^{(\nu)}(x)$, $\nu=0,1,\dots,n$, are continuous functions of x in the closed interval $\langle a,b\rangle$, which have only a finite number of zeros in $\langle a,b\rangle$, and that f_n does not become zero in the open interval $J:=(a,b)$. We assume further that each of the F_ν, formed accordingly to (I.2) for a fixed choice of the r_ν, either identically vanishes in $\langle a,b\rangle$ or has there only a finite number of zeros.

We use further the expressions $N_m(a,b)$ and $N_m(x)$ of which the first signifies the total number of zeros of f_m in the open interval (a,b) and the second one the order of x as zero of $f_m(x)$, which is of course usually $=0$.

Observe that for each x from (a,b) we can choose right-handed and left-handed open neighbourhoods of x in which all f_ν and F_ν have fixed signs, under the convention *) that the identically vanishing F_ν are assumed to be provided with the plus sign.

Consider now for an m with $0 < m \leq n$ the double sequence

$$\text{(I.10)} \qquad \begin{matrix} f_o , & f_1 , & \cdots , & f_m \\ F_o , & F_1 , & \cdots , & F_m \end{matrix} \quad .$$

We denote then with $VP_m(x+0)$ and $VP_m(x-0)$ the numbers of the VP in (I.10) in sufficiently close right-handed and left-handed neighbourhoods of x. In the same way we define the numbers of PP in (I.10), $PP_m(x+0)$ and $PP_m(x-0)$.

We can now define, for $a < x < b$, the interval functions

$$\text{(I.11)} \qquad \Delta'_m(a,b) \; := \; VP_m(a+0) - VP_m(b-0) \quad ,$$

$$\text{(I.12)} \qquad \Delta''_m(a,b) \; := \; PP_m(b-0) - PP_m(a+0) \quad ,$$

$$\text{(I.13)} \qquad \Delta'_m(x) \; := \; VP_m(x-0) - VP_m(x+0) \quad ,$$

$$\text{(I.14)} \qquad \Delta''_m(x) \; := \; PP_m(x+0) - PP_m(x-0) \quad ,$$

with $x \in (a,b)$.

Then, the content of our Main Theorem in §8 is that, if $F_m(x)$ remains in J positive and f_n has no zeros in J, if m=n, the differences

*) This convention is due to Genocchi [4] and de Jonquières [5].

$$\Delta_m'(a,b) - \left(N_o(a,b)-N_m(a,b)\right) \quad , \quad \Delta_m''(a,b) - \left(N_o(a,b)-N_m(a,b)\right)$$

are non-negative even numbers.

1.4. After two introductory lemmas in §2, we discuss in §3 the case when some of the F_ν vanish identically and in §4 the general case when a sequence of consecutive F_ν vanishes at a point. In §5, which is not used in our further considerations, we discuss the behaviour of F_ν with $|x| \to \infty$, under the assumption that a or b become infinite. The proof of the Main Theorem in §8 is prepared by the discussions of so-called f <u>gaps</u> and F <u>gaps</u> in §6 and §7. Finally, in §9, we give the explicit solution of the problem of obtaining the signs of the f_ν and F_ν at a+0 and at b-0 from the values of these functions at a respectively b, and derive the Newton inequality as well as some more general inequalities concerning the numbers of all positive roots and of all negative roots of f_o.

§ 2. Introductory lemmas

2.1. Lemma 1. The most general set of positive r_ν satisfying the relations (I.3) can be given by

$$r_\nu = r_o(1+\alpha\nu) \quad (\nu=0,1,\ldots,n-1) \quad , \quad \alpha > \frac{-1}{n-1} \; , \; r_o > 0. \tag{II.1}$$

2.2. Proof. From (I.3) follows

$$r_{\nu+1} - r_\nu = r_\nu - r_{\nu-1} \quad (\nu=1,\ldots,n-2) \quad ,$$

so that $r_\nu - r_{\nu-1}$ is independent of ν,

$$r_\nu - r_{\nu-1} =: \beta$$

and therefore

$$r_\nu = r_o + \nu\beta = r_o(1+\nu\frac{\beta}{r_o}) \quad , \quad \frac{\beta}{r_o} =: \alpha \; .$$

We obtain then the first formula (II.1), and the second formula follows from the positivity of $r_o,\ldots,r_{n-1}$.

We assume in the following $r_o=1$.

2.3. Lemma 2. Assume

$$f_{\nu+1} \neq 0 \quad , \quad f_{-1} := f_{n+1} := 0 \quad . \tag{II.2}$$

Then, defining the F_ν by (I.2),

$$F' = \frac{f_{\nu+2}}{f_{\nu+1}} F_\nu + \frac{f_\nu}{f_{\nu+1}} F_{\nu+1} \quad (0 \leq \nu \leq n-2) \quad , \tag{II.3}$$

(II.4) $$F'_{n-1} = \frac{f_{n-1}}{f_n}(1+\alpha n) \quad .$$

2.4. Proof. Differentiating (I.2) we obtain for $\nu < n-1$

$$F'_\nu = (2r_\nu - r_{\nu-1})f_\nu f_{\nu+1} - r_{\nu-1}f_{\nu-1}f_{\nu+2}$$

since $f'_{\nu+1} = f_{\nu+2}$. Multiplying with $f_{\nu+1}$ and using (I.2) for ν and for $\nu+1$ we obtain further

$$f_{\nu+1}F'_\nu = f_{\nu+2}F_\nu + f_\nu F_{\nu+1} \quad ,$$

and dividing by $f_{\nu+1}$, (II.3).

For the F_{n-1} we obtain similarly, using (II.1),

$$f_n F'_{n-1} = (2r_{n-1} - r_{n-2})f_{n-1}f_n^2 = (1+\alpha n)f_{n-1}$$

and, dividing by f_n, (II.4).

§ 3. Identically vanishing F_ν

3.1. We assume from now on throughout, that in (II.1),

(III.1) $$\infty > \alpha > \frac{-1}{n-1} .$$

Put for an integer p with $1 \leqslant p \leqslant n-1$, using (II.1),

(III.2) $$\sigma := \frac{r_{p-1}}{r_{p-1}-r_p} = -\frac{1}{\alpha} - p + 1 .$$

If now $\alpha > 0$, $-\frac{1}{\alpha} < 0$, it follows

(III.3) $$\sigma < 1-p \leqslant 0 \quad (\alpha > 0) .$$

On the other hand, if $\alpha < 0$ it follows from (III.1) $-\frac{1}{\alpha} < n-1$ and therefore

(III.4) $$\sigma > n-p \quad (0 > \alpha > \frac{-1}{n-1}).$$

In particular if σ is an integer we have even

(III.5) $$\sigma \geqslant n-p+1 \quad (0 > \alpha > \frac{-1}{n-1}).$$

3.2. Lemma 3. Assume (III.1) and (III.2). Assume $f(x)$ a real function of a real variable x, which has, for an integer p with $1 \leqslant p \leqslant n-1$, continuous derivatives for $a < x < b$ up to the order n and is such that the zeros of $f_{p-1}f_p$ have no limiting points in $J := (a,b)$, while f_n remains $\neq 0$ in J . Then in order that an F_p, defined for $f_o := f$ by (I.2) and (II.1), vanishes identically,

(III.6) $$r_p f_p^2 - r_{p-1} f_{p-1} f_{p+1} = F_p \equiv 0 \ , \ 1 \leqslant p \leqslant n-1 ,$$

it is, if $\alpha \neq 0$, *necessary and sufficient that either* $\alpha = -\frac{1}{n}$, $\sigma = n-p+1$, *and for convenient real constants* $c_o \neq 0$ *and* u *with* $a < u < b$,

$$f_{p-1} = c_o (x-u)^{n-p+1} \quad , \quad a < u < b \quad ; \tag{III.7}$$

or with an $u \leq a$ *and an arbitrary real* σ,

$$f_{p-1} = c_o (x-u)^{\sigma} \quad , \quad u \leq a \quad , \quad c_o = \text{const.}, \ \alpha = \frac{-1}{n}. \tag{III.8}$$

or for an $u \geq b$ *and an arbitrary real* σ,

$$f_{p-1} = c_o (u-x)^{\sigma} \quad , \quad u \geq b \quad , \quad c_o = \text{const.}, \ \alpha = \frac{-1}{n}. \tag{III.9}$$

If $\alpha = 0$, *the necessary and sufficient condition for (III.6) is*

$$f_{p-1} = c_o e^{cx} \quad , \quad c_o c \neq 0 \quad , \quad c_o \wedge c = \text{const.} \quad . \tag{III.10}$$

3.3. *Proof*. Assume first α in (II.1) is $\neq$ 0. Put $\varrho := \frac{r_p}{r_{p-1}}$; then, by (III.2),

$$\sigma = \frac{1}{1-\varrho} \quad . \tag{III.11}$$

Consider a point in J which is no zero of $f_{p-1} f_p$ and consider the greatest open interval, J_o, around this point, lying in J, which does not contain any zeros of $f_{p-1} f_p$. The end points of J_o can only be zeros of $f_{p-1} f_p$ or the points a, b. From (I.2) for $\nu = p$ and (III.6), dividing by $r_{p-1} f_{p-1} f_p$, we can write

$$\varrho \frac{f'_{p-1}}{f_{p-1}} = \frac{f'_p}{f_p} \quad .$$

Integrating in J_o we obtain, denoting by γ an integration constant,

$$\rho \lg|f_{p-1}| = \lg|f_p| - \gamma, \quad |f_{p-1}|^{\rho} = e^{-\gamma}|f_p| \quad .$$

Put

(III.12) $\delta := \operatorname{sgn} f_{p-1}$, $\varepsilon := \operatorname{sgn} f_p$.

Then we can write the last relation in the form

(III.13) $(\delta f_{p-1})^{\rho} = e^{-\gamma}\varepsilon f'_{p-1}$, $e^{\gamma}\varepsilon\delta = \dfrac{(\delta f_{p-1})'}{(\delta f_{p-1})^{\rho}}$

and integrating, with an integration constant u,

(III.14) $\dfrac{1}{1-\rho}(\delta f_{p-1})^{1-\rho} = \varepsilon\delta e^{\gamma}(x-u)$

and resolving with respect to $\delta f_{p-1} = |f_{p-1}|$, by (III.11),

(III.15) $f_{p-1} = \delta\left[\varepsilon\delta e^{\gamma}(1-\rho)(x-u)\right]^{\sigma} = \delta\left[\varepsilon\delta e^{\gamma}(1-\rho)\operatorname{sgn}(x-u)\right]^{\sigma}|x-u|^{\sigma}$,

where $\varepsilon\delta e^{\gamma}(1-\rho)\operatorname{sgn}(x-u)$ is > 0 in J_o. But $1-\rho = 1-\frac{r_p}{r_{p-1}}$ has, by (II.1), a fixed sign both if $x > u$ and if $x < u$, and so have, by (III.14), ε and δ.

3.4. It follows now for a convenient value of the constant c_o, for $u \leq a$, (III.8) and, for $u \geq b$, (III.9), where in both cases $f_{p-1}, f_p, \ldots, f_n$ remain $\neq 0$ in J_o. In particular, if $u=a$ and f_n remains finite for $x \downarrow a$, we must have $\sigma \geq n-p+1$ in (III.8). If $u=b$ and f_n remains finite for $x \uparrow b$, we have in (III.9) again $\sigma \geq n-p+1$. Further, our interval J_o coincides in the cases of (III.8) and (III.9) with the whole interval J, since $u \notin J$.

