

AD-A086 722

AERONAUTICAL RESEARCH LABS MELBOURNE (AUSTRALIA)
STRUCTURAL FATIGUE IN ONE-CRACK MODELS WITH ARBITRARY INSPECTION--ETC(U)
APR 79 D 6 FORD
ARL/STRUC-377

F/G 20/11

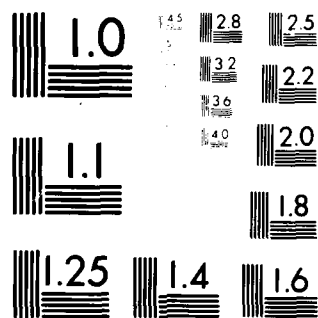
UNCLASSIFIED

NL

[OK]
ABJ
AC-41-1-1



END
DATE
FILMED
8-80
DTIC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

LEVEL 12
D.S.

ADA 086722

DEPARTMENT OF DEFENCE
DEFENCE SCIENCE AND TECHNOLOGY ORGANISATION
AERONAUTICAL RESEARCH LABORATORIES
MELBOURNE, VICTORIA

STRUCTURES REPORT 377

DTIC
ELECTE
JUL 14 1980
S **C**

STRUCTURAL FATIGUE IN ONE-CRACK MODELS
WITH ARBITRARY INSPECTION

by
D. G. FORD

THE UNITED STATES NATIONAL
TECHNICAL INFORMATION SERVICE
IS AUTHORIZED TO
REPRODUCE AND SELL THIS REPORT

Approved for Public Release.



© COMMONWEALTH OF AUSTRALIA 1979

APRIL, 1979

DDG FILE COPY

No 21

80 7 11 006

DEPARTMENT OF DEFENCE
DEFENCE SCIENCE AND TECHNOLOGY ORGANISATION
AERONAUTICAL RESEARCH LABORATORIES

STRUCTURES REPORT 377

14 ARL/STRUC-377

6 STRUCTURAL FATIGUE IN ONE-CRACK MODELS
WITH ARBITRARY INSPECTION,

by

10 D. G. FORD

11 Apr 79

12 29

DTIC
ELECTE
JUL 14 1980

D

C

SUMMARY

The reliability theory for fatigue lives of one-crack, two-stage models with hijacking has been extended to allow for arbitrary inspection intervals with possible arbitrary renewals. The life distribution is similar with a modified initiation density and the associated moment generating function is formally identical. It is possible to identify an unsteady Markov chain formed by the combined fatigue and inspection process.

Two FORTRAN programs have been developed from this formulation, one of which allows for random crack rates and run-time setting of inspections.

POSTAL ADDRESS: Chief Superintendent, Aeronautical Research Laboratories,
Box 4331, P.O., Melbourne, Victoria, 3001, Australia.

This document has been approved
for public release and sale; its
distribution is unlimited.

008650

mt

DOCUMENT CONTROL DATA SHEET

Security classification of this page: Unclassified

1. Document Numbers

- (a) AR Number:
AR-001-728
(b) Document Series and Number:
Structures Report 377 ✓
(c) Report Number:
ARL-Struc-Report-377

2. Security Classification

- (a) Complete document:
Unclassified
(b) Title in isolation:
Unclassified
(c) Summary in isolation:
Unclassified

3. Title: STRUCTURAL FATIGUE IN ONE-CRACK MODELS WITH
ARBITRARY INSPECTION

4. Personal Author(s):
D. G. Ford

5. Document Date:
April, 1979

6. Type of Report and Period Covered:

7. Corporate Author(s):
Aeronautical Research Laboratories ✓

8. Reference Numbers

- (a) Task:
(b) Sponsoring Agency:
DSTO

9. Cost Code:
27 7030

10. Imprint:
Aeronautical Research Laboratories,
Melbourne

11. Computer Program(s)
(Title(s) and language(s)):
Not applicable

12. Release Limitations (of the document)
Approved for Public Release

12-0. Overseas:	N.O.	P.R.	I	A	B	C	D	E
-----------------	------	------	---	---	---	---	---	---

13. Announcement Limitations (of the information on this page):
No Limitation

14. Descriptors:

Fatigue (Materials)
Reliability
Cracking (Fracturing)

Markov processes
Probability theory

15. Cosati Codes:

1404
1201
2012

16.

ABSTRACT

The reliability theory for fatigue lives of one-crack, two-stage models with hijacking has been extended to allow for arbitrary inspection intervals with possible arbitrary renewals. The life distribution is similar with a modified initiation density and the associated moment generating function is formally identical. It is possible to identify an unsteady Markov chain formed by the combined fatigue and inspection process.

Two FORTRAN programs have been developed from this formulation, one of which allows for random crack rates and run-time setting of inspections.

CONTENTS

	Page No.
<i>NOTATION</i>	1
1. INTRODUCTION	1
2. PREVIOUS RESULTS	2
2.1 Effect of Inspections	2
3. LIFE DISTRIBUTION	3
3.1 Renewal Allowance for Single Cracks	3
3.2 Density Function for Attrition	4
4. MOMENT GENERATING FUNCTIONS	5
4.1 Inspection Functions	5
4.2 Step Function Transforms	6
4.3 Complete Moment Generating Function	7
4.4 Transform of f .	9
5. EMBEDDED MARKOV PROCESS	10
5.1 State Probabilities	11
5.2 Transition Probabilities	12
5.2.1 CR	12
5.2.2 $CA_e R_e = M$	13
5.2.3 $C_e A_e$	15
5.3 Transition Matrices	16
6. CONCLUSIONS	16
6.1 Implementation	16
REFERENCES	
TABLES	
DISTRIBUTION	

Accession For	
NTIS GRA&I	☒
DDC TAB	☐
Unannounced	☐
Justification	
By _____	
Distribution/	
Availability Codes	
Dist	Avail and/or special
A	

NOTATION

$a(a)$	Crack size(s).
a_0	Initial crack size.
B	Infinite transition matrix, equation (4.12B).
$B_i = [b_{ij}]$	Fifth order transition matrix for the inspection at time T_i .
E	Expectation.
$f(a t)$	Conditional probability density of crack length (see $\phi(\cdot)$, $0(\cdot)$) at time t .
$f_a(t_a)$	Reliability type density of crack life (see Equation (2.6)).
$f_0(t), f_0(t_0)$	Density of initial life in the absence of any inspection or repair.
$f_i(t)$	Initial life density as affected by inspection or repair.
$f_{it}(t)$	$f_i(t)$ defined in $(T_i, T_{i+1}]$ extended to (T_i, ∞) .
F	Cumulative density of distribution function, not necessarily proper.
F_a	$1 - F_a$.
F_i	$1 - F_{it}$.
$g_i, g_i(t - T_i)$	Density of initial lives in structures repaired after inspection T_i . (This is initial data).
G_i	Distribution corresponding to g_i , not necessarily proper.
$h_a(t_0) = e^{-r_0 t_0} f_a(t_0)$	Used for Equation (5.15A).
H_i	Defined by (5.10A). This is a hypothetical distribution of initial lives from a hijacked distribution for structures repaired at T_i .
I, J, K	Terms of (4.7); see (4.10).
M	Generic form of moment generating function, $E \exp(-u)$.
M_j	Moment generating functions from g_j . See Section 4.4.
N_i	Defined by (5.15) and (5.15A).
$P, P_i(a) = P_i(t_a)$	Probability of rejection for an inspected structure with a crack of length a or at time t_a after initiation (operating characteristic). Normally used without subscripts.
$P_t = P(T_i - T_{i0})$	Used in Section 4.3.
P_{t_0}	Laplace transform of P with respect to t_a when $t_0 = T_{i0} +$ i.e. when the crack starts just after an inspection.
$P_j = P_j(T_j - t + t_a)$	The probability of rejection at an intermediate inspection (Section 3.2)
$P_j = 1 - P_j \text{Pr}(\cdot)$	
$\text{Pr}(\cdot \cdot)$	Probability, absolute or conditional.
$P_a(t_a t_0)$	Defined by (4.2). Overall acceptance probability during growth of some crack.

Q_i	Abbreviation used in I , Equation (4.8) and preceding.
$R = R(a) = R(t_a)$	Average local crack growth rate.
R_i	Probability of rejection at i -th inspection, time T_i . Equation (3.1).
$R = \{R_i i = 1, \dots, \infty\}$	Infinite vector of R_i 's.
$r(a) = r(t_a)$	Risk function.
$r_o = r(0)$	Hijack risk component.
$S(t)$	Arbitrary step function, Section 4.2.
$S(u)$	Laplace transform of $S(t)$.
t	Current equivalent time or cycles.
t_i	Other times according to subscript
T_i	Known inspection times, $T_o = 0$.
u, v, w	Laplace transform variables.
$\Delta T_i = T_{i+1} - T_i$	i -th inspection interval.
γ, δ	Real shifts for Bromwich inversion contours.
$\phi(t)$	Distribution of final lives without inspection, (2.6).
$\phi_o(t)$	Distribution of final lives with inspection and repair, (3.6).
$*$	Convolution function.

Subscripts

The subscripts associate the main symbol with the quantity listed.

o	Crack initiation without renewals.
$*$	Crack initiation affected by renewals.
$*_i$	As above but distinguished by inspection period.
a	Crack growth time.
c	Set complementation.
i	Inspection period $(T_i, T_{i+1}]$.
i_o, i_o	The inspection just before the start of the current crack i.e. $T_{i_o} < t_o \leq T_{i_o+1}$

Sets

Union and intersection are here denoted by "+" and the normal product convention while complements are indicated by the subscript c . Figures 1 and 5 illustrate some of the events.

