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SUBHARMONIC SOLUTIONS OF A FORCED WAVE EQUATION.(U)
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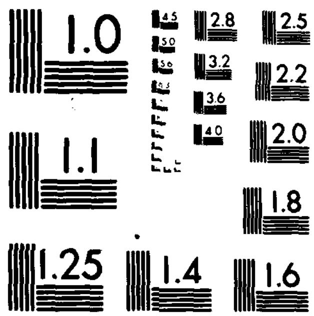

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Subharmonic Solutions of a Forced Wave Equation

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Subharmonic Solutions of a Forced Wave Equation

Introduction

In a recent paper ~~[1]~~, ~~we established~~ the existence of subharmonic solutions of forced Hamiltonian systems of ordinary differential equations WAS ESTABLISHED. The goal of this note is to show that subharmonics also occur for a class of semilinear wave equations.

To be more precise, let $z(t) = (z_1(t),\ldots,z_{2n}(t))$, $H : \mathbb{R}^{2n} \to \mathbb{R}$, and consider the Hamiltonian system of ordinary differential equations:

$$\frac{dz}{dt} = JH_z(t,z), \quad J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \tag{0.1}$$

where I denotes the identity matrix in $\mathbb{R}^n$. Suppose $H(t,0) = 0$, $H(t,z) \geq 0$, and H is T periodic in t. It was shown in [1] that if H satisfies appropriate additional conditions near $z = 0$ and $z = \infty$, then (0.1) possesses an infinite number of distinct subharmonic solutions, i.e. for each $k \in \mathbb{N}$, (0.1) has a solution $z_k(t)$ of period kT and infinitely many of the functions z_k are distinct. For single second order equations of the form

$$v'' + g(t,v) = 0 \tag{0.2}$$

with g T-periodic in t, more delicate such results were obtained earlier under related hypotheses by Jacobowitz [2].

Further work on this question was carried out by Hartman [3] who weakened the hypotheses of [2] and improved the conclusions.

We will show how analogues of some of the results of [1] can be obtained for a family of forced semilinear wave equations. Thus consider

$$(0.3) \qquad \begin{cases} u_{tt} - u_{xx} + f(x,t,u) = 0 & 0 < x < \ell \\ u(0,t) = 0 = u(\ell,t) \end{cases}$$

where f is T periodic in t. It was shown in [4] that (0.3) possesses a nontrivial classical T periodic solution provided that $T \in \ell\mathbb{Q}$, i.e. T is a rational multiple of ℓ, and f satisfies appropriate conditions. Recently a slightly stronger result has been obtained by Brezis, Coron, and Nirenberg [5]. In the following section we will prove that the hypotheses required in [4] for the above existence theorem imply that (0.3) also has subharmonic solutions: for all $k \in \mathbb{N}$, (0.3) possesses a kT periodic solution u_k and infinitely many of these functions are distinct. The proof relies on an amalgam of ideas from [1] and [4].

§1. The existence theorem

Suppose $f : [0,\ell] \times \mathbb{R}^2 \to \mathbb{R}$ and satisfies

(f_1) $f(x,t,0) = 0$, $f_r(x,t,r) > 0$ for $0 \neq r$ near 0, and $f(x,t,r)$ is strictly monotonically increasing in r for all $r \in \mathbb{R}$.

(f_2) $f(x,t,r) = o(|r|)$ at $4 = 0$

(f_3) There are constants $\mu > 2$ and $\bar{r} > 0$ such that

$$0 < \mu F(x,t,r) \equiv \int_0^r f(x,t,s)ds \le rf(x,t,r)$$

for $|r| \ge \bar{r}$

(f_4) There is a constant $T > 0$ such that $f(x,t+T,r) = f(x,t,r)$ for all x,t,r.

Note that (f_3) implies that

$$F(x,t,r) \ge a_1|r|^\mu - a_2 \tag{1.1}$$

for some constants $a_1 > 0$, $a_2 \ge 0$ and for all $r \in \mathbb{R}$, i.e. F grows at a more rapid rate than quadratic at $r = \infty$.