3.5. Consider now the case $a < u < b$. Since, by (III.14), f_{p-1} has in J_o at the most one zero we can replace our J_o by one of the intervals (a,u) or (u,b). In this way we can apply the argument of 3.4. to the corresponding interval (a,u) or (u,b). We obtain one of the *representations*, with constants c_1, c_2

(III.16) $$f_{p-1} = c_1(x-u)^{\sigma} \qquad x \in J_o = (u,b) \quad ,$$

(III.17) $$f_{p-1} = c_2(u-x)^{\sigma} \qquad x \in J_o = (a,u) \quad .$$

Differentiating the corresponding formula (n-p+1)-times we obtain

$$f_n(x) = c_3 |x-u|^{\sigma-n+p-1} \quad , \; c_3 = \text{const. in } J_o \quad .$$

Since $f_n(x)$ is assumed continuous in u this formula remains true up to the point u and as we assumed that f_n has no zeros in J, it follows

(III.18) $$\sigma = n-p+1 \quad .$$

But since f_n has no zeros in J, f_{n-1} has at the most one zero in J and this is, as follows by differentiation from the corresponding formula (III.16) or (III.17), just u.

Now we can apply the above arguments to both intervals (a,u) and (u,b) and obtain both representations (III.16) and (III.17). Since σ is integer the formula (III.17) can be written as

$$f_{p-1} = (-1)^{\sigma} c_2 (x-u)^{\sigma}$$

and it follows, differentiating σ times, $(-1)^{\sigma} c_2 = c_1$, that is (III.7) with $c_o := c_1$.

3.6. Consider now the case $\alpha=0$ where all r_ν have the value 1. Here we obtain from (III.6) as above

$$\frac{f'_{p-1}}{f_{p-1}} = \frac{f'_p}{f_p} \quad , \quad \lg|f_{p-1}| = \lg|f_p| - \lg c \quad ,$$

c being an integration constant. Using (III.12) we obtain

$$\frac{(\delta f_{p-1})'}{\delta f_{p-1}} = \varepsilon\delta c$$

and

$$f_{p-1} = \delta C e^{cx} \quad , \quad C > 0 \quad ,$$

or finally (III.10), in the whole interval J. In particular it follows that if F_p vanishes identically for $\alpha = 0$, then all $F_p, F_{p+1}, F_{p+2}, \ldots, F_{n-1}$ vanish identically. Lemma 3 is proved and we can also formulate generally

Lemma 4. Under the conditions of lemma 3, if (III.2) holds for an F_p, $1 \leq p \leq n-1$, then all $F_p, \ldots, F_{n-1}$ vanish identically.

§4. Vanishing of the F_ν at a point

4.1. The following discussions concern a generalization of the lemma 2. While lemma 2 gives a linear representation of the first derivative of F_ν through F_ν and $F_{\nu+1}$, we consider now higher derivatives $F_\nu^{(\mu)}$ $(\mu \leq n-\nu)$ of F_ν. Here our formulas are slightly different according as $\mu < n-\nu$ or $\mu = n-\nu$. For $\mu < n-\nu$ we will prove the formula

$$F_\nu^{(\mu)} = \frac{f_\nu}{f_{\nu+\mu}} F_{\nu+\mu} + \frac{1}{\Pi(f_{\nu+1}, \ldots, f_{\nu+\mu})} \sum_{\varkappa=\nu}^{\nu+\mu-1} A_{\mu\nu\varkappa}(f_\nu, \ldots, f_{\nu+\mu+1}) F_\varkappa \tag{IV.1}$$

$$(\mu \leq n-\nu-1;\ f_{\nu+1} \cdots f_{\nu+\mu} \neq 0).$$

In this formula we assume that none of the functions $f_{\nu+1}, \ldots, f_{\nu+\mu}$ vanish for the corresponding x and the symbol $\Pi(f_{\nu+1}, \ldots, f_{\nu+\mu})$ signifies generally a product of non-negative powers of $f_{\nu+1}, \ldots, f_{\nu+\mu}$, while the coefficients $A_{\mu\nu\varkappa}$ are some polynomials in their indicated arguments, $f_\nu, \ldots, f_{\nu+\mu+1}$.

4.2. The formula (II.3) is a special case of (IV.1) corresponding to $\mu = 1$. Here is $\Pi(f_{\nu+1}) := f_{\nu+1}$ and $A_{1\nu 1} = f_\nu$.

Since (IV.1) is already proved for $\mu = 1$, we prove it by induction, assuming that this formula is already proved for a $\mu < n-\nu-1$ and proving the corresponding formula for $\mu+1$. Indeed, differentiating (IV.1) we obtain

$$\begin{aligned} F_\nu^{(\mu+1)} = {} & \frac{f_\nu}{f_{\nu+\mu}} F'_{\nu+\mu} + \left(\frac{f_\nu}{f_{\nu+\mu}}\right)' F_{\nu+\mu} + \sum_{\varkappa=\nu}^{\nu+\mu-1} \left(\frac{A_{\mu\nu\varkappa}(f_\nu, \ldots, f_{\nu+\mu+1})}{\Pi(f_{\nu+1}, \ldots, f_{\nu+\mu})}\right)' F_\varkappa \\ & + \frac{1}{\Pi(f_{\nu+1}, \ldots, f_{\nu+\mu})} \sum_{\varkappa=\nu}^{\nu+\mu-1} A_{\mu\nu\varkappa} F'_\varkappa . \end{aligned} \tag{IV.2}$$

Introducing here for $F'_{\nu+\mu}$ the corresponding expression from (II.3) we obtain from the first right-hand term:

$$\frac{f_\nu f_{\nu+\mu}}{f_{\nu+\mu} f_{\nu+\mu+1}} F_{\nu+\mu+1} + \frac{f_\nu f_{\nu+\mu+2}}{f_{\nu+\mu} f_{\nu+\mu+1}} F_{\nu+\mu}$$

where the first term already gives $\frac{f_\nu}{f_{\nu+\mu+1}} F_{\nu+\mu+1}$, that is the first term of (IV.1) for (μ+1)-st derivative of F_ν, as asserted in (IV.1).

The second term of the sum is already of the type corresponding to the terms of the second right-hand expression (IV.1) for μ+1. Further, the second right-hand term of (IV.2) is also of the type corresponding to the second right-hand term of (IV.1) for μ+1. The same holds further for the second right-hand sum in (IV.2) since the derivative $\left(\frac{A_{\mu\nu\chi}(f_\nu, \ldots, f_{\nu+\mu+1})}{\Pi(f_{\nu+1}, \ldots, f_{\nu+\mu})}\right)'$ contains, if written out, in the numerator also $f_{\nu+\mu+2}$.

As to the last right-hand term of (IV.2), it becomes

$$\frac{1}{\Pi(f_{\nu+1}, \ldots, f_{\nu+\mu})} \sum_{\chi=\nu}^{\nu+\mu-1} A_{\mu\nu\chi} \left(\frac{f_{\chi+2}}{f_{\chi+1}} F_\chi + \frac{f_\chi}{f_{\chi+1}} F_{\chi+1} \right)$$

and this consists again of the terms corresponding to (IV.1) written for μ+1. (IV.1) is proved.

4.3. The formula corresponding to (IV.1) in the case $\mu = n-\nu$ is obtained, assuming that

$$f_{\nu+1} \cdots f_n \neq 0$$

and writing out (IV.1) for $\mu = n-\nu-1$:

$$F_\nu^{(n-\nu-1)} = \frac{f_\nu}{f_{n-1}} F_{n-1} + \frac{1}{\Pi(f_{\nu+1}, \ldots, f_{n-1})} \sum_{\chi=\nu}^{n-2} A_{n-\nu-1\,\nu\,\chi}(f_\nu, \ldots, f_n) F_\chi \; . \tag{IV.3}$$

Differentiating (IV.3) we have to use (II.3) and (II.4). From the first right-hand term of (IV.3) we obtain by (II.4)

$$\frac{f_\nu}{f_n} (1+an) F_n + \left(\frac{f_\nu}{f_n}\right)' F_{n-1} \; .$$

As to the further right-hand terms in (IV.3), we have to use for their derivatives only (II.3) and obtain, as above, terms corresponding to the second right-hand term in (IV.1) for $\mu=n-\nu$. Alltogether we obtain

$$F_\nu^{(n-\nu)} = \frac{f_\nu}{f_n}(1+\alpha n)F_n + \frac{1}{W(f_{\nu+1},\ldots,f_n)}\sum_{\chi=\nu}^{n-1} A_{n-\nu\,\nu\,\chi}(f_\nu,\ldots,f_n)F_\chi \tag{IV.4}$$

$$(f_{\nu+1}\cdots f_n \neq 0)\ .$$

Observe that in (IV.4) the first right-hand term vanishes for $\alpha=\frac{-1}{n}$.

4.4. Lemma 5. Assume

$$0 < p < q < n \tag{IV.5}$$

and for a real x

$$F_p(x)=F_{p+1}(x)= \ldots =F_{q-1}(x)=0\ ,\quad F_{p-1}(x)F_q(x)\neq 0\ ,$$
$$f_p(x)\ldots f_q(x)\neq 0\ . \tag{IV.6}$$

Then

$$F_\nu^{(\mu)}(x) = 0 \qquad (p\leq\nu\leq q-1;\mu=0,1,\ldots,q-\nu-1)\ , \tag{IV.7}$$

$$F_\nu^{(q-\nu)}(x) = \frac{f_\nu(x)}{f_q(x)}F_q(x) \qquad (p\leq\nu\leq q-1)\ , \tag{IV.8}$$

$$F_\nu(x+h) = \frac{h^{q-\nu}}{(q-\nu)!}\frac{f_\nu(x)}{f_q(x)}F_q(x) + O(h^{q-\nu+1}) \quad (p\leq\nu\leq q-1;h\neq 0,h\to 0). \tag{IV.9}$$

4.5. Proof. As to the formula (IV.7), observe that in this case we have $\mu+\nu\leq q-1$. Thence all F factors on the right in (IV.1) vanish in x and (IV.7) follows.

If we take now $\mu=q-\nu$ in (IV.1), then all F_χ occurring on the right vanish by (IV.7) and the formula (IV.8) remains.

If we develop then $F_\nu(x+h)$ for $p \leq \nu \leq q-1$ in powers of h, then, by (IV.7), all terms of this development containing h^μ with $\mu < q-\nu$ vanish and the term corresponding to $\mu = q-\nu$ becomes, by (IV.8), that indicated in (IV.9). Lemma 5 is proved.