A	Attrition.
E	Universe.
$M = CA_c R_c$	Mainstream — cracked but still in use.
R	Rejected at inspection.
ϕ	Empty set (see Section 5.1).

1. INTRODUCTION

It is a truism that fatigue life, especially with single cracks, consists of initiation time and crack growth time. The models based on this approach when the initiation is random, have been described in previous Reports^{1,2,3}. In the last of these, considerations of continuity of probability, together with deterministic cracking, led to a first order partial differential equation (true for vector cracks also), which is the same as the continuity condition for compressible flow. It is also a degenerate form of the second order Fokker-Planck equation which it would become for random cracking. Most generally this describes the infinitesimal evolution of probabilities associated with continuous Markov processes^{4,5}.

In an earlier Report³ the density of crack lengths (and thence that of total life) were found from the continuity equation without considering inspections, though these were mentioned briefly. Under the term hijacking it also introduced the effect of losses not due to fatigue. The present Report extends the previous solution to include inspections as well as the hijack risk. It considers the distribution of total life, the moment generating function, and the transition matrix for changes of state between inspections.

2. PREVIOUS RESULTS

Before proceeding, we shall summarise results for the one-crack model without inspection. As before, we shall use the generic notation f , F , ϕ , and M , to denote density, distribution functions and moment generating functions of their arguments. Density is affected by attrition and where necessary, these symbols will be subscripted to avoid confusion.

For vector cracks the continuity equation

$$\frac{Df}{Dt} = f(\text{div } \mathbf{R}(\mathbf{a}) + r(\mathbf{a})) \quad (2.1)$$

holds, where D denotes total derivative and $\mathbf{R}(\mathbf{a}) = d\mathbf{a}/dt$, the previously averaged crack rate, $r(\mathbf{a}) = \text{total risk function including hijack risk}$, and $\text{div } \mathbf{R}(\mathbf{a}) = dR/da$ for a single cracking, a known function of crack length.

When this is expanded, one obtains the degenerate Fokker-Planck equation (for single cracks)

$$\frac{\partial f}{\partial t} + R(a) \frac{\partial f}{\partial a} = -f \left(\frac{d}{da} R(a) + r(a) \right) \quad (2.2)$$

with the characteristic equations

$$dt = da/R(a) = -df/f(dR/da + r(a)). \quad (2.3)$$

In this the crack trajectories are characteristics and the general solution for crack length density is

$$\begin{aligned} f &= f(a_0|t_0) \frac{R(a_0)}{R(a)} \exp - \int_{a_0}^a \frac{r(a)}{R(a)} da \\ &= f(a_0|t_0) \frac{R(a_0)}{R(a)} \exp - \int_0^{t_a} r(t) dt \end{aligned} \quad (2.4)$$

introducing the growth time t_a to reach crack length a . Here a_0 is the initial crack size, a constant, and t_0 , the initial life, corresponds to the crack $a|t$. The growth time

$$t_a = t - t_o.$$

When the hijack risks are denoted r_o , and the boundary conditions are included $r(t) = r(a(t))$

$$f(a|t) = \frac{e^{-r_o t_o} f_o(t_o)}{R(a)} \exp \left(- \int_0^{t_a} r(t) dt \right) \quad (2.5)$$

where the generic f refers to different density functions according to its subscript.

It is now possible to average the risk $r(a)$ at time t for the overall life density.

$$\phi(t) = r_o e^{-r_o t_o} [1 - F_o(t)] + e^{-r_o t_o} f_o(t_o) * f_a(t_a) \quad (2.6)$$

where $f_a(t_a) = r(t_a) \exp - \int_0^{t_a} r(t) dt$, the reliability based crack life density.

The moment generating function, $E_\phi \exp -ut$, follows as

$$M_\phi(u) = M_o(u + r_o) M_a(u) + \frac{r_o}{u + r_o} \left[1 - M_o(u + r_o) \right] \quad (2.7)$$

which differs slightly from the corresponding equation of Reference 3 since $-u$ is used here to assure convergence for positive u .

2.1 Effect of Inspections

This is twofold: in the first place, the density of crack length will be reduced suddenly at a number of steps corresponding to each inspection. Secondly, structures rejected at an inspection may be returned, after repair, to the population. This brings the whole problem into a close relationship with statistical renewal theory. However, the latter does not include hijack risks, nor the two stages involved in fatigue.

We will first consider renewals and then the corresponding moment generating functions. In the following, it is most convenient to have all variables as time, and continue to use t_o , t_a and t to denote initiation time, crack period and their sum. The subscript $*$ will refer to densities or generating functions affected by renewals. Inspection times are T_i , $i = 0, 1, 2$ etc. where $T_o = 0$. The i -th inspection interval and the associated quantities occur after T_i .

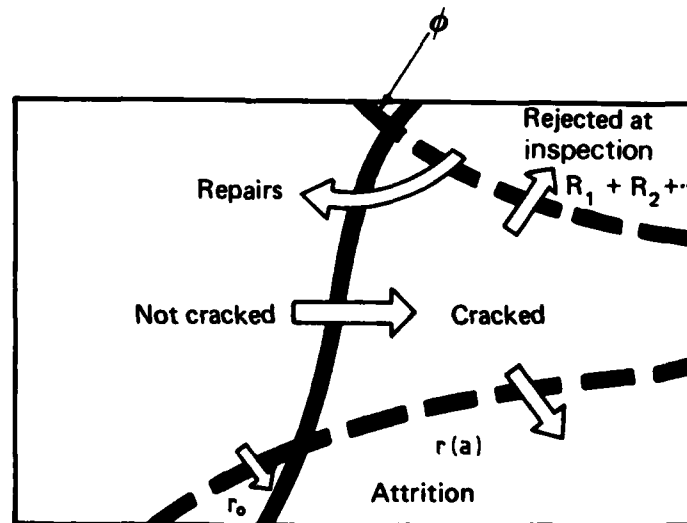


FIG 1. POSSIBLE STATES OF STRUCTURES

3. LIFE DISTRIBUTION

At a particular time the population of structures suffering two-stage fatigue with inspections and attrition may be described by the Venn diagram of Figure 1. In the course of time the arrows indicate the evolution of members of the population; one may equivalently imagine uniform measure density over the rectangle and the subset boundaries moving left and inwards opposite to the arrows. The zero set corresponds to rejection of uncracked structures. This is the general process we intend to describe.

3.1 Renewal Allowance for Single Cracks

At T_i — just before inspection, suppose the partial density of vector crack lengths is $f(a|T_i-)$, which therefore becomes

$$f(a|T_i+) = (1 - P_i(a))f(a|T_i-)$$

just afterwards. With respect to the whole population this gives the probability of rejection at the i -th inspection as

$$R_i = \int_0^{\infty} P_i(a) f(a|T_i-) da \quad (3.1)$$

We also define P_i the operating characteristic of the inspection method, in terms of growth time so that $P_i(a) = P_i(t_a)$. The context will indicate which definition is current.

After T_i , the population will also include repaired or modified structures returned from inspection. It may also be depleted by those retired from service. Statistically, these repairs or retirement affect the initial life distribution. After each inspection we will assume that structures are repaired to the same condition (not necessarily the original as new condition) despite the possibility of differing crack lengths being discovered. (Since it is mathematically easy, the restored condition shall be assumed specific to the particular inspection).

This assumption requires us to first consider the density of crack initiation which may determine boundary conditions for $f(a|t > T_i)$ (see Ref. 3).

For one crack in the absence of inspections, let $e^{-r_0 t}(1 - F_0(t))$ be the survivorship function for initial life when hijack attrition is included (Ref. 3). When inspections are included but not, for the moment, hijacking, let the corresponding survivorship fraction (based on the original population) be $F_{i-1}(t)$ for $t < T_i$ say.

Just after an inspection all the uncracked structures will be retained and also augmented by the fraction $R_i[1 - G_i(0)]$ of inspected and repaired structures. Symbolically

$$1 - F_i(T_i+) = [1 - F_{i-1}(T_i-)] + R_i[1 - G_i(0)] \quad (3.2)$$

where $G_i(t)$ is a general, possibly partial, distribution function of initiation at times $T_i + t$ of structures repaired at time T_i . At time t , $T_i < t < T_{i+1}$ with *hijacking* try

$$e^{-r_0 t}(1 - F_i(t)) = e^{-r_0 t}(1 - F_{i-1}(t)) + R_i e^{-r_0(t-T_i)}(1 - G_i(t - T_i))$$

or

$$\bar{F}_i = \bar{F}_{i-1}(t) + H_i \quad (3.3)$$

where $\bar{F} = 1 - F$ and

$$H_i = e^{r_0 T_i} R_i [1 - G_i(t - T_i)]$$

Recursive substitution then leads to

$$\bar{F}_i(t) = \bar{F}_0 + \sum_{j=1}^i H_j \quad (3.3A)$$

which requires an interpretation of F_0 , G_0 etc.

Initially (3.3) indicates that

$$1 - F_{*o}(T_o) = 1 - F_{*1}(T_o) + e^{r_o T_o} R_o [1 - G_o(0)] \quad (3.3B)$$

and we would expect that

$$F_{*o}(t|t < T_i) = F_o(t),$$

the initial failure distribution. If we define $T_o = 0$ then for $0 < t < T_1$, (3.3) becomes

$$1 - F_{*o}(t) = 1 - F_o(t) = 1 - F_{*1}(t) + R_o [1 - G_o(t - T_o)] \quad (3.3C)$$

The definitions may now be arbitrary. The most convenient for us is $F_{*1}(t) \equiv F_o(t)$ and $R_o = 0$ leaving G_o arbitrary.