We will prove the following theorem:

Theorem 1.2: Let $f \in C^2([0,\ell] \times \mathbb{R}^2, \mathbb{R})$ and satisfy (f_1) - (f_4). If $T \in \ell\mathbb{Q}$, then for all $k \in \mathbb{N}$, the problem

$$\begin{cases} u_{tt} - u_{xx} + f(x,t,u) = 0, & 0 < x < \ell \\ u(0,t) = 0 = u(\ell,t) \end{cases} \tag{1.3}$$

possesses a nonconstant kT periodic solution $u_k \in C^2$. Moreover infinitely many of the functions u_k are distinct.

Before giving the proof of Theorem 1.2, several remarks are in order. Since $T \in \ell\mathbb{Q}$ implies that $kT \in \ell\mathbb{Q}$ for all $k \in \mathbb{N}$, the first assertion of the theorem is a special case of Theorem 4.1 and Corollary 4.14 of [4]. However, since we do not know kT is an minimal period of u_k, the functions u_k may all represent the same T periodic function or possibly a finite number of distinct periodic functions. Thus what is new and of interest here is that in fact infinitely many of the functions u_k must be distinct.

To establish this result we will show that on the one hand, if only finitely many of the functions u_k were distinct, a corresponding variational formulation of (1.3) would have an unbounded subsequence of critical values, c_{k_j}, with corresponding critical points representing reparametrizations of the same function. The growth of the c_{k_j}'s will be like k_j^2. On the other hand it turns out that c_k grows at most linearly in k, a contradiction.

To make this statement, which contains variants of ideas in [1], more precise, a closer inspection must be made of the existence mechanism of [4]. For convenience we take $\ell = \pi$ and $T = 2\pi$. Fixing $k \in \mathbb{N}$, we seek a solution of (1.3) which is $2\pi k$ periodic in t. It is convenient to rescale time $t = k\tau$ so that the period becomes 2π and (1.3) transforms to

$$(1.4)\qquad \begin{cases} u_{\tau\tau} - k^2(u_{xx} - f(x,k\tau,u)) = 0 & 0 < x < \pi \\ u(0,\tau) = 0 = u(\pi,\tau);\ u(x,\tau + 2\pi) = u(x,\tau) \end{cases}$$

The solution of (1.4) is obtained via an approximation argument. Three approximations are made. First observe that the wave operator part of (1.4), $u_{\tau\tau} - k^2u_{xx}$ has an infinite dimensional null space, N, in the class of functions satisfying the periodicity and boundary conditions, namely

$$N = \text{span } \{\sin jx \sin kj\tau, \sin jx \cos kj\tau | j \in \mathbb{N}\}$$

To provide some compactness for the problem in N, we perturb the wave operator by adding a term $-\beta v_{\tau\tau}$ to it where $\beta > 0$ and v denotes the L^2 orthogonal projection of u into N. Secondly the unrestricted rate of growth of $f(x,t,r)$ at $|r| = \infty$ creates technical problems which we bypass by suitably truncating f, i.e., we replace f by $f_K(x,t,r)$ where f_K coincides with f for $|r| \leq K$, satisfies $(f_1) - (f_4)$ with μ replaced by a new constant $\bar{\mu} = \min(4,\mu)$ in (f_3). Moreover f_K grows like r^3 at ∞. (See Eq (5.22) of [4]). Thus we replace (1.4) by

$$(1.5) \quad \begin{cases} u_{\tau\tau} - \beta v_{\tau\tau} - k^2(u_{xx} - f_K(x,k\tau,u)) = 0, & 0 < x < \pi \\ u(0,\tau) = 0 = u(\pi,\tau);\ u(x,\tau+2\pi) = u(x,\tau) \end{cases}$$

Formally (1.5) can be cast as a variational problem, namely that of finding critical points of

$$(1.6) \quad I(u;k,\beta,K) = \int_0^{2\pi} \int_0^{\pi} [\tfrac{1}{2}u_\tau^2 - \tfrac{\beta}{2}v_\tau^2 - k^2(\tfrac{1}{2}u_x^2 + F_K(x,k\tau,u))]dx\, d\tau$$

where F_K is the primitive of f_K. Our final approximation is to pose this variational problem in a finite dimensional space

$$E_m = \text{span}\{\sin jx \sin n\tau, \sin jx \cos n\tau | 0 \leq j, n \leq m\} .$$

A critical point of $I|_{E_m}$ will be a solution of the L^2 orthogonal projection of (1.5) onto E_m.