4.6. While in lemma 5 we assumed $q < n$, we will now consider in the following lemma the case $q=n$.

<u>Lemma 6</u>. <u>If under the hypotheses (IV.6)</u> $0 < p < q = n$, <u>then the formula (IV.7) still persists</u>,

(IV.10) $$F_\nu^{(\mu)}(x) = 0 \qquad (\mu < n-\nu) \quad ,$$

<u>while (IV.8) and (IV.9) become</u>

(IV.11) $$F_\nu^{(n-\nu)}(x) = \frac{f_\nu(x)}{f_n(x)}(1+\alpha n)\, F_n(x) \qquad (p \leq \nu \leq n-1) \quad ,$$

(IV.12) $$F_\nu(x+h) = \frac{h^{n-\nu}}{(n-\nu)!}(1+\alpha n)\, F_n(x) + O(h^{n-\nu+1}) \qquad (p \leq \nu \leq n-1) \quad .$$

<u>In particular, if</u> $\alpha = \frac{-1}{n}$, <u>(IV.10), (IV.11) and (IV.12) become</u>

(IV.13) $$F_\nu^{(\mu)}(x) = 0 \qquad (\mu \geq 0; p \leq \nu \leq n-1; \alpha = \tfrac{-1}{n}) \quad ,$$

<u>so that each</u> f_ν <u>is a polynomial of exact degree</u> ν <u>and</u>

(IV.14) $$F_\nu(x) \equiv 0 \qquad (n > \nu \geq p; \alpha = \tfrac{-1}{n}) \quad .$$

4.7. <u>Proof</u>. (IV.10) follows again from (IV.1) since $\mu+\nu < n = q$.

The formula (IV.11) follows immediately from (IV.4).

Further, in the development of $F_\nu(x+h)$ we have to replace, in virtue of (IV.4), $F_n(x) \equiv 1$ with $1+\alpha n$.

Assume now in particular $\alpha = \frac{-1}{n}$. Then, by (II.4), F_{n-1} is ≡const. and since $F_{n-1}(x)=0$, $F_{n-1}(x)$ vanishes identically. But then it follows from (III.4) in lemma 3 that

$$f_{n-2} \equiv c_1(x-u)^2 \quad , \quad c_1 \neq 0$$

and each $f_\nu(x)$ is a polynomial of exact degree $n-\nu$. On the other hand, by (IV.11), since $F_\nu^{(n-\nu)}$ $(p \leq \nu \leq n-1)$ vanishes identically, each of these F_ν is a polynomial of a degree $<\nu$. Further, differentiating (IV.4), it follows successively

$$F_\nu^{(n-\nu+1)}(x) = F_\nu^{(n-\nu+2)}(x) = \ldots = 0$$

and thence (IV.13). And (IV.14) follows immediately from (IV.13). Lemma 6 is proved.

§5. Behaviour of the F_ν at infinity

5.1. In order to obtain estimate of the total number of real roots of f_o, one can use some formulas containing information on the sign of F_ν at infinity.

Lemma 3. Assume for $-\infty < x < \infty$ and integers $m > 0$ and p:

(V.1)
$$f(x) = x^m + bx^p + O(x^{p-1}) \qquad (m > p,\ |x| \to \infty),$$
$$f_\mu(x) = \frac{m!}{(m-\mu)!}x^{m-\mu} + b\frac{p!}{(p-\mu)!}x^{p-\mu} + O(x^{p-\mu-1}) \quad (\mu = \nu-1, \nu, \nu+1).$$

Put

(V.2)
$$u := m-\nu \qquad (m > \nu).$$

Then, with $|x| \to \infty$:

(V.3)
$$F_\nu = \frac{m!^2(1+\alpha m)}{u!(u+1)!}x^{2u} + O(x^{2u+p-m}) \quad (|x| \to \infty,\ 0<\nu<m).$$

5.2. Proof. By (I.2) and (II.1) we have, neglecting the terms $O(x^{p-m})$:

$$\frac{F_\nu}{x^{2u}} = (1+\alpha\nu)\left[\frac{m!}{u!} + O(x^{p-m})\right]^2 - (1+\alpha\nu-\alpha)\left[\frac{m!}{(u+1)!} + O(x^{p-m})\right]\left[\frac{m!}{(u-1)!} + O(x^{p-m})\right] =$$

$$= \frac{m!^2}{u!(u+1)!}\Big((1+\alpha\nu)(u+1) - (1+\alpha\nu-\alpha)u\Big) + O(x^{p-m}) = \frac{m!^2(1+\alpha m)}{u!(u+1)!} + O(x^{p-m}).$$

(V.3) follows immediately.

We see that F_ν remains positive for sufficiently large $|x|$ as long as $1+\alpha m > 0$.

5.3. It remains now to consider the case

$$\alpha = -\frac{1}{m}. \tag{V.4}$$

In this discussion the symbol of the type $\frac{1}{s!}$ with $s < 0$ will be sometimes used. In this case $\frac{1}{s!}$ has to be considered as 0. We will assume that $f(x)$ for integers $m > 0, p, q$ and real numbers $\neq 0, b, c$, is

$$\text{(V.5)}\quad \begin{aligned} f(x) &= x^m + bx^p + cx^q + O(x^{q-1}) \quad (m > p > q,\ |x| \to \infty), \\ f_\mu(x) &= \frac{m!}{(m-\mu)!}x^{m-\mu} + b\frac{p!}{(p-\mu)!}x^{p-\mu} + c\frac{q!}{(q-\mu)!}x^{q-\mu} + O(x^{q-\mu-1}) \quad (\mu = \nu-1, \nu, \nu+1). \end{aligned}$$

Put

$$u := m-\nu, \qquad v := p-\nu, \qquad w := q-\nu \tag{V.6}$$

and observe that with our notations we always have for integer $s \geq 0$:

$$(x^n)^{(s)} = \frac{n!}{(n-s)!}x^{n-s} \quad (s \lessgtr n).$$

Using (I.2) and (V.5) we obtain, since in our case generally $r_\nu = 1 - \frac{\nu}{m}$,

$$mF_\nu = u\left[\frac{m!}{u!}x^u + b\frac{p!}{v!}x^v + c\frac{q!}{w!}x^w + O(x^{w-1})\right]^2 -$$

$$-(u+1)\left[\frac{m!}{(u+1)!}x^{u+1} + b\frac{p!}{(v+1)!}x^{v+1} + c\frac{q!}{(w+1)!}x^{w+1} + O(x^w)\right]\left[\frac{m!}{(u-1)!}x^{u-1} + \frac{bp!}{(v-1)!}x^{v-1} + \frac{cq!}{(w-1)!}x^{w-1} + O(x^{w-2})\right]$$

and developping:

$$\frac{mF_\nu}{x^{2u}} = u\left[\frac{m!^2}{u!^2} + 2b\frac{m!p!}{u!v!}x^{p-m} + 2c\frac{m!q!}{u!w!}x^{q-m} + b^2\frac{p!^2}{v!^2}x^{2(p-m)}\right] -$$

$$\text{(V.7)}\quad -(u+1)\left[\frac{m!^2}{(u+1)!(u-1)!} + bp!m!x^{p-m}\left(\frac{1}{(v+1)!(u-1)!} + \frac{1}{(v-1)!(u+1)!}\right) + \right.$$

$$\left. + cq!m!x^{q-m}\left(\frac{1}{(w+1)!(u-1)!} + \frac{1}{(w-1)!(u+1)!}\right) + b^2\frac{p!^2}{(v-1)!(v+1)!}x^{2(p-m)}\right] + O(x^{q-m-1}).$$

Here the constant terms are destroyed. The coefficient at x^{p-m} is

$$bm!p!\left(\frac{2u}{u!v!} - \frac{u+1}{(v+1)!(u-1)!} - \frac{u+1}{(v-1)!(u+1)!}\right) =$$

$$\frac{bm!p!}{u!(v+1)!}\Big(2u(v+1)-u(u+1)-v(v+1)\Big) = \frac{bm!p!}{u!(v+1)!}(u-v)(v+1-u)$$

or

$$\frac{bm!p!}{u!(v+1)!}(m-p)(p+1-m) .$$

5.4. We see that in any case

(V.8) $$\frac{mF_\nu}{x^{2\mu}} = -b\frac{m!p!}{u!(v+1)!}(m-p)(m-p-1)x^{p-m} + O(x^{p-m-1}) \quad (|x|\to\infty).$$

The formula (V.8) gives the complete information about the sign of F_ν assuming that

(V.9) $$b\neq 0, \quad m > p+1, \quad v+1 \geq 0 \text{ that is } \nu \leq p+1.$$

In order to obtain a formula valid for $v+1 < 0$, that is $\nu > p+1$, we need the decomposition of f into the sum

(V.10) $$f_0(x) = P(x) + G(x) ,$$

where

$$P(x) = x^m + bx^p + \ldots, m > p \geq 0$$

is a polynomial, while $G(x)$ can be written with a $\mu < 0$:

(V.11) $$G(x) = \frac{d}{x^\mu} + O(x^{-\mu-1}) \quad (|x|\to\infty).$$

We assume further that for positive integers $s \leq m$:

(V.12) $$G^{(s)} = (-1)^s \frac{d\mu(\mu+1)\cdots(\mu+s-1)}{x^{\mu+s}} + O\left(\frac{1}{x^{s+\mu+1}}\right).$$

5.5. As $\nu > p+1 \geq 1$, we can use the notation

(V.13) $$M := \mu(\mu+1)\cdots(\mu+\nu-2)$$

and obtain from (I.2):

$$mF_\nu = u\left[\frac{m!}{u!}x^u + (-1)^\nu \frac{dM(\mu+\nu-1)}{x^{\mu+\nu}} + O(x^{-\mu-\nu-1})\right]^2 -$$

$$- (u+1)\left[\frac{m!}{(u+1)!}x^{u+1} - (-1)^\nu \frac{dM}{x^{\mu+\nu-1}} + O(x^{-\mu-\nu})\right] \times$$

$$\times \left[\frac{m!\,x^{u-1}}{(u-1)!} - (-1)^\nu \frac{dM(\mu+\nu-1)(\mu+\nu)}{x^{\mu+\nu+1}} + O(x^{-\mu-\nu-2})\right].$$

$$mF_\nu = u\left[\frac{m!^2}{u!^2}x^{2u} + (-1)^\nu \frac{2dMm!(\mu+\nu-1)}{u!}x^{u-\mu-\nu}\right] - (u+1) \times$$

$$\times \left[\frac{m!^2\, x^{2u}}{(u+1)!(u-1)!} - (-1)^\nu dM\left(\frac{1}{(u-1)!} + \frac{(\mu+\nu-1)(\mu+\nu)}{(u+1)!}\right)\frac{x^u}{x^{\mu+\nu}}\right] + O(x^{u-\mu-\nu-1}),$$

$$\frac{mF_\nu}{x^{2u}} = (-1)^\nu \frac{dM}{u!}\left(2um!(\mu+\nu-1) - u(u+1) - (\mu+\nu-1)(\mu+\nu)\right)\frac{1}{x^{m+\mu}} + O\left(\frac{1}{x^{m+\mu+1}}\right).$$

Thence, writing

(V.14) $$Q := 2um!(\mu+\nu-1) - u(u+1) - (\mu+\nu)(\mu+\nu-1),$$

we have

(V.15) $$F_\nu = (-1)^\nu \frac{dMQ}{u!} x^{m-2\nu-\mu} + O(x^{m-2\nu-\mu-1}) \quad (m > \nu > p+1).$$

If in particular $G\equiv 0$, that is f is a polynomial, we can assume in (V.15) μ arbitrary large so that the polynomial F_ν identically vanishes for $\nu > p+1$ – as follows also immediately from (I.2).