(Note that all F_{*i} , G_i etc. are defined on $(0, \infty)$; F_{*i} is applied on (T_i, T_{i+1}) and $F_{*i} \equiv F_{*i}$ etc. everywhere with the index suited to t .) To summarise T_o , $R_o = 0$, $F_{*1}(t) = F_{*o}(t) = F_o(t)$ and G_o is arbitrary.

In (3.3) or (3.3A) the restored factor $\exp -r_o t$ allows for hijacking. These equations now provide some of the boundary conditions for $f(a|T_i < t < T_{i+1})$ when the solution has marched to T_{i+1} . Then (3.1) provides R_{i+1} , allowing (3.3C) to operate over the next inspection interval (T_{i+1}, T_{i+2}) .

3.2 Density Function for Attrition

Now include the hijack factor in (3.3C) and consider the density of initiation

$$-\frac{d}{dt} e^{-r_o t} \bar{F}_i = r_o e^{-r_o t} \bar{F}_i + e^{-r_o t} f_{*i}(t) \quad (3.4A)$$

where the first term is obviously the local incidence of hijacking and therefore a direct component of the density $\phi(t)$ of total life.

From (3.3A) the second term is

$$e^{-r_o t} \sum_{j=1}^i e^{r_o T_j} R_j g_j(t - T_j) \quad (3.4B)$$

where g_j is the initiation density corresponding to G_j which may be partial or defective.

Consider $f(a|T_i < t < T_{i+1})$ after the i -th inspection. Such a crack would have begun at $t - t_a(a)$ and would therefore have been inspected an integral number of times. The operating characteristic $P(a)$ of any crack is the defined in terms of crack length. In keeping with our present approach, it is more convenient to define this in terms of the growth time t_a as $P(t_a)$. We further define P_j as the characteristic corresponding to the j -th inspection of a crack.

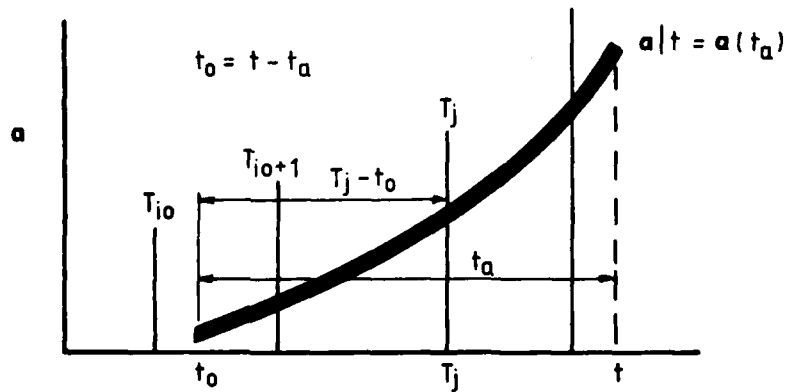


FIG 2. NOTATION FOR INSPECTION DURING CRACK GROWTH

In the notation of the figure

$$P_j = P(T_j - t + t_a).$$

For a long crack, several non-trivial P_j 's are possible. Unless specifically stated, several P_j 's in the same expression refer to the same crack trajectory.

Now consider the basic density $f(a|t)$ allowing attrition of cracked structures and also inspections. If $T_i < t < T_{i+1}$ integration (Ref. 3) along the characteristic and multiplications by $P_j = 1 - P$ produce

$$\begin{aligned} R(a)f(a|t) &= P_{i_0+1} \dots P_i \exp \left(- \int_{t_0}^t r(a) dt \right) e^{-r_0 t_0 f_{*i_0}(t - t_a)} \\ &= \left(\prod_{j=i_0+1}^i P_j \right) \exp \left(- \int_0^{t_a} r(t) dt \right) e^{-r_0 t_0} \sum_{j=0}^{i_0} r_0 T_j R_j g_j(t_0 - T_j) \end{aligned} \quad (3.5)$$

where t_a is the growth time and $T_{i_0} < t - t_a < T_{i_0+1}$. As before, with the hijacking, this produces the attrition density

$$\begin{aligned} \Phi_*(t|T_i < t < T_{i+1}) &= r_0 e^{-r_0 t} F_i + \int_{a_0}^{\infty} r(a)f(a|t) da \\ &= r_0 e^{-r_0 t} [1 - F_{*i}(t)] + \int_0^t e^{-r_0 t_0} f_{*i}(t_0) \prod_{j=i_0+1}^i P_j(T_j - t_0) dF_a(t_a) \end{aligned} \quad (3.6)$$

with $dF_a(t_a)$ from (2.6), the initiation $t_0 = t - t_a$, as always, and P_j is described in full. In (3.5) the finite upper limit t is no restriction because $f_{*i} = 0$ for negative arguments. Recognition of this will aid the manipulations below.

4. MOMENT GENERATING FUNCTIONS

Equation (3.6) appears to be a convolution but the status of P_j is uncertain.

Let us form the moment generating function as

$$\begin{aligned} \sum_{i=0}^{\infty} \int_{T_i}^{T_{i+1}} \phi(t|i) e^{-ut} dt &= \int_0^{\infty} r_0 e^{-(r_0+u)t} [1 - F_{*i}(t)] dt + \\ &\quad \sum_{i=0}^{\infty} \int_{T_i}^{T_{i+1}} \int_0^t e^{-(r_0+u)t_0 - ut} f_{*i}(t_0) \prod_{j=i_0+1}^i P_j(T_j - t_0) dF_a(t_a) dt, \end{aligned} \quad (4.1)$$

where f_{*i} is the global form of f_{*i} , the whole of the modified initial life distribution.

4.1 Inspection Functions

Consider the product of the inspection factors as functions of initiation t_0 and of growth time t_a . For fixed t_0 this term is a step function of t_a (determining $i - i_0$) but it is at least piecewise continuous in t_0 since, almost always, $T_j - t_0$ continuously determines crack sizes at inspection.

Occasionally, through i_0 decreasing, t_0 introduces another factor, but if $P_j(a_0) = 0$, which we now assume, continuity with respect to t_0 will be retained.

Let us abbreviate the inspection factor

$$\prod_{j=i_0+1}^i (1 - P(T_j - t_0)) = P_{*i}(t_a|t_0) \equiv P_{*i} \quad (4.2)$$

where P_{*i} is a step function decreasing from unity. We now intend to treat (4.1) as a convolution by taking some of the inspection terms as part of the crack life distribution. In (4.1) the region of integration is the infinite sector shown in Figure 3. The operation began with integration

along strips such as AB which were then extended to horizontal inspection bands whose contributions were finally summed.

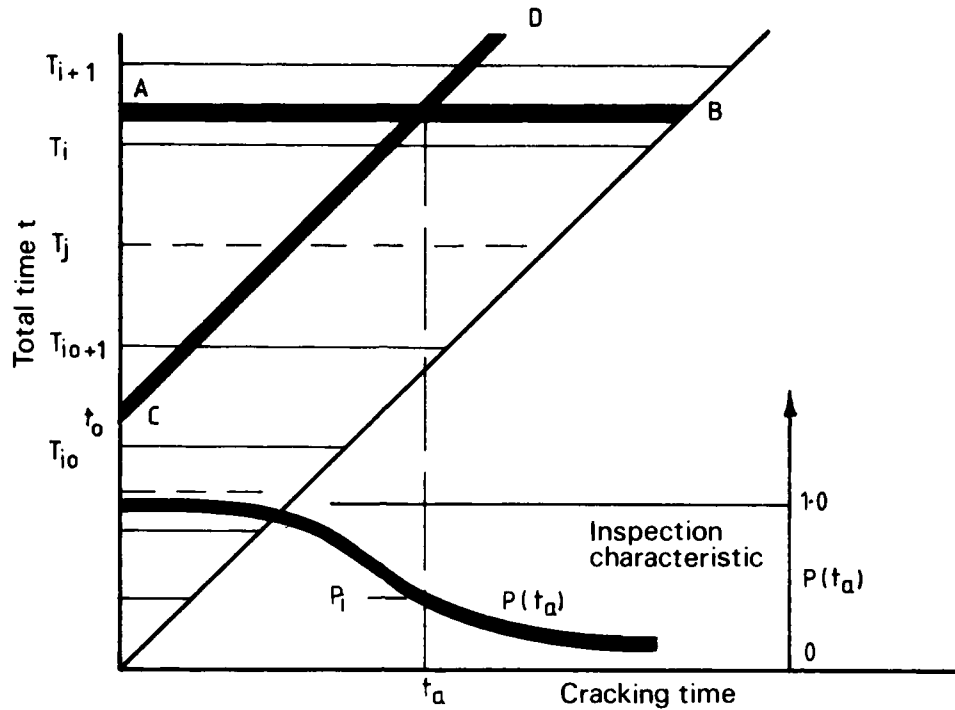


FIG. 3 REGION OF INTEGRATION FOR MOMENT GENERATING FUNCTION

If we change the variable t to t_0 , then we may do the first integration along CD and include the summation by adopting the limits of $(0, \infty)$ for t_a .

Thequus with (4.2) ation (4.1) becomes

$$M_\phi(u) = \frac{r_0}{u + r_0} (1 - M_*(u + r_0)) + \sum_{i=0}^{\infty} \int_{T_i}^{T_{i+1}} \int_0^{\infty} f_*(t_0) e^{-(u+r_0)t_0} P_* e^{-ut_a} dF_a(t_a) dt_0 \quad (4.3)$$

where $M_* = E_* \exp(-ut_0) = \text{MGF of } f_*$.