A series of lemmas in [4] use (f_1) - (f_4) and the form of I to establish the existence of a nontrivial critical point u_{mk} of $I|_{E_m}$ as well as an estimate on the corresponding critical value c_{mk} of the form

$$(1.7) \qquad 0 < c_{mk} = I(u_{mk};k,\beta,K) \le M_k$$

where M_k is a constant independent of $\beta, K,$ and m. Further arguments in [4] allow successively letting $m \to \infty$ and $\beta \to 0$ to get a solution u_k of

$$(1.8) \quad \begin{cases} u_{\tau\tau} - k^2(u_{xx} - f_K(x,k\tau,u)) = 0 & 0 < x < \pi \\ u(0,\tau) = 0 = u(\pi,\tau);\ u(x,\tau + 2\pi) = u(x,\tau) \end{cases}$$

with $c_k \equiv I(u_k,k,0,K) \le M_k$. Moreover for $K = K(k)$ sufficiently large, $||u_k||_{L^\infty} \le K$ so $f_K(x,k\tau,u_k) = f(x,k\tau,u_k)$ and u_k satisfies (1.4). Lastly a separate argument shows $c_k > 0$ so $u_k \not\equiv 0$ via (f_1) and the form of I.

Returning to the question of how many of the functions u_k are distinct, we will first study the dependence of M_k on k. To do so requires a closer look at how the bound M_k is determined. Lemma 1.13 of [4] provides a minimax characterization of $I(u_{mk};k,\beta,K)$ which in turn yields the bound M_k. Let

$$W_{mk} = \text{span}\{\sin jx \sin n\tau,\ \sin jx \cos n\tau \,|\, 0 \le j,n \le m \text{ and } n^2 \le j^2k^2\},$$

$$\varphi_k = \alpha_k \sin x \sin(k+1)\tau$$

and α_k is chosen so that $||\varphi_k||_{L^2} = 1$.
Set $V_{mk} = W_{mk} \oplus \text{span}\ \{\varphi_k\}$. It was shown in [4] that

$$(1.9) \qquad 0 < c_{mk} \le \max_{u \in V_{mk}} I(u\ ;k,\beta,K)$$

(Note that $I \to -\infty$ as $||u||_{L^2} \to \infty$ via (f_3) so we have a max rather than a sup in (1.9)). Let $z = z_{mk}$ denote the point in V_{mk} at which the max is attained. We can write

$$(1.10) \qquad z = ||z||_{L^2}(\gamma\xi + \delta\varphi_k)$$

where $\xi \in W_{mk}$ with $||\xi||_{L^2} = 1$ and $\gamma^2 + \delta^2 = 1$. Substituting (1.10) into (1.9) and using the form of I yields

$$(1.11) \qquad k^2 \int_0^{2\pi}\int_0^{\pi} F_K(x,k\tau,z)\,dx\,d\tau \le \frac{1}{2}\int_0^{2\pi}\int_0^{\pi}(z_\tau^2 - k^2 z_x^2)\,dx\,d\tau$$

$$\le \frac{\delta^2}{2}||z||_{L^2}^2 \int_0^{2\pi}\int_0^{\pi}(\varphi_{k_\tau}^2 - k^2\varphi_{k_x}^2)\,dx\,d\tau$$

$$\le \overline{M}||z||_{L^2}^2\, k$$

where $\overline{M}$ is independent of k and m (as well as β and K). Since F_K satisfies (1.1) with a constant $\overline{\mu}$ independent of K, (1.11) shows that

$$(1.12) \qquad k(a_1||z||_{L^{\overline{\mu}}}^{\overline{\mu}} - a_3) \le \overline{M}||z||_{L^2}^2$$

By the Hölder inequality we find that

$$k(a_4||z||^{\bar{\mu}}_{L^2} - a_3) \leq \bar{M}||z||^2_{L^2} \tag{1.13}$$

which implies that

$$||z||_{L^2} \leq \bar{M}_1 \tag{1.14}$$

with $\bar{M}_1$ independent of m,k,β,K. Returning to (1.9) and using (1.14) yields

$$c_{mk} = I(u_{mk};k,\beta,K) \leq \bar{M}_2 k \tag{1.15}$$

with $\bar{M}_2$ independent of m,k,β,K. It follows that c_k satisfies the same estimate:

$$c_k = I(u_k;k,0,K) \leq \bar{M}_2 k \tag{1.16}$$

To complete the proof of Theorem 1.2, we will show that (1.16) is violated if more than finitely many solutions u_k correspond to the same function in the original t variables. To present the idea in its simplest setting, suppose first that all of the functions $u_k(x,\tau)$ are reparameterizations of $u_1(x,t)$. Then $u_k(x,\tau) = u_1(x,k\tau) = u_1(x,t) \equiv u(x,t)$. For $K = K(k)$ sufficiently large we have

$$(1.17)\quad c_k = \int_0^{2\pi}\int_0^{\pi} [\tfrac{1}{2} u_{k\tau}^2 - k^2(\frac{u_{kx}^2}{2} + F(x,k\tau,u_k))]\, dx\, d\tau$$

$$= k \int_0^{2\pi k} \int_0^{\pi} [\tfrac{1}{2}(u_t^2 - u_x^2) - F(x,t,u)]\, dx\, dt$$

$$= k^2 \int_0^{2\pi} \int_0^{\pi} [\tfrac{1}{2}(u_t^2 - u_x^2) - F(x,t,u)]\, dx\, dt$$

$$= k^2 c_1$$

since u is 2π periodic in t. The positivity of c_1 and (1.17) show that c_k tends to infinity like k^2 contrary to the bound (1.16). This argument shows (1.3) has at least one $2\pi k$ periodic solution distinct from $u_1(x,t)$.

For the general case we argue similarly. Suppose two solutions $u_j(x,\tau)$ and $u_k(x,\tau)$ correspond to the same function of (x,t), i.e. $u_j(x,\tau) = u_j(x,\frac{t}{j}) \equiv v(x,t) \equiv u_k(x,\frac{t}{k})$. Thus $u_j(x,\tau) = v(x,j\tau)$ and $u_k(x,\tau) = v(x,k\tau)$. Since $v(x,t)$ is both $2\pi j$ and $2\pi k$ periodic in t, there are $j_1, k_1, \sigma \in \mathbb{N}$ such that $j = \sigma j_1$, $k = \sigma k_1$ and v is $2\pi\sigma$ periodic in t. (We can take σ to be the greatest common divisor of j and k). Arguing as in (1.17) yields

$$(1.18)\quad c_k = k \int_0^{2\pi k} \int_0^{\pi} [\tfrac{1}{2}(v_t^2 - v_x^2) - F(x,t,v)]\, dx\, dt$$

$$= \frac{k^2}{\sigma} \int_0^{2\pi\sigma} \int_0^{\pi} [\tfrac{1}{2}(v_t^2 - v_x^2) - F(x,t,v)]\, dx\, dt$$

$$\equiv \frac{k^2}{\sigma} A$$

and

(1.19) $$c_j = \frac{j^2}{\sigma} A$$

Thus if there is a sequence u_{k_i} of solutions of (1.4) corresponding to the same function v, by (1.18) - (1.19) we have

(1.20) $$c_{k_i} = \frac{k_i^2}{\sigma} A$$

where $\sigma \in \mathbb{N}$ is the greatest common divisor of $\{k_i\}$. Hence $c_{k_i} \to \infty$ like k_i^2 contrary to (1.16) and the proof of Theorem 1.2 is complete.

<u>Remark 1.21</u>: Note that if $F(x,t,r)$ and F_K satisfy

$$F, F_K \geq a_1 |r|^\nu$$

for some $\nu > 2$, it follows from (1.11) that

$$||z||_{L^2} \leq a_5 \, k^{-\frac{1}{\nu-2}}$$

and therefore

$$c_k \leq a_6 \, k^{1-\frac{2}{\nu-2}} = a_6 \, k^{\frac{\nu-4}{\nu-2}}$$

Thus if $\nu < 4$, $c_k \to 0$ as $k \to \infty$. Further restrictions on F (as in [1]) imply $u_k \to 0$ as $k \to \infty$.

<u>Remark 1.22</u>: Existence of infinitely many distinct subharmonic solutions was also established in [1] for a family of subquadratic Hamiltonian systems, i.e. Hamiltonian systems where H grows less rapidly than quadratically as $|z| \to \infty$. There are several existence theorems for periodic solutions of semilinear wave equations in which the primitive of the forcing term is subquadratic [6-10]. We believe the conclusions of this paper carry over to the subquadratic case via the arguments used here and in [1].

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