5.6. We have to consider now only the cases with $p=m-1$. Here we have $\nu < m=p+1, \nu \leq p$. The formula (V.7) becomes in this case

$$\frac{mF_\nu}{x^{2u}} = u\left[\frac{m!^2}{u!^2} + 2b\frac{m!(m-1)!}{u!(u-1)!}\cdot\frac{1}{x} + b^2\frac{(m-1)!^2}{(u-1)!^2}\cdot\frac{1}{x^2} + 2c\frac{m!q!}{u!w!}x^{q-m}\right] -$$

$$- (u+1)\left[\frac{m!^2}{(u-1)!(u+1)!} + \left(\frac{b(m-1)!m!}{u!(u-1)!} + \frac{b(m-1)!m!}{(u-2)!(u+1)!}\right)\frac{1}{x} + \frac{b^2(m-1)!^2}{(u-2)!u!}\cdot\frac{1}{x^2} +\right.$$

$$\left. + cq!m!\left(\frac{1}{(w+1)!(u-1)!} + \frac{1}{(w-1)!(u+1)!}\right)x^{q-m}\right] + O(x^{q-m-1}).$$

Here the terms with $\frac{1}{x^0}$ and $\frac{1}{x^1}$ are cancelled out and it remains

$$\text{(V.16)} \qquad \frac{u!}{(m-1)!}\cdot\frac{mF_\nu}{x^{2u}} = \frac{(m-1)!}{(u-1)!}\cdot\frac{b^2}{x^2} + \frac{cmq!}{(w+1)!}(u-w)(1+w-u)x^{q-m} + O(x^{q-m-1}) \quad (\nu \leq p=m-1$$

5.7. This becomes, <u>if</u> $q-m < -2$, since then the terms with x^{q-m} and x^{q-m-1} can be neglected,

$$\text{(V.17)} \qquad F_\nu = \frac{b^2(m-1)!^2}{mu!(u-1)!}x^{2m-2\nu-2}\left(1+O\left(\frac{1}{x}\right)\right) \quad (q < m-2, \nu \leq p=m-1),$$

and we see in particular that in the cases of (V.17) F_ν remains positive with $|x| \to \infty$.

5.8. If we consider now the case $q=m-2$, the two first terms in (V.16) can be taken together and we obtain, as $w=u-2$,

$$\frac{u!mF_y}{(m-1)!x^{2u}} = \frac{(m-2)!}{(u-1)!}(b^2(m-1)-2cm)\frac{1}{x^2} + O(\frac{1}{x^3}) ,$$

$$(V.18) \quad F_y = \frac{(m-2)!(m-1)!}{m(u-1)!u!}(b^2(m-1)-2cm)x^{2m-2\nu-2} + O(|x|^{2m-2\nu-3}) \text{ (if } p=m-1, q=m-2\text{)};$$

so that, as long as $\frac{b^2}{2c} \neq \frac{m}{m-1}$, with $|x| \to \infty$,

$$(V.19) \qquad \operatorname{sgn} F_y = \operatorname{sgn} (b^2(m-1)-2cm) .$$

§6. The f gaps

6.1. We call for $0 \leq p < q \leq m+1 \leq n+1$, $m \geq 1$, a sequence of consecutive integers from $0,1,\ldots,m$,

$$g := \{p, p+1, \ldots, q-1\} \tag{VI.1}$$

an f gap at x if, for an x with $a \leq x \leq b$,

$$f_p(x) = f_{p+1}(x) = \ldots = f_{q-1}(x) = 0 \tag{VI.2}$$

and, for $p > 0$, $f_{p-1}(x) \neq 0$ and, for $q \leq m$, $f_q(x) \neq 0$. If in particular $p=0$ we call g open to the left and if $q=m+1$, g is open to the right. If $p > 0$, g will be called closed to the left and if $q \leq m$, g is closed to the right. If $p > 0$ and $q \leq m$, g will be called closed. Further, if for g one the functions

$$F_p, F_{p+1}, \ldots, F_{q-1}$$

vanishes in the whole interval $\langle a,b \rangle$, g will be called singular, otherwise regular. It follows from the lemmas 3 and 4 that for a singular closed f gap (VI.1):

$$F_{p-1} = f_{p-1}^2 > 0 \quad , \quad F_p \equiv \ldots \equiv F_{n-1} \equiv 0 \ , \ q=n. \tag{VI.3}$$

We consider now for our g the double sequence

$$\begin{matrix} f_{p-1} & f_p & f_{p+1} & \ldots & f_{q-1} & f_q \\ F_{p-1} & F_p & F_{p+1} & \ldots & F_{q-1} & F_q \end{matrix} \tag{VI.4}$$

where however the first column $\frac{f_{p-1}}{F_{p-1}}$ is left out for $p=0$ and the last column $\frac{f_q}{F_q}$ is to be left out if $q=m+1$.

6.2. If g is <u>open to the right</u> it follows from (VI.2) that $q-1=m$ and in particular

$$f_{q-1}(x) = 0 \quad , \quad F_{q-1}(x) > 0 \quad .$$

But the, by definition of F_{q-1},

$$F_{q-1}(x) = -r_{q-2}f_{q-2}(x)f_q(x) > 0$$

and thence $f_q(x) \neq 0$, $f_{q-2}(x) \neq 0$, so that $p=m=q-1$. We see that here $f_p(x)$ has x as a <u>simple zero</u> and in particular, since m is assumed positive, p <u>is positive</u>. Thence, <u>if</u> g <u>is open to the left it is then closed to the right</u>.

We denote by $\Delta'(g)$ the number of VP in (VI.4) <u>lost</u> if we go from $x-0$ to $x+0$, and by $\Delta''(g)$ the number of PP in (VI.4) <u>won</u> in going from $x-0$ to $x+0$.

6.3. The behaviour of $F_{p+1},\dots,F_{q-1}$ is described in the

<u>Lemma 8</u>. <u>Assume the</u> f <u>gap (VI.1)</u>. <u>Then in our hypotheses we have</u>, <u>if</u> $q-p \geq 2$,

$$\text{(VI.5)} \qquad F_p = F_{p+1} = \quad . \; . \; . \quad = F_{q-1} = 0 \; , \; F_q > 0 \quad (q \leq m) \; ,$$

<u>Assuming</u> $a < x < b$, <u>we can write with sufficiently small</u> $|h|$ <u>for not identically vanishing</u> F_ν, <u>denoting generally by</u> ε <u>an expression depending on</u> $h \neq 0$ <u>and tending to</u> 0 <u>with</u> $|h| \to 0$:

$$\text{(VI.6)} \; F_\nu(x+h) = f_q^2 \frac{h^{2(q-\nu)}}{(q-\nu)!(q-\nu+1)!}(1+\alpha q+\varepsilon) \quad (\nu=p+1,\dots,q-1;q-p \geq 2);$$

<u>further</u>, <u>if</u> $q \leq m$, <u>for</u> $p > 0$, $a < x < b$,

(VI.7) $$F_p(x+h) = -r_{p-1}(f_{p-1}f_q+\varepsilon)\frac{h^{q-p-1}}{(q-p-1)!} \quad (p>0;q\leq m)$$

<u>and for</u> $p=0$, $q\leq m$, $a<x<b$,

(VI.8) $$F_o(x+h) = \frac{h^{2q}}{q!^2}(f_q^2+\varepsilon) \quad (p=0;q\leq m) \quad ,$$

<u>In the case</u> $\alpha=\frac{-1}{n}$, <u>the relation (VI.6) can be replaced with</u>

(VI.9) $$F_\nu(x) \equiv 0 \quad (\nu=p+1,\dots,n-1;\alpha=\frac{-1}{n}) \ , \ m=n \ .$$

<u>Finally we have generally with</u> $|h|\to 0$, <u>for</u> $q\leq m$,

(VI.10) $$f_\nu(x+h) = \frac{h^{q-\nu}}{(q-\nu)!}(f_q+\varepsilon) \quad (\nu=p,p+1,\dots,q;q\leq m) \ .$$

<u>The formulas (VI.6)-(VI.10) remain true for</u> $x=a$ <u>with</u> $h\downarrow 0$ <u>and</u> $x=b$ <u>with</u> $h\uparrow 0$.

6.4. <u>Proof</u>. Developping $f_\nu(x+h)$, $\nu<q$, in powers of h, it follows from (VI.2), as $f_n(y)$ is assumed continuous, for $p\leq\nu<q$

$$f_\nu(x+h) = \frac{h^{q-\nu}}{(q-\nu)!}(f_q+\varepsilon) \qquad (\nu=p,p+1,\dots,q-1) \ .$$

But this holds also for $\nu=q$, if $q\leq m$, and (VI.10) is proved.

In order to prove (VI.6) introduce into (I.2) the expressions (VI.10) for ν, $\nu-1$ and $\nu+1$. Observe that the symbol ε in different parts of the following relation does not necessarily designate the same quantity. We obtain

$$\frac{F_\nu(x+h)}{h^{2(q-\nu)}} = r_\nu\left(\frac{f_q+\varepsilon}{(q-\nu)!}\right)^2 - r_{\nu-1}\frac{(f_q+\varepsilon)(f_q+\varepsilon)}{(q-\nu+1)!(q-\nu-1)!} =$$

$$(f_q^2+\varepsilon)\left(\frac{1+\alpha\nu}{(q-\nu)!^2} - \frac{1+\alpha\nu-\alpha}{(q-\nu-1)!(q-\nu+1)!}\right) =$$

$$\frac{f_q^2+\varepsilon}{(q-\nu)!(q-\nu+1)!}\Big[(1+\alpha\nu)(q-\nu+1) - (1+\alpha\nu-\alpha)(q-\nu)\Big] = \frac{f_q^2(1+\alpha q+\varepsilon)}{(q-\nu)!(q-\nu+1)!}$$

and (VI.6) follows.

6.5. As to the $F_p(x+h)$ we must distinguish the cases $p > 0$ and $p=0$. In the case $p > 0$ we obtain from (VI.10) and (I.2) for $\nu=p$ and $\nu=p+1$, since $q \leq m$,

$$F_p(x+h) = r_p \frac{h^{2(q-p)}}{(q-p)!^2}(f_q^2+\varepsilon) - r_{p-1}(f_{p-1}+\varepsilon)(f_q+\varepsilon)\frac{h^{q-p-1}}{(q-p-1)!} .$$

But it is $q-p-1 < 2(q-p)$ and $f_{p-1}f_q \neq 0$. Thence, on the right remains only

$$-r_{p-1} \frac{h^{q-p-1}}{(q-p-1)!}(f_{p-1}f_q+\varepsilon)$$

and (VI.7) is proved.

For $p=0$ in (I.2) we have only the first term, $r_o f_o^2$, to consider. Introducing (VI.10) for $\nu=p$ and using $r_o=1$, (VI.8) follows.

Finally, if $\alpha=\frac{-1}{n}$, the relations (VI.9) follow immediately from (IV.14).

6.6. *Lemma 9. Assume the f gap (VI.1) non-singular and open to the left. Then the F_ν in (VI.5) have in the immediate neighbourhood of x the sign plus, and q is $\leq m$ (g is closed to the right),*

(VI.11) $F_\nu(x+0) > 0 \; (\nu \leq q; a \leq x < b) \; , \; F_\nu(x-0) > 0 \; (\nu \leq q; a < x \leq b)$.