When $P_* \equiv 1$ the integral reduces to the uninspected form $M_*(u + r_0)M_a(u)$; in general, these terms introduce convolutions of transforms.

4.2 Step Function Transforms

We have already seen that P_* is a step function for given t_0 and in the figure for (4.1) the first integral along CD traverses an infinite number of inspections. In (4.3) then P_* , given by (4.2) is interpreted for all t_a . Thus $P_*(t_a|t_0)f_a(t_a)$ may be regarded as a defective conditional density of cracking life.

By the convolution theorem for Laplace transforms

$$\int_0^{\infty} P_*(t_a) f_a(t_a) e^{-ut_a} dt_a = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \bar{P}_*(u - v) M_a(v) dv \quad (4.4)$$

where the bar indicates a Laplace transform, as is M_a .

For any step function $S(t)$ such as that below

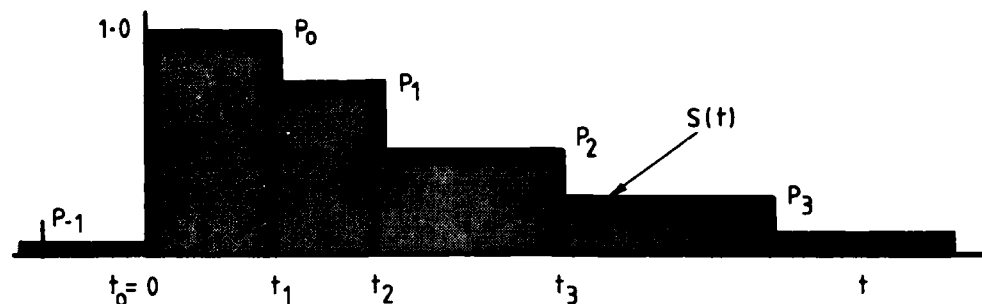


FIG 4. TYPICAL STEP FUNCTION

it is easily shown that, in the notation indicated,

$$S(u) = u^{-1} \sum_{i=0}^{\infty} (P_i - P_{i-1}) e^{-u t_i} \quad (4.5)$$

In the present instance with steps from (4.2), one would have

$$t_1 = T_{i_0+1} - t_0, \quad \dots, \quad t_j = T_{i_0+j} - t_0,$$

with t_0 arbitrary. Part of the t_0 dependence is in the value of $T_{i_0+1} - t_0$.

Let $\bar{P}_{i_0}$ be the transform of P , corresponding to $t_0 = T_{i_0} + \epsilon$. Then for t_0 elsewhere in this inspection interval, the transform of the step function is

$$e^{u(t_0 - T_{i_0})} \bar{P}_{i_0} + u^{-1}(1 - e^{u(t_0 - T_{i_0})}) \quad (4.6)$$

The first term has a simple shift operator but the second "spillage" term is required because the backwards shift truncates $S(t)$ for $t < 0$.

4.3 Complete Moment Generating Function

We now perform the transforms indicated by (4.3) requiring the transform of step functions. From (4.5)

$$\bar{P}_{i_0} = u^{-1} \sum_{i_0}^{\infty} (P_i - P_{i-1}) e^{-u(T_i - T_{i_0})}, \quad P_i = 1, \quad P_{t_0} = 1, \quad P_{t_0-1} = 0,$$

giving (4.6) as

$$e^{u t_0} u^{-1} \sum_{i_0}^{\infty} (P_i - P_{i-1}) e^{-u T_i} + u^{-1}(1 - e^{u t_0 - u T_{i_0}})$$

with

$$P_i = P(T_i - T_{i_0}).$$

The inner integral (4.4) of (4.3) is now

$$\begin{aligned} & \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} v^{-1} e^{v t_0} \sum_{i_0}^{\infty} (P_i - P_{i-1}) e^{-v T_i} M_a(u - v) dv \\ & + \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{1}{u - v} M_a(v) dv - \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} v^{-1} e^{v(T_0 - T_{i_0})} M_a(u - v) dv \quad (4.7) \end{aligned}$$

with the $u - v$ argument variously placed.

All the components of (4.7) are similar so that we now substitute the first into (4.3) to obtain

$$I = \sum_{t_0=0}^{\infty} \int_{T_{t_0}}^{T_{t_0+1}} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} f_s(t_0) e^{-(u-v+r_0)t_0} M_a(u-v) Q_{t_0} dv dt_0$$

$$\text{where } Q_{t_0} = \sum_{t_0}^{\infty} (P_t - P_{t-1}) e^{-vT_t/v}.$$

If the order of integration is reversed then in each interval the inspection factor is constant with respect to t_0 . Thus the t_0 limits of I may be infinite provided the inspection factor is regarded as a step function. The t_0 integral is thus the transform of the products of f_s and this step function. This introduces another convolution whence

$$I = \frac{1}{(2\pi i)^2} \int \int M_a(u-v) M_s(u-v+r_0) \sum_{t=0}^{\infty} (Q_t - Q_{t-1}) \frac{e^{-(u-v-w+r_0)T_t}}{u-v-w+r_0} dv dw$$

where $Q_{-1} = 0$.

In the w -plane there is a pole at $u-v+r_0$. Contour integration then leaves the cancelling residues,

$$I = \frac{1}{2\pi i} \int M_a(u-v) M_s(u-v+r_0) \sum_{t=0}^{\infty} (Q_t - Q_{t-1}) \frac{dv}{v} = 0 \quad (4.8)$$

The next component of (4.7) leads to

$$\begin{aligned} J &= \sum_{t_0=0}^{\infty} \int_{T_{t_0}}^{T_{t_0+1}} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} f_s(t_0) e^{-u+r_0)t_0} M_a(u-v) \frac{dv}{v} dt_0 \\ &= \frac{1}{2\pi i} M_s(u+r_0) \int_{\gamma-i\infty}^{\gamma+i\infty} M_a(u-v) \frac{dv}{v}, \text{ absorbing the summation,} \\ &= M_s(u+r_0) \overline{H(t)f_a(t)} = M_s(u+r_0) M_a(u) \end{aligned} \quad (4.9)$$

using the convolution theorem.

In the third component of (4.7) the presence of T_{t_0} indicates the presence of another step function. In (4.3) it leads to

$$K = - \sum_{t_0=0}^{\infty} \int_{T_{t_0}}^{T_{t_0+1}} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} f_s(t_0) \frac{e^{-(u-v+r_0)t_0}}{v} M_a(u-v) e^{-vT_{t_0}} dv$$

where the step function heights are $v^{-1} \exp -vT_{t_0}$. In the same way as before, using (4.5),

$$K = \frac{1}{2\pi i} \int_{\delta-i\infty}^{\delta+i\infty} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} M_a(u-v) M_s(u-v+r_0) \sum_{t=0}^{\infty} \frac{e^{-(v-w)T_{t_0}} - e^{-(v-w)T_{t_0-1}}}{v-w} dv dw$$

Once again the only w -pole is for $w = v$ in which case the residues again cancel; $K = 0$.

When all these results are assembled ((4.3), (4.8) and (4.9))

$$\begin{aligned} M_\phi(u) &= \text{Hijack term} + I + J + K \\ &= \frac{r_0}{u+r_0} \left[1 - M_s(u+r_0) \right] + M_s(u+r_0) M_a(u) \end{aligned} \quad (4.10)$$

which is exactly the same as it would be without inspections except that $M_s = f_s$ and includes renewals. This last fact means in general that $M_\phi(0) \neq 1$. Other cumulants of the total life are shown in Table 1.

TABLE 1
Cumulants of Fatigue Life in Terms of Initiation and Crack Time Moments

$M_*(u)$ = MGF of cycles to initiation

$$M_*^{(k)}(-r_0) = \mu_k + O(r_0) \quad \text{where } \mu_k = k\text{th moment, } \mu_0 \equiv 1 - \sum_{k=1}^{\infty} R_k[1 - G_k(\infty)]$$

$$M_o(u) = 1 + \alpha_1 u + \frac{1}{2} \alpha_2 u^2 \dots, \quad \alpha_0 \equiv 1,$$

= MGF of cracking time, moments α_k .

Put

$$m_k \equiv M_*^{(k)}(r_0); \quad A_k \equiv \alpha_k - k\alpha_{k-1}/r_0,$$

where

$$r_0 \equiv \text{Risk of loads above ultimate and of hijacking.}$$

Then for total life we have the cumulants:

$$\kappa_1 = m_0 A_1 + r_0^{-1}; \quad (\text{Mean})$$

$$\kappa_2 = 2m_1 A_1 + m_0 A_2 - m_0^2 A_1^2 + r_0^{-2} \quad (\text{Variance})$$

$$\kappa_3 = 3m_2 A_1 + 3m_1 A_2 + m_0 A_3 - 6m_0 m_1 A_1^2 - 3m_0^2 A_1 A_2 + 2m_0^3 A_1^3 + 2/r_0^3$$

$$\begin{aligned} \kappa_4 = & 4m_3 A_1 + 6m_2 A_2 + 4m_1 A_3 + m_0 A_4 - 12m_0 m_2 A_1^2 - 24m_0 m_1 A_1 A_2 + 24m_0^2 m_1 A_1^3 \\ & - 12m_1^2 A_1^2 - m_0^2 \{4A_1 A_3 + 3A_2^2\} + 12m_0^3 A_1^2 A_2 - 6m_0^4 A_1^4 + 6/r_0^4 \end{aligned}$$

4.4 Transform of f .

Though informative, equation (4.10) is still not presented in terms of basic MGFs, relying as it has on M_* . To relate M_* to M_o and other known MGFs define f , piecewise from (3.4). Then its transform

$$\begin{aligned} M_*(u) &= \sum_{i=0}^{\infty} \int_{T_i}^{T_{i+1}} e^{-ut} f_{*i}(t) dt \\ &= M_o(u) + \sum_{i=0}^{\infty} \int_{T_i}^{T_{i+1}} \sum_{j=1}^i e^{r_0 T_j - ut} R_j g_j(t - T_j) dt \end{aligned}$$

in which $M_o(u) = E_o \exp -ut_0$. The R_j , the overall probabilities of rejection at times T_j , are effectively constants.

Changing the order of summation absorbs one sum to produce

$$\begin{aligned} M_*(u) &= M_o(u) + \sum_{j=1}^{\infty} R_j e^{r_0 T_j} \int_{T_j}^{\infty} e^{-ut} g_j(t - T_j) dt \\ &= \sum_{j=0}^{\infty} R_j e^{-(u-r_0)T_j} M_j(u), \quad R_0 = 1 \text{ here,} \end{aligned}$$

defining $M_j(u)$ = MGF of $g_j(t)$.

From (3.1) and (3.4)

$$R_j = \int_{a_0}^{\infty} P_{j_0-1} \dots P_{j-1} P_j e^{-r_0 a_0} [1 - F_a(t_a)] f_{j_0}(t_0) \frac{da}{R(a)} \\ = \int_0^{T_j} \prod_{j_0+1}^{j-1} P_k P_j(t_a) e^{-r_0 a_0} F_a(t_a) / f_{j_0}(t_0) dt_a$$

where $F_a(t_a) = 1 - F_a(t_a)$ and the finite upper limit is set by the fact that initiation $t_0 > 0$. This expands into

$$R_j = \int_0^{T_j} \bar{P}(T_{j_0+1} - t_0) \dots \bar{P}(T_{j-1} - t_0) P(t_a) e^{-r_0 a_0} [1 - F_a(t_a)] f_{j_0}(t_0) dt_a \quad (4.11)$$

where the factors $\bar{P}$, P depend implicitly on t_a also and

$$t_0 = T_j - t_a.$$

Now we know that

$$f_{j_0}(t) = \sum_{k=0}^j e^{-r_0 T_k} R_k g_k(t - T_k), \quad g_0 \equiv f_0(t), \quad (4.11A)$$

whence

$$R_j = \int_0^{T_j} \bar{P}(T_{j_0+1} - T_j + t_a) \dots \bar{P}(T_{j-1} - T_j + t_a) F(t_a) P_a(t_a) \\ \times \sum_{k=0}^{j_0} e^{-r_0 T_k} R_k g_k(T_j - T_k + t_a) dt_a \quad (4.12)$$

This is essentially an infinite set of recursive linear equations for R_j ($R_0 = 1$). If their matrix form is

$$\mathbf{R} = \mathbf{B}\mathbf{R} \quad \mathbf{B} = [b_{ij}]$$

$$\text{then } b_{ij} = \int_0^{T_j} \bar{P}(T_{j_0+1} - T_j + t_a) \dots \bar{P}(T_{j-1} - T_j + t_a) P(t_a) F_a(t_a) \\ \times e^{-r_0 T_i} g_i(T_j - T_i + t_a) dt_a \quad \text{if } i \leq j; t_a, T_j \rightarrow j_0, \\ = 0 \quad \text{if } i > j, \quad (4.12B)$$

all obtainable in principle by simple quadratures. Since R_0 is unity these equations are not homogeneous.

5. EMBEDDED MARKOV PROCESS

In another interpretation the b_{ij} are transition probabilities (rejection and repair) in a transient discrete Markov process embedded in the attrition and fatigue process. The element b_{ij} represents probabilities of structures repaired after time T_i being rejected again at the inspection T_j .

However, in the equation $\mathbf{R} = \mathbf{B}\mathbf{R}$, the matrix $\mathbf{B}$ consists of transition probabilities among an infinite number of states. Nevertheless, these states are still incomplete, not including attrition, hijacking or even new cracking. Furthermore, the infinite vector $\mathbf{R}$ is described by a single transition $\mathbf{B}$ covering all time. It is more convenient to consider transitions just after each inspection time T_i ; the states at such times may be regarded as agglomerations of the states represented by b_{ij} . Thus, structures cracked at T_i include rejections and repairs from previous inspections.

In this way, the process B may be identified with a series of transitions of a smaller process after each inspection. From the Venn diagram of Figure 1, the number of states in this smaller process is six—three for cracked and three for uncracked structures. One of these states has zero probability (although it may be considered an absorbing state) so that a 5×5 transition matrix is required for each inspection time. These will be denoted B for time T_i . The state ignored is the rejection of uncracked structures at inspection.

To find B_i , it is necessary to consider the possible combinations of various states. This amounts to gathering previous results and formalising assumptions, implicit or otherwise. We consider the epochs $T_i +$ just after any inspection and imagine the "mainstream" of structures as having survived the initial hijacking at rate r_0 , becoming cracked by time T_i , and then being subdivided into the mutually exclusive states of "immediate" (i.e. at $T_i +$) rejection, attrition in the i -th interval $(T_i, T_{i+1}]$ and the mainstream for T_{i+1} . The last will be augmented, as at each inspection, by repaired structures cracking again; one of the state transitions to be considered.

5.1 State Probabilities

We begin by considering the state probabilities at $T_i +$ which may also be normalisers for the conditional transition probabilities. Comparison with $T_{i+1} +$ and the use of known results then provide elements of B . Let $\phi(t)$ be the integrated attrition and abbreviate $\phi(T_i)$, $F_i(T_i)$ to ϕ and F_i . In addition C , A and R are sets of respectively cracked, failed or hijacked and rejected structures; before time $T_i +$ in the first instance. Set complements are subscripted c and unions and intersections are most conveniently denoted by $+$ and the product convention.

Among the eight factors of $(C + C_c)(A + A_c)(R + R_c)$ three are empty, namely:

CAR Inspection and rejection of a cracked structure *after* attrition;

C_cAR Inspection and rejection for hijacked structures; and

C_cA_cR Rejection of ordinary uncracked structures.

After obvious condensations this leaves the universe

$$E = CR + CA + CA_cR_c + C_cA + C_cA_c \quad (5.1)$$

in which C , A and their complements describe the end result of past history but R , R_c refer to the one inspection, beginning the interval $(T_i, T_{i+1}]$.

At time $T_i +$ we know that

$$Pr(CR) = R_i \text{ from (3.3) with } t = T_i \quad (5.2)$$

The basic hijack allowance states that the mainstream C_cA_c of uncracked structures is $\exp(-r_0T_i)(1 - F_i)$.

Without hijacking, $Pr(C_cA_c)$ would reduce to $1 - F_i$ so that the difference

$$Pr(C_cA) = (1 - e^{-r_0T_i})(1 - F_i), \quad (5.3)$$

the fraction for hijacked but uncracked structures.

Hence for those already cracked

$$Pr(CR + CA + CA_cR_c) = F_i, \quad (5.4)$$

the cracked mainstream.

By definition $\Phi_i = Pr(A) = Pr(C_cA + CA)$

from which subtraction provides

$$Pr(CA) = \Phi_i - (1 - e^{-r_0T_i})(1 - F_i) \quad (5.5)$$

Now

$$Pr(\text{cracked mainstream}) = Pr(CA + CA_cR_c + CR)$$

and by subtraction

$$Pr(CA_c R_c) = F_{ci} - R_i + (1 - e^{-r_o T_i})[1 - F_{ci}] - \Phi_i - 1 - e^{-r_o T_i}[1 - F_{ci}] - R_i - \Phi_i \quad (5.6)$$

This is *not* the mainstream at T_{i+1} since it will be affected by new cracking and attrition during the interval (T_i, T_{i+1}) .

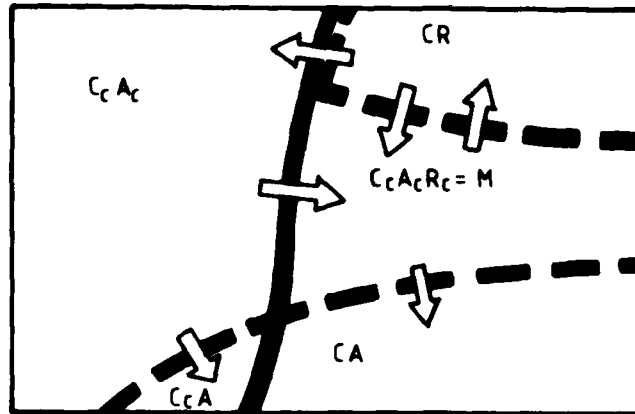


FIG 5. STATES AFTER EACH INSPECTION

5.2 Transition Probabilities

Equations (5.1) to (5.6) describe the five states shown in Figure 5 and their probabilities. Now consider transitions between them during (T_i, T_{i+1}) . The main difference from the previously used infinitesimal Markov process is that the finite inspection interval allows transition from repaired or uncracked structures to any state.

Obviously attrition is always an absorbing state. The process as a whole is therefore transient and the ultimate fate of all structures is either retirement, or attrition by failure or hijacking. We shall now examine transitions out of the five basic conditions using set theory and probability where necessary.

The more difficult elements will require interpretation of various terms in the expressions (3.5), (3.6) or (3.3) for attrition or rejection. Integrals of these may appear as convolution functions.