Further, the double sequence

$$\text{(VI.12)} \qquad \begin{matrix} f_o(y) & f_1(y) & \cdots & f_q(y) \\ F_o(y) & F_1(y) & \cdots & F_q(y) \end{matrix}$$

has q VP and 0 PP at x-0 and q PP and 0 VP at x+0. Correspondingly we have

(VI.13) $\Delta'(g) = \Delta''(g) = N_o(x) \quad (q \leq m)$.

6.7. Proof. In any case $1+\alpha q\neq 0$, since otherwise we had $q=n$, $\alpha=\frac{-1}{n}$ and then, by (VI.9), g were singular. The relation (VI.11) follows, if $q-p=q>1$, for $\nu=0,\ldots,q-1\geq 1$ from (VI.6), while $F_q=r_q f_q^2$ $(q<n)$ or $F_n=1$ $(q=n)$. For $q=1$, that is if x is a simple zero of $f_o(y)$, the relation (VI.11) for $0\leq\nu\leq 1$ follows from $F_o(y)\equiv r_o f_o^2(y)$, $F_1=r_1 f_1^2>0$.

As to the formula (VI.13), it follows from (VI.10) that the sequence $f_o(y)$, $f_1(y)$, ... , $f_q(y)$ presents at $x-0$ $(h<0)$ q variations and at $x+0$ $(h>0)$ q permanences.

On the other hand, since $f_{q-1}(x)=0$ and $f_q(x)\neq 0$, q is exactly the multiplicity, $N_o(x)$, of x as zero of $f_o(x)$. The formula (VI.13) is proved.

The meaning of the lemma 9 which we have proved, is obviously that if we go through a zero of f, with the multiplicity $q\geq 1$, the corresponding double sequence (VI.12) loses q VP and wins q PP. As to the values of the F_ν and f_ν at $a+0$ and $b-0$, those of the F_ν are given by (VI.11). For the values of the f_ν, $\nu\leq q$, we obtain from (VI.10):

$$\text{(VI.14)}\quad \operatorname{sgn} f_\nu(a+0)=\operatorname{sgn} f_q(a)\ ,\ \operatorname{sgn} f_\nu(b-0)=(-1)^{q-\nu}\operatorname{sgn} f_q(b).$$

6.8. In the case of a non-singular closed f gap (VI.1) the situation is completely different.

Lemma 10. Assume a non-singular f gap (VI.1) closed from both sides, so that $q\geq p+1\geq 2$. Then, going through x from the left to the right, the number of the VP in the double sequence

$$\text{(VI.15)}\qquad \begin{matrix} f_{p-1}(y) & f_p(y) & f_{p+1}(y) & \cdots & f_q(y) \\ F_{p-1}(y) & F_p(y) & F_{p+1}(y) & \cdots & F_q(y) \end{matrix}$$

is decreased and the number of the PP increased, both times by non-negative even numbers, $2K'(g)$, $2K''(g)$,

$$\text{(VI.16)}\qquad \Delta'(g)=2K'(g)\geq 0\ ,\ \Delta''(g)=2K''(g)\geq 0\ .$$

6.9. <u>Proof</u>. Consider first the case $q-p \geq 2$. Put sgn $f_q =: \delta$, sgn $f_{p-1} f_q =: \eta$. Using (VI.10), the sequence of the signs of the $f_{q-\nu}$ in (VI.15) can be given by

$$\text{(VI.17)} \qquad \begin{array}{ccccccc} f_q & f_{q-1} & f_{q-2} & \cdots & f_{p+1} & f_p & f_{p-1} \\ \delta & \mp\delta & \delta & \cdots & (\mp 1)^{r-1}\delta & (\mp 1)^r \delta & \eta\delta \end{array}$$

Here, as in the following, the upper signs belong to x-0 and the lower to x+0, while a sign +1 or -1 belongs both to x-0 and x+0. But obviously the variations and permanences of signs do not change, if we multiply all elements of (VI.17) by δ.

As to the signs of the $F_{q-\nu}$ in (VI.15), they are given in virtue of (VI.5), (VI.6) and (VI.7) by

$$\text{(VI.18)} \qquad \begin{array}{ccccccc} F_q & F_{q-1} & F_{q-2} & \cdots & F_{p+1} & F_p & F_{p-1} \\ 1 & 1 & 1 & \cdots & 1 & -\eta(\mp 1)^{r-1} & 1 \end{array}$$

Observe that the sign of F_q is in any case +1, whether $q=n$ (by definition) or $q < n$. We have therefore to study the number of VP and PP in the following table

+1	∓ 1	+1	...	$(\mp 1)^{r-1}$	$(\mp 1)^r$	η
f_q	f_{q-1}	f_{q-2}	...	f_{p+1}	f_p	f_{p-1}
F_q	F_{q-1}	F_{q-2}	...	F_{p+1}	F_p	F_{p-1}
+1	+1	+1	...	+1	$-\eta(\mp 1)^{r-1}$	+1

Table 1

6.10. We have to distinguish four cases according as $\eta=+1$ or $\eta=-1$ and $r\equiv 0 \pmod{2}$ or $r\equiv 1 \pmod{2}$. A glance at the table 1 allows immediately to verify in all cases the results given in the following table 2, where we denote in the last column the lost VP with minus sign and the won PP with plus sign. And all these numbers are indeed <u>even</u> and ≥ 0.

$r\equiv 0$	$\eta=1$	r VP $\rightarrow$ 0 VP	$-r$ VP
		1 PP $\rightarrow$ $(r-1)$ PP	$+(r-2)$ PP
	$\eta=-1$	$(r-1)$ VP $\rightarrow$ 1 VP	$-(r-2)$ VP
		0 PP $\rightarrow$ r PP	$+r$ PP
$r\equiv 1$	$\eta=1$	$(r-1)$ VP $\rightarrow$ 0 VP	$-(r-1)$ VP
		0 PP $\rightarrow$ $(r-1)$ PP	$+(r-1)$ PP
	$\eta=-1$	r VP $\rightarrow$ 1 VP	$-(r-1)$ VP
		1 PP $\rightarrow$ r PP	$+(r-1)$ PP

Table 2

6.11. Consider now the case $q-p=1$. Then in (VI.2) $f_p=0$, $f_{p-1}f_{p+1}\neq 0$. Put $\delta=\operatorname{sgn} f_{p+1}$, $\eta=\operatorname{sgn} f_{p-1}f_{p+1}$. Then

$$F_{p-1} = r_{p-1}f_{p-1}^2 > 0 \;,\; F_p = -r_{p-1}f_{p-1}f_{p+1} \;,$$

$$F_{p+1} = r_{p+1}f_{p+1}^2 > 0 \;(p+1<n) \;,\; F_n = f_n^2 \;(p+1=n) \;,$$

$$F_{p-1}(x\pm 0) > 0 \;,\; F_{p+1}(x\pm 0) > 0 \;,\; \operatorname{sgn} F_p = -\eta \;.$$

Thence the signs in the corresponding double sequence are:

(VI.19)

$f_{p-1}(y)$	$f_p(y)$	$f_{p+1}(y)$
$\delta\eta$	$-\delta \to \delta$	δ
+1	$-\eta$	+1
$F_{p-1}(y)$	$F_p(y)$	$F_{p+1}(y)$

For $\eta=1$ we have here obviously neither VP nor PP. For $\eta=-1$ we have at x-0 1 PP, 1 VP, which become for x+0 1 VP, 1 PP. We see that in both cases the numbers of VP and PP remain unchanged. Lemma 10 is proved.

Observe that the tables (VI.17), (VI.18) and (VI.19) can also be used for x=a and x=b. For x=a we obtain the values at a+0 and for x=b the values at b-0.

6.12. We consider now a non-singular f gap, g, (VI.1), open to the right. By what has been said in section 6.2., we have p=m=q-1 and x is a simple zero of f_m, $N_m(x)=1$. Since f_m vanishes at x, we have certainly $m<n$.

From the definitions it follows

$$F_{m-1} = r_{m-1}f_{m-1}^2 > 0 \ , \ F_m = -r_{m-1}f_{m-1}f_{m+1} > 0 \ .$$

Denoting the sign of f_{m+1} at x by δ it follows that sgn f_{m-1} $=-\delta$. On the other hand, the sign of f_m changes at x from $-\delta$ to δ. We have therefore the following signs, going from x-0 to x+0:

(VI.20)

f_{m-1}	f_m	f_{m+1}
$-\delta$	$-\delta \to \delta$	δ
+1	+1	+1
$F_{m-1} > 0$	$F_m > 0$	$F_{m+1} > 0$

We see that for our gap g, 1 PP, 0 VP at x-0 go over in 0 PP and 1 VP at x+0, that is by our definitions of $\Delta'(g)$, $\Delta''(g)$,

$$\text{(VI.21)} \qquad \Delta'(g) = \Delta''(g) = -1 = -N_m(x) \ .$$

<u>Lemma 11</u>. <u>If</u> g <u>is a non-singular</u> f <u>gap</u>, <u>(VI.1)</u>, <u>open to the right</u>, <u>we have (VI.21)</u>.

6.13. We consider now a <u>singular gap</u>, g. Then $\alpha = \frac{-1}{n}$. By (III.6) in lemma 3 and by lemma 4, we have then, replacing there p with p+1, for a $c \neq 0$:

$$\text{(VI.22)} \quad q=n \ , \ F_{p+1} \equiv \ldots \equiv F_{n-1} \equiv 0 \ , \ f_p(y)=c(y-x)^{n-p} \ , \ f_{p+1}(y)=c(n-p)(y-x)^{n-p-1} .$$

Assume first $p=0$. Then, by lemma 6, $f_o(y)$ is a polynomial of degree n with a zero at x of multiplicity n,

$$f_o(y) = c(y-x)^n \ .$$

But then our gap g at x extends from 1 to n-1 and $\Delta'(g)$ and $\Delta''(g)$ are correspondingly $\Delta'(x)$ and $\Delta''(x)$ in (I.13) and (I.14). Further, it follows immediately that

$$\text{(VI.23)} \quad \operatorname{sgn} f_\nu(x-0) = (-1)^{n-\nu} \operatorname{sgn} c \ , \ \operatorname{sgn} f_\nu(x+0) = \operatorname{sgn} c \ (\nu < n).$$

Now, the sequence of the f_ν has at x+0 exactly n permanences and at x-0 exactly n variations. Since the elements of the second row in (I.10) are positive both at x-0 and x+0 and x is the only zero of the f_ν, we obtain

$$\text{(VI.24)} \qquad \Delta'(a,b) = \Delta''(a,b) = \Delta'(g) = \Delta''(g) = n \ .$$

6.14. Assume now $p \geq 1$. Then we have $f_{p-1}(x) \neq 0$, by lemma 3. Put $\operatorname{sgn} c = \delta$, $\operatorname{sgn}(\delta f_{p-1}(x)) = \varepsilon$. We have then identically

$$F_p(y) = r_p c^2 (y-x)^{2(n-p)} - r_{p-1} c(n-p)(y-x)^{n-p-1} f_{p-1}(y),$$

$$F_p(y) = -\varepsilon\pi(y-x)^{n-p-1} + O\left((y-x)^{n-p}\right) \quad (y \to x) \ , \tag{VI.25}$$

for a certain positive π and therefore

$$\operatorname{sgn} F_p(x-0) = (-1)^{n-p}\varepsilon \ , \ \operatorname{sgn} F_p(x+0) = -\varepsilon \ . \tag{VI.26}$$