5.2.1 CR

In terms of time variates this has the total measure

$$R_i = \int_0^{T_i} \prod_{t_0=1}^i \bar{P}_t P(t_a) e^{-r_o t_0} [1 - F_a(t_a)] f_a(t_0) dt_a, \quad t_0 = T_i - t_a,$$

which leads to the component of initiation density $R_i e^{-r_o(t_0 - T_i)} g_i(t_0 - T_i)$.

The arguments of Section 3 apply equally to all components so that in the interval (T_i, T_{i+1}) this corresponds to the partial density of crack length

$$(R_i/R(a)) e^{-r_o(t_0 - T_i)} [1 - F_a(t_a)] g_i(t - T_i - t_a), \quad t_0 = t - t_a, \quad (5.7)$$

which forms part of (3.5).

Corresponding to (3.6) one finds the attrition rate

$$\begin{aligned} \phi(t) \text{ From } R_i &= R_i r_o e^{-r_o(t - T_i)} [1 - G_i(t - T_i)] \\ &+ R_i \int_0^{t - T_i} e^{-r_o(t_0 - T_i)} g_i(t - T_i - t_a) dF_a(t_a) \end{aligned}$$

from which integration shows

$$R_i Pr(CR_i \rightarrow C_e A | R_i) = R_i \int_{T_i}^{T_{i+1}} r e^{-r_o(t-T_i)} [1 - G_i(t - T_i)] dt \quad (5.8)$$

and

$$R_i Pr(CR_i \rightarrow CA | R_i) = R_i \int_{T_i}^{T_{i+1}} \int_0^{t-T_i} e^{-r_o(t_o-T_i)} g_i(t - T_i - t_a) dF_a(t_a) dt \quad (5.9)$$

The first of these may be integrated by repeating the general argument for (5.3). The transition element is therefore

$$Pr(CR_i \rightarrow C_e A | R_i) = (1 - e^{-r_o \Delta T_i}) (1 - G_i(\Delta T_i)), \quad (5.10)$$

where $\Delta T_i = T_{i+1} - T_i$.

Equation (5.9) has no simpler form but it may be interpreted as a probability. Since

$$1 - H_i(t) = e^{-r_o t} [1 - G_i(t)] \quad (5.10A)$$

is a survival probability under attrition, its complement is a true distribution function and never defective. Then, as a convolution,

$$\begin{aligned} Pr(CR_i \rightarrow A | R_i) &= Pr(t_o + t_a < \Delta T_i) \\ &= H_i * F_a(\Delta T_i), \text{ say,} \end{aligned}$$

where $t_o \sim H_i(t_o)$, H_i from above, and $t_a \sim F_a(t_a)$

for the reliability based crack growth time. This includes $R_i \rightarrow C_i A$ but the time does not allow inspection. By subtraction of (5.10)

$$Pr(CR_i \rightarrow CA | R_i) = H_i * F_a(\Delta T_i) - (1 - e^{-r_o \Delta T_i}) (1 - G_i(\Delta T_i)), \quad (5.11)$$

we have already noted that

$$Pr(CR_i \rightarrow C_e A_e | R_i) = e^{-r_o \Delta T_i} (1 - G_i(\Delta T_i)) \quad (5.12)$$

augmenting the mainstream of uncracked structures.

There remain two more transitions to CR_{i+1} or $CA_e R_e$, in the cracked mainstream. The first of these follows in the same way as R_i itself, but from the partial density (5.7). Thus

$$Pr(CR_i \rightarrow CR | R_i) = \int_0^{\Delta T_i} e^{-r_o(t_o-T_i)} P(t_a) [1 - F_a(t_a)] g_i(\Delta T_i - t_a) dt_a$$

If we recall our assumption $P(0) = 0$, this takes the form

$$\int_0^{\Delta T_i} P(t_a) [1 - F_a(t_a)] dH_i(\Delta T_i - t_a) = (P[1 - F_a]) * H_i(\Delta T_i) \quad (5.13)$$

Finally, for transitions to the cracked mainstream,

$$\begin{aligned} Pr(CR_i \rightarrow CA_e R_e | R_i) &= 1 - \text{Eq. (5.10)} - \dots - (5.13) \\ &= H_i(\Delta T_i) - H_i * ((1 - P)F_a + P)(\Delta T_i) \\ &= ((1 - P)[1 - F_a]) * H_i(\Delta T_i). \end{aligned} \quad (5.14)$$

5.2.2 $CA_e R_e = M$

This is defined by (5.6) for entry to the inspection interval $(T_i, T_{i+1}]$. Just after T_{i+1} , it has changed by rejection and an influx of structures repaired after the previous inspection T_i .

Obviously $M \rightarrow C_e A$ and $M \rightarrow C_e A_e$ are impossible; also the influx of cracked structures, newly repaired and otherwise, complicates $Pr(M_i \rightarrow M_{i+1} | M_i)$.

Equation (5.6) provides the unconditional $Pr(M_{i+1})$ as well as $Pr(M_i)$ but this influx makes it more convenient to start again from the crack length density (3.4). At time T_{i+1} , the only cracks from M_i must be those for which $t_a > \Delta T_i$. If (3.4) is restricted to these at $t = T_{i+1}$, then all attrition during (T_i, T_{i+1}) is accounted for, and also the rejections R_{i+1} . Then, changing the cracked length variate to t_a ,

$$Pr(M_i)Pr(M_{i+1}|M_i) = \int_{\Delta T_i}^{T_{i+1}} \left(\prod_{j=i+1}^{i+1} P_j \right) [1 - F_a(t_a)] e^{-r_a t_a} \sum_{j=0}^{i_0} R_{jg}(t_a - T_j) dt_a$$

with $t_0 = T_{i+1} - t_a$. This differs from $Pr(M_{i+1})$ only by

$$N_i = \int_0^{\Delta T_i} (1 - P(t_a)) [1 - F_a(t_a)] e^{-r_a t_a} f^*(T_{i+1} - t_a) dt_a \quad (5.15)$$

after some obvious simplifications. As in (5.11) it would now be possible to define a distribution $1 - (\exp - r_a t)(1 - F_a(t))$, and thus define this term as another convolution. However, this includes elements of $C_e A$, the attrition of uncracked structures, and to abbreviate N_i we use the defective density $h_a(t_a) = (\exp - r_a t_a) f_a(t_a)$ so that

$$N_i = ((1 - P)[1 - F_a]) * h_a(\Delta T_i) \quad (5.15A)$$

If we are guided by context we may conveniently let $Pr(M_i) = M_i$, using the same symbols for the event and its probability. Then, using (5.6) and (5.15),

$$Pr(M_i \rightarrow M_{i+1}|M_i) = (M_{i+1} - N_i)/M_i = \frac{1 - e^{-r_a T_{i+1}} [1 - F_{a+1}] - R_{i+1} - \Phi_{i+1} - N_i}{1 - e^{-r_a T_i} [1 - F_a] - R_i - \Phi_i} \quad (5.16)$$

The difference $M_i + N_i - M_{i+1}$ must now be divided among $M_i \rightarrow CR_{i+1}$ and $M_i \rightarrow CA_{i+1}$ to find the corresponding transition probabilities. Consider the attrition of cracked mainstream structures during (T_i, T_{i+1}) given by (3.6), and based on (3.5). In the latter, $CA_e R_e$ or M_i correspond to use of the partial density $f_{i-1}(t_a)$ during the i -th inspection interval. This excludes attrition of structures repaired at T_i . If they are cracked however, their absolute probability from (5.11) is known to be

$$R_i H_i * F_a(\Delta T_i) - (5.10) = R_i Pr(CR_i \rightarrow CA|R_i).$$

In (3.6), the second term is the attrition density of previously cracked structures, and those repaired at T_i . Writing the second term as

$$\phi_i - r_a (\exp - r_a t) [1 - F_a(t)]$$

and allowing for (5.16)

$$\begin{aligned} M_i Pr(CA_e R_e \rightarrow CA|T_i < t < T_{i+1}) &= \int_{T_i}^{T_{i+1}} (\phi_i(t) - r_a e^{-r_a t} [1 - F_a(t)]) dt + \\ &+ R_i (1 - e^{-r_a \Delta T_i}) [1 - G_i(\Delta T_i)] - R_i H_i * F_a(\Delta T_i) \end{aligned}$$

The second term of the integral follows as before from the increment of (5.4). After some reduction this becomes

$$\begin{aligned} M_i Pr(M \rightarrow CA|i\text{-th}) &= \Delta \Phi_i + \Delta F_{a_i} (1 - e^{-r_a T_{i+1}}) - R_i H_i * F_a(\Delta T_i) \\ &- (1 - e^{-r_a \Delta T_i}) \{e^{-r_a T_i} [1 - F_a] - R_i [1 - G_i(\Delta T_i)]\} \end{aligned} \quad (5.17)$$

From the remarks below (5.15)

$$M_i Pr(M_i \rightarrow CR_{i+1}|i\text{-th}) = M_i + N_i - M_{i+1} - \text{Eq. (5.17)}$$

which eventually reduces to $M_i Pr(M_i \rightarrow CR_{i+1}|i\text{-th}) =$

$$R_{i+1} - R_i - \Delta F_{a_i} + N_i + R_i \{H_i * F_a(\Delta T_i) - (1 - e^{-r_a T_i}) [1 - G_i(\Delta T_i)]\}$$

with R_i , N_i and H_i defined by (3.1) or (4.11) - (4.12), (5.15) and (5.10A).

5.2.3 $C_c A_c$

This is the only remaining non-trivial state. Any transition is possible from it and its absolute probability at T_i is

$$Pr(C_c A_c) = e^{-r_o T_i} [1 - F_{oi}]$$

During $(T_i, T_{i+1}]$ $Pr(C_c A_c)$ increases by the increment of (5.3) and this comes from $C_c A_c + CR_i$. Thus,

$$\begin{aligned} e^{-r_o T_i} [1 - F_{oi}] Pr(C_c A_c \rightarrow C_c A | i\text{-th}) &= \Delta Pr(C_c A) - R_i Pr(CR_i \rightarrow C_c A | R_i) \\ &= (1 - e^{-r_o \Delta T_i}) \{e^{-r_o T_i} [1 - F_{oi}] - R_i [1 - G_i(\Delta T_i)]\} = \Delta F_{oi} (1 - e^{-r_o \Delta T_i}) \end{aligned} \quad (5.19)$$

using (5.3) and (5.10).

As a whole, C_i comes from $C_c A_c + CR_i$ and of course its increment has measure ΔF_{oi} , the decrease in C_c . For the increment from $C_c A_c$, subtract the transitions from CR_i giving

$$\begin{aligned} Pr(C_c A_c \rightarrow C_{i+1}) &= \Delta F_{oi} - R_i Pr(R_i \rightarrow C(A + R + M) | R_i) \\ &= \Delta F_{oi} - R_i \{H_i(\Delta T_i) - (1 - e^{-r_o \Delta T_i}) [1 - G_i(\Delta T_i)]\} \end{aligned} \quad (5.20)$$

which follows from (5.11), (5.13) and (5.14) respectively.

Consider

$$- \Delta Pr(C_c A_c) = Pr(C_c A_c) \{Pr(C_c A_c \rightarrow C_{i+1} | i\text{-th}) + Pr(C_c A_c \rightarrow C_c(A + A_c) | i\text{-th})\}$$

whence by transposing (5.19) and (5.20)

$$e^{-r_o T_i} [1 - F_{oi}] Pr(C_c A_c \rightarrow C_c A | i\text{-th}) = R_i H_i(\Delta T_i) - \Delta F_{oi} e^{-r_o \Delta T_i} (1 - e^{-r_o T_i}) \quad (5.21)$$

with H_i from (5.10A).

As was done with transitions $CR_i \rightarrow C_{i+1}$ we must now partition (5.20). After subtracting the correction term in (5.19) for $R_i \rightarrow C_{i+1}$, one may imagine the three types of transition during $(T_i, T_{i+1}]$ being driven by f_{oi-1} . Attrition is simplest with a contribution from part of (3.6) minus (5.11). Thus, the appropriate rate is

$$\begin{aligned} \phi(t; T_i \leq t \leq T_{i+1}) &= \int_0^{t-T_i} e^{-r_o(t-t_a)} f_{oi-1}(t-t_a) dF_a(t_a) \\ &= \int_0^{t-T_i} e^{-r_o t_a} (f_a(t_a) - R_i g_i(t_a - T_i)) dF_a(t_a) \end{aligned}$$

Over the inspection interval this integrates to

$$\begin{aligned} e^{-r_o T_i} [1 - F_{oi}] Pr(C_c A_c \rightarrow CA | i\text{-th}) \\ = \Delta \Phi_i - e^{-r_o T_i} (1 - e^{-r_o \Delta T_i}) [1 - F_{oi}] - R_i \{H_i^* F_a(\Delta T_i) - (1 - e^{-r_o \Delta T_i}) [1 - G_i(\Delta T_i)]\} \end{aligned} \quad (5.22)$$

using (5.11) for the subtracted term.

Inspection of (3.5) for $t = T_{i+1}$ indicates the crack length density

$$R(a)f(a|T_{i+1}) = e^{-r_o a} [1 - F_a(a)] (1 - P(a)) f_{oi-1}(t_a)$$

for structures from $C_c A_c$. For rejected structures, we replace $1 - P$ by its complement and integrate to obtain

$$e^{-r_o T_i} [1 - F_{oi}] Pr(C_c A_c \rightarrow CR_{i+1} | i\text{-th}) = (P(1 - F_a))^* H_i(\Delta T_i) - R_i (P(1 - F_a))^* H_i(\Delta T_i) \quad (5.23)$$

using (5.13) with the assumption $P(0) = 0$ and (5.15A), (5.10A) for definitions of H_i and H_i .

We now know probabilities for transition to $CA + CR_{i+1}$ and to C_{i+1} as a whole from equations (5.22), (5.23) and (5.20). Since the transition events are disjoint

$$\begin{aligned} e^{-r_o T_i} [1 - F_{oi}] Pr(C_c A_c \rightarrow M_{i+1} | i\text{-th}) &= \Delta F_{oi} - \Delta \Phi_i + e^{-r_o T_i} (1 - e^{-r_o \Delta T_i}) [1 - F_{oi}] \\ &\quad - R_i N_i - (P(1 - F_a))^* H_i(\Delta T_i) \end{aligned} \quad (5.24)$$

using (5.15A).

5.3 Transition Matrices

It is now possible to arrange the elements above into transition matrices for each epoch $T_i +$. In many cases, normalising to conditional probabilities introduce awkward fractions so that it is most convenient to place the normalisers in a diagonal prefactor. Doing this, and gathering elements from above, leads to the matrices shown in Tables 2 and 3.

Let $P^{-1} = [P_i]^{-1}$ be the prefactor where only P_4 and P_5 differ from unity.

From Section 5.1 we know the absolute state probabilities of Table 2 which provide P_4 and P_5 .

TABLE 2
Normalising Factors

j	State	Name	Absolute Probability	P_j
1	$C_c A$	Hijacked	$(1 - e^{-r_o T_i})[1 - F_{ot}]$	1
2	CA	Failed or hijacked	$\Phi_i - (1 - e^{-r_o T_i})[1 - F_{ot}]$	1
3	CR	Rejected	R_i	1
4	$CA_c R_c = M$	Cracked mainstream	$1 - R_i - \Phi_i - e^{-r_o T_i}[1 - F_{ot}]$	*
5	$C_c A_c$	Uncracked mainstream	$e^{-r_o T_i}[1 - F_{ot}]$	*

* As in previous column.

In previous derivations, all results depend on functions which are fully defined at time $T_i +$ on the range of crack lengths $0 \cup [a_o, \infty)$ which imply the other functions needed from $(T, T_{i+1}]$. This validates the Markovian nature of the discrete transitions.

6. CONCLUSIONS

Using the same procedures, the previous analysis³ for fatigue life distributions in one-crack models with hijacking have been extended to cases with arbitrary renewals at general inspection intervals.

The conclusions are similar to those derived previously if one postulates a defective distribution f_c of crack initiation lives as affected by renewals, and includes factors depending on the operating characteristic of the inspections. The moment generating function has the same form as before³ and with $f_o(t_o)$ replaced by $f_c(t_o)$. The proof of this is facilitated by treating inspection and crack life density together.

The two-stage fatigue process is Markovian in continuous time. At any time a structure must be in one of five states corresponding to combinations of attrition, cracking and rejection at inspection or their complements. Transition between these states is a variable Markov chain embedded in the continuous process with epochs conveniently placed just after inspections.

6.1 Implementation

Life distributions, risk rates and rejection probabilities may be found from two FORTRAN IV programs developed from the preceding theory. The second of these has options for random crack rates and/or run-time setting of inspections. This generalisation and the program will be described in two further reports.

TABLE 3
Transition Matrix

From	To	$C_e A$	CA	CR	$CA_e R_e = M$	$C_e A_e$
$C_e A$	-1	1	.	0	0	0
CA		.	1	0	0	0
CR		$(1 - \rho)G_t$	$H_t * F_a(\Delta T_t) - (1 - \rho)G_t$	$(PF_a) * H_t(\Delta T_t)$	$(PF_a) * H_t(\Delta T_t) = N_t$	ρG_t
$CA_e R_e = M$		0	$\Delta F_a(1 - \tau_{t+1}) - R_t H_t * F_a(\Delta T_t) + \Delta \Phi_t + R_t(1 - \rho)G_t - \tau_t(1 - \rho)F_t$	$R_{t+1} - R_t - \Delta F_a + R_t H_t * F_a(\Delta T_t) - R_t(1 - \rho)G_t + N_t$	$1 - \tau_{t+1}F_{t+1} - R_{t+1} - \Phi_t - N_t$	0
$C_e A_e$		$\tau_t(1 - \rho)F_t - \Delta F_a(1 - \tau_{t+1}) - R_t(1 - \rho)G_t$	$\Delta \Phi_t - \tau_t(1 - \rho)F_t - R_t H_t * F_a(\Delta T_t) + R_t(1 - \rho)G_t$	$(PF_a) * H_t(\Delta T_t) - R_t(PF_a) * H_t(\Delta T_t)$	$\Delta F_a - \Delta \Phi_t + R_t N_t + \tau_t(1 - \rho)F_t - (PF_a) * H_t(\Delta T_t)$	$R_t H_t(\Delta T_t) - \rho \Delta F_a(1 - \tau_t)$

The notation here is

$$F_t = 1 - F_a \quad G_t = 1 - G_t(\Delta T_t) \quad F_a = 1 - F_a(\cdot) \quad P = 1 - P$$

$$\rho = \exp(-r_0 \Delta T_t) \quad \tau_t = \exp(-r_0 T_t)$$

REFERENCES

1. Ford, D. G. *The Analysis of Structural Fatigue*. Ph.D. Thesis, University of London, April 1967.
2. Ford, D. G. The Development of the Theory of Structural Fatigue. Aeronautical Research Laboratories, Structures Tech. Memo. 248, October 1976. Also in *Aircraft Structural Fatigue*. Symposium, Melbourne 19-20 October, 1976. ARL, Structures Report 363, April 1977.
3. Ford, D. G. Reliability and Structural Fatigue in One-Crack Models. Aeronautical Research Laboratories, Structures Report 369, July 1978.
4. Soong, T. T. *Random Differential Equations in Science and Engineering*. Academic Press, New York, 1973.
5. Bailey, N. T. J. *The Elements of Stochastic Processes with Applications to the Natural Sciences*. Wiley, New York, 1964.