6.15. We have to consider the VP and PP in the double sequence

$$\begin{matrix} f_{p-1} & f_p & f_{p+1} & \cdots & f_{n-1} & f_n \\ F_{p-1} & F_p & F_{p+1} & \cdots & F_{n-1} & F_n \end{matrix} \tag{VI.27}$$

at $x-0$ and $x+0$, assuming first $a < x < b$. From (VI.22) it follows that the signs of $f_p, \ldots, f_{n-1}$ are given by

$$\operatorname{sgn} f_{n-\nu}(x-0) = (-1)^{\nu}\delta \ , \quad \operatorname{sgn} f_{n-\nu}(x+0) = \delta \quad (\nu = 1, \ldots, n-p). \tag{VI.28}$$

Using (VI.26) for the signs of $F_p(y)$ and observing that the sign of $F_{p-1}(y)$ at $x \pm 0$ is that of f^2_{p-1}, that is $\varepsilon^2 = 1$, we obtain, putting $r := n-p$, the table of values of the $f_\nu(y)$ and $F_\nu(y)$ in (VI.27) for $x \pm 0$:

(VI.29)

$\delta f_\nu(y)$	p-1	p	p+1	...	n-1	n
x-0	ε	$(-1)^r$	$(-1)^{r-1}$	...	-1	1
x+0	ε	1	1	...	1	1
$F_\nu(y)$	1	$\varepsilon(-1)^r \to -\varepsilon$	1	...	1	1

6.16. In the table (VI.29) let first $\varepsilon=1$. Then, if r is even, we have in the last line of our table first r+1 permanences for x-0 and thence 1 PP and r VP, while for x+0 we have two variations and r-1 permanences. Therefore only the last r columns must be taken into account and we obtain for x+0 exactly r-1 PP and 0 VP. For an odd r, we have for $x \neq 0$ in the last line of our table again two variations and r-1 permanences and we obtain for x-0 exactly r-1 VP and 0 PP and for x+0 r-1 PP and 0 VP. The result is given in

(VI.30)	$\varepsilon=1$	$r \equiv 0$	r VP, 1 PP ⟶ 0 VP, (r-1) PP
		$r \equiv 1$	(r-1) VP, 0 PP ⟶ 0 VP, (r-1) PP .

If we assume now $\varepsilon=-1$, the results can be given in

(VI.31)	$\varepsilon=-1$	$r \equiv 0$	(r-1) VP, 0 PP ⟶ 1 VP, r PP
		$r \equiv 1$	r VP, 1 PP ⟶ 1 VP, r PP .

Indeed, for an even r, we have in the last line of (VI.29) for x-0 only r-1 permanences to which correspond r-1 variations of the $f_\nu(y)$. For x+0 we have only permanences in the last line of (VI.29) to which correspond 1 variation and r permanences. On the other hand, for an odd r, we have in the last line of (VI.29) positive values throughout, while the sequence of the $f_\nu(y)$ has for x-0 1 permanence and r variations, which become at x+0 1 variation and r permanences.

But now it follows from the tables (VI.30) and (VI.31) that in all cases the number of VP lost is even and ≥ 0 as well as the number of PP won.

6.17. We have now proved the

Lemma 12. If g is a singular f gap at x, $a < x < b$, open to the left, we have (VI.24) with $m=n$, while, if g is a closed singular f gap, again with $a < x < b$, we have

$$\text{(VI.32)} \qquad m = n \quad , \quad \Delta'(g) = 2K' \geq 0 \quad , \quad \Delta''(g) = 2K'' \geq 0 \quad .$$

with integers K' and K''.

Observe that at a+0, if x is =a, the second formulas in (VI.23) and (VI.26) remain valid, while, for x=b, the values of the corresponding $F_v(y)$ and $f_v(y)$ at b-0 are obtained from the first formulas in (VI.23) and (VI.26).

§7. The F gaps

7.1. An F *gap* at x in a double sequence

$$\text{(VII.1)}\qquad \begin{matrix} f_{P-1}(y) & f_P(y) & f_{P+1}(y) & \dots & f_{Q-1}(y) & f_Q(y) \\ F_{P-1}(y) & F_P(y) & F_{P+1}(y) & \dots & F_{Q-1}(y) & F_Q(y) \end{matrix} ,$$

$G := \{P, P+1, \dots, Q-1\}$, is defined if

$$\text{(VII.2)}\qquad F_P = F_{P+1} = \dots = F_{Q-1} = 0 \ , \ F_{P-1}F_Q \neq 0 \ , \ R := Q-P \geq 1 \ ,$$

$$\text{(VII.3)}\qquad f_{P-1}f_P \dots f_{Q-1}f_Q \neq 0 \ .$$

If one of the F_ν ($\nu = P+1, P+2, \dots, Q-1$) vanishes identically, the F gap, G, is called *singular*. Then $\alpha = \frac{-1}{n}$ and f_{P-1} is given by (III.7), replacing p with P.

If the general sequence

$$\text{(VII.4)}\qquad \begin{matrix} f_o(y) & f_1(y) & \dots & f_{n-1}(y) & f_n(y) \\ F_o(y) & F_1(y) & \dots & F_{n-1}(y) & F_n(y) \end{matrix}$$

is cut at the index m, where $F_m(y)$ remains positive in J, then obviously the F gap, G, defined by (VII.1) and (VII.2), is either completely contained in (I.10) or completely contained in the part of (VII.4) with the indices $>m$.

We denote by $\Delta'(G)$ the number of VP in (VII.1) *lost* if we go from x-0 to x+0, and by $\Delta''(G)$ the number of PP *won* in going from x-0 to x+0.

Assume first that G is *non-singular*. From $F_{Q-\rho} = 0$ ($\rho = 1, \dots, R$) it follows

$$f_{Q-\rho+1}f_{Q-\rho-1} > 0 \quad (\rho = 1, 2, \dots, R).$$

Thence, if we put

(VII.5) $\operatorname{sgn} F_Q = \beta$, $\operatorname{sgn} f_Q = \gamma$, $\operatorname{sgn} \frac{f_{Q-1}}{f_Q} = \delta$, $\operatorname{sgn} F_{P-1}F_Q = \eta$,

we can write for the sequence $f_Q, f_{Q-1}, \dots, f_P, f_{P-1}$:

(VII.6) $$\operatorname{sgn} f_{Q-\nu} = \gamma\delta^{\nu} \quad (\nu=0,\dots,R+1).$$

As to the $\operatorname{sgn} F_{Q-\varrho}(x\pm0)$ $(\varrho=1,\dots,R)$ we have from (IV.9)

$$F_{\nu}(x+h) = \frac{h^{Q-\nu}}{(Q-\nu)!}\,\frac{f_{\nu}}{f_Q}F_Q + O(h^{Q-\nu+1}) \quad (h\neq0; h\to0)$$

and therefore, replacing ν with $Q-\nu$ and using (VII.5) and (VII.6)

(VII.7) $$\operatorname{sgn} F_{Q-\nu}(x\pm0) = \beta(\pm1)^{\nu}\delta^{\nu} \quad (\nu=0,1,\dots,R).$$

We can now write for the sequences of $\operatorname{sgn} F_{Q-\nu}(x\pm0)$ in the case of <u>even</u> R

(VII.8) $F_{Q-\nu}$,

$\nu =$	0	1	2	. . .	R-1	R	R+1
x-0	β	$-\beta\delta$	β	. . .	$-\beta\delta$	β	$\beta\eta$
x+0	β	$\beta\delta$	β	. . .	$\beta\delta$	β	$\beta\eta$

and in the case of <u>odd</u> R

(VII.9) $F_{Q-\nu}$,

$\nu =$	0	1	2	. . .	R-1	R	R+1
x-0	β	$-\beta\delta$	β	. . .	β	$-\beta\delta$	$\beta\eta$
x+0	β	$\beta\delta$	β	. . .	β	$\beta\delta$	$\beta\eta$

The line (VII.6) gives the signs of $f_{Q-\nu}$ in (VII.1) at x, while the signs of the $F_{Q-\nu}(x\pm0)$ are given in (VII.8) and (VII.9).

7.2. From (VII.6) it follows that for $\delta=1$ we have in the first line of (VII.1) only permanences, so that (VII.1) has no VP at $x \neq 0$ and we have only to count the permanences in (VII.8) and (VII.9) to obtain the numbers of PP at x-0 and at x+0.

On the other hand, for $\delta=-1$, the sequence (VII.6) contains only variation of signs, so that in this case we have no PP, while the numbers of the VP are obtained again counting the permanences in (VII.8) and (VII.9).

Further, replacing δ by $-\delta$, both lines in (VII.8) and similarly in (VII.9) are <u>interchanged</u>, so that, going from $\delta=1$ to $\delta=-1$, we have only to interchange x-0 and x+0 and to replace the PP with the VP.

7.3. The numbers of the permanences in (VII.8) and (VII.9) are obviously independent of β, so that for the counts of the permanences we can take $\beta=1$. We obtain then for $\delta=1$ the numbers of the PP <u>won</u> in passing from x-0 to x+0 from the following table, where, as in the following, $\equiv$ signifies congruence mod.2,

(VII.10) $\delta = 1$,

$\eta=$	$R\equiv 0$	$R\equiv 1$	
1	$1 \to R+1$	$0 \to R+1$	PP
-1	$0 \to R$	$1 \to R$	

We see that this number is R for even R and $R\pm 1$ for odd R, thence in both cases even and ≥ 0. If we replace δ by $-\delta$, we have only to interchange in each couple, $a \to b$, a and b; thus we obtain in this case the table of the VP <u>lost</u>:

(VII.11) $\delta = -1$,

$\eta=$	$R\equiv 0$	$R\equiv 1$	
1	$R+1 \to 1$	$R+1 \to 0$	VP
-1	$R \to 0$	$R \to 1$	

Thence, in this case, the number of the lost VP is either R or $R\pm 1$, both times even and ≥ 0.

7.4. Finally, if we have a singular F gap, G, extending from P to n-1, then $\alpha=\frac{-1}{n}$ and, by (III.7),

$$f_P(y) = c_2(y-u)^{n-P} \quad . \tag{VII.12}$$

Since, by (VII.3), $x \neq u$, the corresponding f_ν from f_P to f_{n-1} do not change their signs if we go through x from x-0 to x+0, we see that the corresponding $\Delta'(G)=\Delta''(G)=0$, that is to say that the contributions of a singular F gap to $\Delta'(x)$ and $\Delta''(x)$ vanish for $x \neq u$.

7.5. We have now proved the

Lemma 13. Assume that in the double sequence (VII.1) we have (VII.2) and (VII.3) at x and no F_ν in (VII.2) vanishes identically. Then, using the notations (VII.5), the signs of the f_ν in (VII.3) are given by (VII.6), where sgn f_{P-1} is γ or $\gamma\delta$ according as R is odd or even. Further, the signs of $F_{Q-\nu}$ at $x \pm 0$ are obtained for even R from (VII.8) and for odd R from (VII.9).

Further, for $\delta=1$ the number of the PP in (VII.1) won in passing from x-0 to x+0 is obtained from (VII.10) and is even and non-negative, while there are no VP involved. For $\delta=-1$ the number of the lost VP is even and ≥ 0 and is given by the table (VII.11) obtained from (VII.10) interchanging x+0 and x-0. Here there are no PP involved.

On the other hand, if the F gap defined by (VII.1) and (VII.3), is singular and (VII.12) holds, there are no contributions of this gap to $\Delta'(x)$ and $\Delta''(x)$.

We can therefore write, denoting by K', K" non-negative integers, in all cases,

$$\Delta'(G) = 2K'(G) \geq 0 \ , \ \Delta''(G) = 2K''(G) \geq 0 \ . \tag{VII.13}$$

§ 8. Main Theorem

8.1. We are going to prove the

Main Theorem. Assume in a closed interval $\bar{J} := \langle a,b \rangle$ a function $f(x) \equiv f_o(x)$ continuous in $\bar{J}$ with its n derivatives,

$$f_o^{(\nu)}(x) =: f_\nu(x) \qquad (\nu = 0,1,\ldots,n) \quad ,$$

which have in $\bar{J}$ a finite number of zeros. Assume further that f_n has no zeros in the open interval $J := (a,b)$. Define the expressions $F_\nu(x)$ by (I.2) where the positive r_ν ($\nu = 1,\ldots,n-1$) are given by (II.1), and assume that those of the F_ν which do not vanish identically have only a finite number of zeros in J.

Using the notations of the introduction and particularly of the section 1.3., assign to those F_ν which vanish identically, the sign plus.

Then we have, if $F_m(x) > 0$ throughout J, the two relations

$$\text{(VIII.1)} \qquad D_m'(a,b) \;:=\; \Delta_m'(a,b) - \Big(N_o(a,b) - N_m(a,b)\Big) = 2K_m' \geq 0,$$

$$\text{(VIII.2)} \qquad D_m''(a,b) \;:=\; \Delta_m''(a,b) - \Big(N_o(a,b) - N_m(a,b)\Big) = 2K_m'' \geq 0,$$

where K_m' and K_m'' are integers.

8.2. Proof. Denote all points of J in which either an f_ν or a non-identically vanishing F_ν ($\nu = 0,1,\ldots,m$) vanishes by

$$\text{(VIII.3)} \qquad a < x_1 < x_2 < \ldots < x_{k-1} < x_k < b \quad .$$

Then, if we consider $D_m'(a,y)$ and $D_m''(a,y)$ for $a < y < b$, these functions can only vary their values if y goes through one of the points $x_1, x_2, \dots, x_k$, and we have therefore

$$\text{(VIII.4)}\quad \begin{aligned} D_m'(a,b) &= \sum_{\varkappa=1}^{k}\left(\Delta_m'(x_\varkappa)-N_o(x_\varkappa)+N_m(x_\varkappa)\right) \quad , \\ D_m''(a,b) &= \sum_{\varkappa=1}^{k}\left(\Delta_m''(x_\varkappa)-N_o(x_\varkappa)+N_m(x_\varkappa)\right) \quad . \end{aligned}$$

The assertion of the theorem will be proved if we show that, y going through any of the points $x_\varkappa$, $D_m'(a,y)$ and $D_m''(a,y)$ can only increase by non-negative even numbers.

Taking now one of the points $x_\varkappa$ and denoting it by x we have to show that the expressions

$$\text{(VIII.5)}\quad \begin{aligned} D_m'(x) &:= VP_m(x-0) - VP_m(x+0) - N_o(x) + N_m(x) \quad , \\ D_m''(x) &:= PP_m(x+0) - PP_m(x-0) - N_o(x) + N_m(x) \end{aligned}$$

are even and non-negative.

8.3. We consider the ordered set of all f gaps in the sequence of indices, $\{0,1,\dots,m\}$:

$$\text{(VIII.6)}\qquad g_1,\dots,g_t,\ g_\tau = \{p_\tau, p_\tau+1,\dots,q_\tau-1\} \quad (\tau=1,\dots,t),$$

where generally $q_\tau < p_{\tau+1}$ $(\tau=1,\dots,t-1)$ and $F_{q_\tau} > 0$ $(\tau=1,\dots,t)$. If there exists a singular f gap it will be the last one, g_t. We delete from the sequence $\{0,\dots,m\}$ all g_τ in (VIII.6). Then for the remaining indices all f_ν are $\neq 0$.

For each $\tau=1,\dots,t$ the expressions $VP_{p_\tau-1}-VP_{q_\tau}$, $PP_{q_\tau}-PP_{p_\tau-1}$ are $\Delta'(g_\tau)$, respectively $\Delta''(g_\tau)$.

Consider further the ordered set of all F gaps in $\{0,\dots,m\}$:

$$\text{(VIII.7)}\qquad G_1,\dots,G_s,\ G_\sigma = \{P_\sigma, P_\sigma+1,\dots,Q_\sigma-1\} \quad (\sigma=1,\dots,s),$$

where $Q_\sigma < P_{\sigma+1}$ $(\sigma=1,\dots,s-1)$. Since each F_{Q_σ} is >0 it is clear that all G_σ lie <u>between the single</u> g_τ, while for the indices from

$$\text{(VIII.8)} \qquad \left\{0,1,\ldots,m\right\} - \sum_{\tau=1}^{t} g_\tau - \sum_{\sigma=1}^{s} G_\sigma$$

no f_ν and F_ν vanish.

For each $\sigma=1,\ldots,s$ the expressions $VP_{P_\sigma-1}-VP_{Q_\sigma}$, $PP_{Q_\sigma}-PP_{P_\sigma-1}$ are $\Delta'(G_\sigma)$, respectively $\Delta''(G_\sigma)$.

Thence the values of $\Delta_m'(x_\varkappa)$, $\Delta_m''(x_\varkappa)$ in (VIII.4) are respectively the sums of the values corresponding to g_τ and G_σ.

$$\text{(VIII.9)} \qquad \Delta_m(x_\varkappa) = \sum_{\tau=1}^{t} \Delta(g_\tau) + \sum_{\sigma=1}^{s} \Delta(G_\sigma) \ ,$$

where we can replace Δ everywhere both with Δ' and with Δ''.

8.4. Observe that the term $N_o(x_\varkappa)$ in (VIII.4) is only $\neq 0$ if for $x=x_\varkappa$ the corresponding gap g_1 is an f gap <u>open to the left</u> (singular or non-singular). Then, by (VI.13) and (VI.21), if g_1 does not include the index m, that is, if $q_1 \leq m$,

$$\text{(VIII.10)} \qquad \Delta'(g_1) = \Delta''(g_1) = q_1 = N_o(x_\varkappa) \ .$$

On the other hand, q_1-1 is certainly $<m$, for $q_1-1=m$ would signify that g_1 is open to the right, contrary to what has been said in 6.2. Thence we can write, both for Δ_m' and Δ_m'',

$$\text{(VIII.11)} \qquad \Delta_m(x_\varkappa) - N_o(x_\varkappa) + N_m(x_\varkappa) = \left[\Delta(g_1)-N_o(x_\varkappa)\right] + \left[\Delta(g_t)+N_m(x_\varkappa)\right] + \left[\sum_{\tau=2}^{t-1}\Delta(g_\tau) + \sum_{\sigma=1}^{s}\Delta(G_\sigma)\right] \ ,$$

where Δ can be replaced both with Δ' and Δ''.

Here the term in the first brackets is either $=0$ or even, in virtue of lemma 10 or lemma 12. As to the sums in the third brackets, they are even and non-negative, in virtue of (VI.16).

8.5. It remains to consider the second bracket term in (VIII.11), that is, assuming that $t \geq 2$,

$$\text{(VIII.12)} \qquad \Delta(g_t) + N_m(x_{\varkappa}) \ .$$

Denote the $N_m(a,b)$ zeros of f_m in J by u_ν,

$$\text{(VIII.13)} \qquad a < u_1 < u_2 < \cdot \ \cdot \ \cdot < u_N < b \quad , \quad N := N_m(a,b) \ .$$

Observe that for any $x_{\varkappa}$ with $N_m(x_{\varkappa})=0$ the expression (VIII.12) is, by lemma 10 and lemma 12, even, since if g_t is singular, it follows from $f_m(x_{\varkappa})\neq 0$ that m=n and lemma 12 can be used. We have therefore only to consider (VIII.12), if $x_{\varkappa}$ is one of the u_ν in (VIII.13). Further, we can assume $m < n$, since f_n has no zeros in J. We can therefore apply the lemma 11 and it follows from (VI.21) that the expression (VIII.12) is =0.

Our Main Theorem is proved.

8.6. In the case that one of the F_ν, $\nu < m$, identically vanishes we have seen in lemma 12 that m=n, so that the last g_τ, g_t is

$$g_t = \left\{p_t, p_t+1, \ldots, n-1\right\} \ .$$

The corresponding f_ν: $f_{p_t}, f_{p_t+1}, \ldots, f_{n-1}$ vanish only in one point u with $a < u < b$. And the corresponding $\Delta(g_t)$ occurs only in the terms of (VIII.4) for x_ν=u. But we have seen that then both $\Delta'(g_t)$ and $\Delta''(g_t)$ are even and non-negative. Further, since m=n, $N_m(u)$=0. We see that the formulas (VIII.1) and (VIII.2) of our Main Theorem remain true if all f_ν beginning with f_{p_t} are disregarded, that is, if we take $m=p_t-1$ and replace $N_m(u)$ by 0. If we then denote by p the first index for which F_p identically vanishes, we have p_t=p-1, m=p-2. And this conclusion holds whether F_m is positive or not. We can therefore formulate the

First Complement to the Main Theorem. If some F_ν identically vanish and F_p is the first identically vanishing F_ν, then (VIII.1) and (VIII.2) hold for $m=p-2$, replacing $N_m(a,b)$ with 0, independently of F_m being positive or not.

Observe that if F_m remains throughout positive, the relations in the First Complement are weaker than the original formulation of the Main Theorem, and this particularly so, since then $N_m(a,b)$ is easy to obtain, as the number of zeros of an equation of the type

$$c_1(x-u)^\ell = c_2$$

for some values of the constants c_1 and c_2. On the other hand, if F_m does not remain positive throughout J, the formulation of our First Complement can be considered as an essential improvement of the Main Theorem.

8.7. The expressions $\Delta_m'(a,b)$, $\Delta_m''(a,b)$, $N_o(a,b)$ and $N_m(a,b)$ in (VIII.1) and (VIII.2) are meant as interval functions defined for the open intervals (a,b). However, it follows from the proof of the Main Theorem that the above relations remain true if b-0 is replaced with b+0, assuming that $f_o(x)$ satisfies the corresponding conditions in (b-0,b+0) and F_m does not vanish there. Similarly a+0 can be replaced with a-0 if the corresponding conditions still holds in (a-0,a+0). We can therefore formulate the

Second Complement to the Main Theorem. The following inequalities hold:

(VIII.14) $$N_o(a,b) - N_m(a,b) \leq \Delta_m'(a,b) \quad ,$$

(VIII.15) $$N_o(a,b) - N_m(a,b) \leq \Delta_m''(a,b) \quad .$$

where a *can be throughout* $=a+0$ *or throughout* $=a-0$, *and similarly* b *can be throughout* $=b+0$ *or throughout* $=b-0$, *assuming that the conditions imposed upon* $f_o(x)$ *hold in* $(a-0,a+0)$ *respectively in* $(b-0,b+0)$. *In all cases the differences between the right-handed and the left-handed expressions in (VIII.14) and in (VIII.15) are even*.