DISTRIBUTION

AUSTRALIA

	Copy No.
Department of Defence	
Central Office	
Chief Defence Scientist	1
Deputy Chief Defence Scientist	2
Superintendent, Science and Technology Programs	3
Australian Defence Scientific and Tech. Rep. (UK)	4
Counsellor, Defence Science	5
Defence Library	6
Joint Intelligence Organization	7
Assistant Secretary, DISB	8-23
Aeronautical Research Laboratories	
Chief Superintendent	24
Superintendent, Structures Division	25
Structures Divisional File	26
Author: D. G. Ford	27-28
Library	29
A. O. Payne	30
G. S. Jost	31
D. Graham	32
J. Q. Clayton	33
Materials Research Laboratories	
Library	34
Defence Research Centre, Salisbury	
Library	35
Central Studies Establishment Information Centre	
Library	36
Engineering Development Establishment	
Library	37
RAN Research Laboratory	
Library	38
Defence Regional Office	
Library	39
Navy Office	
Naval Scientific Adviser	40
Army Office	
Army Scientific Adviser	41
Royal Military College, Library	42
Air Force Office	
Air Force Scientific Adviser	43
D. Air Eng.	44
Engineering (CAFTS) Library	45
HQ Support Command (SENGSO)	46

Department of Productivity		
Government Aircraft Factories, Library		47
Mr M. Cameron		48
Department of Transport		
Director General/Library		49
Airworthiness Group: Mr K. O'Brien		50
Mr C. Torkington		51
Accident Group (Mr I. S. Milligan)		52
Statutory, State Authorities and Industry		
CSIRO Mechanical Engineering Division, Chief		53
CSIRO Tribophysics Division, Director		54
Qantas, Library		55
Trans Australia Airlines, Library		56
Ansett Airlines, Library		57
BHP Central Research Laboratories, NSW		58
BHP Melbourne Research Laboratories		59
Commonwealth Aircraft Corporation, Manager		60
Commonwealth Aircraft Corporation, Manager of Engineering		61
Commonwealth Aircraft Corporation, Mr R. C. Beckett		62
Commonwealth Aircraft Corporation, Dr R. H. Keays		63
Hawker de Havilland Pty. Ltd (Librarian), Bankstown		64
Universities and Colleges		
Adelaide	Barr Smith Library	65
Australian National	Library	66
Deakin	Library	67
	B. Bathgate	68
Flinders	Library	69
James Cook	Library	70
La Trobe	Library	71
Melbourne	Engineering Library	72
	Engineering, Dr J. Williams	73
Monash	Library	74
	Professor I. J. Polmear, Materials Engineering	75
Newcastle	Library	76
New England	Library	77
New South Wales	Physical Sciences Library	78
	Professor Trail-Nash, Structural Engineering	79
Sydney	Engineering Library	80
	Professor G. A. Bird, Aeronautical Engineering	81
Queensland	Engineering Library	82
Tasmania	Engineering Library	83
	Professor R. A. Oliver, Civil and Mech. Eng.	84
Western Australia	Library	85
RMIT	Library	86
	Mr H. Millicer, Aeronautical Engineering	87
	Mr Pugh, Mechanical Engineering	88
CANADA		
Aluminium Labs. Ltd., Library		89
CAARC Co-ordinator (Structures)		90
NRC, National Aeronautical Establishment, Library		91
NRC, National Aeronautical Establishment, Div. of Mech. Eng. (Dr. D. McPhail)		92

Universities		
McGill	Library	93
Toronto	Institute of Aerophysics	94
Waterloo	Dept. of Mech. Eng.: Professor L. F. Topper	95
	Dr D. J. Burns	96
FRANCE		
ONERA, Library		97
Service de Documentation, Technique de L'Aeronautique		98
Mr W. Barrois, Ingenieur en chef de l'air		99
GERMANY		
LBF, Dr D. Schütz		100
DVLR, Dr W. Schütz		101
INTERNATIONAL COMMITTEE ON AERONAUTICAL FATIGUE		
Per Australian Representative, G. S. Jost		102-125
INDIA		
CAARC Co-ordinator Structures		126
Civil Aviation Department, Director		127
Defence Ministry, Aero. Development Est., Library		128
Indian Institute of Science, Library		129
Indian Institute of Technology, Library		130
National Aeronautical Laboratory, Director		131
ITALY		
Associazione Italiana di Aeronautica and Astronautica, Prof. Evla		132
Fiat Co., Dr G. Gabrielli		133
Universita Degli Studi di Pisa, Dr A. Salvetti		134
ISRAEL		
Technion-Israel Institute of Technology, Professor J. Singer		135
Technion-Israel Institute of Technology, Professor A. Buch		136
JAPAN		
National Aerospace Laboratory, Library		137
Universities		
Tokyo Institute of Space and Aerospace, Library		138
Tohoku (Sendai), Library		139
Research Institute of Strength and Fracture of Materials:		
Professor Yamanaka		140
Professor T. Yokobori		141
M. Ichikawa		142
Kobe, Professor T. Nakagawa		143
Kyushu Institute of Technology, Library		144
NETHERLANDS		
Central Org. for Applied Science		145
National Aerospace Laboratory		146
Delft University of Technology		147
NEW ZEALAND		
RNZAF Air Department, Aero. Documents Section		148
Transport Ministry, Civil Aviation Division Library		149
Universities		
Canterbury	Library	150

SWEDEN

Aeronautical Research Institute, Dr Sigge Eggwertz	151
Chalmers Institute of Tech. Library	152
Kungl. Tekniska Hogscholens	153
SAAB, Library	154

SWITZERLAND

F. + W. Ltd.	155
--------------	-----

UNITED KINGDOM

CAARC, Secretary	156
Aeronautical Research Council	157
Royal Aircraft Establishment:	
Farnborough, Library	158
Bedford, Library	159
CATC Secretariat	160
British Library, Science Reference Library	161
British Library, Lending Division	162
Fulmer Research Institute Ltd., Research Director	163
Science Museum Library	164
Welding Institute Library	165
National Engineering Laboratories, Superintendent	166
Naval Construction Research Establishment, Superintendent	167
CAARC Co-ordinator Structures	168
Aircraft Research Association, Library	169
British Ship Research Association	170
Central Electricity Generating Board	171
Motor Industries Research Association, Director	172
Rolls-Royce Aeronautics Division, Library	173
Rolls-Royce Bristol Siddeley Division, Library	174
Hawker Siddeley Aviation Ltd., Brough	175
Hawker Siddeley Aviation Ltd., Greengate	176
Hawker Siddeley Aviation Ltd., Kingston-upon-Thames	177
British Aircraft Corporation (Holdings) Ltd.:	
Commercial Aircraft Division	178
Military Aircraft Division	179
British Hovercraft Corporation Ltd. (E. Cowes)	180
Short Brothers & Harland	181
Westland Helicopters Ltd.	182

Universities and Colleges

Bristol	Library, Engineering Department	183
Cambridge	Library, Engineering Department	184
London, Queen Mary		
College	Library, Department of Aeronautics	185
Manchester	Library, Engineering	186
Nottingham	Library	187
Southampton	Library	188
Strathclyde	Library	189
Cranfield Institute of		
Technology	Library	190
Imperial College	The Head	191
	Professor of Mechanical Engineering	192
	Library, Department of Aeronautics	193

UNITED STATES OF AMERICA

NASA Scientific and Technical Information Facility	194
--	-----

Sandia Group Research Organisation	195
American Institute of Aeronautics and Astronautics	196
Applied Mechanics Reviews	197
The John Crerar Library	198
US Atomic Energy Commission, Division of Tech. Information	199
Cessna Aircraft Co., Executive Engineer	100
Battelle Memorial Institute, Library	201
Battelle Memorial Institute, Dr D. Broek	202
Metals Abstracts	203
Air Force Flight Dynamics Labs., Wright-Patterson AFB	204
NASA Langley (Virginia) John Crews Jnr	205
NASA Langley (Virginia) J. R. Davidson	206
National Bureau of Standards, Library	207
Library of Congress, Gift and Exchange Department	208
US Dept of Transportation, Transportation Systems Centre, Dr Elaine Savage	209
US Naval Research Laboratory, Washington, DC	210
Calspan Corporation	211
Universities and Colleges	
Brown Professor R. E. Meyer	212
Brooklyn Library, Polytechnic Aero. Lab.	213
California Library, Guggenheim Aero. Lab.	214
	Dr M. Holt, Department of Aeronautics
	215
Florida Mark H. Clarkson, Dept of Aeronautical Engineering	216
Harvard Library, Engineering	217
Illinois Professor N. M. Newmark	218
Iowa State Dr G. K. Sekory, Mechanical Engineering	219
Johns Hopkins Department of Mechanical Engineering	220
Massachusetts Library, School of Engineering	221
Institute of Technology Library	222
New Mexico Dept of Nuclear Engineering, Professor J. A. Horak	223
Pennsylvania Library, School of Engineering	224
Stanford Library, Department of Aeronautics	225
Wisconsin Memorial Library, Serials Department	226

Spares

227-236