8.8. A glance at the argument in the §6, §7 and §8 shows that, if $m \leq n-2$, we only used our assumptions about the f_ν and F_ν for $\nu=0,1,\dots,m,m+1$. We can therefore formulate the

Third Complement to the Main Theorem. *For* $m \leq n-2$ *the assumptions of the Main Theorem need only be verified for* $\nu \leq m+1$.

§ 9. Newton's Rule

9.1. In numerical applications the signs of the f_ν and F_ν at a+0 and b-0 are usually not directly given, but must be obtained from the values of these functions at a and b. To do this we have to identify in the formulas of the §6 and §7, x with a respectively b. This implies that the Taylor developments at a and b are one-sided.

$f_\nu(a+0)$ and $F_\nu(a+0)$

9.2. We use the notations of 6.1. and 7.1.

We consider an f gap, $g := \{p, p+1, \ldots, q-1\}$ and assume first that g is non-singular. If g is open to the left, we have from (VI.11)

(IX.1) $$F_\nu(a+0) > 0 \quad (\nu \leq q) \ ,$$

(IX.2) $$\operatorname{sgn} f_\nu(a+0) = \operatorname{sgn} f_q(a) \quad (\nu \leq q) \ ,$$

in virtue of (VI.14).

If g is closed, then, by (VI.17), (VI.18) and (VI.19), putting

(IX.3) $$\delta := \operatorname{sgn} f_q(a) \ , \quad \eta := \operatorname{sgn} f_{p-1}(a) f_q(a) \ :$$

(IX.4) $$\operatorname{sgn} f_{q-\nu}(a+0) = \delta \quad (\nu = 0, \ldots, r) \ ,$$

(IX.5) $$\operatorname{sgn} F_{q-\nu}(a+0) = 1 \ (\nu = 1, \ldots, r-1) \ , \ \operatorname{sgn} F_p(a+0) = -\eta \ (r > 1).$$

If g is open to the right, then q-p=1 and, by (VI.20),

(IX.6) $$\operatorname{sgn} f_p(a+0) = \operatorname{sgn} f_{p+1}(a) \ .$$

If now g is <u>singular</u>, then, by (III.8), replacing there $p-1$ with p, $f_p(y)=c(y-a)^{\sigma}$, $p\geq 0$, and here $\sigma\geq n-p$, as $f_n(y)$ remains finite with $y\downarrow a$. Thence

$$\text{(IX.7)}\qquad \operatorname{sgn} f_\nu(a+0) = \operatorname{sgn} c \quad (\nu=p,p+1,\dots,n)\quad ,$$

where c is the highest coefficient of $f_p(y)$. As to the $F_\nu(y)$, we have by the formula (VI.26), where $p-1$ has to be replaced with p,

$$\text{(IX.8)}\quad F_\nu(a+0) \equiv 0\ (p<\nu<n)\quad ,\quad \operatorname{sgn} F_p(a+0) = \begin{cases} 1 & (p=0) \\ -\operatorname{sgn}(cf_{p-1}(a)) & (p>0). \end{cases}$$

On the other hand, if we have an F <u>gap</u>, $G:=\{P,\dots,Q-1\}$, it follows from (VII.7):

$$\text{(IX.9)}\qquad \operatorname{sgn} F_{Q-\nu}(a+0) = \beta\delta^{\nu}\ (\nu=1,\dots,R)\quad ,$$

where

$$\text{(IX.10)}\qquad \beta := \operatorname{sgn} F_Q(a)\quad ,\quad \delta := \operatorname{sgn} f_{Q-1}(a)f_Q(a)\quad .$$

$f_\nu(b-0)$ and $F_\nu(b-0)$

9.3. Consider a <u>non-singular</u> f gap, $g:=\{p,p+1,\dots,q-1\}$. If g is <u>open to the left</u>, we have from (VI.11) and (VI.14)

$$\text{(IX.11)}\qquad F_\nu(b-0) > 0 \quad (\nu\leq q)\quad ,$$

$$\text{(IX.12)}\qquad \operatorname{sgn} f_\nu(b-0) = (-1)^{q-\nu}\operatorname{sgn} f_q(b)\quad (\nu\leq q)\quad .$$

If g is <u>closed</u>, then, by (VI.17), (VI.18) and (VI.19), putting

(IX.13) $\delta := \text{sgn}\, f_q(b)$, $\eta := \text{sgn}\, f_{p-1}(b) f_q(b)$:

(IX.14) $\text{sgn}\, f_{q-\nu}(b-0) = (-1)^{\nu}\delta \quad (\nu=0,\dots,r)$,

(IX.15) $\text{sgn}\, F_{q-\nu}(b-0)=1 \ (\nu=1,\dots,r-1)$, $\text{sgn}\, F_p(b-0)=(-1)^r\eta \ (r>1)$.

If g is open to the right, $q=m+1$, then $q-p=1$, and, by (VI.20),

(IX.16) $\text{sgn}\, f_m(b-0) = -\text{sgn}\, f_{m+1}(b)$.

If now g is singular, then, by (III.9), replacing there $p-1$ with p, $f_p(y)=c(b-y)^{\sigma}$, $p \geq 0$, and here $\sigma \geq n-p$, as $f_n(y)$ remains finite with $y \uparrow b$. Thence

(IX.17) $\text{sgn}\, f_{p+\nu}(b-0) = (-1)^{\nu}\,\text{sgn}\, c \quad (\nu=0,\dots,n-p)$.

As to the $F_\nu(y)$, we have again by (VI.26),

(IX.18) $F_\nu(b-0) \equiv 0 \ (p<\nu<n)$, $\text{sgn}\, F_p(b-0) = \begin{cases} +1 & (p=0) \\ (-1)^{n-p}\text{sgn}(cf_{p-1}(b)) & (p>0) \end{cases}$

Finally, in the case of an F gap, $G:=\{P,P+1,\dots,Q-1\}$, it follows from (VII.7)

(IX.19) $\text{sgn}\, F_{Q-\nu}(b-0) = \beta(-\delta)^{\nu} \quad (\nu=1,\dots,R)$,

where

(IX.20) $\beta := \text{sgn}\, F_Q(b)$, $\delta := \text{sgn}\, f_{Q-1}(b) f_Q(b)$.

9.4. Denote by $V_m(x)$ the number of variations in the sequence $F_o,\dots,F_m$ and by $P_m(x)$ the number of permanences in this sequence. Here, if some of the F_ν involved vanish, x has to be replaced by x+0 or x-0. Then we have obviously

$$\text{(IX.21)} \qquad VP_m(x) + PP_m(x) = P_m(x) \quad ,$$

for $x=x\pm 0$. On the other hand, it follows from (VIII.14), for $a=a\pm 0$, $b=b\pm 0$,

$$\text{(IX.22)} \qquad N_o(a,b) - N_m(a,b) \leqslant VP_m(a) - VP_m(b) \quad ,$$

under corresponding conditions in a and b.

In this relation, we are going to let b tend to ∞. Denote by $v_m(x\pm 0)$ the number of the variations in the sequence $f_o,\dots,f_m$ at $x\pm 0$ and by $p_m(x\pm 0)$ the number of permanences in this sequence. Assume

$$\text{(IX.23)} \qquad v_m(\infty) = 0 \quad ,$$

a condition, which is satisfied for any polynomial $f_o(x)$ of degree $\geqslant m$. Then it follows

$$\text{(IX.24)} \qquad VP_m(\infty) = 0 \quad .$$

In these relations we have of course to assume the conditions imposed upon $f_o(x)$ satisfied in an interval (a,∞). Replacing in (IX.22) a by x and letting b go to ∞, we obtain

$$\text{(IX.25)} \qquad N_o(x,\infty) - N_m(x,\infty) \leqslant VP_m(x) \quad ,$$

since the minuend on the right in (IX.22) vanishes by (IX.24). In the formula (IX.25) we can put either x=x+0 or x=x-0 assuming our conditions to hold in the corresponding interval between x and ∞.

9.5. On the other hand, if we want to let a tend to $-\infty$, we start from the formula (VIII.15) written for a and x,

$$N_o(a,x) - N_m(a,x) \leq PP_m(x) - PP_m(a) \quad . \tag{IX.26}$$

Assume the conditions imposed upon $f_o(x)$ to hold in the whole interval between $-\infty$ and $x = x \pm 0$. Assume further that for our $f_o(x)$:

$$p_m(-\infty) = 0 \quad , \tag{IX.27}$$

a condition certainly satisfied for any polynomial of degree $\geq m$. Then it follows obviously

$$PP_m(-\infty) = 0 \quad . \tag{IX.28}$$

If we let now a go to $-\infty$, we obtain from (IX.26)

$$N_o(-\infty,x) - N_m(-\infty,x) \leq PP_m(x) \quad . \tag{IX.29}$$

Now assume that the conditions imposed upon $f_o(x)$ hold along the whole interval $(-\infty,\infty)$ and put into (IX.25) and (IX.29) the same $x=x+0$ or $x=x-0$. Then, adding (IX.25) and (IX.29), we have, in virtue of (IX.21)

$$N_o(-\infty,\infty) - N_m(-\infty,\infty) \leq p_m(x \pm 0) \quad . \tag{IX.30}$$

9.6. We specialize now these formulas for the case that $f_o(x)$ is a polynomial of <u>exact degree</u> n.

Denote by κ_o the number of non-real zeros of $f_o(x)$ and by κ_m the corresponding number for $f_m(x)$. Then, obviously

$$N_o(-\infty,\infty) = n - K_o \quad , \quad N_m(-\infty,\infty) = n - m - K_m \quad ,$$

$$N_o(-\infty,\infty) - N_m(-\infty,\infty) = m - (K_o - K_m) \quad .$$

On the other hand, obviously for x=x+0 or x=x-0,

$$P_m(x) + V_m(x) = m \quad , \quad V_m(x) = m - P_m(x) \quad ,$$

since the F_ν vanishing identically are attributed the sign plus.

Introducing this into (IX.30), we obtain

(IX.31) $$K_o - K_m \geqslant V_m(x \pm 0) \quad ,$$

for any polynomial of exact degree n, assuming that $F_m(x)$ remains always positive.

This condition is by definition always satisfied for m=n. It follows therefore

(IX.32) $$K_o \geqslant V(x \pm 0) \quad (-\infty < x < \infty) \; .$$

and in particular for x=+0:

(IX.33) $$K_o \geqslant V(+0) \quad ,$$

Newton's Rule, allthough Newton does not mention the possibility of identically vanishing $F_\nu(x)$.

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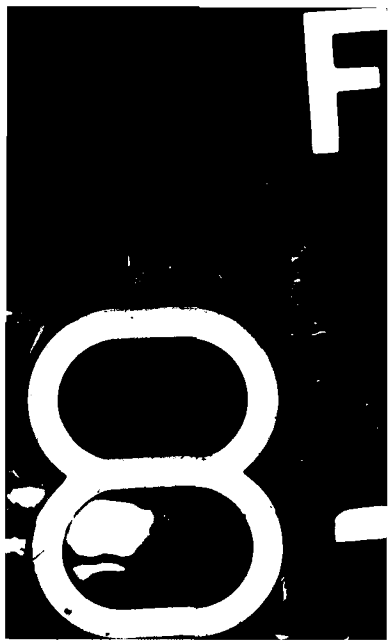
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